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Certain subclasses of starlike functions of complex order involving the Hurwitz–Lerch Zeta function

Abstract. Making use of the Hurwitz–Lerch Zeta function, we define a new subclass of uniformly convex functions and a corresponding subclass of starlike functions with negative coefficients of complex order denoted by $TS_b^{\mu}(\alpha,\beta,\gamma)$ and obtain coefficient estimates, extreme points, the radii of close to convexity, starlikeness and convexity and neighbourhood results for the class $TS_b^{\mu}(\alpha,\beta,\gamma)$. In particular, we obtain integral means inequalities for the function $f(z)$ belongs to the class $TS_b^{\mu}(\alpha,\beta,\gamma)$ in the unit disc.

1. Introduction. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic and univalent in the open disc $U=\{z:z\in\mathcal{C},\ |z|<1\}$. Also denote by T a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n;\quad a_n\geq 0,\ z\in U, \tag{1.2}$$

introduced and studied by Silverman [25]. For functions $f\in\mathcal{A}$ given by (1.1) and $g\in\mathcal{A}$ given by $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$, we define the Hadamard

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product (or convolution) of f and g by

$$(1.3)\qquad (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

We recall here a general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ defined in [28] by

$$(1.4)\qquad \Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

($a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}$; $s \in \mathbb{C}$, $\Re(s) > 1$ and $|z| = 1$) where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, ($\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \dots\}$); $\mathbb{N} := \{1, 2, 3, \dots\}$. Several interesting properties and characteristics of the Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and López [5], Garg et al. [7], Lin and Srivastava [16], Lin et al. [17], and others. Srivastava and Attiya [27] (see also Răducanu and Srivastava [21], and Prajapat and Goyal [20]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$(1.5)\qquad \mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z)$$

($z \in U$; $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}$; $\mu \in \mathbb{C}$; $f \in \mathcal{A}$), where, for convenience,

$$(1.6)\qquad G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U).$$

We recall here the following relationships (given earlier by [20], [21]) which follow easily by using (1.1), (1.5) and (1.6)

$$(1.7)\qquad \mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} C_n(b, \mu) a_n z^n,$$

where

$$(1.8)\qquad C_n = C_n(b, \mu) = \left| \left(\frac{1+b}{n+b} \right)^{\mu} \right|$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}$; $\mu \in \mathbb{C}$.

(1) For $\mu = 0$

$$(1.9)\qquad \mathcal{J}_b^{0}(f)(z) := f(z).$$

(2) For $\mu = 1$; $b = 0$

$$(1.10)\qquad \mathcal{J}_b^{1}(f)(z) := \int_0^z \frac{f(t)}{t} dt := \mathcal{L} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{1}{n} \right) a_n z^n.$$

(3) For $\mu = 1$ and $b = \nu$ $(\nu > -1)$

$$\begin{aligned} \mathcal{J}_\nu^1(f)(z) &:= \mathcal{F}_\nu f(z) = \frac{1+\nu}{z^\nu}\int_0^z t^{\nu-1} f(t)dt \\ &:= z + \sum_{n=2}^{\infty}\left(\frac{1+\nu}{n+\nu}\right) a_n z^n. \end{aligned} \tag{1.11}$$

(4) For $\mu = \sigma$ $(\sigma > 0)$ and $b = 1$

$$\mathcal{J}_1^\sigma(f)(z) := z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\sigma} a_n z^n = \mathcal{I}^\sigma f(z), \tag{1.12}$$

where $\mathcal{L}$ and $\mathcal{F}_\nu$ are the integral operators introduced by Alexander [1] and Bernardi [3], respectively, and $\mathcal{I}^\sigma$ is the Jung–Kim–Srivastava integral operator [11] closely related to some multiplier transformation studied by Flet [6]. Motivated by the study on uniformly convex and uniformly starlike functions (see [9, 10, 12, 13, 14, 15, 22, 23]) and making use of the operator $\mathcal{J}_b^\mu$, we introduce a new subclass of analytic functions with negative coefficients and discuss some usual properties of the geometric function theory of this generalized function class.

For $-1 \le \alpha < 1$, $\beta \ge 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, we let $S_b^\mu(\alpha, \beta, \gamma)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - \alpha\right)\right\} > \beta\left|1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - 1\right)\right|, \tag{1.13}$$

$z \in U$ where $\mathcal{J}_b^\mu f(z)$ is given by (1.7). We also let

$$TS_b^\mu(\alpha, \beta, \gamma) = S_b^\mu(\alpha, \beta, \gamma) \cap T.$$

By suitably specializing the values of μ and b, the class $TS_b^\mu(\alpha, \beta, \gamma)$ reduces to various subclasses as illustrations, we present some examples of the cases.

Example 1. If $\mu = 0$, then

$$\begin{aligned} \mathbb{S}(\alpha, \beta, \gamma) := \Bigg\{ f \in \mathcal{A} : \operatorname{Re}&\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right\} \\ &> \beta\left|1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right|,\ z \in U\Bigg\}. \end{aligned}$$

Further $T\mathbb{S}(\alpha, \beta, \gamma) = \mathbb{S}(\alpha, \beta, \gamma) \cap T$, where T is given by (1.2).

Example 2. If $\mu = 1$; $b = 0$ and $f(z)$ is as defined in (1.10), then

$$\begin{aligned} R_\delta(\alpha, \beta, \gamma) := \Bigg\{ f \in \mathcal{A} : \operatorname{Re}&\left\{1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{L}f(z))'}{\mathcal{L}f(z)} - \alpha\right)\right\} \\ &> \beta\left|1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{L}f(z))'}{\mathcal{L}f(z)} - 1\right)\right|,\ z \in U\Bigg\}. \end{aligned}$$

Also $TR_\delta(\alpha,\beta,\gamma) = R_\delta(\alpha,\beta,\gamma) \cap T$, where T is given by (1.2) and $\mathcal{L}f(z)$ is given by $\mathcal{L}f(z) \coloneqq z - \sum_{n=2}^{\infty}\left(\frac{1}{n}\right)a_n z^n$.

Example 3. If $\mu = 1$, $b = \nu$ $(\nu > -1)$ and $f(z)$ is as defined in (1.11), then

$$B_\mu(\alpha,\beta,\gamma) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{\mathcal{F}_\nu f(z)}{\mathcal{F}_\nu f(z)} - \alpha\right)\right\} \right. \\ \left. > \beta\left|1 + \frac{1}{\gamma}\left(\frac{\mathcal{F}_\nu f(z)}{\mathcal{F}_\nu f(z)} - 1\right)\right|,\ z \in U \right\}.$$

Further, $TB_\mu(\alpha,\beta,\gamma) = B_\mu(\alpha,\beta,\gamma) \cap T$, where T is given by (1.2) and $\mathcal{F}_\nu f(z)$ is given by $\mathcal{F}_\nu f(z) \coloneqq z - \sum_{n=2}^{\infty}\left(\frac{1+\nu}{n+\nu}\right)a_n z^n$.

Example 4. If $\mu = \sigma$ $(\sigma > 0)$, $b = 1$ and $f(z)$ is defined in (1.12), then

$$L_c^a(\alpha,\beta,\gamma) \coloneqq \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - \alpha\right)\right\} \right. \\ \left. > \beta\left|1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - 1\right)\right|,\ z \in U \right\}.$$

Further $TL_c^a(\alpha,\beta,\gamma) = L_c^a(\alpha,\beta,\gamma) \cap T$, where T is given by (1.2) and $\mathcal{I}^\sigma f(z)$ is defined by $\mathcal{I}^\sigma f(z) \coloneqq z - \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\sigma a_n z^n$.

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bound, extreme points, radii of close to convexity, starlikeness and convexity for the class $TS_b^\mu(\alpha,\beta,\gamma)$. Further, we obtain neighbourhood results and integral means inequalities for aforementioned class.

2. Basic properties. In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the class $TS_b^\mu(\alpha,\beta,\gamma)$.

Theorem 2.1. *A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $TS_b^\mu(\alpha,\beta,\gamma)$ is*

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta) - (\alpha-\beta)]C_n a_n \le (1-\alpha) + |\gamma|(1-\beta), \tag{2.1}$$

where $-1 \le \alpha < 1$, $\beta \ge 0$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$.

Proof. Assume that $f(z) \in TS_b^\mu(\alpha,\beta,\gamma)$, then

$$\mathrm{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - \alpha\right)\right\} > \beta\left|1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - 1\right)\right|,$$

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(1-\alpha)-\sum\limits_{n=2}^{\infty}(n-\alpha)C_n a_n z^n}{z-\sum\limits_{n=2}^{\infty}C_n a_n z^n}\right)\right\}$$

$$>\beta\left|1-\frac{1}{\gamma}\left(\frac{\sum\limits_{n=2}^{\infty}(n-1)C_n a_n z^n}{z-\sum\limits_{n=2}^{\infty}C_n a_n z^n}\right)\right|.$$

If we let $z\to 1$ along the real axis, we have

$$\left\{1+\frac{1}{|\gamma|}\left(\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}(n-\alpha)C_n|a_n|}{1-\sum\limits_{n=2}^{\infty}C_n|a_n|}\right)\right\}$$

$$>\beta\left[1-\frac{1}{|\gamma|}\left(\frac{\sum\limits_{n=2}^{\infty}(n-1)C_n|a_n|}{1-\sum\limits_{n=2}^{\infty}C_n|a_n|}\right)\right].$$

The simple computational leads the desired inequality

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]C_n a_n\leq(1-\alpha)+|\gamma|(1-\beta).$$

Conversely, suppose that (2.1) is true for $z\in U$, then

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(\mathcal{J}_b^{\mu}f(z))'}{\mathcal{J}_b^{\mu}f(z)}-\alpha\right)\right\}-\beta\left|1+\frac{1}{\gamma}\left(\frac{z(\mathcal{J}_b^{\mu}f(z))'}{\mathcal{J}_b^{\mu}f(z)}-1\right)\right|>0$$

if

$$1+\frac{1}{|\gamma|}\left(\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}(n-\alpha)C_n a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}C_n a_n|z|^{n-1}}\right)$$

$$-\beta\left[1-\frac{1}{|\gamma|}\left(\frac{\sum\limits_{n=2}^{\infty}(n-1)C_n a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}C_n a_n|z|^{n-1}}\right)\right]\geq 0,$$

that is, if

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]C_n a_n\leq(1-\alpha)+|\gamma|(1-\beta),$$

which completes the proof. $\square$

Corollary 2.2. *Let the function $f(z)$ defined by* (1.2) *belong to* $TS_b^\mu(\alpha,\beta,\gamma)$. *Then*

$$a_n \leq \frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]C_n} \tag{2.2}$$

$n \geq 2$, $-1 \leq \alpha < 1$, $\beta \geq 0$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$, *with equality for*

$$f(z) = z - \frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]C_n} z^n.$$

In the following theorem we give extreme points for the functions of the class $TS_b^\mu(\alpha,\beta,\gamma)$.

Theorem 2.3 (Extreme points). *Let*

$$\begin{aligned} &f_1(z) = z \quad \text{and} \\ &f_n(z) = z - \frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]C_n} z^n \quad \text{for } n = 2,3,4,\ldots. \end{aligned} \tag{2.3}$$

Then $f(z) \in TS_b^\mu(\alpha,\beta,\gamma)$ *if and only if* $f(z)$ *can be expressed in the form* $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, *where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$.

The proof of the Theorem 2.3 follows the lines similar to the proof of the theorem on extreme points given by Silverman [25].

3. Close-to-convexity, starlikeness and convexity. Now, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS_b^\mu(\alpha,\beta,\gamma)$.

Theorem 3.1. *Let* $f \in TS_b^\mu(\alpha,\beta,\gamma)$. *Then* f *is close-to-convex of order* δ $(0 \leq \delta < 1)$ *in the disc* $|z| < r_1$, *that is* $\mathrm{Re}\{f'(z)\} > \delta$, $(0 \leq \delta < 1)$, *where*

$$r_1 = \inf_{n \geq 2} \left[\frac{(1-\delta)}{n} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{[(1-\alpha)+|\gamma|(1-\beta)]} C_n \right]^{\frac{1}{n-1}}.$$

Proof. Given $f \in T$, and f close-to-convex of order δ, we have

$$|f'(z) - 1| < 1 - \delta. \tag{3.1}$$

For the left hand side of (3.1) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n |z|^{n-1} < 1.$$

Using the fact that $f \in TS_b^\mu(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{(1-\alpha)+|\gamma|(1-\beta)} C_n a_n < 1,$$

we can say (3.1) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{(1-\alpha)+|\gamma|(1-\beta)} C_n$$

or, equivalently,

$$|z| \leq \left[\frac{(1-\delta)[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{n[(1-\alpha)+|\gamma|(1-\beta)]} C_n\right]^{\frac{1}{n-1}},$$

which completes the proof. □

Theorem 3.2. *Let $f \in TS_b^\mu(\alpha,\beta,\gamma)$. Then*

(1) *f is starlike of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_2$, that is,* $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, *where*

$$r_2 = \inf_{n\geq 2} \left\{\frac{(1-\delta)}{(n-\delta)} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{[(1-\alpha)+|\gamma|(1-\beta)]} C_n\right\}^{\frac{1}{n-1}} \quad \textit{and}$$

(2) *f is convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z| < r_3$, that is* $\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta$, *where*

$$r_3 = \inf_{n\geq 2} \left\{\frac{(1-\delta)}{n(n-\delta)} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{[(1-\alpha)+|\gamma|(1-\beta)]} C_n\right\}^{\frac{1}{n-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.3).

Proof. Given $f \in T$ such that f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta. \tag{3.2}$$

For the left hand side of (3.2) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\ |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} a_n\ |z|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n\ |z|^{n-1} < 1.$$

Using the fact that $f \in TS_b^\mu(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{(1-\alpha)+|\gamma|(1-\beta)} C_n a_n \ < 1,$$

we can say (3.2) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} < \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{(1-\alpha)+|\gamma|(1-\beta)} C_n.$$

Or, equivalently,

$$|z|^{n-1} < \frac{(1-\delta)[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{(n-\delta)[(1-\alpha)+|\gamma|(1-\beta)]} C_n,$$

which yields the starlikeness of the family.

Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar to the proof of (1). □

4. Integral means. Motivated by Silverman [26], the following subordination result will be required in our present investigation.

Lemma 4.1 ([18]). *If the functions $f(z)$ and $g(z)$ are analytic in U with $g(z) \prec f(z)$, then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta, \quad \eta > 0, \ \ z = re^{i\theta} \ \text{and} \ 0 < r < 1. \tag{4.1}$$

Applying Theorem 2.1 with extremal function and Lemma 4.1, we prove the following theorem.

Theorem 4.2. *Let $\eta > 0$. If $f(z) \in TS_b^\mu(\alpha,\beta,\gamma)$, and $\{\Phi(\alpha,\beta,\gamma,n)\}_{n=2}^{\infty}$ is a non-decreasing sequence, then for $z = re^{i\theta}$ and $0 < r < 1$, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta, \tag{4.2}$$

where

$$f_2(z) = z - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Phi(\alpha,\beta,\gamma,2)} z^2,$$

and $\Phi(\alpha,\beta,\gamma,n) = [(n+|\gamma|)(1-\beta)-(\alpha-\beta)]C_n$.

Proof. Let $f(z)$ be of the form (1.2) and $f_2(z) = z - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Phi(\alpha,\beta,\gamma,2)} z^2$, then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1 - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Phi(\alpha,\beta,\gamma,2)} z\right|^\eta d\theta.$$

By Lemma 4.1, it suffices to show that

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} \prec 1-\frac{(1-\alpha)+|\gamma|(1-\beta)}{\Phi(\alpha,\beta,\gamma,2)} z.$$

Setting

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} = 1-\frac{(1-\alpha)+|\gamma|(1-\beta)}{\Phi(\alpha,\beta,\gamma,2)} w(z). \tag{4.3}$$

From (4.3) and (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,\gamma,n)}{(1-\alpha)+|\gamma|(1-\beta)} a_n z^{n-1} \right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,\gamma,n)}{(1-\alpha)+|\gamma|(1-\beta)} a_n \\ &\le |z| < 1. \end{aligned}$$

This completes the proof of Theorem 4.2. □

5. Inclusion relations involving $N_\delta(e)$. To study the inclusion relations involving $N_\delta(e)$ we need the following definitions. Following [2, 8, 19, 24], we define the n, δ neighbourhood of the function $f(z) \in T$ by

$$N_\delta(f) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|a_n - b_n| \le \delta \right\}. \tag{5.1}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|b_n| \le \delta \right\}. \tag{5.2}$$

Theorem 5.1. *Let*

$$\delta = \frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2}. \tag{5.3}$$

Then $TS_b^\mu(\alpha,\beta,\gamma) \subset N_\delta(e)$.

Proof. For $f \in TS_b^\mu(\alpha,\beta,\gamma)$, Theorem 2.1 yields

$$[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2 \sum_{n=2}^{\infty} a_n \le (1-\alpha)+|\gamma|(1-\beta)$$

so that

$$\sum_{n=2}^{\infty} a_n \le \frac{(1-\alpha)+|\gamma|(1-\beta)}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2}. \tag{5.4}$$

On the other hand, from (2.1) and (5.4) we have

$$\begin{aligned}(1-\beta)C_2\sum_{n=2}^{\infty} na_n &\le (1-\alpha)+|\gamma|(1-\beta)+[(\alpha-\beta)-|\gamma|(1-\beta)]C_2\sum_{n=2}^{\infty} a_n\\ &\le (1-\alpha)+|\gamma|(1-\beta)+[(\alpha-\beta)-|\gamma|(1-\beta)]\\ &\quad\times C_2\frac{(1-\alpha)+|\gamma|(1-\beta)}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2}\\ &\le \frac{[(1-\alpha)+|\gamma|(1-\beta)]2(1-\beta)}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]},\end{aligned}$$

$$\sum_{n=2}^{\infty} na_n \le \frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2}. \tag{5.5}$$ □

Now we determine the neighbourhood for each of the class $TS_b^{\mu}(\alpha,\beta,\gamma)$ which we define as follows. A function $f\in T$ is said to be in the class $TS_b^{\mu}(\alpha,\beta,\gamma,\eta)$ if there exists a function $g\in TS_b^{\mu}(\alpha,\beta,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\eta, \quad (z\in U,\ 0\le\eta<1). \tag{5.6}$$

Theorem 5.2. *If* $g\in TS_b^{\mu}(\alpha,\beta,\gamma)$ *and*

$$\eta=1-\frac{\delta[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2}{2[((2+|\gamma|)(1-\beta)-(\alpha-\beta))C_2-((1-\alpha)+|\gamma|(1-\beta))]}, \tag{5.7}$$

then $N_\delta(g)\subset TS_b^{\mu}(\alpha,\beta,\gamma,\eta)$.

Proof. Suppose that $f\in N_\delta(g)$, then we find from (5.1) that

$$\sum_{n=2}^{\infty} n|a_n-b_n|\le\delta,$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n-b_n|\le\frac{\delta}{2}.$$

Next, since $g\in TS_b^{\mu}(\alpha,\beta,\gamma)$, we have

$$\sum_{n=2}^{\infty} b_n\le\frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2}.$$

So that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum\limits_{n=2}^{\infty}|a_n-b_n|}{1-\sum\limits_{n=2}^{\infty}b_n}\\ &\leq \frac{\delta}{2}\frac{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]C_2}{[((2+|\gamma|)(1-\beta)-(\alpha-\beta))C_2-((1-\alpha)+|\gamma|(1-\beta))]}\\ &\leq 1-\eta\end{aligned}$$

provided that η is given by (5.7). Thus by definition, $f \in TS_b^{\mu}(\alpha,\beta,\gamma,\eta)$ for η given by (5.7), which completes the proof. □

Concluding remarks. By suitably specializing the various parameters involved in Theorem 2.1 to Theorem 5.2, we can state the corresponding results for the new subclasses defined in Example 1 to Example 4 and also for many relatively more familiar function classes.

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