doi: 10.17951/a.2016.70.2.1

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXX, NO. 2, 2016 SECTIO A 1–8

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On $(2-d)$-kernels in the cartesian product of graphs

Abstract. In this paper we study the problem of the existence of $(2-d)$-kernels in the cartesian product of graphs. We give sufficient conditions for the existence of $(2-d)$-kernels in the cartesian product and also we consider the number of $(2-d)$-kernels.

1. Introduction. In general we use the standard terminology and notation of graph theory, see [3]. Graphs $G = (V(G), E(G))$ considered in this paper are undirected, connected and simple. By P_n, $n \geq 2$ and C_n, $n \geq 3$ we mean a path and a cycle on n vertices, respectively. By $d_G(x, y)$ we denote the distance between vertices x and y in G being the length of the shortest path from x to y. Consequently $d_G(X, Y) = \min\{d_G(u, v) : u \in X, v \in Y\}$ means the distance between sets X and Y. A simple graph $G = G(V_1, V_2)$ is called *bipartite* if its vertex set can be partitioned into two disjoint subsets V_1, V_2 such that every edge has the form xy, where $x \in V_1$ and $y \in V_2$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. *The cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = V_1 \times V_2$ and $E(G_1 \times G_2) = \{(x_i, y_p)(x_j, y_q) : (x_i = x_j \text{ and } y_p y_q \in E(G_2)) \text{ or } (y_p = y_q \text{ and } x_i x_j \in E(G_1))\}$.

By the cartesian product of n graphs $G_1, \ldots, G_n$ we mean the cartesian product of G_n and $G_1 \times G_2 \times \ldots \times G_{n-1}$ denoted by $G_1 \times G_2 \times \ldots \times G_n$. If $n = 2$, then we obtain the definition of cartesian product of two graphs.

2010 *Mathematics Subject Classification.* 05C20, 05C69.
Key words and phrases. Independence, domination, cartesian product, $(2-d)$-kernel.

Theorem 1.1 ([12]). *A graph $G_1 \times G_2$ is bipartite if and only if G_1 and G_2 are bipartite.*

We say that a subset $D \subseteq V(G)$ is *dominating*, if every vertex of G is either in D or it is adjacent to at least one vertex of D. A subset $S \subseteq V(G)$ is *independent* if no two vertices of S are adjacent in G. A subset J being dominating and independent is a *kernel* of G.

H. Galeana-Sánchez and C. Hernández-Cruz played an important role in researching kernels in digraphs. During the last decades they studied not only kernels in digraphs but some of its generalizations, mainly kernels by monochromatic paths and (k, l)-kernels. Most of the existing results about kernels and their generalizations in digraphs were related to operations in digraphs and how the kernels are preserved. For results concerning kernels which were obtained quite recently see [4, 6, 7, 5, 8, 9, 10, 11].

In [17] A. Włoch introduced and studied the concept of a 2-*dominating kernel* (for convenience we will write shortly $(2-d)$-kernel). A set J is a $(2-d)$-kernel of a graph G if it is independent and 2-dominating, i.e. J is independent and each vertex from $V(G) \setminus J$ has at least two neighbours in J.

Not every graph possesses a $(2-d)$-kernel, for example a graph P_4 is a graph without $(2-d)$-kernel. In [1] it was proved that the problem of the existence of $(2-d)$-kernel is $\mathcal{NP}$-complete for a general graph.

Some results related to the existence of $(2-d)$-kernels in graphs can be found in [1], [2] and [17]. Moreover, in [1] the number of all $(2-d)$-kernels (denoted by $\sigma_{(2-d)}(G)$) in graphs was studied. In this paper we consider the problem of the existence of $(2-d)$-kernels in the cartesian product of graphs and also we consider the number of $(2-d)$-kernels in this graph product.

The topic of kernels in graphs product was studied for example in [14, 13, 15, 16, 17, 18, 19].

2. Main results. In this section we give some necessary and sufficient conditions for the existence of $(2-d)$-kernels in the cartesian product of graphs.

Theorem 2.1. *If G and H are connected bipartite graphs, then $G \times H$ has two disjoint $(2-d)$-kernels.*

Proof. Let $G = G(V_1, V_2)$ and $H = H(V_3, V_4)$ be bipartite. Then the set $V(G \times H)$ is the union of pairwise disjoint sets $V_1 \times V_3$, $V_1 \times V_4$, $V_2 \times V_3$ and $V_2 \times V_4$. We shall show that sets $J = (V_1 \times V_3) \cup (V_2 \times V_4)$ and $J^* = (V_2 \times V_3) \cup (V_1 \times V_4)$ are $(2-d)$-kernels of the graph $G \times H$. Firstly we shall show that J is a $(2-d)$-kernel. To prove that J is independent let us assume that $(x_i, y_j), (x_p, y_q) \in J$ and consider the following cases:

1.1 $(x_i, y_j), (x_p, y_q) \in V_1 \times V_3$.
If $i = p$, then by the definition of H and the cartesian product

$G \times H$ we have that $y_j, y_q \in V_3$, so $y_j y_q \notin E(H)$ and consequently $(x_i, y_j)(x_p, y_q) \notin E(G \times H)$.
If $j = q$, then we prove analogously with respect to the graph G.
Let $i \neq p$ and $j \neq q$. Then the definition of $G \times H$ immediately gives that $(x_i, y_j)(x_p, y_q) \notin E(G \times H)$.

1.2 $(x_i, y_j), (x_p, y_q) \in V_2 \times V_4$.
Then we prove in the same way as in 1.1.

1.3 $(x_i, y_j) \in V_1 \times V_3$ and $(x_p, y_q) \in V_2 \times V_4$.
Since G and H are bipartite, so $x_i x_p \notin E(G)$ and $y_j y_q \notin E(H)$. Then by the definition of $G \times H$ we have that $(x_i, y_j)(x_p, y_q) \notin E(G \times H)$.

Consequently J is independent. Now we shall prove that J is 2-dominating. Assume that $(x_i, y_j) \notin J$. It suffices to show that there are two vertices from the set J which dominate the vertex (x_i, y_j). We consider the following possibilities:

2.1 $(x_i, y_j) \in V_1 \times V_4$.
Then $x_i \in V_1$ and $y_j \in V_4$. Because H is connected and bipartite so there is a vertex, say $y_q \in V_3$ such that $y_j y_q \in E(H)$. Consequently by the definition of $G \times H$ we have that there is $(x_i, y_q) \in V_1 \times V_3 \subset V(G \times H)$. This means that $(x_i, y_q) \in J$ and $(x_i, y_j)(x_i, y_q) \in E(G \times H)$. Analogously with respect to graph G we can show that there exists a vertex $(x_p, y_j) \in V_2 \times V_3 \subset V(G \times H)$. So $(x_p, y_j) \in J$ and $(x_i, y_j)(x_p, y_j) \in E(G \times H)$. Thus every vertex from the set $V_1 \times V_4$ is at least 2-dominated by the set J.

2.2 $(x_i, y_j) \in V_2 \times V_3$,
Then we prove by using the same method as in 2.1.

Finally, the set J is a $(2-d)$-kernel of $G \times H$. In the same way we can prove that J^* is a $(2-d)$-kernel of $G \times H$ and it is obvious that $J \cap J^* = \emptyset$, which ends the proof. □

Using Theorems 1.1 and 2.1, we can prove the result for the cartesian product of n graphs.

Theorem 2.2. *Let $n \geq 2$ be an integer. If G_i is bipartite, for $i = 1, \ldots, n$, then $G_1 \times G_2 \times \ldots \times G_n$ has two disjoint $(2-d)$-kernels.*

From the above follow results for special bipartite graphs.

Corollary 2.3. *Let $n, m \geq 2$ be integers. A graph $P_n \times P_m$ has exactly two disjoint $(2-d)$-kernels J_1, J_2 and $J_1 \cup J_2 = V(P_n \times P_m)$. Moreover, $|J_1| = |J_2| = \frac{m \cdot n}{2}$ if $m \cdot n$ is even and $|J_1| = \lfloor \frac{m \cdot n}{2} \rfloor$ and $|J_2| = \lceil \frac{m \cdot n}{2} \rceil$, otherwise.*

Corollary 2.4. *Let $n, m \geq 2$ be integers. A graph $C_{2n} \times P_m$ has exactly two disjoint $(2-d)$-kernels J_1, J_2 and $J_1 \cup J_2 = V(C_{2n} \times P_m)$. Moreover, $|J_1| = |J_2| = m \cdot n$.*

Theorem 2.5. *Let $n, m \geq 3$ be integers. A graph $C_n \times C_m$ has a $(2-d)$-kernel if and only if n and m are even or $n = m$.*

Proof. Let n, m be as in the statement of the theorem. If n and m are even, then C_n and C_m are bipartite and $C_n \times C_m$ has a $(2-d)$-kernel by Theorem 2.1. Assume that $n = m$ and n is odd. We will prove that $C_n \times C_n$ has a $(2-d)$-kernel J. Let $V(C_n) = \{x_1, \ldots, x_n\}$, $n \geq 3$ with the numbering of vertices in the natural fashion. We can illustrate the construction of the set J in $C_n \times C_n$ using the matrix $A = [a_{ij}]_{n \times n}$ defined as follows

$$a_{ij} = \begin{cases} 1 \text{ if the vertex } (x_i, x_j) \in J, \\ 0 \text{ otherwise.} \end{cases}$$

Then $a_{ij} = 1$ if and only if

$$i - j \in \left\{-n + 2p; p = 1, \ldots, \frac{n-3}{2}\right\} \cup \left\{n - 2q - 1; q = 1, \ldots, \frac{n-1}{2}\right\}$$

and $a_{ij} = 0$ otherwise. For the explanation if $n = 3$, then

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $n \geq 5$ be odd. Then the matrix A has the form

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \ldots & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & \ldots & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \ldots & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & \ldots & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \ldots & 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Conversely suppose on the contrary that $C_n \times C_m$ has a $(2-d)$-kernel, say J where $m \neq n$ and n is odd.

Let m be even and n be odd. Then the graph C_m has two disjoint $(2-d)$-kernels J_1 and J_2. Let $J_i^{(p)}$, $i = 1, 2$ be a $(2-d)$-kernel in $C_m^{(p)}$, where $C_m^{(p)} \subset C_m \times C_n$ is a p-copy of C_m, $p = 1, \ldots, m$. It is clear that $J = J_1^{(1)} \cup J_2^{(2)} \cup J_1^{(3)} \cup \ldots \cup J_1^{(n)}$, because n is odd, but then $d_{C_m \times C_n}\left(J_1^{(1)}, J_1^{(n)}\right) = 1$, a contradiction with the independence of J.

Let now $m \neq n$ and m, n be odd. Since $m \neq n$, so without loss of the generality let $m > n$. Suppose that $C_m \times C_n$ has a $(2-d)$-kernel J. By definition of $C_m \times C_n$ and $(2-d)$-kernel it is obvious that in each copy $C_m^{(i)}$, $i = 1, \ldots, n$ of $C_m \times C_n$ we have to choose a maximal independent set J_i such that $|J_i| = \lfloor \frac{m}{2} \rfloor$, for all $i = 1, \ldots, n$ and J_i is a subset of $(2-d)$-kernel J. Using given earlier construction of $(2-d)$-kernel J in $C_m \times C_n$ which preserve 2-domination we observe that in copy $C_m^{(n)}$ for every maximal independent set J_n the set $\bigcup_{i=1}^{n} J_i$ is not independent, which is a contradiction with the assumption. □

Corollary 2.6. *Let $m, n \geq 3$ be integers. If m, n are even, then $\sigma_{(2-d)}(C_n \times C_m) = 2$. If n is odd, then $\sigma_{(2-d)}(C_n \times C_n) = 2n$.*

Corollary 2.6 follows by the proof of Theorem 2.5.

Theorem 2.7. *Let $m, n \geq 1$ be integers. A graph $K_n \times K_m$ has a $(2-d)$-kernel if and only if $n = m$.*

Proof. Let $V(K_n) = \{x_1, \ldots, x_n\}$, $n \geq 1$ and $V(K_m) = \{y_1, \ldots, y_m\}$, $m \geq 1$. If $m = n$, then it is easy to observe that the set $J = \{(x_i, y_i), i = 1, \ldots, n\}$ is $(2-d)$-kernel of the graph $K_n \times K_n$.

Suppose on the contrary that a graph $K_n \times K_m$ has a $(2-d)$-kernel J^* and $n > m$. Clearly in each copy of a complete graph K_n and K_m we can choose at most one vertex to the set J^*. Without loss of the generality let $(x_1, y_1) \in J^*$. Then in the copy $K_m^{(2)}$ we choose an arbitrary vertex (x_2, y_i) where $i \neq 1$ and $(x_2, y_i) \in J^*$. Analogously in the copy $K_m^{(3)}$ we choose a vertex $(x_3, y_j) \in J^*$ for $j \neq 1$ and $j \neq i$. Consequently in the copy $K_{n+1}^{(i)}$ for every vertex (x_{n+1}, y_p), $p = 1, \ldots, m$ there is a vertex (x_r, y_p) where $1 \leq r \leq m$. Then $J^* \cup \{(x_{n+1}, y_p), 1 \leq p \leq n\}$ is not independent, which is a contradiction with the assumption that J^* is a $(2-d)$-kernel.

Thus the theorem is proved. □

Corollary 2.8. *Let $n \geq 2$ be integer. Then $\sigma_{(2-d)}(K_n \times K_n) = n!$. Moreover, all $(2-d)$-kernels of $K_n \times K_n$ have the same cardinality n.*

Proof. Let $J \subset V(K_n \times K_n)$ be a $(2-d)$-kernel. Then it is clear that in the copy $K_n^{(1)}$ we can choose the vertex belonging to J on n ways. Moreover, in the copy $K_n^{(p)}$, $2 \leq p \leq n$ we can choose the vertex belonging to J on $(n-p+1)$ ways. This gives that $\sigma_{(2-d)}(K_n \times K_n) = n(n-1) \cdot \ldots \cdot 1 = n!$. □

Let $X \subset V(G)$. A graph G is X-$(2-d)$-*kernel critical* if $G \setminus X$ has a $(2-d)$-kernel.

Let $X, Y \subset V(G)$ be two disjoint subsets of G. A graph G is (X,Y)-$(2-d)$-*kernel critical* if Y is a $(2-d)$-kernel of $G \setminus X$.

If a graph G is (X,Y)-$(2-d)$-kernel critical and (Y,X)-$(2-d)$-kernel critical, then we will write that G is an $\{X,Y\}$-$(2-d)$-kernel critical.

Theorem 2.9. *Let $G = G(J_1, J_2)$ be a bipartite graph such that J_i, $i = 1, 2$ are $(2-d)$-kernels of G. Let H be a graph with $V_1, V_2 \subset V(H)$ such that H is $\{V_1, V_2\}$-$(2-d)$-kernel critical. Then $G \times H$ has at least two $(2-d)$-kernels.*

Proof. Let $G = G(J_1, J_2)$ and H be as in the statement of the theorem. If $V_1 \cup V_2 = V(H)$, then graphs G and H are bipartite and by Theorem 2.1, a graph $G \times H$ has two $(2-d)$-kernels. Suppose that $V_1 \cup V_2 \neq V(H)$. Let $R = V(H) \setminus (V_1 \cup V_2)$. Then the set $V(G \times H)$ is the union of pairwise disjoint sets $J_1 \times V_1$, $J_1 \times V_2$, $J_1 \times R$, $J_2 \times V_1$, $J_2 \times V_2$ and $J_2 \times R$. We will prove that set $J = (J_1 \times V_1) \cup (J_2 \times V_2)$ is $(2-d)$-kernel of the graph $G \times H$. Firstly we show that J is an independent set.

Let $(x_i, y_j), (x_p, y_q) \in J$ and consider the following cases:

1.1 $(x_i, y_j), (x_p, y_q) \in J_1 \times V_1$.
If $j = q$, then by the definition of G and the cartesian product $G \times H$ we have that $x_i, x_p \in J_1$, so $x_i, x_p \notin E(G)$ and consequently $(x_i, y_j)(x_p, y_q) \notin E(G \times H)$.
Let $i = p$. Since a graph H is $\{V_1, V_2\}$-$(2-d)$-kernel critical, then V_1 is $(2-d)$-kernel of $H \setminus V_2$ and this means that V_1 is an independent set of a graph H. So for all $y_j, y_q \in V_1$ we have that $y_j y_q \notin E(H)$ and finally $(x_i, y_j)(x_p, y_q) \notin E(G \times H)$.
Let $i \neq p$ and $j \neq q$. Then the definition of $G \times H$ immediately gives that $(x_i, y_j)(x_p, y_q) \notin E(G \times H)$.

1.2 $(x_i, y_j), (x_p, y_q) \in J_2 \times V_2$.
Then we prove in the same way as in 1.1.

1.3 $(x_i, y_j) \in J_1 \times V_1$ and $(x_p, y_q) \in J_2 \times V_2$.
Since G is bipartite and H is $\{V_1, V_2\}$-$(2-d)$-kernel critical so $x_i x_p \notin E(G)$ and $y_j y_q \notin E(H)$. Then by the definition of $G \times H$ we have that $(x_i, y_j)(x_p, y_q) \notin E(G \times H)$.

Consequently J is independent. Let $(x_i, y_j) \notin J$. We prove that J is 2-dominating. Consider the following cases:

2.1. $(x_i, y_j) \in J_2 \times V_1$,
Then $x_i \in J_2$ and $y_j \in V_1$. Because J_1 is $(2-d)$-kernel of a graph G then there exist at least two vertices, say x_s, x_r such that $x_s x_i, x_r x_i \in E(G)$. By the definition of cartesian product we have that every vertex from $J_2 \times V_1$ is at least 2-dominated by the set $J_1 \times V_1$.

2.2. $(x_i, y_j) \in J_1 \times V_2$,
We can prove this using the same method as in 2.1.

2.3. $(x_i, y_j) \in J_1 \times R$ or $(x_i, y_j) \in J_2 \times R$,
Then $x_i \in V(G)$ and $y_j \in R$. Because the graph H is $\{V_1, V_2\}$-$(2-d)$-kernel critical then V_1 is $(2-d)$-kernel of $H \setminus V_2$ and there exist at least two vertices, say y_w, y_t such that $y_w y_j, y_t y_j \in E(H)$. Moreover every vertex from the set $V_1 \times R$ is at least 2-dominated by

the set $J_1 \times V_1$. Analogously, considering set V_2, we can show that every vertex from $J_2 \times R$ is at least 2-dominated by the set $J_2 \times V_2$.

All this together gives that J is a $(2-d)$-kernel of $G \times H$. In the same way we can prove that $J^* = J_1 \times V_2 \cup J_2 \times V_1$ is a $(2-d)$-kernel of $G \times H$, which ends the proof. □

3. Conclusion and further study. The problem of finding the characterization of the cartesian product $G \times H$ with a $(2-d)$-kernel is still open. However, some sufficient conditions can be found if we add a restriction that a graph G is fixed.

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Received August 9, 2016