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Note about sequences of extremal $(A, 2B)$-edge coloured trees

Abstract. In this paper we determine successive extremal trees with respect to the number of all $(A, 2B)$-edge colourings.

1. Introduction. For concepts not defined here see [4]. Let G be an undirected, connected and simple graph with the vertex set $V(G)$ and the edge set $E(G)$. Then the order (number of vertices) and the size (number of edges) of G is denoted by n and m, respectively. Let $G(m)$ be a graph of size m. Then $P(m)$, $C(m)$, $T(m)$ and $S(m)$ denote a path, a cycle, a tree and a star of size m, respectively.

Let $P(m_1, m - m_1 - 1)$ be a 2-palm of size m, $m \geq 5$ and the diameter 3 with two support vertices $x, y \in V(P(m_1, m - m_1 - 1))$. Suppose that the support vertex x is adjacent to m_1 leaves, then the vertex y is adjacent to $m - m_1 - 1$ leaves.

In a tree, a vertex of degree at least 3 is a *branch vertex*, a vertex of degree 1 is a *leaf*. If a tree has exactly three leaves, then it is named a *tripod*. In other words, every tripod has the unique branch vertex and consequently this branch vertex is the initial vertex of three elementary paths. Let $m \geq 3$, $p \geq 1$, $t \geq 1$ be integers. Then $T(m, p, t)$ denotes a tripod of size m and three paths of length p, t and $m - p - t$ with the branch vertex as the initial vertex of these paths. For convenience a path of length i, $i \geq 1$ we denote shortly by *i-path*.

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Let b_m be the number of all nonisomorphic tripods of size m. Then it is given by the following recurrence relation $b_m = 1 + b_{m-2} + b_{m-3} - b_{m-5}$, for $m \geq 5$ with initial conditions $b_0 = b_1 = b_2 = 0$, $b_3 = b_4 = 1$, see [9], [10].

The nth Fibonacci number F_n is defined recursively as follows: $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ with $F_0 = F_1 = 1$.

The telephone numbers $t(n)$ are defined by the recurrence relation $t(n) = t(n-1) + (n-1)t(n-2)$, for $n \geq 2$ with $t(0) = t(1) = 1$.

Fibonacci and telephone numbers have many interesting applications and interpretations also in graphs. Fibonacci numbers have a graph interpretation known as the Merrifield–Simmons index (i.e. the number of all independent sets) of the n-vertex path P_n, see [5], [6], [7, p. 85–86].

Telephone numbers have also a graph interpretation known as the Hosoya index (i.e. the number of all matchings) of the n-vertex complete graph K_n. For details see [8].

In [1] and [2] there was introduced the graph interpretation of Fibonacci numbers and telephone numbers with respect to the special edge colourings of a graph. We recall this definition.

Let $\mathcal{C} = \{A, B\}$ be the set of two colours. A graph G is $(A, 2B)$-*edge coloured* if for every maximal B-monochromatic subgraph H of G there is a partition of H into edge disjoint paths of length 2. Clearly $(A, 2B)$-edge colouring always exists, since we have no restriction on the colour A.

Let $\mathcal{L}$ be a family of all distinct $(A, 2B)$-edge coloured graphs obtained by $(A, 2B)$-colouring of a graph G. Then $\mathcal{L} = \{G^{(1)}, G^{(2)}, \ldots, G^{(r)}\}$, $r \geq 1$, where $G^{(p)}$, $1 \leq p \leq r$ denotes a graph obtained by $(A, 2B)$-edge colouring of a graph G. For $(A, 2B)$-edge coloured graph $G^{(p)}$, $1 \leq p \leq r$ by $\theta(G^{(p)})$ we denote the number of all partitions of B-monochromatic subgraphs of $G^{(p)}$ into edge disjoint paths of length 2. For the explanation, if $G^{(p)}$ is A-monochromatic, then we put $\theta(G^{(p)}) = 1$. The number of all $(A, 2B)$-edge colourings we define as the graph parameter

$$\sigma_{(A,2B)}(G) = \sum_{p=1}^{r} \theta(G^{(p)}).$$

The parameter $\sigma_{(A,2B)}(G)$ was determined for paths, cycles and bounded for trees, for details see [1], [2], [3]. In this paper we give sequences of $(A, 2B)$-extremal trees, i.e. consecutive trees with extremal values of the parameter $\sigma_{(A,2B)}(T(m))$.

2. Main results. In [2], the lower bound and the upper bound of the parameter $\sigma_{(A,2B)}(T(m))$, $m \geq 1$ were given. Moreover, in [1] it was proved that the upper bound is realized by telephone numbers. This result is presented in the following theorem.

Theorem 1 ([1], [2]). *Let $T(m)$ be a tree of size m, $m \geq 1$. Then*

$$F_m \leq \sigma_{(A,2B)}(T(m)) \leq t(m).$$

Moreover, $\sigma_{(A,2B)}(T(m)) = F_m$ for $T(m) \cong P(m)$ and $\sigma_{(A,2B)}(T(m)) = t(m)$ for $T(m) \cong S(m)$.

Next in [1] the following estimations for the parameter $\sigma_{(A,2B)}(T(m,p,t))$ in the class of tripods were proved:

Theorem 2 ([1]). *Let $m \geq 3$ be an integer. If $T(m) \ncong P(m)$ and $T(m) \ncong T(m,p,t)$ for all $p \geq 1$, $t \geq 1$, then*

$$\sigma_{(A,2B)}(P(m)) \leq \sigma_{(A,2B)}(T(m,p,t)) \leq \sigma_{(A,2B)}(T(m)).$$

From the above theorems it is clear that a path $P(m)$ is the extremal tree achieving the minimum value of $\sigma_{(A,2B)}(T(m))$. Moreover, if we want to find the next successive smallest trees with respect to the parameter $\sigma_{(A,2B)}(T(m))$ we have to study the whole class of tripods. The maximum and minimum value of $\sigma_{(A,2B)}(T(m,p,t))$ were established in [3].

Theorem 3 ([3]). *Let $m \geq 4$, $p \geq 1$, $t \geq 1$ be integers. Then*

$$F_{m-1} + 2F_{m-3} \leq \sigma_{(A,2B)}(T(m,p,t)) \leq 2F_{m-1}.$$

Moreover, $\sigma_{(A,2B)}(T(m,p,t)) = 2F_{m-1}$ if $T(m,p,t) \cong T(m,1,1)$ and $\sigma_{(A,2B)}(T(m,p,t)) = F_{m-1} + 2F_{m-3}$ if $T(m,p,t) \cong T(m,2,2)$.

From Theorem 2 and Theorem 3 we can deduce that the tripod $T(m,2,2)$ is the second smallest tree with respect to the $\sigma_{(A,2B)}(T(m))$. In [1] there was found the second minimal tripod $T(m,4,2)$ with respect to the parameter $\sigma_{(A,2B)}(T(m))$ which is also, by Theorem 2, the third smallest in the class of trees. From Theorem 3 it is obvious that the tripod $T(m,1,1)$ is the largest in the class of tripods with respect to $\sigma_{(A,2B)}(T(m,p,t))$. If we investigate the whole class of tripods, we obtain the sequence of successive extremal tripods from the minimal $T(m,2,2)$ to the maximal $T(m,1,1)$.

Let $T(m,p,t)$ be an arbitrary tripod, where $m \geq 4$, $p \geq 1$, $t \geq 1$. For $t = 2, 4, \ldots, \lfloor \frac{m}{3} \rfloor, \ldots, 3, 1$ we obtain the successive smallest tripods with respect to the parameter $\sigma_{(A,2B)}(T(m))$. The integer t assumes the consecutive even numbers from 2 to $\lfloor \frac{m}{3} \rfloor$, if $\lfloor \frac{m}{3} \rfloor$ is even or to $\lfloor \frac{m}{3} \rfloor - 1$, if $\lfloor \frac{m}{3} \rfloor$ is odd. Then t assumes the consecutive odd numbers from $\lfloor \frac{m}{3} \rfloor$ or $\lfloor \frac{m}{3} \rfloor - 1$ to 1. In other words, the tripod $T(m,p,2)$ always achieves the smallest value of $\sigma_{(A,2B)}(T(m,p,t))$ and the tripod $T(m,p,1)$ always achieves the largest value of this parameter.

Moreover, for each value of t we can construct a sequence of extremal tripods with respect to the integer p.

If $t = 1$, then the successive smallest tripods are given respectively for $p = 2, 4, \ldots, \lfloor \frac{m-1}{2} \rfloor, \ldots, 3, 1$.

If $t = 2$, then $p = 2, 4, \ldots, \lfloor \frac{m-2}{2} \rfloor, \ldots, 5, 3$.

If $t = 3$, then $p = 4, 6, \ldots, \lfloor \frac{m-3}{2} \rfloor, \ldots, 5, 3$.
If $t = 4$, then $p = 4, 6, \ldots, \lfloor \frac{m-4}{2} \rfloor, \ldots, 7, 5$.

$\vdots$

If $t = \lfloor \frac{m}{3} \rfloor$ and $\lfloor \frac{m}{3} \rfloor$ is even, then $p = \lfloor \frac{m}{3} \rfloor, \ldots, \left\lfloor \frac{m - \lfloor \frac{m}{3} \rfloor}{2} \right\rfloor, \ldots, \lfloor \frac{m}{3} \rfloor + 1$, and if $\lfloor \frac{m}{3} \rfloor$ is odd, then $p = \lfloor \frac{m}{3} \rfloor + 1, \ldots, \left\lfloor \frac{m - \lfloor \frac{m}{3} \rfloor}{2} \right\rfloor, \ldots, \lfloor \frac{m}{3} \rfloor$.

For example if we consider the tripods $T(12, p, t)$ for all possible p, t, then the sequence of successive smallest tripods with respect to the increasing parameter $\sigma_{(A,2B)}(T(m, p, t))$ is as follows

$$T(12,2,2), T(12,4,2), T(12,5,2), T(12,3,2), T(12,4,4), T(12,4,3),$$
$$T(12,3,3), T(12,2,1), T(12,4,1), T(12,5,1), T(12,3,1), T(12,1,1).$$

The above considerations give an answer to the question given in [1].

Let $t_i(m)$, $i = 1, \ldots, b_{m+1}$ be the ith minimum tree of size m with respect to the parameter $\sigma_{(A,2B)}(T(m))$. Then

$$t_1(m) \cong P(m), t_2(m) \cong T(m,2,2), t_3(m) \cong T(m,4,2),$$
$$t_4(m) \cong T(m,6,2), \ldots, t_{m-1}(m) \cong T(m,5,1),$$
$$t_m(m) \cong T(m,3,1), t_{m+1}(m) \cong T(m,1,1).$$

Now we determine the successive trees with respect to the decreasing parameter $\sigma_{(A,2B)}(T(m))$. To do this we use among others the following lemma.

Lemma 1 ([2]). *Let $G = H \cup T(l) \cup \{e\}$ be a connected graph, where H is a connected graph, $T(l)$ is a tree of size l, $l \geq 1$ and H and $T(l)$ are vertex disjoint. Assume that $e = uv$, where $u \in V(H)$ and $v \in V(T(l))$. Then*

$$\sigma_{(A,2B)}(H \cup P(l) \cup \{e\}) \leq \sigma_{(A,2B)}(G) \leq \sigma_{(A,2B)}(H \cup S(l) \cup \{e\}), \tag{1}$$

where the vertex v is identified with the center of the star $S(l)$. Moreover, the equality holds if $T(l) \cong P(l)$ or $T(l) \cong S(l)$.

Theorem 4. *Let $m \geq 5$, $m_1 \geq 2$ be integers. Then*

$$\sigma_{(A,2B)}(T(m)) \leq \sigma_{(A,2B)}(P(m_1, m - m_1 - 1)) \leq \sigma_{(A,2B)}(S(m)).$$

Proof. The inequality $\sigma_{(A,2B)}(P(m_1, m-m_1-1)) \leq \sigma_{(A,2B)}(S(m))$ follows immediately from Theorem 1. Let $T(m) \ncong S(m)$ and $T(m) \ncong P(m_1, m - m_1 - 1)$. Then $\operatorname{diam} T(m) \geq 4$. Let $\overline{P} = x - y$ be the path which realizes the diameter $\operatorname{diam} T(m)$. Then $x, y \in V(T(m))$ are leaves. Let $u \in V(T(m))$ be adjacent to the vertex x and the edge $e \in \overline{P}$ be incident with u and e is not incident with a leaf. Then $T(m) = T(m_1) \cup T(m_2) \cup \{e\}$, where $m_1 + m_2 + 1 = m$. Applying Lemma 1, we have

$$\sigma_{(A,2B)}(T(m)) \leq \sigma_{(A,2B)}(S(m_1) \cup S(m_2) \cup \{e\}) = \sigma_{(A,2B)}(P(m_1, m-m_1-1)),$$

which ends the proof. $\square$

Let $T_i(m)$ be the ith maximum tree of size m with respect to the parameter $\sigma_{(A,2B)}(T(m))$.

Theorem 5. *Let $m \geq 5$ be an integer. Then $T_1(m) = S(m)$, $\sigma_{(A,2B)}(S(m)) = t(m)$ and $T_i(m) = P(m-i, i-1)$ for $2 \leq i \leq m - \lceil \frac{m-1}{2} \rceil$.*

Theorem 6. *Let $m \geq 5$, $m_1 \geq 2$ be integers. If $T(m) \ncong S(m)$ and $T(m) \ncong P(m_1, m-m_1-1)$, then $\sigma_{(A,2B)}(T(m)) \leq \sigma_{(A,2B)}(P(m_1, m-m_1-2))$.*

Proof. Since $T(m) \ncong S(m)$ and $T(m) \ncong P(m_1, m-m_1-1)$ then $\operatorname{diam} T(m) \geq 4$. Let $e \in E(T(m))$ is not incident with a leaf. Such edge there exists because $\operatorname{diam} T(m) \geq 4$. Let $T(m) = T(m_1) \cup \{e\} \cup T(m_2)$, where $m = m_1 + m_2 + 1$. Then $\operatorname{diam} T(m_1) \geq 2$ or $\operatorname{diam} T(m_2) \geq 2$. Suppose without loss of generality that $\operatorname{diam} T(m_2) \geq 2$. Let us consider the following possibilities:
1. $c(e) = A$. Then

$$\sigma_{A(e)}(T(m)) = \sigma_{(A,2B)}T(m_1)\sigma_{(A,2B)}T(m_2).$$

2. $c(e) = 2B$. Then

$$\begin{aligned}\sigma_{2B(e)}(T(m)) = {} & \sigma_{2B(e)}(T(m_1) \cup \{e\})\sigma_{(A,2B)}(T(m_2)) \\ & + \sigma_{(A,2B)}(T(m_1))\sigma_{2B(e)}(T(m_2) \cup \{e\}).\end{aligned}$$

Therefore $\sigma_{(A,2B)}(T(m)) = \sigma_{A(e)}(T(m)) + \sigma_{2B(e)}(T(m))$. Hence

$$\begin{aligned}\sigma_{(A,2B)}(T(m)) = {} & \sigma_{(A,2B)}T(m_1)\sigma_{(A,2B)}T(m_2) \\ & + \sigma_{(A,2B)}(T(m_1) \cup \{e\})\sigma_{(A,2B)}(T(m_2)) \\ & + \sigma_{(A,2B)}(T(m_1))\sigma_{(A,2B)}(T(m_2) \cup \{e\}).\end{aligned}$$

Since $\operatorname{diam} T(m_2) \geq 2$ then applying Lemma 1, we have

$$\begin{aligned}\sigma_{(A,2B)}(T(m)) \leq {} & \sigma_{(A,2B)}(P(m_2-1, 1))\sigma_{(A,2B)}(S(m_1)) \\ & + \sigma_{2B(e)}(P(m_2-2, 1))\sigma_{(A,2B)}(S(m_1)) \\ & + \sigma_{(A,2B)}(P(m_2-1, 1))\sigma_{2B(e)}(P(m_2-1, 1)) \\ = {} & \sigma_{(A,2B)}(P(m_2-1) \cup S(m_1) \cup \{e\}) \\ = {} & \sigma_{(A,2B)}(P(m_1, m-m_1-2))\end{aligned}$$

which ends the proof. □

Theorem 7. *Let $m \geq 5$ be an integer. Then*

$$\sigma_{(A,2B)}(P(m_1, m-m_1-2)) = t(m_1+1)t(m-m_1-1) + t(m_1)t(m-m_1-2).$$

Proof. Let $e, e' \in E(P(m_1, m-m_1-2))$ be not incident with a leaf. If $c(e) = c(e') = 2B$ and a 2-path $e - e'$ belongs to a partition of $2B$-monochromatic subgraph into 2-paths, then we have $t(m_1)t(m-m_1-2)$ possibilities in this case. Otherwise the tree $P(m_1, m-m_1-2)$ can be considered as the union of two stars $S(m_1+1)$ and $S(m-m_1-1)$ and the result follows. □

From the above theorems we have

Corollary 8. *Let* $m \geq 5$, $m_1 \geq 2$ *be integers,* $T(m) \not\cong S(m)$ *and* $T(m) \not\cong P(m_1, m - m_1 - 1)$. *Then*

$$\sigma_{(A,2B)}(T(m)) \leq t(m_1 + 1)t(m - m_1 - 1) + t(m_1)t(m - m_1 - 2).$$

In the same way as for the palm $P(m_1, m - m_1 - 1)$, see [1], we can show the behavior of the parameter $\sigma_{(A,2B)}(P(m_1, m - m_1 - 2))$ after moving an edge adjacent to a support vertex to another support vertex. So we omit the proof.

Lemma 2. *Let* $m \geq 6$, $m_1 \geq 2$ *be integers and* $m_1 \geq m - m_1 - 2$. *Then*

$$\sigma_{(A,2B)}(P(m_1 + 1, m - m_1 - 3)) > \sigma_{(A,2B)}(P(m_1, m - m_1 - 2)).$$

Theorem 9. *Let* $T(m)$ *be a tree of size* m, $m \geq 6$, $T(m) \not\cong S(m)$ *and* $T(m) \not\cong P(m_1, m - m_1 - 1)$ *for all* $m_1 \geq 2$. *Then*

$$\sigma_{(A,2B)}(T(m)) \leq \sigma_{(A,2B)}(P(m - 3, 1)).$$

Proof. Let $T(m)$ be a tree of size $m, m \geq 6$ such that $T(m) \not\cong S(m)$ and $T(m) \not\cong P(m_1, m - m_1 - 1)$ for all $m_1 \geq 2$. Then by Theorem 6, $\sigma_{(A,2B)}(T(m)) \geq \sigma_{(A,2B)}(P(m_1, m-m_1-2))$ for all $m \geq 2$. If $m-m_1-2 = 1$, then $m_1 = m - 3$ and $P(m_1, m - m_1 - 2) \not\cong P(m - 3, 1)$, so the result follows. Let $m - m_1 - 2 \geq 2$ and without loss of the generality, suppose that $m_1 \geq m - m_1 - 2$. Applying Lemma 2, we obtain

$$\sigma_{(A,2B)}(P(m_1, m - m_1 - 2)) < \sigma_{(A,2B)}(P(m_1 + 1, m - m_1 - 3)).$$

If $m - m_1 - 3 \geq 2$, then we apply Lemma 2 until we obtain the palm $P(m - 3, 1)$, which ends the proof. □

Let $T_i^*(m)$ be the ith maximum tree of size m in the class of 2-palms $P(m_1, m-m_1-2)$ for $m_1 \geq 2$ with respect to the parameter $\sigma_{(A,2B)}(T(m))$. From the above considerations we have

Theorem 10. *Let* $m \geq 5$ *be an integer. Then* $T_i^*(m) = P(m - i - 2, i)$ *for* $i = 1, 2, \ldots, \lceil \frac{m-2}{2} \rceil$.

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