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**Second Hankel determinant
for a class of analytic functions
of complex order defined by convolution**

ABSTRACT. In this paper, we obtain the Fekete–Szegő inequalities for the functions of complex order defined by convolution. Also, we find upper bounds for the second Hankel determinant $|a_2a_4 - a_3^2|$ for functions belonging to the class $S_\gamma^b(g(z); A, B)$.

1. Introduction. Let \mathcal{A} denote the class of analytic functions of the form:

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\})$$

and \mathcal{S} be the subclass of \mathcal{A} consisting of univalent functions. Furthermore, let \mathcal{P} be a family of functions $p(z) \in \mathcal{A}$.

Let $g(z) \in \mathcal{S}$ be given by

$$(1.2) \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(1.3) \quad (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

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If f and g are analytic functions in \mathbb{U} , we say that f is subordinate to g , written $f \prec g$ if there exists a Schwarz function w , which is analytic in \mathbb{U} with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in \mathbb{U} , then we have the following equivalence (see [6] and [19]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For complex parameters $\alpha_1, \dots, \alpha_q$ and β_1, \dots, β_s ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$; $j = 1, 2, \dots, s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by (see, for example, [29, p. 19])

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k} \cdot \frac{z^k}{k!}$$

($q \leq s + 1$; $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{N} = \{1, 2, \dots\}$; $z \in \mathbb{U}$), where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ , by

$$(1.4) \quad \begin{aligned} (\theta)_\nu &= \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} \\ &= \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \theta(\theta + 1) \dots (\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \end{aligned}$$

It corresponds to the function $h_{q,s}(\alpha_1, \beta_1; z) = h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$, defined by

$$(1.5) \quad \begin{aligned} h_{q,s}(\alpha_1, \beta_1; z) &= z {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ &= z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k, \end{aligned}$$

where

$$(1.6) \quad \Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \dots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_s)_{k-1} (k-1)!}.$$

In [13] El-Ashwah and Aouf defined the operator $I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)$ as follows:

$$\begin{aligned} I_{q,s,\lambda}^{0,\ell}(\alpha_1, \beta_1)f(z) &= f(z) * h_{q,s}(\alpha_1, \beta_1; z); \\ I_{q,s,\lambda}^{1,\ell}(\alpha_1, \beta_1)f(z) &= (1 - \lambda)(f(z) * h_{q,s}(\alpha_1, \beta_1; z)) \\ &\quad + \frac{\lambda}{(1 + \ell)z^{\ell-1}} \left[z^\ell (f(z) * h_{q,s}(\alpha_1, \beta_1; z)) \right]'; \end{aligned}$$

and

$$(1.7) \quad I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z) = I_{q,s,\lambda}^{1,\ell}(I_{q,s,\lambda}^{m-1,\ell}(\alpha_1, \beta_1)f(z)).$$

If $f \in A$, then from (1.1) and (1.7), we can easily see that

$$(1.8) \quad I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1 + \ell + \lambda(k-1)}{1 + \ell} \right]^m \Gamma_k(\alpha_1) a_k z^k,$$

where $m \in \mathbb{Z} = \{0, \pm 1, \dots\}$, $\ell \geq 0$ and $\lambda \geq 0$.

We note that when $\ell = 0$, the operator

$$I_{q,s,\lambda}^{m,0}(\alpha_1, \beta_1)f(z) = D_{\lambda}^m(\alpha_1, \beta_1)f(z)$$

was studied by Selvaraj and Karthikeyan [28]. We also note that:

- (i) $I_{q,s,\lambda}^{0,\ell}f(z) = H_{q,s}(\alpha_1, \beta_1)f(z)$ (see Dziok and Srivastava [11, 12]);
- (ii) For $q = s + 1$, $\alpha_i = 1$ ($i = 1, \dots, s + 1$) and $\beta_j = 1$ ($j = 1, \dots, s$), we get the operator $I(m, \lambda, \ell)$ (see Catas [7], Prajapat [24] and El-Ashwah and Aouf [14]);
- (iii) For $q = s + 1$, $\alpha_i = 1$ ($i = 1, \dots, s + 1$), $\beta_j = 1$ ($j = 1, \dots, s$), $\lambda = 1$ and $\ell = 0$, we obtain the Sălăgean operator D^m (see Sălăgean [27]);
- (iv) For $q = s + 1$, $\alpha_i = 1$ ($i = 1, \dots, s + 1$), $\beta_j = 1$ ($j = 1, \dots, s$) and $\lambda = 1$, we get the operator I_{ℓ}^m (see Cho and Srivastava [8] and Cho and Kim [9]).
- (v) For $q = s + 1$, $\alpha_i = 1$ ($i = 1, \dots, s + 1$), $\beta_j = 1$ ($j = 1, \dots, s$) and $\ell = 0$, we obtain the operator D_{λ}^m (see Al-Oboudi [2]).

By specializing the parameters $m, \lambda, \ell, q, s, \alpha_i$ ($i = 1, \dots, q$) and β_j ($j = 1, \dots, s$) we obtain:

$$(i) \quad I_{2,1,\lambda}^{m,\ell}(n+1, 1; 1)f(z) = I_{\lambda}^{m,\ell}(n)f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell} \right]^m \frac{(n+1)_{k-1}}{(1)_{k-1}} a_k z^k$$

$(n > -1)$;

$$(ii) \quad I_{2,1,\lambda}^{m,\ell}(a, 1; c)f(z) = I_{\lambda}^{m,\ell}(a; c)f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell} \right]^m \frac{(a)_{k-1}}{(c)_{k-1}} a_k z^k$$

$(a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-)$;

$$(iii) \quad I_{2,1,\lambda}^{m,\ell}(2, 1; n+1)f(z) = I_{\lambda,n}^{m,\ell}f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell} \right]^m \frac{(2)_{k-1}}{(n+1)_{k-1}} a_k z^k$$

$(n \in \mathbb{Z}; n > -1)$.

In 1976, Noonan and Thomas [23] discussed the q th Hankel determinant of a locally univalent analytic function $f(z)$ for $q \geq 1$ and $n \geq 1$ which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

For our present discussion, we consider the Hankel determinant in the case $q = 2$ and $n = 2$, i.e. $H_2(2) = a_2a_4 - a_3^2$. This is popularly known as the second Hankel determinant of f .

In this paper, we define the following class $S_{\gamma}^b(g(z); A, B)$ ($0 \leq \gamma \leq 1, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$) as follows:

Definition 1. Let $0 \leq \gamma \leq 1$, $b \in \mathbb{C}^*$. A function $f(z) \in \mathcal{A}$ is said to be in the class $S_\gamma^b(g(z); A, B)$ if

$$(1.9) \quad 1 + \frac{1}{b} \left((1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) \prec \frac{1 + Az}{1 + Bz}$$

($b \in \mathbb{C}^*$; $0 \leq \gamma \leq 1$; $-1 \leq B < A \leq 1$; $z \in \mathbb{U}$), which is equivalent to say that

$$\left| \frac{(1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1}{b(A - B) - B \left[(1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right]} \right| < 1.$$

We note that for suitable choices of b , γ and $g(z)$ we obtain the following subclasses:

- (i) $S_\gamma^b\left(\frac{z}{1-z}; A, B\right) = S_\gamma^b(A, B)$ ($0 \leq \gamma \leq 1$, $b \in \mathbb{C}^*$, $-1 \leq B < A \leq 1$) (see Bansal [5]);
- (ii) $S_0^{(1-\rho)e^{-i\theta} \cos \theta} \left(z + \sum_{k=2}^{\infty} \frac{(\alpha)_{k-1}}{(\beta)_{k-1}} z^k; 1, -1 \right) = \mathcal{R}_{\alpha, \beta}(\theta, \rho)$ ($-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $0 \leq \rho < 1$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$) (see Mishra and Kund [21]);
- (iii) $S_0^{(1-\rho)e^{-i\alpha} \cos \alpha} \left(z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(m)_{k-1}} k^n z^k; 1, -1 \right) = S_m^{\lambda, n}(\alpha, \sigma)$ ($m \in \mathbb{N}$; $n, \lambda \in \mathbb{N}_0$; $|\alpha| < \frac{\pi}{2}$; $0 \leq \sigma < 1$) (see Mohammed and Darus [22]);
- (iv) $S_1^1 \left(z + \sum_{k=2}^{\infty} [1 + (\alpha\mu k + \alpha - \mu)(k-1)]^\sigma (\rho)_{k-1} z^k; 1, -1 \right) = R_{\alpha, \mu}(\sigma, \rho)$ ($0 \leq \mu \leq \alpha \leq 1$; $\rho, \sigma \in \mathbb{N}_0$) (see Abubaker and Darus [1]);
- (v) $S_\gamma^b \left(z + \sum_{k=2}^{\infty} k^m z^k; A, B \right) = G_m(\gamma, b)$ ($b \in \mathbb{C}^*$, $0 \leq \gamma \leq 1$, $m \in \mathbb{N}_0$) (see Aouf [3]).

Also, we note that:

$$(i) \quad S_\gamma^b \left(z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell} \right]^m \Gamma_k(\alpha_1) z^k; A, B \right) = S_\gamma^b(\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B)$$

$$= \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b} \left((1 - \gamma) \frac{I_{q, s, \lambda}^{m, \ell}(\alpha_1, \beta_1) f(z)}{z} + \gamma \left(I_{q, s, \lambda}^{m, \ell}(\alpha_1, \beta_1) f(z) \right)' - 1 \right) \prec \frac{1 + Az}{1 + Bz}, \right. \\ \left. (b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; m \in \mathbb{N}_0; \ell \geq 0; \lambda \geq 0; q \leq s + 1; q, s \in \mathbb{N}_0; z \in \mathbb{U}) \right\};$$

$$(ii) \quad S_\gamma^b \left(z + \sum_{k=2}^{\infty} \left[\frac{1+\ell}{1+\ell+\lambda(k-1)} \right]^m z^k; A, B \right) = S_\gamma^b(\lambda, \ell, m; A, B)$$

$$= \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b} \left((1 - \gamma) \frac{J^m(\lambda, \ell) f(z)}{z} + \gamma (J^m(\lambda, \ell) f(z))' - 1 \right) \prec \frac{1 + Az}{1 + Bz}, \right. \\ \left. (b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; m \in \mathbb{N}_0; \ell \geq 0; \lambda \geq 0; z \in \mathbb{U}) \right\};$$

$$\begin{aligned}
 \text{(iii)} \quad & S_\gamma^{(1-\rho)\cos\eta e^{-i\eta}}(g(z); A, B) = S^\gamma[\rho, \eta, A, B, g(z)] \\
 & = \left\{ f(z) \in \mathcal{A} : e^{i\eta} \left[(1-\gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) \right] \right. \\
 & \quad \prec (1-\rho)\cos\eta \cdot \frac{1+Az}{1+Bz} + \rho\cos\eta + i\sin\eta, \\
 & \quad \left. (|\eta| < \frac{\pi}{2}; 0 \leq \gamma \leq 1; 0 \leq \rho < 1; -1 \leq B < A \leq 1; z \in \mathbb{U}) \right\}
 \end{aligned}$$

In this paper, we obtain the Fekete–Szegő inequalities for the functions in the class $S_\gamma^b(g(z); A, B)$. We also obtain an upper bound to the functional $H_2(2)$ for $f(z) \in S_\gamma^b(g(z); A, B)$. Earlier Janteng et al. [16], Mishra and Gochhayat [20], Mishra and Kund [21], Bansal [4] and many other authors have obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions.

2. Preliminaries. To prove our results, we need the following lemmas.

Lemma 1 ([26]). *Let*

$$(2.1) \quad h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \prec 1 + \sum_{n=1}^{\infty} C_n z^n = H(z) \quad (z \in \mathbb{U}).$$

If the function H is univalent in \mathbb{U} and $H(\mathbb{U})$ is a convex set, then

$$(2.2) \quad |c_n| \leq |C_1|.$$

Lemma 2 ([10]). *Let a function $p \in \mathcal{P}$ be given by*

$$(2.3) \quad p(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (z \in \mathbb{U}),$$

then, we have

$$(2.4) \quad |c_n| \leq 2 \quad (n \in \mathbb{N}).$$

The result is sharp.

Lemma 3 ([17, 18]). *Let $p \in \mathcal{P}$ be given by the power series (2.3), then for any complex number ν*

$$(2.5) \quad |c_2 - \nu c_1^2| \leq 2 \max\{1; |2\nu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

Lemma 4 ([15]). *Let a function $p \in \mathcal{P}$ be given by the power series (2.3), then*

$$(2.6) \quad 2c_2 = c_1^2 + \varkappa(4 - c_1^2)$$

for some \varkappa , $|\varkappa| \leq 1$, and

$$(2.7) \quad 4c_3 = c_1^3 + 2(4 - c_1^2)c_1\varkappa - c_1(4 - c_1^2)\varkappa^2 + 2(4 - c_1^2)(1 - |\varkappa|^2)z,$$

for some z , $|z| \leq 1$.

3. Main results. We give the following result related to the coefficient of $f(z) \in S_\gamma^b(g(z); A, B)$.

Theorem 1. Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(g(z); A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, then

$$(3.1) \quad |a_k| \leq \frac{(A-B)|b|}{[1+\gamma(k-1)]b_k} \quad (k \in \mathbb{N} \setminus \{1\}).$$

Proof. If $f(z)$ of the form (1.1) belongs to the class $S_\gamma^b(g(z); A, B)$, then

$$1 + \frac{1}{b} \left((1-\gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) \prec \frac{1 + Az}{1 + Bz} = h(z)$$

($b \in \mathbb{C}^*$; $0 \leq \gamma \leq 1$; $-1 \leq B < A \leq 1$; $z \in \mathbb{U}$), where $h(z)$ is convex univalent in \mathbb{U} and we have

$$(3.2) \quad \begin{aligned} & 1 + \frac{1}{b} \left((1-\gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) \\ &= 1 + \sum_{k=1}^{\infty} \frac{(1+k\gamma)}{b} b_{k+1} a_{k+1} z^k \prec 1 + (A-B)z - B(A-B)z^2 + \dots \end{aligned}$$

($z \in \mathbb{U}$). Now, by applying Lemma 1, we get the desired result. \square

Remark 1. Putting $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the result obtained by Bansal [5, Theorem 2.1].

It is easy to derive a sufficient condition for $f(z)$ to be in the class $S_\gamma^b(m, \lambda, \ell; A, B)$ using standard techniques (see [25]). Hence we state the following result without proof.

Theorem 2. Let $f(z) \in \mathcal{A}$, then a sufficient condition for $f(z)$ to be in the class $S_\gamma^b(g(z); A, B)$ is

$$(3.3) \quad \sum_{k=2}^{\infty} [1 + \gamma(k-1)] b_k |a_k| \leq \frac{(A-B)|b|}{1+B}.$$

In the next two theorems, we obtain the result concerning Fekete–Szegő inequality and an upper bound for the Hankel determinant for the class $S_\gamma^b(g(z); A, B)$.

Remark 2. Putting $g(z) = \frac{z}{1-z}$ in Theorem 2, we obtain the result obtained by Bansal [5, Theorem 2.2].

Theorem 3. Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(g(z); A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, then

$$(3.4) \quad |a_3 - \mu a_2^2| \leq \frac{(A-B)|b|}{(1+2\gamma)b_3} \cdot \max \left\{ 1, \left| B + \frac{\mu b b_3 (A-B)(1+2\gamma)}{(1+\gamma)^2 b_2^2} \right| \right\}.$$

This result is sharp.

Proof. Let $f(z) \in S_\gamma^b(g(z); A, B)$, then there is a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$(3.5) \quad 1 + \frac{1}{b} \left((1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) = \Phi(w(z))$$

($z \in \mathbb{U}$), where

$$(3.6) \quad \begin{aligned} \Phi(z) &= \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 - \dots \\ &= 1 + B_1z + B_2z^2 + B_3z^3 + \dots \end{aligned}$$

($z \in \mathbb{U}$). If the function $p_1(z)$ is analytic and has positive real part in U and $p_1(0) = 1$, then

$$(3.7) \quad p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots$$

($z \in \mathbb{U}$), since $w(z)$ is a Schwarz function. Define

$$(3.8) \quad \begin{aligned} h(z) &= 1 + \frac{1}{b} \left((1 - \gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) \\ &= 1 + d_1z + d_2z^2 + \dots \end{aligned}$$

($z \in \mathbb{U}$). In view of the equations (3.5) and (3.7), we have

$$p(z) = \Phi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right).$$

Since

$$(3.9) \quad \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left[c_1z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1c_2 \right) z^3 + \dots \right],$$

we have

$$(3.10) \quad \Phi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2 \right] z^2 + \dots,$$

and from this equation and (3.8), we obtain

$$(3.11) \quad d_1 = \frac{1}{2}B_1c_1, \quad d_2 = \frac{1}{2}B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2$$

and

$$(3.12) \quad d_3 = \frac{B_1}{2} \left(c_3 - c_1c_2 + \frac{c_1^3}{4} \right) + \frac{B_2c_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_3c_1^3}{8}.$$

Then, from (3.6), we see that

$$(3.13) \quad d_1 = \frac{(1 + \gamma)b_2a_2}{b} \quad \text{and} \quad d_2 = \frac{(1 + 2\gamma)b_3a_3}{b}.$$

Now from (3.6), (3.8) and (3.13), we have

$$(3.14) \quad a_2 = \frac{(A-B)bc_1}{2(1+\gamma)b_2}, \quad a_3 = \frac{b(A-B)}{4(1+2\gamma)b_3} \{2c_2 - c_1^2(1+B)\}$$

and

$$(3.15) \quad a_4 = \frac{b(A-B)}{8(1+3\gamma)b_4} \{4c_3 - 4c_1c_2(1+B) + c_1^3(1+B)^2\}$$

Therefore, we have

$$(3.16) \quad a_3 - \mu a_2^2 = \frac{b(A-B)}{2(1+2\gamma)b_3} \{c_2 - \nu c_1^2\},$$

where

$$(3.17) \quad \nu = \frac{1}{2} \left[1 + B + \frac{\mu b(A-B)(1+2\gamma)b_3}{(1+\gamma)^2 b_2^2} \right].$$

Our result now follows by an application of Lemma 3. The result is sharp for the functions

$$(3.18) \quad 1 + \frac{1}{b} \left((1-\gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) = \Phi(z^2)$$

and

$$(3.19) \quad 1 + \frac{1}{b} \left((1-\gamma) \frac{(f * g)(z)}{z} + \gamma (f * g)'(z) - 1 \right) = \Phi(z).$$

This completes the proof of Theorem 3. \square

Remark 3. Putting $g(z) = \frac{z}{1-z}$ in Theorem 3, we obtain the result due to Bansal [5, Theorem 2.3].

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell} \right]^m \Gamma_k(\alpha_1)$ ($m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$, $q \leq s+1$, $q, s \in \mathbb{N}_0$), where $\Gamma_k(\alpha_1)$ is given by (1.6) in Theorem 3, we obtain the following corollary.

Corollary 1. Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$, $q \leq s+1$, $q, s \in \mathbb{N}_0$ and $b \in \mathbb{C}^*$, then

$$(3.20) \quad |a_3 - \mu a_2^2| \leq \frac{(A-B)(1+\ell)^m |b|}{(1+2\gamma)(1+\ell+2\lambda)^m \Gamma_3(\alpha_1)} \times \max \left\{ 1, \left| B + \frac{\mu b \left[\frac{1+\ell+2\lambda}{1+\ell} \right]^m \Gamma_3(\alpha_1)(A-B)(1+2\gamma)}{(1+\gamma)^2 \left[\frac{1+\ell+\lambda}{1+\ell} \right]^{2m} \Gamma_2^2(\alpha_1)} \right| \right\}.$$

This result is sharp.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell}{1+\ell+\lambda(k-1)} \right]^m z^k$ ($m \in \mathbb{N}_0$; $\ell \geq 0$; $\lambda \geq 0$) in Theorem 3, we obtain the following corollary.

Corollary 2. Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(\lambda, \ell, m; A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$ and $b \in \mathbb{C}^*$, then

$$(3.21) \quad |a_3 - \mu a_2^2| \leq \frac{(A - B) |b|}{(1 + 2\gamma)} \left[\frac{1 + \ell + 2\lambda}{1 + \ell} \right]^m \times \max \left\{ 1, \left| B + \frac{\mu b \left[\frac{1 + \ell}{1 + \ell + 2\lambda} \right]^m (A - B) (1 + 2\gamma)}{(1 + \gamma)^2 \left[\frac{1 + \ell}{1 + \ell + \lambda} \right]^{2m}} \right| \right\}.$$

This result is sharp.

Putting $b = (1 - \rho) e^{-i\eta} \cos \eta$ ($|\eta| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Theorem 3, we obtain the following corollary.

Corollary 3. Let $f(z)$ given by (1.1) belong to the class $S^\gamma[\rho, \eta, A, B, g(z)]$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, then

$$(3.22) \quad |a_3 - \mu a_2^2| \leq \frac{(A - B) (1 - \rho) \cos \eta}{(1 + 2\gamma) b_3} \times \max \left\{ 1, \left| B + \frac{\mu b_3 (A - B) (1 + 2\gamma) (1 - \rho) e^{-i\eta} \cos \eta}{(1 + \gamma)^2 b_2^2} \right| \right\}.$$

This result is sharp.

Theorem 4. Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(g(z); A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, then

$$(3.23) \quad |a_2 a_4 - a_3^2| \leq \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 b_3^2}.$$

Proof. Using (3.14) and (3.15), we have

$$(3.24) \quad |a_2 a_4 - a_3^2| = \frac{(A - B)^2 |b|^2}{16 (1 + \gamma) (1 + 3\gamma) b_2 b_4} \left| 4c_1 c_3 - 4c_1^2 c_2 (1 + B) + c_1^4 (1 + B)^2 - \frac{(1 + \gamma) (1 + 3\gamma) b_2 b_4}{(1 + 2\gamma)^2 b_3^2} [4c_2^2 - 4c_1^2 c_2 (1 + B) + c_1^4 (1 + B)^2] \right| \\ = M \left| 4c_1 c_3 - 4c_1^2 c_2 (1 + B) + c_1^4 (1 + B)^2 - N [4c_2^2 - 4c_1^2 c_2 (1 + B) + c_1^4 (1 + B)^2] \right|,$$

where

$$(3.25) \quad M = \frac{(A - B)^2 |b|^2}{16 (1 + \gamma) (1 + 3\gamma) b_2 b_4} \quad \text{and} \quad N = \frac{(1 + \gamma) (1 + 3\gamma) b_2 b_4}{(1 + 2\gamma)^2 b_3^2}.$$

The above equation (3.24) is equivalent to

$$(3.26) \quad |a_2 a_4 - a_3^2| = M |4c_1 c_3 + d_2 c_1^2 c_2 + d_3 c_2^2 + d_4 c_1^4|,$$

where

$$(3.27) \quad d_1 = 4, \quad d_2 = -4(1+B)(1-N), \quad d_3 = -4N, \quad d_4 = (1-N)(1+B)^2.$$

Since the functions $p(z)$ and $p(re^{i\theta})$ ($\theta \in \mathbb{R}$) are members of the class \mathcal{P} simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ ($c \in [0, 2]$, see (2.4)). Also, substituting the values of c_2 and c_3 , respectively, from (2.6) and (2.7) in (3.26), we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{M}{4} \left| c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2\kappa c^2(4 - c^2)(d_1 + d_2 + d_3) \right. \\ &\quad \left. + (4 - c^2)\kappa^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2) \left(1 - |\kappa|^2 z\right) \right|. \end{aligned}$$

An application of triangle inequality, replacement of $|\kappa|$ by ν and substituting the values of d_1, d_2, d_3 and d_4 from (3.27), we have

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{M}{4} [4c^4(1-N)B^2 + 8|B|(1-N)\nu c^2(4 - c^2) \\ &\quad + (4 - c^2)\nu^2(4c^2 + 4N(4 - c^2)) + 8c(4 - c^2)(1 - \nu^2)] \\ (3.28) \quad &= M [c^4(1-N)B^2 + 2c(4 - c^2) + 2\nu|B|(1-N)c^2(4 - c^2) \\ &\quad + \nu^2(4 - c^2)(c^2(1-N) - 2c + 4N)] \\ &= F(c, \nu). \end{aligned}$$

Next, we assume that the upper bound for (3.28) occurs at an interior point of the rectangle $[0, 2] \times [0, 1]$. Differentiating $F(c, \nu)$ in (3.28) partially with respect to ν , we have

$$(3.29) \quad \frac{\partial F(c, \nu)}{\partial \nu} = M [2|B|(1-N)c^2(4 - c^2) + 2\nu(4 - c^2)(c^2(1-N) - 2c + 4N)].$$

For $0 < \nu < 1$ and for any fixed c with $0 < c < 2$, from (3.29), we observe that $\frac{\partial F}{\partial \nu} > 0$. Therefore, $F(c, \nu)$ is an increasing function of ν , which contradicts our assumption that the maximum value of $F(c, \nu)$ occurs at an interior point of the rectangle $[0, 2] \times [0, 1]$. Moreover, for fixed $c \in [0, 2]$,

$$(3.30) \quad \max F(c, \nu) = F(c, 1) = G(c).$$

Thus

$$(3.31) \quad \begin{aligned} G(c) &= M [c^4(1-N)(B^2 - 2|B| - 1) \\ &\quad + 4c^2(2|B|(1-N) + 1 - 2N) + 16N]. \end{aligned}$$

Next,

$$\begin{aligned} G'(c) &= 4Mc [c^2(1-N)(B^2 - 2|B| - 1) + 2(2|B|(1-N) + 1 - 2N)] \\ &= 4Mc [c^2(1-N)(B^2 - 2|B| - 1) + 2\{(1-N)(2|B| + 1) - N\}]. \end{aligned}$$

So $G'(c) < 0$ for $0 < c < 2$ and has a real critical point at $c = 0$. Also $G(c) > G(2)$. Therefore, maximum of $G(c)$ occurs at $c = 0$. Therefore, the upper bound of $F(c, \nu)$ corresponds to $\nu = 1$ and $c = 0$. Hence,

$$|a_2a_4 - a_3^2| \leq 16MN = \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 b_3^2}.$$

This completes the proof of Theorem 4. □

Remark 4. (i) Putting $g(z) = \frac{z}{1-z}$ in Theorem 4, we obtain the result due to Bansal [5, Theorem 2.4];

(ii) Putting

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha)_{k-1}}{(\beta)_{k-1}} z^k$$

($\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$), $b = (1 - \rho)e^{-i\theta} \cos \theta$ ($|\theta| < \frac{\pi}{2}, 0 \leq \rho < 1$), $\gamma = 0, A = 1$ and $B = -1$ in Theorem 4, we obtain the result due to Mishra and Kund [21, Theorem 3.1];

(iii) Putting

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda + 1)_{k-1}}{(m)_{k-1}} k^n z^k$$

($m \in \mathbb{N}; \lambda, n \in \mathbb{N}_0$), $b = (1 - \rho)e^{-i\alpha} \cos \alpha$ ($|\alpha| < \frac{\pi}{2}; 0 \leq \sigma < 1$), $\gamma = 0, A = 1$ and $B = -1$ in Theorem 4, we obtain the result due to Mohammed and Darus [22, Theorem 2.1];

(iv) Putting

$$g(z) = z + \sum_{k=2}^{\infty} [1 + (\alpha\mu k + \alpha - \mu)(k - 1)]^\sigma (\rho)_{k-1} z^k$$

($0 \leq \mu \leq \alpha \leq 1, \rho, \sigma \in \mathbb{N}_0$), $b = \gamma = A = 1$ and $B = -1$ in Theorem 4, we obtain the result due to Abubaker and Darus [1, Theorem 3.1].

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell} \right]^m \Gamma_k(\alpha_1)$ ($m \in \mathbb{N}_0, \ell \geq 0, \lambda \geq 0, q \leq s+1, q, s \in \mathbb{N}_0$), where $\Gamma_k(\alpha_1)$ is given by (1.6) in Theorem 4, we obtain the following corollary.

Corollary 4. Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(\lambda, \ell, m, q, s, \alpha_1, \beta_1; A, B)$, $0 \leq \gamma \leq 1, -1 \leq B < A \leq 1, m \in \mathbb{N}_0, \ell \geq 0, \lambda \geq 0, q \leq s+1, q, s \in \mathbb{N}_0$ and $b \in \mathbb{C}^*$, then

$$(3.32) \quad |a_2a_4 - a_3^2| \leq \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 \left[\frac{1+\ell+2\lambda}{1+\ell} \right]^{2m} \Gamma_3^2(\alpha_1)}.$$

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell}{1+\ell+\lambda(k-1)} \right]^m z^k$ ($m \in \mathbb{N}_0; \ell \geq 0; \lambda \geq 0$) in Theorem 4, we obtain the following corollary.

Corollary 5. Let $f(z)$ given by (1.1) belong to the class $S_\gamma^b(\lambda, \ell, m; A, B)$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$, $\ell \geq 0$, $\lambda \geq 0$ and $b \in \mathbb{C}^*$, then

$$(3.33) \quad |a_2 a_4 - a_3^2| \leq \frac{(A - B)^2 |b|^2}{(1 + 2\gamma)^2 \left[\frac{1+\ell}{1+\ell+2\lambda} \right]^{2m}}.$$

Putting $b = (1 - \rho) e^{-i\eta} \cos \eta$ ($|\eta| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Theorem 4, we obtain the following corollary.

Corollary 6. Let $f(z)$ given by (1.1) belong to the class $S^\gamma[\rho, \eta, A, B, g(z)]$, $0 \leq \gamma \leq 1$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, then

$$(3.34) \quad |a_2 a_4 - a_3^2| \leq \frac{(A - B)^2 (1 - \rho)^2 \cos^2 \eta}{(1 + 2\gamma)^2 b_3^2}.$$

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