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An extension of typically-real functions and associated orthogonal polynomials

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. Two-parameters extension of the family of typically-real functions is studied. The definition is obtained by the Stjeltjes integral formula. The kernel function in this definition serves as a generating function for some family of orthogonal polynomials generalizing Chebyshev polynomials of the second kind. The results of this paper concern the exact region of local univalence, bounds for the radius of univalence, the coefficient problems within the considered family as well as the basic properties of obtained orthogonal polynomials.

1. Introduction. Let $\mathcal{H}(\mathbb{D})$ denote the class of holomorphic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. The class of typically-real functions in $\mathcal{H}(\mathbb{D})$ is denoted by $T_{\mathbb{R}}$. This class is characterized by the condition $\operatorname{Im} z \cdot \operatorname{Im} f(z) \geq 0$, $z \in \mathbb{D}$, and has the integral representation:

$$T_{\mathbb{R}} = \left\{ f : f(z) = \int_0^{\pi} \frac{z}{(1 - ze^{i\theta})(1 - ze^{-i\theta})} d\mu(\theta), \quad \mu \in \mathcal{P}_{[0,\pi]} \right\}, \tag{1}$$

where $\mathcal{P}_{[0,\pi]}$ denotes the set of probability measures on $[0, \pi]$, and was studied by many authors, e.g. [2], [6], [7], [9], [11].

From the above representation we see that the class $T_{\mathbb{R}}$ is closely connected with the generating function Ψ for the Chebyshev polynomials of

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the second kind, $U_n(x)$, $x = \cos\theta$, $\theta \in [0, \pi]$, namely:

$$\Psi(e^{i\theta}; z) = \frac{1}{(1 - ze^{i\theta})(1 - ze^{-i\theta})} = \sum_{n=0}^{+\infty} U_n(x) z^n, \quad z \in \mathbb{D},$$

where

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta, \quad \theta \in [0, \pi], \quad n = 0, 1, \dots .$$

An important role in the extremal problems for univalent function as well as for $T_{\mathbb{R}}$ plays the Koebe function

$$k^{(1)}(z) = z\Psi(1; z) = \frac{z}{(1-z)^2} = z \cdot {}_1F_0 \left[\begin{matrix} 2 \\ \ \end{matrix} ; z \right], \quad z \in \mathbb{D},$$

where ${}_1F_0$ is the hypergeometric series. Studying the q-extension of the above formula and of the Löwner differential equation, Gasper [3] observed the important role of the q-Koebe function:

$$k^{(q)}(z) = \frac{z}{(1-z)(1-qz)} = z \cdot {}_1\Phi_0 \left[\begin{matrix} q^2 \\ \ \end{matrix} ; q, z \right], \quad z \in \mathbb{D}, \ q \in [-1, 1],$$

where by ${}_r\Phi_s$ we denote the basic hypergeometric series [4]:

$${}_r\Phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} \frac{z^n}{(q;q)_n}, \ z \in \mathbb{D},$$

where in general $q \in (-1, 1)$ and

$$(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_r; q)_n,$$

$r, \ s \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $(a; q)_n$ is the q-shifted factorial defined by

$$(a; q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad n \in \mathbb{N}, \ (a, q)_0 = 1.$$

(In the situations which we consider below the cases $q = \pm 1$ will be allowed as well.)

The form of $k^{(q)}(z)$ gives us the motivation for studying the "more symmetric" (p, q)-Koebe function:

$$k^{(p,q)}(z) = \frac{z}{(1-pz)(1-qz)} = z \cdot {}_1\Phi_0 \left[\begin{matrix} \frac{q^2}{p^2} \\ \ \end{matrix} ; \frac{q}{p}, pz \right], \ z \in \mathbb{D}, \tag{2}$$

where $(p, q) \in \Delta = \{(p, q) : -1 \le q \le p \le 1\}$, and the class of holomorphic functions $T^{(p,q)}$ in $\mathbb{D}$ is defined below. In what follows we assume $pq \neq 0$. The case $p = 0$ or $q = 0$ is easy and can be treated separately. We omit it.

The form of the (p,q)-Koebe function and the kernel function in the integral representation formula (1) for the class $T_{\mathbb{R}}$ suggest the study of the following class of functions:

$$T^{(p,q)} = \Big\{ f : f(z) = z + a_2 z^2 + \cdots \in \mathcal{H}(\mathbb{D}) : \\ f(z) = \int_{-\pi}^{\pi} \frac{z}{(1-pze^{i\theta})(1-qze^{-i\theta})} d\mu(\theta), \quad \mu \in \mathcal{P}_{[-\pi,\pi]} \Big\}, \tag{3}$$

where $\mathcal{P}_{[-\pi,\pi]}$ denotes the set of probability measures on $[-\pi,\pi]$. Observe that contrary to $T_{\mathbb{R}}$ the coefficients of $f \in T^{(p,q)}$ are not real in general.

Remark. We observe two important special cases of $T^{(p,q)}$:

- $T^{(1,1)} = T_{\mathbb{R}}$,
- $T^{(1,0)} = T^{(0,-1)} = \overline{co}(S^c) = \overline{co}\left(S^*(\frac{1}{2})\right)$, where S^c denotes the class of convex univalent function in $\mathbb{D}$, and $S^*(\frac{1}{2})$ denotes the class of $\frac{1}{2}$-starlike functions in $\mathbb{D}$.

The class $T^{(1,-1)}$ is of special interest and will be studied elsewhere. We denote

$$k^{(p,q)}(\theta;z) = \frac{z}{(1-pze^{i\theta})(1-qze^{-i\theta})} = z\sum_{n=0}^{\infty} U_n(p,q;e^{i\theta})z^n,$$

$z \in \mathbb{D}$, $(p,q) \in \Delta$, $\theta \in [-\pi,\pi]$.

This paper consists of two parts. In the first part we observe a few properties of $k^{(p,q)}(z)$ given by (2) and solve some extremal problems within the class $T^{(p,q)}$, namely: the coefficients problem, the sharp bound for $|f(z)|$, the exact domain of local univalence and the bound for the radius of univalence. In the second part we prove some properties of the "polynomials" $U_n(p,q,e^{i\theta})$ and related "polynomials" $T_n(p,q,e^{i\theta})$. These results extend and generalize the corresponding ones for the class $T_{\mathbb{R}}$ (the case $p=q=1$), as well as for $\overline{co}(S^C)$ $(p=1,\ q=0)$ and for $T^{(1,q)}$ which has been studied in [5]. The results for the "polynomials" $U_n(p,q;e^{i\theta})$ and $T_n(p,q;e^{i\theta})$ extend known results for classical Chebyshev polynomials of the second and first kind. In what follows we assume $pq \neq 0$, because if $p=0$ or $q=0$, then the result follows from general cases, taking the limit ($p \to 0$ or $q \to 0$).

2. Statements of the results – the class $T^{(p,q)}$. When studying the extremal problems for $T^{(p,q)}$, especially coefficient problems, we meet "the trigonometric polynomials" $U_n(p,q;e^{i\theta})$ which are defined by the generating function

$$\Psi^{(p,q)}(e^{i\theta};z) = \frac{1}{(1-pze^{i\theta})(1-qze^{-i\theta})} = \sum_{n=0}^{\infty} U_n(p,q;e^{i\theta})z^n,\ z \in \mathbb{D}, \tag{4}$$

$\theta \in [-\pi, \pi]$, $(p,q) \in \Delta$, where

$$U_0(p,q;e^{i\theta}) = 1,\ U_1(p,q;e^{i\theta}) = pe^{i\theta} + qe^{-i\theta},$$
$$U_n(p,q;e^{i\theta}) = \frac{p^{n+1}e^{i(n+1)\theta} - q^{n+1}e^{-i(n+1)\theta}}{pe^{i\theta} - qe^{-i\theta}}, \quad n \geq 2. \tag{5}$$

The function $k^{(p,q)}(z)$ is of course starlike in $\mathbb{D}$. But moreover, we have the following result, which is sharp if $pq > 0$.

Proposition 1. *The function $k^{(p,q)}(z)$ is α-starlike in $\mathbb{D}$ with*

$$\alpha = \alpha(p,q) = \frac{1}{2}\left(\frac{1-|p|}{1+|p|} + \frac{1-|q|}{1+|q|}\right),$$

and convex in the disk $|z| < r_c(p,q)$, where

$$r_c(p,q) = \frac{2}{t + \sqrt{t^2 - 4|p||q|}}$$

and $t = \frac{|p|+|q|+\sqrt{|p|^2+|q|^2+34|p||q|}}{2}$.

Proof. We recall that $f \in \mathcal{H}(\mathbb{D})$ is α-starlike in $\mathbb{D}$ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}, \quad 0 \leq \alpha < 1,$$

and convex in $\mathbb{D}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We find that

$$\frac{zk'^{(p,q)}(z)}{k^{(p,q)}(z)} = 1 + \frac{pz}{1-pz} + \frac{qz}{1-qz},$$
$$1 + \frac{zk''^{(p,q)}(z)}{k'^{(p,q)}(z)} = \frac{1+pz}{1-pz} + \frac{1+qz}{1-qz} - \frac{1+pqz^2}{1-pqz^2}.$$

Using obvious inequality

$$\frac{1-r}{1+r} \leq \operatorname{Re}\frac{1+z}{1-z} \leq \frac{1+r}{1-r}, \quad |z| = r < 1,$$

we find that

$$\operatorname{Re}\frac{zk'^{(p,q)}(z)}{k^{(p,q)}(z)} = \frac{1}{2}\operatorname{Re}\frac{1+pz}{1-pz} + \frac{1}{2}\operatorname{Re}\frac{1+qz}{1-qz} \geq \frac{1}{2}\frac{1-|p|}{1+|p|} + \frac{1}{2}\frac{1-|q|}{1+|q|}, \ z \in \mathbb{D},$$

and

$$\operatorname{Re}\left\{1 + \frac{zk''^{(p,q)}(z)}{k'^{(p,q)}(z)}\right\} \geq \frac{1-|p|r}{1+|p|r} + \frac{1-|q|r}{1+|q|r} - \frac{1+|p||q|r^2}{1-|p||q|r^2}.$$

The last expression is positive if $0 < r < r_c(p,q)$, where $r_c(p,q)$ is the last positive root of the equation

$$|p|^2|q|^2r^4 - |p||q|r^3(|p|+|q|) - 6|p||q|r^2 - (|p|+|q|)r + 1 = 0.$$

The substitution $\frac{1}{r} + |p||q|r = t$ gives the result. $\square$

Proposition 2. *If $f \in T^{(p,q)}$, then we have the following sharp bound*

$$|a_n| \le \begin{cases} \frac{|p|^n - |q|^n}{|p|-|q|} & \text{if} \quad |p| \neq |q| \\ n|p|^{n-1} & \text{if} \quad |p| = |q|. \end{cases}$$

The extremal functions have the form:

$$k^{(p,q)}(0;z) \text{ if } pq > 0 \text{ and } k^{(p,q)}\left(\frac{\pi}{2};z\right) \text{ if } pq < 0.$$

Proof.

$$\begin{aligned} |a_n| &= \left|\int_{-\pi}^{\pi} U_{n-1}(p,q;e^{i\theta})d\mu(\theta)\right| \le \int_{-\pi}^{\pi} \left|\frac{p^ne^{in\theta} - q^ne^{-in\theta}}{pe^{i\theta} - qe^{-i\theta}}\right| d\mu(\theta) \\ &= \int_{-\pi}^{\pi} |p^{n-1}e^{i(n-1)\theta} + p^{n-1}qe^{i(n-3)\theta} + \dots \\ &\qquad + pq^{n-3}e^{-i(n-3)\theta} + q^ne^{-i(n-1)\theta}| d\mu(\theta) \\ &\le \frac{|p|^n - |q|^n}{|p|-|q|} = U_{n-1}(|p|,|q|;1). \end{aligned}$$

$\square$

Proposition 3. *For any $f \in T^{(p,q)}$ and $z = re^{it} \in \mathbb{D}$, we have the sharp bound:*

$$|f(z)| \le \frac{r}{(1-|p|r)(1-|q|r)}.$$

The extremal functions have the form: $k^{(p,q)}(0;r)$ if $p > 0$ and $q > 0$; $k^{(p,q)}(0;-r)$ if $p < 0$ and $q < 0$; $k^{(p,q)}(\frac{\pi}{2};r)$ if $p > 0$ and $q < 0$.

This result follows directly from the integral representation (3) and the triangle inequality.

Remark. Observe that if $|p| < 1$ and $|q| < 1$, then $f \in T^{(p,q)}$ is bounded.

The set of local univalence for $T_{\mathbb{R}}$ has been found in [2], [7] and has lens-shape bounded by two arcs of the symmetric circles. For the class $T^{(p,q)}$ this set is more complicated. Moreover, the method of calculations is completely different.

Theorem 1. *Let $f \in T^{(p,q)}$, $(p,q) \in \Delta$, $z = re^{it} \in \mathbb{D}$ and*

$$A = \left\{2(p-q)^2 + 4pq\sin^2 t\right\}^{\frac{1}{2}}.$$

The equation of the boundary in polar coordinates $z = r(t)e^{it}$ *of the set* $D'(T^{(p,q)})$ *of the local univalence is given by the formula:*

$$r = r(t) = \begin{cases} 1 & \text{if} \quad A < 1 - pq \\ \frac{2}{\sqrt{A^2+4pq}+A} & \text{if} \quad A \geq 1 - pq. \end{cases}$$

In the proof we will use the following practical result of Koczan and Szapiel [6].

Lemma 1. *Denote*

$$\mathcal{K} = \left\{ f(z) \in \ \mathcal{H}(\mathbb{D}) : \ f(z) = \int_a^b S(z,\theta) d\mu(\theta), \ z \in \mathbb{D}, \ \mu \in \mathcal{P}[a,b] \right\},$$

where $S(z,\cdot)$ *is holomorphic in* $\mathbb{D}$ *and* $S(\cdot,\theta)$ *is continuous in* $[a,b]$, *and* $\mathcal{P}_{[a,b]}$ *denotes the set of probability measures on* $[a,b]$. *The set of local univalence is given by the formula*

$$D'(\mathcal{K}) = \bigcap_{f\in\mathcal{K}} \{f'(z) \neq 0\} = \left\{ z \in \mathbb{D} : \left| \Delta_{a\leq\theta_1<\theta_2\leq b} \arg \frac{d}{dz} S(z,\theta) \right| < \pi \right\}.$$

Proof. For $f \in T^{(p,q)}$ we have

$$\frac{d}{dz} S(z,\theta) = \frac{d}{dz} k^{(p,q)}(e^{i\theta}; z) = \frac{\frac{1}{z^2} - pq}{[(\frac{1}{z} + pqz) - (pe^{i\theta} + qe^{-i\theta})]^2}$$

and

$$\arg \left[\frac{d}{dz} k^{(p,q)}(e^{i\theta}; z) \right] = \arg \left(\frac{1}{z^2} - pq \right) - 2 \arg \left[\left(\frac{1}{z} + pqz \right) - \left(pe^{i\theta} + qe^{-i\theta} \right) \right].$$

Let us put $z = re^{it}$, $r \in (0,1)$, $t \in [-\pi, \pi]$, and $z_0 = x_0 + iy_0 = \frac{1}{z} + pqz$, and consider the ellipse:

$$\begin{aligned} \mathcal{E} : w = w(\theta) &= u + iv \\ &= [x_0 - (p+q)\cos\theta] + i[y_0 - (p-q)\sin\theta], \ \theta \in [-\pi, \pi], \end{aligned} \tag{6}$$

where

$$x_0 = \frac{1}{r}(1 + pqr^2)\cos t, \ y_0 = -\frac{(1 - pqr^2)}{r} \sin t. \tag{7}$$

Denoting

$$\psi(\theta) \coloneqq \arg \left[\frac{d}{dz} w(\theta) \right] = \arctan \frac{y_0 - (p-q)\sin\theta}{x_0 - (p+q)\cos\theta},$$

we see that the problem

$$\Delta_{-\pi\leq\theta_1<\theta_2\leq\pi} \arg \frac{d}{dz} k^{(p,q)}(e^{i\theta}; z)$$

is equivalent to the problem of finding

$$\max_{-\pi\leq\theta\leq\pi} \psi(\theta) - \min_{-\pi\leq\theta\leq\pi} \psi(\theta).$$

From the geometrical point of view this is nothing else but finding the biggest angle with the vertex at the origin in which lies the ellipse $\mathcal{E}$ given by (6).

The equations of the tangent lines ($v = mu$) from the origin to $\mathcal{E}$ have the form

$$v = m_1 u,\ v = m_2 u,\ m_1 = \tan\alpha_1,\ m_2 = \tan\alpha_2,$$

where m_1 and m_2 are the roots of the equation:

$$\{x_0^2 - (p+q)^2\}m^2 - 2x_0 y_0 m + \{y_0^2 - (p-q)^2\} = 0. \tag{8}$$

Using the formula

$$\tan(\alpha_2 - \alpha_1) = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

and the fact that

$$|\Delta \arg[k^{(p,q)}(e^{i\theta}; z)]'| = 2|\max\psi - \min\psi| = 2|\alpha_2 - \alpha_1|,$$

we see that the equation of the boundary of $D'(z)$ is given by the condition $m_1 m_2 = -1$ or

$$\frac{y_0^2 - (p-q)^2}{x_0^2 - (p+q)^2} = -1 \quad \text{or} \quad x_0^2 + y_0^2 = 2(p^2 + q^2). \tag{9}$$

The equation (9) with notation (7) is equivalent to

$$p^2 q^2 r^4 - 2\{p^2 + q^2 - pq\cos 2t\}r^2 + 1 = 0$$

or

$$(pqr^2 + Ar - 1)(pqr^2 - Ar - 1) = 0$$

where

$$A^2 = \{2(p-q)^2 + 4pq \cdot \sin^2 t\}. \tag{10}$$

One can verify that the expression

$$(pqr^2 - Ar - 1)$$

is negative for $r \in (0, 1)$. Therefore, the equation $r = r(t)$ of the boundary of $D'(z)$ is given by the equation

$$pqr^2 + Ar - 1 = 0 \quad \text{if} \quad r(t) \le 1 \quad \text{and} \quad r = 1 \quad \text{elsewhere.} \tag{11}$$

Because $r(-t) = r(t)$ one can consider only the case $t \in [0, \pi]$. The solution of (11) is less than 1, if $A \ge 1 - pq$ and is given by the formula

$$r = r(t) = \frac{1}{2pq}(\sqrt{A^2 + 4pq} - A), \tag{12}$$

(if $A < 1 - pq$ we put $r(t) = 1$).

The inequality $A \ge 1 - pq$ is equivalent to

$$4pq\sin^2 t \ge (1 - pq)^2 - 2(p - q)^2. \tag{13}$$

We have to consider two cases: (α) $pq > 0$ and (β) $pq < 0$.
(α) $pq > 0,\ (p,q) \in \Delta$.
Inequality (13) holds for any $t \in [0,\pi]$ if

$$(1-pq)^2 - 2(p-q)^2 \le 0$$

and then $r = r(t)$; and does not hold for any $t \in [0,\pi]$ if

$$(1-pq)^2 - 2(p-q)^2 \ge 4pq$$

and then $r = 1$. If

$$0 \le (1-pq)^2 - 2(p-q)^2 \le 4pq$$

then (13) holds if $t \in [t_0, \pi - t_0]$, where

$$t_0 = \arcsin\left\{\frac{(1-pq)^2 - 2(p-q)^2}{4pq}\right\}^{\frac{1}{2}}.$$

Therefore, in this case the equation of the boundary of $D'(z)$ is

$$r = \begin{cases} r(t) & \text{if } t \in [0,t_0] \cup [\pi - t_0, \pi], \\ 1 & \text{if } t \in [t_0, \pi - t_0]. \end{cases} \tag{14}$$

Denote: $q_1(p) = \frac{\sqrt{2}p-1}{p-\sqrt{2}}$, $\quad q_2(p) = -\frac{\sqrt{2}p+1}{p+\sqrt{2}}$.

A simple analysis shows that for any $t \in [0,\pi]$: In the case: (α) $pq > 0$, $(p,q) \in \Delta$, we have:

$$r = r(t) \qquad \text{if} \quad q \le q_1(p),\ p \in [-1,0] \cup \left[\frac{1}{\sqrt{2}}, 1\right];$$

$$r = 1 \qquad \text{if} \quad q_2 \le q \le p,\ p \in [1-\sqrt{2}, 0] \text{ or } 0 \le q \le p,\ p \in [0, \sqrt{2}-1]$$

$$\text{or } 0 \le q \le -q_1(p),\ p \in \left[\sqrt{2}-1, \frac{1}{\sqrt{2}}\right].$$

In an analogous way one can prove that for any $t \in [0,\pi]$ in the case:
(β) $pq < 0$, $(p,q) \in \Delta$, we have:

$$r = 1 \quad \text{if} \quad q_1(p) \le q \le 0 \text{ for } p \in [0, \sqrt{2}-1]$$

and

$$r = r(t) \quad \text{if} \quad -q_1(p) \le q \le 0 \text{ for } p \in \left[\frac{1}{\sqrt{2}}, 1\right]$$

$$\text{or } -1 \le q \le q_2(p) \text{ for } p \in [0,1].$$

In the part of triangle Δ which is not mentioned above we have the equality (14). □

From the above consideration we come to the following conclusion about the radius of local univalence of $T^{(p,q)}$.

Theorem 2. *The sharp value of the radius $r_0^{(p,q)}$ of local univalence of the class $T^{(p,q)}$ is given by the formula:*

$$r_0^{(p,q)} = \begin{cases} \frac{\sqrt{2}}{\sqrt{p^2+q^2+|p+q|}} & \text{if} \quad (p,q) \in D_1 \cup D_2 \cup D_3 \cup D_4, \\ \frac{\sqrt{2}}{\sqrt{p^2+q^2+(p-q)}} & \text{if} \quad (p,q) \in D_5 \cup D_6, \\ 1 & \text{if} \quad (p,q) \in \Delta \setminus \bigcup_{k=1}^{6} D_k, \end{cases} \tag{15}$$

where the sets D_j, $j = 1, 2, \ldots, 6$ are defined as follows:

$$D_1 = \left\{ (p,q) : \sqrt{2} - 1 \le p \le \frac{1}{\sqrt{2}}, \quad \frac{\sqrt{2}p - 1}{p - \sqrt{2}} \le q \le p \right\},$$

$$D_2 = \left\{ (p,q) : \frac{1}{\sqrt{2}} \le p \le 1, \quad 0 \le q \le p \right\},$$

$$D_3 = \left\{ (p,q) : -1 \le p \le 1 - \sqrt{2}, \quad -1 \le q \le p \right\},$$

$$D_4 = \left\{ (p,q) : 1 - \sqrt{2} \le p \le 0, \quad -1 \le q \le \frac{-\sqrt{2}p - 1}{p + \sqrt{2}} \right\},$$

$$D_5 = \left\{ (p,q) : 0 \le p \le 1/\sqrt{2}, \quad -1 \le q \le \frac{1 - \sqrt{2}p}{p - \sqrt{2}} \right\},$$

$$D_6 = \left\{ (p,q) : \frac{1}{\sqrt{2}} \le p \le 1, \quad -1 \le q \le 0 \right\}.$$

The extremal functions are:
(α) if $(p,q) \in D_1 \cup D_2 \cup D_3 \cup D_4$

$$f_0(z) = \frac{1}{2}\left(\frac{z}{(1 - pze^{i\theta})(1 - qze^{-i\theta})} + \frac{z}{(1 + pze^{-i\theta})(1 + qze^{i\theta})} \right) \text{ at } z = \pm i r_0,$$

where $\cos\theta = \frac{|p+q|}{\sqrt{2(p^2+q^2)}}$, $\sin\theta = -\frac{(p-q)}{\sqrt{2(p^2+q^2)}}$.

(β) if $(p,q) \in D_5 \cup D_6$

$$f_0(z) = \frac{1}{2}\left(\frac{z}{(1 - pze^{i\theta})(1 - qze^{-i\theta})} + \frac{z}{(1 - pze^{-i\theta})(1 - qze^{i\theta})} \right) \text{ at } z = \pm r_0,$$

where $\cos\theta = \frac{|p+q|}{\sqrt{2(p^2+q^2)}}$, $\sin\theta = \frac{(p-q)}{\sqrt{2(p^2+q^2)}}$.

Proof. The radius $r_0^{(p,q)}$ of the biggest disk with the center at the origin which is contained in $D'(z)$ for any $t \in [-\pi, \pi]$ is the radius of local univalence of the class $T^{(p,q)}$. Finding the maximal value of $r(t)$ given by (12), which is attained for $t = 0$ if $pq < 0$ and for $t = \frac{\pi}{2}$ if $pq > 0$ we find (15).

The form of the extremal functions follows from (7) and (8). □

Remark. To find the exact value of the radius of univalence $r_u^{(p,q)}$ of the class $T^{(p,q)}$, we meet some technical difficulties. However, we can prove some bound from below (the bound from above is evident).

Theorem 3. *The radius of univalence $r_u^{(p,q)}$ of the class $T^{(p,q)}$ satisfies the inequality*

$$r_0^{(p,q)} \geq r_u^{(p,q)} \geq \hat{r}^{(p,q)}$$

where $\hat{r}^{(p,q)}$ is the unique root of the equation

$$1 - |p||q|r^2 = 2r^2 \left(|p|\sqrt{1-q^2r^2} + |q|\sqrt{1-p^2r^2}\right)^2. \tag{16}$$

Proof. We will use the sufficient condition for univalence: $\operatorname{Re} f'(z) > 0$. From (3) we have for $f \in T^{(p,q)}$,

$$f'(z) = \int_{-\pi}^{\pi} \frac{1 - pqz^2}{(1 - pze^{i\theta})^2(1 - qze^{-i\theta})^2} d\mu(\theta) = \int_{-\pi}^{\pi} [k^{(p,q)}(e^{i\theta}; z)]' d\mu(\theta).$$

We see that $\operatorname{Re} f'(z) > 0$ if and only if $|\arg[k^{(p,q)}(e^{i\theta}; z)]'| < \frac{\pi}{2}$.

Putting $z = re^{it}$, $r \in (0,1)$, $t \in [-\pi, \pi]$, we find that

$$\arg[k^{(p,q)}(e^{i\theta}; z)]' = \Big\{ -\arctan \frac{pqr^2 \sin 2t}{1 - pqr^2 cos2t} + 2\arctan \frac{pr\sin(t+\theta)}{1 - pr\cos(t+\theta)} + 2\arctan \frac{qr\sin(t-\theta)}{1 - qr\cos(t-\theta)} \Big\}.$$

Because

$$\max(\min)_{-\pi \leq \varphi \leq \pi} \arctan \frac{\tau \sin\varphi}{1 - \tau\cos\varphi} = \pm \frac{|\tau|}{\sqrt{1-\tau^2}}, \ |\tau| < 1,$$

we conclude that

$$|\arg[k^{(p,q)}(e^{i\theta}; z)]'| < \arctan \frac{|p||q|r^2}{\sqrt{1 - p^2q^2r^2}} + 2\left(\arctan \frac{|p|r}{\sqrt{1-p^2r^2}} + \arctan \frac{|q|r}{\sqrt{1-q^2r^2}}\right)$$
$$= \arcsin |p||q|r^2 + 2(\arcsin |p|r + \arcsin |q|r),$$

because $\arctan \frac{\tau}{\sqrt{1-\tau^2}} = \arcsin \tau$. Using the formula

$$\arcsin x + \arcsin y = \eta \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2}) + \varepsilon\pi,$$

where

$$\eta = 1,\ \varepsilon = 0, \quad \text{if and only if} \quad xy < 0 \text{ or } x^2 + y^2 \leq 1,$$
$$\eta = -1,\ \varepsilon = -1, \quad \text{if and only if} \quad x^2 + y^2 > 1,\ x < 0,\ y < 0,$$
$$\eta = -1,\ \varepsilon = 1, \quad \text{if and only if} \quad x^2 + y^2 > 1,\ x > 0,\ y > 0,$$

we come to the conclusion that

$$|\arg[k^{(p,q)}(e^{i\theta};z)]'| < \frac{\pi}{2}$$

if and only if

$$1-|p||q|r^2 > 2r^2\left(|p|\sqrt{1-q^2r^2}+|q|\sqrt{1-p^2r^2}\right)^2,$$

which ends the proof. □

Observe that by formula (16) we have $\hat{r}^{(1,1)} = \frac{\sqrt{2}}{4} = 0.35\ldots$; which is not sharp ($\hat{r}^{(1,1)} = \sqrt{2}-1$) [2], however $\hat{r}^{(1,0)} = \frac{\sqrt{2}}{2}$ is the exact value, [10].

3. Statements of the results – the "polynomials" $U_n(p,q;e^{i\theta})$ and $T_n(p,q;e^{i\theta})$. In this chapter we collect some properties of the polynomials $U_n(p,q;e^{i\theta})$ which are defined by the generating function (4) or explicit formulas (5). We can observe, moreover, that the "polynomials" $U_n(p,q;e^{i\theta})$ can be expressed via classical Chebyshev polynomials of the second kind $U_n(x)$, where the variable x is now complex and has a special form. Namely, putting in the generating function (4) instead of z the value $\frac{z}{\sqrt{pq}}$, $pq \neq 0$ and comparing the result with the generating function for $U_n(x)$, we conclude that

$$U_n(p,q;e^{i\theta}) = (\sqrt{pq})^n U_n\left(\frac{pe^{i\theta}+qe^{-i\theta}}{2\sqrt{pq}}\right), \quad pq \neq 0. \tag{17}$$

As we see, if $\theta \in [-\pi,\pi]$, then the variable

$$\omega(\theta) = \frac{pe^{i\theta}+qe^{-i\theta}}{2\sqrt{pq}}$$

is describing an ellipse E with semi-axes: $a = \left|\frac{(p+q)}{2\sqrt{pq}}\right|$ and $b = \left|\frac{(p-q)}{2\sqrt{pq}}\right|$.

Using the representation (5) and (17) after same calculations, we can get the following basic results for the "polynomials" $U_n(p,q;e^{i\theta})$.

Theorem 4. (*a*) *The "trigonometric polynomials" $U_n(p,q;e^{i\theta})$ satisfy the three-term recurrence relation*

$$U_{n+2}(p,q;e^{i\theta}) - (pe^{i\theta}+qe^{-i\theta})U_{n+1}(p,q;e^{i\theta}) + pqU_n(p,q;e^{i\theta}) = 0,$$
$$n = 0,1,\ldots,$$
$$U_0(p,q;e^{i\theta}) = 1, \quad U_1(p,q;e^{i\theta}) = pe^{i\theta}+qe^{-i\theta}.$$

(*b*) *The function $y(\theta) = U_n(p,q;e^{i\theta})$ satisfies the following differential equation of the second order:*

$$y''(\theta)(pe^{i\theta}-qe^{-i\theta}) + 2i(pe^{i\theta}+qe^{-i\theta})y'(\theta) + n(n+2)(pe^{i\theta}-qe^{-i\theta})y(\theta) = 0.$$

(*c*) *The "polynomials"* $U_n(p,q;e^{i\theta})$ *satisfy the "quasi-Rodrigues formula"*

$$U_n(p,q;e^{i\theta})=\frac{1}{[(n+1)i]^n}\cdot\frac{1}{pe^{i\theta}-qe^{-i\theta}}\left[p^{n+1}e^{i(n+1)\theta}-(-1)^n q^{n+1}e^{-i(n+1)\theta}\right]^{(n)}.$$

(*d*) *The "polynomials"* $U_n(p,q;e^{i\theta})$, *satisfy the following orthogonality relation:*

$$\int_E U_n(p,q;e^{i\theta})\overline{U_m}(p,q;e^{i\theta})\rho(\theta)d\theta=\begin{cases}0 & \text{if}\quad m\neq n,\\ \frac{\pi}{4pq}\left(p^{2(n+1)}+q^{2(n+1)}\right) & \text{if}\quad m=n,\end{cases}$$

where $\rho(\theta)=\frac{-1}{2i\sqrt{pq}}(pe^{-i\theta}-qe^{i\theta})$.

Proof. (*a*) and (*b*) follow directly from (5) by obvious calculations and differentiation from formula (5).

We verify directly the property (*c*) and for the property (*d*) we have

$$\begin{aligned}&\int_E U_n(p,q;e^{i\theta})\overline{U_m}(p,q;e^{i\theta})\rho(\theta)d\omega\\
&=\int_{-\pi}^{\pi}U_n(p,q;e^{i\theta})\overline{U_m}(p,q;e^{i\theta})\frac{1}{2i\sqrt{pq}}(pe^{-i\theta}-qe^{i\theta})\Big(\frac{pe^{i\theta}-qe^{-i\theta}}{2\sqrt{pq}}\Big)id\theta\\
&=\frac{1}{4pq}\int_{-\pi}^{\pi}\frac{p^{n+1}e^{i(n+1)\theta}-q^{n+1}e^{-i(n+1)\theta}}{pe^{i\theta}-qe^{-i\theta}}\cdot\frac{p^{m+1}e^{-i(m+1)\theta}-q^{m+1}e^{i(m+1)\theta}}{pe^{-i\theta}-qe^{i\theta}}\\
&\qquad\times(pe^{-i\theta}-qe^{i\theta})(pe^{i\theta}-qe^{-i\theta})d\theta\\
&=\frac{1}{4pq}\int_{-\pi}^{\pi}(p^{n+m+2}e^{i(n-m)\theta}-p^{n+1}q^{m+1}e^{i(n+m+2)\theta}-p^{m+1}q^{n+1}e^{-i(n+m+2)\theta}\\
&\qquad+q^{n+m+2}e^{i(m-n)\theta})d\theta=\begin{cases}0 & \text{if}\quad m\neq n,\\ \frac{\pi}{4pq}\left(p^{2(n+1)}+q^{2(n+1)}\right) & \text{if}\quad m=n.\end{cases}\end{aligned}$$

□

Remark. One can observe that the "trigonometric polynomials" $U_n(p,q;e^{i\theta})$ can be considered as the boundary values for $z=e^{i\theta}$ of the following symmetric Laurent polynomials:

$$U_n(p,q;z)=p^nz^n+p^{n-1}qz^{n-2}+p^{n-2}q^2z^{n-4}+\cdots+pq^{n-1}\frac{1}{z^{n-2}}+\frac{q^n}{z^n},\ z\neq 0$$

$$U_n(p,q;z)=U_n\left(p,q;\frac{q}{pz}\right).$$

Remark. Together with the $U_n(p,q;e^{i\theta})$ "polynomials" one can consider the related family of "trigonometric polynomials"

$$T_n(p,q;e^{i\theta})=\frac{1}{2}(p^ne^{in\theta}+q^ne^{-in\theta}),$$

which for $p=q=1$ and $\theta\in[-\pi,\pi]$, give the Chebyshev polynomials of the first kind.

As the classical Chebyshev polynomials $U_n(x)$ and $T_n(x)$ are connected by several relations [1], [8] the "polynomials" $U_n(p,q;e^{i\theta})$ and $T_n(p,q;e^{i\theta})$ are connected as well by some relations, for example:

$$U_n(p,q;e^{i\theta}) = \frac{-2i}{(n+1)(pe^{i\theta}-qe^{-i\theta})} T'_{n+1}(p,q;e^{i\theta}).$$

$$T''_n(p,q;e^{i\theta}) + n^2 \cdot T_n(p,q;e^{i\theta}) = 0.$$

$$\frac{1}{2}(n^2-1)(p^{n+1}e^{i(n+1)\theta} + q^{n+1}e^{-i(n+1)\theta}) \\ + (n-1)U_n(pe^{i\theta}+qe^{-i\theta}) + n^2 \cdot T_n(p,q;e^{i\theta}) = 0.$$

$$T_n^2(p,q;e^{i\theta}) = \frac{1}{4}U_{n-1}^2(p,q;e^{i\theta}) \cdot (pe^{i\theta}-qe^{-i\theta})^2 + p^n q^n.$$

$$T_{2n}(p,q;e^{i\theta}) = \frac{1}{2}U_{n-1}^2(p,q;e^{i\theta}) \cdot (pe^{i\theta}-qe^{-i\theta})^2 + 2p^n q^n.$$

$$T_n(p,q;e^{i\theta}) = 2[(pe^{i\theta}-qe^{-i\theta}) \cdot U_n(p,q;e^{i\theta})]'.$$

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