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On almost polynomial structures from classical linear connections

Abstract. Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and local diffeomorphisms and let T be the tangent functor on $\mathcal{M}f_m$. Let $\mathcal{V}$ be the category of real vector spaces and linear maps and let $\mathcal{V}_m$ be the category of m-dimensional real vector spaces and linear isomorphisms. Let w be a polynomial in one variable with real coefficients. We describe all regular covariant functors $F\colon \mathcal{V}_m \to \mathcal{V}$ admitting $\mathcal{M}f_m$-natural operators $\tilde{P}$ transforming classical linear connections ∇ on m-dimensional manifolds M into almost polynomial w-structures $\tilde{P}(\nabla)$ on $F(T)M = \bigcup_{x\in M} F(T_xM)$.

1. Introduction. All manifolds considered in the paper are assumed to be Hausdorff, finite dimensional, second countable, without boundaries and smooth (i.e. of class C^∞). Maps between manifolds are assumed to be of class C^∞.

The category of m-dimensional manifolds and local diffeomorphisms is denoted by $\mathcal{M}f_m$. The category of vector bundles and vector bundle homomorphisms between them is denoted by $\mathcal{VB}$. The category of m-dimensional real vector spaces and linear isomorphisms is denoted by $\mathcal{V}_m$. The category of finite dimensional real vector spaces and linear maps is denoted by $\mathcal{V}$.

Let w be a polynomial in one variable. A tensor field P of type $(1,1)$ on a manifold N is called an almost polynomial w-structure on N if $w(P) = 0$ (i.e. $w(P_{|x}) = 0$ for any $x \in N$).

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In the present paper we solve the following problem.

Problem 1. *Let w be a polynomial in one variable with real coefficients. We characterize all covariant regular functors $F\colon \mathcal{V}_m \to \mathcal{V}$ admitting $\mathcal{M}f_m$-natural operators $\tilde{P}$ transforming classical linear connections ∇ on m-manifolds M into almost polynomial w-structures $\tilde{P}(\nabla)$ on $F(T)M = \bigcup_{x\in M} F(T_xM)$, where $T\colon \mathcal{M}f_m \to \mathcal{VB}$ denotes the tangent functor on the category $\mathcal{M}f_m$.*

If $w(t) = t^2 + 1$, then we reobtain the result from [5] on the characterization of covariant regular functors $F\colon \mathcal{V}_m \to \mathcal{V}$ admitting $\mathcal{M}f_m$-natural operators $\tilde{J}$ transforming classical linear connections ∇ on m-manifolds M into almost complex structures $\tilde{J}(\nabla)$ on $F(T)M$.

If $w(t) = t^2 - 1$, then we characterize covariant regular functors $F\colon \mathcal{V}_m \to \mathcal{V}$ admitting $\mathcal{M}f_m$-natural operators $\tilde{J}$ transforming classical linear connections ∇ on m-manifolds M into almost para-complex structures $\tilde{J}(\nabla)$ on $F(T)M$.

2. Basic definitions. The concept of natural bundles and natural operators can be found in the fundamental monograph [3].

Let $F\colon \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor. The regularity of the functor F means that F transforms smoothly parametrized families of isomorphisms into smoothly parametrized families of linear maps. Let $T\colon \mathcal{M}f_m \to \mathcal{VB}$ be the tangent functor sending any m-dimensional manifold M into the tangent bundle TM of M and any $\mathcal{M}f_m$-map $\varphi\colon M_1 \to M_2$ into the tangent map $T\varphi\colon TM_1 \to TM_2$. Applying F to fibers T_xM of TM, one can define a natural vector bundle $F(T)$ of order 1 over m-manifolds by

$$F(T)M = \bigcup_{x\in M} F(T_xM) \text{ and } F(T)\varphi = \bigcup_{x\in M} F(T_x\varphi)\colon F(T)M_1 \to F(T)M_2$$

for any m-manifold M and any $\mathcal{M}f_m$-map $\varphi\colon M_1 \to M_2$ between m-manifolds M_1 and M_2. In particular, if F is the identity functor, then $F(T) = T$.

A classical linear connection on an m-manifold M is an $\mathbb{R}$-bilinear map $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:

(1) $\nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y$
(2) $\nabla_X(Y_1 + Y_2) = \nabla_XY_1 + \nabla_XY_2$
(3) $\nabla_X(fY) = Xf \cdot Y + f \cdot \nabla_XY$,

where X, X_1, X_2, Y, Y_1, $Y_2 \in \mathfrak{X}(M)$ are any vector fields on M and $f, f_1, f_2\colon M \to \mathbb{R}$ are any smooth functions on M. Equivalently, a classical linear connection on M is a right invariant decomposition $TLM = H^\nabla \oplus VLM$ of the tangent bundle TLM of LM, where LM is the principal bundle with the structural group $GL(m)$ of linear frames over M and VLM is the vertical bundle of LM, see [2].

Let $w(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ be the polynomial in one variable with real coefficients $a_{m-1}, \ldots, a_0$.

A polynomial w-structure on a real vector space W is a linear endomorphism $P \colon W \to W$ such that $w(P) = P^m + a_{m-1}P^{m-1} + \cdots + a_1 P + a_0 I = 0$, where P^k denotes the composition $\underbrace{P \circ \cdots \circ P}_{k\text{-times}}$ and I denotes the identity map on W.

An almost polynomial w-structure on manifold N is a tensor field $\tilde{P} \colon TN \to TN$ on N of type $(1,1)$ (affinor) such that $P_x \colon T_xN \to T_xN$ is a polynomial w-structure on T_xN for any $x \in N$. In other words, an almost polynomial w-structure is a tensor field P of type $(1,1)$ on manifold N satisfying a polynomial equation $P^m + a_{m-1}P^{m-1} + \cdots + a_1 P + a_0 I = 0$, where $a_{m-1}, \ldots, a_0$ are real numbers, at every point of N.

The general concept of natural operators can be found in the fundamental monograph [3]. In particular, we have the following definition.

Definition 1. Let $F \colon \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor. An $\mathcal{M}f_m$-natural operator transforming classical linear connections ∇ on m-manifolds M into almost polynomial w-structures $\tilde{P}(\nabla) \colon TF(T)M \to TF(T)M$ on $F(T)M$ is an $\mathcal{M}f_m$-invariant family $\tilde{P} \colon Q \rightsquigarrow (AwS)F(T)$ of operators

$$\tilde{P} \colon Q(M) \to (AwS)(F(T)M)$$

for m-manifolds M, where $Q(M)$ is the set of classical linear connections on M and $(AwS)(F(T)M)$ is the set of almost polynomial w-structures on $F(T)M$. The invariance of $\tilde{P}$ means that if $\nabla_1 \in Q(M_1)$ and $\nabla_2 \in Q(M_2)$ are φ-related by an embedding $\varphi \colon M_1 \to M_2$ (i.e. if φ is (∇, ∇_1)-affine embedding), then $\tilde{P}(\nabla_1)$ and $\tilde{P}(\nabla_2)$ are $F(T)\varphi$-related (i.e. $TF(T)\varphi \circ \tilde{P}(\nabla_1) = \tilde{P}(\nabla_2) \circ TF(T)\varphi$).

Let $F \colon \mathcal{V}_m \to \mathcal{V}$ be as above. A $\mathcal{V}_m$-canonical polynomial w-structure on $V \oplus FV$ is a $\mathcal{V}_m$-invariant system P of polynomial w-structures

$$P \colon V \oplus FV \to V \oplus FV$$

on vector spaces $V \oplus FV$ for m-dimensional real vector spaces V. The invariance of P means that $(\varphi \oplus F\varphi) \circ P = P \circ (\varphi \oplus F\varphi)$ for any linear isomorphism $\varphi \colon V_1 \to V_2$ between m-dimensional vector spaces.

3. The main result. The main result of the present note is the following theorem.

Theorem 1. *Let $F \colon \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor and w be a polynomial in one variable with real coefficients. The following conditions are equivalent:*

(*i*) *There exists an $\mathcal{M}f_m$-natural operator $\tilde{P} \colon Q \rightsquigarrow (AwS)F(T)$.*

(*ii*) *There exists a $\mathcal{V}_m$-canonical polynomial w-structure P on $V \oplus FV$.*

Proof. $(i) \Rightarrow (ii)$. Let $\tilde{P}\colon Q \rightsquigarrow (AwS)F(T)$ be an $\mathcal{M}f_m$-natural operator in question. Let V be an m-dimensional vector space from the category $\mathcal{V}_m$ and let ∇^V be the $\mathcal{V}_m$-canonical torsion free flat classical linear connection on V. Then the almost polynomial w-structure $\tilde{P}(\nabla^V)\colon TF(T)V \to TF(T)V$ on $F(T)V$ restricts to the polynomial w-structure

$$P := \tilde{P}(\nabla^V)_{0_{0_V}} : T_{0_{0_V}}F(T)V \to T_{0_{0_V}}F(T)V$$

on the tangent space $T_{0_{0_V}}F(T)V$ of $F(T)(V)$ at $0_{0_V} \in F(T)V$, where 0_V is the zero in V and 0_{0_V} is the zero in $F(T)_{0_V}V$. Since $TV = V \oplus V$, we have $F(T)V = V \oplus FV$. Therefore $T_{0_{0_V}}F(T)V = V \oplus FV$ modulo above identifications. So,

$$P\colon V \oplus FV \to V \oplus FV$$

is the polynomial w-structure on $V \oplus FV$ for any $\mathcal{V}_m$-object V. Because of the canonical character of the construction of P, the structure P is $\mathcal{V}_m$-canonical.

$(ii) \Rightarrow (i)$. Suppose $P\colon V \oplus FV \to V \oplus FV$ is a $\mathcal{V}_m$-canonical polynomial w-structure. Let $\nabla \in Q(M)$ be a classical linear connection on an m-manifold M. Let $v \in F(T)_xM$, $x \in M$. Since $F(T)$ is of order 1, $F(T)M = LM[F(T)_0\mathbb{R}^m]$ (the associated space). Then ∇-decomposition $TLM = H^\nabla \oplus VLM$ induces (in obvious way) ∇-decomposition $TF(T)M = \tilde{H}^\nabla \oplus VF(T)M$. Then we have the identification

$$T_vF(T)M = \tilde{H}^\nabla_v \oplus V_vF(T)M \cong T_xM \oplus F(T)_xM = T_xM \oplus F(T_xM)$$

canonically depending on ∇, where the equality is the connection decomposition, the identification $\cong$ is the usual one (namely, $\tilde{H}^\nabla_v = T_xM$ modulo the tangent of the projection of $F(T)M$ and $V_vF(T)M = T_v(F(T)_xM) = F(T)_xM$ modulo the standard identification) and the second equality is by the definition of $F(T)M$. We define $\tilde{P}(\nabla)_{|v}\colon T_vF(T)M \to T_vF(T)M$ by

$$\tilde{P}(\nabla)_{|v} := P\colon T_xM \oplus F(T_xM) \to T_xM \oplus F(T_xM)$$

modulo the above identification $T_vF(T)M \cong T_xM \oplus F(T_xM)$. Then $\tilde{P}(\nabla)\colon TF(T)M \to TF(T)M$ is an almost polynomial w-structure on $F(T)M$. By the canonical character of $\tilde{P}(\nabla)$, the resulting family $\tilde{P}\colon Q \rightsquigarrow (AwS)F(T)$ is an $\mathcal{M}f_m$-natural operator. □

4. An application to para-complex structures. Let $w(t) = t^2 - 1$. Let J be a polynomial w-structure on a vector space W. Then $W = W_+ \oplus W_-$, where $W_\pm = \{v \in W\colon J(v) = \pm v\}$. If additionally $\dim(W_+) = \dim(W_-)$, then J is called a para-complex structure on W, see [6].

An almost para-complex structure on a manifold N is an affinor $J\colon TN \to TN$ on N such that $J_x\colon T_xN \to T_xN$ is a para-complex structure on T_xN for any $x \in N$. In other words, an almost para-complex structure is a smooth

$(1,1)$-tensor field on the manifold N of even dimension m, if the following conditions are satisfied:

(1) $J^2 = id_{TN}$

(2) for each point $x \in N$, the eigenspaces $T_x^+ N$ and $T_x^- N$ of J_x (the value of J at x) are both $\frac{m}{2}$-dimensional subspaces of the tangent space $T_x N$ at x, [1], [7].

Corollary 1. *Let $F\colon \mathcal{V}_m \to \mathcal{V}$ be a regular covariant functor. The following conditions are equivalent:*

(a) *There is an $\mathcal{M}f_m$-natural operator $\tilde{J}\colon Q \rightsquigarrow (APC)F(T)$ transforming classical linear connections ∇ on m-manifolds M into almost para-complex structures $\tilde{J}(\nabla)$ on $F(T)M$.*

(b) *There exists a $\mathcal{V}_m$-canonical para-complex structure J on $V \oplus FV$.*

Proof. This is a simple consequence of Theorem 1. □

Lemma 1. *Let p be a positive integer. Let $F\colon \mathcal{V}_m \to \mathcal{V}$ be a covariant regular functor given by $FV = V \times \cdots \times V$ ($(p-1)$ times of V) and $F\varphi = \varphi \times \cdots \times \varphi$ ($(p-1)$ times of φ). If p is even, there is a $\mathcal{V}_m$-canonical para-complex structure on $V \oplus FV$.*

Proof. If p is even, we have the $\mathcal{V}_m$-canonical para-complex structure on $V \times \cdots \times V$ (p times of V) . Namely, we have the $\frac{p}{2}$ copies of the canonical para-complex structure on $V \times V$ given by $(v, w) \to (v, -w)$. □

A Weil algebra A is a finite dimensional, commutative, associative and unital algebra of the form $A = \mathbb{R} \times N$, where N is the ideal of all nilpotent elements of A.

Lemma 2 (Lemma 5.1 in [4]). *Let A be a p-dimensional Weil algebra and let T^A be the corresponding Weil functor. For any classical linear connection ∇ on an m-manifold M, we have the base-preserving fibred diffeomorphism $I^A_\nabla\colon T^A M \to TM \otimes \mathbb{R}^{p-1}$ canonically depending on ∇.*

We see that $TM \otimes \mathbb{R}^{p-1} = TM \times_M \cdots \times_M TM$ ($(p-1)$ times of TM) $= F(T)M$, where $F\colon \mathcal{V}_m \to \mathcal{V}$, $FV = V \times \cdots \times V$ ($(p-1)$ times of V), $F\varphi = \varphi \times \cdots \times \varphi$ ($(p-1)$ times of φ). So, from Corollary 1, Lemma 1 and Lemma 2 we obtain

Proposition 1. *Let A be a Weil algebra. If A is even dimensional, there exists an $\mathcal{M}f_m$-natural operator $\tilde{J}\colon Q \rightsquigarrow (APC)T^A$ sending classical linear connections ∇ on m-manifolds M into almost para-complex structures $\tilde{J}(\nabla)$ on $T^A M$.*

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