doi: 10.2478/v10062-012-0021-1

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVII, NO. 1, 2013 SECTIO A 45–56

IWONA NARANIECKA, JAN SZYNAL and ANNA TATARCZAK

Linearly-invariant families and generalized Meixner–Pollaczek polynomials

Abstract. The extremal functions $f_0(z)$ realizing the maxima of some functionals (e.g. $\max|a_3|$, and $\max \arg f'(z)$) within the so-called universal linearly invariant family U_α (in the sense of Pommerenke [10]) have such a form that $f_0'(z)$ looks similar to generating function for Meixner–Pollaczek (MP) polynomials [2], [8]. This fact gives motivation for the definition and study of the generalized Meixner–Pollaczek (GMP) polynomials $P_n^\lambda(x;\theta,\psi)$ of a real variable x as coefficients of

$$G^\lambda(x;\theta,\psi;z)=\frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1-ze^{i\psi})^{\lambda+ix}}=\sum_{n=0}^{\infty}P_n^\lambda(x;\theta,\psi)z^n,\ |z|<1,$$

where the parameters λ, θ, ψ satisfy the conditions: $\lambda>0$, $\theta\in(0,\pi)$, $\psi\in\mathbb{R}$. In the case $\psi=-\theta$ we have the well-known (MP) polynomials. The cases $\psi=\pi-\theta$ and $\psi=\pi+\theta$ leads to new sets of polynomials which we call quasi-Meixner–Pollaczek polynomials and strongly symmetric Meixner–Pollaczek polynomials. If $x=0$, then we have an obvious generalization of the Gegenbauer polynomials.

The properties of (GMP) polynomials as well as of some families of holomorphic functions $|z|<1$ defined by the Stieltjes-integral formula, where the function $zG^\lambda(x;\theta,\psi;z)$ is a kernel, will be discussed.

1. Linearly-invariant families of holomorphic functions

$$f(z)=z+a_2z^2+\dots,\quad z\in\mathbb{D}\tag{1.1}$$

2010 *Mathematics Subject Classification.* 30C45, 30C70, 42C05, 33C45.

Key words and phrases. Typically-real functions, univalent functions, local univalence, orthogonal polynomials, Meixner–Pollaczek polynomials.

in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ were introduced by Pommerenke in [10], and then were intensively studied by several authors (e.g. [14], [15] and [17]).

A family $\mathfrak{M}$ of holomorphic functions of the form (1.1) is linearly-invariant if it satisfies two conditions:

(a) $f'(z) \neq 0$ for any z in $\mathbb{D}$ (local univalence),

(b) for any linear fractional transformation

$$\phi(z) = e^{i\theta}\frac{z+a}{1+\bar{a}z}, \quad a, z \in \mathbb{D}, \ \theta \in \mathbb{R},$$

of $\mathbb{D}$ onto itself, the function

$$\Lambda[f](z) = F(z) = \frac{f(\phi(z)) - f(\phi(0))}{f'(\phi(0))\phi'(0)} = z + \ldots \in \mathfrak{M}.$$

The order of the linearly-invariant family $\mathfrak{M}$ is defined as

$$\operatorname{ord}\mathfrak{M} = \sup_{f\in\mathfrak{M}} |a_2(f)|.$$

Universal invariant family U_α is defined as

$$U_\alpha = \bigcup_{\operatorname{ord}\mathfrak{M}\le\alpha} \mathfrak{M}.$$

It is well known that $\alpha \geq 1$ and $U_1 \equiv S^c =$ the class of convex univalent functions in $\mathbb{D}$, and the familiar class S of all univalent functions is strictly included in U_2. Moreover, for every $\alpha > 1$, the class U_α contains functions which are infinitely valent in $\mathbb{D}$ [10], for example:

$$f_0(z) = \frac{1}{2i\gamma}\left[\left(\frac{1+z}{1-z}\right)^{i\gamma} - 1)\right],$$

$$f_0'(z) = \frac{1}{(1+z)^{1-i\gamma}(1-z)^{1+i\gamma}}, \quad \gamma = \sqrt{\alpha^2-1}.$$

Another example of such a function was presented in [15]:

$$f_0(z) = \frac{1}{(e^{it_2} - e^{it_1})i\sqrt{\alpha^2-1}}\left[\left(\frac{1-ze^{it_1}}{1-ze^{it_2}}\right)^{i\sqrt{\alpha^2-1}} -1\right], \ t_1 \neq t_2 + 2k\pi, \tag{1.2}$$

for which

$$f_0'(z) = \frac{1}{(1-ze^{it_1})^{1-i\sqrt{\alpha^2-1}}(1-ze^{it_2})^{1+i\sqrt{\alpha^2-1}}}. \tag{1.3}$$

Functions of the form (1.2) appear to be extremal for the long lasting problems:

$$\max_{f\in U_\alpha} |a_3| \quad \text{and} \quad \max_{f\in U_\alpha} |\arg f'(z)|,$$

recently solved by Starkov [14], [15], who proved that the extremal function for $\max|a_3|$ is of the form (1.2) with $t_1 = \theta, \ t_2 = -\theta$, where

$$e^{i\theta} = \sqrt{\frac{(3-\alpha^2) + 3i\sqrt{\alpha^2-1}}{\alpha\sqrt{\alpha^2+3}}}, \ f \in U_\alpha \ ([14]).$$

However, the extremal function f_0 for $\max_{f\in U'_\alpha} |\arg f'(z)|$ is of the form (1.2) with

$$t_1 = \pi - \arctan\frac{1}{\alpha} - \arctan\frac{r}{\alpha},$$
$$t_2 = -\pi + \arcsin\frac{1}{\alpha} - \arcsin\frac{r}{\alpha},\ r = |z| < 1,\ t_1 \neq -t_2\ ([15]).$$

We see that the extremal function for $\max_{f\in U_\alpha} |a_3|$ has a special form leading to (MP) polynomials, but the extremal function for $\max_{f\in U_\alpha} |\arg f'(z)|$ leads to (GMP) polynomials, defined below.

2. Comparing (1.3) with the generating function for Meixner–Pollaczek polynomials $P_n^\lambda(x;\theta)$ ([2]):

$$G^\lambda(x;\theta,-\theta;z) = \frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1-ze^{-i\theta})^{\lambda+ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x;\theta)z^n, \quad z\in\mathbb{D},$$

where $\lambda > 0$, $\theta \in (0,\pi)$, $x\in\mathbb{R}$, we are motivated to introduce the generalized Meixner–Pollaczek (GMP) polynomials $P_n^\lambda(x;\theta,\psi)$ of variable $x\in\mathbb{R}$ and parameters $\lambda > 0$, $\theta\in(0,\pi)$, $\psi\in\mathbb{R}$ via the generating function

$$G^\lambda(x;\theta,\psi;z) = \frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1-ze^{i\psi})^{\lambda+ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x;\theta,\psi)z^n, \tag{2.1}$$

$z\in\mathbb{D}$. Of course, we have $P_n^\lambda(x;\theta,-\theta) = P_n^\lambda(x;\theta)$. We will find the three-term recurrence relation, the explicite formula, the hypergeometric representation and the difference equation for (GMP) polynomials $P_n^\lambda(x;\theta,\psi)$.

Theorem 2.1. (i) *The polynomials $P_n^\lambda = P_n^\lambda(x;\theta,\psi)$ satisfy the three-term recurrence relation:*

$$\begin{aligned} P_{-1}^\lambda &= 0, \\ P_0^\lambda &= 1, \\ nP_n^\lambda &= [(\lambda-ix)e^{i\theta} + (\lambda+ix)e^{i\psi} + (n-1)(e^{i\theta}+e^{i\psi})]P_{n-1}^\lambda \\ &\quad - [(2\lambda+n-2)e^{i(\theta+\psi)}]P_{n-2}^\lambda,\ n\geq 1. \end{aligned} \tag{2.2}$$

(ii) *The polynomials $P_n^\lambda(x;\theta,\psi)$ are given by the formula:*

$$P_n^\lambda(x;\theta,\psi) = e^{in\theta}\sum_{j=0}^{n}\frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}e^{ij(\psi-\theta)}, \quad n\in\mathbb{N}\cup\{0\}. \tag{2.3}$$

(iii) *The polynomials $P_n^\lambda(x;\theta,\psi)$ have the hypergeometric representation*

$$n!P_n^\lambda(x;\theta,\psi) = (2\lambda)_n e^{in\theta} F\left(-n, \lambda+ix, 2\lambda; 1-\frac{e^{i\psi}}{e^{i\theta}}\right). \tag{2.4}$$

(iiii) *Let* $y(x) = P_n^\lambda(x;\theta,\psi)$. *The function* $y(x)$ *satisfies the following difference equation*

$$
\begin{aligned}
(2.5)\qquad & e^{i\theta}(\lambda - ix)y(x+i) + [ix(e^{i\theta}+e^{i\psi}) - (n+\lambda)(e^{i\theta}-e^{i\psi})]y(x) \\
& \qquad - e^{i\psi}(\lambda + ix)y(x-i) = 0.
\end{aligned}
$$

Proof. (i) We differentiate the formula (2.1) with respect to z and after multiplication by $(1-ze^{i\theta})(1-ze^{i\psi})$ we compare the coefficients at the power z^{n-1}.
(ii) The Cauchy product for the power series

$$(1-ze^{i\theta})^{-(\lambda-ix)} = \sum_{n=0}^{\infty} \frac{(\lambda-ix)_n e^{in\theta}}{n!} z^n$$

and

$$(1-ze^{i\psi})^{-(\lambda+ix)} = \sum_{n=0}^{\infty} \frac{(\lambda+ix)_n e^{in\psi}}{n!} z^n$$

gives (2.3).
(iii) We apply the formula from ([4], vol. 1, p. 82):

$$(1-s)^{a-c}(1-s+sz)^{-a} = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} F(-n,a;c;z)s^n, \quad |s|<1,\ |s(1-z)|<1,$$

with $s = ze^{i\theta}$, $a = \lambda + ix$, $c = 2\lambda$, $z = 1 - e^{i(\psi-\theta)}$ and obtain

$$
\begin{aligned}
&(1-ze^{i\theta})^{-(\lambda-ix)}(1-ze^{i\psi})^{-(\lambda+ix)} \\
&\qquad = \sum_{n=0}^{\infty} \frac{z^n e^{in\theta}(2\lambda)_n}{n!} F(-n,\lambda+ix,2\lambda;1-e^{i(\psi-\theta)}).
\end{aligned}
$$

Comparing the coefficients at the power z^n, we get (2.4).
(iiii) Putting $(x+i)$ and $(x-i)$ instead of x into the generating function (2.1), we find that

$$
\begin{aligned}
y(x+i) &= \sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)[e^{i(n-k)\theta} - e^{i[(n-k-1)\theta+\psi]}] + P_n^\lambda \\
y(x-i) &= \sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)[e^{i(n-k)\psi} - e^{i[(n-k-1)\psi+\theta]}] + P_n^\lambda,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&e^{i\theta}(\lambda - ix)y(x+i) - e^{i\psi}(\lambda + ix)y(x-i)\\
&= (e^{i\theta} - e^{i\psi})\sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)[(\lambda - ix)e^{i(n-k)\theta} + (\lambda + ix)e^{i(n-k)\psi}]\\
&\quad + [e^{i\theta}(\lambda - ix) - e^{i\psi}(\lambda + ix)]P_n^\lambda.
\end{aligned} \tag{2.6}
$$

Differentiation of the generating function (2.1) with respect to z and comparison of the coefficients at z^{n-1} yields:

$$nP_n^\lambda(x;\theta,\psi) = \sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)[(\lambda - ix)e^{i(n-k)\theta} + (\lambda + ix)e^{i(n-k)\psi}]$$

which together with (2.6) gives (2.5). □

The first four polynomials P_n^λ are given by the formulas:

Corollary 1.

$$
\begin{aligned}
P_0^\lambda &= 1,\\
P_1^\lambda &= ix(e^{i\psi} - e^{i\theta}) + \lambda(e^{i\theta} + e^{i\psi}),\\
2P_2^\lambda &= -x^2(e^{i\psi} - e^{i\theta})^2 + ix(2\lambda+1)(e^{2i\psi} - e^{2i\theta}) + \lambda[(1+\lambda)e^{2i\psi}\\
&\quad + 2\lambda e^{i(\psi+\theta)} + (1+\lambda)e^{2i\theta}],\\
6P_3^\lambda &= ix^3[3e^{i\theta}e^{i\psi}(e^{i\psi} - e^{i\theta}) - (e^{3i\psi} - e^{3i\theta})]\\
&\quad + 3(1+\lambda)x^2[e^{i\theta}e^{i\psi}(e^{i\psi} + e^{i\theta})\\
&\quad - (e^{3i\psi} + e^{3i\theta})] + +ix[3\lambda^2 e^{i\theta}e^{i\psi}(e^{i\psi} - e^{i\theta})\\
&\quad + (3\lambda^2 + 6\lambda + 2)(e^{3i\psi} - e^{3i\theta})]\\
&\quad + \lambda(1+\lambda)[3\lambda e^{i\theta}e^{i\psi}(e^{i\psi} + e^{i\theta}) + (\lambda+2)(e^{3i\psi} + e^{3i\theta})],\\
24P_4^\lambda &= x^4[(e^{i\psi} - e^{i\theta})^4 + 4e^{2i\psi}e^{2i\theta}] + 2ix^3(2\lambda+3)(e^{2i\psi} - e^{2i\theta})(e^{i\psi} + e^{i\theta})^2\\
&\quad + x^2[-(6\lambda^2 + 18\lambda + 11)(e^{4i\psi} + e^{4i\theta}) + 4(3\lambda+2)e^{i\psi}e^{i\theta}(e^{2i\psi} + e^{2i\theta})\\
&\quad + 6(2\lambda^2 + 2\lambda + 1)e^{2i\psi}e^{2i\theta}] + 2ix(e^{2i\psi} - e^{2i\theta})[(4\lambda^3 + 9\lambda^2 + 11\lambda + 3)\\
&\quad \times (e^{2i\psi} + e^{2i\theta}) + 2\lambda(2\lambda+3)e^{i\psi}e^{i\theta}]\\
&\quad + \lambda(1+\lambda)[(\lambda+2)(\lambda+3)(e^{4i\psi} + e^{4i\theta})\\
&\quad + 4\lambda(\lambda+2)e^{i\psi}e^{i\theta}(e^{2i\psi} + e^{2i\theta}) + 6\lambda(\lambda+1)e^{2i\psi}e^{2i\theta}].
\end{aligned}
$$

The four special cases of $P_n^\lambda(x;\theta,\psi)$ corresponding to the choice:

$$(a)\quad \psi = -\theta,\quad (b)\quad \psi = \pi - \theta,\quad (c)\quad \psi = \pi + \theta,\quad (d)\quad \psi = \theta$$

lead to some interesting families of polynomials. Namely, we define:

$$(a)\ \ G^{\lambda}(x;\theta,-\theta;z)=\frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1-ze^{-i\theta})^{\lambda+ix}}=\sum_{n=0}^{\infty}P_n^{\lambda}(x;\theta)z^n,\ \ z\in\mathbb{D},$$

and of course $P_n^{\lambda}(x;\theta)$ are the well-known (MP) polynomials of variable $x\in\mathbb{R}$ with parameters $\lambda>0$, $\theta\in(0,\pi)$;

$$(b)\ \ G^{\lambda}(x;\theta,\pi-\theta;z)=\frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1+ze^{-i\theta})^{\lambda+ix}}=\sum_{n=0}^{\infty}Q_n^{\lambda}(x;\theta)z^n,\ \ z\in\mathbb{D},$$

where $Q_n^{\lambda}(x;\theta)$ we call quasi-Meixner–Pollaczek (QMP) polynomials;

$$(c)\ \ G^{\lambda}(x;\theta,\pi+\theta;z)=\frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1+ze^{i\theta})^{\lambda+ix}}=\sum_{n=0}^{\infty}S_n^{\lambda}(x;\theta)z^n,\ \ z\in\mathbb{D},$$

where $S_n^{\lambda}(x;\theta)$ we call strongly symmetric Meixner–Pollaczek (SSMP) polynomials.

Observe that the special cases: $i^{-n}Q_n^{\lambda}(x;0)$ and $S_n^{\lambda}(x;\frac{\pi}{2})$ represent symmetric (MP) polynomials studied in [1], [6], [8] and [9].

$$(d)\ \ G^{\lambda}(x;\theta,\theta;z)=\frac{1}{(1-ze^{i\theta})^{2\lambda}}=\sum_{n=0}^{\infty}H_n^{\lambda}(\theta)z^n,\quad z\in\mathbb{D},$$

where $H_n^{\lambda}(\theta)=\frac{(2\lambda)_n}{n!}e^{in\theta}$.

From Theorem 2.1 we have as the corollaries the following formulas for the polynomials

$$P_n^{\lambda}(x;\theta)=P_n^{\lambda}(x;\theta,-\theta),\ Q_n^{\lambda}(x;\theta)=P_n^{\lambda}(x;\theta,\pi-\theta),\ S_n^{\lambda}(x;\theta)=P_n^{\lambda}(x;\theta,\pi+\theta).$$

Corollary 2. (i) *The (MP) polynomials $P_n^{\lambda}(x;\theta)$ satisfy the three-term recurrence relation:*

$$\begin{aligned}P_{-1}^{\lambda}(x;\theta)&=0,\\ P_0^{\lambda}(x;\theta)&=1,\\ nP_n^{\lambda}(x;\theta)&=2[xsin\theta+(n-1+\lambda)\cos\theta]P_{n-1}^{\lambda}(x;\theta)\\ &\qquad-(2\lambda+n-2)P_{n-2}^{\lambda}(x;\theta),\quad n\geq 1.\end{aligned}$$

(ii) *The polynomials $P_n^{\lambda}(x;\theta)$ are given by the formula:*

$$P_n^{\lambda}(x;\theta)=e^{in\theta}\sum_{j=0}^{n}\frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}e^{-2ij\theta},\quad n\in\mathbb{N}\cup\{0\}.$$

(iii) *The polynomials $P_n^{\lambda}(x;\theta)$ have the hypergeometric representation*

$$P_n^{\lambda}(x;\theta)=e^{in\theta}\frac{(2\lambda)_n}{n!}F\big(-n,\lambda+ix,2\lambda;1-e^{-2i\theta}\big).$$

(iiii) *The polynomials $y(x)=P_n^{\lambda}(x;\theta)$ satisfy the following difference equation*

$$e^{i\theta}(\lambda - ix)y(x+i) + 2i[x\cos\theta - (n+\lambda)sin\theta]y(x) - e^{-i\theta}(\lambda + ix)y(x-i) = 0.$$

Corollary 3. (i) *The (QMP) polynomials* $Q_n^\lambda = Q_n^\lambda(x;\theta)$ *satisfy the three-term recurrence relation:*

$$Q_{-1}^\lambda = 0,$$
$$Q_0^\lambda = 1,$$
$$nQ_n^\lambda = 2i[(\lambda+n-1)\sin\theta - x\cos\theta]Q_{n-1}^\lambda + (2\lambda+n-2)Q_{n-2}^\lambda, \quad n \geq 1.$$

(ii) *The polynomials* $Q_n^\lambda = Q_n^\lambda(x;\theta)$ *are given by the formula:*

$$Q_n^\lambda(x;\theta) = e^{in\theta}\sum_{j=0}^{n}(-1)^j \frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}e^{-2ij\theta}, \quad n \in \mathbb{N}\cup\{0\}.$$

(iii) *The polynomials* $Q_n^\lambda = Q_n^\lambda(x;\theta)$ *have the hypergeometric representation*

$$Q_n^\lambda(x;\theta) = e^{in\theta}\frac{(2\lambda)_n}{n!}F\big(-n, \lambda+ix, 2\lambda; 1+e^{-2i\theta}\big).$$

(iiii) *The polynomials* $y(x) = Q_n^\lambda(x;\theta)$ *satisfy the following difference equation*

$$e^{i\theta}(\lambda - ix)y(x+i) - 2[xsin\theta + (n+\lambda)\cos\theta]y(x) + e^{-i\theta}(\lambda + ix)y(x-i) = 0.$$

Corollary 4. (i) *The (SSMP) polynomials* $S_n^\lambda = S_n^\lambda(x;\theta)$ *satisfy the three-term recurrence relation:*

$$S_{-1}^\lambda = 0,$$
$$S_0^\lambda = 1,$$
$$nS_n^\lambda = -2ixe^{i\theta}S_{n-1}^\lambda + (2\lambda+n-2)e^{2i\theta}S_{n-2}^\lambda, \quad n \geq 1.$$

(ii) *The polynomials* $S_n^\lambda = S_n^\lambda(x;\theta)$ *are given by the formula:*

$$S_n^\lambda(x;\theta) = e^{in\theta}\sum_{j=0}^{n}(-1)^j \frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}, \quad n \in \mathbb{N}\cup\{0\}.$$

(iii) *The polynomials* $S_n^\lambda = S_n^\lambda(x;\theta)$ *have the hypergeometric representation*

$$S_n^\lambda(x;\theta) = e^{in\theta}\frac{(2\lambda)_n}{n!}F(-n, \lambda+ix, 2\lambda; 2).$$

(iiii) *The polynomials* $y(x) = S_n^\lambda(x;\theta)$ *satisfy the following difference equation*

$$(\lambda - ix)y(x+i) - 2(n+\lambda)y(x) + (\lambda+ix)y(x-i) = 0.$$

Theorem 2.2. *The polynomials* $Q_n^\lambda(x;\theta)$ *are orthogonal on* $(-\infty, +\infty)$ *with the weight*

$$w_\theta^\lambda(x) = \frac{1}{2\pi}e^{2\theta x}|\Gamma(\lambda+ix)|^2 \ \ if \ \ \lambda > 0 \ \ and \ \ \theta \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{2\theta x}|\Gamma(\lambda+ix)|^2 Q_n^\lambda(x;\theta)\overline{Q_m^\lambda(x;\theta)}dx = \delta_{mn}\frac{\Gamma(n+2\lambda)}{(2\cos\theta)^{2\lambda}n!}.$$

In the proof we use the following lemmas.

Lemma 1 ([4], vol. I p. 12). *If $\alpha > 0$ and $p > 0$, then*

$$\int_0^{+\infty} u^{\alpha-1}e^{-pu}e^{-iqu}du = \Gamma(\alpha)(p^2+q^2)^{-\frac{\alpha}{2}}e^{-i\alpha\arctan(\frac{p}{q})}.$$

Lemma 2 ([11]). *Let $F(s)$ and $G(s)$ be Mellin transforms of $f(x)$ and $g(x)$, i.e.*

$$F(s) = \int_0^{+\infty} f(x)x^{s-1}dx, \quad G(s) = \int_0^{+\infty} g(x)x^{s-1}dx.$$

Then the following formula (Parseval's identity) holds:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)G(1-s)ds = \int_0^{+\infty} f(x)g(x)dx.$$

Corollary 5. *If $f(x) = x^{2(\lambda+j)}e^{-x^2}$ and $g(x) = x^{2(\lambda+k)-1}e^{-x^2}$, then*

$$F(s) = \Gamma\left(\lambda+j+\frac{s}{2}\right),\ G(s) = \Gamma\left(\lambda+k+\frac{s-1}{2}\right).$$

Lemma 3. *For any $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$, $\lambda > 0$, $j,k = 1,2,\dots$ we have*

$$I = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\lambda+ix)_j(\lambda-ix)_k|\Gamma(\lambda+ix)|^2e^{2\theta x}dx = \frac{e^{i(j-k)\theta}\Gamma(2\lambda+k+j)}{(2\cos\theta)^{2\lambda+k+j}}.$$

Proof. Putting $x = \frac{t}{2}$ and next $it = s$ we have:

$$\begin{aligned} I &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\lambda+i\frac{t}{2}\right)_j\left(\lambda-i\frac{t}{2}\right)_k\left|\Gamma\left(\lambda+i\frac{t}{2}\right)\right|^2e^{\theta t}dt \\ &= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}\left(\lambda+\frac{s}{2}\right)_j\left(\lambda-\frac{s}{2}\right)_k\left|\Gamma\left(\lambda+\frac{s}{2}\right)\right|^2e^{-i\theta s}ds, \end{aligned}$$

where we use the well-known formula for Pochammer symbol: $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$, $j = 1,2,\dots$. □

Lemma 4. *For arbitrary polynomial $Q_n^\lambda(x;\theta)$, $\lambda > 0$, $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$; $k,n = 1,2,\dots$ we have*

$$J = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{2\theta x}(\lambda-ix)_k|\Gamma(\lambda+ix)|^2Q_n^\lambda(x;\theta)dx = \frac{e^{in\theta}\Gamma(2k+\lambda)e^{-i\theta k}}{(2\cos\theta)^{2\lambda+k}}(-k)_n.$$

Proof. Using hypergeometric representation for $Q_n^\lambda(x;\theta)$ we can write

$$\begin{aligned} Q_n^\lambda(x;\theta) &= \frac{e^{in\theta}(2\lambda)_n}{n!} F\big(-n, \lambda+ix; 2\lambda; 1+e^{-2i\theta}\big) \\ &= \frac{e^{in\theta}(2\lambda)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j(\lambda+ix)_j}{(2\lambda)_j j!} \big(1+e^{-2i\theta}\big)^j. \end{aligned}$$

Therefore

$$\begin{aligned} J &= \frac{e^{in\theta}(2\lambda)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j(1+e^{-2i\theta})^j}{(2\lambda)_j j!} \cdot I \qquad \text{(by Lemma 3)} \\ &= \frac{e^{in\theta}(2\lambda)_n}{n!} e^{-ik\theta}\Gamma(2\lambda+k) \frac{1}{(4\cos^2\theta)^{\frac{2\lambda+k}{2}}} \\ &\quad \times \sum_{j=0}^{n} \frac{(-n)_j(2\lambda+k)_j}{(2\lambda)_j j!} \left(\frac{1+e^{-2i\theta}}{(4\cos^2\theta)^{\frac{1}{2}}\cdot e^{-i\theta}}\right)^j \\ &= \frac{e^{in\theta}(2\lambda)_n\Gamma(2\lambda+k)e^{-ik\theta}}{n!(2\cos\theta)^{2\lambda+k}} \cdot F(-n; 2\lambda+k; 2\lambda; 1). \end{aligned}$$

Using the well-known formula:

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we obtain

$$J = \frac{e^{i(n-k)\theta}\Gamma(2\lambda+k)}{n!(2\cos\theta)^{2\lambda+k}} \cdot (-k)_n. \tag{2.7}$$

□

Proof of Theorem 2.2. Let $m \le n$. Observe that $(-k)_n = 0$ if $k < n$. Therefore by (2.7)

$$J = \frac{\Gamma(2\lambda+n)(-n)_n}{n!(2\cos\theta)^{2\lambda+n}}, \quad \text{if} \quad k = n$$

and

$$J = 0, \quad \text{if} \quad k < n.$$

Using hypergeometric representation for $Q_n^\lambda(x;\theta)$ we can write

$$Q_n^\lambda(x;\theta) = \frac{e^{in\theta}(2\lambda)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j(1+e^{-2i\theta})^j}{(2\lambda)_j j!} (\lambda+ix)_j = \sum_{j=0}^{n} A_j(\lambda+ix)_j.$$

Therefore

$$\int_{-\infty}^{+\infty} Q_n^\lambda(x;\theta)\overline{Q_m^\lambda(x;\theta)} w_\theta^\lambda(x)dx = \delta_{nm}\overline{A_n}\frac{\Gamma(2\lambda+n)(-n)_n}{n!(2\cos\theta)^{2\lambda+n}},$$

where
$$\overline{A_n} = \frac{(-n)_n(1+e^{2i\theta})^n}{(2\lambda)_n n!} \cdot \frac{e^{-in\theta(2\lambda)_n}}{n!},$$
which ends the proof after some obvious simplifications. □

Remark 1. In the case $x = 0$ we can obtain "more pleasant" sets of "polynomials":
$$\begin{aligned} &Q_n^\lambda(0;\theta) = Q_n^\lambda(\theta), \\ &S_n^\lambda(0,\theta) = S_n^\lambda(\theta), \quad \theta \in [0,\pi],\ \lambda > 0. \end{aligned}$$
for which one can prove the following.

Theorem 2.3. *The function $y = y(\theta) = Q_n^\lambda(\theta) = Q_n^\lambda(0;\theta)$, $\lambda > 0$ satisfies the following second order differential equation:*
$$\cos\theta (Q_n^\lambda)'' - 2\lambda \sin\theta (Q_n^\lambda)' + n(n+2\lambda)\cos\theta Q_n^\lambda = 0.$$
In particular, if $\lambda = 1$ we have:
$$\cos\theta (Q_n^1)'' - 2\sin\theta (Q_n^1)' + n(n+2)\cos\theta Q_n^1 = 0.$$

Theorem 2.4. *The sets of functions $Q_{2k}^\lambda(\theta)$ and $Q_{2k-1}^\lambda(\theta)$ form (separately) the orthogonal systems with the weight function $w^\lambda(\theta) = \cos^{2\lambda}\theta$, $\theta \in [0,\pi]$, $\lambda > 0$.*

3. The generating function for (MP) polynomials allows us to define the generalization of the well-known class $\mathbb{T}$ of holomorphic function (1.1) which are typically-real in $\mathbb{D}$ ($\mathrm{Im} f(z) \cdot \mathrm{Im} z \geq 0$, $z \in \mathbb{D}$) and have the following integral representation
$$f(z) = \int_0^\pi \frac{z}{(1-ze^{i\theta})(1-ze^{-i\theta})} d\mu(\theta),$$
where μ is a probability measure on $[0,\pi]$ (e.g. [3], [5], [12], [13]).

Namely, we are going to study the extremal problems within the class $\mathbb{T}(\lambda,\tau)$, $\lambda > 0, \tau \in \mathbb{R}$ of holomorphic functions f of the form (1.1) given by the following integral representation
$$f(z) = \int_0^\pi \frac{z}{(1-ze^{i\theta})^{\lambda - i\tau}(1-ze^{-i\theta})^{\lambda+i\tau}} d\mu(\theta),$$
where μ is a probability measure on $[0,\pi]$.

We have in particular $\mathbb{T}(\lambda,0) = \mathbb{T}(\lambda)$ (e.g. [16], [7]) and $\mathbb{T}(1,0) = \mathbb{T}(\lambda,\tau)$. In parallel way we are going to study the extremal problems within the classes $\mathbf{T}(\lambda,\tau)$ and $\mathcal{T}(\lambda,\tau)$, $\lambda > 0$, $\tau \in \mathbb{R}$ of holomorphic functions of the form (1.1) which have the integral representation
$$f(z) = \int_0^\pi \frac{z}{(1-ze^{i\theta})^{\lambda - i\tau}(1+ze^{-i\theta})^{\lambda+i\tau}} d\mu(\theta),$$

and

$$f(z)=\int_0^\pi \frac{z}{(1-ze^{i\theta})^{\lambda-i\tau}(1+ze^{i\theta})^{\lambda+i\tau}}d\mu(\theta),$$

where μ is a probability measure on $[0,\pi]$.

The classes $\mathbb{T}(\lambda,\tau)$, $\mathbf{T}(\lambda,\tau)$ and $\mathcal{T}(\lambda,\tau)$ differ pretty much, for instance all coefficients a_k of $f\in\mathbb{T}(\lambda,\tau)$ are real, however the odd coefficients of $f\in\mathbb{T}(\lambda,\tau)$ are real and even coefficients of $f\in\mathbf{T}(\lambda,\tau)$ are purely imaginary.

In special case $\tau=0$, $\lambda=1$, i.e. $\mathbf{T}=\mathbf{T}(1,0)$, we are able to find explicitly the radius of local univalence and the radius of univalence of $\mathbf{T}$ which differ from the corresponding values in the class $\mathbb{T}=\mathbb{T}(1,0)$.

The classes $\mathbb{T}(0,\tau)$, $\mathbf{T}(0,\tau)$ and $\mathcal{T}(0,\tau)$ appear to be of special interest when $\lambda\to 0^+$.

The same remarks concern also the sets of polynomials

$$S^0(x,\theta)=\lim_{\lambda\to 0^+}S^\lambda(x,\theta)\quad\text{and}\quad Q^0(x,\theta)=\lim_{\lambda\to 0^+}Q^\lambda(x,\theta),$$

which generalize the special symmetric Pollaczek polynomials [1].

Remark 2. Due to definition (2.1) of the polynomials $P_n^\lambda(\tau;\theta,\psi)$, $\tau\in\mathbb{R}$, $\theta\in(0,\pi)$, $\psi\in\mathbb{R}$ we can as well consider the extremal problems for more general class of the holomorphic function f of the form (1.1) which have the integral representation

$$f(z)=\int\int_\Delta zG^\lambda(\tau;\theta,\psi;z)d\mu(\theta,\psi),$$

where μ is a probability measure on $\Delta=(0,\pi)\times\mathbb{R}$.

References

[1] Araaya, T. K., *The symmetric Meixner–Pollaczek polynomials*, Uppsala Dissertations in Mathematics, Department of Mathematics, Uppsala University, 2003.

[2] Chihara, T. S., *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[3] Duren, P. L., *Univalent Functions,* Springer, New York, 1983.

[4] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G., *Higher Transcendental Functions*, vol. I, McGraw-Hill Book Company, New York, 1953.

[5] Golusin, G., *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs, no. 26, Amer. Math. Soc., Providence, R.I., 1969.

[6] Ismail, M., *On sieved ultraspherical polynomials I: Symmetric Pollaczek analogues*, SIAM J. Math. Anal. **16** (1985), 1093–1113.

[7] Kiepiela, K., Naraniecka, I., Szynal, J., *The Gegenbauer polynomials and typically real functions*, J. Comp. Appl. Math **153** (2003), 273–282.

[8] Koekoek, R., Swarttouw, R. F., *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report 98-17, Delft University of Technology, 1998.

[9] Koornwinder, T. H., *Meixner–Pollaczek polynomials and the Heisenberg algebra*, J. Math. Phys. **30** (4) (1989), 767–769.

[10] Pommerenke, Ch., *Linear-invariant Familien analytischer Funktionen*, Mat. Ann. **155** (1964), 108–154.

[11] Poularikas, A. D., *The Mellin Transform, The Handbook of Formulas and Tables for Signal Processing*, CRC Press LLC, Boca Raton, 1999.

[12] Robertson, M. S., *On the coefficients of typically-real functions*, Bull. Amer. Math. Soc. **41** (1935), 565–572.

[13] Rogosinski, W. W., *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Math. Z. **35** (1932), 93–121.

[14] Starkov, V. V., *The estimates of coefficients in locally-univalent family* $U_{\alpha}^{'}$, Vestnik Lenin. Gosud. Univ. **13** (1984), 48–54 (Russian).

[15] Starkov, V. V., *Linear-invariant families of functions*, Dissertation, Ekatirenburg, 1989, 1–287 (Russian).

[16] Szynal, J., *An extension of typically-real functions*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **48** (1994), 193–201.

[17] Szynal, J., Waniurski, J., *Some problems for linearly invariant families*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **30** (1976), 91–102.

Iwona Naraniecka
Department of Mathematics
Faculty of Economics
Maria Curie-Skłodowska University
20-031 Lublin
Poland
e-mail: inaraniecka@gmail.com

Jan Szynal
University of Economics and Innovation in Lublin
20-209 Lublin
Poland
e-mail: jan.szynal3@gmail.com

Anna Tatarczak
Department of Mathematics
Faculty of Economics
Maria Curie-Skłodowska University
20-031 Lublin
Poland
e-mail: antatarczak@gmail.com

Received September 28, 2012