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On a two-parameter generalization of Jacobsthal numbers and its graph interpretation

Abstract. In this paper we introduce a two-parameter generalization of the classical Jacobsthal numbers ((s,p)-Jacobsthal numbers). We present some properties of the presented sequence, among others Binet's formula, Cassini's identity, the generating function. Moreover, we give a graph interpretation of (s,p)-Jacobsthal numbers, related to independence in graphs.

1. Introduction. The Jacobsthal sequence $\{J_n\}$ is defined by the second order linear recurrence

(1) $$J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2$$

with $J_0 = 0$, $J_1 = 1$. The Binet's formula of this sequence has the following form

$$J_n = \frac{1}{3}(2^n - (-1)^n) \quad \text{for } n \geq 0.$$

Moreover, the explicit closed form expression for numbers J_n is

$$J_n = \sum_{r=0}^{[\frac{n-1}{2}]} \binom{n-1-r}{r} 2^r \quad \text{for } n \geq 0.$$

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Other interesting properties of Jacobsthal numbers are given in [6]. There are many generalizations of this sequence in the literature. The second order recurrence (1) has been generalized in two ways: first, by preserving the initial conditions and second, by preserving the recurrence relation. We recall some of such generalizations:

1) k-Jacobsthal sequence $\{j_{k,n}\}$ [5], $j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $j_{k,0} = 0$, $j_{k,1} = 1$,
2) k-Jacobsthal sequence $\{J_{k,n}\}$ [3], $J_{k,n+1} = J_{k,n} + kJ_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $J_{k,0} = 0$, $J_{k,1} = 1$,
3) generalized Jacobsthal p-sequence $\{J_p\}$ [1], for any $p \in \mathbb{Z}^+$ and $n > p+1$ $J_p(n) = J_p(n-1) + 2J_p(n-p-1)$ with initial conditions $J_p(1) = J_p(2) = \ldots = J_p(p+1) = 1$,
4) (s,t)-Jacobsthal sequence $\{\hat{j}_n(s,t)\}$ [8], $\hat{j}_n(s,t) = s\hat{j}_{n-1}(s,t) + 2t\hat{j}_{n-2}(s,t)$ for $n \geq 2$ with $\hat{j}_0(s,t) = 0$ and $\hat{j}_1(s,t) = 1$, for real numbers s,t, $s > 0$, $t \neq 0$ and $s^2 + 8t > 0$,
5) Jacobsthal sequence $\{J(d,t,n)\}$ [7], $J(d,t,n) = J(d,t,n-1) + tJ(d,t,n-d)$ for $n \geq d$ with $J(d,t,0) = 1$, $J(d,t,n) = 1$ for $n = 1,\ldots,d$, $t \geq 1, d \geq 2$.

In this paper we introduce a new generalization of the classical Jacobsthal numbers. Unlike other variations, this generalization depends on two integer parameters used in the recurrence relation (1). Let $n,s,p \geq 0$ be integers. We define (s,p)-Jacobsthal sequence $\{J_n(s,p)\}$ by the following recurrence

(2) $$J_n(s,p) = 2^{s+p}J_{n-1}(s,p) + (2^{2s+p} + 2^{s+2p})J_{n-2}(s,p) \text{ for } n \geq 2$$

with initial conditions $J_0(s,p) = 1$, $J_1(s,p) = 2^s + 2^p + 2^{s+p}$.

For $s = p = 0$ we obtain $J_n(0,0) = J_{n+2}$.

We will describe the terms of the sequence $\{J_n(s,p)\}$ explicitly by using a generalization of Binet's formula. Moreover, we will present some identities for (s,p)-Jacobsthal numbers, which generalize known results for the classical Jacobsthal numbers.

2. A graph interpretation of (s,p)-Jacobsthal numbers. In general we use the standard terminology and notation of graph theory, see [2]. In this section, we will present an interpretation of (s,p)-Jacobsthal numbers related to independence in graphs. Let G be a finite, undirected, simple graph with vertex set $V(G)$ and edge set $E(G)$. Recall that a subset S of $V(G)$ is an independent set of G if no two vertices of S are adjacent in G. Moreover, every one-element subset of $V(G)$ and the empty set are independent sets of G. The number of independent sets of a graph G is denoted by $NI(G)$. In the chemical literature the number of independent sets of a graph G is called the Merrifield–Simmons index of G and is denoted by $\sigma(G)$ ([4]). The numbers $J_n(s,p)$ have the graph interpretation directly related to the Merrifield–Simmons index.

Consider a graph $H_n^{s,p}$ (Figure 1), where $n \geq 1$, $s, p \geq 0$.

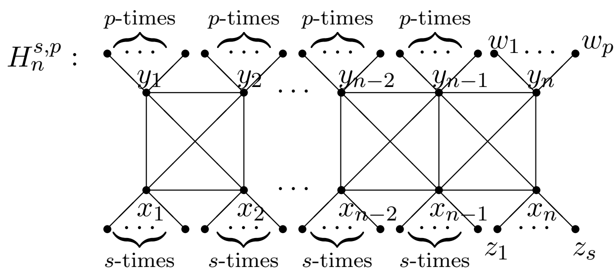

Figure 1.

Theorem 1. *Let n, s, p be integers, $n \geq 1$, $s, p \geq 0$. Then*

$$\sigma(H_n^{s,p}) = J_n(s,p).$$

Proof. In the beginning we will determine the number of independent sets of graphs $H_1^{s,p}$ and $H_2^{s,p}$. Assume that vertices of the graphs are numbered as in Figure 1. Denote by $L(x)$ the set of pendant vertices attached to the vertex x. Let $n = 1$. Assume that S is any independent set of $H_1^{s,p}$. Consider two cases.

Case 1. $y_1 \in S$.
Then $x_1, w_1, \dots, w_p \notin S$. Hence $S = \{y_1\} \cup Z$, where Z is any subset of the set $\{z_1, \dots, z_s\}$.

Case 2. $y_1 \notin S$. Consider two possibilities.

2.1. $x_1 \in S$.
Then $S = \{x_1\} \cup W$, where W is any subset of the set $\{w_1, \dots, w_p\}$.

2.2. $x_1 \notin S$.
Then $S = Z \cup W$.

Finally, we have $\sigma(H_1^{s,p}) = 2^s + 2^p + 2^{s+p} = J_1(s,p)$.

In the same manner we can obtain

$$\begin{aligned}\sigma(H_2^{s,p}) &= 2^{2p+s} + 2^{p+2s} + 2^{s+p}(2^s + 2^p + 2^{s+p}) \\ &= 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p} = J_2(s,p).\end{aligned}$$

Let $n \geq 3$. Assume that S is any independent set of $H_n^{s,p}$. Consider two cases.

Case 1. $y_n \in S$.
Let $\mathcal{S}_1$ be a family of all independent sets S of the graph $H_n^{s,p}$ such that $y_n \in S$. Then $x_n, x_{n-1}, y_{n-1}, w_1, \dots, w_p \notin S$. Hence $S = S' \cup \{y_n\} \cup S_1 \cup S_2 \cup S_3$, where S' is any independent set of the graph $H_n^{s,p} \setminus \{x_n, x_{n-1}, y_n, y_{n-1}\} \setminus (L(x_n) \cup L(x_{n-1}) \cup L(y_n) \cup L(y_{n-1}))$, isomorphic to $H_{n-2}^{s,p}$, $S_1 \subset L(x_n)$, $S_2 \subset L(x_{n-1})$, $S_3 \subset L(y_{n-1})$. Hence by the fundamental combinatorial statements we have $|\mathcal{S}_1| = 2^p \cdot (2^s)^2 \sigma(H_{n-2}^{s,p})$.

Case 2. $y_n \notin S$.

Let $\mathcal{S}_2$ be a family of all independent sets S of the graph $H_n^{s,p}$ such that $y_n \notin S$. Consider two possibilities.

2.1. $x_n \notin S$.

Then $S = S'' \cup S_1 \cup S_4$, where S'' is any independent set of the graph $H_n^{s,p} \setminus \{x_n, y_n\} \setminus (L(x_n) \cup L(y_n))$, isomorphic to $H_{n-1}^{s,p}$, $S_1 \subset L(x_n)$, $S_4 \subset L(y_n)$.

2.2. $x_n \in S$.

Then $S = S' \cup \{x_n\} \cup S_2 \cup S_3 \cup S_4$, where S' is any independent set of the graph $H_n^{s,p} \setminus \{x_n, x_{n-1}, y_n, y_{n-1}\} \setminus (L(x_n) \cup L(x_{n-1}) \cup L(y_n) \cup L(y_{n-1}))$, isomorphic to $H_{n-2}^{s,p}$.

Consequently, $|\mathcal{S}_2| = 2^s \cdot 2^p \sigma(H_{n-1}^{s,p}) + (2^p)^2 \cdot 2^s \sigma(H_{n-2}^{s,p})$.

Finally, for $n \geq 3$ we obtain

$$\sigma(H_n^{s,p}) = |\mathcal{S}_1| + |\mathcal{S}_2| = 2^{s+p}\sigma(H_{n-1}^{s,p}) + (2^{2s+p} + 2^{2p+s})\sigma(H_{n-2}^{s,p})$$

with $\sigma(H_1^{s,p}) = 2^s + 2^p + 2^{s+p}$ and $\sigma(H_2^{s,p}) = 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p}$, which ends the proof. □

Corollary 2. *Let* $n \geq 1$. *Then* $\sigma(H_n^{0,0}) = J_n(0,0) = J_{n+2}$.

3. Some identities for (s,p)-Jacobsthal numbers. The characteristic equation, associated with the recurrence relation (2) is

$$r^2 - 2^{s+p}r - (2^{2s+p} + 2^{s+2p}) = 0 \tag{3}$$

with roots

$$r_1 = 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}, \tag{4}$$

$$r_2 = 2^{s+p-1} - \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}. \tag{5}$$

Note that

$$r_1 + r_2 = 2^{s+p}, \tag{6}$$

$$r_1 r_2 = -(2^{2s+p} + 2^{s+2p}), \tag{7}$$

$$r_1 - r_2 = \sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}. \tag{8}$$

The general formula of (s,p)-Jacobsthal sequence can be written by the following identity

$$J_n(s,p) = c_1 r_1^n + c_2 r_2^n$$

for some constants c_1, c_2. Using initial conditions $J_0(s,p) = 1$, $J_1(s,p) = 2^s + 2^p + 2^{s+p}$, we get the system of two linear equations

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 r_1 + c_2 r_2 = 2^s + 2^p + 2^{s+p}. \end{cases}$$

Solving the system, we obtain

$$(9)\qquad \begin{aligned} c_1 &= \frac{2^s + 2^p + 2^{s+p} - 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}, \\ c_2 &= \frac{2^{s+p-1} - 2^s - 2^p - 2^{s+p} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}. \end{aligned}$$

Hence we get the following result.

Proposition 3 (Binet's formula). *Let $n, s, p \geq 0$. Then the n-th (s,p)-Jacobsthal number is given by*

$$(10)\qquad \begin{aligned} J_n(s,p) = &\frac{(2^s + 2^p + 2^{s+p} - 2^{s+p-1} + \frac{1}{2}\sqrt{\Delta})r_1^n}{\sqrt{\Delta}} \\ &+ \frac{(2^{s+p-1} - 2^s - 2^p - 2^{s+p} + \frac{1}{2}\sqrt{\Delta})r_2^n}{\sqrt{\Delta}}, \end{aligned}$$

where $\Delta = 4^{s+p} + 2^{s+p+2}(2^s + 2^p)$, $r_1 = 2^{s+p-1} + \frac{1}{2}\sqrt{\Delta}$, $r_2 = 2^{s+p-1} - \frac{1}{2}\sqrt{\Delta}$.

Using Binet's formula, we can get some identities for (s,p)-Jacobsthal numbers.

Theorem 4 (Cassini's identity). *Let n, s, p be integers, $n \geq 1$, $s, p \geq 0$. Then*

$$J_{n+1}(s,p)J_{n-1}(s,p) - J_n^2(s,p) = (-1)^n(2^s + 2^p)^2(2^{2s+p} + 2^{s+2p})^{n-1}.$$

Proof. By formula (10) we get

$$\begin{aligned} J_{n+1}(s,p)&J_{n-1}(s,p) - J_n^2(s,p) \\ &= (c_1r_1^{n+1} + c_2r_2^{n+1})(c_1r_1^{n-1} + c_2r_2^{n-1}) - (c_1r_1^n + c_2r_2^n)^2 \\ &= c_1c_2r_1^{n+1}r_2^{n-1} + c_1c_2r_2^{n+1}r_1^{n-1} - 2c_1c_2r_1^nr_2^n \\ &= c_1c_2(r_1r_2)^n(\frac{r_1}{r_2} + \frac{r_2}{r_1} - 2) = c_1c_2(r_1r_2)^{n-1}(r_1 - r_2)^2. \end{aligned}$$

By simple calculations we obtain

$$(11)\qquad c_1c_2 = \frac{-(2^s + 2^p)^2}{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}.$$

Using formulas (7), (8) and (11), we have

$$J_{n+1}(s,p)J_{n-1}(s,p) - J_n^2(s,p) = (-1)^n(2^s + 2^p)^2(2^{2s+p} + 2^{s+2p})^{n-1}.$$

□

Proposition 5. *Let n, s, p be integers, $n \geq 1$, $s, p \geq 0$. Then*

$$\lim_{n\to\infty} \frac{J_{n+1}(s,p)}{J_n(s,p)} = 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}.$$

Proof. By formula (10) we have

$$\lim_{n\to\infty}\frac{J_{n+1}(s,p)}{J_n(s,p)}=\lim_{n\to\infty}\frac{c_1{r_1}^{n+1}+c_2r_2^{n+1}}{c_1{r_1}^n+c_2r_2^n}=\lim_{n\to\infty}\frac{c_1r_1+c_2r_2(\frac{r_2}{r_1})^n}{c_1+c_2(\frac{r_2}{r_1})^n}.$$

Since $\lim\limits_{n\to\infty}(\frac{r_2}{r_1})^n=0$, we have

$$\lim_{n\to\infty}\frac{J_{n+1}(s,p)}{J_n(s,p)}=r_1=2^{s+p-1}+\frac{1}{2}\sqrt{4^{s+p}+2^{s+p+2}(2^s+2^p)}.$$

□

Theorem 6 (summation formula).

$$\sum_{i=0}^{n-1}J_i(s,p)=\frac{J_n(s,p)+(2^{2s+p}+2^{s+2p})J_{n-1}(s,p)-1-2^s-2^p}{2^{s+p}(1+2^s+2^p)-1}. \tag{12}$$

Proof. By Binet's formula (10) we have

$$\begin{aligned}\sum_{i=0}^{n-1}J_i(s,p)&=\sum_{i=0}^{n-1}(c_1r_1^n+c_2r_2^n)=c_1\frac{1-r_1^n}{1-r_1}+c_2\frac{1-r_2^n}{1-r_2}\\&=\frac{c_1+c_2-(c_1r_2+c_2r_1)-(c_1r_1^n+c_2r_2^n)+r_1r_2(c_1r_1^{n-1}+c_2r_2^{n-1})}{(1-r_1)(1-r_2)}\\&=\frac{c_1+c_2-(c_1r_2+c_2r_1)-J_n(s,p)+r_1r_2J_{n-1}(s,p)}{1-(r_1+r_2)+r_1r_2}.\end{aligned}$$

By formulas (4), (5) and (9) we obtain

$$c_1r_2+c_2r_1=-(2^s+2^p). \tag{13}$$

Using (6), (7) and (13), we get

$$\sum_{i=0}^{n-1}J_i(s,p)=\frac{1+2^s+2^p-J_n(s,p)-(2^{2s+p}+2^{s+2p})J_{n-1}(s,p)}{1-2^{s+p}-2^{2s+p}-2^{s+2p}}.$$

Hence

$$\sum_{i=0}^{n-1}J_i(s,p)=\frac{J_n(s,p)+(2^{2s+p}+2^{s+2p})J_{n-1}(s,p)-1-2^s-2^p}{2^{s+p}(1+2^s+2^p)-1}.$$

□

Corollary 7. *For $s=p=0$ we get the well-known identity for the classical Jacobsthal numbers*

$$\sum_{i=0}^{n-1}J_i=\frac{J_{n+2}+2J_{n+1}-3}{2}.$$

The next theorem presents the generating function of (s,p)-Jacobsthal sequence.

Theorem 8. *The generating function of the sequence* $\{J_n(s,p)\}$ *has the following form*

$$f(x) = \frac{1+(2^s+2^p)x}{1-2^{s+p}x-(2^{2s+p}+2^{s+2p})x^2}.$$

Proof. Let $f(x) = \sum_{n=0}^{\infty} J_n(s,p)x^n$. Then, by recurrence relation (2), we have

$$\begin{aligned}
f(x) &= J_0(s,p) + J_1(s,p)x + \sum_{n=2}^{\infty} J_n(s,p)x^n \\
&= 1 + (2^s+2^p+2^{s+p})x \\
&\qquad + \sum_{n=2}^{\infty} \left(2^{s+p}J_{n-1}(s,p) + (2^{2s+p}+2^{s+2p})J_{n-2}(s,p)\right) x^n \\
&= 1 + (2^s+2^p+2^{s+p})x \\
&\qquad + 2^{s+p}\sum_{n=2}^{\infty} J_{n-1}(s,p)x^n + (2^{2s+p}+2^{s+2p})\sum_{n=2}^{\infty} J_{n-2}(s,p)x^n \\
&= 1 + (2^s+2^p+2^{s+p})x \\
&\qquad + 2^{s+p}x\sum_{n=1}^{\infty} J_n(s,p)x^n + (2^{2s+p}+2^{s+2p})x^2\sum_{n=0}^{\infty} J_n(s,p)x^n \\
&= 1 + (2^s+2^p+2^{s+p})x \\
&\qquad + 2^{s+p}x\sum_{n=0}^{\infty} J_n(s,p)x^n - 2^{s+p}x + (2^{2s+p}+2^{s+2p})x^2 f(x).
\end{aligned}$$

Thus

$$f(x) = 1 + (2^s+2^p)x + 2^{s+p}xf(x) + (2^{2s+p}+2^{s+2p})x^2 f(x).$$

Hence

$$f(x) = \frac{1+(2^s+2^p)x}{1-2^{s+p}x-(2^{2s+p}+2^{s+2p})x^2},$$

which ends the proof. □

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