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Trace parameters for Teichmüller space of genus 2 surfaces and mapping class group

Abstract. We obtain a representation of the mapping class group of genus 2 surface in terms of a coordinate system of the Teichmüller space defined by trace functions.

1. Introduction. We identify $PSL(2,\mathbb{R})$ with the group of orientation-preserving isometries of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ equipped with the hyperbolic metric $|dz|/(\text{Im } z)$.

A Fuchsian subgroup G of $PSL(2,\mathbb{R})$ is said to be of type $(2;-;-;-)$ ([5, p. 38]) if $\mathbb{H}/G$ is a closed surface of genus 2 and the projection $\pi : \mathbb{H} \to \mathbb{H}/G$ is an unbranched covering. G has a canonical generator system or a *marking* $E = (A,B,C,D)$ which satisfies

$$[A,B][C,D] = 1,$$

where $[a,b] = aba^{-1}b^{-1}$ is the commutator of a and b, and 1 stands for the unit matrix. We call the pair (G,E) a *marked Fuchsian group of type* $(2;-;-;-)$. Two marked Fuchsian groups (G_1,E_1) and (G_2,E_2) are *equivalent* if there exists a matrix $P \in PSL(2,\mathbb{R})$ such that

$$A_2 = P^{-1}A_1P,\ B_2 = P^{-1}B_1P,\ C_2 = P^{-1}C_1P,\ D_2 = P^{-1}D_1P,$$

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where $E_j = (A_j, B_j, C_j, D_j)$, $j = 1, 2$. The *Teichmüller space* $\mathcal{T}_2$ *of type* $(2; -; -; -)$ is the space of all equivalence classes of marked Fuchsian groups of type $(2; -; -; -)$. Each marked Fuchsian group (G, E) can be represented by a tuple (A, B, C, D) of matrices in $SL(2, \mathbb{R})$ such that

$$\mathrm{tr}A > 0,\ \mathrm{tr}B > 0, \mathrm{tr}C > 0 \text{ and } \mathrm{tr}D > 0. \tag{1.1}$$

Therefore, for the rest of this paper, we always assume that $E = (A, B, C, D)$ consists of matrices satisfying (1.1). In this case $\mathrm{tr}AB$ and $\mathrm{tr}CD$ are both positive (this follows from [5, 33.17 (b)]). In [3] we considered the following traces as functions of $[(G, E = (A, B, C, D)]$ in $\mathcal{T}_2$:

$$\begin{aligned} &a = \mathrm{tr}A, b = \mathrm{tr}B, z = \mathrm{tr}AB, u = -\mathrm{tr}ACDC^{-1}, \\ &v = -\mathrm{tr}ACD^2, w = -\mathrm{tr}ACD, t = \mathrm{tr}CD. \end{aligned} \tag{1.2}$$

Since all non trivial elements of G are hyperbolic, their traces take values in $\mathbb{R}_{>2} = \{x : x > 2\}$. It is shown in [3] (see also [4]) that the mapping $\Phi : \mathcal{T}_2 \to \mathbb{R}^7_{>2}$ defined by $\Phi([G, E]) = (a, b, z, u, v, w, t)$ is an embedding and a, b, z, u, v, w, t satisfy the identity

$$awt + a^2 + w^2 + t^2 + K^2 + S^2 + 4 - w\sqrt{(K^2 + 4)(S^2 + 4)} = 0, \tag{1.3}$$

where

$$K = \sqrt{abz - a^2 - b^2 - z^2} \text{ and } S = \sqrt{uvt - u^2 - v^2 - t^2}.$$

The mapping class group $\mathcal{MC}_2$ is the group of isotopy classes of orientation-preserving homeomorphisms of the orientable closed surface S of genus 2. It is a subgroup of outer automorphisms of the fundamental group of S (see [5]). $\mathcal{MC}_2$ acts on the Teichmüller space $\mathcal{T}_2$ by changing the marking. The purpose of this paper is to describe a generating system of $\mathcal{MC}_2$ by using the coordinate-system (a, b, z, u, v, w, t). It is an interesting observation that $\mathcal{MC}_2$ acts on $\mathcal{T}_2$ as a group of rational transformations.

2. Trace identities.

2.1. Basic trace identities. The matrices A, B and C in $SL(2, \mathbb{R})$ satisfy the following identities (see [2, §3.4]):

(I1) $\mathrm{tr}A = \mathrm{tr}A^{-1}$,
(I2) $\mathrm{tr}AB + \mathrm{tr}AB^{-1} = \mathrm{tr}A\mathrm{tr}B$,
(I3) $\mathrm{tr}ABC = \mathrm{tr}A\mathrm{tr}BC + \mathrm{tr}B\mathrm{tr}CA + \mathrm{tr}C\mathrm{tr}AB - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}C - \mathrm{tr}ACB$.

We shall use repeatedly the following identities, which are consequences of (I1), (I2) and (I3) above:

$$\begin{aligned} \mathrm{tr}[A, B] &= \mathrm{tr}ABA^{-1}B^{-1} \\ &= (\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB - 2, \end{aligned} \tag{2.1a}$$

$$\mathrm{tr}ABCB = \mathrm{tr}AB\mathrm{tr}BC + \mathrm{tr}AC - \mathrm{tr}A\mathrm{tr}C, \tag{2.1b}$$

$$\mathrm{tr}ABCB^{-1} = \mathrm{tr}A\mathrm{tr}C - \mathrm{tr}AC - \mathrm{tr}AB\mathrm{tr}BC + \mathrm{tr}B\mathrm{tr}ABC. \tag{2.1c}$$

Let G be a group generated by a finite number of matrices $A_1,\ldots, A_n \in SL(2,\mathbb{R})$ and

(2.2) $\quad \mathcal{S} = \{\mathrm{tr}(A_{i_1}A_{i_2}\cdots A_{i_r}) : 1 \le i_1 < i_2 < \cdots < i_r \le n, 1 \le r \le n\}.$

Then the following fact is well known (see [2, §3.5]).

Lemma 2.1. *Let $g \in G$. Then* tr$g$ *is an integer polynomial in $\mathcal{S}$.*

2.2. Trace identities for genus 2 surface. Let $E = (A,B,C,D)$ be a marking of a Fuchsian group G of type $(2;-;-;-)$. Let $c = x_1 = \mathrm{tr}C$ and $d = x_2 = \mathrm{tr}D$, $x_3 = \mathrm{tr}AC$, $x_4 = \mathrm{tr}AD$, $x_5 = \mathrm{tr}BC$, $x_6 = \mathrm{tr}BD$, $x_7 = \mathrm{tr}ABC$, $x_8 = \mathrm{tr}ABD$, $x_9 = \mathrm{tr}BCD$ and $x_{10} = \mathrm{tr}ABCD$. Then the set $\mathcal{S}$ for G with respect to (A,B,C,D) is

$$\mathcal{S} = \{a,b,c,d,z,x_3,x_4,x_5,x_6,t,x_7,x_8,x_9,x_{10}\}.$$

The purpose of this section is to find expressions of $x_1,\ldots, x_{10}$ in $\{a,b,z,u,v,w,t\}$ of (1.2). Then by Lemma 2.1 we can express the trace of any element of G in $\{a,b,z,u,v,w,t\}$. We shall apply this fact to obtain a representation of the mapping class group $\mathcal{MC}_2$ via rational transformations.

(1) Since $[A,B] = [C,D]^{-1}$, we obtain by (2.1a)

(2.3) $$abz - a^2 - b^2 - z^2 = cdt - c^2 - d^2 - t^2.$$

Note that $\mathrm{tr}[A,B] = a^2 + b^2 + z^2 - abz - 2 < -2$, since G is discrete (see, for example [5, 33 D]). In what follows $K = \sqrt{abz - a^2 - b^2 - z^2}$.

(2) From $BAB^{-1} = CDC^{-1}D^{-1}A$ and the basic identity (I3) we obtain

$$a = \mathrm{tr}((ACD)\cdot C^{-1}\cdot D^{-1}) = -wt + cx_3 - ud + wcd - a.$$

and hence

(2.4) $$2a + wt - cx_3 + ud - wcd = 0.$$

(3) From (I2), $v = -\mathrm{tr}ACD\cdot D = -(\mathrm{tr}ACD\mathrm{tr}D - \mathrm{tr}AC) = wd + x_3$ and so

(2.5) $$x_3 = v - dw.$$

From this and (2.4) it follows that

(2.6) $$2a + wt - cv + ud = 0.$$

(4) From (I3),

$$\begin{aligned} -u = \mathrm{tr}A\cdot CD\cdot C^{-1} &= ad + t(\mathrm{tr}AC^{-1}) - wc - atc - x_4 \\ &= ad + t(ac - x_3) - wc - atc - x_4. \end{aligned}$$

It follows from this and (2.5) that

(2.7) $$x_4 = u + ad - tx_3 - wc = u + ad - tv + twd - cw.$$

By substituting $d = u^{-1}(cv - 2a - wt)$ (see (2.6)) into (2.3) we obtain

$$(uvt - u^2 - v^2)c^2 - (2a + wt)(tu - 2v)c - (K^2 + t^2)u^2 - (2a + tw)^2 = 0.$$

If this identity is regarded as a quadratic equation in c, it always has a negative root because

$$uvt - u^2 - v^2 = (-\mathrm{tr}[CD^{-1}C^{-1}A^{-1}, ACD^2] - 2) + t^2 > t^2 > 0$$

(see [5, 33 D]) and $-(K^2+t^2)u^2 - (2a+tw)^2 < 0$. Hence the condition $c = \mathrm{tr}C > 2$ yields

$$\begin{aligned} c &= \frac{(2a+tw)(ut-2v) + u\sqrt{(2a+tw)^2(t^2-4) + 4(K^2+t^2)(S^2+t^2)}}{2(S^2+t^2)}, \\ d &= \frac{cv - 2a - wt}{u} \end{aligned} \tag{2.8}$$

where $S = \sqrt{uvt - u^2 - v^2 - t^2}$. By using (1.3) we see that $(2a+tw)^2(t^2-4) + 4(K^2+t^2)(S^2+t^2)$ equals

$$\begin{aligned} &\left((t^2-4)w + 2\sqrt{(S^2+4)(K^2+4)}\right)^2 \\ &\qquad = \left((t^2-4)w + \frac{2(awt + a^2 + t^2 + K^2 + S^2 + 4)}{w}\right)^2. \end{aligned}$$

Now from (2.8) we obtain

$$\begin{aligned} c &= \frac{(K^2+S^2+t^2+a^2+4)u + w(2atu - 2av - uw + t^2uw - tvw)}{w(S^2+t^2)}, \\ d &= \frac{(K^2+S^2+t^2+a^2+4)v + w(2au + twu - vw)}{w(S^2+t^2)}. \end{aligned} \tag{2.9}$$

By (2.5), (2.7) and (2.9), we can obtain the expressions of $x_3 = \mathrm{tr}AC$ and $x_4 = \mathrm{tr}AD$ in (a, b, z, u, v, w, t),

$$\begin{aligned} x_3 &= -\frac{uw(2a+tw) + v(4 + a^2 + K^2 - w^2)}{S^2+t^2} \\ x_4 &= (ad + u - cw) + t\frac{(4 + a^2 + K^2 - w^2)v + wu(2a+tw)}{S^2+t^2}. \end{aligned} \tag{2.10}$$

(5) From (I2) and (2.1c) applied to $BCDC^{-1}$ we obtain

$$\begin{aligned} \mathrm{tr}B^{-1}(CDC^{-1}) &= bd - \mathrm{tr}BCDC^{-1} \\ &= bd - (bd - x_6 - x_5t + cx_9) = x_6 + tx_5 - cx_9. \end{aligned} \tag{2.11}$$

From (I3), $\mathrm{tr}B^{-1}CD = bt - x_9$. Then, from the trace of $AB^{-1}A^{-1} = B^{-1}CD \cdot C^{-1} \cdot D^{-1}$, (I2), (I3) and (2.11),

$$\begin{aligned} b &= (\mathrm{tr}B^{-1}CD)t + c\mathrm{tr}B^{-1}C + d\mathrm{tr}(B^{-1}CD \cdot C^{-1}) - (\mathrm{tr}B^{-1}CD)cd - b \\ &= (bt - x_9)(t - cd) + c(bc - x_5) + d(x_6 + tx_5 - cx_9) - b. \end{aligned}$$

Hence

$$(dt - c)x_5 + dx_6 - tx_9 = 2b - bt^2 + bcdt - bc^2.$$

(6) From (I2), $\mathrm{tr}A^{-1}CD = at + w$, and from (I2) and (I3),

$$\begin{aligned}\mathrm{tr}B^{-1}A^{-1} \cdot C \cdot D &= zt + c\mathrm{tr}ABD^{-1} + d\mathrm{tr}ABC^{-1} - zcd - \mathrm{tr}B^{-1}A^{-1}DC \\ &= zt + c(zd - x_8) + d(zc - x_7) - zcd - \mathrm{tr}B^{-1}A^{-1}DC \\ &= zt + cdz - dx_7 - cx_8 - \mathrm{tr}B^{-1}A^{-1}DC.\end{aligned}$$

Substituting these into the next equation obtained from $B^{-1}A^{-1}DC = A^{-1} \cdot B^{-1} \cdot CD$ and (I3),

$$\begin{aligned}\mathrm{tr}B^{-1}A^{-1}DC &= a\mathrm{tr}B^{-1}CD + b\mathrm{tr}A^{-1}CD + zt - abt - \mathrm{tr}B^{-1}A^{-1}CD \\ &= a(bt - x_9) + b(at + w) + zt - abt \\ &\quad - zt - cdz + dx_7 + cx_8 + \mathrm{tr}B^{-1}A^{-1}DC,\end{aligned}$$

we obtain

$$dx_7 + cx_8 - ax_9 = -abt - bw + cdz.$$

(7) From $B^{-1}CDC^{-1} = \mathrm{tr}AB^{-1}A^{-1}D$, $\mathrm{tr}B^{-1}(CDC^{-1})$ equals

$$\begin{aligned}\mathrm{tr}AB^{-1}A^{-1}D &= \mathrm{tr}B\mathrm{tr}AA^{-1}D - \mathrm{tr}ABA^{-1}D = bd - \mathrm{tr}DABA^{-1} \\ &= bd - (\mathrm{tr}B\mathrm{tr}D - \mathrm{tr}BD - \mathrm{tr}BA\mathrm{tr}AD + \mathrm{tr}A\mathrm{tr}ABD) \\ &= x_6 + zx_4 - ax_8.\end{aligned}$$

Here we have used (I2) and (2.1c). Then from (2.11),

$$tx_5 + ax_8 - cx_9 = zx_4.$$

(8) From $BA^{-1}B^{-1}C = A^{-1}DCD^{-1}$ and (I2), we have

$$ac - \mathrm{tr}BAB^{-1}C = \mathrm{tr}BA^{-1}B^{-1}C = \mathrm{tr}A^{-1}DCD^{-1} = ac - \mathrm{tr}ADCD^{-1},$$

and hence $\mathrm{tr}CBAB^{-1} = \mathrm{tr}ADCD^{-1}$. We have by using (2.1c)

$$\begin{aligned}\mathrm{tr}CBAB^{-1} &= \mathrm{tr}C\mathrm{tr}A - \mathrm{tr}AC - \mathrm{tr}BC\mathrm{tr}AB + \mathrm{tr}B\mathrm{tr}CBA \\ &= ac - x_3 - zx_5 + b(\mathrm{tr}C\mathrm{tr}BA + \mathrm{tr}B\mathrm{tr}CA + \mathrm{tr}A\mathrm{tr}CB \\ &\quad - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}C - \mathrm{tr}ABC) \\ &= ac - x_3 - zx_5 + bcz + b^2x_3 + abx_5 - ab^2c - bx_7\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}ADCD^{-1} &= \mathrm{tr}A\mathrm{tr}C - \mathrm{tr}AC - \mathrm{tr}AD\mathrm{tr}DC + \mathrm{tr}D\mathrm{tr}ADC \\ &= ac - x_3 - tx_4 + d(\mathrm{tr}A\mathrm{tr}CD + \mathrm{tr}D\mathrm{tr}AC + \mathrm{tr}C\mathrm{tr}AD \\ &\quad - \mathrm{tr}A\mathrm{tr}D\mathrm{tr}C - \mathrm{tr}ACD) \\ &= ac - x_3 - tx_4 + adt + d^2x_3 + cdx_4 - ad^2c + wd.\end{aligned}$$

Thus we obtain

$$(z - ab)x_5 + bx_7 = (b^2 - d^2)x_3 + (t - cd)x_4 + bcz - ab^2c - adt + ad^2c - wd.$$

(9) We use $C^{-1}BA = \mathrm{tr}DC^{-1}D^{-1}AB$. Then from (I2) and (I3),

$$\begin{aligned}\mathrm{tr}C^{-1}BA &= zc - \mathrm{tr}CBA \\ &= zc - (cz + bx_3 + ax_5 - abc - x_7) = -bx_3 - ax_5 + abc + x_7.\end{aligned}$$

From (I2) and (2.1c) this equals

$$\begin{aligned}\mathrm{tr}(DC^{-1}D^{-1})AB &= cz - \mathrm{tr}ABDCD^{-1} \\ &= cz - (\mathrm{tr}AB\mathrm{tr}C - \mathrm{tr}ABC - \mathrm{tr}ABD\mathrm{tr}CD \\ &\quad + \mathrm{tr}D\mathrm{tr}(AB \cdot D \cdot C)) \\ &= x_7 + tx_8 - d(zt + dx_7 + cx_8 - zcd - x_{10}).\end{aligned}$$

Hence we obtain

$$-ax_5 + d^2x_7 + (cd - t)x_8 - dx_{10} = -abc + bx_3 - dtz + cd^2z.$$

(10) We use $D^{-1}C^{-1}B = C^{-1}D^{-1}ABA^{-1}$. From (I2), $\mathrm{tr}D^{-1}C^{-1}B = bt - x_9$ and from (I2), (2.1c) and (I3),

$$\begin{aligned}\mathrm{tr}C^{-1}D^{-1}ABA^{-1} &= tb - \mathrm{tr}(DC)ABA^{-1} \\ &= tb - (tb - \mathrm{tr}DCB - \mathrm{tr}DCA\mathrm{tr}AB + \mathrm{tr}A\mathrm{tr}(D{\cdot}C{\cdot}AB)) \\ &= (dx_5{+}cx_6{+}bt{-}bcd{-}x_9){+}z(dx_3{+}cx_4{+}at{-}acd{+}w) \\ &\quad - a(zt + dx_7 + cx_8 - zcd - x_{10})\end{aligned}$$

we obtain

$$dx_5 + cx_6 - adx_7 - acx_8 + ax_{10} = bcd - zdx_3 - zcx_4 - zw.$$

Let

$$M = \begin{pmatrix} dt-c & d & 0 & 0 & -t & 0 \\ 0 & 0 & d & c & -a & 0 \\ t & 0 & 0 & a & -c & 0 \\ z-ab & 0 & b & 0 & 0 & 0 \\ -a & 0 & d^2 & cd-t & 0 & -d \\ d & c & -ad & -ac & 0 & a \end{pmatrix}, \ \vec{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix}$$

and

$$\vec{v} = \begin{pmatrix} 2b - bt^2 + bcdt - bc^2 \\ -abt - bw + cdz \\ zx_4 \\ (b^2 - d^2)x_3 + (t - cd)x_4 + bcz - ab^2c - adt + acd^2 - wd \\ -abc + bx_3 - dzt + cd^2z \\ bcd - dzx_3 - czx_4 - zw \end{pmatrix}.$$

From the results (5)–(10) we obtain $M\vec{x} = \vec{v}$. The matrix M is singular, if $a = c$. However, by using (2.4) and (2.7) we can deduce:
(2.12)

$$x_5 = \frac{c(2b + a^2b - 2az + bK^2) - tuz + dw(ab + z + zK^2) - v(ab + zK^2)}{K^2 + a^2},$$

$$x_6 = \frac{2(adz-bd)-u(ab+K^2z)+tv(ab+z+K^2z)+(c-dt)w(ab+z+K^2z)}{K^2 + a^2},$$

$$x_7 = \frac{-2cz - btu + avz + wd(b - az)}{K^2 + a^2},$$

$$x_8 = \frac{d(K^2 + a^2 + 2) + auz + vt(b - az) + w(bc - bdt - acz + adtz)}{K^2 + a^2},$$

$$x_9 = \frac{t(2b + a^2b - 2az + bK^2) + dvz + w(ab + K^2z) + u(cz - dtz)}{K^2 + a^2},$$

$$x_{10} = \frac{-2tz + b(c - dt)u + bdv - awz}{K^2 + a^2}.$$

Expressions for x_3 and x_4 are obtained in (2.10).

3. Mapping class group. Let G be a group of type $(2; -; -; -)$ and $E = (A, B, C, D)$ a marking (or a canonical generator system) of G. We consider the following changes of marking:

$$\begin{aligned} &\omega_1(E) = (AB^{-1}, B, C, D), && \omega_2(E) = (B, BA, C, D),\\ &\omega_3(E) = (B^{-1}CA, B, C, B^{-1}CD), &&\\ &\omega_4(E) = (A, B, CD^{-1}, D), && \omega_5(E) = (A, B, C, DC). \end{aligned} \tag{3.1}$$

Each ω_j induces an automorphism of G, which is also denoted by ω_j. The table below shows the images of the elements in the leftmost column under ω_j.

	ω_1	ω_2	ω_3	ω_4	ω_5
A	AB^{-1}	A	$B^{-1}CA$	A	A
B	B	BA	B	B	B
AB	A	ABA	$B^{-1}CAB$	AB	AB
$ACDC^{-1}$	$AB^{-1}CDC^{-1}$	$ACDC^{-1}$	$B^{-1}CACB^{-1}CDC^{-1}$	$ACDC^{-1}$	ACD
ACD^2	$AB^{-1}CD^2$	ACD^2	$B^{-1}CAC(B^{-1}CD)^2$	ACD	$AC(DC)^2$
ACD	$AB^{-1}CD$	ACD	$B^{-1}CACB^{-1}CD$	AC	$ACDC$
CD	CD	CD	$CB^{-1}CD$	C	CDC

Let $\omega_{j*} \in \mathcal{M}C_2$ denote the mapping class induced by ω_j. Then $\omega_{1*},\dots,$ ω_{5*} generate $\mathcal{M}C_2$ and satisfy the following relations [1, Theorem 4.8]:

$$\omega_{i*}\omega_{j*} = \omega_{j*}\omega_{i*} \text{ if } |i-j| \geq 2,\ 1 \leq i, j \leq 5,$$

$$\omega_{j*}\omega_{j+1*}\omega_{j*} = \omega_{j+1*}\omega_{j*}\omega_{j+1*}\ (j = 1, 2, 3, 4),$$

$$(\omega_{1*}\omega_{2*}\omega_{3*}\omega_{4*}\omega_{5*})^6 = 1,$$

$$\omega_{1*}\omega_{2*}\omega_{3*}\omega_{4*}\omega_{5*}^2\omega_{4*}\omega_{3*}\omega_{2*}\omega_{1*} = 1.$$

In this section we represent the action of ω_{j*} on $\mathcal{T}_2$ in the variables $a, b, z, u, v,$ w, t. More precisely, when $(A_j, B_j, C_j, D_j) = \omega_j(A, B, C, D)$, we express

$$a_j = \mathrm{tr}A_j, \quad b_j = \mathrm{tr}B_j, \quad z_j = \mathrm{tr}A_jB_j, \ u_j = -\mathrm{tr}A_jC_jD_jC_j^{-1},$$
$$v_j = -\mathrm{tr}A_jC_jD_j^2, \ w_j = -\mathrm{tr}A_jC_jD_j, \ t_j = \mathrm{tr}C_jD_j$$

by using a, b, z, u, v, w, t. However, for the case of ω_3 we modify the signs of some traces to obtain positive values.
(Case of ω_{1*}) By using basic trace identities we have $\mathrm{tr}AB^{-1} = \mathrm{tr}A\mathrm{tr}B -$ $\mathrm{tr}AB = ab - z$,

$$w_1 = -\mathrm{tr}AB^{-1}CD = -\mathrm{tr}B\mathrm{tr}ACD + \mathrm{tr}ABCD = bw + x_{10},$$

$$\begin{aligned} u_1 = -\mathrm{tr}AB^{-1}CDC^{-1} &= -\mathrm{tr}B\mathrm{tr}ACDC^{-1} + \mathrm{tr}(AB)CDC^{-1} \quad (\because \text{(I2)}) \\ &= bu + (\mathrm{tr}AB\mathrm{tr}D - \mathrm{tr}ABD \\ &\quad - \mathrm{tr}ABC\mathrm{tr}CD + \mathrm{tr}C\mathrm{tr}ABCD) \ (\because (2.1c)) \\ &= bu + zd - x_8 - tx_7 + cx_{10}, \end{aligned}$$

and

$$\begin{aligned} v_1 = -\mathrm{tr}AB^{-1}CD^2 &= -\mathrm{tr}B\mathrm{tr}ACD^2 + \mathrm{tr}ABCD^2 \quad (\because \text{(I2)}) \\ &= bv + (\mathrm{tr}ABCD\mathrm{tr}D - \mathrm{tr}ABC) \quad (\because \text{(I2)}) \\ &= bv + dx_{10} - x_7. \end{aligned}$$

Hence

$$\omega_{1*}(a, b, z, u, v, w, t) = (ab - z, b, a, u_1, v_1, w_1, t).$$

(Case of ω_{2*}) Since $\mathrm{tr}ABA = \mathrm{tr}AB\mathrm{tr}A - \mathrm{tr}B = za - b$,

$$\omega_{2*}(a, b, z, u, v, w, t) = (a, z, az - b, u, v, w, t).$$

(Case of ω_{3*}) First we remark that $\mathrm{tr}B^{-1}CA < 0$ and $\mathrm{tr}B^{-1}CD < 0$. To see $\mathrm{tr}B^{-1}CA < 0$, for example, note that (AB^{-1}, B) is a marking for a group of type $(1; 0; 0; 1)$ and $\mathrm{tr}A$ and $\mathrm{tr}B$ are positive. Then we have $\mathrm{tr}AB^{-1} > 0$. Then (AB^{-1}, C) is a marking for a group of type $(0; 0; 0; 3)$. Since $\mathrm{tr}AB^{-1}$ and $\mathrm{tr}C$ are positive, $\mathrm{tr}AB^{-1}C < 0$ (see [5, Section 33 A and D]). The calculation for ω_{3*} is the most complicated: By using the basic trace identities we have

$$a_3 = \mathrm{tr}B^{-1}CA = \mathrm{tr}B\mathrm{tr}AC - \mathrm{tr}ABC = bx_3 - x_7.$$

$$\begin{aligned}
w_3 &= -\mathrm{tr}(B^{-1}C)(AC)(B^{-1}C)D \\
&= -\mathrm{tr}(AC)(B^{-1}C)D(B^{-1}C) \\
&= -\mathrm{tr}ACB^{-1}C\mathrm{tr}B^{-1}CD - \mathrm{tr}ACD + \mathrm{tr}AC\mathrm{tr}D \\
&= -(\mathrm{tr}B\mathrm{tr}AC^2 - \mathrm{tr}ACBC)(\mathrm{tr}B\mathrm{tr}CD - \mathrm{tr}BCD) + w + dx_3 \\
&= -[b(cx_3 - a) - (x_3x_5 + z - ab)](bt - x_9) + w + dx_3 \\
&= (x_3x_5 + z - bcx_3)(bt - x_9) + w + dx_3,
\end{aligned}$$

$$\begin{aligned}
u_3 &= -\mathrm{tr}(B^{-1}C)(AC)(B^{-1}C)(DC^{-1}) \\
&= -\mathrm{tr}(AC)(B^{-1}C)(DC^{-1})(B^{-1}C) \\
&= -\mathrm{tr}ACB^{-1}C\mathrm{tr}B^{-1}CDC^{-1} - \mathrm{tr}ACDC^{-1} + \mathrm{tr}AC\mathrm{tr}DC^{-1} \\
&= -(\mathrm{tr}AC\mathrm{tr}B^{-1}C - \mathrm{tr}AB)(\mathrm{tr}B\mathrm{tr}D - \mathrm{tr}BCDC^{-1}) + u + x_3(cd - t) \\
&= -(x_3(bc - x_5) - z)[bd - (bd - x_6 - tx_5 + cx_9)] + u + x_3(cd - t) \\
&= (x_3x_5 + z - bcx_3)(x_6 + tx_5 - cx_9) + u + x_3(cd - t),
\end{aligned}$$

$$\begin{aligned}
v_3 &= -\mathrm{tr}B^{-1}CAC(B^{-1}CD)^2 \\
&= -\mathrm{tr}B^{-1}CD\mathrm{tr}B^{-1}CACB^{-1}CD + \mathrm{tr}B^{-1}CAC \\
&= (bt - x_9)[(x_3x_5 + z - bcx_3)(bt - x_9) + w + dx_3] + (bc - x_5)x_3 - z,
\end{aligned}$$

$$t_3 = \mathrm{tr}CB^{-1}CD = \mathrm{tr}CB^{-1}\mathrm{tr}CD - \mathrm{tr}BD = (bc - x_5)t - x_6.$$

In this case a_3, x_3, v_3 and t_3 are negative. We modify the sign of these parameters and obtain

$$\omega_{3*}(a, b, z, u, v, w, t) = (-a_3, b, -x_3, u_3, -v_3, w_3, -t_3).$$

(Case of ω_{4*}) For the expression of ω_{4*} we have easily

$$\omega_{4*}(a, b, z, u, v, w, t) = (a, b, z, u, w, -x_3, c).$$

(Case of ω_{5*}) Since $-\mathrm{tr}ACDC = -\mathrm{tr}C\mathrm{tr}ACD + \mathrm{tr}ACDC^{-1} = cw - u$,

$$\begin{aligned}
v_5 &= -\mathrm{tr}AC(DC)^2 = -\mathrm{tr}CD\mathrm{tr}ACDC + \mathrm{tr}AC \\
&= -t(\mathrm{tr}C\mathrm{tr}ACD - \mathrm{tr}ACDC^{-1}) + x_3 \\
&= cwt - tu + x_3,
\end{aligned}$$

and $\mathrm{tr}CDC = ct - d$, we have

$$\omega_{5*}(a, b, z, u, v, w, t) = (a, b, z, w, cwt - tu + x_3, cw - u, ct - d).$$

Now we conclude

Theorem 3.1. *The mapping classes ω_{1*}, ω_{2*}, ω_{3*}, ω_{4*}, ω_{5*} are represented by the following rational maps in variables a, b, z, u, v, w, t:*
(3.2)
$$\begin{aligned}
&\omega_{1*}(a, b, z, u, v, w, t) = (ab - z, b, a, u_1, v_1, w_1, t)\\
&\omega_{2*}(a, b, z, u, v, w, t) = (a, z, az - b, u, v, w, t)\\
&\omega_{3*}(a, b, z, u, v, w, t) = (-bx_3 + x_7, b, -x_3, u_3, -v_3, w_3, -bct + x_5t + x_6)\\
&\omega_{4*}(a, b, z, u, v, w, t) = (a, b, z, u, w, -x_3, c)\\
&\omega_{5*}(a, b, z, u, v, w, t) = (a, b, z, w, cwt - tu + x_3, cw - u, ct - d),
\end{aligned}$$
where c, d, x_3, x_4, x_5, x_6 and x_7 are given in (2.9) *and* (2.10) *and* (2.12).

As it is shown in Section 2, $x_1 = c$, $x_2 = d$,..., x_{10} are all rational functions in (a, b, z, u, v, w, t). Hence the inverse mappings of ω_{j*} $(j = 1, ..., 5)$ are also rational mappings. The expressions in (3.2) in (a, b, z, u, v, t), especially the one for ω_{3*}, are very complicated.

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