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Inequalities concerning polar derivative of polynomials

ABSTRACT. In this paper we obtain certain results for the polar derivative of a polynomial $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu \le n$, having all its zeros on $|z| = k$, $k \le 1$, which generalizes the results due to Dewan and Mir, Dewan and Hans. We also obtain certain new inequalities concerning the maximum modulus of a polynomial with restricted zeros.

1. Introduction and statement of results. Let $p(z)$ be a polynomial of degree n and $p'(z)$ its derivative, then according to Bernstein's inequality (for reference see [1]), we have

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is sharp and equality holds in (1.1) for $p(z) = \lambda z^n$, where $|\lambda| = 1$.

For the class of polynomials not vanishing in $|z| < k$, $k \ge 1$, Malik [8] proved

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is sharp and the extremal polynomial is $p(z) = (z + k)^n$.

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \le 1$, Govil [5] proved that if $p(z)$ has all its zeros on

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$|z| = k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-1} + k^n} \max_{|z|=1} |p(z)|. \tag{1.3}$$

While seeking for a better bound in the inequality (1.3), Dewan and Mir [4] proved the following result.

Theorem A. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^n} \left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|} \right) \max_{|z|=1} |p(z)|. \tag{1.4}$$

Dewan and Hans [3] generalized the above result to the class of polynomials of the type $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ and proved the following result.

Theorem B. *If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\begin{aligned} &\max_{|z|=1} |p'(z)| \\ &\leq \frac{n}{k^{n-\mu+1}} \left(\frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1}) + n|c_n|k^{\mu-1}(1+k^{\mu+1})} \right) \max_{|z|=1} |p(z)|. \end{aligned} \tag{1.5}$$

Let α be a complex number. If $p(z)$ is a polynomial of degree n, then polar derivative of $p(z)$ with respect to the point α, denoted by $D_\alpha p(z)$, is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \tag{1.6}$$

Clearly $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left[\frac{D_\alpha p(z)}{\alpha} \right] = p'(z)\,. \tag{1.7}$$

In this paper, we first prove the following result which extends Theorem A and Theorem B to the polar derivative of a polynomial having all its zeros on $|z| = k$, $k \leq 1$.

Theorem 1. *If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$, we have*

$$\begin{aligned} &\max_{|z|=1} |D_\alpha p(z)| \\ &\leq \frac{n(|\alpha|+k^\mu)}{k^{n-\mu+1}} \left(\frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1}) + n|c_n|k^{\mu-1}(1+k^{\mu+1})} \right) \max_{|z|=1} |p(z)|. \end{aligned} \tag{1.8}$$

Instead of proving Theorem 1 we prove the following theorem which gives a better bound than the above theorem. Briefly, we prove:

Theorem 2. *If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge k$, we have*

$$\begin{aligned} &\max_{|z|=1} |D_\alpha p(z)| \\ &\le \frac{n(|\alpha|+S_\mu)}{k^{n-\mu+1}} \left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})} \right) \max_{|z|=1} |p(z)|, \end{aligned} \tag{1.9}$$

where

$$S_\mu = \frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1} + \mu|c_{n-\mu}|}. \tag{1.10}$$

To prove that the bound obtained in the above theorem is better than the bound obtained in Theorem 1, we show that

$$S_\mu \le k^\mu$$

or

$$\frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}| + n|c_n|k^{\mu-1}} \le k^\mu$$

which is equivalent to

$$n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1} \le \mu|c_{n-\mu}|k^\mu + n|c_n|k^{2\mu-1},$$

which implies

$$n|c_n| \left(k^{2\mu} - k^{2\mu-1}\right) \le \mu|c_{n-\mu}| \left(k^\mu - k^{\mu-1}\right)$$

or

$$\frac{n}{\mu} \left| \frac{c_n}{c_{n-\mu}} \right| \ge \frac{1}{k^\mu},$$

which is always true (see Lemma 6).

Remark 1. Dividing both sides of inequalities (1.8) and (1.9) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get Theorem B due to Dewan and Hans [3].

If we choose $\mu = 1$ in Theorem 2 , we have the following result.

Corollary 1. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge k$, we have*

$$\begin{aligned} &\max_{|z|=1} |D_\alpha p(z)| \\ &\le \frac{n(|\alpha| + S_1)}{k^n} \left(\frac{n|c_n|k^2 + |c_{n-1}|}{2|c_{n-1}| + n|c_n|(1 + k^2)} \right) \max_{|z|=1} |p(z)|, \end{aligned} \tag{1.11}$$

where

$$S_1 = \left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n| + |c_{n-1}|} \right). \tag{1.12}$$

Remark 2. Dividing both sides of (1.11) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain Theorem A due to Dewan and Mir [4].

We next prove the following interesting results for the maximum modulus of polynomials.

Theorem 3. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge k$ and $0 \le r \le k \le R$, we have*

$$\begin{aligned}\max_{|z|=R} |D_\alpha p(z)| \le \frac{nR^{n-1}(|\alpha| + RS_1')}{k^n} &\left(\frac{n|c_n|k^2 + R|c_{n-1}|}{2R|c_{n-1}| + n|c_n|(R^2 + k^2)}\right) \\ &\times \left(\frac{R^n + kR^{n-1}}{r^n + kr^{n-1}}\right) \max_{|z|=r} |p(z)|,\end{aligned} \tag{1.13}$$

where

$$S_1' = \frac{1}{R}\frac{n|c_n|k^2 + R|c_{n-1}|}{nR|c_n| + |c_{n-1}|}. \tag{1.14}$$

Dividing both sides of (1.13) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain the following result.

Corollary 2. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for $0 \le r \le k \le R$, we have*

$$\begin{aligned}\max_{|z|=R} |p'(z)| \le \frac{nR^{n-1}}{k^n} &\left(\frac{n|c_n|k^2 + R|c_{n-1}|}{2R|c_{n-1}| + n|c_n|(R^2 + k^2)}\right) \\ &\times \left(\frac{R^n + kR^{n-1}}{r^n + kr^{n-1}}\right) \max_{|z|=r} |p(z)|.\end{aligned} \tag{1.15}$$

By involving the coefficients c_0 and c_1 of $p(z) = \sum_{j=0}^{n} c_j z^j$, we prove the following generalization of Theorem 3.

Theorem 4. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge k$ and $0 \le r \le k \le R$, we have*

$$\begin{aligned}\max_{|z|=R} |D_\alpha p(z)| \le &\frac{nR^{n-1}(|\alpha| + RS_1')}{k^n} \left(\frac{n|c_n|k^2 + R|c_{n-1}|}{2R|c_{n-1}| + n|c_n|(R^2 + k^2)}\right) \\ &\times \left(\frac{2k^2R^n|c_1| + R^{n-1}(R^2 + k^2)n|c_0|}{2k^2r^n|c_1| + r^{n-1}(r^2 + k^2)n|c_0|}\right) \max_{|z|=r} |p(z)|,\end{aligned} \tag{1.16}$$

where S_1' is the same as defined in Theorem 3.

On dividing both sides of (1.16) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 3. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leq 1$, *then for* $0 \leq r \leq k \leq R$, *we have*

$$\begin{aligned}\max_{|z|=R} |p'(z)| \leq & \frac{nR^{n-1}}{k^n} \left(\frac{n|c_n|k^2 + R|c_{n-1}|}{2R|c_{n-1}| + n|c_n|(R^2 + k^2)} \right) \\ & \times \left(\frac{2k^2 R^n |c_1| + R^{n-1}(R^2 + k^2) n |c_0|}{2k^2 r^n |c_1| + r^{n-1}(r^2 + k^2) n |c_0|} \right) \max_{|z|=r} |p(z)|. \end{aligned} \tag{1.17}$$

2. Lemmas.

We need the following lemmas for the proof of these theorems.

Lemma 1. *If* $p(z)$ *is a polynomial of degree* n, *then for* $|z| = 1$

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{2.1}$$

where here and throughout this paper $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

This is a special case of a result due to Govil and Rahman [6].

Lemma 2. *Let* $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$, *be a polynomial of degree* n *having no zero in the disk* $|z| < k$, $k \leq 1$. *Then for* $|z| = 1$

$$k^{n-\mu+1} \max_{|z|=1} |p'(z)| \leq \max_{|z|=1} |q'(z)|. \tag{2.2}$$

The above lemma is due to Dewan and Hans [3].

Lemma 3. *Let* $p(z) = c_0 + \sum_{\upsilon=\mu}^{n} c_\upsilon z^\upsilon$, $1 \leq \mu \leq n$, *be a polynomial of degree* n *having no zero in the disk* $|z| < k$, $k \geq 1$. *Then for* $|z| = 1$

$$k^\mu |p'(z)| \leq |q'(z)|. \tag{2.3}$$

The above lemma is due to Chan and Malik [2].

Lemma 4. *Let* $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, *be a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leq 1$. *Then for* $|z| = 1$

$$k^\mu |p'(z)| \geq |q'(z)|. \tag{2.4}$$

Proof of Lemma 4. If $p(z)$ has all its zeros on $|z| = k$, $k \leq 1$, then $q(z)$ has all its zeros on $|z| = \frac{1}{k}$, $\frac{1}{k} \geq 1$. Now applying Lemma 3 to the polynomial $q(z)$, the result follows. □

Lemma 5. *Let* $p(z) = c_0 + \sum_{\upsilon=\mu}^{n} c_\upsilon z^\upsilon$, $1 \leq \mu \leq n$, *be a polynomial of degree* n *having no zero in the disk* $|z| < k$, $k \geq 1$. *Then for* $|z| = 1$,

$$k^{\mu+1} \left\{ \frac{\mu |c_\mu| k^{\mu-1} + n|c_0|}{\mu |c_\mu| k^{\mu+1} + n|c_0|} \right\} \left| p'(z) \right| \leq \left| q'(z) \right|, \tag{2.5}$$

and

$$\frac{\mu}{n} \left| \frac{c_\mu}{c_0} \right| k^\mu \leq 1. \tag{2.6}$$

The above lemma was given by Qazi [10, Remark 1].

Lemma 6. *Let $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$. Then for $|z| = 1$,*

$$k^{\mu-1}\left\{\frac{\mu|c_{n-\mu}| + n|c_n|k^{\mu+1}}{\mu|c_{n-\mu}| + n|c_n|k^{\mu-1}}\right\}\left|p'(z)\right| \ge \left|q'(z)\right| \tag{2.7}$$

and

$$\frac{\mu}{n}\left|\frac{c_{n-\mu}}{c_n}\right| \le k^{\mu}. \tag{2.8}$$

Proof of Lemma 6. Since $p(z)$ has all its zeros on $|z| = k$, $k \le 1$, then $q(z)$ has all its zeros on $|z| = \frac{1}{k}$, $\frac{1}{k} \ge 1$. Now applying Lemma 5 to the polynomial $q(z)$, Lemma 6 follows. □

Lemma 7. *If $p(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ be a polynomial of degree n, having all its zeros in the disk $|z| \ge k$, $k > 0$, then for $r \le k$ and $R \ge k$*

$$\frac{M(p,r)}{r^n + kr^{n-1}} \ge \frac{M(p,R)}{R^n + kR^{n-1}}. \tag{2.9}$$

The above lemma is due to Jain [7].

Lemma 8. *If $p(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ be a polynomial of degree n, having all its zeros in the disk $|z| \ge k$, $k > 0$, then for $r \le k$ and $R \ge k$*

$$\frac{M(p,r)}{2k^2r^n|c_1| + r^{n-1}(r^2+k^2)n|c_0|} \ge \frac{M(p,R)}{2k^2R^n|c_1| + R^{n-1}(R^2+k^2)n|c_0|}. \tag{2.10}$$

The above lemma is due to Mir [9].

3. Proofs of the theorems.

Proof of Theorem 1. The proof of this theorem follows on the same lines as that of Theorem 2, but instead of using Lemma 6, we use Lemma 4. We omit the details. □

Proof of Theorem 2. Since $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, then it can be easily verified that

$$|q'(z)| = |np(z) - zp'(z)| \quad \text{for} \quad |z| = 1\,.$$

Now for every real or complex number α, we have

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

This implies with the help of Lemma 6 that

$$\begin{aligned} |D_\alpha p(z)| &\le |\alpha p'(z)| + |np(z) - zp'(z)| \\ &= |\alpha||p'(z)| + |q'(z)| \\ &\le (|\alpha| + S_\mu)|p'(z)|. \end{aligned} \tag{3.1}$$

Let z_0 be a point on $|z|=1$, such that $|q'(z_0)| = \max_{|z|=1}|q'(z)|$, then by Lemma 1, we get

$$|p'(z_0)| + \max_{|z|=1}|q'(z)| \le n \max_{|z|=1}|p(z)|, \tag{3.2}$$

which on using Lemma 6, gives

$$\frac{1}{k^{\mu-1}}\left(\frac{\mu|c_{n-\mu}| + n|c_n|k^{\mu-1}}{n|c_n|k^{\mu+1} + \mu|c_{n-\mu}|}\right)|q'(z_0)| + \max_{|z|=1}|q'(z)| \le n \max_{|z|=1}|p(z)|$$

or

$$\left(\frac{\mu|c_{n-\mu}|(1+k^{\mu-1}) + n|c_n|k^{\mu-1}(1+k^{\mu+1})}{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}\right)\max_{|z|=1}|q'(z)| \le n \max_{|z|=1}|p(z)|.$$

The above inequality when combined with Lemma 2, gives

$$\begin{aligned} k^{n-\mu+1}\left(\frac{\mu|c_{n-\mu}|(1+k^{\mu-1}) + n|c_n|k^{\mu-1}(1+k^{\mu+1})}{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}\right)&\max_{|z|=1}|p'(z)| \\ &\le n \max_{|z|=1}|p(z)|. \end{aligned} \tag{3.3}$$

On combining the inequalities (3.1) and (3.3), we get the desired result. □

Proof of Theorem 3. Let $0 \le r \le k \le R$. Since $p(z)$ has all its zero on $|z| = k$, $k \le 1$, then the polynomial $p(Rz)$ has all its zeros on $|z| = \frac{k}{R}$, $\frac{k}{R} \le 1$, therefore, applying Corollary 1 to the polynomial $p(Rz)$ with $|\alpha| \ge k$, we get

$$\begin{aligned} &\max_{|z|=1}|D_{\frac{\alpha}{R}}p(Rz)| \\ &\quad \le \frac{n\left(\frac{|\alpha|}{R} + S_1'\right)}{\frac{k^n}{R^n}}\left(\frac{nR^n|c_n|\frac{k^2}{R^2} + R^{n-1}|c_{n-1}|}{2R^{n-1}|c_{n-1}| + nR^n|c_n|\left(1+\frac{k^2}{R^2}\right)}\right)\max_{|z|=1}|p(Rz)| \end{aligned}$$

or

$$\begin{aligned} &\max_{|z|=1}\left|np(Rz) + \left(\frac{\alpha}{R} - z\right)Rp'(Rz)\right| \\ &\quad \le \frac{n\left(\frac{|\alpha|}{R} + S_1'\right)}{\frac{k^n}{R^n}}\left(\frac{nR^n|c_n|\frac{k^2}{R^2} + R^{n-1}|c_{n-1}|}{2R^{n-1}|c_{n-1}| + nR^n|c_n|\left(1+\frac{k^2}{R^2}\right)}\right)\max_{|z|=R}|p(z)| \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\max_{|z|=R}|D_\alpha p(z)| \\ &\le \frac{nR^{n-1}(|\alpha| + RS_1')}{k^n}\left(\frac{nR^{n-2}|c_n|k^2 + R^{n-1}|c_{n-1}|}{2R^{n-1}|c_{n-1}| + nR^{n-2}|c_n|(R^2+k^2)}\right)\max_{|z|=R}|p(z)|. \end{aligned}$$

For $0 \leq r \leq k \leq R$, the above inequality in conjunction with Lemma 7, yields

$$\begin{aligned}\max_{|z|=R} |D_\alpha p(z)| \\ &\leq \frac{nR^{n-1}(|\alpha| + RS_1')}{k^n}\left(\frac{n|c_n|k^2 + R|c_{n-1}|}{2R|c_{n-1}| + n|c_n|(R^2+k^2)}\right) \\ &\quad\times\left(\frac{2R^n + kR^{n-1}}{r^n + kr^{n-1}}\right)\max_{|z|=r}|p(z)|,\end{aligned}$$

which completes the proof of Theorem 3. □

Proof of Theorem 4. The proof follows on the same lines as that of Theorem 3, but instead of using Lemma 7 we use Lemma 8. □

Remark 3. For $\mu = n$, Theorems 1 and 2 hold if the polynomial satisfies the condition $|c_0| \leq k|c_n|$.

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