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Estimates for polynomials in the unit disk with varying constant terms

To the memory of Professor Jan Krzyż, friend and teacher.

Abstract. Let $||\cdot||$ be the uniform norm in the unit disk. We study the quantities $M_n(\alpha) := \inf(||zP(z)+\alpha|| - \alpha)$ where the infimum is taken over all polynomials P of degree $n-1$ with $||P(z)|| = 1$ and $\alpha > 0$. In a recent paper by Fournier, Letac and Ruscheweyh (Math. Nachrichten **283** (2010), 193–199) it was shown that $\inf_{\alpha>0} M_n(\alpha) = 1/n$. We find the exact values of $M_n(\alpha)$ and determine corresponding extremal polynomials. The method applied uses known cases of *maximal ranges* of polynomials.

1. Introduction. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. $\mathcal{P}_n$ denotes the set of complex polynomials of degree n and $||\cdot||$ stands for the uniform norm in $\mathbb{D}$. In a recent paper, R. Fournier, G. Letac and S. Ruscheweyh [4] proved the following theorem.

Theorem A. *For $P \in \mathcal{P}_{n-1}$ and $\alpha > 0$ we have*

$$||P|| \leq n\,(||z\,P(z)+\alpha|| - \alpha)\,. \tag{1.1}$$

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For no n and no α the factor n can be replaced by anything smaller without violating the conclusion. On the other hand, the only polynomial for which we have equality in (1.1) *is $P \equiv 0$.*

In this paper we are interested in a refinement of (1.1), namely in obtaining a bound replacing the factor n by something that depends on α. The statement in Theorem A, namely that the bound n cannot be replaced by anything smaller for any α, is only due to the fact that the inequality does not change if we multiply it by $\varepsilon > 0$, replacing P by εP and α by $\varepsilon\alpha$. This, however, does not change at all the class of polynomials in question, but it changes the value of α! Therefore, if we want to fix α efficiently we have to impose another restriction as well, for instance on $||P||$. This will be assumed in the sequel.

We write $\mathcal{P}_n^*$ for the set of polynomials $P \in \mathcal{P}_n$ satisfying $||P|| = 1$. Then to pose the following problem makes sense.

Problem. *For $\alpha \geq 0$ and $n \in \mathbb{N}$ determine the numbers*

$$M_n(\alpha) \coloneqq \inf_{P \in \mathcal{P}_{n-1}^*} (||zP(z) + \alpha|| - \alpha).$$

Remark 1.1. It follows from Theorem A that

$$\inf_{\alpha > 0} M_n(\alpha) = \frac{1}{n}. \tag{1.2}$$

Using this result and the definition of M_n, we obtain the following general inequality satisfied by the function M_n:

$$\frac{1}{n} \leq M_n\left(\frac{|P(0)|}{||P(z) - P(0)||}\right) \leq \frac{||P|| - |P(0)|}{||P(z) - P(0)||} \leq 1,$$

valid for every non-constant polynomial $P \in \mathcal{P}_n$. By definition it is clear that in this inequality the function M_n cannot be replaced by any larger one. The real problem of course remains: the explicit determination of this function M_n.

By using special choices for the polynomial P, namely

$$P(z) \equiv 1 \text{ respectively } P(z) = F_n(z) \coloneqq \frac{-2}{n(n+1)} \sum_{k=0}^{n-1} (n-k) z^k,$$

one easily verifies (using the relation $\operatorname{Re} z\, F_n(z) < \frac{1}{n}$ for $z \in \mathbb{D}$) that

$$M_n(\alpha) \leq \min\left\{1, \frac{1}{n} + \frac{1}{2\alpha}\right\}. \tag{1.3}$$

This estimate is reasonable for large values of α, but not satisfactory for small ones. A simple inspection of the definition gives

$$M_n(\alpha) \geq |1 - \alpha| - \alpha, \tag{1.4}$$

and, surprisingly, it turns out that for small values of $\alpha \geq 0$ this is indeed sharp.

In the sequel we are going to determine the precise values of $M_n(\alpha)$ for all $\alpha \geq 0$ and $n \in \mathbb{N}$.

The numbers

$$\rho_n := \left(\cos\frac{\pi}{n+1}\right)^{-n-1}, \quad n \in \mathbb{N}, \tag{1.5}$$

will be used throughout this discussion. A simple calculation gives

$$1 < \frac{n+1}{n-1} < \rho_n = 1 + \frac{\pi^2}{2n} + \mathcal{O}(n^{-2}). \tag{1.6}$$

Theorem 1.1. *Let $n \in \mathbb{N}$ be fixed. Then the following statements are valid.*

(i) *$M_n(\alpha)$ is a differentiable, strictly decreasing and convex function of α in $0 \leq \alpha < \infty$ with $M_n(0) = 1$ and $\lim_{\alpha\to\infty} M_n(\alpha) = \frac{1}{n}$.*

(ii) *Let $\alpha > \frac{1}{1+\rho_n}$. Then we have*

$$M_n(\alpha) = \alpha\,(s_n(\alpha) - 1), \tag{1.7}$$

where $s = s_n(\alpha)$ is the unique solution of the equation

$$s\,T_{n+1}(s^{-1/(n+1)}) = 1 - \frac{1}{\alpha}, \quad 1 < s < \rho_n\,, \tag{1.8}$$

Here T_{n+1} denotes the Chebychev polynomial of the first kind of degree $n+1$. In this range of α the only extremal polynomials $P \in \mathcal{P}^*_{n-1}$ with $M_n(\alpha) = ||zP(z) + \alpha|| - \alpha$ are

$$P(z) = \alpha\frac{Q_{n,\rho}(xz) - 1}{z}\,, \qquad |x| = 1, \tag{1.9}$$

where $\rho = \cos(s_n(\alpha))^{-n-1}$ and

$$Q_{n,\rho}(z^2) = \frac{-\rho\, z^{2n+3}}{n+1}\,\frac{d}{dz}\left\{z^{-n-1}\,T_{n+1}\left(\rho^{-1/(n+1)}\,\frac{1+z^2}{2z}\right)\right\}. \tag{1.10}$$

(iii) *For $0 \leq \alpha \leq \frac{1}{1+\rho_n}$ we have*

$$M_n(\alpha) = 1 - 2\alpha.$$

Extremal polynomials for this case are for instance

$$P(z) = \frac{Q_{n,\rho_n}(xz) - 1}{(1+\rho_n)z}, \quad |x| = 1, \tag{1.11}$$

but there are others as well.

The functions $Q_{n,\rho}(z)$ are indeed polynomials of degree n and satisfy $Q_{n,\rho}(0) = 1$. They have been introduced in [5] by Ruscheweyh and Varga and were later used by Córdova and Ruscheweyh [2].

An immediate consequence of Theorem 1.1 is the following explicit evaluation.

Corollary 1.1. *For $n \in \mathbb{N}$ we have*

$$M_n(1) = \left(\cos\left(\frac{\pi}{2n+2}\right)\right)^{-n-1} - 1 = \frac{\pi^2}{8n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{1.12}$$

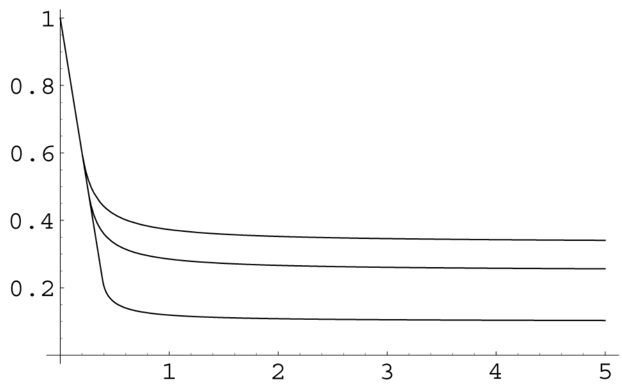

FIGURE 1. $M_n(\alpha)$ for $n = 3, 4, 10$

Remark 1.2. It is clear from Theorem 1.1 that the function

$$\tilde{M}_n(\alpha) := \inf_{P \in \mathcal{P}^*_{n-1}} ||zP(z) + \alpha|| = M_n(\alpha) + \alpha$$

is also convex and differentiable in $\alpha \in [0, \infty)$, and has a minimum at a point α_n slightly larger than $1/(1+\rho_n)$ with a value of $\tilde{M}_n(\alpha_n)$ slightly larger than $\frac{1}{2}$. Using Theorem 1.1, these values can be evaluated numerically.

2. The method: maximal polynomial ranges. The results in this paper are based on previous ones on the maximal polynomial ranges for circular domains, see [2]. We briefly explain what this method is about. Let Ω be a domain in the complex plane $\mathbb{C}$ with $1 \in \Omega$. For $n \in \mathbb{N}$ we define

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \,:\, P(0) = 1,\ P(\mathbb{D}) \subset \Omega\}$$

and then

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D})$$

as the *nth maximal (polynomial) range* of Ω. This concept and basic results are due to Córdova and Ruscheweyh [3], for a survey see [1]. For the special circular domains

$$\Omega_\rho : \{z \,:\, |z| < \rho\}, \quad \rho > 1,$$

the maximal ranges $\Omega_{\rho,n}$ were described in [2]. We state the essential parts of those results. Let $Q_{\rho,n}$ be the polynomials defined in (1.10) and ρ_n as in (1.5).

Theorem B ([2, Theorem 1]). *Let $n \in \mathbb{N}$ be fixed. Then the following statements hold.*

(i) *For $1 < \rho \le \rho_n$ the set $\Omega_{\rho,n}$ consists of the interior domain of the Jordan curve consisting of the two arcs*

$$\mathcal{C}_1 := \left\{Q_{\rho,n}(e^{i\tau}) \,:\, |\tau| \le \tau_1\right\}, \tag{2.1}$$

$$\mathcal{C}_2 := \left\{\rho\, e^{i\tau} \,:\, |\tau| < \pi - \frac{n+1}{2}\tau_1\right\}, \tag{2.2}$$

where

$$\tau_1 := 2 \arccos\left(\rho^{1/(n+1)} \cos\left(\frac{\pi}{n+1}\right)\right). \tag{2.3}$$

(ii) *For $\rho > \rho_n$ we have*

$$\Omega_{\rho,n} = \Omega_\rho. \tag{2.4}$$

It follows from [5, (3.24)] and [2, Lemma 2.2] that for $n \in \mathbb{N}$

$$Q_{\rho,n} \in \mathcal{P}_n(\Omega_\rho), \quad 1 < \rho \le \rho_n. \tag{2.5}$$

We refer to [2] for some graphics showing the mapping properties of the polynomials $Q_{\rho,n}$ (which happen to be univalent in $\mathbb{D}$). In the sequel we write $K(c,s)$ for the open disk with center c and radius s.

Basic for the proof of Theorem 1.1 are the following inclusions.

Lemma 2.1. *For $n \in \mathbb{N}$ and $1 < \rho \le \rho_n$ we have*

$$\Omega_{\rho,n} \subset K(1, |Q_{\rho,n}(1) - 1|). \tag{2.6}$$

Actually, as we will see in the proof below, (2.6) is equivalent to

$$|Q_{\rho,n}(z) - 1| \le |Q_{\rho,n}(1) - 1|, \quad |z| \le 1. \tag{2.7}$$

and there is numerical evidence for the following

Conjecture 2.1. *Let*

$$Q_{\rho,n}(z) = 1 + \sum_{k=1}^{n} a_k(\rho,n) z^k.$$

Then, for $k = 1, \dots, n$,

$$a_k(\rho,n) < 0, \quad 1 < \rho \le \rho_n.$$

This would immediately imply (2.7) and therefore Lemma 2.6. Unfortunately, we have not been able to verify the conjecture so far, so we have to use another method to prove Lemma 2.1.

Proof. Actually we only need to prove $\partial\Omega_{\rho,n} \in \overline{K(1, |Q_{\rho,n}(1) - 1|)}$. This boundary consists of the points $z = e^{i\tau}$, $|\tau| \le |\tau_1|$, and the circular arc $\mathcal{C}_2$.

However, our claim concerning this latter arc is actually contained in the first one: one needs to show that

$$|\rho e^{i\tau} - 1| \leq |Q_{\rho,n}(1) - 1|, \quad |\tau| \leq |\tau_1|,$$

and it is clear that the worst case in this inequality belongs to the choice $\tau = \tau_1$, which corresponds to the endpoint of $\mathcal{C}_1$, and will be established in that context.

It remains to prove

$$|Q_{\rho,n}(e^{i\tau}) - 1| \leq |Q_{\rho,n}(1) - 1|, \quad 0 \leq \tau \leq \tau_1, \tag{2.8}$$

for all $n \in \mathbb{N}$ (there exists symmetry w.r.t. $\tau = 0$).

Let

$$f(\tau) := T_{n+1}\left(\rho^{-1/(n+1)} \cos\frac{\tau}{2}\right).$$

Then (see [2, (2.10)])

$$Q_{\rho,n}(e^{i\tau}) = \rho e^{i\frac{n+1}{2}\tau}\left[f(\tau) + \frac{2i}{n+1} f'(\tau)\right]$$

and

$$e^{i\tau} Q'_{\rho,n}(e^{i\tau}) = \frac{(n+1)\rho}{2} e^{i\frac{n+1}{2}\tau}\left[f(\tau) + \frac{4}{(n+1)^2} f''(\tau)\right].$$

In a point τ where $|Q_{\rho,n}(e^{i\tau}) - 1|$ takes its maximum, we must have

$$\frac{e^{i\tau} Q'_{\rho,n}(e^{i\tau})}{Q_{\rho,n}(e^{i\tau}) - 1} = \frac{n+1}{2} \frac{f(\tau) + \frac{4}{(n+1)^2} f''(\tau)}{f(\tau) + \frac{2i}{n+1} f'(\tau) - \frac{1}{\rho} e^{-i\frac{n+1}{2}\tau}} > 0,$$

and this clearly implies

$$\frac{2}{n+1} f'(\tau) + \frac{1}{\rho} \sin\left(\frac{n+1}{2}\tau\right) = 0.$$

Hence we can restrict ourselves to such points τ which satisfy this latter condition. In those points we have

$$\begin{aligned} |Q_{\rho,n}(e^{i\tau}) - 1| &= \left|\rho\, f(\tau) - \cos\frac{n\tau}{2}\right| \\ &= \left|\rho\, T_{n+1}\left(\rho^{-1/(n+1)} \cos\frac{\tau}{2}\right) - T_{n+1}\left(\cos\frac{\tau}{2}\right)\right| \end{aligned} \tag{2.9}$$

Set $y := \rho^{-1/(n+1)}$ so that $y \in (\cos(\pi/(n+1)), 1)$. Writing $x = \cos(\tau/2)$, our restriction $0 \leq \tau \leq \tau_1$ becomes

$$y \geq xy \geq \cos\frac{\pi}{n+1}, \tag{2.10}$$

values for which we wish to establish

$$|y^{-n-1} T_{n+1}(xy) - T_{n+1}(x)| \leq |y^{-n-1} T_{n+1}(y) - T_{n+1}(1)|. \tag{2.11}$$

We rewrite this as follows.

$$\begin{aligned}
y^{-n-1}T_{n+1}(xy) - T_{n+1}(x) &= \int_1^y \left(\frac{d}{dt} t^{-n-1} T_{n+1}(xt)\right) dt \\
&= \int_1^y t^{-n-1}(-(n+1)T_{n+1}(xt) + xtT'_{n+1}(xt))dt \\
&= \int_1^y t^{-n-1} U_{n-1}(xt) dt,
\end{aligned} \tag{2.12}$$

where we used the general relation

$$-(n+1)T_{n+1}(z) + zT'_{n+1}(z) = U_{n-1}(z)$$

with U_{n-1} the Chebychev polynomial of the second kind of degree $n-1$. We are now left with the inequality

$$\left|\int_1^y t^{-n-1} U_{n-1}(xt) dt\right| \le \left|\int_1^y t^{-n-1} U_{n-1}(t) dt\right|, \tag{2.13}$$

subject to the restriction (2.10). The polynomial $U_{n-1}(z)$ has only real zeros, the largest being $z_0 = \cos(\pi/n)$, and is positive and strictly increasing for $z \ge z_0$. Therefore, we have

$$U_{n-1}(xt) \le U_{n-1}(t)$$

for all the values of t and xt in question. This establishes (2.13) and completes the proof of Lemma 2.1. □

3. Proof of Theorem 1.1.

Proof of part (ii). Let $n \in \mathbb{N}$ and $\alpha \ge \frac{1}{1+\rho_n}$ be fixed. Then, using the abbreviation K_α for $K(1, 1/\alpha)$,

$$\begin{aligned}
M &:= 1 + M_n(\alpha)/\alpha \\
&= \min\{||zP(z)+1|| : P \in \mathcal{P}_{n-1},\ ||P|| = 1/\alpha\} \\
&= \min\{||Q|| : Q \in \mathcal{P}_n, Q(0) = 1, Q(\mathbb{D}) \subset K_\alpha, \partial Q(\mathbb{D}) \cap \partial K_\alpha \ne \emptyset\} \\
&= \min\{\rho : \rho > 1,\ \Omega_{\rho,n} \subset K_\alpha,\ \partial\Omega_{\rho,n} \cap \partial K_\alpha \ne \emptyset\}.
\end{aligned} \tag{3.1}$$

$\Omega_{\rho,n}$ is growing with $\rho > 1$. By Lemma 2.1 this implies that the minimal ρ in (3.1) is the one for which

$$1 - Q_{\rho,n}(1) = \frac{1}{\alpha}, \tag{3.2}$$

and this makes sense only for $1 < \rho \le \rho_n$. Since

$$Q_{\rho_n,n}(1) = -\rho_n, \quad Q_{1,n}(1) = 1,$$

we have solutions in (3.2) only for $1 + \rho_n \ge 1/\alpha$, which coincides with our initial condition on α. Note that

$$Q_{\rho,n}(1) = \rho\, T_{n+1}\left(\rho^{\frac{-1}{n+1}}\right),$$

so that after a little calculation using (3.2) one arrives at (1.8) and then finally at the representation (1.7). Furthermore it follows from the proof of Lemma 2.1 that the all points of $\overline{\Omega_{\rho,n}}$ belong to the interior of $K(1, |Q_{\rho,n}(1)-1|)$, except for the point $Q_{\rho,n}(1)$ which lies on the boundary of that disk. It follows from the results in [2] that the polynomials $Q_{\rho,n}(xz)$ with $|x| = 1$ are the only ones within $\mathcal{P}_n(\Omega_\rho)$ reaching that point on the boundary of $K(1, |Q_{\rho,n}(1)-1|)$. Whenever another polynomial $P \in \mathcal{P}(\Omega_\sigma)$ assumes that same value we must have $\sigma > \rho$. This shows the uniqueness claimed in (1.9).

Proof of part (iii). Let $n \in \mathbb{N}$ and $\alpha \leq \frac{1}{1+\rho_n}\,(< \frac{1}{2})$ be fixed. Since $||zP(z)+\alpha|| \geq 1-\alpha$ for every polynomial in $P \in \mathcal{P}^*_{n-1}$, it is clear that we must have

$$M_n(\alpha) \geq 1 - 2\alpha.$$

To see that this is sharp, we show that $||zP(z)+\alpha|| - \alpha = 1-2\alpha$ for the functions (1.11). Since $||Q_{\rho_n,n}|| = \rho_n$ with $Q_{\rho_n,n}(1) = -\rho_n,\ Q_{\rho_n,n}(0) = 1$, we find

$$||P(z)|| = \frac{|Q_{\rho_n,n}(z)-1|}{1+\rho_n} \leq 1 = |P(1)|,$$

so that $P \in \mathcal{P}^*_{n-1}$ and

$$||zP(z)+\alpha|| - \alpha = \left|\left|\frac{Q_{\rho,n}-1}{1+\rho_n} + \alpha\right|\right| \leq \frac{\rho_n}{1+\rho_n} + \left|\alpha - \frac{1}{1+\rho_n}\right| - \alpha = 1-2\alpha$$

where the admissible range for α has been used. Therefore, $M_n(\alpha) = 1-2\alpha$ for that range.

Proof of part (i). $M_n(\alpha)$ is obviously continuous for $\alpha \geq 0$. First we show that it is decreasing with α. Indeed, for any $P \in \mathcal{P}_{n-1}$ and $\alpha, \varepsilon > 0$ we have

$$|zP(z)+\alpha+\varepsilon| - (\alpha+\varepsilon) \leq |zP(z)+\alpha| + \varepsilon - (\alpha+\varepsilon) = |zP(z)+\alpha| - \alpha, \quad z \in \mathbb{D},$$

and therefore,

$$||zP(z)+\alpha+\varepsilon|| - (\alpha+\varepsilon) \leq ||zP(z)+\alpha|| - \alpha,$$

which implies $M_n(\alpha+\varepsilon) \leq M_n(\alpha)$.

Let $n \in \mathbb{N}$ be fixed. It is clear that $M_n(\alpha) = 1-2\alpha$ is decreasing, differentiable and convex in the range $0 < \alpha < 1/(1+\rho_n)$ with

$$\lim_{\alpha \to 1/(1+\rho_n)-0} M_n'(\alpha) = -2.$$

Now let $\alpha \geq 1/(1+\rho_n)$. In this case, by Theorem 1.1 (ii), we have

$$M_n(\alpha) = \alpha(s_n(\alpha) - 1) \tag{3.3}$$

with $s := s_n(\alpha)$ the unique solution of

$$Q_{s,n}(1) = sT_{n+1}(\sigma) = 1 - \frac{1}{\alpha}, \quad 1 \leq s \leq \rho_n, \tag{3.4}$$

where $\sigma := s^{-1/(n+1)}$. That $s_n(\alpha)$ is uniquely determined follows immediately from the fact that $Q_{s,n}(1)$ is strictly decreasing in $1 \le s \le \rho_n$. This also implies that $s_n(\alpha)$ (and then also $M_n(\alpha)$) is differentiable in that range of α, and that $s_n'(\alpha) < 0$. To prove the differentiability of $M_n(\alpha)$ in the remaining point $\alpha = 1/(1+\rho_n)$, it will be enough to show that

$$\lim_{\alpha \to 1/(1+\rho_n)+0} M_n'(\alpha) = -2. \tag{3.5}$$

However, a differentiation of (3.3) w.r.t. α yields

$$\lim_{\alpha \to 1/(1+\rho_n)+0} s_n'(\alpha) = -(1+\rho_n)^2,$$

which gives (3.5) after differentiation of (3.3).

For the convexity of $M_n(\alpha)$ we only have to show $M_n''(\alpha) > 0$ for $\alpha > 1/(1+\rho_n)$. Differentiation of (3.4) gives

$$s_n'(\alpha) F(s_n(\alpha)) = \alpha^{-2},$$

where

$$F(s) := T_{n+1}(\sigma) - \sigma U_n(\sigma).$$

After another differentiation we obtain

$$M_n''(\alpha) = \alpha s_n''(\alpha) + 2 s_n'(\alpha) = \frac{-s_n'(\alpha)}{\alpha} \frac{F'(s_n(\alpha))}{F^2(s_n(\alpha))},$$

so that we are left with a proof of $F'(s) \ge 0$ in $1 \le s \le \rho_n$. Writing $G(y) := F(s)$ with $y = \arccos(\sigma(s))$ this transforms into the condition $G'(y) \ge 0$ for $0 \le y \le \frac{\pi}{n+1}$, and G has the following representation:

$$G(y) := \cos((n+1)y) - \cos(y) \frac{\sin((n+1)y}{\sin(y)}.$$

Some manipulation shows that the required inequality is equivalent to

$$(n+1)\cos(ny)\sin(y) \le \sin((n+1)y), \quad 0 \le y \le \frac{\pi}{n+1},$$

and to

$$\frac{1}{n+1} \sin((n+1)y) \le \frac{1}{n-1} \sin((n-1)y), \quad 0 \le y \le \frac{\pi}{n+1}.$$

This, however, is easily verified. □

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