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On inclusion relationships of certain subclasses of meromorphic functions involving integral operator

Abstract. In this paper, we introduce some subclasses of meromorphic functions in the punctured unit disc. Several inclusion relationships and some other interesting properties of these classes are discussed.

1. Introduction. Let $\mathcal{M}$ denote the class of functions $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured open unit disc

$$E^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = E \backslash \{0\}.$$

If $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k, \tag{1.2}$$

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we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f\star g)(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kb_kz^k=(g\star f)(z)\quad (z\in E)\,. \tag{1.3}$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0)=1$ and

$$\int_0^{2\pi}\left|\frac{\operatorname{Re}p(z)-\rho}{1-\rho}\right|d\theta\le k\pi,\ z=re^{i\theta}, \tag{1.4}$$

where $k\geqslant 2$ and $0\le\rho<1$. This class was introduced by Padmanbhan et al. in [16]. We note that $P_k(0)=P_k$, see [17], $P_2(\rho)=P(\rho)$, the class of analytic functions with positive real part greater than ρ and $P_2(0)=P$, the class of functions with positive real part. From (1.4) we can easily deduce that $p(z)\in P_k(\rho)$ if and only if, there exists $p_1(z)$, $p_2(z)\in P(\rho)$ such that for $z\in E$,

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z). \tag{1.5}$$

In recent years, several families of integral operators and differential operators were introduced using Hadamard product (or convolution). For example, we choose to mention the Rushcheweyh derivative [18], the Carlson–Shaffer operator [1], the Dziok–Srivastava operator [4], the Noor integral operator [14], also see [3, 5, 6, 11]. Motivated by the work of N. E. Cho and K. I. Noor [2, 9], we introduce a family of integral operators defined on the space of meromorphic functions in the class $\mathcal{M}$. By using these integral operators, we define several subclasses of meromorphic functions and investigate various inclusion relationships and some other properties for the meromorphic function classes introduced here.

For complex parameters $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_s$ ($\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $j=1,\dots,s$; $\mathbb{Z}_0^-=\{0,-1,-2,\dots\}$) we now define the function $\phi(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ by

$$\phi(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)=\frac{1}{z}+\sum_{k=0}^{\infty}\frac{(\alpha_1)_{k+1}\dots(\alpha_q)_{k+1}}{(\beta_1)_{k+1}\dots(\beta_s)_{k+1}(k+1)!}z^k,$$

$(q\le s+1;\ s\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ \mathbb{N}=\{1,2,\dots\};\ z\in E)$,

where $(v)_k$ is the Pochhammer symbol (or shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k=\frac{\Gamma(v+k)}{\Gamma(v)}=\begin{cases}1 & \text{if } k=0 \text{ and } v\in\mathbb{C}\backslash\{0\}\\ v(v+1)\dots(v+k-1) & \text{if } k\in\mathbb{N} \text{ and } v\in\mathbb{C}.\end{cases}$$

Now we introduce the following operator

$$I_\mu^p(\alpha_1,\dots,\alpha_q,\beta_1,\dots,\beta_s):\mathcal{M}\longrightarrow\mathcal{M}$$

as follows:

Let $F_{\mu,p}(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+\mu+1}{\mu}\right)^p z^k$, $p \in \mathbb{N}_0$, $\mu \neq 0$ and let $F_{\mu,p}^{-1}(z)$ be defined such that

$$F_{\mu,p}(z) * F_{\mu,p}^{-1}(z) = \phi(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z).$$

Then

$$I_\mu^p(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_s) f(z) = F_{\mu,p}^{-1}(z) * f(z). \tag{1.6}$$

From (1.6) it can be easily seen

$$\begin{aligned} &I_\mu^p(\alpha_1, \dots \alpha_q, \beta_1, \dots \beta_s) f(z) \\ &= \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\mu}{k+\mu+1}\right)^p \frac{(\alpha_1)_{k+1} \dots (\alpha_q)_{k+1}}{(\beta_1)_{k+1} \dots (\beta_s)_{k+1} (k+1)!} \, a_k z^k. \end{aligned} \tag{1.7}$$

For conveniences, we shall henceforth denote

$$I_\mu^p(\alpha_1, \dots \alpha_q, \beta_1, \dots \beta_s) f(z) = I_\mu^p(\alpha_1, \beta_1) f(z). \tag{1.8}$$

For the choices of the parameters $p = 0$, $q = 2$, $s = 1$, the operator $I_\mu^p(\alpha_1, \beta_1) f(z)$ is reduced to an operator by N. E. Cho and K. I. Noor [2] and K. I. Noor [9] and when $p = 0$, $q = 2$, $s = 1, \alpha_1 = \lambda$, $\alpha_2 = 1$, $\beta_1 = (n+1)$, the operator $I_\mu^p(\alpha_1,\ \beta_1) f(z)$ is reduced to an operator recently introduced by S.-M. Yuan et al. in [20].

It can be easily verified from the above definition of the operator $I_\mu^p(\alpha_1, \beta_1)$ that

$$z(I_\mu^{p+1}(\alpha_1, \beta_1) f(z))' = \mu I_\mu^p(\alpha_1, \beta_1) f(z) - (\mu+1) I_\mu^{p+1}(\alpha_1, \beta_1) f(z) \tag{1.9}$$

and

$$z(I_\mu^p(\alpha_1, \beta_1) f(z))' = \alpha_1 I_\mu^p(\alpha_1+1, \beta_1) f(z) - (\alpha_1+1) I_\mu^p(\alpha_1, \beta_1) f(z). \tag{1.10}$$

By using the operator $I_\mu^p(\alpha_1, \beta_1)$, we now introduce the following subclasses of meromorphic functions:

Definition 1.1 ([9]). A function $f \in \mathcal{M}$ is said to belong to the class $MR_k(\gamma)$ for $z \in E^*$, $0 \leq \gamma < 1, k \geq 2$, if and only if

$$-\frac{zf'(z)}{f(z)} \in P_k(\gamma),$$

and $f \in MV_k(\gamma)$, for $z \in E^*$, $0 \leq \gamma < 1, k \geq 2$, if and only if

$$-\frac{(zf(z))'}{f'(z)} \in P_k(\gamma).$$

We call $f \in MR_k(\gamma)$ a meromorphic function with bounded radius rotation of order γ and $f \in MV_k(\gamma)$ a meromorphic function with bounded boundary rotation.

Definition 1.2. Let $f \in \mathcal{M}$, $0 \leq \gamma < 1$, $k \geq 2$, $z \in E^*$. Then

$$f \in MR^p_{k,\mu}(\alpha_1, \beta_1, \gamma) \text{ if and only if } I^p_\mu(\alpha_1, \beta_1)f \in MR_k(\gamma).$$

Also

$$f \in MV^p_{k,\mu}(\alpha_1, \beta_1, \gamma) \text{ if and only if } I^p_\mu(\alpha_1, \beta_1)f \in MV_k(\gamma), \quad z \in E^*.$$

We observe that, for $z \in E^*$,

$$f \in MV^p_{k,\mu}(\alpha_1, \beta_1, \gamma) \Leftrightarrow -zf' \in MR^p_{k,\mu}(\alpha_1, \beta_1, \gamma).$$

Definition 1.3. Let $\lambda \geq 0$, $f \in \mathcal{M}$, $p \in \mathbb{N}_0$, $0 \leq \gamma$, $\rho < 1$, $\mu > 0$ and $z \in E^*$. Then $f \in B^{\lambda,p}_{k,\mu}(\alpha_1, \beta_1, \gamma, \rho)$, if and only if there exists a function $g \in MV^p_{2,\mu}(\alpha_1, \beta_1, \gamma)$, such that

$$\left\{(1-\lambda)\frac{(I^p_\mu(\alpha_1, \beta_1)f(z))'}{(I^p_\mu(\alpha_1, \beta_1)g(z))'} + \lambda\left[-\frac{(z(I^p_\mu(\alpha_1, \beta_1)f(z))')'}{(I^p_\mu(\alpha_1, \beta_1)g(z))'}\right]\right\} \in P_k(\rho).$$

In particular, for $\lambda = 0 = p$, $k = q = \mu = 2$ and $s = 1$, we obtain the class of meromorphic close-to-convex function, see [7], see also K. I. Noor [9]. For $\lambda = 1$, $p = 0$, $k = q = \mu = 2$, $s = 1$, we have the class of meromorphic quasi-convex functions defined for $z \in E^*$. We note that the class C^* of quasi-convex univalent functions, analytic in E, was first introduced and studied in [12], see also [13, 15].

In order to establish our main results, we need the following lemma, which is properly known as the Miller–Mocanu Lemma.

Lemma 1.1 ([8]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u, v)$ be a complex valued function satisfying the conditions:*
(i) $\Psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,
(ii) $(1, 0) \in D$ and $\operatorname{Re}\Psi(1, 0) > 0$,
(iii) $\operatorname{Re}\Psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}\left(1 + u_2^2\right)$.

If $h(z) = 1 + c_1z + c_2z^2 + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re}(\Psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E.

2. Main results.

Theorem 2.1. *Let $\operatorname{Re}\alpha_1 > 0$, $\mu > 0$ and $0 \leq \gamma < 1$. Then*

$$MR^p_{k,\mu}(\alpha_1 + 1, \beta_1, \gamma) \subset MR^p_{k,\mu}(\alpha_1, \beta_1, \rho) \subset MR^{p+1}_{k,\mu}(\alpha_1, \beta_1, \eta).$$

Proof. We prove the first part of the Theorem 2.1 and the second part follows by using similar techniques. Let

$$f \in MR^p_{k,\mu}(\alpha_1 + 1, \beta_1, \gamma), \quad z \in E^*$$

and set

$$-\frac{(I^p_\mu(\alpha_1, \beta_1)f(z))'}{(I^p_\mu(\alpha_1, \beta_1)f(z))} = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z) = H(z). \tag{2.1}$$

Simple computation together with (2.1) and (1.10) yields

$$(2.2)\quad -\frac{(I^p_\mu(\alpha_1+1,\beta_1)f(z))'}{(I^p_\mu(\alpha_1+1,\beta_1)f(z))}=\left[H(z)+\frac{zH'(z)}{-H(z)+\alpha_1+1}\right]\in P_k(\gamma),\ \ z\in E.$$

Let

$$\Phi_{\alpha_1}(z)=\frac{1}{\alpha_1+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty}z^k\right]+\frac{\alpha_1}{\alpha_1+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty}kz^k\right],$$

then

$$\begin{aligned}H(z)*z\Phi_{\alpha_1}(z)&=\left[H(z)+\frac{zH'(z)}{-H(z)+\alpha_1+1}\right]\\&=\left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+\frac{zh_1'(z)}{-h_1(z)+\alpha_1+1}\right]\\&\quad-\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+\frac{zh_2'(z)}{-h_2(z)+\alpha_1+1}\right].\end{aligned}\tag{2.3}$$

Since $f\in MR^p_{k,\mu}(\alpha_1+1,\beta_1,\gamma)$, it follows from (2.2) and (2.3) that

$$\left[h_i(z)+\frac{zh_i'(z)}{-h_i(z)+\alpha_1+1}\right]\in P(\gamma),\quad i=1,2,\quad z\in E.$$

Let $h_i(z)=(1-\rho)p_i(z)+\rho$. Then

$$\left\{(1-\rho)p_i(z)+\rho-\gamma+\frac{(1-\rho)zp_i'(z)}{-(1-\rho)p_i(z)-\rho+\alpha_1+1}\right\}\in P,\quad z\in E.$$

We shall show that $p_i(z)\in P$, $i=1,2$.

We form the functional $\Psi(u,v)$ by taking $u=u_1+iu_2=p_i(z)$, $v=v_1+iv_2=zp_i'(z)$. The first two conditions of Lemma 1.1 can be easily verified. We need to verify condition (iii) as follows:

$$\Psi(u,v)=\left\{(1-\rho)u+\rho-\gamma+\frac{(1-\rho)v}{-(1-\rho)u-\rho+\alpha_1+1}\right\},$$

implies that

$$\operatorname{Re}\Psi(iu_2,v_1)=\rho-\gamma+\frac{(1-\rho)(\alpha_1+1-\rho)v_1}{(1-\rho)^2u_2^2+(-\rho+\alpha_1+1)^2}.$$

By taking $v_1\le-\frac{1}{2}(1+u_2^2)$, we have

$$\operatorname{Re}\Psi(iu_2,v_1)\le\frac{A+Bu_2^2}{2C},$$

where

$$\begin{aligned}A&=2(\rho-\gamma)(\alpha_1+1-\rho)^2-(1-\rho)(\alpha_1+1-\rho),\\B&=2(\rho-\gamma)(1-\rho)^2-(1-\rho)(\alpha_1+1-\rho),\\C&=(\alpha_1+1-\rho)^2+(1-\rho)^2u_2^2>0.\end{aligned}$$

We note that $\operatorname{Re}\Psi(iu_2, v_1) \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain

$$\rho = \frac{1}{4}\left\{(3 + 2\alpha_1 + 2\gamma) - \sqrt{(3 + 2\alpha_1 + 2\gamma)^2 - 8}\right\}, \tag{2.4}$$

and $B \leq 0$ gives us $0 \leq \rho < 1$.

Now using Lemma 1.1, we see that $p_i(z) \in P$ for $z \in E$, $i = 1, 2$ and hence $f \in MR^p_{k,\mu}(\alpha_1, \beta_1, \rho)$ with ρ given by (2.4). □

In particular, we note that

$$\rho = \frac{1}{4}\left\{(3 + 2\alpha_1) - \sqrt{(12\alpha_1 + 4\alpha_1^2) + 1}\right\}.$$

Theorem 2.2. *Let* $\operatorname{Re}\alpha_1$, $\mu > 0$. *Then*

$$MV^p_{k,\mu}(\alpha_1 + 1, \beta_1, \gamma) \subset MV^p_{k,\mu}(\alpha_1, \beta_1, \rho) \subset MV^{p+1}_{k,\mu}(\alpha_1, \beta_1, \eta).$$

Proof. We observe that

$$\begin{aligned} f(z) \in MV^p_{k,\mu}(\alpha_1 + 1, \beta_1, \gamma) &\Leftrightarrow -zf'(z) \in MR^p_{k,\mu}(\alpha_1 + 1, \beta_1, \gamma) \\ &\Rightarrow -zf'(z) \in MR^p_{k,\mu}(\alpha_1, \beta_1, \rho) \\ &\Leftrightarrow f(z) \in MV^p_{k,\mu}(\alpha_1, \beta_1, \rho), \end{aligned}$$

where ρ is given by (2.4).

The second part can be proved by means of similar arguments. □

Theorem 2.3. *Let* $\operatorname{Re}\alpha_1$, $\mu > 0$.*Then*

$$B^{\lambda,p}_{k,\mu}(\alpha_1 + 1, \beta_1, \gamma_1, \rho_1) \subset B^{\lambda,p}_{k,\mu}(\alpha_1, \beta_1, \gamma_2, \rho_2) \subset B^{\lambda,p+1}_{k,\mu}(\alpha_1, \beta_1, \gamma_3, \rho_3),$$

where $\gamma_i = \gamma_i(\rho_i, \mu)$, $i = 1, 2, 3$ *are given in the proof.*

Proof. We prove the first inclusion of this result and the other part follows along similar lines.

Let $f \in B^{\lambda,p}_{k,\mu}(\alpha_1 + 1, \beta_1, \gamma_1, \rho_1)$. Then by Definition 1.3, there exists a function $g \in MV^p_{2,\mu}(\alpha_1 + 1, \beta_1, \gamma_1)$ such that

$$\begin{aligned} &\left\{(1 - \lambda)\frac{(I^p_\mu(\alpha_1 + 1, \beta_1)f(z))'}{(I^p_\mu(\alpha_1 + 1, \beta_1)g(z))'}\right. \\ &\qquad \left. + \lambda\left[-\frac{(z(I^p_\mu(\alpha_1 + 1, \beta_1)f(z))')'}{(I^p_\mu(\alpha_1 + 1, \beta_1)g(z))'}\right]\right\} \in P_k(\rho_1). \end{aligned} \tag{2.5}$$

Set

$$h(z) = \left\{(1 - \lambda)\frac{(I^p_\mu(\alpha_1, \beta_1)f(z))'}{(I^p_\mu(\alpha_1, \beta_1)g(z))'} + \lambda\left[-\frac{(z(I^p_\mu(\alpha_1, \beta_1)f(z))')'}{(I^p_\mu(\alpha_1, \beta_1)g(z))'}\right]\right\}, \tag{2.6}$$

where $h(z)$ is an analytic function in E with $h(0) = 1$.

Now, $g \in MV^{p}_{2,\mu}(\alpha_1+1,\beta_1,\gamma_1) \subset MV^{p}_{2,\mu}(\alpha_1,\beta_1,\gamma_2)$, where γ_2 is given by the equation

$$2\gamma_2^2 + (3+2\alpha_1-2\gamma_1)\gamma_2 - \{2\gamma_1(1+\alpha_1)+1\} = 0. \tag{2.7}$$

Therefore,

$$q(z) = -\frac{(zI^p_\mu(\alpha_1,\beta_1)g(z))'}{(I^p_\mu(\alpha_1,\beta_1)g(z))'} \in P(\gamma_2), \quad z \in E.$$

By using (1.10), (2.5), (2.6) and (2.7), we have

$$\left\{h(z) + \frac{\lambda z h'(z)}{-q(z)+\alpha_1+1}\right\} \in P_k(\rho_1),\ q(z) \in P(\gamma_2), \quad z \in E. \tag{2.8}$$

With

$$h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)[(1-\rho_2)h_1(z)+\rho_2] - \left(\frac{k}{4}-\frac{1}{2}\right)[(1-\rho_2)h_2(z)+\rho_2],$$

(2.8) can be written as

$$\left(\frac{k}{4}+\frac{1}{2}\right)\left\{(1-\rho_2)h_1(z)+\rho_2+\frac{(1-\rho_2)\lambda z h_1'(z)}{-q(z)+\alpha_1+1}\right\}$$
$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(1-\rho_2)h_2(z)+\rho_2+\frac{(1-\rho_2)\lambda z h_2'(z)}{-q(z)+\alpha_1+1}\right\},$$

where

$$\left\{(1-\rho_2)h_i(z)+\rho_2+\frac{(1-\rho_2)\lambda z h_i'(z)}{-q(z)+\alpha_1+1}\right\} \in P(\rho_1),\ z \in E,\ i=1,2.$$

That is

$$\left\{(1-\rho_2)h_i(z)+\rho_2-\rho_1+\frac{(1-\rho_2)\lambda z h_i'(z)}{-q(z)+\alpha_1+1}\right\} \in P,\ z \in E,\ i=1,2.$$

We form the functional $\Psi(u,v)$ by choosing $u = u_1+iu_2 = h_i(z)$, $v = v_1+iv_2 = zh_i'(z)$, and

$$\Psi(u,v) = \left\{(1-\rho_2)u+\rho_2-\rho_1+\frac{(1-\rho_2)\lambda v}{-q(z)+\alpha_1+1}\right\}, \quad (q = q_1+iq_2).$$

The first two conditions of Lemma 1.1 are clearly satisfied. We verify (iii), with $v_1 \le -\frac{1}{2}(1+u_2^2)$ as follows:

$$\operatorname{Re}\Psi(iu_2,v_1) = \rho_2-\rho_1+\operatorname{Re}\left\{\frac{\lambda(1-\rho_2)v_1\{(-q_1+\alpha_1+1)+iq_2\}}{(-q_1+\alpha_1+1)^2+q_2^2}\right\}$$
$$\le \frac{2(\rho_2-\rho_1)\left|-q+\alpha_1+1\right|^2 - \lambda(1-\rho_2)\{(-q_1+\alpha_1+1)(1+u_2^2)}{2\left|-q+\alpha_1+1\right|^2}$$
$$= \frac{A+Bu_2^2}{2C} \le 0,$$

if $A \le 0$ and $B \le 0$, where

$$A = 2(\rho_2 - \rho_1)\left|-q+\alpha_1+1\right|^2 - \lambda(1-\rho_2)\{(-q_1+\alpha_1+1),$$
$$B = -\lambda(1-\rho_2)\{(-q_1+\alpha_1+1) \le 0,$$
$$C = \left|-q+\alpha_1+1\right|^2 > 0.$$

From $A \le 0$, we obtain

$$\rho_2 = \frac{2\rho_1\left|-q+\alpha_1+1\right|^2 + \lambda\,\mathrm{Re}(-q(z)+\alpha_1+1)}{2\left|-q+\alpha_1+1\right|^2 + \lambda\,\mathrm{Re}(-q(z)+\alpha_1+1)}.$$

Hence, using Lemma 1.1, it follows that $h(z)$, defined by (2.6), belongs to $P_k(\rho_2)$ and thus $f \in B^{\lambda,p}_{k,\mu}(\alpha_1,\beta_1,\gamma_2,\rho_2)$ for $z \in E^*$. This completes the proof of the first part. The second part of this result can be obtained by using similar techniques and the relation (1.9). □

Theorem 2.4. *Let* $\mathrm{Re}\,\alpha_1,\ \mu > 0$. *Then*
(i) $B^{\lambda,p}_{k,\mu}(\alpha_1,\beta_1,\gamma,\rho) \subset B^{0,p}_{k,\mu}(\alpha_1,\beta_1,\gamma,\rho_4)$.
(ii) $B^{\lambda_1,p}_{k,\mu}(\alpha_1,\beta_1,\gamma,\rho) \subset B^{\lambda_2,p}_{k,\mu}(\alpha_1,\beta_1,\gamma,\rho)$, *for* $0 \le \lambda_2 < \lambda_1$.

Proof. (i). Let

$$h(z) = \frac{(I^p_\mu(\alpha_1,\beta_1)f(z))'}{(I^p_\mu(\alpha_1,\beta_1)g(z))'},$$

$h(z)$ is analytic in E and $h(0)=1$. Then

$$\left\{(1-\lambda)\frac{(I^p_\mu(\alpha_1,\beta_1)f(z))'}{(I^p_\mu(\alpha_1,\beta_1)g(z))'} + \lambda\left[-\frac{(z(I^p_\mu(\alpha_1,\beta_1)f(z))')'}{(I^p_\mu(\alpha_1,\beta_1)g(z))'}\right]\right\} = h(z) + \lambda\frac{zh'(z)}{-h_0(z)}, \tag{2.10}$$

where

$$h_0(z) = -\frac{(z(I^p_\mu(\alpha_1,\beta_1)f(z))')'}{(I^p_\mu(\alpha_1,\beta_1)g(z))'} \in P(\gamma).$$

Since $f \in B^{\lambda,p}_{k,\mu}(\alpha_1,\beta_1,\gamma,\rho)$, it follows that

$$\left[h(z) + \lambda\frac{zh'(z)}{-h_0(z)}\right] \in P_k(\rho),\ h_0 \in P(\gamma),\ \text{for } z \in E.$$

Let

$$h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z).$$

Thus (2.10) implies that

$$\left[h_i(z) + \lambda\frac{zh_i'(z)}{-h_0(z)}\right] \in P(\rho),\quad z \in E,\ i = 1,2.$$

and using similar techniques, together with Lemma 1.1, it follows that $h_i(z) \in P(\rho_4)$, $i = 1, 2$, where

$$\rho_4 = \frac{2\rho\,|h_0(z)|^2 + \lambda \operatorname{Re} h_0(z)}{2\,|h_0(z)|^2 + \lambda \operatorname{Re} h_0(z)}.$$

Therefore $h(z) \in P_k(\rho_4)$, and $f \in B^{0,p}_{k,\mu}(\alpha_1, \beta_1, \gamma, \rho_4)$, for $z \in E^*$. In particular, it can be shown that $h_i(z) \in P(\rho)$, $i = 1, 2$. Consequently $h \in P_k(\rho)$ and $f \in B^{0,p}_{k,\mu}(\alpha_1, \beta_1, \gamma, \rho)$ in E^*.

For $\lambda_2 = 0$, we have part (i). Therefore, we let $\lambda_2 > 0$ and $f \in B^{\lambda_1,p}_{k,\mu}(\alpha_1, \beta_1, \gamma, \rho)$. There exist two functions $H_1(z)$, $H_2(z) \in P_k(\rho)$ such that

$$\left\{(1-\lambda_1)\frac{(I^p_\mu(\alpha_1+1,\beta_1)f(z))'}{(I^p_\mu(\alpha_1+1,\beta_1)g(z))'} + \lambda_1\left[-\frac{(z(I^p_\mu(\alpha_1+1,\beta_1)f(z))')'}{(I^p_\mu(\alpha_1+1,\beta_1)g(z))'}\right]\right\} = H_1(z)$$

$$\frac{(I^p_\mu(\alpha_1+1,\beta_1)f(z))'}{(I^p_\mu(\alpha_1+1,\beta_1)g(z))'} = H_2(z),$$

where $g(z) \in MV^p_{2,\mu}(\alpha_1, \beta_1, \gamma)$.

Now

$$\begin{aligned}&\left\{(1-\lambda_2)\frac{(I^p_\mu(\alpha_1+1,\beta_1)f(z))'}{(I^p_\mu(\alpha_1+1,\beta_1)g(z))'} + \lambda_2\left[-\frac{(z(I^p_\mu(\alpha_1+1,\beta_1)f(z))')'}{(I^p_\mu(\alpha_1+1,\beta_1)g(z))'}\right]\right\}\\ &= \frac{\lambda_2}{\lambda_1}H_1(z) + \left(1-\frac{\lambda_2}{\lambda_1}\right)H_2(z).\end{aligned} \tag{2.11}$$

Since the class $P_k(\rho)$ is convex, see [10], it follows that the right hand side of (2.11) belongs to $P_k(\rho)$ and this shows that $f \in B^{\lambda_2,p}_{k,\mu}(\alpha_1, \beta_1, \gamma, \rho)$ for $z \in E^*$. This completes the proof. □

Inclusion properties involving the integral operator F_c. Consider the operator F_c, defined by

$$F_c(f)(z) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)dt \quad (f \in \mathcal{M};\ c > 0). \tag{2.12}$$

From the Definition of F_c defined by (2.12), we observe that

$$z((I^p_\mu(\alpha_1,\beta_1)F_c f(z))' = c(I^p_\mu(\alpha_1,\beta_1)f(z) - (c+1)(I^p_\mu(\alpha_1,\beta_1)F_c f(z). \tag{2.13}$$

Using (2.12), (2.13) with similar arguments as used earlier, we can prove the following theorem.

Theorem 2.5. *Let $f \in MR^p_{k,\mu}(\alpha_1, \beta_1, \gamma)$ or $f \in MV^p_{k,\mu}(\alpha_1, \beta_1, \gamma)$ or $f \in B^{\lambda,p}_{k,\mu}(\alpha_1, \beta_1, \gamma, \rho)$, for $z \in E$. Then $F_c(f)$ defined by (2.12) is also in the same class for $z \in E^*$.*

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