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The Fekete–Szegö problem for a class of analytic functions defined by Carlson–Shaffer operator

Abstract. In the present investigation we solve Fekete–Szegö problem for the generalized linear differential operator. In particular, our theorems contain corresponding results for various subclasses of strongly starlike and strongly convex functions.

1. Introduction. Let $\mathcal{A}$ be the family of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose S is a subfamily of $\mathcal{A}$ consisting of functions that are univalent in $\mathcal{U}$. For functions $f, g \in \mathcal{A}$, given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or *convolution*) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \ \ z \in \mathcal{U}. \tag{1.2}$$

Carlson and Shaffer in [4] introduced a linear operator $L(a, c) : \mathcal{A} \to \mathcal{A}$ defined by $L(a, c)f(z) = \phi(a, c; z) * f(z)$, where the symbol $*$ denotes the

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convolution of two functions in $\mathcal{A}$ and where $\phi(a,c;z)$ is the well-known incomplete beta function given by

$$\phi(a,c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}z^n,\ \ z\in\mathcal{U}.$$

Here a and c are nonzero complex parameters and $a,c\neq -1,-2,-3,\ldots$. Also, $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & n=0,\\ \lambda(\lambda+1)\ldots(\lambda+n-1), & n\in\{1,2,3,\ldots\}.\end{cases}$$

We also note that $L(a,a)f(z)=f(z)$, $L(2,1)f(z)=zf'(z)$ and $L(\delta+1,1)f(z)=D^\delta f(z)$, where

$$D^\delta f(z)=\frac{z}{(1-z)^{\delta+1}}*f(z),\ \ \delta>-1,$$

is the generalized Ruscheweyh derivative of function f in $\mathcal{A}$ [22]. The operator $L(a,c)$ is analytic in $\mathcal{U}$ and plays an important role in Geometric Functions Theory; see for example [24], [14], [21] and [9].

The *linear multiplier differential operator* $D^m(\lambda,\varphi)f$ was defined by the authors in [7] as follows:

$$\begin{aligned}D^0(\lambda,\varphi)f(z)&=f(z),\\ D^1(\lambda,\varphi)f(z)&=D(\lambda,\varphi)f(z)\\ &=\lambda\varphi z^2(f(z))''+(\lambda-\varphi)z(f(z))'+(1-\lambda+\varphi)f(z),\\ D^2(\lambda,\varphi)f(z)&=D(\lambda,\varphi)\left(D^1(\lambda,\varphi)f(z)\right),\\ &\ \ \vdots\\ D^m(\lambda,\varphi)f(z)&=D(\lambda,\varphi)\left(D^{m-1}(\lambda,\varphi)f(z)\right),\end{aligned}$$

where $\lambda\geq\varphi\geq 0$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

If f is given by (1.1), then from the definition of the operator $D^m(\lambda,\varphi)f(z)$ it is easy to see that

$$D^m(\lambda,\varphi)f(z)=z+\sum_{n=2}^{\infty}[1+(\lambda\varphi n+\lambda-\varphi)(n-1)]^m a_n z^n. \tag{1.3}$$

It should be remarked that the $D^m(\lambda,\varphi)$ is a generalization of many other linear operators considered earlier. In particular, for $f\in\mathcal{A}$ we have the following:

- $D^m(1,0)f(z)\equiv D^m f(z)$, the operator investigated by Sălăgean (see [23]).
- $D^m(\lambda,0)f(z)\equiv D^m(\lambda)f(z)$, the operator studied by Al-Oboudi (see [2]).
- $D^m(\lambda,\varphi)f(z)$, the operator firstly considered for $0\leq\varphi\leq\lambda\leq 1$, by Răducanu and Orhan (see [20]). The operator $D^m(\lambda,\varphi)f(z)$ is called Răducanu–Orhan operator.

Definition 1.1. The generalized linear operator $L(m,\lambda,\varphi;a,c):\mathcal{A}\to\mathcal{A}$ is given as

$$L(m,\lambda,\varphi;a,c)f(z)=\phi(a,c;z)*D^m(\lambda,\varphi)f(z)$$
$$=z+\sum_{n=2}^{\infty}\Phi_n^m(\lambda,\varphi)\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^n$$

where $\Phi_n^m(\lambda,\varphi)=[1+(\lambda\varphi n+\lambda-\varphi)(n-1)]^m$, $\lambda\ge\varphi\ge 0$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $a,c\neq -1,-2,-3,\dots$.

We note here some special cases:

(1) $L(0,\lambda,\varphi;a,c)f(z)=L(a,c)f(z)$ is the Carlson–Shaffer linear operator [4].
(2) $L(0,\lambda,\varphi;\delta+1,1)f(z)$, $\delta\in\mathbb{N}_0$, is the Ruscheweyh derivative operator [22].
(3) $L(m,\lambda,\varphi;1,1)f(z)$, $\lambda\ge\varphi\ge 0$, $m\in\mathbb{N}_0$, is extended Raducanu–Orhan operator [7].
(4) $L(m,\lambda,0;1,1)f(z)$, $m\in\mathbb{N}_0$, is the Al-Oboudi linear operator [2].
(5) $L(m,1,0;1,1)f(z)$, $m\in\mathbb{N}_0$, is the Sălăgean derivative operator [23].

Now, by making use of the extended linear differential operator $L(m,\lambda,\varphi;a,c)$, we define a new subclass $Q(m,\lambda,\varphi,\beta;a,c)$ of analytic functions.

Definition 1.2. Let a,c be nonzero complex parameters such that $a,c\neq -1,-2,-3,\dots$, $\lambda\ge\varphi\ge 0$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Also, suppose $0<\beta\le 1$. A function f given by (1.1) is said to be in the class $Q(m,\lambda,\varphi,\beta;a,c)$ if

$$\left|\arg\left(\frac{z(L(m,\lambda,\varphi;a,c)f(z))'}{L(m,\lambda,\varphi;a,c)f(z)}\right)\right|<\frac{\pi}{2}\beta,\ z\in\mathcal{U}. \tag{1.4}$$

This class includes a variety of well-known subclasses of $\mathcal{A}$. For example,

$$Q(0,\lambda,\varphi,\beta;a,a)\equiv S_1^*(\beta)$$
$$=\left\{z\in A:\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2}\beta,\ z\in\mathcal{U}\right\};\ [3]$$
$$Q(0,\lambda,\varphi,\beta;2,1)\equiv K_1(\beta)$$
$$=\left\{f\in A:\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\pi}{2}\beta,\ z\in\mathcal{U}\right\};\ [3]$$
$$Q(0,\lambda,\varphi,\beta,\delta+1,1)\equiv\tilde{R}_\delta(\beta)$$
$$=\left\{f\in A:\left|\arg\left(\frac{z(D^\delta f(z))'}{D^\delta f(z)}\right)\right|<\frac{\pi}{2}\beta,\ z\in\mathcal{U}\right\},\ \delta\ge 0;\ [6].$$

A function f in $S_1^*(\beta)$ is called strongly starlike of order β. The class $K_1(\beta)$ consists of strongly convex functions of order β. These observations help us to conclude that the differential-integral representation given by (1.4) is a generalization of the Carlson–Shaffer operator in [4] and includes $S_1^*(\beta)$ and $K_1(\beta)$ studied by Brannan and Kirwan in [3].

In 1933, Fekete and Szegö [10] found the maximum value of $\left|a_3 - \mu a_2^2\right|$ as a function of the real parameters μ, for functions belonging to the class S. Since then, several researchers solved the Fekete–Szegö problem for various sublasses of the class of S and related subclasses of functions in $\mathcal{A}$. See, for example [1], [5], [6], [7], [8], [11], [12], [13], [15], [16], [17], [18], [25]. In the present paper, we solve Fekete–Szegö problem for functional $\left|a_3 - \mu a_2^2\right|$, where μ is real or complex when f is in the family $Q(m, \lambda, \varphi, \beta; a, c)$. In particular, our theorems contain corresponding results for various subclasses of strongly starlike and strongly convex and other several subclasses of $\mathcal{A}$.

2. Preliminary results. Let P be the class of all analytic functions P given by $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathcal{U}$. To prove our main results we need the following lemmas.

Lemma 2.1 ([19]). *If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is in P, then*
(i) $|c_n| \leq 2$ *for* $n \geq 1$,
(ii) $\left|c_2 - \frac{1}{2}c_1^2\right| \leq 2 - \frac{|c_1|^2}{2}$.

Lemma 2.2. *Let a and c be nonzero complex numbers with $a, c \neq -1, -2, -3, \dots$, $\lambda \geq \varphi \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $f \in Q(m, \lambda, \varphi, \beta; a, c)$ is given by* (1.1) *then*

$$\text{(i)} \quad |a_2| \leq \frac{2\beta\,|c|}{\Phi_2^m(\lambda,\varphi)\,|a|},$$

$$\text{(ii)} \quad |a_3| \leq \begin{cases} \dfrac{\beta\,|c|\,|c+1|}{\Phi_3^m(\lambda,\varphi)\,|a|\,|a+1|}, & \beta \leq \dfrac{1}{3}, \\[2ex] \dfrac{3\beta^2\,|c|\,|c+1|}{\Phi_3^m(\lambda,\varphi)\,|a|\,|a+1|}, & \beta \geq \dfrac{1}{3}. \end{cases}$$

Proof. Let $F(z) := L(m, \lambda, \varphi; a, c)f(z) := z + A_2 z^2 + A_3 z^3 + \dots$. Since

$$\frac{zF'(z)}{F(z)} = p^\beta(z), \;\; p \in P$$

and so,

$$\frac{z(1 + 2A_2 z + 3A_3 z^2 + \dots)}{z + A_2 z^2 + A_3 z^3 + \dots} = (1 + c_1 z + c_2 z^2 + \dots)^\beta,$$

which implies that

$$z + 2A_2 z^2 + 3A_3 z^3 + \dots = z + (\beta c_1 + A_2) z^2 + \left(\beta c_2 + \frac{\beta(\beta-1)c_1^2}{2} + \beta c_1 A_2 + A_3\right) z^3 + \dots.$$

Equating the coefficients of z^2 and z^3, we have

$$A_2 = \beta c_1, \tag{2.1}$$

since

$$A_3 = \frac{\beta}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{3}{4}\beta^2 c_1^2. \tag{2.2}$$

$$\begin{aligned} F(z) = \phi(a,c;z) * D^m(\lambda,\varphi)f(z) &= z + \sum_{n=2}^{\infty} \Phi_n^m(\lambda,\varphi)\frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \Phi_n^m(\lambda,\varphi)\frac{\Gamma(a+n-1)\Gamma(c)}{\Gamma(c+n-1)\Gamma(a)} a_n z^n, \end{aligned} \tag{2.3}$$

so we have

$$\beta c_1 = \Phi_2^m(\lambda,\varphi)\frac{\Gamma(a+1)\Gamma(c)}{\Gamma(c+1)\Gamma(a)} a_2.$$

This yields

$$a_2 = \frac{\beta c c_1}{a\Phi_2^m(\lambda,\varphi)}. \tag{2.4}$$

In view of Lemma 2.1 (i) we have

$$|a_2| \le \frac{2\beta\,|c|}{|a|\,\Phi_2^m(\lambda,\varphi)}.$$

On comparing the coefficients of z^3 in (2.3), we get

$$A_3 = \Phi_3^m(\lambda,\varphi)\frac{\Gamma(a+2)\Gamma(c)}{\Gamma(a)\Gamma(c+2)} a_3 = \Phi_3^m(\lambda,\varphi)\frac{a(a+1)}{c(c+1)} a_3.$$

Using (2.2), we obtain

$$a_3 = \frac{c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)}\left(\frac{\beta}{2}(c_2 - \frac{c_1^2}{2}) + \frac{3}{4}\beta^2 c_1^2\right). \tag{2.5}$$

Therefore, by applying Lemma 2.1 (ii), it follows that

$$|a_3| \le \frac{|c|\,|(c+1)|\,\beta}{4\Phi_3^m(\lambda,\varphi)\,|a|\,|(a+1)|}\left\{4 - |c_1|^2 + 3\beta\,|c_1|^2\right\}.$$

This inequality immediately proves the result. □

3. Main results. We first consider the functional $\left|a_3 - \mu a_2^2\right|$ for complex parameter μ.

Theorem 3.1. *Let a and c be complex parameters such that $a, c \ne 0, -1, -2, -3, \dots,$ $\lambda \ge \varphi \ge 0$ and $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$. If $f \in Q(m,\lambda,\varphi,\beta;a,c)$, $\beta \in (0,1]$ and μ is a complex parameter, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{\beta\,|c|\,|c+1|}{\Phi_3^m(\lambda,\varphi)\,|a|\,|a+1|}\max\left\{1, \frac{\beta v(\Phi,\mu;a,c)}{\Phi_2^{2m}(\lambda,\varphi)\,|a|\,|c+1|}\right\}, \tag{3.1}$$

where $v(\Phi,\mu;a,c) = 3\Phi_2^{2m}(\lambda,\varphi)a(c+1) - 4\Phi_3^m(\lambda,\varphi)\mu c(a+1)$.

Proof. From (2.4) and (2.5), it follows that

$$a_3-\mu a_2^2=\frac{\beta c(c+1)}{2\Phi_3^m(\lambda,\varphi)a(a+1)}\left(c_2-\frac{1}{2}c_1^2\right) + \frac{\beta^2 c[3\Phi_2^{2m}(\lambda,\varphi)a(c+1)-4\mu\Phi_3^m(\lambda,\varphi)c(a+1)]}{4\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)}c_1^2. \tag{3.2}$$

Therefore,

$$\left|a_3-\mu a_2^2\right|\le\frac{\beta\,|c|\,|c+1|}{2\Phi_3^m(\lambda,\varphi)\,|a|\,|a+1|}\left|c_2-\frac{1}{2}c_1^2\right| + \frac{\beta^2\,|c|\,|v(\Phi,\mu;a,c)|}{4\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)\,|a|^2\,|a+1|}\,|c_1|^2\,.$$

In view of Lemma 2.1 (ii), we obtain

$$\left|a_3-\mu a_2^2\right|\le\frac{\beta\,|c|\,|c+1|}{\Phi_3^m(\lambda,\varphi)\,|a|\,|a+1|} + \frac{\beta\,|c|\,[\beta\,|v(\Phi,\mu;a,c)|-\Phi_2^{2m}(\lambda,\varphi)\,|a|\,|c+1|]}{4\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)\,|a|^2\,|a+1|}\,|c_1|^2\,. \tag{3.3}$$

Suppose $\beta\,|v(\Phi,\mu;a,c)|\le\Phi_2^{2m}(\lambda,\varphi)\,|a|\,|c+1|$. Then it immediately follows that

$$\left|a_3-\mu a_2^2\right|\le\frac{\beta\,|c|\,|c+1|}{\Phi_3^m(\lambda,\varphi)\,|a|\,|a+1|}. \tag{3.4}$$

On the other hand, if $\beta\,|v(\Phi,\mu;a,c)|\ge\Phi_2^{2m}(\lambda,\varphi)\,|a|\,|c+1|$, then using Lemma 2.1 (i), we have

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\le\frac{\beta\,|c|\,|c+1|}{\Phi_3^m(\lambda,\varphi)\,|a|\,|a+1|} + \frac{\beta\,|c|\,[\beta\,|v(\Phi,\mu;a,c)|-\Phi_2^{2m}(\lambda,\varphi)\,|a|\,|c+1|]}{\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)\,|a|^2\,|a+1|}\\ &=\frac{\beta\,|a|\,|c|\,|c+1|\,\Phi_2^{2m}(\lambda,\varphi)+\beta^2\,|c|\,|v(\Phi,\mu;a,c)|-\beta\,|a|\,|c|\,|c+1|\,\Phi_2^{2m}(\lambda,\varphi)}{\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)\,|a|^2\,|a+1|}\\ &=\frac{\beta^2\,|c|\,|v(\Phi,\mu;a,c)|}{\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)\,|a|^2\,|a+1|}.\end{aligned} \tag{3.5}$$

The result immediately follows from (3.4) and (3.5). □

Equality in (3.4) and (3.5) is attained, respectively, for functions in $Q(m,\lambda,\varphi,\beta;a,c)$ given by

$$\frac{z\,(L(m,\lambda,\varphi;a,c)f(z))'}{L(m,\lambda,\varphi;a,c)f(z)}=\left(\frac{1+z^2}{1-z^2}\right)^\beta,\quad \frac{z\,(L(m,\lambda,\varphi;a,c)f(z))'}{L(m,\lambda,\varphi;a,c)f(z)}=\left(\frac{1+z}{1-z}\right)^\beta.$$

In the next result we consider the cases where μ is a real parameter.

Theorem 3.2. *Let $a, c \in (0, \infty)$, $\beta \in (0, 1]$, $\lambda \geq \varphi \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $f \in Q(m, \lambda, \varphi, \beta; a, c)$ and f is given by* (1.1) *then for real μ we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\beta^2 c[3a(c+1)\Phi_2^{2m}(\lambda,\varphi) - 4\mu c(a+1)\Phi_3^m(\lambda,\varphi)]}{\Phi_3^m(\lambda,\phi)\Phi_2^{2m}(\lambda,\phi)a^2(a+1)}, \\ \qquad \text{if } \mu \leq \frac{(3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)}, \\ \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)}, \\ \qquad \text{if } \frac{(3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)} \leq \mu \leq \frac{(3\beta+1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)}, \\ \frac{\beta^2 c\left(4\mu c(a+1)\Phi_3^m(\lambda,\varphi) - 3a(c+1)\Phi_2^{2m}(\lambda,\varphi)\right)}{\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)}, \\ \qquad \text{if } \mu \geq \frac{(3\beta+1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)}. \end{cases}$$

Proof. In view of (3.3), we need to consider two main cases.

Case 1. Let $\mu \leq \frac{3\Phi_2^{2m}(\lambda,\varphi)a(c+1)}{4\Phi_3^m(\lambda,\varphi)c(a+1)}$. Then (3.3) gives

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)} \\ &+ \frac{\beta c[(3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi) - 4\beta\mu c(a+1)\Phi_3^m(\lambda,\varphi)]}{4\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)} |c_1|^2 \end{aligned} \tag{3.6}$$

and by using the fact that $|c_1| \leq 2$, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\beta^2 c[3a(c+1)\Phi_2^{2m}(\lambda,\varphi) - 4\mu c(a+1)\Phi_3^m(\lambda,\varphi)]}{\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)},$$

provided that

$$\mu \leq \frac{(3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)}.$$

On the other hand, if

$$\mu \geq \frac{(3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)},$$

then the inequality (3.6) reduces to

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)} \\ &\quad - \frac{\beta c[4\mu\beta c(a+1)\Phi_3^m(\lambda,\varphi) - (3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)]}{4\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)} |c_1|^2 \\ &\leq \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)}. \end{aligned}$$

Case 2. Assume that $\mu \geq \frac{3\Phi_2^{2m}(\lambda,\varphi)a(c+1)}{4\Phi_3^m(\lambda,\varphi)c(a+1)}$. In this case, note that

$$v(\Phi, \mu; a, c) = 4\Phi_3^m(\lambda,\varphi)\mu c(a+1) - 3\Phi_2^{2m}(\lambda,\varphi)a(c+1)$$

and (3.3) reduces to

$$
\begin{aligned}
\left|a_3-\mu a_2^2\right| &\le \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)} \\
&+\frac{\beta c[4\beta\mu c(a+1)\Phi_3^m(\lambda,\varphi)-(3\beta+1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)]}{4\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)}\left|c_1\right|^2 .
\end{aligned}
\tag{3.7}
$$

Again, using the fact that $|c_1|\le 2$, we obtain

$$
\left|a_3-\mu a_2^2\right| \le \frac{\beta^2 c[4\mu c(a+1)\Phi_3^m(\lambda,\varphi)-3a(c+1)\Phi_2^{2m}(\lambda,\varphi)]}{\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)},
$$

where we have also used the condition that

$$
\mu \ge \frac{(3\beta+1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)}.
$$

On the other hand, if

$$
\mu \le \frac{(3\beta+1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)},
$$

then (3.7) yields

$$
\begin{aligned}
\left|a_3-\mu a_2^2\right| &\le \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)} \\
&\quad-\frac{\beta c[(3\beta+1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)-4\mu\beta c(a+1)\Phi_3^m(\lambda,\varphi)]}{4\Phi_3^m(\lambda,\varphi)\Phi_2^{2m}(\lambda,\varphi)a^2(a+1)}\left|c_1\right|^2 \\
&\le \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)}.
\end{aligned}
$$

Finally, we observe that

$$
\begin{aligned}
\frac{(3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)} \le \mu &\le \frac{3a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4c(a+1)\Phi_3^m(\lambda,\varphi)} \\
&\le \frac{(3\beta+1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4\beta c(a+1)\Phi_3^m(\lambda,\varphi)}.
\end{aligned}
$$

Thus the proof is complete. □

Corollary 3.3. *Let* $a,c\in(0,\infty)$, $\lambda\ge\varphi\ge 0$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ *and*

$$
0<\beta\le \frac{3a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{9a(c+1)\Phi_2^{2m}(\lambda,\varphi)-8c(a+1)\Phi_3^m(\lambda,\varphi)}.
$$

If $f\in Q(m,\lambda,\varphi,\beta;a,c)$ *and* f *is given by* (1.1), *then*

$$
|a_3|-|a_2|\le \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)}.
$$

Proof. Since

$$\frac{(3\beta-1)a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{4c(a+1)\beta\Phi_3^m(\lambda,\varphi)} \leq \frac{2}{3}$$

for

$$\beta \leq \frac{3a(c+1)\Phi_2^{2m}(\lambda,\varphi)}{9a(c+1)\Phi_2^{2m}(\lambda,\varphi) - 8c(a+1)\Phi_3^m(\lambda,\varphi)}$$

and

$$|a_3| - |a_2| \leq \left| a_3 - \frac{2}{3}a_2^2 \right| + \frac{2}{3}\left|a_2\right|^2 - \left|a_2\right|,$$

from Theorem 3.2 it follows that

$$|a_3| - |a_2| \leq \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)} + \frac{2}{3}\left|a_2\right|^2 - \left|a_2\right|.$$

Setting $|a_2| := x \in [0, 2\beta c/a]$, we can write

$$|a_3| - |a_2| \leq \frac{\beta c(c+1)}{\Phi_3^m(\lambda,\varphi)a(a+1)} + \frac{2}{3}x^2 - x := \Omega(x).$$

Since $\Omega(x)$ attains its maximum value at $x = 0$, the result follows. □

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