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# Inclusion properties of certain subclasses of analytic functions defined by generalized Sălăgean operator

Abstract. Let $A$ denote the class of analytic functions with the normalization $f(0) = f'(0) - 1 = 0$ in the open unit disc $U = \{z : |z| < 1\}$. Set

$$f_\lambda^n(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n z^k \quad (n \in N_0;\ \lambda \geq 0;\ z \in U),$$

and define $f_{\lambda,\mu}^n$ in terms of the Hadamard product

$$f_\lambda^n(z) * f_{\lambda,\mu}^n = \frac{z}{(1-z)^\mu} \quad (\mu > 0;\ z \in U).$$

In this paper, we introduce several subclasses of analytic functions defined by means of the operator $I_{\lambda,\mu}^n : A \longrightarrow A$, given by

$$I_{\lambda,\mu}^n f(z) = f_{\lambda,\mu}^n(z) * f(z) \quad (f \in A;\ n \in N_0;\ \lambda \geq 0;\ \mu > 0).$$

Inclusion properties of these classes and the classes involving the generalized Libera integral operator are also considered.

**1. Introduction.** Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

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which are analytic in the open unit disc $U = \{z : |z| < 1\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. For $0 \leq \eta < 1$, we denote by $S^*(\eta)$, $K(\eta)$ and $C$ the subclasses of $A$ consisting of all analytic functions which are, respectively, starlike of order $\eta$, convex of order $\eta$ and close-to-convex of order $\eta$ in $U$ (see, e.g., Srivastava and Owa [11]).

For $n \in N_0 = N \cup \{0\}$, where $N = \{1, 2, \dots\}$, $\lambda \geq 0$ and $f$ given by (1.1), we consider the generalized Sălăgean operator defined as follows:

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k \quad (z \in U)\,. \tag{1.2}$$

The operator $D_\lambda^n$ was introduced and studied by Al-Oboudi [1] which reduces to the Sălăgean differential operator [10] for $\lambda = 1$.

Let $S$ be the class of all functions $\phi$ which are analytic and univalent in $U$ and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\operatorname{Re}\{\phi(z)\} > 0$ $(z \in U)$. The Hadamard product (or convolution) $f * g$ of two analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is given by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S^*(\eta; \phi)$, $K(\eta; \phi)$ and $C(\eta, \delta; \phi, \psi)$ of the class $A$ for $0 \leq \eta,\ \delta < 1$ and $\phi, \psi \in S$ (cf., [3], [5] and [7] ), which are defined by

$$S^*(\eta; \phi) = \left\{ f \in A : \frac{1}{1-\eta} \left( \frac{z f'(z)}{f(z)} - \eta \right) \prec \phi(z) \ \ (z \in U) \right\},$$

$$K(\eta; \Phi) = \left\{ f \in A : \frac{1}{1-\eta} \left( 1 + \frac{z f''(z)}{f'(z)} - \eta \right) \prec \phi(z) \ \ (z \in U) \right\}$$

and

$$C(\eta, \delta; \phi, \psi) = \left\{ f \in A : \exists g \in S^*(\eta; \phi) \text{ s. t. } \frac{1}{1-\delta} \left( \frac{z f'(z)}{g(z)} - \delta \right) \prec \psi(z) \right. \\ \left. (z \in U) \right\}.$$

We note that, for special choices for the functions $\phi$ and $\psi$ involved in these definitions, we can obtain the well-known subclasses of $A$. For example, we have

$$S^*\left(\eta; \frac{1+z}{1-z}\right) = S^*(\eta),\ \ K\left(\eta; \frac{1+z}{1-z}\right) = K(\eta)$$

and

$$C\left(0,0;\frac{1+z}{1-z},\frac{1+z}{1-z}\right)=C.$$

Setting

$$f_{\lambda}^{n}(z)=z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}z^{k}\quad\left(n\in N_{0};\ \lambda\geq 0\right),$$

we define the function $f_{\lambda,\mu}^{n}$ in terms of the Hadamard product by

$$f_{\lambda}^{n}(z)*f_{\lambda,\mu}^{n}(z)=\frac{z}{(1-z)^{\mu}}\quad\left(\mu>0;\ z\in U\right). \tag{1.3}$$

We now introduce the operator $I_{\lambda,\mu}^{n}:A\longrightarrow A$, which is defined here by

$$\begin{gathered}I_{\lambda,\mu}^{n}f(z)=f_{\lambda,\mu}^{n}(z)*f(z)=z+\sum_{k=2}^{\infty}\tfrac{(\mu)_{k-1}}{(k-1)![1+\lambda(k-1)]^{n}}a_{k}z^{k}\\(f\in A;\ n\in N_{0};\ \lambda\geq 0;\ \mu>0),\end{gathered} \tag{1.4}$$

where $(\theta)_{k}$ is the Pochhammer symbol defined, in terms of the Gamma function $\lceil$, by

$$(\theta)_{k}=\frac{\lceil(\theta+k)}{\lceil(\theta)}=\begin{cases}1 & (k=0,\ \theta\in C\backslash\{0\}),\\ \theta(\theta+1)\ldots(\theta+k-1) & (k\in N,\ \theta\in C).\end{cases}$$

We note that $I_{1,2}^{1}f(z)=f(z)$ and $I_{0,2}^{0}f(z)=zf'(z)$.

From (1.4), we obtain the following relations:

$$\lambda z(I_{\lambda,\mu}^{n+1}f(z))'=I_{\lambda,\mu}^{n}f(z)-(1-\lambda)I_{\lambda,\mu}^{n+1}f(z)\quad(\lambda>0) \tag{1.5}$$

and

$$z(I_{\lambda,\mu}^{n}f(z))'=\mu I_{\lambda,\mu+1}^{n}f(z)-(\mu-1)I_{\lambda,\mu}^{n}f(z). \tag{1.6}$$

Next, by using the operator $I_{\lambda,\mu}^{n}$, we introduce the following classes of analytic functions for $\phi,\psi$:

$$S_{\lambda,\mu}^{n}(\eta;\phi)=\left\{f\in A:I_{\lambda,\mu}^{n}f(z)\in S^{*}(\eta;\phi)\right\},$$

$$K_{\lambda,\mu}^{n}(\eta;\phi)=\left\{f\in A:I_{\lambda,\mu}^{n}f(z)\in K(\eta;\phi)\right\}$$

and

$$C_{\lambda,\mu}^{n}(\eta,\delta;\phi,\psi)=\left\{f\in A:I_{\lambda,\mu}^{n}f(z)\in C(\eta,\delta;\phi,\psi)\right\}.$$

We also note that

$$f(z)\in K_{\lambda,\mu}^{n}(\eta;\phi)\Longleftrightarrow zf'(z)\in S_{\lambda,\mu}^{n}(\eta;\phi). \tag{1.7}$$

In particular, we set

$$S_{\lambda,\mu}^{n}\left(\eta;\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right)=S_{\lambda,\mu}^{n}(\eta;A,B;\alpha)\quad(0<\alpha\leq 1;\ -1\leq B<A\leq 1)$$

and

$$K^n_{\lambda,\mu}\left(\eta; \left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right) = K^n_{\lambda,\mu}(\eta; A, B; \alpha) \quad (0 < \alpha \le 1;\ -1 \le B < A \le 1).$$

We note that for $\lambda = 1$ in the above classes, we obtain the following classes $S^n_{\mu}(\eta;\phi)$, $K^n_{\mu}(\eta;\phi)$ and $C^n_{\mu}(\eta,\delta;\phi,\psi)$.

In this paper, we investigate several inclusion properties of the classes $S^n_{\lambda,\mu}(\eta;\phi)$, $K^n_{\lambda,\mu}(\eta;\phi)$ and $C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$ associated with the operator $I^n_{\lambda,\mu}$. Some applications involving these and other classes of integral operators are also considered.

**2. Inclusion properties involving the operator $I^n_{\lambda,\mu}$.** The following lemmas will be required in our investigation.

**Lemma 1** ([4]). *Let $\phi$ be convex univalent in $U$ with $\phi(0) = 1$ and* $\text{Re}\{\mu\phi(z) + \nu\} > 0$ *$(\mu, \nu \in C)$. If $p$ is analytic in $U$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\mu p(z) + \nu} \prec \phi(z) \quad (z \in U)$$

*implies that*

$$p(z) \prec \phi(z) \quad (z \in U).$$

**Lemma 2** ([8]). *Let $\phi$ be convex univalent in $U$ and $w$ be analytic in $U$ with* $\text{Re}\{w(z)\} \ge 0$*. If $p$ is analytic in $U$ and $p(0) = \phi(0)$, then*

$$p(z) + w(z)zp'(z) \prec \phi(z) \quad (z \in U)$$

*implies that*

$$p(z) \prec \phi(z) \quad (z \in U).$$

At first, with the help of Lemma 1, we obtain the following theorem.

**Theorem 1.** *Let $n \in N_0$, $\lambda > 0$, $\mu \ge 1$ and* $\text{Re}\{(1-\eta)\phi(z) + \frac{1}{\lambda} - 1 + \eta\} > 0$*. Then we have*

$$S^n_{\lambda,\mu+1}(\eta;\phi) \subset S^n_{\lambda,\mu}(\eta;\phi) \subset S^{n+1}_{\lambda,\mu}(\eta;\phi)$$

*$(0 \le \eta < 1$; $\phi \in S)$.*

**Proof.** First of all, we will show that

$$S^n_{\lambda,\mu+1}(\eta;\phi) \subset S^n_{\lambda,\mu}(\eta;\phi).$$

Let $f \in S^n_{\lambda,\mu+1}(\eta;\phi)$ and put

$$p(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu}f(z)\right)'}{I^n_{\lambda,\mu}f(z)} - \eta\right), \tag{2.1}$$

where $p(z)$ is analytic in $U$ with $p(0) = 1$. Using the identity (1.6) in (2.1), we obtain

$$\mu \frac{I^n_{\lambda,\mu+1} f(z)}{I^n_{\lambda,\mu} f(z)} = (1-\eta)p(z) + \mu - 1 + \eta. \tag{2.2}$$

Differentiating (2.2) logarithmically with respect to $z$ and multiplying by $z$, we obtain

$$\frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu+1} f(z)\right)'}{I^n_{\lambda,\mu+1} f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{(1-\eta)p(z) + \mu - 1 + \eta} \tag{2.3}$$

$(z \in U)$. Applying Lemma 1 to (2.3), we see that $p(z) \prec \phi(z)$, that is, $f \in S^n_{\lambda,\mu}(\eta;\phi)$.

To prove the second part, let $f \in S^n_{\lambda,\mu}(\eta;\phi)$ and put

$$h(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^{n+1}_{\lambda,\mu} f(z)\right)'}{I^{n+1}_{\lambda,\mu} f(z)} - \eta\right),$$

where $h$ is analytic in $U$ with $h(0) = 1$. Then, by using the arguments similar to these detailed above with (1.5), it follows that $h \prec \phi$ $(z \in U)$, which implies that $f \in S^{n+1}_{\lambda,\mu}(\eta;\phi)$. This completes the proof of Theorem 1. □

**Theorem 2.** *Let $n \in N_0$, $\lambda > 0$ and $\mu \geq 1$. Then we have*

$$K^n_{\lambda,\mu+1}(\eta;\phi) \subset K^n_{\lambda,\mu}(\eta;\phi) \subset K^{n+1}_{\lambda,\mu}(\eta;\phi)$$

*$(0 \leq \eta < 1;\ \phi \in S)$.*

**Proof.** Applying (1.7) and Theorem 1, we observe that

$$\begin{aligned}
f(z) \in K^n_{\lambda,\mu+1}(\eta;\phi) &\iff I^n_{\lambda,\mu+1} f(z) \in K(\eta;\phi) \\
&\iff z(I^n_{\lambda,\mu+1} f(z))' \in S^*(\eta;\phi) \\
&\iff I^n_{\lambda,\mu+1}(zf'(z)) \in S^*(\eta;\phi) \\
&\iff zf'(z) \in S^n_{\lambda,\mu+1}(\eta;\phi) \\
&\implies zf'(z) \in S^n_{\lambda,\mu}(\eta;\phi) \\
&\iff I^n_{\lambda,\mu}(zf'(z)) \in S^*(\eta;\phi) \\
&\iff z(I^n_{\lambda,\mu} f(z))' \in S^*(\eta;\phi) \\
&\iff I^n_{\lambda,\mu} f(z) \in K(\eta;\phi) \\
&\iff f(z) \in K^n_{\lambda,\mu}(\eta;\phi)
\end{aligned}$$

and

$$\begin{aligned}
f(z) \in K^n_{\lambda,\mu}(\eta;\phi) &\iff zf'(z) \in S^n_{\lambda,\mu}(\eta;\phi) \\
&\implies zf'(z) \in S^{n+1}_{\lambda,\mu}(\eta;\phi) \\
&\iff z(I^{n+1}_{\lambda,\mu} f(z))' \in S^*(\eta;\phi) \\
&\iff I^{n+1}_{\lambda,\mu} f(z) \in K(\eta;\phi) \\
&\iff f(z) \in K^{n+1}_{\lambda,\mu}(\eta;\phi),
\end{aligned}$$

which evidently proves the theorem. □

**Remark.** Taking

$$\phi(z)=\left(\frac{1+Az}{1+Bz}\right)^{\alpha} \quad (-1\le B<A\le 1;\ 0<\alpha\le 1;\ z\in U)$$

in Theorems 1 and 2, we have the following corollary.

**Corollary 1.** *Let $n\in N_0$, $\lambda>0$ and $\mu\ge 1$. Then we have*

$$S^{n}_{\lambda,\mu+1}(\eta;\ A,B;\ \alpha)\subset S^{n}_{\lambda,\mu}(\eta;\ A,B;\ \alpha)\subset S^{n+1}_{\lambda,\mu}(\eta;\ A,B;\ \alpha)$$

*$(0\le\eta<1;\ -1\le B<A\le 1;\ 0<\alpha\le 1)$, and*

$$K^{n}_{\lambda,\mu+1}(\eta;\ A,B;\ \alpha)\subset K^{n}_{\lambda,\mu}(\eta;\ A,B;\ \alpha)\subset K^{n+1}_{\lambda,\mu}(\eta;\ A,B;\ \alpha)$$

*$(0\le\eta<1;\ -1\le B<A\le 1;\ 0<\alpha\le 1)$.*

Next, by using Lemma 2, we obtain the following inclusion relation for the class $C^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi)$.

**Theorem 3.** *Let $n\in N_0$, $\lambda>0$ and $\mu\ge 1$. Then we have*

$$C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi)\subset C^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi)\subset C^{n+1}_{\lambda,\mu}(\eta,\delta;\phi,\psi)$$

*$(0\le\eta,\ \delta<1;\ \phi,\psi\in S)$.*

**Proof.** We begin by proving that

$$C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi)\subset C^{n}_{\lambda,\mu}(\eta,\delta;\phi,\psi).$$

Let $f\in C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi)$. Then, in view of the definition of the class $C^{n}_{\lambda,\mu+1}(\eta,\delta;\phi,\psi)$, there exists a function $g\in S^{n}_{\lambda,\mu+1}(\eta;\phi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z\left(I^{n}_{\lambda,\mu+1}f(z)\right)'}{I^{n}_{\lambda,\mu+1}g(z)}-\delta\right)\prec\psi(z)\quad (z\in U).$$

Now let

$$p(z)=\frac{1}{1-\delta}\left(\frac{z\left(I^{n}_{\lambda,\mu}f(z)\right)'}{I^{n}_{\lambda,\mu}g(z)}-\delta\right),$$

where $p(z)$ is analytic in $U$ with $p(0)=1$. Using (1.6), we have

$$[(1-\delta)p(z)+\delta]I^{n}_{\lambda,\mu}g(z)+(\mu-1)I^{n}_{\lambda,\mu}f(z)=\mu I^{n}_{\lambda,\mu+1}f(z). \tag{2.4}$$

Differentiating (2.4) with respect to $z$ and multiplying by $z$, we obtain

$$\begin{aligned}(1-\delta)zp'(z)I^{n}_{\lambda,\mu}g(z)+[(1-\delta)p(z)+\delta]z(I^{n}_{\lambda,\mu}g(z))'\\ =\mu z(I^{n}_{\lambda,\mu+1}f(z))'-(\mu-1)\,z(I^{n}_{\lambda,\mu}f(z))'.\end{aligned} \tag{2.5}$$

Since $g(z) \in S^n_{\lambda,\mu+1}(\eta;\phi)$, by Theorem 1, $g \in S^n_{\lambda,\mu}(\eta;\phi)$. Let

$$q(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu}g(z)\right)'}{I^n_{\lambda,\mu}g(z)} - \eta\right).$$

Then, using (1.6) once again, we have

$$\mu\frac{I^n_{\lambda,\mu+1}g(z)}{I^n_{\lambda,\mu}g(z)} = (1-\eta)q(z) + \mu - 1 + \eta. \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\frac{1}{1-\delta}\left(\frac{z\left(I^n_{\lambda,\mu+1}f(z)\right)'}{I^n_{\lambda,\mu+1}g(z)} - \delta\right) = p(z) + \frac{zp'(z)}{(1-\eta)\,q(z) + \mu - 1 + \eta}.$$

Since $0 \le \eta < 1$, $\mu \ge 1$ and $q(z) \prec \phi(z)$ $(z \in U)$, we have

$$\mathrm{Re}\{(1-\eta)q(z) + \mu - 1 + \eta\} > 0 \quad (z \in U).$$

Hence, applying Lemma 2, we can show that $p(z) \prec \psi(z)$, so that $f \in C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$.

For the second part, by using the arguments similar to these detailed above with (1.5), we obtain

$$C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi) \subset C^{n+1}_{\lambda,\mu}(\eta,\delta;\phi,\psi).$$

This completes the proof of Theorem 3. □

**3. Inclusion properties involving the integral operator $F_c$.** In this section, we consider the generalized Libera integral operator $F_c$ (see [2], [6] and [9]) defined by

$$F_c(f) = F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (f \in A;\ c > -1). \tag{3.1}$$

We first prove the following theorem.

**Theorem 4.** *Let $c \ge 0$, $n \in N_0$, $\lambda > 0$ and $\mu > 0$. If $f \in S^n_{\lambda,\mu}(\eta;\phi)$ $(0 \le \eta < 1;\ \phi \in S)$, then we have $F_c(f) \in S^n_{\lambda,\mu}(\eta;\phi)$ $(0 \le \eta < 1;\ \phi \in S)$.*

**Proof.** Let $f \in S^n_{\lambda,\mu}(\eta;\phi)$ and put

$$p(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu}F_c(f)(z)\right)'}{I^n_{\lambda,\mu}F_c(f)(z)} - \eta\right), \tag{3.2}$$

where $p(z)$ is analytic in $U$ with $p(0) = 1$. From (3.1), we have

$$z(I^n_{\lambda,\mu}F_c(f)(z))' = (c+1)I^n_{\lambda,\mu}f(z) - cI^n_{\lambda,\mu}F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we have

$$(c+1)\frac{I^n_{\lambda,\mu}f(z)}{I^n_{\lambda,\mu}F_c(f)(z)} = (1-\eta)p(z) + c + \eta. \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to $z$ and multiplying by $z$, we obtain

$$p(z) + \frac{zp'(z)}{(1-\eta)p(z)+c+\eta} = \frac{1}{1-\eta}\left(\frac{z(I^n_{\lambda,\mu}f(z))'}{I^n_{\lambda,\mu}f(z)} - \eta\right) \quad (z \in U).$$

Hence from Lemma 1, we conclude that $p(z) \prec \phi(z)$ $(z \in U)$, which implies $F_c(f) \in S^n_{\lambda,\mu}(\eta;\phi)$. □

Next, we derive an inclusion property involving $F_c$, which is given by the following theorem.

**Theorem 5.** *Let $c \geq 0$, $\lambda > 0$, $n \in N_0$ and $\mu > 0$. If $f \in K^n_{\lambda,\mu}(\eta;\phi)$ $(0 \leq \eta < 1;\ \phi \in S)$, then we have*

$$F_c(f) \in K^n_{\lambda,\mu}(\eta;\phi) \quad (0 \leq \eta < 1;\ \phi \in S).$$

**Proof.** By applying Theorem 4, we have

$$\begin{aligned} f(z) \in K^n_{\lambda,\mu}(\eta;\phi) &\iff zf'(z) \in S^n_{\lambda,\mu}(\eta;\phi) \\ &\implies F_c(zf'(z)) \in S^n_{\lambda,\mu}(\eta;\phi) \\ &\iff z(F_c(f)(z))' \in S^n_{\lambda,\mu}(\eta;\phi) \\ &\iff F_c(f)(z) \in K^n_{\lambda,\mu}(\eta;\phi) \end{aligned}$$

which proves Theorem 5. □

From Theorems 4 and 5, we have the following corollary.

**Corollary 2.** *Let $c \geq 0$, $\lambda > 0$, $n \in N_0$ and $\mu > 0$. If $f(z)$ belongs to the class $S^n_{\lambda,\mu}(\eta;A,B;\alpha)$ (or $K^n_{\lambda,\mu}(\eta;A,B;\alpha)$) $(0 \leq \eta < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$, then $F_c(f)$ belongs to the class $S^n_{\lambda,\mu}(\eta;A,B;\alpha)$ (or $K^n_{\lambda,\mu}(\eta;A,B;\alpha)$) $(0 \leq \eta < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$.*

Finally, we prove the following theorem.

**Theorem 6.** *Let $c \geq 0$, $\lambda > 0$, $n \in N_0$ and $\mu > 0$. If $f \in C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$ $(0 \leq \eta,\ \delta < 1;\ \phi,\psi \in S)$, then we have $F_c(f) \in C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$ $(0 \leq \eta,\ \delta < 1;\ \phi,\psi \in S)$.*

**Proof.** Let $f \in C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$. Then, in view of the definition of the class $C^n_{\lambda,\mu}(\eta,\delta;\phi,\psi)$, there exists a function $g \in S^n_{\lambda,\mu}(\eta;\phi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z(I^n_{\lambda,\mu}f(z))'}{I^n_{\lambda,\mu}g(z)}-\delta\right) \prec \psi(z) \quad (z \in U). \tag{3.5}$$

Thus, we put

$$p(z) = \frac{1}{1-\delta}\left(\frac{z\left(I^n_{\lambda,\mu}F_c(f)(z)\right)'}{I^n_{\lambda,\mu}F_c(g)(z)}-\delta\right),$$

where $p(z)$ is analytic in $U$ with $p(0)=1$. Since $g(z) \in S^n_{\lambda,\mu}(\eta;\phi)$, we see from Theorem 4 that $F_c(g) \in S^n_{\lambda,\mu}(\eta;\phi)$. Using (3.3), we have

$$[(1-\delta)p(z)+\delta]I^n_{\lambda,\mu}F_c(g)(z) + cI^n_{\lambda,\mu}F_c(f)(z) = (c+1)I^n_{\lambda,\mu}f(z). \tag{3.6}$$

Differentiating (3.6) with respect to $z$ and multiplying by $z$, we obtain

$$(c+1)\frac{z(I^n_{\lambda,\mu}f(z))'}{I^n_{\lambda,\mu}F_c(g)(z)} = [(1-\delta)p(z)+\delta][(1-\eta)q(z)+c+\eta] + (1-\delta)zp'(z),$$

where

$$q(z) = \frac{1}{1-\eta}\left(\frac{z\left(I^n_{\lambda,\mu}F_c(g)(z)\right)'}{I^n_{\lambda,\mu}F_c(g)(z)}-\eta\right).$$

Hence, we have

$$\frac{1}{1-\delta}\left(\frac{z\left(I^n_{\lambda,\mu}f(z)\right)'}{I^n_{\lambda,\mu}g(z)}-\delta\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z)+c+\eta}.$$

The remaining part of the proof in Theorem 6 is similar to that of Theorem 3 and so we omit it. □

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