doi: 10.2478/v10062-012-0005-1

UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVI, NO. 1, 2012 SECTIO A 41–48

ANDRZEJ MIERNOWSKI

Cartan connection of transversally Finsler foliation

Abstract. The purpose of this paper is to define transversal Cartan connection of Finsler foliation and to prove its existence and uniqueness.

1. Introduction. Let $(M, \mathcal{F})$ be a smooth n-dimensional manifold equipped with a foliation $\mathcal{F}$ of codimension q. We put $n = p + q$. We denote by (x^i, y^α), $i = 1, 2, \dots, p$, $\alpha = 1, 2, \dots, q$ the foliated (or distinguished) coordinates with respect to the foliation $\mathcal{F}$. If $(x^{i'}, y^{\alpha'})$, $i' = 1, 2, \dots, p$, $\alpha' = 1, 2, \dots, q$ is another foliated coordinate system, then

$$x^{i'} = x^{i'}(x, y),$$
$$y^{\alpha'} = y^{\alpha'}(y).$$

Let TM be a tangent bundle of M. We consider an induced coordinate system $(x^i, y^\alpha, a^i, b^\alpha)$ in TM, where (a^i, b^α) are coordinates of the vector $a^i \frac{\partial}{\partial x^i} + b^\alpha \frac{\partial}{\partial y^\alpha}$ tangent to M at the point $p = (x, y)$. Let $Q(M)$ denotes the normal bundle of the foliation $\mathcal{F}$ with the projection $\delta : TM \longrightarrow Q(M)$. In $Q(M)$ we have the coordinates $(x^i, y^\alpha, b^\alpha)$, where b^α are coordinates of the vector $b^\alpha \frac{\partial}{\partial y^\alpha}$. Here $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^q}$ is a local frame of Q. The coordinates in Q transform as follows

$$x^{i'} = x^{i'}(x, y),$$

2000 *Mathematics Subject Classification.* 53C12.
Key words and phrases. Finsler metric, normal bundle, Cartan connection.

$$y^{\alpha'} = y^{\alpha'}(y),$$
$$b^{\alpha'} = J^{\alpha'}_{\alpha}(y),$$

where $J^{\alpha'}_{\alpha}(y) = \frac{\partial y^{\alpha'}}{\partial y^{\alpha}}(y)$. If $\frac{\partial}{\partial y^{\alpha'}}$, $\alpha' = 1, \ldots, q$ are the vectors of a local frame in new coordinates in Q, then

$$\frac{\partial}{\partial y^{\alpha'}} = J^{\alpha}_{\alpha'} \frac{\partial}{\partial y^{\alpha}}.$$

Let us recall some basic facts from the theory of Riemannian foliations ([5]). Let g^T be a Riemannian metric in the normal bundle Q. The metric g^T is called adapted to the foliation $\mathcal{F}$ if for any vector field X tangent to the leaves of $\mathcal{F}$ and any sections Y, Z of the normal bundle

$$Xg^T(Y, Z) - g^T(\delta([X, \widehat{Y}]), Z) - g^T(Y, \delta([X, \widehat{Z}]) = 0,$$

where $\widehat{Y}, \widehat{Z}$ are any vector fields on M such that $\delta(\widehat{Y}) = Y$, $\delta(\widehat{Z}) = Z$.

The vector field V on M is called foliated if for any vector field X tangent to the leaves of $\mathcal{F}$ the vector field $[X, V]$ is also tangent to the leaves. Locally in the foliated coordinate system foliated vector fields are of the form

$$V = a^i(x, y)\frac{\partial}{\partial x^i} + b^{\alpha}(y)\frac{\partial}{\partial y^{\alpha}}.$$

The section Y of the normal bundle is called a transverse vector field if $Y = \delta(V)$ with V foliated. It is clear that the metric g^T is adapted if the function $g^T(Y, Z)$ is constant along the leaves for any transverse vector fields Y, Z.

Let (x^i, y^a) be a foliated coordinate system on $U \subset M$. Denote by $\overline{U}$ the local quotient manifold and let $\pi : U \longrightarrow \overline{U}$ be a local projection $\pi(x^i, y^a) = (y^a)$. The adapted metric g^T induces the metric $\overline{g}$ on $\overline{U}$ such that for each point $u \in U$, $\pi_\star$ is an isometry between the transversal space at u and the tangent space at $\pi(u)$.

Let $B_T(M)$ be the bundle of transversal frames of M and θ_T be the canonical form on $B_T(M)$ with values in $\mathbb{R}^q$. P. Molino ([5]) has proved that p-dimensional distribution P_T on $B_T(M)$ such that

$$P_T(e) = \{X_e \in T_eB_T : i_{X_e}\theta_T = i_{X_e}d\theta_T = 0\} \tag{1.1}$$

is completely integrable and the associated foliation (the lifted foliation) is invariant by the right translations. Let $B_T(U)$ be the bundle of transversal frames on U and denote by $B(\overline{U})$ the bundle of linear frames of local quotient manifold $\overline{U}$. Let $\pi_T : B_T(U) \longrightarrow B(\overline{U})$ be the natural projection. Then locally $X_e \in P_T(e) \subset T_eB_T(U)$ if and only if $\pi_{T\star}(X) = 0$.

Using the metric g^T, we can define the bundle E^1_T of the orthonormal transversal frames. The bundle E^1_T is saturated by the leaves of the lifted foliation. The connection H in E^1_T is called transverse if the distributions tangent to the leaves of the lifted foliation are horizontal with respect to

H. The following theorem is fundamental in the theory of transversally Riemannian foliations.

Theorem ([5]). *For any transversal metric g^T there exists in E_T^1 exactly one torsion-free transverse connection.*

A. Spiro in [6] has given the characterization of Cartan connection of Finsler manifold (M, F) in terms of a bundle of Chern frames. The purpose of this paper is to prove the similar theorem for the transversally Finsler foliation.

2. Transversally Finsler foliations. We start with the definition of the transverse Finsler metric.

Definition 2.1. A Finsler metric F^T in the normal bundle of the foliation $\mathcal{F}$ is called transverse if for any transverse vector field X the function $F^T(X)$ is basic.

Consider a foliated coordinate system (x^i, y^α), where $y^1, \ldots, y^q$ are transverse coordinates. If $V = a^i(x,y)\frac{\partial}{\partial x^i} + b^\alpha(y)\frac{\partial}{\partial y^\alpha}$ is a foliated vector field and $b^\alpha(y)\frac{\partial}{\partial y^\alpha}$ is a corresponding transverse vector field, then F^T is a transverse Finsler metric if and only if the function $F^T(x, y, b)$ does not depend on x. Let $\pi : U \longrightarrow \overline{U}$ be a local projection. Then we have the Finsler metric $\overline{F}$ on $\overline{U}$ defined by the formula $\overline{F}(y, b) = F^T(y, b)$ such that π induces an isometry between Q_u and $T_{\pi(u)}\overline{U}$, for any $u \in U$.

A. Spiro in [6] has defined the bundle of spheres of the Finsler metric F. In our case we define the bundle of the transversal spheres.

Definition 2.2. The set $S_u^T = \{V \in Q_u : F^T(u, V) = 1\}$ is called the transversal sphere at u. The manifold $\bigcup_{u \in M} S_u^T$ is called the transversal spheres bundle.

Let us fix a vector $V \in Q_u$, $u \in M$, $u = (x, y)$, $V = b^\alpha \frac{\partial}{\partial y^\alpha}(u)$ and put $g^T_{\alpha\beta}(x, y, b) = \frac{1}{2}\frac{\partial^2 (F^T)^2}{\partial b^\alpha \partial b^\beta}(x, y, b)$. In this way we obtained a bilinear form g^T on the tangent space $T_V Q_u$ for any $u \in M$ and $V \in Q_u$.

Definition 2.3. $(M, \mathcal{F}, F)$ is called transversally Finsler foliation if F is a transverse metric and g is a positively definite scalar product.

If $\pi : U \longrightarrow \overline{U}$ and $\overline{u} = \pi(u)$, then $\overline{S}_{\overline{u}} = \pi_*(S_u^T)$, where $\overline{S}_{\overline{u}}$ is a sphere at $\overline{u}$ with respect to $\overline{F}$.

A. Spiro in [6] has constructed a bundle of Chern orthogonal frames for the Finsler space. In the case of the transverse Finsler metric we can define in a similar way a bundle of transverse orthogonal Chern frames.

For fixed $V \in Q_u$ there is the natural identification of the vector space Q_u with the space $T_V Q_u$ tangent to Q_u at V.

Definition 2.4. The frame $E_0, E_1, \ldots, E_{q-1}$ of the vector space Q_u is called the transverse orthogonal Chern frame if

(1) $F^T(u, E_0) = 1$.
(2) The vectors $E_1, \ldots, E_{q-1}$ are tangent to S_u^T at E_0.
(3) $g_{E_0}(E_\alpha, E_\beta) = \delta_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, q-1$.

Denote by $O_{g^T}(S^T(M))$ the bundle of the transverse orthogonal Chern frames. For a distinguished open set U the bundle $O_{g^T}(S^T(U))$ is a pull-back of the bundle $O_g(S(\overline{U}))$ of orthogonal Chern frames of the local quotient manifold $\overline{U}$ under the restriction $\widehat{\pi}_T$ of π_T to $O_{g^T}(S^T(U))$. There is a natural right action of the group $\mathrm{O}(\mathbb{R}, q-1)$ on $O_{g^T}(S^T(M))$.

Proposition 2.1. *The bundle $O_{g^T}(S^T(M))$ is saturated by the leaves of the lifted foliation and foliation of $O_{g^T}(S^T(M))$ is invariant under the action of* $\mathrm{O}(\mathbb{R}, q-1)$.

Proof. Let $X_e \in P_T(e)$ be a vector tangent at e to the leave of the lifted foliation. Consider distinguished open set U and the projection

$$\widehat{\pi}_T : O_g^T(S^T(U)) \longrightarrow O_g(S(\overline{U})).$$

Then $\widehat{\pi}_{T\star}(X_e) = 0$. But $\dim O_g^T(S^T(U)) - \dim O_g(S(\overline{U}) = p$, which means that $\dim\ker \widehat{\pi}_{T\star} = p$. From (1.1) it follows that the foliation of $O_{g^T}(S^T(M))$ is invariant under the action of $\mathrm{O}(\mathbb{R}, q-1)$. □

Definition 2.5. A local section $\sigma \longrightarrow O_{g^T}(S^T(M))$ is called foliated if for all $u \in U$ the distribution $P_T(\sigma(u))$ is tangent to $\sigma(U)$.

Equivalently σ is a foliated section if it sends locally the leaves of $\mathcal{F}$ into the leaves of the lifted foliation.

Let $p_T : O_{g_T}(S^T(M)) \longrightarrow M$ be the natural projection.

Proposition 2.2. *For any $e \in O_{g_T}(S^T(M))$ there exists a local foliated section $\overline{\sigma} : U \longrightarrow O_{g_T}(S^T(U))$ defined in a neighborhood of $u_0 = p_T(e)$ such that $\overline{\sigma}(u_0) = e$.*

Proof. Let $\overline{u}_0 = \pi(u_0)$, where π is a projection onto a local quotient manifold $\overline{U}$. The projection $\pi_T : B_T(U) \longrightarrow B(\overline{U})$ induces an isometry of the transversal sphere $S^T_{p_T(e)}$ onto the sphere $\overline{S}_{\pi(u)}$ of the Finsler space $(\overline{U}, \overline{F})$. The image of the transversal orthonormal frame e is an orthonormal frame $\overline{e}$ at $\overline{u}$ with respect to $\overline{F}$. Let $\overline{e} = (\overline{e}_0, \overline{e}_1, \ldots, \overline{e}_{q-1})$, where $\overline{F}(\overline{u}_0, \overline{e}_0) = 1$ and $\overline{e}_1, \ldots, \overline{e}_{q-1}$ is an orthonormal basis of $T_{\overline{e}_0}\overline{S}_{\overline{u}_0}$. Denote the transversal coordinates by $(y_0, \ldots, y_{q-1})$. We can suppose that $\overline{e}_0 = \frac{\partial}{\partial y_0}_{|\overline{u}_0}$. Let $\overline{z}_0 = \frac{1}{\overline{F}(\overline{u}, \frac{\partial}{\partial y_0})}\frac{\partial}{\partial y_0}$. We can use the scalar product $\overline{g}(\overline{u}, \overline{z}_0)$ to project the vectors $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{q-1}}$ onto the tangent space $T_{\overline{z}_0}\overline{S}_{\overline{u}}$ and next applying the Gram–Schmidt orthonormalization process, we obtain an orthonormal frame $\overline{z}_1, \ldots, \overline{z}_{q-1}$ of $T_{\overline{z}_0}\overline{S}_{\overline{u}}$. In this way we obtain a section $\widehat{\sigma} : \overline{U} \longrightarrow O_{\overline{g}}(\overline{S}(\overline{U}))$ such that $\widehat{\sigma}(\overline{u}_0) = (\overline{z}_0, \ldots, \overline{z}_{q-1})$ and $\overline{z}_0 = \overline{e}_0$. Using an appropriate element $g \in \mathrm{O}(\mathbb{R}, q-1)$ we get a section $\overline{\sigma} : \overline{U} \longrightarrow O_{\overline{g}}(\overline{S}(\overline{U}))$ such that $\overline{\sigma}(\overline{u}_0) = \overline{e} = (\overline{e}_0, \ldots, \overline{e}_{q-1})$. The section $\sigma : U \longrightarrow 0_{g^T}(S^T(U))$ is the unique section defined by the condition $p_T \circ \sigma = \overline{\sigma} \circ \pi$.

The fibre $V_u^T = p_T^{-1}(u)$ consists of the orthonormal frames of the transversal sphere S_u^T. Denote by V_e^T the subspace of $T_u O_{g^T}(S^T(M))$ of the vectors tangent at e to the fibre V_u^T. Let $A^\star$ be the fundamental vector field on $B_T(M, \mathcal{F})$ associated to the element $A \in \mathfrak{gl}(q, \mathbb{R})$. We put

$$\mathfrak{g}_e^T = \{A \in \mathfrak{gl}(q, \mathbb{R}) : A_e^\star \in V_e^T\}.$$

For any open $U \in M$ adapted to the foliation $\mathcal{F}$ and any $g \in Gl(\mathbb{R}, q)$

$$\pi_T \circ R_g = \overline{R}_g \circ \pi_T, \tag{2.1}$$

where R_g (resp. $\overline{R}_g$) denotes the right translation of $B_T(U)$ (resp. $B(\overline{U})$). □

Example 2.1. Let U be a distinguished open set and $\pi : U \longrightarrow \overline{U}$. Denote by (x^i, y^β) the coordinates of $u \in U$. For any non-zero vectors $V, W \in T_{\overline{u}}\overline{U}$ put $V \equiv W$ if and only if there exists $\lambda > 0$ such that $V = \lambda W$. Let $P_{\overline{u}} = T_{\overline{u}}\overline{U}/\equiv$ and $P_{\overline{U}} = \bigcup_{\overline{u} \in \overline{U}} P_{\overline{u}}$. Then the bundle of spheres $\overline{S}(\overline{U})$ is diffeomorphic to $P_{\overline{U}}$ and we can use the positively homogeneous coordinates to get the coordinates in $\overline{S}(\overline{U})$. For $V \in \overline{S}_{\overline{u}}(\overline{U})$, $V = v^\beta \frac{\partial}{\partial y^\beta}$, (y^β, w^β), where $w^\beta = \lambda v^\beta$, $\lambda > 0$ are called homogeneous coordinates of V. Let $\widetilde{\pi} : S^T(U) \longrightarrow \overline{S}(\overline{U})$ be a natural projection. Consider an open subset $\overline{S}^q(\overline{U})$ of $\overline{S}(\overline{U})$ such that $V \in \overline{S}^q(\overline{U})$ if and only if $w^q > 0$. Then (y^β, z^α), $z^\alpha = \frac{w^\alpha}{w^q}$, are coordinates in $\overline{S}^q(\overline{U})$, (x^i, y^β, z^α) are coordinates in $S^q(U) = \widetilde{\pi}^{-1}(\overline{S}^q(\overline{U}))$, $\widetilde{\pi}(x^i, y^\beta, z^\alpha) = (y^\beta, z^\alpha)$ and (x^i) are coordinates along the plaques of the lifted foliation. Let $e \in O_g^T(S^T(U)$, $e = (x^i, y^\beta, z^\alpha, A_\gamma^\alpha)$ where $A_\gamma^\alpha \in O(\mathbb{R}, q-1)$. Then $\widehat{\pi}_T(x^i, y^\beta, z^\alpha, A_\gamma^\alpha) = (y^\beta, z^\alpha, A_\gamma^\alpha)$ and if $g = G_\gamma^\alpha \in O(\mathbb{R}, q-1)$, then $R_g(x^i, y^\beta, z^\alpha, A_\gamma^\alpha) = (x^i, y^\beta, z^\alpha, A_\kappa^\alpha G_\gamma^\kappa)$, $\overline{R}_g(y^\beta, z^\alpha, A_\gamma^\alpha) = (y^\beta, z^\alpha, A_\kappa^\alpha G_\gamma^\kappa)$.

Proposition 2.3. *The subspace $\mathfrak{g}_e^T$ is constant along the plaques of the lifted foliation restricted to $O_{g^T}(S^T(U))$ and $\mathfrak{g}_e^T = \overline{\mathfrak{g}}_{\overline{e}}$, where $\overline{\mathfrak{g}}_{\overline{e}}$ corresponds to the vertical subspace at $\overline{e}$ of the bundle $O_{\overline{g}}(\overline{S}(\overline{U}))$ of the Finsler space $(\overline{U}, \overline{F})$.*

Proof. Proposition 2.3 is a direct consequence of (2.1). □

Let $A = (A_\beta^\alpha)_{\alpha,\beta=0,\ldots,q-1} \in \mathfrak{g}_e^T$. Then $A \in \overline{g}_{\overline{e}}$ and from [6] we know that

$$A_0^o = 0, \; A_\beta^0 = -A_0^\beta, \tag{2.2}$$

$$H_{(u,e_o)}(e_\alpha, e_\beta, e_\gamma)A_0^\gamma + A_\beta^\alpha + A_\alpha^\beta = 0, \tag{2.3}$$

where H is the Hessian at the point (u, e_0) of the transversal Finsler metric.

Definition 2.6 ([6]). A non-linear connection in $O_g^T(S^T(M))$ is a distribution H such that H is complementary to the vertical distribution and for any $h \in O(q-1, \mathbb{R})$, $H_{eh} = (R_h)_\star H_e$.

Equivalently a non-linear connection is defined by a $\mathfrak{g}_e$-valued 1-form ω on $O_g^T(S^T(M))$ which vanishes on H and $\omega(A_e^\star) = A$ for any $A \in g_e$.

A non-linear connection H is called adapted to the transverse Finsler sphere bundle if the vectors tangent to the lifted foliation are horizontal. The R^q valued 2-form $\Sigma_T = d\theta_T + \omega \wedge \theta_T$ is called the torsion of H. In the similar way as in [5] we can prove the following proposition.

Proposition 2.4. *A non-linear connection H is adapted to the transverse Finsler sphere bundle if and only if $i_{X_e}\Sigma_T = 0$ for any X_e tangent to the lifted foliation and $e \in O_g^T(S^T(M))$.*

Proposition 2.5. *Let F be a transverse Finsler metric on a foliated manifold $(M, \mathcal{F})$. Then there exists on $O_g^T(S^T(M))$ an adapted non-linear connection with zero torsion.*

Proof. Let U be a distinguished open set and

$$\overline{\pi}_T : O_g^T(S^T(U)) \longrightarrow O_g(S(\overline{U})).$$

There exists in $O_g(S(\overline{U}))$ a unique torsion free connection $\overline{\omega}$. Then $\overline{\pi}_T^\star(\overline{\omega})$ is a torsion free connection in $O_g^T(S^T(U))$ adapted to the lifted foliation restricted to $O_g^T(S^T(U))$. Consider a covering $\{U_i : i \in I\}$ of M by the distinguished open sets and let $\pi_i : U_i \longrightarrow \overline{U}_i$ be a local projection and $\overline{\omega}_i$ denotes a unique torsion free connection on $O_g(S(\overline{U}_i))$. Let $\{f_i : i \in I\}$ be a partition of unity subordinate to the covering $\{U_i : i \in I\}$. Then $\omega = \sum f_i \circ p \overline{\pi}_T^\star(\overline{\omega}_i)$ is a torsion free connection adapted to the lifted foliation. □

Theorem 2.1. *On the bundle $O_g^T(S^T(M))$ of the transversal Chern orthonormal frames there exists a unique torsion-free non-linear connection.*

Proof. We need to prove the uniqueness of torsion-free non-linear connection. Let ω and $\widehat{\omega}$ be the torsion-free non-linear connections. It is enough to prove that for any $e \in O_g^T(S^T(M))$ ω and $\widehat{\omega}$ agreed on on the section $\overline{\sigma} : U \longrightarrow O_g^T(S^T(U))$ such as in Proposition 2.2. Let $\overline{\sigma}^\star(\theta_T) = (\theta^0, \theta^1, \ldots, \theta^{q-1})$ and $\overline{\sigma}^\star(\omega) = A^\alpha_{\gamma\beta}\theta^\gamma$, $\overline{\sigma}^\star(\widehat{\omega}) = B^\alpha_{\gamma\beta}\theta^\gamma$, where $\omega_\gamma = (A^\alpha_{\gamma\beta})$ and $\widehat{\omega}_\gamma = (B^\alpha_{\gamma\beta})$ are the elements of g_e. Since ω and $\widehat{\omega}$ are torsion-free it follows that $(\omega - \widehat{\omega}) \wedge \theta_T = 0$. We have

$$A^\alpha_{\gamma\beta} - A^\alpha_{\beta\gamma} = B^\alpha_{\gamma\beta} - B^\alpha_{\beta\gamma}$$

or

$$A^\alpha_{\gamma\beta} - B^\alpha_{\gamma\beta} = A^\alpha_{\beta\gamma} - B^\alpha_{\beta\gamma}.$$

From (2.2) and (2.3) we get

$$A^{\alpha}_{00} - B^{\alpha}_{00} = -A^{0}_{0\alpha} + B^{0}_{0\alpha} = -A^{0}_{\alpha 0} + B^{0}_{\alpha 0} = 0.$$

For $\alpha, \beta = 1, \dots, q-1$ we have

$$\begin{aligned}
A^{0}_{\alpha\beta} - B^{0}_{\alpha\beta} &= -A^{\beta}_{\alpha 0} + B^{\beta}_{\alpha 0} = -A^{\beta}_{0\alpha} - B^{\beta}_{0\alpha} \\
&= A^{\alpha}_{0\beta} + H_{(u,e_0)}(e_\alpha, e_\beta, e_\gamma)A^{\gamma}_{00} - B^{\alpha}_{0\beta} - H_{(u,e_0)}(e_\alpha, e_\beta, e_\gamma)B^{\gamma}_{00} \\
&= A^{\alpha}_{0\beta} - B^{\alpha}_{0\beta} = A^{\alpha}_{\beta 0} - B^{\alpha}_{\beta 0} = -A^{0}_{\beta\alpha} + B^{0}_{\beta\alpha} = -A^{0}_{\alpha\beta} + B^{0}_{\alpha\beta}. \\
A^{\beta}_{\alpha 0} - B^{\beta}_{\alpha 0} &= -A^{0}_{\alpha\beta} + B^{0}_{\alpha\beta} = 0. \\
A^{\alpha}_{\beta\gamma} - B^{\alpha}_{\beta\gamma} &= -A^{\gamma}_{\beta\alpha} - H_{(u,e_0)}(e_\alpha, e_\gamma, e_\kappa)A^{\kappa}_{\beta 0} + B^{\gamma}_{\beta\alpha} - H_{(u,e_0)}(e_\alpha, e_\gamma, e_\kappa)B^{\kappa}_{\beta 0} \\
&= -A^{\gamma}_{\beta\alpha} + B^{\gamma}_{\beta\alpha} - H_{(u,e_0)}(e_\alpha, e_\gamma, e_\kappa)(A^{\kappa}_{\beta 0} - B^{\kappa}_{\beta 0}) = -A^{\gamma}_{\beta\alpha} + B^{\gamma}_{\beta\alpha} \\
&= -A^{\gamma}_{\alpha\beta} + B^{\gamma}_{\alpha\beta} = -A^{\beta}_{\alpha\gamma} + B^{\beta}_{\alpha\gamma} - H_{(u,e_0)}(e_\gamma, e_\beta, e_\kappa)(A^{\kappa}_{\alpha 0} - B^{\kappa}_{\alpha 0}) \\
&= A^{\beta}_{\alpha\gamma} - B^{\beta}_{\alpha\gamma} = A^{\beta}_{\gamma\alpha} - B^{\beta}_{\gamma\alpha} \\
&= -A^{\alpha}_{\gamma\beta} + B^{\alpha}_{\gamma\beta} - H_{(u,e_0)}(e_\beta, e_\alpha, e_\kappa)(A^{\kappa}_{\gamma 0} - B^{\kappa}_{\gamma 0}) \\
&= -A^{\alpha}_{\gamma\beta} + B^{\alpha}_{\gamma\beta} = -A^{\alpha}_{\beta\gamma} + B^{\alpha}_{\beta\gamma}.
\end{aligned}$$

□

Example 2.2. Let F be a transversal Finsler metric in Q and g an arbitrary Riemannian metric on M. Denote by $(T_uL)^\perp$ an orthogonal complement of T_uM with respect to g. The projection $\rho_u : T_uM \longrightarrow Q_u$ induces an isomorphism of $(T_uL)^\perp$ onto Q_u. Put for $X \in T_uM$, $X = X_L + X_L^\perp$, $X_L \in T_uL$, $X_L^\perp \in (T_uL)^\perp$

$$\widehat{F}(u, X) = \sqrt{g_u(X_L, X_L) + F^2(u, \rho_u(X))}.$$

Then $\widehat{F}$ is a Finsler metric on M adapted to $\mathcal{F}$ in the sense of [3], [4], $(T_uL)^\perp$ is its transversal cone at u ([3]) and the metric $\widehat{F}$ induces the metric F on the bundle Q.

References

[1] Álvarez Paiva, J. C., Duran, C. E., *Isometric submersions of Finsler manifolds*, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2409–2417 (electronic).

[2] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*, vol. I, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[3] Miernowski, A., *A note on transversally Finsler foliation*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **60** (2006), 57–64.

[4] Miernowski, A., Mozgawa, W., *Lift of the Finsler foliation to its normal bundle*, Differential Geom. Appl. **24** (2006), no. 2, 209–214.

[5] Molino, P., *Riemannian Foliations*, Progress in Mathematics, 73, Birkhäuser Boston, Inc., Boston, MA, 1988.

[6] Spiro, A., *Chern's orthonormal frame bundle of a Finsler space*, Houston J. Math. **25** (1999), no. 4, 641–659.

Andrzej Miernowski
Institute of Mathematics
Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin
Poland
e-mail: mierand@hektor.umcs.lublin.pl

Received March 23, 2011