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Solution of a class of the first kind singular integral equation with multiplicative Cauchy kernel

Abstract. In the present paper, we give the exact solutions of a singular equation with logarithmic singularities in two classes of functions and construct formulae for the approximate solutions.

1. Introduction. Let us consider a singular integral equation of the form

$$\frac{1}{\pi^2}\iint\limits_D \frac{\varphi(\xi,\eta)}{(\xi-x)(\eta-y)}d\xi d\eta = f(x,y)\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}, \quad (x,y)\in D, \tag{1}$$

where $D=(-1,1)\times(-1,1)$, $f(x,y)$ is a given Hölder continuous function in $\overline{D}$, and $\varphi(x,y)$ is an unknown function. The equation (1) has applications in the theory of aeroelasticity [1].

Note that the equation without logarithmic singularities was many times considered in different classes of functions. In the literature the solutions of the equation (1) in bounded domains [2, 5, 6, 9] as well as unbounded [3, 4, 7, 8, 10], are known for both single and multiple integrals.

Let us introduce the function classes that will be used here.

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Definition 1. We write $\varphi(x,y) \in h(-1,1) \times h(-1,1)$, if it satisfies the Hölder inequality

$$|\varphi(x,y) - \varphi(x',y')| \leq K_1 |x-x'|^{\mu_1} + K_2 |y-y'|^{\mu_2}, \tag{2}$$

where $0 < \mu_1, \mu_2 \leq 1$, and $K_1, K_2 > 0$ are constants independent of the choice of points $(x,y),\ (x',y') \in \overline{D}$.

Definition 2. We write $\varphi(x) \in h_0$, if it satisfies the Hölder inequality in each point of the interval $(-1,1)$, and has the following representation

$$\varphi(x) = \frac{\varphi_1^*(x)}{(1+x)^{\alpha_1}}, \quad \varphi(x) = \frac{\varphi_2^*(x)}{(1-x)^{\alpha_2}} \tag{3}$$

in a neighborhood of the points $x = -1,\ x = 1$ respectively, with $0 \leqslant \alpha_1, \alpha_2 < 1$, and $\varphi_1^*(x)$, $\varphi_2^*(x)$ satisfying the Hölder inequality in the interval $[-1,1]$.

Definition 3. We write $\varphi(x,y) \in h_0 \times h_0$, if it satisfies the Hölder inequality (2) in each interior point of the set D, and has the following representation

$$\begin{aligned} \varphi(x,y) &= \frac{\varphi_1^*(x,y)}{(1+x)^{\alpha_1}}, \quad \varphi(x,y) = \frac{\varphi_2^*(x,y)}{(1-x)^{\alpha_2}}, \\ \varphi(x,y) &= \frac{\varphi_3^*(x,y)}{(1+y)^{\alpha_3}}, \quad \varphi(x,y) = \frac{\varphi_4^*(x,y)}{(1-y)^{\alpha_4}} \end{aligned} \tag{4}$$

in a neighborhood of points of the lines $x = -1,\ x = 1,\ y = -1,\ y = 1$, respectively, with $0 \leqslant \alpha_k < 1$, and $\varphi_k^*(x,y)$ belonging to the class $h(-1,1) \times h(-1,1)$, for $k = 1, \ldots, 4$.

2. Exact solution in the class $h_0 \times h_0$.

Theorem 1. *Let the function $f(x,y) \in h(-1,1) \times h(-1,1)$. Then the general solution of the equation (1) in the class $h_0 \times h_0$ has the form*

$$\varphi(x,y) = \frac{1}{\sqrt{(1-x^2)(1-y^2)}} R(f;x,y) + \frac{\gamma_1(x)}{\sqrt{1-y^2}} + \frac{\gamma_2(y)}{\sqrt{1-x^2}}, \tag{5}$$

where

$$R(f;x,y) = \frac{1}{\pi^2} \iint\limits_D \frac{\sqrt{(1-\xi^2)(1-\eta^2)} f(\xi,\eta) \ln\frac{1-\xi}{1+\xi} \ln\frac{1-\eta}{1+\eta}}{(\xi-x)(\eta-y)} d\xi d\eta, \tag{6}$$

and $\gamma_1(x),\ \gamma_2(y)$ are arbitrary functions from the class h_0.

If the solution $\varphi(x,y)$ *satisfies the conditions*

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(x,\eta)d\eta = g(x),\ -1<x<1, \tag{7}$$

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi,y)d\xi = h(y),\ -1<y<1, \tag{8}$$

where $g(x)$ *and* $h(x)$ *are given functions of the class* h_0 *such that*

$$\frac{1}{\pi}\int_{-1}^{1}g(\xi)d\xi = \frac{1}{\pi}\int_{-1}^{1}h(\eta)d\eta = A, \tag{9}$$

then the equation (1) *has the unique solution given by the following formula:*

$$\begin{aligned}\varphi(x,y) = {} & \frac{R\,(f;x,y)}{\sqrt{(1-x^2)(1-y^2)}} + \frac{g(x)}{\sqrt{1-y^2}} + \frac{h(y)}{\sqrt{1-x^2}} \\ & - \frac{A}{\sqrt{(1-x^2)\,(1-y^2)}}.\end{aligned} \tag{10}$$

Proof. Denoting

$$\Psi_1(x,y) = \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(x,\eta)}{\eta-y}d\eta, \tag{11}$$

$$\Psi_2(x,y) = \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(\xi,y)}{\xi-x}d\xi, \tag{12}$$

one can express the equation (1) in the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Psi_1(\xi,y)}{(\xi-x)}d\xi = f(x,y)\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}, \tag{13}$$

or

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Psi_2(x,\eta)}{(\eta-y)}d\eta = f(x,y)\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}. \tag{14}$$

Solving the equation (13) in the class h_0, we obtain [9]

$$\Psi_1(x,y) = -\frac{\ln\frac{1-y}{1+y}}{\sqrt{1-x^2}}\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}\ln\frac{1-\xi}{1+\xi}f(\xi,y)}{(\xi-x)}d\xi + \frac{c_1(y)}{\sqrt{1-x^2}}, \tag{15}$$

where $c_1(y)$ is an arbitrary function from h_0.

Next, solving (11), we have

$$\begin{aligned} &\varphi(x,y) \\ &= \frac{R(f;x,y)}{\sqrt{(1-x^2)(1-y^2)}} - \frac{1}{\sqrt{1-x^2}}\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\eta^2}c_1(\eta)}{\sqrt{1-y^2}(\eta-y)}d\eta + \frac{c_2(x)}{\sqrt{1-y^2}} \qquad (16) \\ &= \frac{R(f;x,y)}{\sqrt{(1-x^2)(1-y^2)}} + \frac{\gamma_1(x)}{\sqrt{1-y^2}} + \frac{\gamma_2(y)}{\sqrt{1-x^2}}, \end{aligned}$$

where $R(f;x,y)$ is given by the formula (6), and $c_2(x)$, $\gamma_1(x)$, $\gamma_2(y)$ are arbitrary functions from h_0. We get the same result solving the equations (14) and, consequently, (12).

In order to determine the functions $\gamma_1(x)$, $\gamma_2(y)$, we substitute the general solution given by (5) to the conditions (7)–(9). Then using the Poincaré–Bertrandt formula, we prove that the unique solution of the equation (1) in the class $h_0 \times h_0$ is given by the formula (10). □

3. Exact solution in the class $h(-1,1) \times h(-1,1)$.

Theorem 2. *Let the function* $f(x,y) \in h(-1,1) \times h(-1,1)$. *Then the unique solution of the equation* (1) *in the class* $h(-1,1) \times h(-1,1)$ *exists if and only if the following conditions:*

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(x,\eta)}{\sqrt{1-\eta^2}}\ln\frac{1-\eta}{1+\eta}d\eta = 0, \qquad (17)$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(\xi,y)}{\sqrt{1-\xi^2}}\ln\frac{1-\xi}{1+\xi}d\xi = 0, \qquad (18)$$

are fulfilled and it is given by the following formula:

$$\begin{aligned} &\varphi(x,y) \\ &= \sqrt{(1-x^2)(1-y^2)}\frac{1}{\pi^2}\iint_D \frac{f(\xi,\eta)\ln\frac{1-\xi}{1+\xi}\ln\frac{1-\eta}{1+\eta}}{\sqrt{(1-\xi^2)(1-\eta^2)}(\xi-x)(\eta-y)}d\xi d\eta. \qquad (19) \end{aligned}$$

Proof. Similarly to the previous proof we introduce denotations (11), (12) and express the equation (1) in the form (13) or (14), respectively. Solving the equation (13) in the class of bounded functions, having the condition (18) fulfilled, we get [9]

$$\Psi_1(x,y) = -\sqrt{1-x^2}\frac{1}{\pi}\int_{-1}^{1}\frac{f(\xi,y)\ln\frac{1-\xi}{1+\xi}\ln\frac{1-y}{1+y}}{\sqrt{1-\xi^2}(\xi-x)}d\xi. \qquad (20)$$

Next, from (14) and condition (17) we get

$$\Psi_2(x,y) = -\sqrt{1-y^2}\frac{1}{\pi}\int\limits_{-1}^{1}\frac{f(x,\eta)\ln\frac{1-x}{1+x}\ln\frac{1-\eta}{1+\eta}}{\sqrt{1-\eta^2}(\eta-y)}d\eta. \tag{21}$$

It is easy to verify that the conditions (17), (18) are also sufficient for solvability of the equations (11) and (12), respectively in the class of bounded functions, and the solution of the equation (1) is given by the formula (19). □

4. Approximate solution in the class $h_0 \times h_0$. To find an approximate solution of the equation (1) in the class $h_0 \times h_0$ we introduce a new unknown function $u(x,y)$ by the relation

$$\begin{aligned} &\frac{u(x,y)}{\sqrt{(1-x^2)(1-y^2)}} \\ &\quad = \varphi(x,y) - \frac{g(x)}{\sqrt{1-y^2}} - \frac{h(y)}{\sqrt{1-x^2}} + \frac{A}{\sqrt{(1-x^2)(1-y^2)}}. \end{aligned} \tag{22}$$

Then it is easy to show that the problem (1), (7), (8), (9) takes the form

$$\frac{1}{\pi^2}\iint\limits_D \frac{u(\xi,\eta)}{\sqrt{(1-\xi^2)(1-\eta^2)}}\frac{d\xi d\eta}{(\xi-x)(\eta-y)} = f(x,y)\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}, \tag{23}$$

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{u(x,\eta)}{\sqrt{1-\eta^2}}d\eta = 0, \tag{24}$$

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{u(\xi,y)}{\sqrt{1-\xi^2}}d\xi = 0. \tag{25}$$

Now we find an approximate solution of the problem (23), (24), (25). For this purpose we approximate the function $f(x,y)$ by the interpolating polynomial of the form

$$f_{mn}(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}x^k y^j, \tag{26}$$

and define the approximate solution $u_{mn}(x,y)$ as a solution of the following problem:

$$\begin{aligned} &\frac{1}{\pi^2}\iint\limits_D \frac{u_{mn}(\xi,\eta)}{\sqrt{(1-\xi^2)(1-\eta^2)}}\frac{d\xi d\eta}{(\xi-x)(\eta-y)} \\ &\qquad = f_{mn}(x,y)\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}, \end{aligned} \tag{27}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u_{mn}(x,\eta)}{\sqrt{1-\eta^2}}d\eta = 0, \tag{28}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u_{mn}(\xi,y)}{\sqrt{1-\xi^2}}d\xi = 0. \tag{29}$$

To find the form of an approximate solution, we use the relation (22) and the exact solution in the class $h_0 \times h_0$ given by (5) with the kernel (6). Next, substituting the function $f_{m,n}(x,y)$ defined in (26) in place of the given function $f(x,y)$ and using the formula

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\ln\frac{1-t}{1+t}\frac{t^k}{t-x}\,dt = -\pi\sqrt{1-x^2}x^k - P_k(x), \ -1 < x < 1, \tag{30}$$

we obtain

$$\begin{aligned} u_{mn}(x,y) &= \pi^2\sqrt{(1-x^2)(1-y^2)}f_{mn}(x,y) + \pi\sqrt{1-x^2}\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}x^kQ_j(y) \\ &+ \pi\sqrt{1-y^2}\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}y^jP_k(x) + \sum_{k=0}^{m}\sum_{j=0}^{n} c_{kj}x^ky^j, \end{aligned} \tag{31}$$

where $P_k(x)$, $Q_j(y)$ are the polynomials of degree k and j, respectively, defined as the principal part of the Laurent expansion of the following functions:

$$z^k\sqrt{z^2-1}\ln\frac{z-1}{z+1} = P_k(z) + \frac{\widetilde{p}_1^{(k)}}{z} + \frac{\widetilde{p}_2^{(k)}}{z^2} + \dots, \tag{32}$$

$$w^j\sqrt{w^2-1}\ln\frac{w-1}{w+1} = Q_j(w) + \frac{\widetilde{q}_1^{(j)}}{w} + \frac{\widetilde{q}_2^{(j)}}{w^2} + \dots, \tag{33}$$

in a neighborhood of the infinity, c_{kj} are the unknown coefficients.

To determine the coefficients c_{kj} we substitute the right-hand side of the formula (31) to the equation (27), getting

$$
\begin{aligned}
&\sum_{k=1}^{m}\sum_{j=1}^{n} c_{kj}\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\xi^k}{\sqrt{1-\xi^2}}\frac{d\xi}{\xi-x}\right)\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\eta^j}{\sqrt{1-\eta^2}}\frac{d\eta}{\eta-y}\right)\\
&=\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}x^k y^j \ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}\\
&\qquad-\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\int_{-1}^{1}\frac{\xi^k}{\xi-x}d\xi\right)\left(\int_{-1}^{1}\frac{\eta^j}{\eta-y}d\eta\right)\\
&\qquad-\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\int_{-1}^{1}\frac{\xi^k}{\xi-x}d\xi\right)\left(\frac{1}{\pi}\int_{-1}^{1}\frac{Q_j(\eta)}{\sqrt{1-\eta^2}}\frac{d\eta}{\eta-y}\right)\\
&\qquad-\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\int_{-1}^{1}\frac{\eta^j}{\eta-y}d\eta\right)\left(\frac{1}{\pi}\int_{-1}^{1}\frac{P_k(\xi)}{\sqrt{1-\xi^2}}\frac{d\xi}{\xi-x}\right).
\end{aligned}
\tag{34}
$$

Denoting

$$P_k(z)=p_0^{(k)}+p_1^{(k)}z+\ldots+p_k^{(k)}z^k,\quad Q_j(z)=q_0^{(j)}+q_1^{(j)}z+\ldots+q_j^{(j)}z^j, \tag{35}$$

and taking into account the following formulae:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{t^n}{\sqrt{1-t^2}(t-x)}dt=\begin{cases}0, & n=0,\\ \sum_{k=1}^{\left[\frac{n+1}{2}\right]}\frac{2k(2k)!}{(2k-1)(k!)^2 4^k}x^{n-2k+1}, & n\geq 1,\end{cases} \tag{36}$$

$$\int_{-1}^{1}\frac{t^n}{t-x}dt=\begin{cases}\log\frac{1-x}{1+x}, & n=0,\\ \sum_{k=1}^{\left[\frac{n+1}{2}\right]}\frac{2}{2k-1}x^{n-2k+1}, & n\geq 1,\end{cases} \tag{37}$$

we compare the corresponding coefficients getting formulae for coefficients c_{kj}, $k,j=1,\ldots,n$. Then substituting the right-hand side of the formula (31) to the conditions (28), (29), we get

$$
\begin{aligned}
&\sum_{k=0}^{m}\sum_{j=0}^{n} c_{kj}\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\eta^j}{\sqrt{1-\eta^2}}d\eta\right)x^k\\
&\quad= -\pi\sqrt{1-x^2}\sum_{k=0}^{m} f_{kj}\left(\int_{-1}^{1}\eta^j d\eta\right)x^k\\
&\qquad -\pi\sqrt{1-x^2}\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\frac{1}{\pi}\int_{-1}^{1}\frac{Q_j(\eta)}{\sqrt{1-\eta^2}}d\eta\right)x^k\\
&\qquad -\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\int_{-1}^{1}\eta^j d\eta\right)P_k(x)
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{m}\sum_{j=0}^{n} c_{kj}\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\xi^k}{\sqrt{1-\xi^2}}d\xi\right)y^j\\
&\quad= -\pi\sqrt{1-y^2}\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\int_{-1}^{1}\xi^k d\xi\right)y^j\\
&\qquad -\pi\sqrt{1-y^2}\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\frac{1}{\pi}\int_{-1}^{1}\frac{P_k(\xi)}{\sqrt{1-\xi^2}}d\xi\right)y^k\\
&\qquad -\sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj}\left(\int_{-1}^{1}\xi^k d\xi\right)Q_j(y),
\end{aligned}
\tag{39}
$$

respectively. From the above equations, using formula

$$
\frac{1}{\pi}\int_{-1}^{1}\frac{t^n}{\sqrt{1-t^2}}dt = \begin{cases} 0, & n = 1,3,5,\dots,\\ \frac{(n+2)(n+2)!}{(n+1)(\frac{n+2}{2}!)^2 2^{n+2}}, & n = 0,2,4,\dots, \end{cases}
\tag{40}
$$

we get the coefficients c_{00}, c_{0j}, c_{k0}, $j,k = 1,\dots,n$.

Finally,

$$
c_{kj} = \sum_{r=k}^{m}\sum_{s=j}^{n} f_{rs}p_k^{(r)}q_j^{(s)}, \quad k = 0,1,\dots,m,\ j = 0,1,\dots,n,
\tag{41}
$$

where $p_k^{(r)}$, $q_j^{(s)}$ are the coefficients of the polynomials $P_r(x)$, $Q_s(y)$ given in (32) and (33).

5. Approximate solution in the class $h(-1,1) \times h(-1,1)$. As previously, we introduce a new unknown function $u(x,y)$ defined by

$$\varphi(x,y) = \sqrt{(1-x^2)\,(1-y^2)}\,u(x,y). \tag{42}$$

Then the equation (1) takes the form

$$\frac{1}{\pi^2}\iint\limits_D \sqrt{(1-\xi^2)\,(1-\eta^2)}\,\frac{u\,(\xi,\eta)\,d\xi d\eta}{(\xi-x)\,(\eta-y)} = f(x,y)\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}, \tag{43}$$

where the given function $f(x,y)$ satisfies the conditions (17), (18). Next, we approximate the function $f(x,y)$ by the polynomial (26) and define the approximate solution $u_{mn}(x,y)$ as a solution of the following equation:

$$\begin{aligned} &\frac{1}{\pi^2}\iint\limits_D \sqrt{(1-\xi^2)\,(1-\eta^2)}\,\frac{u_{mn}\,(\xi,\eta)\,d\xi d\eta}{(\xi-x)\,(\eta-y)} \\ &\quad = f_{mn}(x,y)\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y} + Q_1^*(x) + Q_2^*(y). \end{aligned} \tag{44}$$

Note that the function $f_{mn}(x,y)$ does not have to satisfy the conditions (17), (18). Therefore the sum $Q_1^*(x) + Q_2^*(y)$ is added. Substituting the right-hand side of the equation (44) to the conditions (17), (18), we get the relations

$$\ln\frac{1-x}{1+x}\frac{1}{\pi}\int\limits_{-1}^{1}\frac{f_{mn}(x,\eta)\ln\frac{1-\eta}{1+\eta}}{\sqrt{1-\eta^2}}d\eta + Q_1^*(x) + \frac{1}{\pi}\int\limits_{-1}^{1}\frac{Q_2^*(\eta)}{\sqrt{1-\eta^2}}d\eta = 0, \tag{45}$$

$$\ln\frac{1-y}{1+y}\frac{1}{\pi}\int\limits_{-1}^{1}\frac{f_{mn}(\xi,y)\ln\frac{1-\xi}{1+\xi}}{\sqrt{1-\xi^2}}d\xi + \frac{1}{\pi}\int\limits_{-1}^{1}\frac{Q_1^*(\xi)}{\sqrt{1-\xi^2}}d\xi + Q_2^*(y) = 0. \tag{46}$$

Dividing (45) by $\sqrt{1-x^2}$, integrating respect to x, and adding both sides of the equations (45) and (46), we get

$$\begin{aligned} Q_1^*(x) + Q_2^*(y) = {} & \frac{1}{\pi^2}\iint\limits_D \frac{f_{mn}(\xi,\eta)\ln\frac{1-\xi}{1+\xi}\ln\frac{1-\eta}{1+\eta}}{\sqrt{(1-\xi^2)(1-\eta^2)}}d\xi d\eta \\ & - \ln\frac{1-x}{1+x}\frac{1}{\pi}\int\limits_{-1}^{1}\frac{f_{mn}(x,\eta)\ln\frac{1-\eta}{1+\eta}}{\sqrt{1-\eta^2}}d\eta \\ & - \ln\frac{1-y}{1+y}\frac{1}{\pi}\int\limits_{-1}^{1}\frac{f_{mn}(\xi,y)\ln\frac{1-\xi}{1+\xi}}{\sqrt{1-\xi^2}}d\xi. \end{aligned} \tag{47}$$

Similarly to the class $h_0 \times h_0$, it can be proved that the approximate solution $u_{mn}(x,y)$ has the form

$$
\begin{aligned}
u_{mn}(x,y) &= \sum_{k=0}^{m-2}\sum_{j=0}^{n-2} c_{kj}x^k y^j + \frac{\pi^2}{\sqrt{(1-x^2)(1-y^2)}} f_{mn}(x,y) \\
&\quad - \frac{\pi}{\sqrt{1-x^2}} \sum_{k=0}^{m}\sum_{j=0}^{n-2} f_{k,\,j+2} x^k Q_j(y) \\
&\quad - \frac{\pi}{\sqrt{1-y^2}} \sum_{k=0}^{m-2}\sum_{j=0}^{n} f_{k+2,\,j} P_k(x) y^j,
\end{aligned} \tag{48}
$$

where $P_{k-2}(x)$, $Q_{j-2}(y)$ are polynomials of degree $k-2$ i $j-2$, respectively, defined as the leading part of the Laurent expansion of the following functions:

$$
\frac{z^k}{\sqrt{z^2-1}} \ln\frac{z-1}{z+1} = P_{k-2}(z) + \frac{\widetilde{p}_1^{(k)}}{z} + \frac{\widetilde{p}_2^{(k)}}{z^2} + \ldots, \tag{49}
$$

$$
\frac{w^j}{\sqrt{w^2-1}} \ln\frac{w-1}{w+1} = Q_{j-2}(w) + \frac{\widetilde{q}_1^{(j)}}{w} + \frac{\widetilde{q}_2^{(j)}}{w^2} + \ldots \tag{50}
$$

in a neighborhood of the infinity. Here c_{kj} are the unknown coefficients.

Substituting the right-hand side of the formula (48) to equation (44), we get

$$
\begin{aligned}
&\sum_{k=0}^{m-2}\sum_{j=0}^{n-2} c_{kj} \left(\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\xi^2}\xi^k}{\xi - x} d\xi\right) \left(\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\eta^2}\eta^j}{\eta - y} d\eta\right) \\
&\quad + \sum_{k=0}^{m}\sum_{j=0}^{n} f_{kj} \left(\int_{-1}^{1} \frac{\xi^k}{\xi - x} d\xi\right) \left(\int_{-1}^{1} \frac{\eta^j}{\eta - y} d\eta\right) \\
&\quad - \sum_{k=0}^{m}\sum_{j=0}^{n-2} f_{k,\,j+2} \left(\int_{-1}^{1} \frac{\xi^k}{\xi - x} d\xi\right) \left(\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\eta^2}Q_j(\eta)}{\eta - y} d\eta\right) \\
&\quad - \sum_{k=0}^{m-2}\sum_{j=0}^{n} f_{k+2,\,j} \left(\int_{-1}^{1} \frac{\eta^j}{\eta - y} d\eta\right) \left(\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\xi^2}P_k(\xi)}{\xi - x} d\xi\right) \\
&= f_{mn}(x,y) \ln\frac{1-x}{1+x} \ln\frac{1-y}{1+y} + Q_1^*(x) + Q_2^*(y),
\end{aligned} \tag{51}
$$

where $Q_1^*(x) + Q_2^*(y)$ is given by the formula (47). Using (37) and

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}t^k}{t-x}dt = \sum_{k=0}^{\left[\frac{n+1}{2}\right]}\frac{(2k)!}{(2k-1)(k!)^2 4^k}x^{n-2k+1}, \tag{52}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{t^n \log\frac{1-t}{1+t}}{\sqrt{1-t^2}}dt = -\operatorname*{Res}_{z=\infty}\frac{z^n \ln\frac{z-1}{z+1}}{\sqrt{z^2-1}}, \tag{53}$$

we compare the coefficients and get

$$c_{kj} = \sum_{r=k}^{m-2}\sum_{s=j}^{n-2} f_{r+2,\,s+2}\,p_k^{(r)} q_j^{(s)}, \tag{54}$$

$k = 0, 1, \dots, m-2$, $j = 0, 1, \dots, n-2$, where $p_k^{(r)}$, $q_j^{(s)}$ are the coefficients of the polynomials $P_r(x)$, $Q_s(y)$ given in (49) and (50), respectively.

6. Example. We give the approximate solution of (1) in the class $h_0 \times h_0$. Let

$$f(x,y) = \frac{1}{(x^2-25)(y^2-16)}, \quad g(x) = 0, \quad h(x) = 0. \tag{55}$$

Then the exact solution $u(x,y)$ of the problem (23)–(25) has the form

$$u(x,y) = \frac{(\pi\sqrt{1-x^2} + 2\sqrt{6}\log\frac{2}{3})(\pi\sqrt{1-y^2} + \sqrt{15}\log\frac{3}{5})}{(x^2-25)(y^2-16)} \tag{56}$$

and the problem (1), (7), (8), (9) has the solution given by the function

$$\varphi(x,y) = \frac{(\pi\sqrt{1-x^2} + 2\sqrt{6}\log\frac{2}{3})(\pi\sqrt{1-y^2} + \sqrt{15}\log\frac{3}{5})}{\sqrt{(1-x^2)(1-y^2)}(x^2-25)(y^2-16)}. \tag{57}$$

The function $f(x,y)$ is approximated by the polynomial

$$f(x,y) \approx f_{mn}(x,y) = \frac{1}{400}\sum_{p=1}^{m}\left(\frac{x}{5}\right)^{2p}\sum_{q=1}^{n}\left(\frac{y}{4}\right)^{2q}. \tag{58}$$

Taking $m = n = 10$, we compare values of the exact and approximate solutions for some points $(x,y) \in D$. The results are shown in Table 1.

x	y	$u_{mn}(x,y) - u(x,y)$
0.9999	0.9999	$-2.8045206 \times 10^{-16}$
0.8976	0.3504	$2.8428276 \times 10^{-18}$
0.4576	0.7234	$-8.6580677 \times 10^{-18}$
0.0026	0.0211	$-4.9440984 \times 10^{-18}$
-0.0015	0.9986	$1.32194482 \times 10^{-16}$
-0.5523	0.6686	$-5.5595619 \times 10^{-18}$
-0.9853	-0.0006	$6.6776657 \times 10^{-18}$
-0.3247	-0.8954	$-3.0507894 \times 10^{-17}$
-0.0247	-0.2354	$-5.260122 \times 10^{-18}$
0.4247	-0.7554	$-1.0698026 \times 10^{-17}$
0.9487	-0.1554	$4.2669215 \times 10^{-18}$

Table 1. Comparison of the exact and approximate solutions in the class $h_0 \times h_0$.

The authors are now working on estimating errors of the approximate solutions. The results will be presented in the next papers.

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