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## On certain general integral operators of analytic functions


#### Abstract

In this paper, we obtain new sufficient conditions for the operators $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ and $G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ to be univalent in the open unit disc $\mathcal{U}$, where the functions $f_{1}, f_{2}, \ldots, f_{n}$ belong to the classes $\mathcal{S}^{\star}(a, b)$ and $\mathcal{K}(a, b)$. The order of convexity for the operators $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ and $G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ is also determined. Furthermore, and for $\beta=1$, we obtain sufficient conditions for the operators $F_{n}(z)$ and $G_{n}(z)$ to be in the class $\mathcal{K}(a, b)$. Several corollaries and consequences of the main results are also considered.


1. Introduction and definitions. Let $\mathcal{A}$ denote the class of functions of the form

$$
f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n}
$$

which are analytic in the open unit $\operatorname{disc} \mathcal{U}=\{z:|z|<1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of order $\gamma(0 \leq \gamma<1)$ if it satisfies

$$
\begin{equation*}
\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)>\gamma \quad(z \in \mathcal{U}) \tag{1.1}
\end{equation*}
$$

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Also, we say that a function $f(z) \in \mathcal{A}$ is said to be convex of order $\gamma(0 \leq$ $\gamma<1$ ) if it satisfies

$$
\begin{equation*}
\operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>\gamma \quad(z \in \mathcal{U}) \tag{1.2}
\end{equation*}
$$

We denote by $\mathcal{S}^{\star}(\gamma)$ and $\mathcal{K}(\gamma)$, respectively, the usual classes of starlike and convex functions of order $\gamma(0 \leq \gamma<1)$ in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^{\star}(a, b)$ if

$$
\begin{equation*}
\left|\frac{z f^{\prime}(z)}{f(z)}-a\right|<b \quad(z \in \mathcal{U} ;|a-1|<b \leq a) \tag{1.3}
\end{equation*}
$$

and a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}(a, b)$ if

$$
\begin{equation*}
\left|1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}-a\right|<b \quad(z \in \mathcal{U} ;|a-1|<b \leq a) \tag{1.4}
\end{equation*}
$$

From (1.3) and (1.4), we have

$$
\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)>a-b \quad(z \in \mathcal{U} ;|a-1|<b \leq a)
$$

and

$$
\operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>a-b \quad(z \in \mathcal{U} ;|a-1|<b \leq a)
$$

The class $\mathcal{S}^{\star}(a, b)$ was introduced by Jakubowski [12]. It is clear that $a>\frac{1}{2}$, $\mathcal{S}^{\star}(a, b) \subset \mathcal{S}^{\star}(a-b) \subset \mathcal{S}^{\star}(0) \equiv \mathcal{S}^{\star}$ and $\mathcal{K}(a, b) \subset \mathcal{K}(a-b) \subset \mathcal{K}(0) \equiv \mathcal{K}$. Further, applying the Briot-Bouquet differential subordination [9], we can easily see that $\mathcal{K}(a, b) \subset \mathcal{S}^{\star}(a, b)$.

Several authors (e.g., see $[4,5,6,8,10,11,15,16]$ ), obtained many sufficient conditions for the univalency of the integral operators

$$
\begin{equation*}
F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)=\left\{\beta \int_{0}^{z} t^{\beta-1} \prod_{i=1}^{n}\left(\frac{f_{i}(t)}{t}\right)^{\alpha_{i}} d t\right\}^{\frac{1}{\beta}} \tag{1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)=\left\{\beta \int_{0}^{z} t^{\beta-1} \prod_{i=1}^{n}\left(f_{i}^{\prime}(t)\right)^{\alpha_{i}} d t\right\}^{\frac{1}{\beta}} \tag{1.6}
\end{equation*}
$$

where the functions $f_{1}, f_{2}, \ldots, f_{n}$ belong to the class $\mathcal{A}$ and the parameters $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$, and $\beta$ are complex numbers such that the integrals in (1.5) and (1.6) exist. Here and throughout in the sequel every many-valued function is taken with the principal branch.

For $\beta=1$, we obtain the integral operators

$$
\begin{equation*}
F_{n}(z)=\int_{0}^{z}\left(\frac{f_{1}(t)}{t}\right)^{\alpha_{1}} \ldots\left(\frac{f_{n}(t)}{t}\right)^{\alpha_{n}} d t \tag{1.7}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{n}(z)=\int_{0}^{z}\left(f_{1}^{\prime}(t)\right)^{\alpha_{1}} \ldots\left(f_{n}^{\prime}(t)\right)^{\alpha_{n}} d t \tag{1.8}
\end{equation*}
$$

introduced and studied by Breaz and Breaz [5] and Breaz et al. [7], respectively.

In this paper, we obtain new sufficient conditions for the operators $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ and $G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ defined by (1.5) and (1.6) to be univalent in the open unit $\operatorname{disc} \mathcal{U}$, where the functions $f_{1}, f_{2}, \ldots, f_{n}$ belong to the above classes $\mathcal{S}^{\star}(a, b)$ and $\mathcal{K}(a, b)$. The order of convexity for the operators $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ and $G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ is also determined. Furthermore, we obtain sufficient conditions for the operators $F_{n}(z)$ and $G_{n}(z)$ defined by (1.5) and (1.6) to be in the class $\mathcal{K}(a, b)$.

In the proofs of our main results we need the following univalence criteria. The first result, i.e. Lemma 1.1 is a generalization of the wellknown univalence criterion of Becker [2] (which in fact corresponds to the case $\beta=\delta=1$ ), while the second, i.e. Lemma 1.2 is a generalization of Ahlfors' and Becker's univalence criterion [1,3] (which corresponds to the case $\beta=1$ ).

Lemma 1.1 ([13]). Let $\delta$ be a complex number with $\operatorname{Re}(\delta)>0$. If $f \in \mathcal{A}$ satisfies

$$
\frac{1-|z|^{2 \operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}\left|\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right| \leq 1
$$

for all $z \in \mathcal{U}$, then, for any complex number $\beta$ with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\delta)$, the integral operator

$$
F_{\beta}(z)=\left\{\beta \int_{0}^{z} t^{\beta-1} f^{\prime}(t) d t\right\}^{\frac{1}{\beta}}
$$

is in the class $\mathcal{S}$.
Lemma 1.2 ([14]). Let $\beta$ be a complex number with $\operatorname{Re}(\beta)>0$ and $c$ be a complex number with $|c| \leq 1, c \neq-1$. If $f \in \mathcal{A}$ satisfies

$$
\left.\left.|c| z\right|^{2 \beta}+\left(1-|z|^{2 \beta}\right) \frac{z f^{\prime \prime}(z)}{\beta f^{\prime}(z)} \right\rvert\, \leq 1
$$

for all $z \in \mathcal{U}$, then the integral operator

$$
F_{\beta}(z)=\left\{\beta \int_{0}^{z} t^{\beta-1} f^{\prime}(t) d t\right\}^{\frac{1}{\beta}}
$$

is in the class $\mathcal{S}$.
2. Univalence conditions for $\boldsymbol{F}_{\boldsymbol{\alpha}_{1}, \boldsymbol{\alpha}_{2}, \ldots, \boldsymbol{\alpha}_{\boldsymbol{n}}, \boldsymbol{\beta}}(\boldsymbol{z})$. We first prove

Theorem 2.1. Let $f_{i}(z) \in \mathcal{S}^{\star}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}, \alpha_{i} \in \mathbb{C}$ for all $i=1, \ldots, n$, and $\delta \in \mathbb{C}$ with

$$
\begin{equation*}
\operatorname{Re}(\delta) \geq 2 \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i} \tag{2.1}
\end{equation*}
$$

Then for any $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\delta)$, the integral operator $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ defined by (1.5) is analytic and univalent in $\mathcal{U}$.

Proof. Defining

$$
h(z)=\int_{0}^{z} \prod_{i=1}^{n}\left(\frac{f_{i}(t)}{t}\right)^{\alpha_{i}} d t
$$

we observe that $h(0)=h^{\prime}(0)-1=0$, where

$$
\begin{equation*}
h^{\prime}(z)=\prod_{i=1}^{n}\left(\frac{f_{i}(z)}{z}\right)^{\alpha_{i}} \tag{2.2}
\end{equation*}
$$

Differentiating both sides of (2.2) logarithmically, we obtain

$$
\frac{z h^{\prime \prime}(z)}{h^{\prime}(z)}=\sum_{i=1}^{n} \alpha_{i}\left(\frac{z f_{i}^{\prime}(z)}{f_{i}(z)}-1\right)
$$

which is equivalent to

$$
\begin{equation*}
\frac{z h^{\prime \prime}(z)}{h^{\prime}(z)}=\sum_{i=1}^{n} \alpha_{i}\left(\frac{z f_{i}^{\prime}(z)}{f_{i}(z)}-a_{i}\right)+\sum_{i=1}^{n} \alpha_{i} a_{i}-\sum_{i=1}^{n} \alpha_{i} \tag{2.3}
\end{equation*}
$$

Since $f_{i}(z) \in \mathcal{S}^{\star}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}$ for all $i=1,2, \ldots, n$, it follows from (2.3) that

$$
\begin{align*}
\left|\frac{z h^{\prime \prime}(z)}{h^{\prime}(z)}\right| & \leq \sum_{i=1}^{n}\left|\alpha_{i}\right|\left|\frac{z f_{i}^{\prime}(z)}{f_{i}(z)}-a_{i}\right|+\sum_{i=1}^{n}\left|\alpha_{i}\right|\left|a_{i}-1\right|  \tag{2.4}\\
& <2 \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i}
\end{align*}
$$

Multiplying both sides of (2.4) by $\frac{1-|z|^{2 \operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}$ and making use of (2.1), we obtain

$$
\begin{aligned}
\frac{1-|z|^{2 \operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}\left|\frac{z h^{\prime \prime}(z)}{h^{\prime}(z)}\right| & \leq 2\left(\frac{1-|z|^{2 \operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}\right) \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i} \\
& <\frac{2}{\operatorname{Re}(\delta)} \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i} \leq 1
\end{aligned}
$$

Applying Lemma 1.1 for the function $h(z)$, we prove that $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z) \in$ $\mathcal{S}$.

Letting $n=1, \alpha_{1}=\alpha, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 2.1, we have

Corollary 2.2. Let $f(z) \in \mathcal{S}^{\star}(a, b) ;|a-1|<b \leq a, \alpha \in \mathbb{C}$ and $\delta \in \mathbb{C}$ with $\operatorname{Re}(\delta)>2|\alpha| b$. Then for any $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\delta)$, the integral operator

$$
\begin{equation*}
F_{\alpha, \beta}(z)=\left\{\beta \int_{0}^{z} t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\alpha} d t\right\}^{\frac{1}{\beta}} \tag{2.5}
\end{equation*}
$$

is analytic and univalent in $\mathcal{U}$.
Making use of Lemma 1.2, we prove the following theorem:
Theorem 2.3. Let $f_{i}(z) \in \mathcal{S}^{\star}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}, \alpha_{i} \in \mathbb{C}$ for all $i=1,2, \ldots, n$, and $\beta \in \mathbb{C}$ with

$$
\operatorname{Re}(\beta) \geq 2 \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i}
$$

and

$$
\begin{equation*}
|c| \leq 1-\frac{2}{\operatorname{Re}(\beta)} \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i} \quad(c \in \mathbb{C}) \tag{2.6}
\end{equation*}
$$

Then the integral operator $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ defined by (1.5) is analytic and univalent in $\mathcal{U}$.

Proof. Let $f_{i}(z) \in \mathcal{S}^{\star}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}$ for all $i=1,2, \ldots, n$, it follows from (2.4) that

$$
\begin{aligned}
\left.\left.|c| z\right|^{2 \beta}+\left(1-|z|^{2 \beta}\right) \frac{z h^{\prime \prime}(z)}{\beta h^{\prime}(z)} \right\rvert\, & \leq|c|+\left|\frac{1-|z|^{2 \beta}}{\beta}\right|\left|\frac{z h^{\prime \prime}(z)}{h^{\prime}(z)}\right| \\
& \leq|c|+2\left|\frac{1-|z|^{2 \beta}}{\beta}\right| \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i} \\
& <|c|+\frac{2}{|\beta|} \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i} \\
& <|c|+\frac{2}{\operatorname{Re}(\beta)} \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i}
\end{aligned}
$$

which, in the light of the hypothesis (2.6), yields

$$
\left.\left.|c| z\right|^{2 \beta}+\left(1-|z|^{2 \beta}\right) \frac{z h^{\prime \prime}(z)}{\beta h^{\prime}(z)} \right\rvert\, \leq 1
$$

Finally, by applying Lemma 1.2 , we conclude that $F_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z) \in \mathcal{S}$.

Letting $n=1, \alpha_{1}=\alpha, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 2.3, we have

Corollary 2.4. Let $f(z) \in \mathcal{S}^{\star}(a, b) ;|a-1|<b \leq a, \alpha \in \mathbb{C}$, and $\beta \in \mathbb{C}$ with

$$
\operatorname{Re}(\beta) \geq 2|\alpha| b
$$

and

$$
|c| \leq 1-\frac{2}{\operatorname{Re}(\beta)}|\alpha| b \quad(c \in \mathbb{C})
$$

Then the integral operator $F_{\alpha, \beta}(z)$ defined by (2.5) is analytic and univalent in $\mathcal{U}$.
3. Univalence conditions for $G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \boldsymbol{\beta}}(z)$. Now, we prove

Theorem 3.1. Let $f_{i}(z) \in \mathcal{K}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}, \alpha_{i} \in \mathbb{C}$ for all $i=1, \ldots, n$, and $\delta \in \mathbb{C}$ with

$$
\operatorname{Re}(\delta) \geq 2 \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i}
$$

Then for any $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\delta)$, the integral operator $G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ defined by (1.6) is analytic and univalent in $\mathcal{U}$.

Proof. Defining

$$
h(z)=\int_{0}^{z} \prod_{i=1}^{n}\left(f_{i}^{\prime}(t)\right)^{\alpha_{i}} d t
$$

we observe that $h(0)=h^{\prime}(0)-1=0$. On the other hand, it is easy to see that

$$
\begin{equation*}
h^{\prime}(z)=\prod_{i=1}^{n}\left(f_{i}^{\prime}(z)\right)^{\alpha_{i}} \tag{3.1}
\end{equation*}
$$

Differentiating both sides of (3.1) logarithmically, we obtain

$$
\frac{z h^{\prime \prime}(z)}{h^{\prime}(z)}=\sum_{i=1}^{n} \alpha_{i}\left(\frac{z f_{i}^{\prime \prime}(z)}{f_{i}^{\prime}(z)}\right)
$$

Thus, we have

$$
\begin{equation*}
\frac{z h^{\prime \prime}(z)}{h^{\prime}(z)}=\sum_{i=1}^{n} \alpha_{i}\left(1+\frac{z f_{i}^{\prime \prime}(z)}{f_{i}^{\prime}(z)}-a_{i}\right)+\sum_{i=1}^{n} \alpha_{i}\left(a_{i}-1\right) \tag{3.2}
\end{equation*}
$$

Let $f_{i}(z) \in \mathcal{K}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}$, for all $i=1,2, \ldots, n$, and following the same steps in the proof of Theorem 2.1, we get the required result.

Letting $n=1, \alpha_{1}=\alpha, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 3.1, we have

Corollary 3.2. Let $f(z) \in \mathcal{K}(a, b) ;|a-1|<b \leq a, \alpha$ and $\delta \in \mathbb{C}$ with $\operatorname{Re}(\delta) \geq 2|\alpha| b$. Then for any $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\delta)$, the integral operator

$$
\begin{equation*}
G_{\alpha, \beta}(z)=\left\{\beta \int_{0}^{z} t^{\beta-1}\left(f^{\prime}(t)\right)^{\alpha} d t\right\}^{\frac{1}{\beta}} \tag{3.3}
\end{equation*}
$$

is analytic and univalent in $\mathcal{U}$.
Using (3.1), (1.4) and applying Lemma 1.2, we prove the following theorem:
Theorem 3.3. Let $f_{i}(z) \in \mathcal{K}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}, \alpha_{i} \in \mathbb{C}$ for all $i=1, \ldots, n$ and $\beta \in \mathbb{C}$ with

$$
\operatorname{Re}(\beta) \geq 2 \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i}
$$

and

$$
|c| \leq 1-\frac{2}{\operatorname{Re}(\beta)} \sum_{i=1}^{n}\left|\alpha_{i}\right| b_{i} \quad(c \in \mathbb{C})
$$

Then the integral operator $G_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, \beta}(z)$ defined by (1.6) is analytic and univalent in $\mathcal{U}$.

Letting $n=1, \alpha_{1}=\alpha, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 3.3, we have

Corollary 3.4. Let $f(z) \in \mathcal{K}(a, b) ;|a-1|<b \leq a, \alpha$ and $\beta \in \mathbb{C}$ with

$$
\operatorname{Re}(\beta) \geq 2|\alpha| b
$$

and

$$
|c| \leq 1-\frac{2}{\operatorname{Re}(\beta)}|\alpha| b
$$

Then the integral operator $G_{\alpha, \beta}(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$.
4. Order of convexity. Now, we prove

Theorem 4.1. Let $f_{i}(z) \in \mathcal{S}^{\star}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}$, and $\alpha_{i}>0$ for all $i=1, \ldots, n$, with

$$
0 \leq 1-\sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right)<1 \quad \text { and } \quad \sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right) \leq 1
$$

Then the integral operator $F_{n}(z)$ defined by (1.7) is in the class

$$
\mathcal{K}\left(1-\sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right)\right)
$$

Proof. From (1.7), it follows that

$$
\begin{equation*}
F_{n}^{\prime}(z)=\prod_{i=1}^{n}\left(\frac{f_{i}(z)}{z}\right)^{\alpha_{i}} \tag{4.1}
\end{equation*}
$$

Differentiating both sides of (4.1) logarithmically, we obtain

$$
1+\frac{z F_{n}^{\prime \prime}(z)}{F_{n}^{\prime}(z)}=\sum_{i=1}^{n} \alpha_{i}\left(\frac{z f_{i}^{\prime}(z)}{f_{i}(z)}\right)-\sum_{i=1}^{n} \alpha_{i}+1
$$

Since $f_{i}(z) \in \mathcal{S}^{\star}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}$ and $a_{i}>\frac{1}{2}$ for all $i=1,2, \ldots, n$, we have

$$
\begin{align*}
\operatorname{Re}\left(1+\frac{z F_{n}^{\prime \prime}(z)}{F_{n}^{\prime}(z)}\right) & =\sum_{i=1}^{n} \alpha_{i} \operatorname{Re}\left(\frac{z f_{i}^{\prime}(z)}{f_{i}(z)}\right)-\sum_{i=1}^{n} \alpha_{i}+1 \\
& \geq \sum_{i=1}^{n} \alpha_{i}\left(a_{i}-b_{i}-1\right)+1  \tag{4.2}\\
& >1-\sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right)
\end{align*}
$$

Therefore, $F_{n}(z)$ is convex of order $1-\sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right)$ in $\mathcal{U}$.

Letting $n=1, \alpha_{1}=\alpha, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 4.1, we have

Corollary 4.2. Let $f(z) \in \mathcal{S}^{\star}(a, b) ;|a-1|<b \leq a$, and $\alpha>0$ with $0 \leq$ $1-\alpha\left(b+\frac{1}{2}\right)<1$ and $\alpha\left(b+\frac{1}{2}\right) \leq 1$. Then $\int_{0}^{z}\left(\frac{f(t)}{t}\right)^{\alpha} d t \in \mathcal{K}\left(1-\alpha\left(b+\frac{1}{2}\right)\right)$.

Next, we prove
Theorem 4.3. Let $f_{i}(z) \in \mathcal{K}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i}$, and $\alpha_{i}>0$ for all $i=1, \ldots, n$, with

$$
0 \leq 1-\sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right)<1 \quad \text { and } \quad \sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right) \leq 1
$$

Then the integral operator $G_{n}(z)$ defined by (1.8) is in the class

$$
\mathcal{K}\left(1-\sum_{i=1}^{n} \alpha_{i}\left(b_{i}+\frac{1}{2}\right)\right)
$$

Proof. From (1.8), we have

$$
\begin{equation*}
1+\frac{z G_{n}^{\prime \prime}(z)}{G_{n}^{\prime}(z)}=\sum_{i=1}^{n} \alpha_{i}\left(1+\frac{z f_{i}^{\prime \prime}(z)}{f_{i}^{\prime}(z)}\right)-\sum_{i=1}^{n} \alpha_{i}+1 \tag{4.3}
\end{equation*}
$$

Let $f_{i}(z) \in \mathcal{K}\left(a_{i}, b_{i}\right) ;\left|a_{i}-1\right|<b_{i} \leq a_{i} ; a_{i}>\frac{1}{2}$ for all $i=1,2, \ldots, n$, and following the same steps in the proof of Theorem 4.1, we get the required result.

Letting $n=1, \alpha_{1}=\alpha, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 4.3, we have

Corollary 4.4. Let $f(z) \in \mathcal{K}(a, b) ;|a-1|<b \leq a$, and $\alpha>0$ with $0 \leq$ $1-\alpha\left(b+\frac{1}{2}\right)<1$ and $\alpha\left(b+\frac{1}{2}\right) \leq 1$. Then $\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha} d t \in \mathcal{K}\left(1-\alpha\left(b+\frac{1}{2}\right)\right)$.
5. Sufficient conditions for the operators $F_{n}(z)$ and $G_{n}(z)$.

Theorem 5.1. Let $f_{i}(z) \in \mathcal{S}^{\star}\left(\gamma_{i}\right) ; 0 \leq \gamma_{i}<1$, for all $i=1,2, \ldots, n$. Then the integral operator $F_{n}(z)$ defined by (1.7) is in the class $\mathcal{K}\left(a_{i}, b_{i}\right)$, where $a_{i}=\sum_{i=1}^{n} \alpha_{i} \gamma_{i}+1, b_{i}=\sum_{i=1}^{n} \alpha_{i}$ and $\sum_{i=1}^{n} \alpha_{i}\left(1-\gamma_{i}\right) \leq 1$ for all $i=1,2, \ldots, n$.

Proof. Let $f_{i}(z) \in \mathcal{S}^{\star}\left(\gamma_{i}\right) ; 0 \leq \gamma_{i}<1$, for all $i=1,2, \ldots, n$. Then it follows from (4.2) that

$$
\begin{aligned}
\operatorname{Re}\left(1+\frac{z F_{n}^{\prime \prime}(z)}{F_{n}^{\prime}(z)}\right) & =\sum_{i=1}^{n} \alpha_{i} \operatorname{Re}\left(\frac{z f_{i}^{\prime}(z)}{f_{i}(z)}\right)+1-\sum_{i=1}^{n} \alpha_{i} \\
& >\sum_{i=1}^{n} \alpha_{i} \gamma_{i}+1-\sum_{i=1}^{n} \alpha_{i}
\end{aligned}
$$

which proves that $F_{n}(z) \in \mathcal{K}\left(a_{i}, b_{i}\right)$, where $a_{i}=\sum_{i=1}^{n} \alpha_{i} \gamma_{i}+1$ and $b_{i}=$ $\sum_{i=1}^{n} \alpha_{i}$ for all $i=1,2, \ldots, n$.

Letting $n=1, \alpha_{1}=\alpha, \gamma_{1}=\gamma, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 5.1, we have

Corollary 5.2. Let $f(z) \in \mathcal{S}^{\star}(\gamma) ; 0 \leq \gamma<1$. Then $\int_{0}^{z}\left(\frac{f(t)}{t}\right)^{\alpha} d t \in$ $\mathcal{K}(\alpha \gamma+1, \alpha)$, where $0<\alpha(1-\gamma) \leq 1$.

Using (4.3), we can prove the following theorem:
Theorem 5.3. Let $f_{i}(z) \in \mathcal{K}\left(\gamma_{i}\right) ; 0 \leq \gamma_{i}<1$, for all $i=1,2, \ldots, n$. Then the integral operator $G_{n}(z)$ defined by (1.8) is in the class $\mathcal{K}\left(a_{i}, b_{i}\right)$, where $a_{i}=\sum_{i=1}^{n} \alpha_{i} \gamma_{i}+1, b_{i}=\sum_{i=1}^{n} \alpha_{i}$ and $\sum_{i=1}^{n} \alpha_{i}\left(1-\gamma_{i}\right) \leq 1$ for all $i=1,2, \ldots, n$.

Letting $n=1, \alpha_{1}=\alpha, \gamma_{1}=\gamma, a_{1}=a, b_{1}=b$ and $f_{1}=f$ in Theorem 5.3, we have

Corollary 5.4. Let $f(z) \in \mathcal{K}(\gamma) ; 0 \leq \gamma<1$. Then $\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha} d t \in \mathcal{K}(\alpha \gamma+$ $1, \alpha)$, where $0<\alpha(1-\gamma) \leq 1$.

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