Domain Filling Circle Packings

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submitted by **Dipl.-Math. David Krieg**

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9th March 2018 Dipl.-Math. David Krieg

Acknowledgment

Dear reader, before you jump right into this work, I would like to ask you to read these few lines first. They are about some amazing people, who all contributed a major part to the creation of this thesis.

Above all, I would like to express my deep gratitude to my supervisor and mentor Prof. Elias Wegert. His invaluable advice and inexhaustible help have not only shaped my work, they made it possible in the first place. Whether I had questions or problems concerning mathematics, language or personal topics, I could always rely on you during all these years. I could not ask for a better supervisor. Thank you so much.

The next person I want to thank warmly is the author of a wonderful book, which opened for me the fantastic world of circle packing: *"Introduction to Circle Packing"* by Ken Stephenson. Thank you very much Ken for writing an opus that I frequently (half jokingly, half seriously) call the one and only necessary entry of my bibliography. And let me also thank you for your free(!) software CirclePack. I really enjoyed doing spectacular circle packing experiments with it, and making impressive pictures. And to every reader, who is not familiar with circle packings, I warmly recommend to read *"Circle Packings: A Mathematical Tale"* ([30]). Really, put this thesis aside, take a comfortable seat, close the curtains, light a candle and be infected by Stephenson's joy in circle packings.

At this point, I am tempted to list many more fabulous mathematicians, colleagues and friends, who all supported me in their own unique ways. In my eyes, this thesis is the last link of a long chain of great experiences, which I was able to gather during my time at the Technical University Bergakademie Freiberg, and in which I do not want to miss any of you. Be it a fellow student, I could compete against. Be it a professor, I could look up to. Or be it a tutor, the combination of the former two. You are awesome.

Last but not least, particular thanks are due to my beautiful wife Jule. Thank you for reminding me to eat healthily when my head was filled with circles. I love you.

Contents

1. Introduction

The circle: one of the oldest geometrical objects known by mankind, yet studied to this day. It is so simple that a child can draw it, yet it is a symbol of perfection. It is a rich source of interesting questions without easy answers – all the more if circles appear in a pack.

1.1. Motivation

Roughly speaking, a *circle packing* – in this work usually denoted by $\mathcal{P}$ – is an ensemble of circles which touch each other in a prescribed way. The touching pattern of the circles in the packing is encoded in its *complex* K – also called the *nerve* or *contact graph* of $\mathcal{P}$. An example is depicted in Figure 1.1.

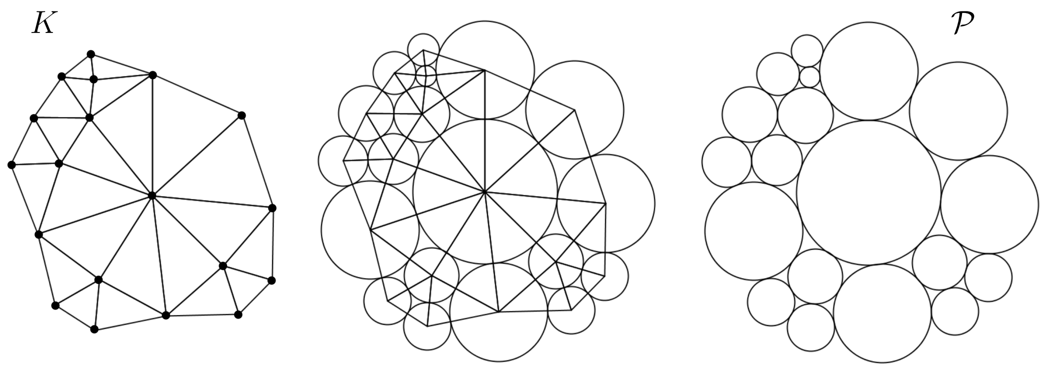


Fig. 1.1.: A circle packing $\mathcal{P}$ and its complex K

While it is easy to identify the complex of a circle packing, it is not clear if a given complex has an associated circle packing and how this can be constructed. The following result is a slightly simplified version of Paul Koebe's famous Circle Packing Theorem ([18]), which was rediscovered by William Thurston ([32],[33]) using a result of Andreev ([2],[3]), and presented to a wide audience in Thurston's talk at the Purdue conference celebrating the proof of the Bieberbach conjecture in 1985.

Theorem 1.1 (Koebe-Andreev-Thurston Circle Packing Theorem)**.** *For any finite planar graph K there exists a circle packing $\mathcal{P}$ with contact graph K. If K is a (finite combinatoric) triangulation of a disk there exists a circle packing with complex K which fills the unit disk $\mathbb{D}$. This packing is uniquely determined up to a conformal automorphism of the unit disk.*

That $\mathcal{P}$ *fills the unit disk* means that all circles of $\mathcal{P}$ are *contained* in the closure of $\mathbb{D}$, and every boundary circle of $\mathcal{P}$ *touches* the unit circle $\partial\mathbb{D}$. A circle packing that fills $\mathbb{D}$ is said to be a *maximal packing.*

In his talk Thurston also proposed to model conformal mappings between two given domains G and D in the framework of circle packing. If $D = \mathbb{D}$ is the unit disk, the most popular approach to construct such a *discrete conformal mapping* is based on cookie-cutting (see Stephenson [31], Chapter 19). Here one covers the given domain G by a (usually hexagonal) circle packing and removes circles which do not lie in G. This yields a *domain packing* $\mathcal{P}_G$ with some complex K. The *range packing* $\mathcal{P}_{\mathbb{D}}$ is a maximal packing with the same complex K. Together the packings $\mathcal{P}_G$ and $\mathcal{P}_{\mathbb{D}}$ define a *discrete analytic function*, which is a piecewise affine-linear function emerging from the correspondence of associated pairs of circles in $\mathcal{P}_G$ and $\mathcal{P}_{\mathbb{D}}$ (for details of this construction see Stephenson [31]).

The cookie-cutting approach to discrete conformal mapping has the disadvantage that the complex K is determined by the geometry of the domain G and cannot be prescribed. So one cannot construct a mapping of G onto another domain D as composition of the discrete conformal mapping of G onto $\mathbb{D}$ with the inverse mapping of D onto $\mathbb{D}$ since, in general, the circle packings $\mathcal{P}_G$ and $\mathcal{P}_D$ will have different combinatorics.

In order to overcome this deficiency it is desirable to fill *arbitrary bounded simply connected domains* with circle packings having a *prescribed* complex K. Existence and uniqueness of such domain filling circle packings $\mathcal{P}$ and more general *circle agglomerations* are the main topics of this thesis.

In what follows we will often speak of the *disks* in $\mathcal{P}$, which are the open disks bounded by the circles in $\mathcal{P}$. Using this terminology has some advantages in our general setting, especially if the domain G to be filled by $\mathcal{P}$ is not Jordan.

Very general types of domain-filling packings have been studied in a series of papers by Oded Schramm ([24],[25],[26],[27]). The objects of his packings are not necessarily circles or disks, but so-called *packable sets* characterized by a few special properties, which for instance include smooth strictly convex sets. If these sets are specialized to be disks, Schramm's packings are classical circle packings or, more generally, *circle agglomerations* (which are allowed to have more general complexes than circle packings). His stunning existence results comprise the existence of domain-filling circle packings for smooth Jordan domains G and, more generally, for Jordan domains without inward cusps. As Schramm observes, these assumptions are essential to prevent that sets of the packing may degenerate to points, but he did not investigate this possibility further.

As in the Koebe-Andreev-Thurston Theorem, circle packings filling general domains cannot be expected to be unique. Schramm therefore proposed several types of normalization, which are related to the three common normalizations of conformal mappings $f : G \to D$. If G is a *Jordan domain* these are the following:

(i) $f(z_j) = w_j$ for $j = 1, 2, 3$ with $z_j \in \partial G$ and $w_j \in \partial\mathbb{D}$,

(ii) $f(z_0) = 0$ and $f(z_1) = w_1$ with $z_0 \in G$, $z_1 \in \partial G$ and $w_1 \in \partial\mathbb{D}$,

(iii) $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}_+$ with $z_0 \in G$.

Before we describe the corresponding normalizations of circle packings filling a *Jordan domain* G, we observe that a disk D which lies in G may be unable to touch a point p on the boundary ∂G because of geometrical obstructions (curvature). Following Schramm, we therefore adapt the concept of touching, saying that a disk $D \in \mathcal{P}$ *meets a point* $p \in \partial G$ if p lies on boundary of D, or if D separates p from the rest of the packing (see Chapter 3 for more details). Note that this and some of the following definitions require some subtle modifications for domains G which are not Jordan.

The first normalization, which we call *Alpha-Beta-Gamma normalization*, requires that three distinguished boundary disks D_1, D_2 and D_3 of $\mathcal{P}$ meet three prescribed boundary points p_1, p_2 and p_3 of G, respectively.

For the *Alpha-Gamma normalization* we assume that the center of some interior disk D_a is fixed at some point $A \in G$, while a boundary disk D_c meets a given boundary point C of G.

The *Alpha-Beta normalization* fixes the center of some interior disk D_a at $A \in G$ and requires that the center of a neighboring disk D_b lies on a given open segment $\Gamma \subset G$ which connects A with a boundary point B of G.

For all three normalizations (and a fourth one which we do not consider in this thesis) Schramm proved the *existence* of a domain-filling circle packing $\mathcal{P}$ under the assumption that G is a Jordan domain without inward cusps.

The *uniqueness* of domain-filling circle packings (and more general circle agglomerations) was proved by Schramm only for the Alpha-Beta-Gamma normalization and under the additional restrictive assumption that the normalization points A, B, C are *untouchable*, i.e., no disk lying in the domain can touch one of these points. For the proofs of uniqueness with the other normalizations he gave only some vague hints.

Considering the case of general domains only briefly, Schramm observes that under certain circumstances domain filling circle packings may degenerate (which is possible even for Jordan domains, see Figure 1.2), i.e., some "circles" have radius zero. Since Schramm did not develop techniques for studying degenerate circle packings, he could not seriously investigate when this may happen and which conditions exclude degeneracy. To the best of the authors knowledge, there is no subsequent work that provides existence and/or uniqueness statements for (finite) domain filling (generalized) circle packings or circle agglomerations that exceeds Schramm's results. Thus we are highly encouraged to close this gap by studying circle packings *filling general bounded, simply connected domains* in full detail.

1.2. Objectives and Results

The primary goal of this work is to extend Schramm's results to *arbitrary domains* without any restrictions on the regularity of their boundary and, as a natural consequence, to packings which may contain degenerate disks. In particular we study existence and uniqueness of domain-filling circle packings and circle agglomerations for several types of *normalizations*.

Moreover, we want to keep our proofs independent of Schramm's work and as elementary as possible. A fundamental ingredient of our existence proof is Sperner's Lemma (see [1] or [17] for example). This will be combined with induction, compactness arguments, continuous dependence of solutions on parameters, combinatorial constructions and some elementary geometry. Detailed investigations of possible degenerations play a crucial role.

Since already very simple examples show that for some normalizations the packings *must* contain degenerate circles with radius zero (which we call *dots*) we include this possibility from the very beginning and study *generalized* circle packings. On the other hand, trying to exclude degeneration whenever possible, we also provide specific conditions which guarantee the existence of classical circle packings involving only circles with positive radii.

In order to achieve our goals we have to overcome several obstacles:

- Since the boundary of general domains does not need to be a curve, we must work with *prime ends* as natural substitutes for boundary points.
- As mentioned above, we investigate *generalized circle packings* which may contain *disks* and *dots*. Working with dots requires special techniques, in particular when the domain to be filled is not Jordan.
- In order to apply induction, we sometimes remove one vertex from a complex K. The resulting complex K' is not necessarily a triangulation. We therefore enlarge the class of *admissible complexes*, which is traditionally used in circle packing, and work with *acceptable complexes*, consisting of more general connected planar graphs. This set is closed with respect to the removal of (appropriate) vertices.
- Though (or since) we try to keep proofs elementary, some of them are quite technical and longish. In particular a number of combinatorial problems require detailed constructions and careful case-by-case studies, which makes part of our presentation rather demanding (and perhaps boring) for readers.

The following theorem describes our most general existence result for *generalized* circle packings, involving disks and dots. It summarizes the main statements of Theorem 3.45, Theorem 4.39 and Theorem 5.11 as well as of Lemma 5.3 and Lemma 5.4.

Theorem 1.2 (Existence of generalized domain-filling circle packings)**.** *For every bounded, simply connected domain G, for every admissible complex K, and for each of the following normalizations there is a* generalized circle packing *$\mathcal{P}$ for K that fills G:*

Alpha-Beta-Gamma. *For $i = 1, 2, 3$ let v_i be a boundary vertex of K (the number of different v_i is called the* degree *of the normalization). For $i = 1, 2, 3$ let $P_i \in \mathcal{P}$ be the disk or dot that is associated with v_i, and let X_i be a prime end of G. Then P_i meets X_i (for a definition see p. 59).*

Alpha-Gamma. *Let a be a vertex of K and let $A \in G$ be a point. Let $c \neq a$ be a boundary vertex of K and let C be a prime end of G. Let $P_a, P_c \in \mathcal{P}$ be the disks or dots associated with a and b, respectively. Then the center of P_a is A and P_c meets C.*

Alpha-Beta. *Let $a \neq b$ be two vertices of K. If a is a boundary vertex, suppose that b is a neighbor of a. Let $A \in G$ be a point, and let Γ be a ray that emerges from A. Let $P_a, P_b \in \mathcal{P}$ be the disks or dots associated with a and b, respectively. Then the center of P_a is A and the center of P_b lies on Γ.*

Note that we implicitly assumed that the triples of boundary vertices v_1, v_2, v_3 and of prime ends X_1, X_2, X_3 are equi-oriented for the Alpha-Beta-Gamma normalization.

Existence and *uniqueness* of normalized *non-degenerate* domain-filling circle packings depend on the underlying complex and on specific properties of prime ends. In the following we do not try to present our most general results, but consider a somewhat simplified setting. In particular we assume that the complex K is a *strongly connected* triangulation of a disk, i.e., every boundary vertex of K has exactly two boundary vertices of K as neighbors. This is often a standing assumption in the circle packing literature.

A prime end X of a domain G is called *regular* if for every two disks $D_1, D_2 \subset G$ that touch X we have $D_1 \subset D_2$ or $D_2 \subset D_1$. A prime end X of a domain G is said to be an *inward spike* if it can be touched by two disjoint (open) disks (see Chapter 2 for a detailed explanation).

The next theorem summarizes typical existence and uniqueness statements for domain-filling circle packings with different normalizations.

Theorem 1.3 (Normalized domain-filling circle packings)**.** *For any bounded, simply connected domain G and any strongly connected combinatorial triangulation K of a disk there exists a circle packing $\mathcal{P}$ with complex K which fills G. Moreover, the existence and uniqueness of a* normalized *domain-filling circle packing $\mathcal{P}$ are guaranteed in the following cases (compare the normalization conditions of Theorem 1.2):*

(i) *Let the boundary vertices v_1, v_2, v_3 be pairwise different. If none of the prime ends X_1, X_2, X_3 of G is an inward spike, then there exists a circle packing $\mathcal{P}$ satisfying the Alpha-Beta-Gamma normalization. If the prime ends X_1, X_2, X_3 are regular, then $\mathcal{P}$ is uniquely determined.*

(ii) *Let a be an interior vertex of K. If the prime end C is not an inward spike then there exists a circle packing $\mathcal{P}$ satisfying the Alpha-Gamma normalization. If the prime end C is regular, then $\mathcal{P}$ is uniquely determined.*

(iii) *Let a be an interior vertex of K and denote by D_A the* maximal disk *in G with center at A. If $\partial D_A \cap \Gamma \in G$, then there exists a circle packing $\mathcal{P}$ satisfying the Alpha-Beta normalization.*

Somewhat surprisingly, the Alpha-Beta normalization does in general *not guarantee uniqueness* of the circle packing $\mathcal{P}$, even for smooth Jordan domains. Counterexamples are given in Section 5.2 and AppendixA.2.

In the next table we describe in more detail which conditions on the domain G, the involved prime ends, and the complex K ensure the existence of a (*non-degenerate*) domain-filling *circle packing* subject to different normalizations. For the Alpha-Gamma and Alpha-Beta normalizations we also admit that the disk D_a with fixed center A is a boundary disk of $\mathcal{P}$, i.e. $a \in \partial V$. Recall that D_A denotes the maximal disk which is contained in G and centered at the point A in G. Further explanations of the properties are given below (for details we refer to the corresponding sections).

The answers "yes", or "yes if" tell us that a corresponding circle packing always exists. We point out that these entries can be interpreted in an even stronger sense. Any *generalized circle packing* which solves the corresponding problem is automatically *non-degenerate*.

If the answer is "no" there are specific *counterexamples* for which no circle packing solution exists, but we do not claim that problems of this type do *never* have a solution in the class of circle packings.

Existence of Circle Packings

		G has no inward spikes, critical PE are untouchable	**G has no inward spikes**	**G has no inward spikes at critical PE**	**G has inward spikes at critical PE**
Alpha-Beta-Gamma	**degree = 3**	yes (Lemma 3.11)	yes (Lemma 3.12)	yes if K strongly connected (Lemma 3.13)	no
	degree < 3	yes (Lemma 3.11)	no	no	no
Alpha-Gamma	$\mathbf{a \in intV}$, $G(A, C)$ **is not dubious**	yes (Lemma 4.10)	yes (Lemma 4.10)	yes if K strongly connected (Corollary 4.9)	yes if K strongly connected (Corollary 4.9)
	$\mathbf{a \in \partial V}$, $G(A, C)$ **is not dubious**	yes (Lemma 4.4)	yes (Lemma 4.4)	yes if K strongly connected (Lemma 4.4)	yes if K strongly connected (Lemma 4.4)
Alpha-Beta	$\mathbf{a \in intV}$, $\mathbf{\partial D_A \cap \Gamma}$ **in G**	yes (Lemma 5.12)	yes (Lemma 5.12)	yes if K strongly connected (Lemma 5.12)	yes if K strongly connected (Lemma 5.12)
	$\mathbf{a \in \partial V}$, $\mathbf{\partial D_A \cap \Gamma}$ **in G**	yes (Lemma 5.5)	yes (Lemma 5.5)	yes if K strongly connected (Theorem 5.1)	yes if K strongly connected (Theorem 5.1)

By *critical PE* we mean such prime ends (PE) that are relevant in the corresponding normalization. So for the Alpha-Beta-Gamma normalization the critical prime ends are X_1, X_2 and X_3, and for the Alpha-Gamma normalization it is C. For the Alpha-Beta normalization it is the prime end Y that is associated by Γ with the first intersection point of Γ with ∂G. In order to explain the meaning of "$G(A, C)$ is *not dubious*", let D_A be the maximal disk in G with center at A. Then the *pinned domain* $G(A, C)$ is said to be *dubious* if there exists a disk or dot P_C in $G \setminus D_A$ which meets the prime end C such that $\partial D_A \cap \partial P_C \cap \partial G \neq \emptyset$ (see Figure 1.2). Note that this property can be checked a priory without knowing the domain filling packing.

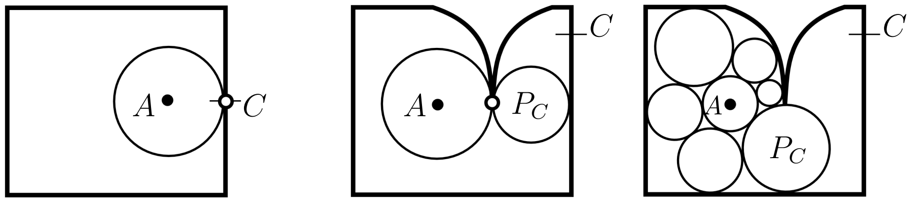


Fig. 1.2.: Generalized circle packings for dubious pinned domains $G(A, C)$

In the last table we summarize *uniqueness results* for the largest class of normalized domain-filling *generalized circle agglomerations* and their subset of *generalized circle packings*.

Conditions Guaranteeing Uniqueness

Normalization	**Generalized Circle Agglomeration**	**Generalized Circle Packing**
Alpha-Beta-Gamma	X_1, X_2 and X_3 are regular (Theorem 3.31)	X_1, X_2 and X_3 are regular (Theorem 3.31)
Alpha-Gamma	not studied	C is regular (Theorem 4.33)
Boundary Alpha-Beta	not studied	Y is no inward spike (Lemma 5.3, Lemma 5.4)
Interior Alpha-Beta	not studied	in general not unique even for smooth domains (Theorem 5.8)

In those cases which were not studied for circle agglomerations the normalizations often do not make sense or the corresponding problems are not well-posed.

Comparing the above *uniqueness* results for *generalized* circle packings (agglomerations), where degenerate circles are admitted, with the non-degenerate case, reveals that these results are *not influenced by the presence of dots.*

On the other hand, existence fails in general if we restrict ourselves to non-degenerate packings. So admitting circles with radius zero has only positive but no negative impact at all. Therefore we strongly believe that generalized circle packings (agglomerations) are the natural choice for filling general bounded, simply connected domains.

2. Basic Notation and Concepts

In order to keep this work self contained and to avoid misunderstandings, this work contains a vast amount of definitions of all kinds. Especially this chapter, but also the beginning of every new one, develops step by step the vocabulary needed. Some definitions, phrases, concepts, etc. are common knowledge, some are slightly different to the literature and some are completely new.

We point out that a major part of this chapter has already been developed within the papers [19], [20] and [37]. Nevertheless, we do not explicitly refer every definition to its corresponding source(s), but only the key ideas of the theorems, lemmas and their proofs. A reason for doing so is the fact that over time some minor or even major details have changed. Thus, there is no guarantee that the phrasing or meaning of the papers matches the version presented here.

An overview of the most important terms can be found in the glossary. Please don't get lost in the labyrinth of notation.

2.1. Circle Packings

2.1.1. Combinatorics

Let $p_0, p_1, \ldots, p_n$ be $n+1$ affinely independent points in $\mathbb{R}^n$, i.e., $\{p_i - p_0,\}_{i=1}^n$ is a set of linearly independent vectors. Let σ_1 be the convex hull of all those points while σ_2 shall be the convex hull of a subset of them. Then σ_1 is denoted a *simplex of dimension* n (short n*-simplex*) and σ_2 is a *facet* of σ_1. Clearly facets of simplexes are simplexes themselves (maybe of lower dimension).

Definition 2.1. A *simplicial complex* K is a set of simplexes so that (i) any facet of a simplex from K is again in K, and (ii) the intersection of any two simplexes from K is either a facet of both or empty.

The *carrier* of a simplicial complex is the union of all its simplexes. Those simplexes of a simplicial complex K which themselves are no facets of any other (larger) simplex of K are called *faces.* Note that some authors may use the two terms "facet" and "face" with interchanged meaning.

The *dimension* of a simplicial complex equals the maximal dimension of its simplexes (maybe it is infinite), so for example a *simplicial* 2*-complex* K contains only 2-simplexes (triangles), 1-simplexes (line segments called *edges*) or 0-simplexes (points called *vertices*).

Definition 2.2. An *admissible complex* K is a simplicial 2-complex with the following properties (i)-(viii) (see again Figure 1.1, p.11).

(i) The carrier of K is simply connected.

(ii) Every edge of K belongs to either one or two triangles (the former are called *boundary edges*, the latter *interior edges*).

(iii) Every vertex v of K belongs to at most finitely many triangles, and these form an ordered chain in which each triangle shares an edge from v with the next one.

(iv) Every vertex of K belongs either to no boundary edge or to exactly two boundary edges (the former are called *interior vertices*, the latter *boundary vertices*).

(v) Any two triangles of K are either disjoint, share a single vertex or share a single edge.

(vi) Let t_1 and t_2 be two triangles of K sharing the edge e with the two vertices v and w. Let e be walked through in the direction from v to w when we walk along the boundary of t_1 with positive orientation. Then e is walked through in the opposite direction (i.e., from w to v) when we walk along the boundary of t_2 with positive orientation.

(vii) The number of triangles, edges and vertices within K is finite.

(viii) The set of boundary edges (hence of boundary vertices) is not empty.

The set of all admissible complexes is denoted $\mathcal{K}$. The members of $\mathcal{K}$ having at most n vertices form the class $\mathcal{K}_n$. Clearly, $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ and $\mathcal{K}_1 = \mathcal{K}_2 = \emptyset$. Simply speaking of a *complex* we always mean a simplicial 2-complex of the class $\mathcal{K}$.

Due to Lemma 3.2 of [31] we can interpret every admissible complex $K \in \mathcal{K}$ as finite triangulation of the closure of a Jordan domain (a topological closed disk, definition see p.30). In this sense we may also call K a finite *combinatorial closed disk*.

Note that, in order to define admissible complexes, one could also use so called *abstract* simplicial complexes instead of our (*geometrically*) simplicial complexes. To keep things simple we won't do this, in particular since we are only interested in the very specific situation of circles lying in the complex plane $\mathbb{C}$.

Since (ii) and (iii) of Definition 2.2 imply that all faces of an admissible complex K must be triangles (and vice versa) we may and will use the term *faces* instead of triangles; we will even redefine the meaning of a triangle of K on the next page.

We denote the sets of vertices, edges and faces of K by V, E, F, respectively, and we write $K = K(V, E, F)$. The edge adjacent to the vertices u and v is denoted by $e(u, v)$ or $\langle u, v\rangle$, where the first version stands for the non-oriented edge while the second means the oriented edge from u to v. Similarly, a face of K with vertices u, v, w is written as $f(u, v, w)$ (non-oriented) and $\langle u, v, w\rangle$ (oriented).

Two vertices u and v are said to be *neighbors* if they are connected by an edge $e(u, v)$ in E. The number of neighbors of v is the *degree* of v. With $N(v)$ we denote the set of all neighbors of v in K. By property (iii) of Definition 2.2 this set is endowed with a cyclic (counterclockwise) ordering so that for $w_1, w_2 \in N(v)$ definitions like $\{w \in N(v) :$

$w_1 \prec w \preceq w_2\}$ make sense. Let the degree of v be n. If v is a boundary vertex, then it is well defined to speak of an ordered set of neighbors $N(v) = \{w_1, w_2, \ldots, w_n\}$. If v is an interior vertex, then this ordering depends only on the choice of the starting vertex w_1.

Similar to $N(v)$ we denote the set of edges and faces containing v by $E(v)$ and $F(v)$, respectively. Since both sets $E(v)$ and $F(v)$ correspond to $N(v)$ they are ordered in the same sense. We use this behavior to define chains of vertices, edges or faces.

Speaking of a *chain*, we mean a finite sequence $(c_1, \ldots, c_n)$ of vertices, edges or faces so that neighboring elements c_j and c_{j+1} are adjacent to a common edge (if the c_j are vertices or faces) or a common vertex (if the c_j are edges), respectively.

While some readers may know the union of $N(v)$, $E(v)$ and $F(v)$ as the *star* of v, we are more interested in its closure. Let $B(v)$ denote the smallest admissible sub-complex of K which contains the vertex v and all its neighbors. If v is an interior vertex, then $B(v)$ is said to be the *(combinatorial) flower* of v (see Figure 2.1, middle). If v is a boundary vertex, then we speak of an *incomplete flower* (see Figure 2.1, left).

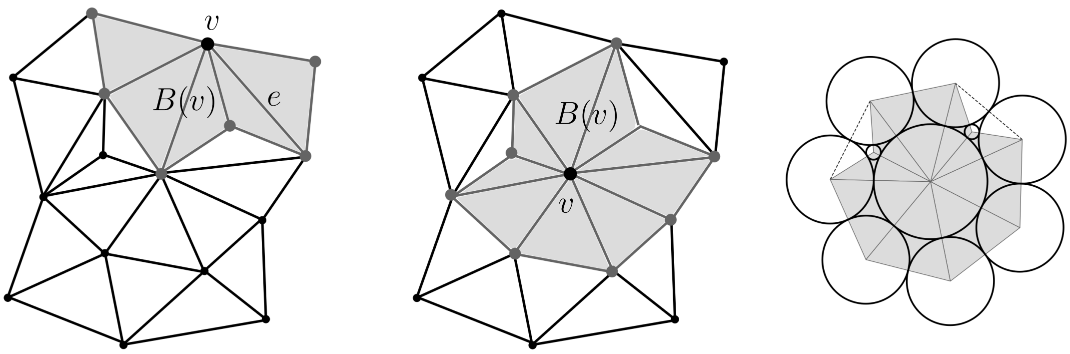


Fig. 2.1.: Incomplete flower $B(v)$ (left); complete flower $B(v)$ and corresponding disks (right)

Note that $B(v)$ does not need to contain all edges which connect neighbors of v (see Figure 2.1). The *extended (incomplete) flower* $B^*(v)$ of v is the smallest admissible sub-complex of K containing $B(v)$ and all edges $e(u, w)$ between vertices $u, w \in B(v)$. In the following we investigate under which conditions we have $B^*(v) = B(v)$.

If $e(u, v)$, $e(v, w)$ and $e(w, u)$ are pairwise different edges of K, then we say that u, v, w form a *triangle* $\triangle = \triangle(u, v, w)$. Please be not confused with the term "triangle" we used in Definition 2.2. Still, any face $f(u, v, w)$ of K is a triangle, but now maybe there are triangles which are no faces.

Definition 2.3. If $e(u, v)$, $e(v, w)$ and $e(w, u)$ form a triangle $\triangle(u, v, w)$ of an admissible complex $K \in \mathcal{K}$ but not a face, and if at least one of these edges is an interior edge, then we call $\triangle$ *reducible*. Moreover, K is said to be *reducible* if it contains a reducible triangle.

If an admissible complex K is not reducible, then we call it *irreducible*. The set of irreducible complexes in $\mathcal{K}$ is denoted $\mathcal{K}^i$.

We call K *boundary irreducible* if all of its reducible triangles belong to the interior of the complex, i.e., they do not contain any boundary edge. In particular, an irreducible admissible complex is also boundary irreducible.

The additional requirement that at least one edge of a reducible triangle must be an interior edge of K excludes one and only one case: Let K be an admissible complex with non-empty interior and exactly three boundary vertices a, b and c. Then $\triangle(a, b, c)$ is a triangle but not a face of K that we do not want to be reducible.

This is motivated by the fact that otherwise every such *tri-complex* $K = T(a, b, c)$ (introduced in more detail in Chapter 3) would be reducible. Moreover, up to this exception, the *inner reduction* (as defined below) of a tri-complex always yields again a tri-complex, what becomes very important later on.

If we are interested in some interior vertex v of K, as in the following lemma, we need not to worry about this pitfall since then v has automatically only interior edges (see also [20] Lemma 1).

Lemma 2.1 (Irreducible Extended Flower)**.** *Let $K \in \mathcal{K}$ be irreducible. Let v be an interior vertex of K and let $N(v) = \{v_1, \dots, v_n\}$ be the ordered set of its neighbors. Then we have $B^*(v) = B(v)$. Moreover, v_j is a neighbor of v_k if and only if $|j - k| = 1$ $(\operatorname{mod} n)$.*

Proof. Assume that $B(v)^* \neq B(v)$. By the definitions of $B^*(v)$ and $B(v)$ there is an edge e between two vertices of $B(v)$ such that e is not in $B(v)$ but in $B^*(v)$. All the vertices within $B(v)$ are v and $N(v) = \{v_1, \dots, v_n\}$. All the edges within $B(v)$ are $e(v, v_j)$ and $e(v_j, v_{j+1})$ for $j = 1, \dots, n$ (with $v_{n+1} := v_1$). Since, by definition of $B(v)$, every edge containing v is in $B(v)$, there exist two vertices $v_j \neq v_k$ with $|j - k| > 1$ $(\operatorname{mod} n)$ that are connected by an edge e in $B^*(v)$. This implies that the (geometric) interior of the triangle $\triangle(v, v_j, v_k)$ contains one of the vertices v_{j+1} or v_{k+1} so that K would be reducible (see Figure 2.2). $\square$

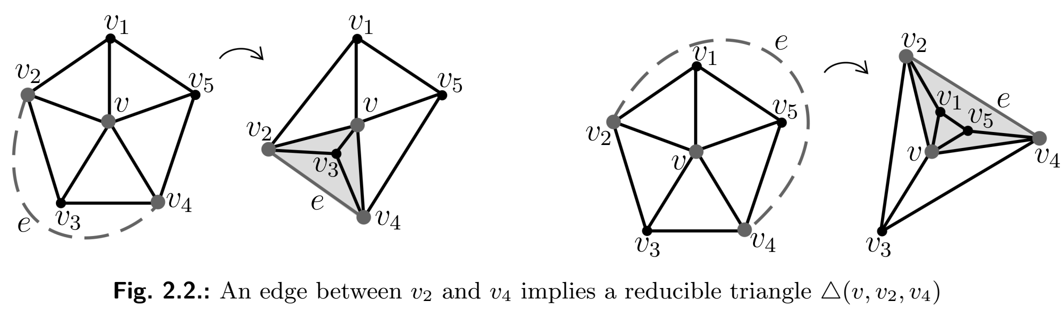


Fig. 2.2.: An edge between v_2 and v_4 implies a reducible triangle $\triangle(v, v_2, v_4)$

The following operations remove either the exterior or the interior of a reducible triangle $\triangle = \triangle(u, v, w)$ in an admissible complex $K \in \mathcal{K}$.

Definition 2.4. The *inner reduction* $\varrho_+(K, \triangle)$ with respect to $\triangle$ is the unique admissible complex obtained from K by removing the interior of $\triangle$ and making $f(u, v, w)$ a face of $\varrho_+(K, \triangle)$. The *outer reduction* $\varrho_-(K, \triangle)$ of K with respect to $\triangle$ is the complex formed by the union of $\triangle$ with $K \setminus \varrho_+(K, \triangle)$ (see Figure 2.3).

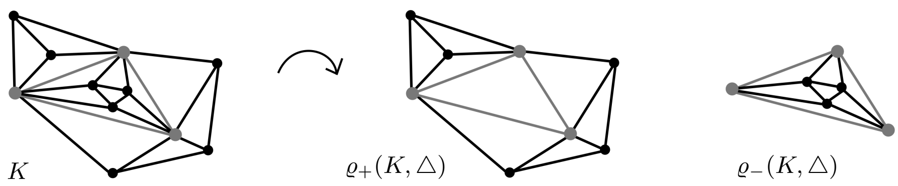


Fig. 2.3.: Inner and outer reduction of a complex K with respect to $\triangle$

For reducible K both types of reduction ϱ_+ and ϱ_- remove at least one vertex, so we have

$$\varrho_+ : \mathcal{K}_n \setminus \mathcal{K}^i \to \mathcal{K}_{n-1}, \quad \varrho_- : \mathcal{K}_n \setminus \mathcal{K}^i \to \mathcal{K}_{n-1}.$$

We point out that neither the inner nor the outer reduction needs to be irreducible. Nevertheless, applying the inner reduction as many times as possible eventually yields an irreducible admissible complex $\sigma(K)$ called the *skeleton* of K. By Lemma 2.1 the interior of this *complete reduction* $\varrho_+(K) = \sigma(K)$ of K behaves locally like a flower, i.e., $B^*(v) = B(v)$ for all v in $\sigma(K)$.

Lemma 2.2 (Skeleton Uniqueness)**.** *The result of the complete reduction $\varrho_+(K) = \sigma(K)$ is independent of the order in which the reductions are performed.*

Proof. Assume contrary there would be two different skeletons $S_1 \neq S_2$ for the same admissible complex K. Then w.l.o.g. S_2 contains a vertex u, an edge e or a face f that does not lie in S_1. Thus, there is a triangle $\triangle(v_1, v_2, v_3)$ in K that is a face in S_1 so that u, e or f lies in the interior of $\triangle(v_1, v_2, v_3)$ within K. Since the inner reduction always removes edges and faces together with at least one of its vertices, we may and will assume the existence of the vertex u.

Clearly, S_2 cannot contain all three vertices v_1, v_2 and v_3 otherwise it would be reducible (since u lies in S_2). Let v_1 do not lie in S_2. Thus, there is a triangle $\triangle(u_1, u_2, u_3)$ in K that is a face in S_2 so that v_1 lies in the interior of $\triangle(u_1, u_2, u_3)$ within K.

We look at K (see Figure 2.4). Since $\triangle(u_1, u_2, u_3)$ encloses v_1, we conclude that the whole triangle $\triangle(v_1, v_2, v_3)$ lies in the closure of $\triangle(u_1, u_2, u_3)$, i.e., v_2 either lies in the interior of $\triangle(u_1, u_2, u_3)$ or $v_2 \in \{u_1, u_2, u_3\}$. The same goes for v_3. Thus, u lies in the interior of $\triangle(u_1, u_2, u_3)$ since it is contained in the interior of $\triangle(v_1, v_2, v_3)$.

Now, all the vertices u, u_1, u_2 and u_3 lie in S_2. Hence, the triangle $\triangle(u_1, u_2, u_3)$ cannot be a face of S_2, which is a contradiction. $\square$

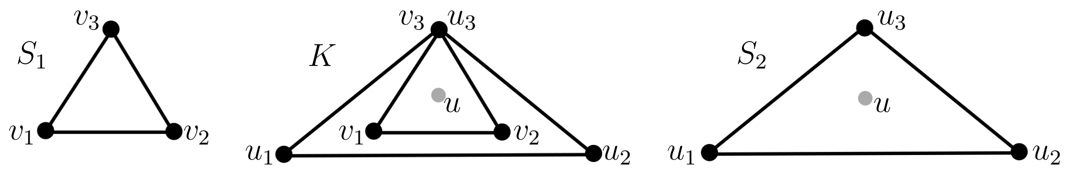


Fig. 2.4.: Construction for the proof of Lemma 2.2

On the boundary the situation is quite similar. A *boundary face* is a face containing a boundary edge. The sets of boundary edges, vertices and faces are denoted ∂E, ∂V and ∂F, respectively. There is a natural cyclic ordering of boundary edges, the *boundary chain*, corresponding to the orientation of the boundary of the triangulated surface. With respect to the boundary chain, any boundary vertex (boundary face) has a *precursor* and a *successor* that are well-defined and induce a cyclic ordering of all boundary vertices (boundary faces). Nevertheless, a boundary vertex can be adjacent to many other boundary vertices, and an edge that connects two boundary vertices does not need to be a boundary edge (as vertex v and edge e in Figure 2.1, left).

Definition 2.5. An admissible complex K is *strongly connected* if every boundary vertex has exactly two other boundary vertices as neighbors. The class of all strongly connected complexes is $\mathcal{K}^0$.

In order to understand the characteristics of a strongly connected admissible complex K, we introduce the *kernel* of K. On the first glance these two things do not seem to be very related but it turns out that the kernel provides a characterization for strongly connectedness.

To start with, let $K \in \mathcal{K}$ have a distinguished interior vertex $a \in V$, the *alpha vertex.* A vertex $v \in V$ is called *accessible from a* if there is a chain of vertices $(v, v_1, \ldots, v_n, v_{n+1} = a)$ so that $v_1, \ldots, v_n \in V$ are interior vertices (maybe $n = 0$). The vertex a itself is accessible in a trivial sense. The set of all accessible vertices of K is denoted by V^*, the set of all edges $e(u, v) \in E$ with $u, v \in V^*$ by E^* and the set of all faces $f(u, v, w) \in F$ with $u, v, w \in V^*$ by F^*.

Definition 2.6. The *kernel* K^* of K is defined as the simplicial 2-complex arising from V^*, E^* and F^*, i.e., $K^*(V^*, E^*, F^*) \subset K(V, E, F)$.

Lemma 2.3 (Kernel)**.** *With respect to some alpha-vertex, let $K^*(V^*, E^*, F^*)$ be the kernel of $K(V, E, F)$. Then K^* is a strongly connected admissible complex with $\partial V^* = \partial V \cap V^*$.*

Proof. We first investigate what happens with interior and boundary vertices of the original complex K, then we show that K^* fulfills the properties (i)-(viii) of Definition 2.2, eventually we prove that K^* is strongly connected. The property $\partial V^* = \partial V \cap V^*$ is shown en passant.

Case A. Let $v \in (\mathrm{int}V \cap V^*)$. Since v is accessible and an interior vertex of K, there is a chain $C = (v, \ldots, a)$ connecting v with a that only contains interior vertices. So every neighbor u of v in K is accessible via the chain $C_u = (u, v, \ldots, a)$. Hence, the flower $B(v)$ of v in K must be a sub-complex of K^*. We write $B(v) \subset K^*$ (see Figure 2.5, left).

Case B. Let $v \in (\partial V \cap V^*)$. Let $N(v) = \{v_1, \ldots, v_n\}$ be the ordered set of all neighbors of v in K. Since v is accessible and a boundary vertex of K, there is a neighbor v_j of v in K so that v_j is an interior vertex of K connecting v with a by a chain $C = (v, v_j, \ldots, a)$. According to Case A every neighbor of v_j is contained in K^*. If such a neighbor is again an interior vertex of K, then all its neighbors also lie in K^*, and so forth, until we reach a boundary vertex of K. Hence, there is an ordered subset $\{v_i, v_{i+1}, \ldots, v_j, \ldots, v_{k-1}, v_k\} \subset N(v)$ of neighbors of v so that $v_i, \ldots, v_k \in (\mathrm{int}V \cap V^*)$ while $v_{i-1}, v_{k+1} \in (\partial V \cap V^*)$. Clearly, we have $1 < i \leq j \leq k < n$, otherwise v_i or v_k would be a boundary vertex in K. We collect all these vertices within the set $N_j(v) = \{v_{i-1}, v_i, \ldots, v_k, v_{k+1}\}$.

Let $u \notin N_j(v)$ be a neighbor of v that does not lie in $N_j(v)$. We show that $u \notin V^*$. Assume contrary that u is accessible. Then there is a chain $C_u = (u, \ldots, a)$ connecting u with a. The concatenation of C_u with the inversion of the chain $C = (v, v_j, \ldots, a)$ yields a closed chain $S = (v, u, \ldots, a, \ldots, v_j, v)$. The only vertices in S that are boundary vertices of K are v and u. So especially v_{i-1} and v_{k+1} do not lie in S. According to construction, this implies that v_{i-1} or v_{k+1} must be enclosed by S (see Figure 2.5, middle), which is a contradiction to $v_{i-1}, v_{k+1} \in \partial V$. Hence, $u \notin V^*$.

We showed that out of all vertices in $N(v)$ exactly those in $N_j(v)$ are contained in K^*. One consequence of this is that v has exactly two other boundary vertices of K as neighbors in K^*. Another implication is that the (different) edges $e(v, v_{i-1})$ and $e(v, v_{k+1})$ are boundary edges in K^* while the remaining edges $e(v, v_m)$ with $i \leq m \leq k$ are interior edges of K^*.

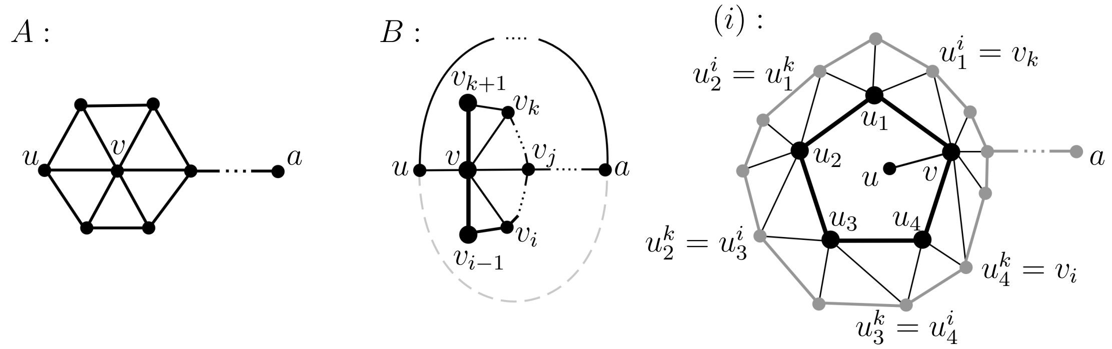


Fig. 2.5.: Constructions for Case A, Case B and Property (i) of the proof of Lemma 2.3

Proving that K^* is an admissible complex is now an easy task. We verify the properties of Definition 2.2 one by one, but we move the proof of (i) to the end.

Proof of (ii). We show that every edge of K^* is contained in at least one triangle; more than 2 is impossible since K is admissible. To do so, let $e(u, v) \in E^*$ be an edge of K^*. If u or v is an interior vertex of K, then e is an interior edge of K and both of its faces are contained in K^* by Case A. If $u, v \in \partial V$, then we use Case B to show that w.l.o.g. $u = v_{i-1}$. So the face $f(v, u, v_i)$ must be contained in K^*.

Proof of (iii). For an interior vertex v the property (iii) is fulfilled since $B(v) \subset K^*$ by Case A. For a boundary vertex v property (iii) is fulfilled by the set $N_j(v)$ together with the fact that out of all vertices in $N(v)$ exactly those in $N_j(v)$ are contained in K^* (Case B).

Proof of (iv). A vertex $v \in \mathrm{int}V$ cannot belong to any boundary edge of K^* by Case A. If $v \in \partial V$ is a boundary vertex of K, then it is contained in exactly two boundary edges of K^* by Case B. This also proves $\partial V^* = \partial V \cap V^*$.

Proof of (v)-(vii). Since K is an admissible complex, those properties are trivially fulfilled for K^*.

Proof of (viii). At least one boundary vertex of K must be contained in K^* (there are even at least three), and since every such vertex has exactly two boundary edges in K^*, the boundary of K^* cannot be empty.

Proof of (i). Let v be a boundary vertex of K^* and let u_1, u_n be its two boundary neighbors within K^*, i.e., $v, u_1, u_n \in \partial V^*$. Then u_1 has exactly one boundary neighbor within K^* that is different to v, say $u_2 \in \partial V^*$. Now again, the vertex u_2 has exactly one new boundary neighbor u_3, whose next boundary neighbor is u_4, and so on, until we arrive at u_n and eventually back at v. The edges $e(v, u_1), e(u_1, u_2), \dots, e(u_n, v) \in \partial E^*$ form a Jordan curve Γ (for a definition of Jordan curve see p.30).

According to Case B, either the interior or the exterior of Γ must be disjoint to K^*. In order to exclude the former case, we start at the interior vertex $v_i \in V^*$ and follow its interior neighbors $v_i, \dots, v_k \in V^*$ – the interior neighbors of v within K^* as constructed in Case B. From v_k we follow the interior neighbors of u_1 within K^*, which are $v_k = u_1^i, \dots, u_1^k \in V^*$. Then we walk through $u_1^k = u_2^i, \dots, u_2^k \in V^*$ as interior neighbors of u_2, and so on, until the interior neighbors $u_{n-1}^k = u_n^i, \dots, u_n^k = v_i$ of u_n within K^* lead us back to v_i (see Figure 2.5, right). Since the associated edge chain cannot enclose any boundary vertex, we conclude that K^* is disjoint to the exterior of Γ.

For the same reason we have $V^* = \{v, u_1, \dots, u_n\}$ since any additional boundary vertex would always be enclosed by the constructed edge chain. Thus, by Case A, the carrier of K^* is exactly the closure of the domain bounded by Γ, hence, simply connected.

Strongly connectedness. Clearly, every boundary vertex $v \in \partial V^*$ has at least two neighbors in K^* that are also boundary vertices of K^*. From Case B we further know that v has exactly two neighbors in K^* that are boundary vertices of K. Since $\partial V^* \subset \partial V$, we conclude that v has at most (thus exactly) two neighbors in K^* that are boundary vertices of K^*. Hence, K^* is a strongly connected admissible complex. □

The Lemma 2.3 and especially the constructions within the Case B allows us to characterize strongly connectedness as stated in the following result (see also [19] Lemma 1).

Note that in terms of graph theory a connected graph (with more than k vertices) is *k-connected* if it remains connected whenever fewer than k arbitrary vertices (and their edges) are removed.

Lemma 2.4 (Strongly Connectedness)**.** *The following statements (i)-(iv) are equivalent for every $K \in \mathcal{K} \setminus \mathcal{K}_3$.*

(i) $K = K^*$.

(ii) *K is strongly connected.*

(iii) *Every boundary vertex of K has an interior neighbor, and every two interior vertices can be connected by a chain of interior vertices.*

(iv) *K (interpreted as a graph) is 3-connected.*

Proof. We first show that (i) implies (ii) and (iii), then that (ii) and (iii) both imply (i), and finally that (iv) is equivalent to (ii).

To start with, we notice that in all three cases (i)-(iii) the admissible complex K must have an interior vertex. In the last case it is directly demanded, in the first case indirectly since otherwise the kernel K^* would not be defined. For Case (ii) it is easy to check that the only strongly connected complex without any interior vertex must be the unique complex in $\mathcal{K}_3$, which is excluded. Thus, K has at least 4 vertices and especially an (arbitrary) alpha vertex $a \in \mathrm{int}V \neq \emptyset$.

Assume $K = K^*$. Then K is strongly connected and every boundary vertex of K has an interior neighbor by Case B from the proof of Lemma 2.3. Moreover, since every vertex of K is accessible, we can connect any two interior vertices u and v by connecting u with a and then a with v. So (i) implies (ii) and (iii).

Assume that K is strongly connected. Then every boundary vertex $v \in \partial V$ of K has exactly two boundary neighbors $u_1, u_n \in \partial V$ within K. At least one boundary vertex of K must be in K^*, too, say v. According to Case B from the proof of Lemma 2.3 v has exactly two boundary neighbors $w_1, w_n \in \partial V^*$ within K^*. Since $\partial V^* \subset \partial V$, we conclude $w_1 = u_1$ and $w_n = u_n$. Repeating this argumentation for u_1 we see that its only other boundary neighbor $u_2 \in \partial V$ within K must also be its boundary neighbor $w_2 = u_2$ with $w_2 \in \partial V^*$ in K^*. Proceeding up to $w_{n-1} = u_{n-1}$ we get $\partial V^* = \partial V$. Since K^* and K are simply connected, we get $V^* = V$ by Case A of the proof of Lemma 2.3. Hence, $K^* = K$ by the definition of the kernel K^*. So (ii) implies (i).

Assume that every boundary vertex of K has an interior neighbor and that every two interior vertices can be connected by a chain of interior vertices. Then we can especially connected every vertex $v \in V$ with a by a chain of interior vertices. Hence, every vertex of K is accessible, i.e., $V = V^*$ and $K = K^*$. So (iii) implies (i).

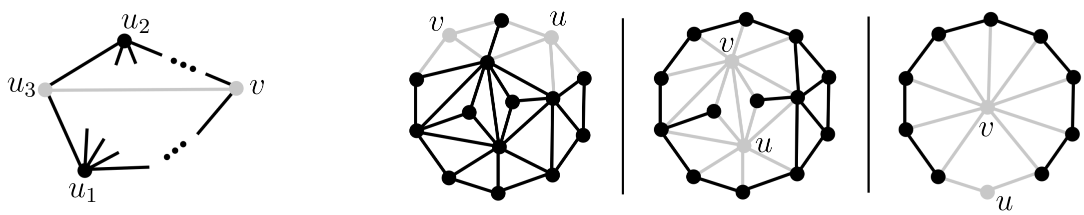


Fig. 2.6.: Constructions for the 3-connected part of the proof of Lemma 2.4

Assume that K is 3-connected. Assume further that K is not strongly connected, i.e., there is a boundary vertex $v \in \partial V$ that has at least three other boundary vertices $u_1, u_2, u_3 \in \partial V$ as neighbors. Following the boundary chain of K with positive orientation, we walk through those vertices w.l.o.g. with the ordering u_1, v, u_2, u_3 (see Figure 2.6, left). If we remove v and u_3 as well as all their edges, then u_2 cannot be connected with u_1 anymore, which is a contradiction to K being 3-connected. Hence, K must be strongly connected.

Assume that K is strongly connected. Then we can especially rely on (iii). Let $u, v \in V$ be two vertices of K which we want to remove, and let $\widetilde{K}$ be the reduced complex. We distinguish whether u, v are boundary or interior vertices (see Figure 2.6, right).

If u, v are both boundary vertices, then all the remaining boundary vertices of K still have an interior neighbor in $\widetilde{K}$. Since the interior of K is connected and since no interior vertex of K was removed, the whole $\widetilde{K}$ is connected.

If u, v are both interior vertices, then all the in $\widetilde{K}$ remaining interior vertices are still connected and at least one of them has a boundary neighbor because u and v alone cannot enclose any other vertex(es) in K. Since the boundary chain stays intact, all the vertices of $\widetilde{K}$ are connected.

This leaves the case $u \in \partial V$ and $v \in \mathrm{int}V$. By removing only one interior (boundary) vertex the remaining interior (boundary chain) stays of course connected. Since every boundary vertex has an interior neighbor, $\widetilde{K}$ could only fall apart if all the boundary vertices in $\partial V \setminus \{u\}$ would have v as one and only interior neighbor. But this implies that v would be the one and only interior vertex of K at all and therefore $\widetilde{K}$ would be the connected boundary chain of K without u. So in all cases $\widetilde{K}$ is connected. Hence, K is 3-connected. □

For strongly connected admissible complexes $K \in \mathcal{K}^0$ we can now extend the idea of Lemma 2.1 to the boundary of K.

Lemma 2.5. *Let $K \in (\mathcal{K} \setminus \mathcal{K}_3)$ be irreducible and strongly connected. Let v be a vertex of K. Then $B^*(v) = B(v)$ holds true for all $v \in V$ if and only if K has at least 4 boundary vertices (that is $|\partial V| \geq 4$).*

Proof. Since Lemma 2.1 covers the case of an interior vertex, let v be a boundary vertex of K. Let $a \neq b$ be two boundary neighbors of v in K. Since K is strongly connected, v has no other boundary neighbors.

Assume $|\partial V| = 3$ (see Figure 2.7, left). If K would be the one and only complex in $\mathcal{K}_3$, then we trivially would have $B^*(v) = B(v)$. Since this is excluded, there must be an interior vertex w of K which forms together with a and b the face $f = f(a, b, w)$ of K. On the one side, $B^*(v)$ contains f since $e(a, b)$ is an edge of f and $a, b \in N(v)$ are neighbors of v. On the other side, f is not contained in $B(v)$ since every face of $B(v)$ must be associated with the vertex v which is not part of f. Hence, $B^*(v) \neq B(v)$.

Assume $|\partial V| \geq 4$ (see Figure 2.7, right). We embed K in a larger complex by adding a new vertex u, the edges $e(u, v)$, $e(u, a)$ and $e(u, b)$, and the faces $f(u, v, a)$ and $f(u, v, b)$.

The resulting admissible complex R is (due to it's construction) strongly connected. Since K is strongly connected and $|\partial V| \neq 3$, the vertices a and b do not have a common edge neither in K nor in R. Therefore, R is irreducible since K is irreducible. Now v is an inner vertex of R, hence, by Lemma 2.1, the flowers $B(v)$ and $B^*(v)$ coincide in R. Returning to K, we keep $B(v) = B^*(v)$. $\square$

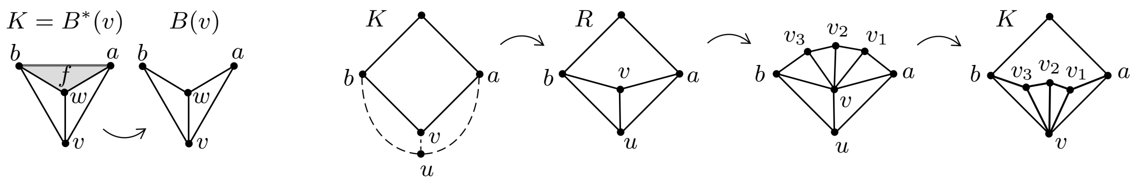


Fig. 2.7.: The structure of neighbors of v in K for $|\partial V| = 3$ (left) and $|\partial V| \geq 4$ (right)

Since in the following statement the vertices v_1 and v_n can not be neighbors, its proof follows directly from Lemma 2.5 (see also [20] Lemma 2).

Lemma 2.6. *Let $K \in (\mathcal{K} \setminus \mathcal{K}_3)$ be irreducible and strongly connected with at least 4 boundary vertices. Let v be a boundary vertex of K and let $N(v) = \{v_1, \dots, v_n\}$ be the ordered set of its neighbors. Then v_i and v_j are neighbors in K if and only if $|i - j| = 1$.*

The case where K has exactly three boundary vertices is an exception. Here we need the periodic continuation of $v_{n+1} := v_1$ also for boundary vertices, as shown next.

Lemma 2.7. *Let $K \in \mathcal{K}$ be irreducible and strongly connected with exactly 3 boundary vertices. Let v be any vertex of K and let $N(v) = \{v_1, \dots, v_n\}$ be the ordered set of its neighbors. Then v_i and v_j are neighbors if and only if $|i - j| = 1 \pmod n$.*

Proof. For every interior vertex of K the Lemma 2.1 applies and we are done. So let v be one of the three boundary vertices of $\partial V = \{v, a, b\}$. Since the case $K \in \mathcal{K}_3$ is trivial, we may and will assume that there is an interior vertex w of K. Say w forms the face $f(a, b, w)$. We remove the edge $e(a, b)$ as well as the face $f(a, b, w)$ from K and instead we add the new vertex u, the three edges $e(u, a)$, $e(u, b)$ and $e(u, w)$, and the two faces $f(a, u, w)$ and $f(u, b, w)$ (see Figure 2.8). Due to it's construction, the resulting admissible complex R is strongly connected, irreducible and has 4 boundary vertices. For $N(v) = \{v_1, v_2, \dots, v_n\}$ the Lemma 2.6 provides that v_i and v_j are neighbors in R if and only if $|i - j| = 1$. When returning from R to K, this is only altered by the fact that we have to identify v_1 with v_{n+1} since now $a = v_1$ neighbors $b = v_n$. $\square$

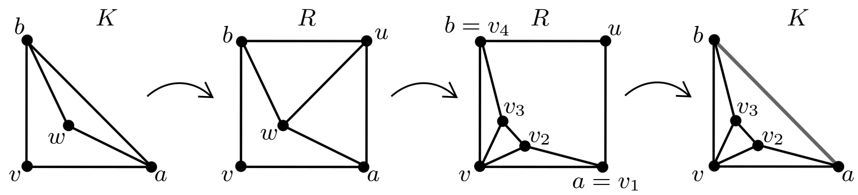


Fig. 2.8.: Construction for the proof of Lemma 2.7

2.1.2. Geometry

Let us fix some very basic terminology first. Saying that two subsets A and B of the complex plane $\mathbb{C}$ *intersect* (each other) means that $A \cap B \neq \emptyset$. An open set A *touches* a set B if $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

Speaking simply of *disks*, we usually mean (topologically) open disks. A *circle* ∂D is said to touch a set B if the disk D touches B. As usual, the symbol ∂ denotes the boundary operator. If p and q are different points of ∂D, then we define $\delta(p, q)$ as the *positively oriented open subarc* of ∂D from p to q, and $\delta[p, q] := \overline{\delta(p, q)}$. In addition we set $\delta(p, p) := \emptyset$ and $\delta[p, p] := \{p\}$. Note that $\delta(p, q)$ and $\delta[q, p]$ are complementary subarcs of ∂D provided that $p \neq q$.

By a *curve* γ we understand the image of a continuous mapping $\varphi : [a, b] \to \mathbb{C}$. The points $\varphi(a)$ and $\varphi(b)$ are said to be the *initial point* and the *terminal point* of γ, respectively. Both are referred to as *endpoints* of γ. A *Jordan arc* and a *Jordan curve* are the homeomorphic images of a segment and a circle, respectively. By an *open Jordan arc* we mean a Jordan arc without its endpoints.

Let J be an *oriented* Jordan curve. For $p, q \in J$ with $p \neq q$ we denote by $J(p, q)$ the (oriented) open subarc of J with initial point p and terminal point q. If p, q, r are three pairwise different points on J, then we say that q *lies between* p *and* r *on* J if $q \in J(p, r)$. Corresponding to whether q lies between p and r, or q lies between r and p, the *orientation of the triplet* (p, q, r) *with respect to* J is said to be *positive* or *negative*, respectively.

The Jordan curve theorem asserts that every Jordan curve J divides the complex plane into exactly two regions. The unbounded component of $\mathbb{C} \setminus J$ is called the *exterior* of J, while the bounded component is its *interior* or *Jordan domain.* Every Jordan domain is a bounded, simply connected domain, but the converse does not hold true.

Let G be a bounded, simply connected domain in $\mathbb{C}$. A conformal mapping $f : \mathbb{D} \to G$ of $\mathbb{D}$ onto G has a continuous extension f^* to $\overline{\mathbb{D}}$ if and only if ∂G is a *closed curve*, i.e. a continuous image of the unit circle $\mathbb{T}$ (see [21] Theorem 2.1). The extension f^* is a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{G}$ if (and only if) G is a *Jordan domain* (see [21] Theorem 2.6).

In general, the conformal mapping f induces a one-to-one correspondence between the points on $\mathbb{T}$ and certain equivalence classes (*prime ends*) of so called *crosscuts* of G. The whole Section 2.2 is dedicated to this very important topic.

The following result is a neat statement whose significance must not be underestimated. For example, it forms the nucleus of *incompressibility* as studied in Section 3.4. (see also [37] Lemma 1 and [20] Lemma 5)

Lemma 2.8 (Two-Disk-Lemma)**.** *Let D and D' be disks and assume that there are points $p, q \in \overline{D} \setminus D'$ and $p', q' \in \overline{D'} \setminus D$ so that the closed segments $\sigma := [p, q]$ and $\sigma' := [p', q']$ intersect each other at a point $s \in D \cap D'$. Then $\sigma = \sigma'$ or $D = D'$.*

Proof. If σ and σ' lie on a line g, then a straightforward discussion of the relative positions of p, p', q, q' on g leads to the conclusion that either $p = p'$ and $q = q'$, or $p = q'$ and $q = p'$. In both cases $\sigma = \sigma'$ (see Figure 2.9, left).

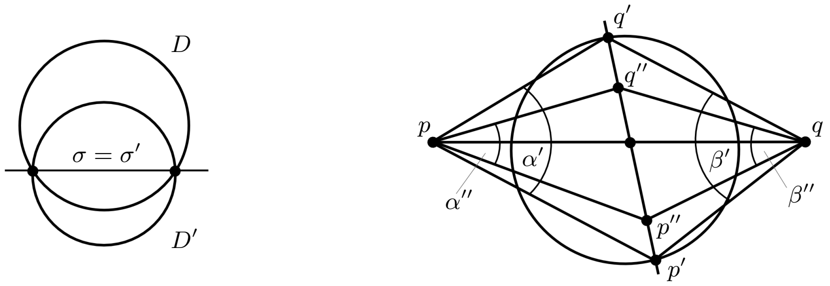


Fig. 2.9.: Illustrations to the proof of the Two-Disk-Lemma

Suppose now that σ and σ' have a single intersection point s. Then the line g through p' and q' divides the plane in two half-planes, each containing one of the points p or q, respectively.

Without loss of generality we may assume that σ and σ' are chords of D and D', respectively. Since $p', q' \notin D$, there are points $p'', q'' \in \partial D$ such that $p'' \in [p', q'']$ and $q'' \in [p'', q']$ (see Figure 2.9).

Let α', α'', β' and β'' denote the (non-oriented) angles $\angle(p'pq')$, $\angle(p''pq'')$, $\angle(p'qq')$, and $\angle(p''qq'')$, respectively (see Figure 2.9, right). Obviously,

$$\alpha'' \leq \alpha', \qquad \beta'' \leq \beta'. \tag{2.1}$$

Since p, p'', q, q'' form an inscribed quadrilateral of D, while $[p', q']$ is a chord of D' and p, q lie outside D' on different sides of g, the (extended) inscribed angle theorem tells us that

$$\alpha' + \beta' \leq \pi, \qquad \alpha'' + \beta'' = \pi. \tag{2.2}$$

Combining (2.1) and (2.2), we conclude that equality must hold in all cases. This implies $p' = p''$, $q' = q''$, and hence $p', q' \in \partial D \cap \partial D'$. Since neither p nor q can lie in D', we finally get $D = D'$. □

A last comment and we are finally going to define circle packings. Note that many statements are simplified when they are rephrased for disks instead for their boundary circles. Thus, we shall always speak of the disks in a circle packing.

Definition 2.7. A collection $\mathcal{P}$ of open disks D_v is said to be a *circle packing* for the complex $K = K(V, E, F)$ if it satisfies the following conditions (i)–(iii):

(i) Each vertex $v \in V$ has an associated disk $D_v \in \mathcal{P}$ so that $\mathcal{P} = \{D_v : v \in V\}$.

(ii) If $e(u, v) \in E$ is an edge of K, then the disks D_u and D_v touch each other.

(iii) If $\langle u, v, w, \rangle \in F$ is a positively oriented face of K, then the centers of the disks D_u, D_v, D_w form a positively oriented triangle in the plane.

A circle packing is called *univalent* if its disks are non-overlapping, i.e., $D_u \cap D_v = \emptyset$ for all $u, v \in V$ with $u \neq v$. In this work all circle packings are assumed to be univalent.

Since the structure of the underlying complex K carries over to the associated packing $\mathcal{P}$, all related attributes can be applied to the disks D_v as well – so we shall speak of boundary disks, interior disks, neighboring disks, etc. In particular, the subset of $\mathcal{P}$ that only contains disks associated with the kernel K^* of K shall be denoted as the *main part* $\mathcal{P}^*$ of $\mathcal{P}$.

The *contact point* of two neighboring disks D_u, D_v is defined by $c(u,v) := \overline{D}_u \cap \overline{D}_v$. The *contact points of a packing* $\mathcal{P}$ for the complex $K(V,E,F)$ are the points $c(u,v)$ with $e(u,v) \in E$. Note that two (boundary) disks may touch each other at a point which is not a contact point of $\mathcal{P}$ (see Figure 2.15, right).

If $\langle u,v,w,\rangle$ is a face of K, then the *interstice* $I(u,v,w)$ of $\mathcal{P}$ is the Jordan domain bounded by the arcs $\delta_u := \delta\big(c(u,v),c(u,w)\big)$, $\delta_v := \delta\big(c(v,w),c(v,u)\big)$ and $\delta_w := \delta\big(c(w,u),c(w,v)\big)$ (see Figure 2.10, left).

We say that a circle packing $\mathcal{P}$ is *contained in* a Jordan domain G (or *lies in* G) if every disk of $\mathcal{P}$ is a subset of G, i.e., $D_v \subset G$ for all $v \in V$. A packing $\mathcal{P}$ contained in G is said to *fill* G if every boundary disk of $\mathcal{P}$ touches ∂G.

Recall that a packing which fills the unit disk $\mathbb{D}$ is called *maximal*, and that the Koebe-Andreev-Thurston-Theorem tells us that any admissible complex K has an associated maximal packing, which is unique up to conformal automorphisms of $\mathbb{D}$.

For general non-Jordan domains the condition $D_v \subset G$ for all $v \in V$ is too weak to describe when a packing lies in G since then it could happen that "spikes" of ∂G (think of G as a slit disk) penetrate into the packing, sneaking through between two boundary disks at their contact point. In the next section we will provide a more general definition using prime ends, but for the remaining part of this section this is not yet necessary.

In accordance with the boundary chain $(v_1,\dots,v_m)$ of its complex K, the boundary disks of a circle packing $\mathcal{P}$ form a cyclic ordered chain $(D_1,\dots,D_m)$, which we label modulo m. In particular let $D_0 := D_m$ and $D_{m+1} := D_1$. For $k \in \{1,\dots,m\}$, we denote by η_k the closed segment which connects the centers of D_k and D_{k+1}. These *boundary segments* form a (polygonal) Jordan curve η.

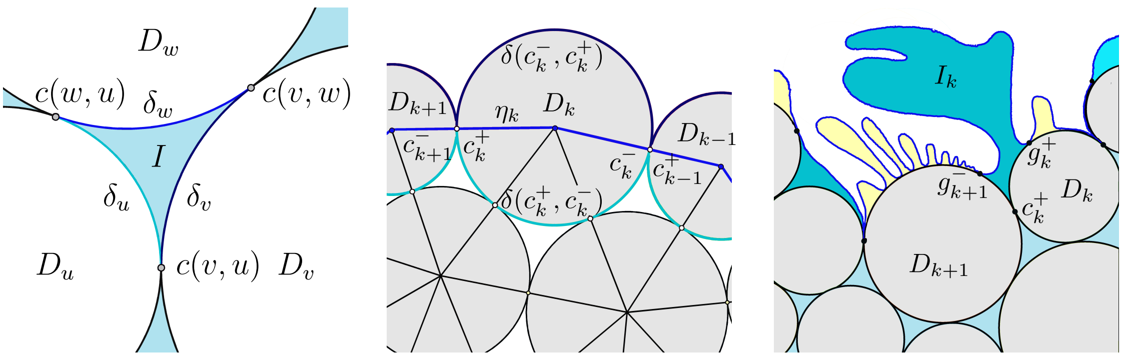


Fig. 2.10.: Definition of the interstice $I := I(u,v,w)$, boundary arcs and boundary interstices

If D_{k-1}, D_k and D_{k+1} are three consecutive boundary disks, then the contact points $c_k^- := \overline{D}_{k-1} \cap \overline{D}_k$ and $c_k^+ := \overline{D}_k \cap \overline{D}_{k+1}$ split ∂D_k into two arcs. We call $\delta(c_k^-, c_k^+)$ the *exterior boundary arc* and $\delta(c_k^+, c_k^-)$ the *interior boundary arc* of D_k, respectively (see Figure 2.10, middle).

The following result corresponds to Lemma 3 of [19].

Lemma 2.9. *Let D_k be a boundary disk of a circle packing $\mathcal{P}$. Then the exterior boundary arc of D_k contains no contact points of disks in $\mathcal{P}$.*

Proof. The polygonal line η which connects consecutive centers of the boundary disks is a Jordan curve which separates the exterior boundary arcs from the interior boundary arcs. The interior of η contains the closures $\overline{D}_v$ of all interior disks. Any contact point c of $\mathcal{P}$ is either a contact point of two boundary disks or it lies on the boundary of an interior disk. In both cases c does not belong to any exterior boundary arc. □

To provide even more notation, let $\mathcal{P}$ be a circle packing that fills a Jordan domain G. By definition, every boundary disk D_k touches ∂G in a non-void (possibly uncountable) set G_k of points, and G_k must be contained in the closure $\delta[c_k^-, c_k^+]$ of the exterior boundary arc $\delta(c_k^-, c_k^+)$ of D_k. Let $\delta_k := \delta[g_k^-, g_k^+]$ be the smallest subarc (we allow the possibility that this 'arc' degenerates to a point) of $\delta[c_k^-, c_k^+]$ which contains G_k. Since G_k is a closed set, we have $g_k^-, g_k^+ \in G_k$.

In order to define the *boundary interstice* I_k between two consecutive boundary disks D_k and D_{k+1} (see Figure 2.10, right) we distinguish two cases. If $g_k^+ = c_k^+$, then we set $I_k := \emptyset$. Otherwise, we let δ be the union of the arcs $\delta(g_k^+, c_k^+]$ (a subarc of ∂D_k) and $\delta[c_k^+, g_{k+1}^-)$ (a subarc of ∂D_{k+1}). The open Jordan arc δ is contained in G with different endpoints on ∂G (so it's a crosscut). Hence, the set $G \setminus \delta$ consists of two simply connected components G_1 and G_2 (Proposition 2.12 in [21]). One of these components contains D_k and D_{k+1}, the other one is (by definition) the boundary interstice I_k.

We end this section with an easy-to-see but nevertheless a bit technically-to-prove result on boundary interstices (see also [19] Lemma 4).

Lemma 2.10. $I_k \cap D_v = \emptyset$ *for all $k = 1, \ldots, m$ and all $v \in V$.*

Proof. Let $k \in \{1, \ldots, m\}$ be fixed. If $I_k = \emptyset$, then assertion is trivially fulfilled. Let $I_k \neq \emptyset$ and let δ be the crosscut defined above so that $G \setminus \delta$ consists of exactly two simply connected domains $G_1 = I_k$ and G_2.

Clearly, each disk of $\mathcal{P}$ is contained either in G_1 or G_2. We assume that there is a disk D_u in G_1 (remember $D_k \subset G_2$). Because K is connected, there is a chain C of vertices $\{u, \ldots, v\}$ where v is the vertex associated with D_k. Because $D_u \subset G_1$ and $D_k \subset G_2$, there have to be two consecutive vertices w_1, w_2 in C so that D_{w_1} is contained in G_1 and D_{w_2} in G_2. The contact point $c(w_1, w_2)$ must lie on $\partial G_1 \setminus \delta$, because there are no contact points of $\mathcal{P}$ on δ according to Lemma 2.9.

Let w_3 be a vertex so that $f(w_1, w_2, w_3)$ is a face of K. The interstice $I := I(w_1, w_2, w_3)$ is contained either in G_1 or G_2 because it is disjoint from ∂G. Moreover, both arcs $\partial D_{w_1} \cap \partial I$ and $\partial D_{w_2} \cap \partial I$ (up to their endpoints) lie in the same domain as I without being contained in the boundary of G. This implies that both disks D_{w_1} and D_{w_2} are contained either in G_1 or G_2, which is a contradiction. Hence, $I_k \cap D_v = \emptyset$ for all $k = 1, \ldots, m$ and all $v \in V$. $\square$

2.2. Prime Ends

So far we investigated the combinatorics and the geometry of circle packings. Now we take a closer look at the domains which we want to fill. As mentioned earlier, when studying bounded, simply connected domains that are not Jordan, the definition of some concepts needs more care. Here, the idea of *prime ends* becomes important. Note that there are several alternative (equivalent) definitions. We mainly rely on the approach of Pommerenke as stated for example in [21].

2.2.1. General Definition

A *crosscut* J of a domain G with boundary ∂G is an open Jordan arc in G so that $\overline{J} = J \cup \{a, b\}$ with $a, b \in \partial G$. If G is simply connected - which we always assume - then $G \setminus J$ consists of two simply connected components. A *null-chain* (J_n) is a sequence of crosscuts J_n of G which satisfies the following conditions (i)–(iii):

(i) $\overline{J}_n \cap \overline{J}_{n+1} = \emptyset \quad \text{for} \quad n = 0, 1, 2, \ldots,$

(ii) J_0 and J_{n+1} lie in different components of $G \setminus J_n$ for $n = 1, 2, 3, \ldots,$

(iii) $\operatorname{diam} J_n \to 0$ as $n \to \infty$.

For $n = 1, 2, \ldots$ the component of $G \setminus J_n$ which does not contain J_0 is denoted U_n. The domains U_n, called the *tails* of the null-chain (J_n), form a nested family,

$$U_{n+1} \subset U_n, \qquad n = 0, 1, 2, \ldots.$$

Two null-chains (J_n) and (J'_n) are said to be *equivalent* if for every n there exists an m such that

$$U_m \subset U'_n, \qquad U'_m \subset U_n.$$

The equivalence classes of null-chains with respect to this equivalence relation are the *prime ends* of G.

The set of all prime ends of a bounded, simply connected domain G will be denoted by ∂G^* and called the *intrinsic boundary* of G. The set $G^* := G \cup \partial G^*$ can be considered as compactification of G.

In order to make this more explicit, let $f : \mathbb{D} \to G$ be a (univalent) conformal mapping of the unit disk $\mathbb{D}$ onto G. It is well known ([21], Theorem 2.6) that f has a continuous and injective extension f^* to $\overline{\mathbb{D}}$ if and only if ∂G is a Jordan curve. For arbitrary bounded, simply connected domains, the *Prime End Theorem* says that $f^* : \overline{\mathbb{D}} \to G^*$ is a bijection which maps $\partial \mathbb{D}$ onto ∂G^* (see [21], Theorem 2.15). For $t \in \partial \mathbb{D}$ we call $f^*(t)$ the *prime end associated with* t (via f).

The inverses of f and f^* will be denoted by g and g^*, respectively. We shall refer to f and g as *canonical parameterization* and *canonical embedding* of G, respectively, and to f^* and g^* as their extensions.

An (oriented) *arc of prime ends* σ^* on ∂G^* is the image $f^*(\sigma)$ of an (oriented) arc $\sigma \subset \mathbb{T}$. The *initial and terminal points* of σ^* (i.e., its *endpoints*) are the the images of the initial and terminal points of σ under f^*, respectively.

In order to describe the relative positions of prime ends $X, Y, Z \in \mathbb{G}^*$ we write $X \prec Y \prec Z$ to express that Y lies on the closed positively oriented subarc of ∂G^* with initial point X and terminal point Z (which is interpreted as ∂G^* if $X = Z$). The symbol "$\preceq$" stands for "$\prec$ or $=$". The chained relations

$$X_1 \prec X_2 \prec \ldots \prec X_n \prec X_1$$

mean that the prime ends $X_1, \ldots, X_n \in \partial G^*$ are arranged counter-clockwise on ∂G^* in this order (see Figure 2.11, left).

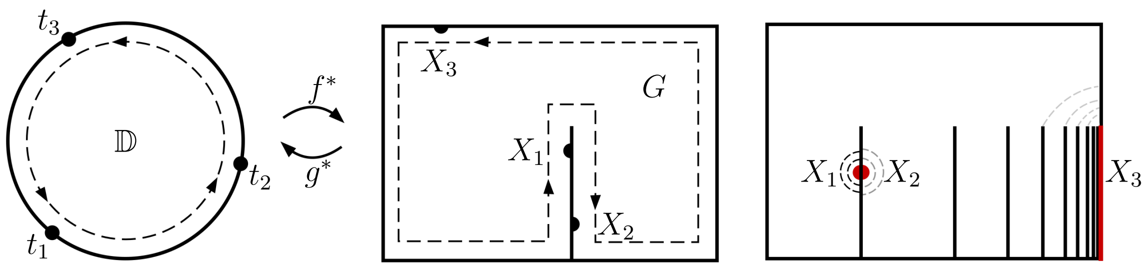


Fig. 2.11.: Orientation of $\partial\mathbb{D}$ applies to ∂G^* (left); the impressions of some prime ends (right)

The following result is Proposition 2.14 in Pommerenke [21]. It will often be used without special notice.

Lemma 2.11 (Images of Jordan arcs)**.** *Assume that g maps G conformally onto $\mathbb{D}$, and let $J \subset G$ be an open Jordan arc with endpoints $p \in G$ and $q \in \partial G$. Then the image $g(J)$ of J is an open Jordan arc in $\mathbb{D}$ with endpoints in $\mathbb{D}$ and on $\partial\mathbb{D}$, respectively. Two such arcs with distinct endpoints on ∂G have images with distinct endpoints on $\partial\mathbb{D}$.*

Combining this with the prime end theorem, we conclude that a Jordan arc J as in Lemma 2.11 is associated with a prime end $J^* \in \partial G^*$. The endpoints of $g(J)$ are $g(p) \in \mathbb{D}$ and $g^*(J^*) \in \partial\mathbb{D}$.

Let X be a prime end represented by a null-chain (J_n) with tails U_n. The compact sets $\overline{U}_n$ form a nested family and their intersection is a non-void compact connected subset of ∂G. This set does not depend on the choice of the representing null-chain (J_n) of X. It is called the *impression* of X and denoted by $I(X)$. Note that different prime ends may have the same impression (as X_1, X_2 in Figure 2.11, right).

A point $p \in I(X)$ is called *accessible via X* if, for each tail U_n of one (and then any) null-chain (J_n) in the equivalence class X, p can be joined with some interior point of U_n by an open Jordan arc that lies entirely in U_n. A classical result (Goluzin [10], Theorem 3 of Sect. 2.3) tells us that the impression $I(X)$ of every prime end X contains *at most one* (boundary) point which is accessible via X. If such a point $p(X)$ exists, then the prime end X is called *accessible*.

In Figure 2.11 (on the right) the upper endpoint of the red segment is accessible via X_3, while in Figure 2.14 the prime end X_1 is not accessible.

In order to define subordinate domains later on, we must be able to compare prime ends of a domain G with prime ends of a sub-domain G' (Figure 2.12 shows some examples).

Definition 2.8. Let G and G' be bounded, simply connected domains with $G' \subset G$. A prime end X' of G' is said to be *subordinate* to a prime end X of G (written as $X' \subset X$) if for every null-chain (J'_n) representing X' there is a null-chain (J_n) representing X such that the corresponding tails U'_n and U_n satisfy $U'_n \subset U_n$. If $\alpha \subset \partial G^*$ is a set of prime ends of G and X' is a prime end of G', then we say that X' *lies in* α and write $X' \in \alpha$ if there is some $X \in \alpha$ such that $X' \subset X$. Finally, $\alpha' \subset \alpha$ means that all X' of α' lie in α.

Some interesting examples are depicted in Figure 2.12. There are prime ends in G that do not have subordinate prime ends in G' (such as X_3). Others may have exactly one (e.g. X_1) or more than one subordinate prime end in G' (like X_4). The other way around, there are prime ends in G' that are not subordinate to any prime end of G (e.g. X'_3) or to exactly one (e.g. X'_1). We find prime ends in G that are determined uniquely by their impressions while their subordinate prime ends in G' are not determined this way (e.g. X_4, X'_4, X''_4). Also the converse may happens (as for X'_2, X_2, X_3). However, a prime end of G' that is subordinate to more than one prime end in G cannot exist.

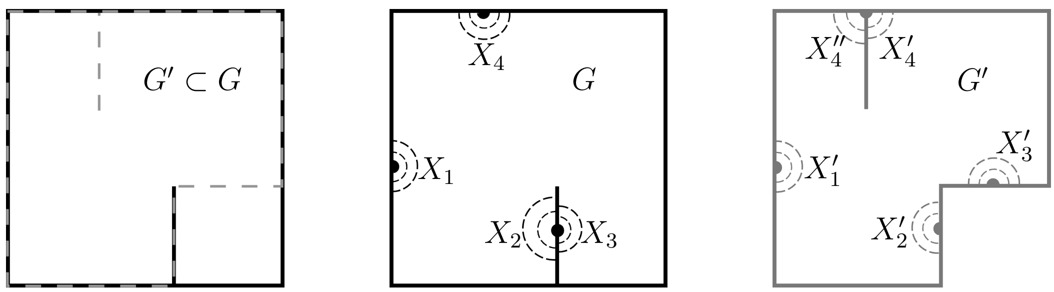


Fig. 2.12.: Examples illustrating subordination of prime ends

2.2.2. Applying Circle Packings

When a disk D is contained in a domain G that is not Jordan, it is not a-priori clear what the image of D under a canonical embedding of G looks like. Nevertheless, it is not surprising that it turns out to be a Jordan domain, as shown below (see also [20] Lemma 7). Note that the assertion of Lemma 2.12 remains valid when D is an arbitrary Jordan domain contained in G.

Lemma 2.12 (Embedded disks)**.** *Let G be a simply connected domain, let $D \subset G$ be a disk in G. Then the restriction of any canonical embedding $g : G \to \mathbb{D}$ to D extends continuously to an injective mapping $g_D : \overline{D} \to \mathbb{C}$. In particular, $g(D)$ is a Jordan domain and $\overline{g(D)}$ is homeomorphic to $\overline{\mathbb{D}}$.*

Proof. We prove that (the restriction of) g (to D) has limits at all points p on ∂D. This is trivial if $p \in G$. So let $p \in \partial G$.

Using Lemma 2.11, we conclude that $g(x)$ has a limit, say a, as x tends to p along a radius ϱ of D. We prove that a is the unrestricted limit of $g(z)$ if $z \in D$ and $z \to p$. If this were not so, there would exist a sequence $(y_k) \subset D$ with $y_k \to p$ and $g(y_k) \to b$ and $b \neq a$. From this sequence, we shall construct a sequence of segments $[z_n, z_{n+1}]$ in D forming a Jordan arc J with endpoint p. The points z_{2n} are points on the radius ϱ, while the points z_{2n+1} form a subsequence of (y_k). Since both sequences tend to p, we have $g(z_{2n}) \to a$ and $g(z_{2n+1}) \to b$, so that $g(J)$ cannot be a Jordan arc, which is a contradiction to Lemma 2.11. The construction of J is described in the next step.

To simplify notation, we assume that $p = 0$ and that the center of D lies at the point 1 on the positive real axis. In order to construct the sequence (z_k), we set $z_1 := y_1$ and define z_k recursively so that $z_{2n} := \operatorname{Re} z_{2n-1}$ and z_{2n+1} is in $Y := \{y_k\}$ with $|z_{2n+1}| \le \frac{1}{4}\,|z_{2n}|$ for $n = 1, 2, \ldots$. This ensures that $|z_{n+2}| \le 1/4\,|z_n|$, and from $|z_1| \le 2 \le 4$, $|z_2| \le |z_1| \le 2$, we get $|z_n| \le 2^{3-n}$ by induction. As can easily be seen (Figure 2.13),

$$|z_{2n} - z_{2n-1}| \le |z_{2n-1}| \le 2^{4-2n}, \qquad |z_{2n+1} - z_{2n}| \le 2|z_{2n}| \le 2^{4-2n}.$$

This shows that the total length of the segments $[z_n, z_{n+1}]$ is bounded. So their concatenation forms a (rectifiable) curve J from y_1 to p. Clearly, J is a simple curve, i.e., J is a Jordan arc.

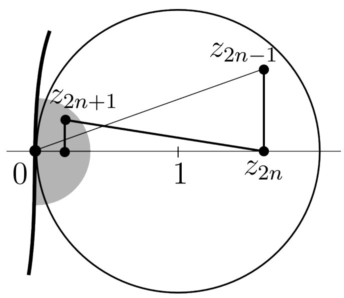


Fig. 2.13.: Construction of the Jordan arc J

Extending g by its limits on ∂D, we get a continuous function g_D on $\overline{D}$. It remains to prove that g_D is injective on $\overline{D}$. Obviously, $p \neq q$ implies $g_D(p) \neq g_D(q)$ if $p \in G$ or $q \in G$. So let $p, q \in \partial G$. The segments σ_p and σ_q which connect the center of D with p and q, respectively, are associated with different prime ends of G. Hence, $g(\sigma_p)$ and $g(\sigma_q)$ have different endpoints $g_D(p)$ and $g_D(q)$ on $\partial\mathbb{D}$ (see Lemma 2.11). $\square$

When working with general domains, the usual concept of disks touching a boundary point (in its usual meaning $p \in \overline{D}$) is too rough. Instead, we better use *prime ends*.

Definition 2.9. Let X be a prime end of G represented by a null-chain (J_n) with tails U_n. We say that a disk D *touches* X if $D \cap U_n \neq \emptyset$ for all n. Moreover, D *touches a set* α of prime ends if it touches a member of α.

Clearly, Definition 2.9 does not depend on the choice of the null-chain (J_n). It can be rephrased using the canonical embedding $g : G \to \mathbb{D}$ and its extension g^* as follows: D touches X if and only if $\overline{g(D \cap G)} \cap g^*(X) \neq \emptyset$. We point out that, so far, D was not assumed to be contained in G (see Figure 2.14 for some examples).

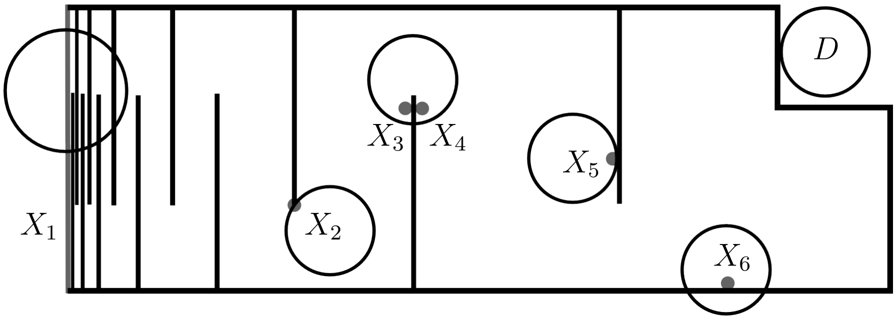


Fig. 2.14.: All disks except D touch (at least) one prime end of G

If $D \subset G$, then we get the following result (see also [20] Lemma 8 and [37] Lemma 2).

Lemma 2.13. *If an open disk D is contained in G and touches a prime end X of G, then $\overline{D} \cap I(X) = \{p\}$ for some $p \in \partial G$.*

Proof. From Definition 2.9 follows that $\overline{D} \cap \overline{U}_n \neq \emptyset$ for all n. Hence, $\overline{D} \cap I(X) \neq \emptyset$. If p is a point in $\overline{D} \cap I(X)$ and U_n is a tail of X, then the crosscut J_n defining U_n has a positive distance from p. Hence, a (convex) neighborhood V_n of p in D belongs to U_n. The open segment connecting p with a point in V_n lies in U_n. So p is accessible via X. Since $I(X)$ has at most one accessible point, we have $\overline{D} \cap I(X) = \{p\}$. ☐

The assertion of Lemma 2.13 does not need to hold true if $D \not\subset G$. For a counterexample, let X be a prime end whose impression is not a single point. Now, choose any point $p \in I(X)$ as the center of a disk D (as is illustrated by the disk touching X_1 in Figure 2.14). Then $\overline{D} \cap I(X)$ contains more than a single point.

Nevertheless, a disk D that is contained in G and touches the prime end X of G touches ∂G at some point p in the usual geometric sense. We shall say that D *touches (the prime end) X (as well as $\alpha \ni X$ and ∂G^*) at (a point) p*, and we call p a *contact point* of D with X (α and ∂G^*).

Note that by Lemma 2.12 the set C of contact points of D with (closed arcs of) ∂G^* is closed, which means that $g^*(C)$ is a closed subset of $\partial\mathbb{D}$.

If an open disk D in G touches the boundary of G at some point p, then the straight line connecting the center of D with p (as well as any open Jordan arc in D ending at

p) represents a unique prime end X of G, and we say that *p is associated with X by D*. The following statement makes this more explicit.

Lemma 2.14. *Let $X, Y \in \partial G^*$ be prime ends and let $q, p, p_1, p_2 \in \partial G$ be boundary points of a bounded, simply connected domain G. Let $D, D_1, D_2 \subset G$ be disks contained in G.*

(i) Assume D touches X, Y at p, q, respectively. Then $X = Y$ if and only if $p = q$.

(ii) If D_1 and D_2 touches X at p_1 and p_2, respectively, then $p_1 = p_2$.

Proof. Regarding assertion (i), we already know that $X = Y$ implies $p = q$ (see Lemma 2.13). In order to verify the reverse implication, assume that D intersects prime ends $X, Y \in \partial G^*$ so that $\overline{D} \cap I(X) = \overline{D} \cap I(Y) = p$. Let $x, y \in \mathbb{T}$ be the pre-images of X, Y, respectively, under the canonical parameterization f of G (or rather it's extension f^*), i.e. $f^*(x) = X$ and $f^*(y) = Y$. Let m be the center of D. According to Lemma 2.11, the preimage $S := f^{-1}(\sigma)$ of $\sigma := (p, m]$ is a semi-closed Jordan arc in $\mathbb{D}$ and $\{x\} = \overline{S} \cap \mathbb{T} = \{y\}$. Hence, $X = Y$ by the Prime End Theorem.

The assertion (ii) holds true since the points in $\overline{D}_1 \cap I(X)$ and $\overline{D}_2 \cap I(X)$ are accessible and $I(X)$ contains no more than one such point. □

Finally, we are ready to define when a circle packing $\mathcal{P}$ is *contained in* or *fills* a bounded, simply connected domain G. Recall that even if two disks $D_1, D_2 \in \mathcal{P}$ are contained in G, i.e., $D_1, D_2 \subset G$, it may happen that ∂G "cuts" through their contact point $\overline{D}_1 \cap \overline{D}_2$ (see Figure 2.15, left). This behavior is excluded by condition (ii) of the next definition.

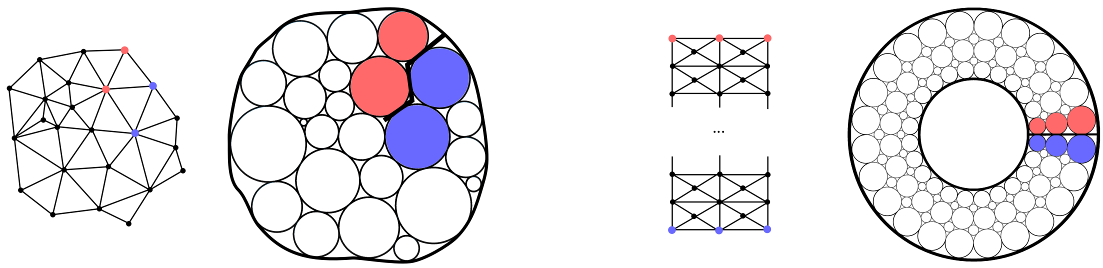


Fig. 2.15.: One packing is not contained in its domain (left), but the other one is (right)

Definition 2.10. A circle packing $\mathcal{P} = \{D_v\}$ with complex $K(V, E, F)$ *is contained in* a bounded, simply connected domain G if the following conditions (i) and (ii) are satisfied:

(i) $D_v \subset G$ for every $v \in V$.

(ii) A contact point $c(u, v)$ of two disks D_u and D_v that lies on ∂G is associated with the same prime end of G by both D_u and D_v.

The second condition is automatically satisfied for Jordan domains. So Definition 2.10 is consistent with the former one. An equivalent form of condition (ii) requires that $g(D_u)$ touches $g(D_v)$ whenever $u, v \in V$ are neighbors in K.

Note that this does not exclude that two boundary disks D_u and D_v touch the same boundary point (and each other) at different prime ends provided that u and v are not neighbors in K (see Figure 2.15, right).

With Definition 2.10 at hand, we can now say when a circle packing *fills a domain.*

Definition 2.11. Let G be a bounded, simply connected domain and let $\mathcal{P}$ be a circle packing. We say $\mathcal{P}$ *fills* G if (i) $\mathcal{P}$ is contained in G and (ii) every boundary disk of $\mathcal{P}$ touches a prime end of G.

As we know, a domain filling circle packing provides a set of *boundary interstices* I_k formed by ∂G and two boundary disks. The definition of I_k for general bounded, simply connected domains runs analog to the Jordan case.

Let D_k be the ordered set of all boundary disks. Then every D_k touches ∂G^* in a non-void (possibly uncountable) set G_k^* of *prime ends.* The associated contact points G_k of G_k^* must be contained in the closure $\delta[c_k^-, c_k^+]$ of the exterior boundary arc $\delta(c_k^-, c_k^+)$ of D_k (definition of the exterior boundary and the points c_k^-, c_k^+ see p.32).

Let $\delta_k := \delta[g_k^-, g_k^+]$ be the smallest subarc (maybe it is a singleton) of $\delta[c_k^-, c_k^+]$ that contains G_k. Since G_k is a closed set – using the canonical embedding $g : G \to \mathbb{D}$, clearly $\overline{g(D_k)} \cap \mathbb{T}$ is a closed set, thus G_k is one, too – we have $g_k^-, g_k^+ \in G_k$. If $g_k^+ = c_k^+$, then we set $I_k := \emptyset$. Otherwise, the concatenation of the arcs $\delta(g_k^+, c_k^+]$ and $\delta[c_k^+, g_{k+1}^-)$ is a crosscut of G. One connected component of $G \setminus \delta$ contains D_k and D_{k+1}, the other one is (by definition) the boundary interstice I_k.

Lemma 2.15. *Let $\mathcal{P}$ be a circle packing for K filling G. Let I_1, I_2 be any interior, boundary interstice, respectively. Then $I_1 \cap \partial G = \emptyset$, and $I_2 \cap D_v = \emptyset$ for every v in K.*

The first assertion of Lemma 2.15 follows by (ii) of Definition 2.10, and the second assertion can be proven exactly the same way as for Lemma 2.10.

In some sense, the Lemma 2.9 implies that the boundary arcs $\delta_1, \ldots, \delta_m$ of a circle packing $\mathcal{P}$ appear with the same interlacing order as the corresponding boundary vertices $v_1, \ldots, v_m$ within the boundary chain of K. This property is passed on to the prime ends of ∂G^* touched by the boundary disks of $\mathcal{P}$.

Lemma 2.16. *Let K be an admissible complex and let G be a bounded, simply connected domain. Let $\mathcal{P}$ be a circle packing for K filling G. Let $C = (v_1, \ldots, v_m)$ be the boundary chain of K. For $j = 1, \ldots, m$, let X_j be a prime end of G touched by the disk D_j associated with v_j. Then*

$$X_1 \preceq X_2 \preceq \ldots \preceq X_m \preceq X_1.$$

Proof. Assume contrarily that $X_i \preceq X_j \prec X_{i+1} \preceq X_i$ for some $i, j \in \{1, \ldots, m\}$ with $j \neq i$ and $j \neq i+1$. Up to an index shift, we may and will assume $i = 1$, so

$$X_1 \preceq X_j \prec X_2 \preceq X_1$$

for some $3 \leq j \leq m$. Note that this implies $X_j \neq X_2$.

If $X_1 = X_2$, then we simply exchange their roles in order to get $X_2 \prec X_j \prec X_1 \preceq X_2$. By reformulation, this yields $X_1 \preceq X_2 \prec X_j \prec X_1$, so this is not a contradiction to the assertion of the lemma. If $X_1 = X_j$, then an exchange of their roles yields $X_j \preceq X_1 \prec X_2 \prec X_j$, so by reformulation we get $X_1 \prec X_2 \prec X_j \preceq X_1$. Again, this sustains the assertion of the lemma. Therefore, assume X_1, X_2, X_j are pairwise different, i.e., we have

$$X_1 \prec X_j \prec X_2 \prec X_1.$$

Let $p_1, p_2 \in \partial G$ be the contact points of D_1, D_2 with X_1, X_2, respectively. Let c be the contact point between D_1 and D_2. Clearly, D_j cannot touch c. We distinguish whether $c \in G$ is an interior point or $c \in \partial G$ is a boundary point of G.

Case 1. Assume $c \in G$. Let σ be the concatenation of the two chords $(p_1, c]$ and $[c, p_1)$. Then σ is a crosscut of G. Only one of the two connected components G_1 and G_2 of $G \setminus \sigma$ contains X_j as (subordinate) prime end, say G_1. Thus, we have $D_j \subset G_1$.

Note that every two neighboring disks of $\mathcal{P} \setminus \{D_1, D_2\}$ must be contained either both in G_1 or both in G_2. Otherwise, their contact point would lie in σ or it would be p_1 or p_2. Clearly, this is impossible since $\mathcal{P}$ is univalent. Therefore, every disk of the chain $(D_3, \ldots, D_j, \ldots, D_m)$ must be contained in G_1 since $D_j \subset G_1$ (see Figure 2.16, left).

Using $X_1 \prec X_j \prec X_2 \prec X_1$, we conclude that the contact point between D_1 and D_m lies on the positive oriented circular arc $\delta[p_1, c] \subset \partial D_1$, i.e., $\delta(c, p_1)$ belongs to the boundary arc of D_1.

Now, let w be that vertex of K that forms the positive oriented face $\langle v_1, v_2, w \rangle$ in K, and let D_w be its associated disk in $\mathcal{P}$. Since the interstice formed by the three disks D_1, D_2 and D_w must have positive orientation, we conclude $D_w \subset G_2$. By Lemma 2.9, this is only possible if the contact point between D_1 and D_w is p_1 or c. By definition of the boundary arc and since $w \neq v_2$, this forces $w = m$, thus $D_w = D_m \subset G_1$, which is a contradiction.

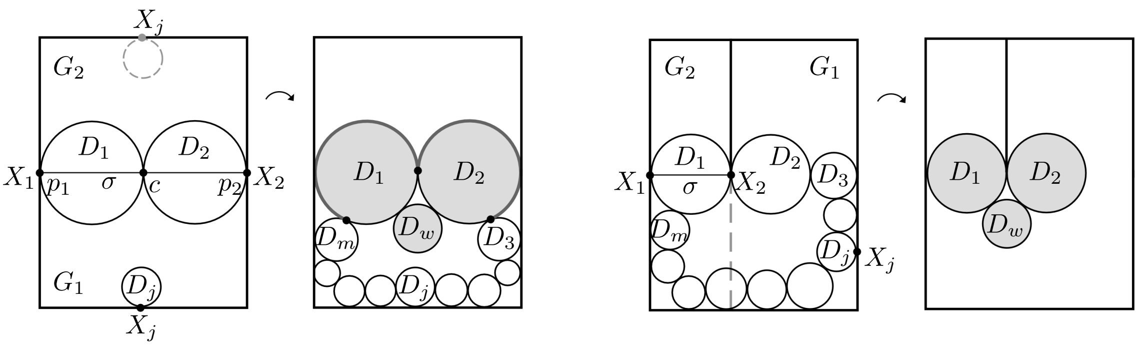


Fig. 2.16.: Constructions for the proof of Lemma 2.16; dotted elements yield $X_1 \prec X_2 \prec X_j \prec X_1$

Case 2. Assume $c \in \partial G$. Let Y be the associated prime end of ∂G^* so that c is the contact point of D_1, and thus of D_2, with Y. Maybe we have $Y = X_1$ or $Y = X_2$, but since $X_1 \neq X_2$ we may and will assume $Y \neq X_1$. Moreover, since D_j cannot touch c, we also have $Y \neq X_j$.

Let σ be the chord $(p_1, c) \subset D_1$. Then σ is a crosscut of G (see Figure 2.16, right). Proceeding as in Case 1, we obtain the same contradiction. Consequently, there is no X_j with $X_i \prec X_j \prec X_{i+1} \prec X_i$ for any $i, j \in \{1, \ldots, m\}$ with $j \neq i$ and $j \neq i+1$. $\square$

As a final obstacle, we want to investigate prime ends for which we do not know whether they are accessible or not. A priori, such prime ends cannot be connected with any interior point of the domain. So one cannot speak of their distance to, say, a center of a disk. The following result will be helpful in this situation.

Lemma 2.17. *Let X be a prime end of a bounded, simply connected domain G. Let α be a closed set of prime ends of G, but different to X, i.e., $X \notin \alpha$. Let (J_n) be a null-chain with tails (U_n) representing X. Let $D_n \in G$ be disks in G with radii r_n and centers $c_n \in U_n$ lying in U_n. Then $r_n \to 0$ for $n \to \infty$. Moreover, for every $m \in \mathbb{N}$ and every sufficiently large $n \geq m$ the disk D_n lies in U_m, i.e., $D_n \subset U_m$, and D_n cannot touch α.*

Proof. We assume the contrary. Let r_n do not tend to zero for $n \to \infty$. Then there is a sub-sequence, which we also denote r_n, so that $r_n \geq R > 0$ is bounded below by some positive constant R.

By the Bolzano-Weierstraß-Theorem, the associated sequence of centers c_n contains a sub-sequence converging to a point $c \in \overline{G}$, since $c_n \in U_n \subset U_0$ and U_0 is bounded. Moreover, the distance between c and ∂G is at least $R > 0$, i.e., c is an interior point of G. Let $g(c) = a \in \mathbb{D}$ be the image of c in $\mathbb{D}$ under the canonical embedding $g : G \to \mathbb{D}$. Note that a has positive distance to $\partial\mathbb{D} = \mathbb{T}$.

By Theorem 2.15 of [21], we know that the null-chain (J_n) is mapped onto a null-chain (Γ_n) of the boundary point $g^*(X) =: t \in \mathbb{T}$ by $g(J_n) =: \Gamma_n$. Let W_n be the tails of Γ_n. According to construction, for every $n = 0, 1, \ldots$ we have $a \in \overline{W_n}$. For sufficiently large $N \in \mathbb{N}$, every W_n with $n \geq N$ is contained in an arbitrarily small neighborhood of $t \in \mathbb{T}$, which is a contradiction for a having positive distance to $\partial\mathbb{D}$. Hence, we need to have $r_n \to 0$ for $n \to \infty$.

Now, assume that D_n touches α for all $n \in \mathbb{N}$, say in $Y_n \in \alpha$. Let $a_n \in \partial G$ be the contact point between D_n and Y_n. Let $\sigma_n = \sigma(c_n, a_n)$ be a segment in D_n connecting its center c_n with its contact point a_n. After embedding everything into $\mathbb{D}$ via g, it is an easy observation that $g(\sigma_n)$ has to intersect the crosscut Γ_n in an interior point $q_n \in \mathbb{D}$ since Γ_n separates $g^*(X)$ from $g^*(\alpha)$ for sufficiently large n. Therefore, for sufficiently large $N \in \mathbb{N}$ and every $n \geq N$, the segment σ_n intersects J_N and J_{N-1} in interior points $p_n, p_{n-1} \in G$ (the pre-images of q_n and q_{n-1} via g), respectively.

Now (per definition), $\overline{J_N} \cap \overline{J_{N-1}} = \emptyset$ and $U_N \subset U_{N-1}$. So for sufficiently small $\varepsilon > 0$ every ε-neighborhood $U_\varepsilon(p)$ of every point $p \in J_N$ lies in U_{N-1}, i.e., $U_\varepsilon(p) \subset U_{N-1}$. This implies that the distance between p_n and p_{n-1} must be at least ε (independent of n), i.e., $\operatorname{dist}(p_n, p_{n-1}) \geq \varepsilon > 0$. This contradicts the fact that $\operatorname{dist}(p_n, p_{n+1}) < 2r_{n+1} \to 0$ for $n \to \infty$. Hence, for every sufficiently large n the disk D_n cannot touch α.

For the same reason, we have $D_n \subset U_m$ for every m and every sufficiently large $n \geq N$ since otherwise the positive distance between U_N and U_{N-1} would prevent $r_n \to 0$. $\square$

2.3. Degeneration

Since in certain situations one cannot maintain the classical setting of circles having positive radius, one major task of this work is to handle degeneration, i.e., circles with radius zero. This section provides the necessary framework, but it also introduces some properties that prevent degeneration. Additional properties that depend on the used normalization are stated at the corresponding Chapters 3 and 4.

2.3.1. Generalized Circle Packings

A circle with radius zero is nothing but a single point $p \in \mathbb{C}$ of the complex plane. We denote it a *degenerate* circle. Since the interior of a degenerate circle is the empty set, it makes no sense to speak of a degenerate disk. In order to overcome this obstacle and at the same time to not mix the terminology of disks and circles, we define a *dot* $S = \{p\}$ to be a set consisting of a single point $p \in \mathbb{C}$.

We say that a dot S_1 *touches* a disk D, if it is contained in its boundary, i.e., $S_1 \subset \partial D$. Moreover, S_1 *touches* another dot S_2 if both dots are the same, i.e., $S_1 = S_2$. Allowing circle packings to contain dots leads to the next term.

Definition 2.12. A collection $\mathcal{P}$ of dots S_u and disks D_v is said to be a *generalized circle packing* for the admissible complex $K(V, E, F) \in \mathcal{K}$ if it satisfies the following conditions (i)–(iii):

(i) Each vertex $v \in V$ has either an associated dot $S_v \in \mathcal{P}$ or disk $D_v \in \mathcal{P}$ so that $\mathcal{P} = \{S_u : u \in U \subset V\} \cup \{D_v : v \in V \setminus U\}$.

(ii) If $e(v, w) \in E$ is an edge of K, then S_v and S_w, or S_v and D_w, or D_v and S_w, or D_v and D_w, respectively, touch each other (as explained above).

(iii) If $f(u, v, w) \in F$ is a positively oriented face of K, then the centers of D_u, D_v, D_w form a positively oriented triangle in the plane provided that all three sets are disks.

The set of all dots within a generalized circle packing $\mathcal{P}$ shall be denoted $\mathcal{S}$ whereas the set of all disks shall be $\mathcal{D}$, i.e., $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$. If $\mathcal{S} \neq \emptyset$, then $\mathcal{P}$ is said to be a *degenerate* circle packing, and if even $\mathcal{D} = \emptyset$, then we call $\mathcal{P}$ *collapsed.* If $\mathcal{S} = \emptyset$, then $\mathcal{P}$ is a circle packing as we already knew it.

Since terms like *main part* of $\mathcal{P}$, *boundary* dots, *interior* dots, *neighboring* dots, etc root in the combinatorics of a generalized circle packing, they are well defined. Also the *contact point* between a dot $S_u = \{p\}$ and a neighboring (touched) disk D_v or dot S_v is still defined as $c(u, v) := S_u \cap \overline{D}_v = p$ or $c(u, v) := S_u \cap S_v = p$, respectively.

Similar to the classical case, a generalized circle packing is called *univalent* if its disks are non-overlapping, i.e., $D \cap D' = \emptyset$ for all $D, D' \in \mathcal{D}$ with $D \neq D'$. In this paper all generalized circle packings are assumed to be univalent.

2.3.2. Behavior of Degeneracy

We show that as soon as there is a disk in $\mathcal{P}$ every dot S is "glued" to at least one disk, maybe even to two, which then acts as a sort of a representative for S.

Let $\mathcal{P}$ be a generalized but non-collapsed circle packing with admissible complex K. Let V be the set of all vertices of K and let $U \subset V$ be the subset associated with the dots $\mathcal{S}$ of $\mathcal{P}$. Let $w \in U$ be associated with $\{p\} = S_w \in \mathcal{S}$. Then we define $W \subset U$ as the subset of all vertices of U that can be connected with w by a chain in U.

Let W' be the subset of vertices of $V \setminus U$ that are neighbors of W. We say every $v \in W'$ has combinatorial distance 1 from W.

Since $|\mathcal{D}| \geq 1$, we have $W \neq V$. Hence, there is a vertex $v_1 \in W' \neq \emptyset$. Let D_1 be the disc associated with v_1 and let $u \in W$ be a neighbor of v_1 in W. By definition of W, there is a chain $C' = (u, \ldots, w)$ of vertices in W connecting v_1 with w. Let $C = (v_1, u \ldots, w)$ be the extension of C' to v_1. Since D_1 touches S_u, and since all dots associated with C' must be the same singleton $S_u = \ldots = S_w = \{p\}$, the disk D_1 is also touched by S_w.

Assume $|\mathcal{D}| \geq 2$. Since a triangulation trivially is 2-connected, the simplicial 2-complex we get by deleting v_1 and all its edges and faces from K is still connected. Since $|\mathcal{D}| \geq 2$, there is a second vertex $v_2 \neq v_1$ in W'. By the same argumentation as above, S_w touches D_2, the disk associated with v_2. Hence, we have $S_u = \overline{D}_1 \cap \overline{D}_1$.

The following definition and the subsequent lemma consolidate this properties.

Definition 2.13. A dot $S_w = \{p\}$ is *attached* to a disk D_1, whenever there is a chain of vertices $C = (v_1, u_1, \ldots, u_k)$ so that every u_i is associated with a dot and $u_k = w$. If S_w is attached to two disks $D_1 \neq D_2$, then we denote p a *pseudo contact point* of D_1 with D_2.

Lemma 2.18. *Let $\mathcal{P}$ be a degenerate circle packing. Let $S_w \in \mathcal{S}$ be a dot of $\mathcal{P}$. Then there is a chain of dots $(S_1, \ldots, S_k)$ in $\mathcal{P}$, such that $S_1 = S_w$ and S_k is a boundary dot. Furthermore,*

(i) if $|\mathcal{D}| \geq 1$, then S_w is attached to a disk in $\mathcal{P}$, and

(ii) if $|\mathcal{D}| \geq 2$, then S_w is a pseudo contact point of two disks of $\mathcal{P}$.

Proof. Since (i) and (ii) were shown above, we only need to prove the very first assertion of the lemma. If w is a boundary vertex of K, then we are done.

Assume w is an interior vertex. Let W, W' be the sets defined above. Since $|W'| > 2$ is impossible, otherwise three disks would touch each other at a single point, there are at most two vertices $v_1, v_2 \in W'$.

Let K^* be the kernel of K with respect to w. By Lemma 2.4, K^* is 3-connected. So removing v_1, v_2 together with their edges and faces from K^* still yields a connected simplicial 2-complex. Thus, at least one boundary vertex of K^* lies in W. By Lemma 2.3, we have $\partial V^* \subset \partial V$, what completes the proof. $\square$

Lemma 2.18 has a very interesting implication for strongly connected complexes: Either they are almost collapsed, or not even degenerate. The following statement makes this more explicit (see also [37] Lemma 4).

Lemma 2.19. *Let $K \in \mathcal{K}^0$ be strongly connected. Let $\mathcal{P}$ be a generalized circle packing for K. Then either $\mathcal{P}$ is a circle packing, or it consists of at most two disks.*

Proof. If $\mathcal{P}$ is not degenerate, then it trivially contains at least three disks. So all we must consider is the case of $\mathcal{P}$ being degenerate. If $|\mathcal{D}| < 2$, then we are done. So assume $|\mathcal{D}| \geq 2$.

Let $w \in V$ be associated with a dot $S_w = \{p\}$ of $\mathcal{P}$. By Lemma 2.18, there are two disks $D_1, D_2 \in \mathcal{D}$ so that p is a pseudo contact point of them. Let v_1 and v_2 be the associated vertices of D_1 and D_2, respectively.

Let W and W' be defined as above. Clearly, $|W'| > 2$ is impossible since then there would be a third disk D_3 touching D_1 and D_2 at p. So $W' = \{v_1, v_2\}$. Since K is strongly connected, Lemma 2.4 states that we can delete v_1, v_2 and all their edges and faces from K and the resulting simplicial 2-complex still is (edge-)connected. By definition of W, we conclude $W = (V \setminus \{v_1, v_2\})$, thus $|\mathcal{D}| = 2$.

Hence, if $\mathcal{P}$ is degenerate, then we have $|\mathcal{D}| \leq 2$. □

The different normalizations of Chapter 3 and 4 yield different (additional) behavior of degeneracy. These will be treated separately in the Sections 3.2 and 4.3.

2.3.3. Generalized Filling

The idea of filling is for generalized circle packings the same as before: $\mathcal{P}$ must be contained in its domain G and every boundary disks and dots must touch ∂G. The first part reads as follows.

Definition 2.14. A non-collapsed generalized circle packing $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ with admissible complex K *is contained in* a bounded, simply connected domain G if the following conditions (i) and (ii) are satisfied.

(i) Every disk $D \in \mathcal{D}$ lies in G, i.e., $D \subset G$.

(ii) If p is a contact point or pseudo contact point of $D_v, D_w \in \mathcal{D}$ that lies on the boundary of G, then the two disks D_v and D_w touch the same prime end at p.

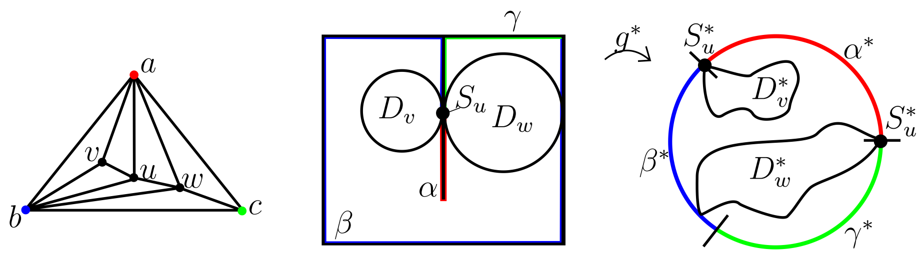


Fig. 2.17.: A packing contradicting (ii) of Definition 2.14, and its embedding in $\mathbb{D}$

Figure 2.17 illustrates the importance of condition (ii): The disks D_v and D_w have the pseudo contact point p on ∂G, but the two disks associate p with different prime ends of G. The dot $S_u = \{p\}$ is attached to both disks. The images of D_v and D_w under the conformal embedding $g : G \to \mathbb{D}$ do not touch each other, and S_u has no well-defined image under the extended mapping $g^* : G^* \to \overline{\mathbb{D}}$.

Since condition (ii) of Definition 2.14 guarantees that $g^*(S)$ is well defined for all dots of a non-collapsed generalized circle agglomeration that lies in G, it is natural to say that a dot S touches a prime end $X \in \partial G^*$ if $g^*(S) = g^*(X)$. The following equivalent definition is more intuitive and can be verified directly without reference to g^*.

Definition 2.15. Let $\mathcal{P}$ be a non-collapsed generalized circle packing contained in G. A dot $\{p\} = S \in \mathcal{P}$ *touches a prime end* X of G if one (and then all) of its attached disks touches X at p. A dot S *touches a set* α of prime ends if it touches a member of α.

Due to condition (ii) of Definition 2.14, a dot S can touch at most one prime end. But the concept of touching must be used with care: Without an attached disk, a dot $S = \{p\}$ may lie on the boundary ∂G of G (i.e., $p \in \partial G$) without touching ∂G^*. Thus, the following definition implies indirectly, that $\mathcal{P}$ is not collapsed.

Definition 2.16. Let G be a bounded, simply connected domain, and let $\mathcal{P}$ be a generalized circle packing. We say $\mathcal{P}$ *fills* G if

(i) $\mathcal{P}$ is contained in G, and

(ii) every boundary disk and boundary dot of $\mathcal{P}$ touches a prime end of G.

2.4. Sperner's Lemma

The last section of this chapter is devoted to a quite remarkable statement known as Sperner's Lemma. In general, it is a combinatorial analog of the Brouwer Fixed Point Theorem. In particular, Sperner colorings have been used for effective computation of fixed points within in a variety of algorithms, solving for example root-finding or fair division problems. (see Henle [17], for some more information)

The one-dimensional case of Sperner's Lemma is a combinatorial version of the Intermediate Value Theorem. But it is the two-dimensional case that we are interested in. In some sense, this is the core of every existence proof we will do so its importance for this work is not to be underestimated. For the convenience of the reader, we state Sperner's Lemma in its two-dimensional form. Since its proof is very neat we will state it, too (alternatively see [1] for example).

Let $K \in \mathcal{K}$ be an admissible complex with at least three distinguished (counterclockwise ordered) boundary vertices $r, g, b \in \partial V$. We associate K with a *Sperner coloring* in the following way (see Figure 2.18, left). Let r, g and b be associated with the colors red, green and blue, respectively. Let every boundary vertex that lies on the boundary chain (with positive orientation) between r and g be colored either red or green. Analogously, every boundary vertex between g and b (b and r) shall be either green or blue (either blue or red). For interior vertices there is no further restriction, but to be colored red, green or blue. Then the following holds true.

Lemma 2.20 (Sperner's Lemma)**.** *Every Sperner coloring of an admissible complex K contains a face $f(u,v,w) \in F$ so that the vertices u,v,w are colored all different.*

Proof. Assume w.l.o.g. that K has the form of a triangle (see Figure 2.18). We derive a graph H' from K as follows. Every face of K shall be associated with exactly one vertex of H' and vice verse. Two vertices of H' get a common edge if and only if the associated faces of K share an edge e and e is a red-blue-edge, that means one vertex of e is colored red and the other one blue. Since the number of red-blue-edges within a face of K can be either 0, 1 or 2, the degree of a vertex of H' can be also either 0, 1 or 2. Moreover, a degree of 1 corresponds to a face f in K colored in all three colors. We are done if there is at least one vertex in H' with odd degree.

Now, let H arise from H' by adding the new vertex v_0 to H'. The vertex v_0 gets an edge to every vertex w of H' that is associated with a boundary face in K with red-blue-*boundary*-edge. Since the number of red-blue-boundary-edges of K must be odd (easy to see) the degree of v_0 is odd. According to the Handshaking Lemma (a very basic statement of graph theory; see [34] for instance), the number of vertices with odd degree in H (as in every finite graph) is even. So H' has an odd number of vertices with odd degree, hence at least one. $\square$

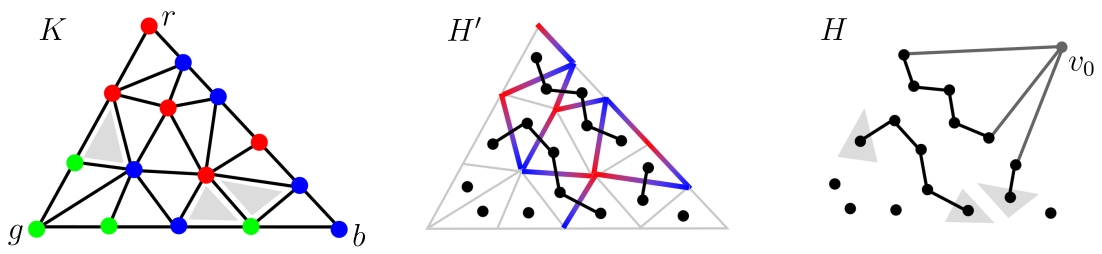


Fig. 2.18.: Sperner coloring for K and the associated graphs H' and H

Recall that originally Sperner's Lemma is way more powerful since it deals with n-dimensional simplicial complexes. The version stated here is merely the induction base of the true thing.

3. Alpha-Beta-Gamma Normalization

The first normalization we will look at, and maybe the most versatile one, is the *alpha-beta-gamma normalization.* Roughly speaking, we associate three boundary points of a domain G – the *vertices* of G – with three boundary disks of a domain filling packing $\mathcal{P}$ – the *leadings disks* of $\mathcal{P}$.

To be more precise, we use *prime ends* instead of points (for reasons mentioned in Chapter 2), and there are only up to three leading disks, since a single boundary disk can be used for the normalization multiple times. Furthermore, since we allow degeneration, there can also be *leading dots* instead of disks.

Combinatorial, we express the normalization by framing an admissible complex K with a so called *tri-complex* T. Roughly speaking, T is an admissible complex with exactly three boundary vertices, and $K = \text{int}T$. This correlation leads to an even more general class of *acceptable complexes* instead of admissible ones. The associated packings $\mathcal{P}$ are called (generalized) *circle agglomerations.*

In order to prove that two generalized circle agglomerations $\mathcal{P}_1$ and $\mathcal{P}_2$ are unique under the alpha-beta-gamma normalization, we reduce $\mathcal{P}_i$ to its skeleton, show that this must be a non-degenerate circle packing, and use an *incompressibility* argument to see that at least the skeletons of $\mathcal{P}_1$ and $\mathcal{P}_2$ must be equal. An induction on the number of disks deals with the remaining sub-packings in $\mathcal{P}_1$ and $\mathcal{P}_2$.

En passant we give a classification of several types of degeneration that can occur, and eventually we prove existence by an application of Sperner's Lemma (for regular domains) and an exhaustion argument (for the general case).

The following statement is the main result, which we prove in Section 3.7.2.

Theorem 3.1 (Alpha-Beta-Gamma Theorem)**.** *Let $T \in \mathcal{T}^*$ be a tri-complex and let $G(\alpha, \beta, \gamma)$ be a trilateral for a bounded, simply connected domain G. Then there exists a generalized circle agglomeration $\mathcal{P}$ for T filling $G(\alpha, \beta, \gamma)$. If the trilateral $G(\alpha, \beta, \gamma)$ is tame, then $\mathcal{P}$ is unique. If the tri-complex $T \notin \mathcal{T}_1^*$ is boundary irreducible and the trilateral $G(\alpha, \beta, \gamma)$ is not spiky, then $\mathcal{P}$ is a circle packing.*

We conclude this chapter by some consequences of Theorem 3.1, such as a generalized Incompressibility Theorem and a discrete version of the conformal modulus.

We point out that key parts of this chapter have already been developed within the papers [20] and [37]. Since over time some minor or even major details have changed, we do not explicitly refer every definition to its corresponding source(s), but only the major ideas of the theorems, lemmas and their proofs. There is no guarantee that all terms and definitions written in the papers are consistent with the corresponding versions presented here.

3.1. Additional Notation and Concepts

To begin with, we provide (even more) notations and some special constructions that are related solely or especially to the alpha-beta-gamma normalization.

3.1.1. Acceptable Complexes and Frames

In order to specify in which way the boundary disks of a domain-filling circle packing touch the boundary of the domain it is convenient to *frame* the complex with three (or four) additional vertices. To do so, we first consider strongly connected admissible complexes with exactly three or exactly four boundary vertices.

Definition 3.1. Let $T = T(a,b,c)$ and $Q = Q(a,b,c,d)$ be strongly connected admissible complexes with $\partial V = \{a,b,c\}$ and $\partial V = \{a,b,c,d\}$, respectively (arranged counter-clockwise on the boundary chain). We denote T a *trilateral complex* and Q a *quadrilateral complexes*, a *tri-* and *quad-complex*, respectively, for short.

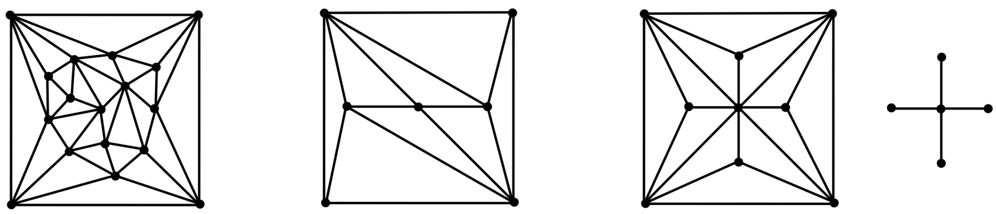


Fig. 3.1.: proper quad-complex in $\mathcal{Q}_{13}$; quad-complexes in $\mathcal{Q}_3^*$ and $\mathcal{Q}_5^*$; acceptable complex in $\mathcal{K}_5^*$

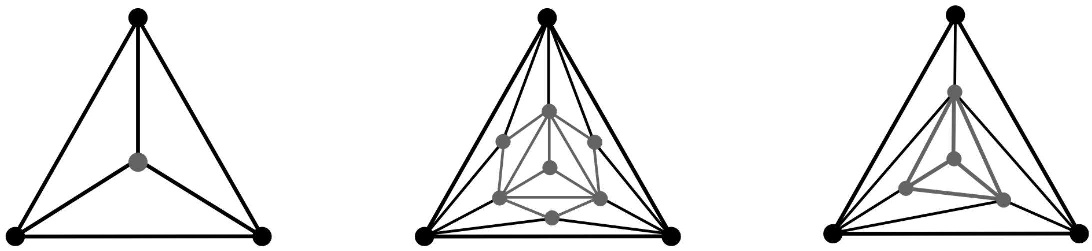


Fig. 3.2.: Tri-complexes: boundary irreducible but not proper (left), boundary irreducible but not intrinsic strongly connected (middle), intrinsic strongly connected but boundary reducible (right)

The class of all tri- and quad-complexes is denoted $\mathcal{T}^* \subset \mathcal{K}^0$ and $\mathcal{Q}^* \subset \mathcal{K}^0$, respectively. If in addition the interior $\operatorname{int} T$ of T (or $\operatorname{int} Q$ of Q) is an admissible complex, then T (or Q) is called *proper*. The proper tri-complexes form the set $\mathcal{T} \subset \mathcal{T}^*$, and the proper quad-complexes form $\mathcal{Q} \subset \mathcal{Q}^*$.

Moreover, we introduce the sets $\mathcal{T}_n^*$, $\mathcal{T}_n$ and $\mathcal{Q}_n^*$, $\mathcal{Q}_n$, which stand for the corresponding subsets of $\mathcal{T}^*, \mathcal{T}$ and $\mathcal{Q}^*, \mathcal{Q}$, respectively, that have complexes with at most n interior vertices.

Up to some simple geometric transformations, we may and will assume (w.l.o.g.) that the boundary edges of a tri- or quad-complex always form an equilateral triangle or a square, respectively. See Figure 3.1 and Figure 3.2 for some examples.

Note that the restriction to strongly connected complexes implies an implicit property of quad-complexes: Within $Q(a, b, c, d)$ the vertices a and c as well as b and d cannot be neighbors. This is not an unwanted side product but our firm intention.

Moreover, strongly connectedness implies for tri- and quad-complexes that there is at least one interior vertex, so $\mathcal{T}_0^* = \mathcal{Q}_0^* = \emptyset$.

By Lemma 2.4, the interior of a strongly connected admissible complex is a finite, simply connected, simplicial complex (possibly of dimension less than 2). We will use this as a generalization for admissible complexes.

Definition 3.2. Let $K = \operatorname{int} Q$ be the interior of some quad-complex $Q \in \mathcal{Q}^*$. Then we say that Q *frames* K. A simplicial complex K is said to be *acceptable* if it can be framed by some $Q \in \mathcal{Q}^*$ (see Figure 3.1, right). The set of acceptable complexes is $\mathcal{K}^* := \{\operatorname{int} Q : Q \in \mathcal{Q}^*\}$. The members of $\mathcal{K}^*$ that have at most n vertices form the class $\mathcal{K}_n^*$.

It can easily be verified that any admissible complex $K \in \mathcal{K}_n$ can be framed by some quad-complex $Q \in \mathcal{Q}_n$. Just use the boundary chain of K and proceed as depicted in Figure 3.3. Thus, the set of acceptable complexes comprises the set of admissible complexes, i.e. $\mathcal{K}^* \supset \mathcal{K}$.

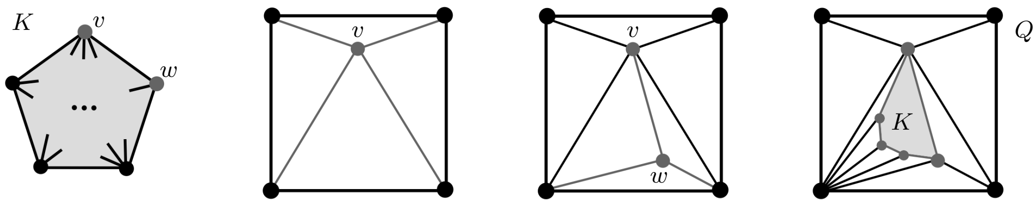


Fig. 3.3.: Framing an admissible complex K by a quad-complex Q

Let $K = \operatorname{int} T$ for some $T \in \mathcal{T}^*$. Then analogously we say that T *frames* K. Since it is no problem to frame K also by a quad-complex (just use any boundary face of T and proceed as depicted in Figure 3.4), we get $K \in \mathcal{K}^*$. Hence, tri- and quad-complexes give rise to acceptable complexes.

Note that the inverse does not hold true since for example the acceptable complex depicted in Figure 3.1 (right) cannot be framed by a tri-complex, but only by a quad-complex.

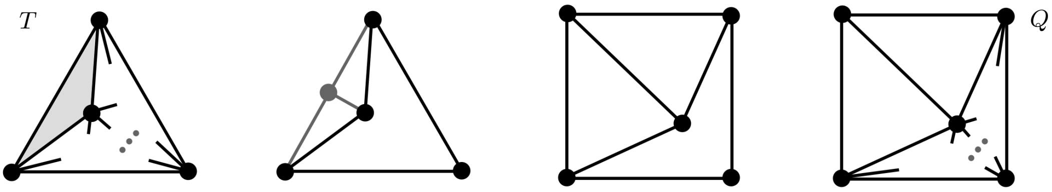


Fig. 3.4.: Transforming a tri-complex T into a quad-complex Q with the same interior

Definition 3.3. A tri- or quad-complex T or Q is said to be *intrinsic* strongly connected if its *interior* is a strongly connected admissible complex, so $\mathrm{int}T \in \mathcal{K}^0$ or $\mathrm{int}Q \in \mathcal{K}^0$, respectively. The class of all intrinsic strongly connected tri- or quad-complexes is $\mathcal{T}^0$, $\mathcal{Q}^0$, respectively.

All in all we have

$$\mathcal{T}^0 \subset \mathcal{T} \subset \mathcal{T}^* \subset \mathcal{K}^0, \quad \mathcal{Q}^0 \subset \mathcal{Q} \subset \mathcal{Q}^* \subset \mathcal{K}^0, \quad \mathcal{K}^0 \subset \mathcal{K} \subset \mathcal{K}^*.$$

If $T(a, b, c)$ frames $K(V, E, F)$, then there are vertices $v_1, v_2, v_3 \in V$ that form faces $f(a, b, v_1)$, $f(b, c, v_2)$ and $f(c, a, v_3)$ of T. We call v_1, v_2 and v_3 the *leading vertices* of T. In the same sense we may and will also speak of the leading vertices v_1, v_2, v_3 and v_4 of a quad-complex $Q(a, b, c, d)$. Note that those vertices do not need to be pairwise different. The *degree* of T and Q refers to the number of pairwise different leading vertices of T and Q, respectively.

3.1.2. Reduction and Merging (combinatorial)

Some of the mechanics from Section 2.1.1 have a special interpretation for tri- and quad-complexes. First of all we can apply the idea of (irr-)reducibiliy and the associated transformations, i.e., the inner reduction ϱ_+ and the outer reduction ϱ_-.

Recall the subtle exception we implanted into Definition 2.3. The only reason why a tri-complex $T(a, b, c)$ is not reducible in a trivial sense by its boundary vertices a b and c is the fact that we prohibited $\triangle(a, b, c)$ to be reducible. Therefore, $\varrho_+(T, .) = \triangle(a, b, c)$ is excluded, as well as $\varrho_-(T, .) = T$.

In general, ϱ_+ removes only interior vertices and ϱ_- converts at least one interior vertex into a boundary vertex. So we get

$$\varrho_+ : \mathcal{T}^*_n \to \mathcal{T}^*_{n-1}, \quad \varrho_- : \mathcal{T}^*_n \to \mathcal{T}^*_{n-1}, \quad \varrho_+ : \mathcal{Q}^*_n \to \mathcal{Q}^*_{n-1}, \quad \varrho_- : \mathcal{Q}^*_n \to \mathcal{T}^*_{n-1}.$$

The outer reduction ϱ_- always creates a tri-complex so the latter T is not a typo.

Note that $K \in \mathcal{T}_n$ or $K \in \mathcal{Q}_n$ does not imply $\varrho_+(K, .) \in \mathcal{T}_{n-1}$ or $\varrho_+(K, .) \in \mathcal{Q}_{n-1}$. This observation is one reason why we cannot restrict ourselves to circle packings with admissible complexes: Starting with an *admissible* complex $K \in \mathcal{K}$ framed by a tri-(or quad-)complex $T \in \mathcal{T}$ (or $Q \in \mathcal{Q}$), and applying inner reduction, we end up with an *acceptable* complex $K^* \in \mathcal{K}^*$ framed by a tri-(or quad-)complex $T^* \in \mathcal{T}^*$ (or $Q^* \in \mathcal{Q}^*$)

that does not need to be proper anymore. This applies in particular to the skeleton $\sigma(T)$ of T (or $\sigma(Q)$ of Q).

The following very useful statement connects boundary reducibility with proper tri-complexes.

Lemma 3.2. *Let $T \in \mathcal{T}^*$ be a boundary irreducible tri-complex. Then either $T \in \mathcal{T}$, i.e., $\mathrm{int}T = K \in \mathcal{K}$ is an admissible complex, or $\{T\} = \mathcal{T}_1^*$. Moreover, in the first case T has the degree $\mathrm{deg}T = 3$.*

Proof. Clearly, if $\{T\} = \mathcal{T}_1^*$, then T is (boundary) irreducible. Thus, let $T \in \mathcal{T}^* \setminus \mathcal{T}_1^*$. Then the vertex set V of T has at least 5 elements, i.e., $|V| \geq 5$. The interior $K = \mathrm{int}T$ of T is a (non-void) simplicial complex of dimension $k \leq 2$ that contains at least two vertices. Since T is strongly connected, Lemma 2.4 states that K must be (edge-)connected. So there is at least one edge in K.

Let T be boundary irreducible. Let a b and c be the three boundary vertices of $T(a, b, c)$, and let v be a vertex of K. Then v cannot be a neighbor of all three vertices a, b and c since otherwise at least one of the triangles $\triangle(v, a, b)$, $\triangle(v, b, c)$ or $\triangle(v, c, a)$ would contain a vertex of $V \backslash \{a, b, c, v\}$ since $|V| \geq 5$, i.e., T would be boundary reducible. This already shows that T has the degree $\mathrm{deg}T = 3$.

We now prove that K is an admissible complex, i.e., $k = 2$ and $T \in \mathcal{T}$, what proves the lemma. To do so, we verify the properties (i)–(viii) demanded in Definition 2.2, but we move the proof of (i) to the end.

Fig. 3.5.: Constructions for the proof of Lemma 3.2

Property (ii). Assume contrarily that assertion (ii) does not hold true. Since no edge can belong to more than two faces, there is an edge $e := e(v_1, v_2)$ in K that is not contained in any face of K. Since e is an interior edge of T, there are two faces $f(u, v_1, v_2)$ and $f(w, v_1, v_2)$ in T with $u, w \in \{a, b, c\}$ and $u \neq w$. Because all three vertices a, b and c are pairwise neighbors, we have $v_2 \in \mathrm{int}\triangle(u, w, v_1)$ or $v_1 \in \mathrm{int}\triangle(u, w, v_2)$ (see Figure 3.5 left), which is impossible since T is boundary irreducible.

Properties (iii)-(iv). Let's prove (iii) and (iv). We already know that every vertex v of K has at least one edge e_1 in K. According to assertion (ii), such an edge is contained in a face of K. So there is a neighboring edge e_2 and even a chain of neighboring edges $e_1, e_2, \ldots, e_k, e_{k+1}$ in K with $k \geq 2$. If $e_{k+1} = e_1$, then v belongs to no boundary edge of K and (iii) and (iv) are fulfilled. If $e_{k+1} \neq e_1$, then v belongs to at least two boundary edges of K. Showing that three or more such boundary edges are impossible proves (iii) and (iv).

Assume v has at least three boundary edges $e(v, w_1)$, $e(v, w_2)$ and $e(v, w_3)$ in ∂K. The edges $e(v, w_1)$, $e(v, w_2)$ and $e(v, w_3)$ are inner edges of T and boundary edges of K. So they form faces with a boundary vertex of T. Since v can have at most two of the three vertices a, b and c as neighbors, we can assume (w.l.o.g.) that $f(v, w_1, a)$ and $f(v, w_2, a)$ are faces of T. Since $e(v, w_3)$ cannot belong to these faces, we may further assume (w.l.o.g.) that $f(v, w_3, b)$ is a face of T. But then $\triangle(v, a, b)$ is a triangle that either contains w_1 or w_2 (see Figure 3.5, right), i.e., T is boundary reducible, which is a contradiction.

Properties (v)-(vii). The assertions (v)–(vii) are trivially fulfilled since T is an admissible complex, and also (viii) holds true since the boundary of K clearly cannot be empty.

Property (i)). Assume contrarily that (i) does not hold true. We already know that the carrier of K is connected. So there must be a vertex u in T enclosed by a chain of vertices $(v_1, \dots, v_n, v_1)$ so that all v_i lie in K but u does not. This is impossible since K contains all vertices of T, but only the boundary vertices a, b and c not. $\square$

Note that Lemma 3.2 does not hold true for quad-complexes since the second quad-complex depicted in Figure 3.1 is (boundary) irreducible while its interior is neither admissible nor a single vertex.

If a tri- or quad-complex is irreducible, then we can rely on its well known local interior as described in by Lemma 2.1. Therefore, we can apply a very useful transformation. Roughly speaking, *merging* a vertex v with a vertex w yields a new complex resulting from the old one by replacing all simplexes containing v with the “corresponding“ simplexes containing w.

Although we only merge within tri- or quad-complexes, we wand to make this construction more explicit for the general case of arbitrary irreducible admissible complexes (for the sake of completeness).

Let $K = K(V, E, F)$ be an irreducible admissible complex, $v, w \in V$ with $v \in \text{int}V$ and $e(v, w) \in E$. Denote by $E(v)$ and $F(v)$ the set of all edges and the set of all faces in K, respectively, that contain v. Let $N = \{v_1, \dots, v_n\}$ be the ordered set of all neighbors of v so that $v_1 = w$. Define the sets of new edges and new faces by $E'(w) := \{e(w, v_i) : i = 3, \dots, n-1\}$ and $F'(w) := \{f(w, v_i, v_{i+1}) : i = 2, \dots, n-1\}$, respectively, and let

$$V' := V \setminus \{v\}, \quad E' := \big(E \setminus E(v)\big) \cup E'(w), \quad F' := \big(F \setminus F(v)\big) \cup F'(w).$$

Definition 3.4. *Merging* v *with* w *within* K *yields the admissible complex* $K'(V', E', F')$ *denoted by* $\mu(K; v, w)$.

That $K' = \mu(K; v, w)$ is a simplicial complex follows directly by it’s definition. Irreducibility of K together with Lemma 2.1 implies that $\mu(K; v, w)$ is also admissible (as suggested in Figure 3.6, left).

Unfortunately, there is a small hidden hurdle: We defined admissible complexes to be *geometrical* constructions, but merging is a *combinatoric* operation. So it may be impossible to perform the merging of v and w in such a way that all new faces of $F'(w)$ are geometrically triangles (see for example Figure 3.6, right). To be precise, we need to interpret K and K' as *abstract* simplicial complexes and then we can use Lemma 2.1 to prove admissibility.

Fortunately, this is not really a problem since (as any abstract admissible complex) K' can be represented as the contact graph $\widetilde{K}$ of a maximal packing $\mathcal{P}$ for K' (as explained in the previous section). Eventually, $\widetilde{K}$ can be reinterpreted as (geometrically) admissible complex. Hence, all in all we assume that w.l.o.g. $\mu(K; v, w)$ is an admissible complex (up to it's representation).

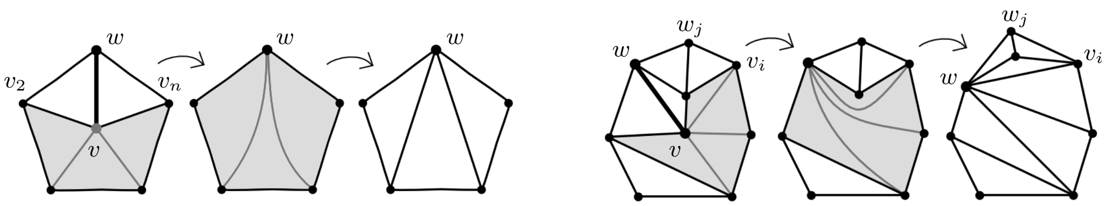


Fig. 3.6.: Definition of merging v with w (left) and a simple example (right)

Note that $\mu(K; ., .)$ need not be irreducible any more (see Figure 3.6, right). We get the following mapping property of the merging operator:

$$\mu : \mathcal{K}_n \cap \mathcal{K}^i \to \mathcal{K}_{n-1}.$$

Nevertheless, even if merging results in a reducible complex, then the reducible triangles of $\mu(K; ., .)$ have a notable structure (see also [20] Lemma 3).

Lemma 3.3. *Let $K \in \mathcal{K}^i$, let $v, w \in V$ with $v \in \operatorname{int}V$ and $e(v, w) \in E$. Let $N(v)$ and $N(w)$ be the sets of all neighbors of v and w, respectively. Then every reducible triangle $\triangle$ of $\mu(K; v, w)$ has the form $\triangle = \triangle(w, u_v, u_w)$ with*

$$u_v \in N(v) \setminus N(w), \qquad u_w \in N(w) \setminus N(v).$$

Proof. Let u_1, u_2, u_3 be the vertices of a reducible triangle $\triangle(u_1, u_2, u_3)$. Then one of the edges $e(u_1, u_2)$, $e(u_2, u_3)$ or $e(u_3, u_1)$ cannot be an edge of K otherwise K would be reducible. Assuming that $e(u_1, u_2)$ is not in K, we conclude from the definition of $\mu(K; v, w)$ that either u_1 or u_2 must be the vertex w while the other one must be a neighbor of v in K, say $u_1 = w$ and $u_2 =: u_v$, i.e., $u_v \in N(v) \setminus \{w\}$. Moreover, we have $u_v \notin N(w)$ otherwise $e(u_1, u_2) = e(w, u_v) \in E$ would be an edge in K, which was excluded. Hence $u_v \in N(v) \setminus N(w)$. For the third vertex u_3 we clearly have $u_3 \neq w$ and $u_3 \neq v$. Moreover, u_3 cannot be a neighbor of v otherwise $\triangle$ would be one of the new (non-reducible) faces of $\mu(K; v, w)$. So it only remains that u_3 is a neighbor of w, and thus $u_3 =: u_w \in N(w) \setminus N(v)$. $\square$

Whenever merging in an *irreducible* complex K produces a *reducible* triangle, this triangle is associated with an *irreducible* quad-complex contained in K. In the following lemma we adopt the notations from the proof of Lemma 3.3 (see also [20] Lemma 4).

Lemma 3.4. *Let $K \in \mathcal{K}^i$. Assume that $\mu(K; v, w)$ has a reducible triangle $\triangle(w, u_v, u_w)$. Then the minimal admissible sub-complex of K containing the edges $e(w, v)$, $e(v, u_v)$, $e(u_v, u_w)$ and $e(u_w, w)$ is an irreducible quad-complex $Q(w, v, u_v, u_w)$ or (depending on the orientation) $Q(w, u_w, u_v, v)$. Every vertex in the interior of $\triangle$ is contained in Q.*

Proof. Using Lemma 3.3 one can verify that the four vertices w, v, u_v and u_w as well as the four edges $e(v, u_v)$, $e(u_v, u_w)$, $e(u_w, w)$ and $e(w, v)$ are pairwise different. Moreover, again by Lemma 3.3, u_v is not a neighbor of w, and u_w is not a neighbor of v. So Q is a strongly connected admissible complex with exactly four boundary vertices, a quad-complex, that is irreducible because K is irreducible (for an example see Figure 3.6, right). The orientation of Q depends on the orientation of $\triangle(w, u_v, u_w)$, but either $Q(w, v, u_v, u_w)$ or $Q(w, u_w, u_v, v)$ must be positively oriented. By definition of Q, every vertex in the interior of $\triangle$ is also a vertex of Q. □

3.1.3. Trilaterals and Quadrilaterals

Since we want to normalize a circle agglomeration filling a domain G by prescribing the positions of three (or four) boundary disks, we establish so called *trilaterals* and *quadrilaterals*. To keep things simple we consider Jordan domains first.

A *Jordan trilateral* $G = G(\alpha, \beta, \gamma)$ is a Jordan domain G whose boundary is split into three (topologically closed) arcs α, β, γ. We assume that the *edges* α ,β and γ of G are arranged in counter-clockwise order and refer to their common endpoints $\alpha \cap \beta$, $\beta \cap \gamma$ and $\gamma \cap \alpha$ as *vertices* of G.

A similar definition leads to *Jordan quadrilaterals* $G = G(\alpha, \beta, \gamma, \delta)$. Here, the boundary of the Jordan domain G is split into the four (counter-clockwise oriented) arcs α, β, γ and δ. The *standard quadrilaterals* $R(\alpha, \beta, \gamma, \delta)$ are rectangles with sides parallel to the coordinate-axes so that the distinguished arcs α, β, γ and δ correspond to the four sides ("lower", "right", "upper" and "left") of R, respectively.

In order to define the concept of tri- and quadrilaterals for general bounded, simply connected domains G, we use an (arbitrary) extended canonical embedding $g^* : G^* \to \overline{\mathbb{D}}$ of G (as introduced in Section 2.2.1).

Definition 3.5. A *trilateral* $G(\alpha, \beta, \gamma)$ is a bounded, simply connected domain G with three distinguished sets α, β and γ of prime ends so that $\mathbb{D}\big(g^*(\alpha), g^*(\beta), g^*(\gamma)\big)$ is a Jordan trilateral. The sets α, β, γ and the prime ends $\alpha \cap \beta$, $\beta \cap \gamma$, $\gamma \cap \alpha$ are referred to as the *edges* and the *vertices* of the trilateral, respectively. A similar definition leads to *quadrilaterals* $G(\alpha, \beta, \gamma, \delta)$.

Besides the canonical embedding $g : G \to \mathbb{D}$ of the quadrilateral $G(\alpha, \beta, \gamma, \delta)$ there is an alternative mapping $g_R : G \to R$, which is sometimes more convenient. It maps G conformally onto a standard quadrilateral R (rectangle with specific aspect ratio) so that its extension $g_R^* : G^* \to \overline{R}$ sends α, β, γ andδ to the four sides α^*, β^*, γ^* and δ^* of R, respectively.

3.1.4. Circle Agglomerations and Generalized Circle Agglomerations

Since we established new types of complexes, we also need suitable packings. Moreover, as the similarity within the notation of trilateral and tri-complex (quadrilateral and quad-complex) already suggests, we are going to associate both terms with one another. To do so, we first generalize the definition of circle packings in order to fit with acceptable complexes.

Definition 3.6. A collection $\mathcal{P}$ of disks D_v is said to be a *circle agglomeration* for the acceptable complex $K(V, E, F) \in \mathcal{K}^*$ if it satisfies the following conditions (i)–(iii):

(i) Every vertex $v \in V$ has an associated disk $D_v \in \mathcal{P}$ so that $\mathcal{P} = \{D_v : v \in V\}$.

(ii) If $e(u, v) \in E$ is an edge of K, then the disks D_u and D_v touch each other.

(iii) If $f(u, v, w) \in F$ is a positively oriented face of K, then the centers of the disks D_u, D_v and D_w form a positively oriented triangle in the plane.

Since we are also interested in degenerate packings, we provide an even more generalized version of circle agglomerations, which is in accordance with the definition of generalized circle packings.

Definition 3.7. A collection $\mathcal{P}$ of dots S_u and disks D_v is said to be a *generalized circle agglomeration* for the acceptable complex $K(V, E, F) \in \mathcal{K}^*$ if it satisfies the following conditions (i)–(iii):

(i) Every vertex $v \in V$ has either an associated dot $S_v \in \mathcal{P}$ or disk $D_v \in \mathcal{P}$ so that $\mathcal{P} = \{S_u : u \in U \subset V\} \cup \{D_v : v \in V \setminus U\}$.

(ii) If $e(v, w) \in E$ is an edge of K, then S_v and S_w, or S_v and D_w, or D_v and S_w, or D_v and D_w, respectively, touch each other (as explained in Section 2.3).

(iii) If $f(u, v, w) \in F$ is a positively oriented face of K so that all three associated sets are disks D_u, D_v and D_w, then the centers of D_u, D_v and D_w form a positively oriented triangle in the plane.

Again, we denote by $\mathcal{S}$ the set of all dots within a generalized circle agglomeration $\mathcal{P}$ whereas the set of all disks shall be $\mathcal{D}$, i.e., $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$. If $\mathcal{S} \neq \emptyset$, then $\mathcal{P}$ is said to be *degenerate*, and if even $\mathcal{D} = \emptyset$, then we call $\mathcal{P}$ *collapsed*. If $\mathcal{S} = \emptyset$, then $\mathcal{P}$ is a circle agglomeration as defined before.

In accordance with the definitions of Chapter 2, a (generalized) circle agglomeration $\mathcal{P}$ for an admissible complex $K \in \mathcal{K}$ is a (generalized) circle packing. Also all combinatorial deduced terms can be applied to (generalized) circle agglomerations as well like *boundary* disks (dots), *interior* disks (dots), *neighboring* disks (dots), etc.

A (generalized) circle agglomeration is called *univalent* if its *disks* are non-overlapping, i.e., $D \cap D' = \emptyset$ for all $D, D' \in \mathcal{D}$ with $D \neq D'$. Note that in this paper all (generalized) circle agglomeration are assumed to be univalent.

The *contact point* $c(u, v)$ between two neighboring disks or dots is still defined as $\overline{D}_u \cap \overline{D}_v$, $S_u \cap \overline{D}_v$, $\overline{D}_u \cap S_v$ or $S_u = S_v$, respectively. Moreover (for details see Definition 2.13), a dot $S = \{p\}$ is *attached* to D_1 if there is a chain of dots connecting S with D_1, and p is a *pseudo contact point* of D_1 and D_2 if both disks are attached to S.

The following generalization of Lemma 2.18 forms the nucleus of generalized domain filling circle agglomerations.

Lemma 3.5. *Let $\mathcal{P}$ be a degenerate circle agglomeration for an acceptable complex K. Let $S_w \in \mathcal{S}$ be a dot of $\mathcal{P}$. Then there is a chain of dots $(S_1, \ldots, S_k)$ in $\mathcal{P}$ so that $S_1 = S_w$ and S_k is a boundary dot. Furthermore, if $|\mathcal{D}| \geq 1$, then S_w is attached to a disk in $\mathcal{P}$.*

Proof. Since K is (edge) connected, the second assertion of the lemma trivially holds true. The same goes for the first assertion whenever w is a boundary vertex.

Assume w is an interior vertex. Let K^* be the kernel of K with respect to w, and let $\mathcal{P}^*$ be the associated main part of $\mathcal{P}$. By Lemma 2.18 there is a chain of dots $(S_1, \ldots, S_k)$ in $\mathcal{P}^*$ so that $S_1 = S_w$ and S_k is a boundary dot of $\mathcal{P}^*$, and thus of $\mathcal{P}$ by Lemma 2.3. $\square$

Using Lemma 3.5, we define when a non-collapsed generalized circle agglomeration *lies in* a bounded, simply connected domain, or even *fills it* (without normalization). Note that for generalized circle packings the following definitions coincide with those of Section 2.3.3 (indeed, substitute “agglomeration” with “packing” and they are literally the same).

Definition 3.8. A non-collapsed generalized circle agglomeration $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ with acceptable complex K *is contained in* a bounded, simply connected domain G if the following conditions (i) and (ii) are satisfied.

(i) Every disk $D \in \mathcal{D}$ lies in G, i.e., $D \subset G$.

(ii) If p is a contact point or pseudo contact point between $D_v, D_w \in \mathcal{D}$ that lies on the boundary of G, then the two disks D_v and D_w touch the same prime end at p.

Definition 3.9. Let $\mathcal{P}$ be a non-collapsed generalized circle agglomeration contained in G. A dot $\{p\} = S \in \mathcal{P}$ *touches a prime end* X of G if one (and then all) of its attached disks touches X at p. A dot S *touches a set* α of prime ends if it touches a member of α.

Definition 3.10. Let G be a bounded, simply connected domain, and let $\mathcal{P}$ be a generalized circle agglomeration. We say $\mathcal{P}$ *fills* G if

(i) $\mathcal{P}$ is contained in G, and

(ii) every boundary disk and every boundary dot of $\mathcal{P}$ touches a prime end of G.

Finally, we define when a (generalized) circle agglomeration $\mathcal{P}$ fills a trilateral. The interplay of domain and tri-complex specifies the *normalization* of $\mathcal{P}$. Please note that the normalization with respect to quadrilaterals and quad-complexes works analogously.

Definition 3.11. Let $G(\alpha, \beta, \gamma)$ be a trilateral. Let $T(a, b, c) \in \mathcal{T}^*$ be a tri-complex with boundary vertices a, b and c. We say that a generalized circle agglomeration $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ is *associated with T and fills G* if the following conditions (i)-(iii) are satisfied:

(i) The generalized circle agglomeration $\mathcal{P}$ has complex $K(V, E, F) := \operatorname{int} T$.

(ii) $\mathcal{P}$ fills G (Definition 3.10).

(iii) If $u = a$, $u = b$ or $u = c$, and if $v \in V$ is a neighbor of u in T, then S_v or D_v (respectively) touches the corresponding edge α, β or γ, respectively.

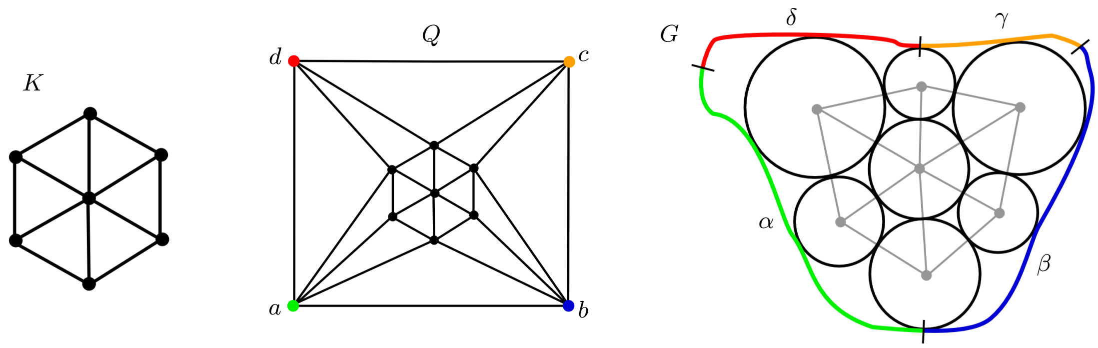


Fig. 3.7.: A complex K, a framing of K, and an associated circle packing filling a quadrilateral

According to (iii), each *leading disk* D_v in $\mathcal{P}$ associated with a leading vertex of a tri-complex touches (at least) two of the edges of the associated trilateral G, say α and β. The common end point X of α and β is referred to as a *meeting point* of D_v with ∂G, and we say that D_v *meets* ∂G at X.

Analogously, we define *leading dots*, and we apply those terms also to quad-complexes with associated quadrilaterals. Note that – despite its name – a meeting point is in general not a point but a prime end.

Note further that a dot S that meets a vertex prime end X of G always has to touch X while a disk D meeting X does not necessarily need to do so (see Figure 3.7). Nevertheless, under an additional assumption on G one can assure that D *separates* X from all disks in $\mathcal{P}$, namely if the tri- or quadrilateral G is *not spiky*. This is investigated in the next section; in particular see Lemma 3.9.

3.1.5. Regularity and Tameness

As we shall see, uniqueness (and other aspects) of domain-filling (generalized) circle agglomerations depends on local properties of the boundary of the domain. To describe these effects, we provide the following characterization of prime ends. Readers only interested in Jordan domains may substitute prime ends by boundary points.

Definition 3.12. A prime end $X \in \partial G^*$ is said to be *regular* if for any two open disks D_1 and D_2 that lie in G and touch X necessarily $D_1 \subset D_2$ or $D_2 \subset D_1$. A trilateral (quadrilateral) is said to be *tame* if all its vertices are regular.

Note that, according to this definition, a prime end that cannot be touched by a disk in G is regular. Such a prime end is said to be *untouchable*. This happens for all inaccessible prime ends, but also for boundary points of Jordan domains that are "outward corners" (in a naive meaning).

A prime end that is not regular is called *inward corner*. An *inward spike* is a prime end that can be touched by two disjoint disks D_1 and D_2 in G. Obviously, an inward spike is also an inward corner.

Trilaterals and quadrilaterals are called *spiky* if at least one of its vertices is an inward spike. Conversely, a domain with solely regular prime ends is called *regular*, e.g. if ∂G is of class $\mathcal{C}^1$.

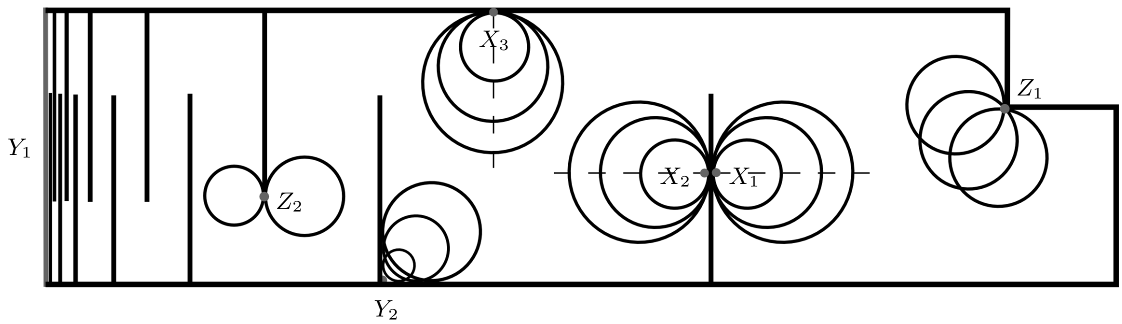


Fig. 3.8.: The prime ends X_i, Y_i are regular, Z_i are inward corners, and Z_2 is a spike

The following auxiliary result provides a criterion for regularity of prime ends (see also [20] Lemma 9).

Lemma 3.6 (Three touching disks)**.** *Let G be a bounded, simply connected domain and assume that open disks $D_1, D_2 \subset G$ touch a prime end X of G at the point $p \in \partial G$. If there is a disk D so that $p \in \overline{D}$, $D_1 \cap D = \emptyset$ and $D_2 \cap D = \emptyset$, then $D_1 \subset D_2$ or $D_2 \subset D_1$.*

Proof. We point out that the disk D does not have to lie in G. Since $D_j \cap D = \emptyset$, the disks D_j and D must touch the point p, have a common tangent τ at p, and lie on opposite sides of τ. But then D_1 and D_2 lie on the same side of τ so that one disk is contained in the other. □

Here is a first application of Lemma 3.6. Again the disk D does not have to lie in G (see also [20] Lemma 11).

Lemma 3.7 (Disk removal)**.** *Let D be a disk and assume that $\overline{D}$ intersects the boundary of a bounded, simply connected domain G, i.e., $\overline{D} \cap \partial G \neq \emptyset$. If X is a prime end of G that D does not touch, then there is a (unique) connected component G_X of $G \setminus \overline{D}$ with $X \in \partial G^*_X$. The domain G_X is simply connected and all accessible prime ends Y of G_X with accessible point $p(Y) \in \overline{D}$ are regular.*

Proof. According to Definition 2.9, we find a tail U of a null-chain (J_n) representing X so that $U \cap D = \emptyset$. Since U is open and connected, this implies that U is contained in a connected component G_X of $G \setminus \overline{D}$. Then $U_n \subset G_X$ for every sufficiently large n. So (J_n) is also a null-chain within G_X representing a prime end of G_X. In what follows we identify this prime end with X and write $X \in \partial G^*_X$.

We show that G_X is simply connected. As open and connected subset of $\mathbb{C}$ the domain G_X is path-connected. So it is simply connected if and only if the interior G_Γ of every Jordan curve $\Gamma \subset G_X$ is contained in G_X ([28], p.29) — which we are going to show next.

Assume that there is a point $p \in G_\Gamma \setminus G_X$. Connecting p by a curve in G_Γ with $\Gamma \subset G_X$, we see that there must exist a point $q \in G_\Gamma \cap \partial G_X \neq \emptyset$. Now $\partial G_X \subset \partial G \cup \partial D$ and $q \in G_\Gamma \subset G$ whence $q \in G_\Gamma \cap \partial D \neq \emptyset$. Since $\Gamma \subset G_X$ and $G_X \cap \overline{D} = \emptyset$, we have $\Gamma \cap \partial D = \emptyset$, i.e., $\partial D \subset G_\Gamma$. Since G_Γ is simply connected, this implies that $\overline{D} \subset G_\Gamma \subset G$, which finally yields the contradiction $\overline{D} \cap \partial G = \emptyset$.

Assume that an (accessible) prime end Y of G_X with an accessible point $p \in \overline{D}$ is touched by two disks $D_1, D_2 \subset G_X$. Since the impression $I(Y)$ of Y contains at most one point p that is accessible via Y, both disks touch Y at p. Since D is disjoint to G_X, we have $D \cap D_j = \emptyset$, and hence Lemma 3.6 tells us that $D_1 \subset D_2$ or $D_2 \subset D_1$, i.e., Y is regular. □

The exceptional role of spikes is made clear by the next result (see also [20] Lemma 10).

Lemma 3.8 (Touching boundary arcs). *Let $\mathcal{P}$ be a generalized circle agglomeration that fills a non-spiky trilateral $G(\alpha, \beta, \gamma)$ for a tri-complex $T(a, b, c)$. Alternatively let $\mathcal{P}$ fill a non-spiky quadrilateral $G(\alpha, \beta, \gamma, \delta)$ for a quad-complex $Q(a, b, c, d)$. Let v be an interior vertex of T or Q that is associated with a disk $D_v \in \mathcal{P}$. Then D_v touches α, β, γ or δ if and only if v is a neighbor of a, b, c or d, respectively.*

Proof. If $\mathcal{P}$ fills G, then every disk must touch its associated edge(s). So we only need to prove that it cannot touch an additional one. In the first case we look at vertices that are neighbors of a boundary vertex of a tri- and then of a quad-complex (second case). Afterwards, in Case 3, we deal with the vertices that are not neighbors of a boundary vertex.

Case 1. Let the complex of $\mathcal{P}$ be a tri-complex $T(a, b, c) \in \mathcal{T}^*$, and let G be a trilateral $G(\alpha, \beta, \gamma)$. Let $K := \operatorname{int} T$ and let v be an arbitrary vertex of K that is a neighbor of a, b or c in T, say a, and that is associated with a disk $D_v \in \mathcal{P}$. Since there is nothing to prove if v is a neighbor of all three boundary vertices of T, we assume w.l.o.g. that v and b are no neighbors in T.

Assume D_v touches β. Let $p, q \in \partial G$ (maybe $p = q$) be contact points between D_v and α, β, respectively. Let $a, v_1, \ldots, v_k, c$ be the neighbors of b in T, arranged in consecutive order. Since $\alpha \cap \beta \cap \gamma = \emptyset$, not every vertex v_i (for $i = 1, \ldots, k$) can be associated with one (and thus the same) dot S_i in $\mathcal{P}$. So let $1 \leq j \leq k$ be the smallest index so that $v_j =: w$ is associated with a disk D_w. According to its definition, the disk D_w touches α and β. Let $p', q' \in \partial G$ (maybe $p' = q'$) be the corresponding contact points, respectively.

In order to show the contrary, assume that $p = q$ and $p' = q'$. Then D_v associates p and q with one and the same prime end X, and D_w associates p' and q' with a common prime end Y. By definition we have $X \in \alpha \cap \beta$ and $Y \in \alpha \cap \beta$. Thus, we conclude $X = Y$. Since $v \neq w$, two disks of $\mathcal{P}$ touch the same vertex of G, which contradicts the fact that G is not spiky. So $p \neq q$ or $p' \neq q'$.

At least one of the chords $\sigma_v = [p, q]$ or $\sigma_w = [p', q']$ is a crosscut of G. For $u \in \{v, w\}$, if σ_u is a crosscut, then let G_u be that component of $G \setminus \sigma_u$ that contains the prime end $\alpha \cap \beta$. Clearly, either $D_w \subset G_v$ or $D_v \subset G_w$.

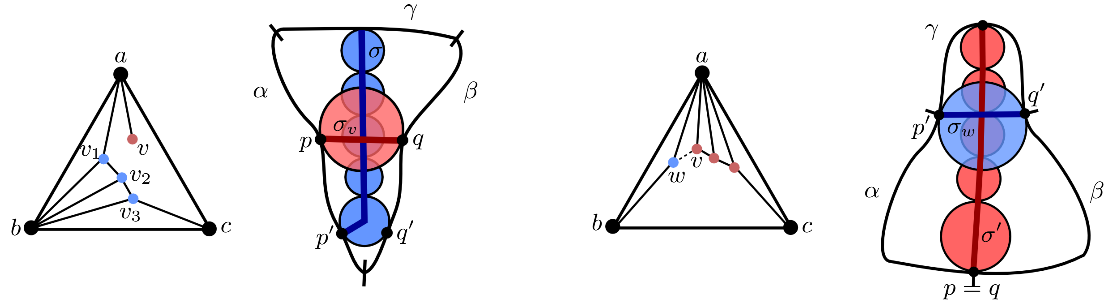


Fig. 3.9.: Schematic constructions for Case 1 of the proof of Lemma 3.8

Assume $D_w \subset G_v$ (see Figure 3.9, left). We look at the neighbors $w = v_j, \ldots, v_k, c$ of b in T. From this vertices we only keep those that are associated with disks. We denote them $w = u_1, \ldots, u_l$ (maybe $l = 1$) and their associated disks $D_1, \ldots, D_l$. The disks $D_1 = D_w$ and D_l have contact points with α and γ, respectively, say at $p = c_0$ and some $c_l \in \partial G$, respectively. For $i = 1, \ldots, l-1$ let c_i be the contact point of D_i and D_{i+1}. Then the concatenation σ of chords $\sigma_i := [c_{i-1}, c_i] \subset \overline{D_i}$ $(i = 1, \ldots, l)$ connects prime ends in α and γ. Now $\partial G_v^* \cap \gamma$ contains at most two elements, namely the vertex prime ends $\alpha \cap \gamma$ or $\beta \cap \gamma$, which then must be touched by the two disjoint disks D_v and D_l. Since G is not spiky, this is impossible, i.e., $D_w \not\subset G_v$.

Assume $D_v \subset G_w$ (see Figure 3.9, right). The leading vertex v_1 within the face $f(a, b, v_1)$ cannot be associated with a dot since then D_w would touch $\alpha \cap \beta$, i.e., $G_w = \emptyset$, which is a contradiction to $D_v \subset G_w$. Therefore we have $w = v_1$. Let $b, w_1, \ldots, w_m, c$ be the neighbors of a in T, arranged in consecutive order. We have $w_1 = w = v_1$ and $v = w_n$ for some $2 \le n \le m$. Let $\{v = u_1, \ldots, u_l\}$ be the ordered subset of those vertices w_j that are associated with disks in $\mathcal{P}$ and that have an index of at least $j \ge n$. Let $D_1, \ldots, D_l$ be their associated disks. Then $D_w \ne D_i$ for all $i = 1, \ldots, l$. Let c_i denote the contact point of D_i with D_{i+1} $(i = 1, \ldots, l-1)$, and let c_0 and c_l be a contact point of D_1 and D_l with β and γ, respectively. Then the concatenation σ' of chords $\sigma_i := [c_{i-1}, c_i]$ $(i = 1, \ldots, l)$ connects prime ends in β and γ. Again $\partial G_w^* \cap \gamma$ contains at most $\alpha \cap \gamma$ or $\beta \cap \gamma$, which then must be touched by D_w and D_l. Since G is not spiky, this is impossible, i.e., $D_v \not\subset G_w$. Hence, the assumption that D_v touches β was wrong. This concludes the Case 1.

Case 2. Let $Q(a, b, c, d) \in \mathcal{Q}^*$ be the complex of $\mathcal{P}$ and let G be a quadrilateral $G(\alpha, \beta, \gamma, \delta)$. Let $K := \operatorname{int} Q$ and let v be an arbitrary vertex v of K that is associated with a disk $D_v \in \mathcal{P}$ and that is a neighbor of a boundary vertex of Q. Denoting $N \subset \{a, b, c, d\}$ the set of boundary vertices that are neighbors of v, we may suppose (without loss of generality) that $a \in N$. Since there is nothing to prove if $N = \{a, b, c, d\}$, we distinguish the cases that b, c or d are not contained in N.

Assume that $b \notin N$ but that D_v touches β. This case runs nearly the same as for tri-complexes, thus we will keep the arguments rather brief (see Figure 3.10, left and middle). Let the disk D_w, the points $p, p', q', q \in \partial G$, the chords $\sigma_v = [p, q]$ and $\sigma_w = [p', q']$ as well as the domains G_v and G_w be defined as above. If $D_w \subset G_v$, then the same argumentation as before provides that D_l must touch γ in $\partial G_v^* \cap \gamma$, which (here) only consists of the vertex prime end $\beta \cap \gamma$, which then must be an inward spike. If $D_v \subset G_w$, then we also proceed as above but with the single difference that here d and not c is the last neighbor of a. Therefore, it turns out that D_l (together with D_w) must touch δ in $\partial G_w^* \cap \delta$, which contains at most $\delta \cap \alpha$, hence G needs to be spiky.

The case where $d \notin N$ can be treated the same way (by symmetric arguments).

The proof for $c \notin N$ is even simpler (see Figure 3.10, right). Assume that D_v touches α and γ. Then there is a chord σ_v of D_v that connects prime ends in α and γ. Now we take the neighbors $b, v_1, \ldots, v_k, d$ of c in Q, arranged in consecutive order, and select all vertices that are associated with disks. Since $\beta \cap \delta = \emptyset$ there must be at least one. Their associated disks shall be $D_1, \ldots, D_l$, and they define again a chain σ of chords $\sigma_i \subset \overline{D_i}$. This time σ connects prime ends in β and δ. Clearly, σ_v and σ must intersect at some point p, and since $D_j \neq D_v$ for every i, the point p must be a common endpoint of σ and σ'. This is only possible if p is the contact point of D_v and D_1 or D_l with one of the four vertices of the quadrilateral G, i.e., G needs to be spiky. This concludes the Case 2.

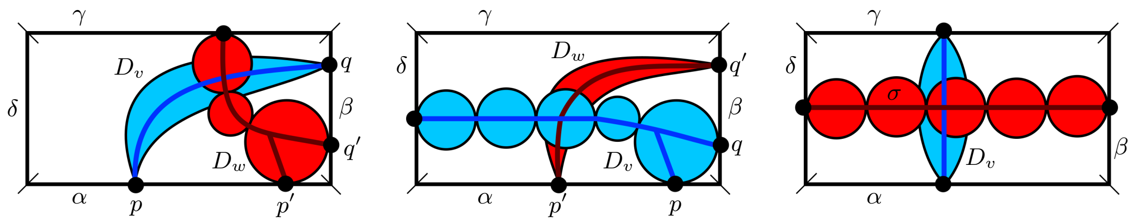


Fig. 3.10.: Schematic constructions for the proof of Lemma 3.8

Case 3. Consider a vertex v in $K = \text{int}T$ (analogously for $K = \text{int}Q$) so that v is associated with a disk while it is not a neighbor of any boundary vertex of T (or Q). If all the neighbors $\{u_1, \ldots, u_k\}$ of v are associated with disks $D_1, \ldots, D_k$ in $\mathcal{P}$, then D_v trivially cannot touch any edge of G since the chain of the disks D_i separates D_v from ∂G. Therefore, assume that there is a neighbor u of v so that u is associated with a dot S_u.

By Lemma 3.5, there is a boundary dot $S_w = S_u$ so that its associated vertex w is a neighbor of a boundary vertex of T (or Q), say of b. Thus, the disk D_v touches ∂G in $\{p\} = S_w$. Moreover, D_v touches β via the attached S_w.

Let $N(b) = \{a, v_1, \ldots, v_m, c\}$ be the ordered set of all neighbors of b, then $w = v_j$ for some $1 \leq j \leq m$. Since except D_v at most one other disk of $\mathcal{P}$ can touch p, at least one of the vertices v_1 and v_m (as neighbors or neighbors-neighbors of w; maybe $w = v_1$ or $w = v_m$) must be associated with a dot $S = S_w$. Nevertheless, at least one v_i must be associated with a disk in $\mathcal{P}$, otherwise $S_1 = S_m$ would touch $\alpha \cap \beta \cap \gamma = \emptyset$, which is impossible.

Let w.l.o.g. v_1 be associated with the dot $S_1 = S_w = \{p\}$, while n (with $1 < n \leq m$) is the smallest index so that v_n is associated with a disk D_n. According to construction, p is a pseudo contact point of D_n and D_v. Since S_1 touches α while S_w touches β, both disks D_n and D_v touch $\alpha \cap \beta$, which is only possible if G is spiky. □

Note that Lemma 3.8 does not hold for the dots of a degenerate circle agglomeration. For a counter example see Figure 3.12. There, almost all vertices of the interior of the tri-complex T_2 are associated with a dot touching two edges of a trilateral, although many of them are not neighbors of both corresponding boundary vertices of T_2.

Lemma 3.8 is a powerful tool that will make several upcoming proofs much easier. But there is even more to understand about the behavior of (boundary) disks touching the boundary of their domain. The following result consolidates the idea of meeting points as a generalization of touching points.

Definition 3.13. Let G be a bounded, simply connected domain. Let $\Gamma \subset G$ be a crosscut of G and let G_1 and G_2 be the two connected components of $G \setminus \Gamma$. Let $X \in \partial G^*$ be a prime end of G that is not touched by Γ. Let $\mathcal{D}$ be a set of pairwise disjoint disks contained in G. Then we say that Γ *separates* X from $\mathcal{D}$ if X is a subordinate prime end of G_1 and every disk of $\mathcal{D}$ is contained in G_2.

Lemma 3.9 (Separation by leading disks)**.** *Let $\mathcal{P}$ fill the trilateral (or quadrilateral) G for T (or Q). Let G be non-spiky. Let D_v be a leading disk of $\mathcal{P}$ that meets its vertex prime end $X \in \partial G^*$. Then D_v touches X or there is a crosscut $\Gamma \subset D_v$ separating X from all disks of $\mathcal{P} \setminus \{D_v\}$.*

Proof. Let (w.l.o.g.) v form the face $f(a, b, v)$ with two boundary vertices of $T(a, b, c)$ (or $Q(a, b, c, d)$). Let X be the vertex prime end of $G(\alpha, \beta, \gamma)$ (or $G(\alpha, \beta, \gamma, \delta)$) in $\alpha \cap \beta$. If D_v touches X, then there is nothing to prove.

Assume that D_v does not touch X. This implies contact points $p, q \in \partial G$ between D_v and α, β, respectively. Moreover, we have $p \neq q$ and neither p nor q is a contact point between D_v and X. Thus, the open chord $\sigma = (p, q) \subset D_v$ is a crosscut of G, and exactly one of the components G_1 and G_2 of $G \setminus \sigma$ contains X as a subordinate prime end, say G_1. We want to show that every disk of $\mathcal{P} \setminus \{D_v\}$ is contained in G_2, thus we assume the contrary.

Assume $D_u \in \mathcal{P} \setminus \{D_v\}$ is a disk in G_1, i.e. $D_u \subset G_1$. Since T (or Q) are 3-connected (Lemma 2.4), we can remove v and all its edges and faces and still there is a chain $C = (u = u_1, \ldots, u_k, w)$ so that every u_i is an interior vertices while w is a boundary vertex.

If $w \in \{a, b\}$, then we may and will assume w.l.o.g. that $w = b$. If $w = c$ or $w = d$, then we may and will assume w.l.o.g. that $w = c$. In the latter case we simply set $C' := C$. In the former case let $N(b) = \{a, v_1, \ldots, v_m, c\}$ be the ordered set of neighbors of b. Clearly $u_k = v_j$ for some $1 \leq j \leq m$. Using this, we extend the chain C to $C' = (u_1, \ldots, u_k = v_j, \ldots, v_m)$.

In both cases C' connects u with a neighbor of c. We now remove all vertices from C' that are associated with dots, and we denote the remaining vertices $w_1, \ldots, w_l$ with

$w_1 = u$ (and maybe $l = 1$). The associated disks shall be $D_1, \ldots, D_l$. By construction, D_l touches γ.

Let c_i denote the contact point of D_i with D_{i+1} ($i = 1, \ldots, l-1$) and let c_l be a contact point of D_l with γ. Let c_0 be the center of D_1. Then the concatenation σ' of chords $\sigma_i := [c_{i-1}, c_i]$ ($i = 1, \ldots, l$) connects a point in G_1 with a prime end in γ.

Since $D_1 = D_u \subset G_1$ and $D_v \neq D_i$ for every $i = 1, \ldots, l$, we conclude $D_i \subset G_1$. Now, $\partial G_1^* \cap \gamma$ contains at most two elements, namely the two endpoints of γ, which then must be touched by the two disjoint disks D_v and D_l. This is only possible if G is spiky. □

Finally, we state a generalized version of Lemma 2.16 that guarantees that the ordering of prime ends touched by boundary disks or boundary dots of $\mathcal{P}$ corresponds (in some sense) to the ordering of the boundary vertices of int$T$ or rather int$Q$. Roughly speaking, the orientation of boundary interstices equals the orientation of the associated boundary faces.

Lemma 3.10 (Ordering of touched prime ends)**.** *Let $T(a, b, c)$ be a tri-complex and let $G(\alpha, \beta, \gamma)$ be a non-spiky trilateral. Alternatively let $Q(a, b, c, d)$ be a quad-complex and let $G(\alpha, \beta, \gamma, \delta)$ be a non-spiky quadrilateral. Let $\mathcal{P}$ be a generalized circle agglomeration for T or Q filling the tri- or quadrilateral G, respectively. Let u be any boundary vertex of T or Q, say $u = b$. Let $N(b) = \{a = v_0, v_1 \ldots, v_m, v_{m+1} = c)$ be the ordered set of all neighbors of b. For every $j = 1, \ldots, m$, let X_j be a prime end of β touched by the disk or dot P_j that is associated with v_j. Then the ordering of these prime ends on β is*

$$X_1 \preceq X_2 \preceq \ldots \preceq X_m.$$

Proof. In order to prove the lemma, we assume contrary that there is some $j > i$ with $X_j \prec X_i$. Let j be minimal. Then we can choose $i = j - 1$. We distinguish whether P_1 is a disk or a dot, and whether it touches $X := \alpha \cap \beta$ or not. Note that P_1 is a leading disk or a leading dot of $\mathcal{P}$ that meets X.

Case 1. Assume $P_1 = D_1$ is a disk that does not touch X. Then Lemma 3.9 states that there is a crosscut $\Gamma \subset D_1$ separating X from all disks of $\mathcal{P} \setminus \{D_1\}$. The endpoints of Γ are contact points of D_1 with a prime end $Y \in \alpha$ and a different prime end $Z \in \beta$. Thus, on $\alpha \cup \beta$, we have the ordering

$$Y \prec Z \preceq X_j \prec X_i, \quad \text{and in particular} \quad Y \prec X_j \prec X_i.$$

We will use this result later, but before we do so we show that the other cases lead to similar results.

Case 2. Assume $P_1 = D_1$ is a disk touching X. Then none of the sets $P_2, \ldots, P_m$ can touch X. In order to see this, assume some $P_k \neq D_1$ touches X. Since X is no inward spike and D_1 already touches it, the set P_k must be a dot $P_k = S_k$. Thus, the set P_{k+1} must be a dot $S_{k+1} = S_k$, too, in order to touch S_k. For the same reason, we get that all sets $P_k, \ldots, P_m$ are dots $S_k = \ldots = S_m$. Since v_m is a neighbor of c, its associated dot S_m must touch γ. Furthermore (by assumption), the dot $S_k = S_m$ touches X. Since a dot always touches at most one prime end, we conclude $X \in \gamma$, hence, $X \in (\alpha \cap \beta \cap \gamma) = \emptyset$. This contradiction yields $X \neq X_k$ for $k = 2, \ldots, m$, so in particular $X \neq X_j$. Let $Y := X$. On β and in particular on $\alpha \cup \beta$, we then have the ordering

$$Y \prec X_j \prec X_i.$$

Case 3. Assume $P_1 = S_1$ is a dot. Then S_1 touches X automatically. Let $l \in \{2, \ldots, m\}$ be the smallest number so that $P_l = D_l$ is associated with a disk. Clearly l exists, otherwise $S_1 = \ldots = S_m$ would touch a prime end in $(\alpha \cap \beta \cap \gamma) = \emptyset$, what is impossible. Moreover, we have $l \leq i$, otherwise all dots $S_1 = \ldots = S_j$ (recall that $j = i + 1$) would touch one and the same prime end $X_1 = \ldots = X_j$, what contradicts $X_j \prec X_i$. Since D_l is a disk touching X, none of the sets $P_{l+1}, \ldots, P_m$ can touch X, too, for the same reason as stated in Case 2. We conclude $X \neq X_k$ for every $k = l+1, \ldots, m$. In particular we have $X \neq X_j$. Let $Y := X$. On β and in particular on $\alpha \cup \beta$, we then have the ordering

$$Y \prec X_j \prec X_i.$$

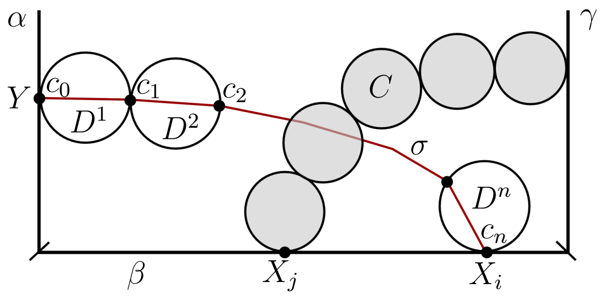


Fig. 3.11.: Schematic constructions for the proof of Lemma 3.10

Case 1-3. Since all three cases lead (essentially) to the same situation, we treat them simultaneously (see Figure 3.11). Recall that in every case at least one of the sets $P_1, \ldots, P_i$ is a disk. Furthermore, $Y \prec X_j \prec X_i$ implies that X_j cannot lie on γ, i.e., $X_j \notin \gamma$, since it is clamped between $Y \in \alpha$ and $X_i \in \beta$.

Let $\{D^1, \ldots, D^n\}$ be the ordered subset of all disks in $\{P_1, \ldots, P_i\}$ – please be not confused about the fact that now the indices do not correspond any more to the vertices $v_1, \ldots, v_m$. Let c_0 c_n be the contact points of P_1 and P_i, and thus of D^1 and D^n with Y and X_i, respectively. If $n > 1$, then let c_k be the (pairwise different) contact points between D^k and D^{k+1} for $k = 1, \ldots, n-1$. If $c_k \in \partial G$ is a boundary point of G, with $k \in \{0, \ldots, n\}$, then let Z_k be the prime end associated with c_k by D^k (we set $D^0 := D^1$). If $c_k \in G$, then we simply set $Z_k := \emptyset$.

For $k = 1, \ldots, n$ we connect c_{k-1} with c_k by chords $\sigma_k = [c_{k-1}, c_k]$. Let σ be the concatenation of all σ_k. Then σ is a polygonal Jordan arc connecting Y with X_i. Moreover, σ is a (concatenation of) crosscut(s) of G. We assume first that $X_j \neq Z_k$ for every $k = 0, \ldots, n$.

Then σ contains at least a sub-arc σ' that is a crosscut of G connecting two prime ends Z_{k_1} and Z_{k_2} so that $Y \preceq Z_{k_1} \prec X_j \prec Z_{k_2} \preceq X_i$. Let C be the corresponding chain of all disks in $\{P_j, \dots, P_m\}$. In order to be in general able to connect $X_j \notin \gamma$ with some prime end in γ (recall that v_m is a neighbor of c) we must have $C \neq \emptyset$. Since we have $Z_{k_1} \prec X_j \prec Z_{k_2}$, at least one disk of C must touch a disk of $\{D^1, \dots, D^n\}$ in one of the contact points c_k with $k \in \{0, \dots, n\}$. Since three disjoint disks cannot touch a common point, we even have $k \in \{0, n\}$. Thus, exactly one disk of C, which then must be the final one, touches D^1 or D^n in Y or X_i, respectively, which then must yield $Y = \alpha \cap \gamma$ or $X_i = \beta \cap \gamma$, respectively. This is only possible if G is spiky, what is excluded by the assumptions of the lemma. Hence, we have $X_j = Z_k$ for some $k = 0, \dots, n$.

From $Y \prec X_j$ follows $Y \neq X_j$, i.e., $k \neq 0$. Analogously, from $X_j \prec X_i$ we conclude $k \neq n$. For every $k \in \{1, \dots, n-1\}$ the prime end Z_k is already touched by the two disks D^k and D^{k+1} so P_j must be a dot $P_j = S_j = \{c_k\}$ in order to touch $X_j = Z_k$. For the same reason also the neighbors and neighbors-neighbors $P_{j+1}, \dots, P_m$ of P_j must be dots $S_{j+1} = \ldots = S_m = S_j$. Since at least S_m must touch γ, this implies $X_j \in \gamma$, which is a contradiction to the result $X_j \notin \gamma$ shown before.

Hence, $i < j$ implies $X_i \preceq X_j$, and this proves the lemma. $\square$

3.2. Characterization of Degeneracy

This subsection investigates the degeneracy of generalized circle agglomerations. In particular, we are interested in criteria that keep packings non-degenerate. This can be achieved by restrictions on the complex or on the boundary of the domain. We provide some combinations of both. Recall, a domain filling packing cannot be collapsed (by definition).

Let T (or Q) be a tri-(or quad-)complex, and let $K := \text{int}T$ (or $K := \text{int}Q$). Let G be a trilateral (or quadrilateral) and let $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ be a generalized circle agglomeration for T (or Q) filling G. Assume $\mathcal{S} \neq \emptyset$.

Let V be the set of all vertices of T (or Q) and let $U \subset V$ be the subset associated with the dots $\mathcal{S}$ of $\mathcal{P}$. Let $w \in U$ be associated with $\{p\} = S_w \in \mathcal{S}$. Then we define $W \subset U$ as the subset of all vertices of U that can be connected with w by a chain in U. Let W' be the subset of vertices of $V \setminus U$ that are neighbors of W. We say every $v \in W'$ has combinatorial distance 1 from W.

Assume $W' = \emptyset$. On the one hand, since T (or Q) is connected, we have $W = V$. On the other hand, since $|\mathcal{D}| \geq 1$, we have $W \neq V$, which is a contradiction. Thus, there is a vertex $v_1 \in W'$ and v_1 is associated with a disk D_1.

Assume $|W'| = 1$. By Lemma 2.4 every tri- and quad-complex is 3-connected. So we can remove v_1 together with all its edges and faces from T (or Q), and the remaining complex T' (or Q') is still connected. Thus, on the one hand, $|W'| = 1$ implies $W = V'$ for $V' = V \setminus \{v_1\}$. On the other hand, since the boundary vertices of T (or Q), which of course lie in V', cannot be contained in W, we have $W \neq V'$, which is a contradiction. Thus, there is a vertex $v_2 \neq v_1$ in W' and v_2 is associated with a disk of $\mathcal{P}$ or an edge of G.

Assume $|W'| = 2$. Let T'' (or Q'') emerge from T (or Q) by removing v_1 and v_2 together with all their edges and faces. Then T'' (or Q'') is connected again by Lemma 2.4. So $|W'| = 2$ implies $W = V''$ for $V'' = V \setminus \{v_1, v_2\}$. But since there is still a boundary vertex of T (or Q) contained in V'', we have $W \neq V''$, which is a contradiction. Hence, $|W'| \geq 3$.

We now prove that W' is exactly one of the following types I–III.

Type I: $W' = \{v_1, v_2, v_3\}$ so that v_1 is associated with a disk of $\mathcal{P}$ that touches two edges of G that are associated with v_2 and v_3, respectively.

Type II: $W' = \{v_1, v_2, v_3\}$ so that v_1 and v_2 are associated with two disks of $\mathcal{P}$ that both touch an edge of G that is associated with v_3.

Type III: $W' = \{v_1, v_2, v_3, v_4\}$ so that v_1 and v_2 are associated with two disks of $\mathcal{P}$ that both touch two edges of G that are associated with v_3 and v_4, respectively.

Assume v_1, v_2 and v_3 are associated with disks D_1, D_2 and D_3, respectively. By construction, S_w is attached to all three of them, i.e., D_1, D_2 and D_3 all touch p, which is impossible. Thus, at most two vertices of W' can be associated with disks of $\mathcal{P}$.

Assume v_2, v_3 and v_4 are associated with edges of G, say α, β and γ, respectively. Recall that v_1 is per definition always associated with a disk D_1 of $\mathcal{P}$. By construction, S_w is attached to D_1, i.e., D_1 touches α, β and γ in a single prime end. Since $\alpha \cap \beta \cap \gamma = \emptyset$, this is impossible. Thus, at most two vertices of W' can be associated with edges of G.

This proves $|W'| \leq 4$. Since we already showed $3 \leq |W'|$, the types I–III are all possible combinations of 1–2 disks with 1–2 edges. Note that we did not assume neither T (or Q) nor G to have any special property. In this sense the types I–III characterize all possibilities of degeneracy.

In order to exclude degeneration, we state some assertions that itself exclude the types I–III from above. The first lemma needs no combinatorial restrictions but therefore some very strict geometrical ones.

Lemma 3.11 (No inward spikes and untouchable vertices)**.** *Let G be a trilateral (or quadrilateral) and let T (or Q) be a tri-(or quad-)complex. Let $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ be a generalized circle agglomeration for T (or Q) filling G. If the edges of G have no inward spikes while the vertices of G are even untouchable, then $\mathcal{S} = \emptyset$.*

Proof. Since Type I implies that a disk of $\mathcal{P}$ touches a vertex of G, the untouchable assertion excludes this case. Since Type II and III imply that two disk of $\mathcal{P}$ touch an inward spike of G, the lack of any inward spikes excludes also this cases. Hence, $\mathcal{P}$ cannot be degenerate. □

An example for a suitable trilateral for Lemma 3.11 would be a polygon with three exterior corners as chosen vertices (see for instance Figure 3.12, left).

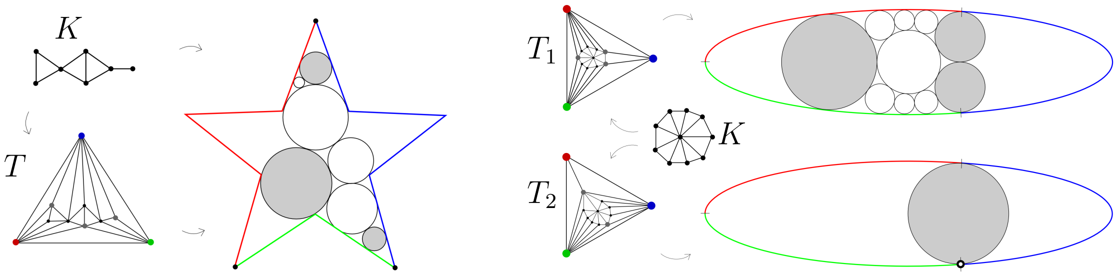


Fig. 3.12.: Left, example for Lemma 3.11; right, examples for Lemma 3.12

In what follows we replace the geometrical untouchable assertion by a combinatorial one: The complex needs to be proper. Since this alone turns out to be not sufficient to secure non-degeneracy, we also need to consider the degree of the complex.

Lemma 3.12 (No inward spikes and a proper complex)**.** *Let G be a trilateral (or quadrilateral) and let T (or Q) be a tri-(or quad-)complex. Let $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ be a generalized circle agglomeration for T (or Q) filling G. Let T (or Q) be proper and let the edges of G have no inward spikes.*

(i) If the degree of T (or Q) is at least three*, then $\mathcal{S} = \emptyset$.*

(ii) If $\mathcal{S} \neq \emptyset$, then there are exactly two leading vertices w_1 and w_2. The vertex w_1 is a neighbor of every boundary vertex, and it is associated with the one and only disk D_1 in $\mathcal{D} = \{D_1\}$. The vertex w_2 is a neighbor of exactly two boundary vertices, and its associated leading dot S_2 equals every dot $S \in \mathcal{S}$. Hence, D_1 touches every edge of G, every dot S is attached to D_1, and D_1 touches (via S_2) the vertex prime end X associated with w_2.

Proof. Since there are no inward spikes, the Type II and III from above cannot occur. So assume that Type I applies. Let w.l.o.g. $v_2 = a$ and $v_3 = b$ be associated with α and β, respectively. Let D_1 be the disk associated with v_1, and let $S_0 = \{p\}$ be a dot that is attached to D_1 and that touches $X = \alpha \cap \beta$. Let $K := \mathrm{int}T$ (or $K := \mathrm{int}Q$) be the admissible interior of T (or Q).

By Lemma 2.18 we conclude $\mathcal{D} = \{D_1\}$ otherwise p would be a pseudo contact point between D_1 and some disk D_2 making X an inward spike, which is a contradiction. Moreover, there is a boundary vertex w of K being associated with a boundary dot $S_w = S_0$. Starting at w and following the boundary chain of K first with positive then with negative orientation, we see that all boundary vertices are associated with dots $S = S_w$ but at most v_1 not. Since every dot equals some boundary dot (Lemma 2.18) that itself equals S_w, we have $S_v = S_w$ for all $v \neq v_1$ in K.

Assume the degree of T (or Q) is at least three. Then there are at least two leading dots S_1 and S_2, and $S_1 = S_2 = S_w$ touches at least three edges of G, which is impossible. So in this case $\mathcal{P}$ cannot be degenerate.

Assume the degree of T (or Q) is two. Let w_1 and w_2 be the two leading vertices. By the argumentation from above only one leading vertex, say w_2, can be associated with a dot. So the other one fulfills $w_1 = v_1$. Moreover, w_2 cannot be a neighbor of more than two boundary vertices of T (or Q) otherwise three edges of G would share a common prime end, which is impossible. Hence, w_1 is a neighbor of every boundary vertex, and since the degree of any proper T (or Q) cannot be less than two, all properties that are stated in the lemma are fulfilled. □

An example for a suitable trilateral for Lemma 3.12 would be a smooth Jordan domain with any three pairwise different chosen boundary points (see for instance Figure 3.12, right). Note that the properties described in (*ii*) allows us to check a priori whether a degree of 2 can lead to degeneration or not. Using the notation of the upcoming section, the question is whether the *incircle* of G touches X, or not.

If the domain has interior spikes, then we need even more combinatorial restrictions. Recall that a tri- or quad-complex is strongly intrinsic connected if its interior is a strongly connected admissible complex.

Lemma 3.13 (Non-spiky and strongly intrinsic connected)**.** *Let G be a trilateral (or quadrilateral) and let T (or Q) be a tri-(or quad-)complex. Let $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ be a generalized circle agglomeration for T (or Q) filling G. Let T (or Q) be strongly intrinsic connected and let G be non-spiky. If the degree of T (or Q) is at least three, then $\mathcal{S} = \emptyset$.*

Proof. Since no vertex prime end of G is an inward spike, the Type III from above cannot occur. Type I can be excluded by the argumentation of the proof of Lemma 3.12. So assume that Type II applies. Let v_1 and v_2 be associated with the disks D_1 and D_2, and let v_3 be associated (w.l.o.g.) with the edge α.

By Lemma 2.19 we instantly get $\mathcal{D} = \{D_1, D_2\}$, and by Lemma 2.18 every dot of $\mathcal{P}$ equals the pseudo contact point p of D_1 and D_2. Therefore, at least one of the leading vertices of T (or Q) is associated with a dot $S = \{p\}$. Hence, the vertex prime end X of G that is touched by S is an inward spike, which is a contradiction. So $\mathcal{P}$ cannot be degenerate. □

Roughly speaking, Lemma 3.13 states: Starting with a good complex, i.e., K is strongly connected, one only has to take care that neither the frame is degenerate, i.e., $\deg T = 3$, nor the trilateral, i.e., G is non-spiky, then the resulting packing is not degenerate, too. For practically usage these are no crucial restrictions – some readers may only consider strongly connected complexes to begin with – in particular compared to the advantage of being able to fill arbitrary bounded, simply connected domains.

Nevertheless, we want to provide yet another non-degeneracy statement. Since the assertions of the following lemma directly emerge from situations we will encounter later on, it is a crucial tool for the uniqueness and existence proofs of not only this but also the upcoming chapter(s). Recall that a tri- or quad-complex is boundary irreducible if all its reducible triangles lie in its interior.

Lemma 3.14 (Non-spiky and boundary irreducible)**.** *Let G be a trilateral (or quadrilateral) and let T (or Q) be a tri-(or quad-)complex. Let $\mathcal{P} = \mathcal{S} \cup \mathcal{D}$ be a generalized circle agglomeration for T (or Q) filling G. Let T (or Q) be boundary irreducible and let G be non-spiky. Then $\mathcal{S} = \emptyset$.*

Proof. Since no vertex prime end of G is an inward spikes, the Type III from above cannot occur. So assume $W' = \{v_1, v_2, v_3\}$. Since at least one boundary vertex of T (or Q) lies in W', we set w.l.o.g. $v_3 = b$. Let $N(b) = \{a = u_0, u_1, \ldots, u_n, u_{n+1} = c\}$ be the ordered set of all neighbors of b (see Figure 3.13). By construction, there is a vertex $u_j \in W$ with $j \in \{1, \ldots, n\}$.

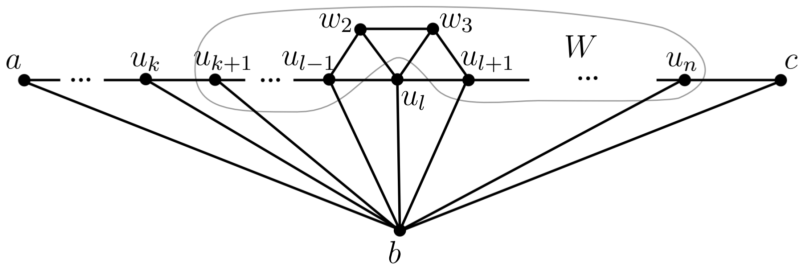


Fig. 3.13.: Constructions for the proof of Lemma 3.14

Since $\alpha \cap \beta \cap \gamma = \emptyset$, not all vertices $u_1, \ldots, u_n$ can be associated with dots. So let u_l with $l \in \{1, \ldots, n\}$ be associated with a disk in $\mathcal{P}$. Clearly, we have $j \neq l$. Moreover, we may and will assume w.l.o.g. that $l = j + 1$ and $u_l = v_1 \in W'$. Since at least a does not lie in W, there is yet another vertex $u_k = v_2 \in W'$ so that $k \in \{0, \ldots, l - 2\}$.

Assume v_1 is not a neighbor of v_2. Let $N(v_1) = \{b = w_0, w_1, \ldots, w_m, b\}$ be the ordered set of all neighbors of v_1. By construction, and since we already found all members of $|W'| = 3$, none of the vertices $w_1, \ldots, w_m$ lies in W' and $w_1 = u_{l-1} \in W$. Therefore, all $w_1, \ldots, w_m$ lie in W, in particular $w_m = u_{l+1} \in W$. For the same reason, now all $u_{l+1}, \ldots, u_n$ lie in W, too. So c has combinatorial distance 1 from W, i.e., $c \in W'$, which is a contradiction to $|W'| = 3$ since $c \notin \{v_1, v_2, v_3\}$.

We showed that v_1 is a neighbor of v_2. Moreover, both vertices v_1 and v_2 are neighbors of v_3 by construction. Thus, $\triangle = \triangle(v_1, v_2, v_3)$ is a triangle in T (or Q). Since we have $v_1 = u_l$, $v_2 = u_k$ and $k < l - 1 < l$, the vertex u_{l-1} lies in the interior of $\triangle(v_1, v_2, v_3)$. Since v_1 is an interior vertex, the triangle $\triangle$ is reducible, and since v_3 is a boundary vertex, T (or Q) is boundary reducible, which is a contradiction. So $\mathcal{P}$ cannot be degenerate. $\square$

The final statement of this section is a simple, yet essential, application of Lemma 3.14 and Lemma 3.2. In some sense this is the natural setting of both lemmas.

Corollary 3.15 (Non-degenerate skeleton)**.** *Let $G(\alpha, \beta, \gamma)$ be a non-spiky trilateral. Let T be a tri-complex. Let $\mathcal{P}$ be a generalized circle agglomeration for T filling G. Let $\sigma(T)$ be the skeleton of T and let $\mathcal{P}_\sigma$ be the associated sub-packing of $\mathcal{P}$. Then either $\mathcal{P}_\sigma$ is a circle packing filling $G(\alpha, \beta, \gamma)$, or we have $\mathcal{P}_\sigma = \{D\}$ so that D touches α, β and γ.*

The case of a single disk touching every edge of G is explored next.

3.3. Incircles

The most simple (non-empty) circle agglomeration $\mathcal{P}$ consists of exactly one disk D, i.e., $\mathcal{P} = \{D\}$, and already this neat special case is very important for our proofs later on. By proving the existence of incircles we demonstrate our approach for the existence proofs in the general case, and by showing uniqueness we provide the induction base for the uniqueness proofs in the general case.

The results of this section were published in the paper [37].

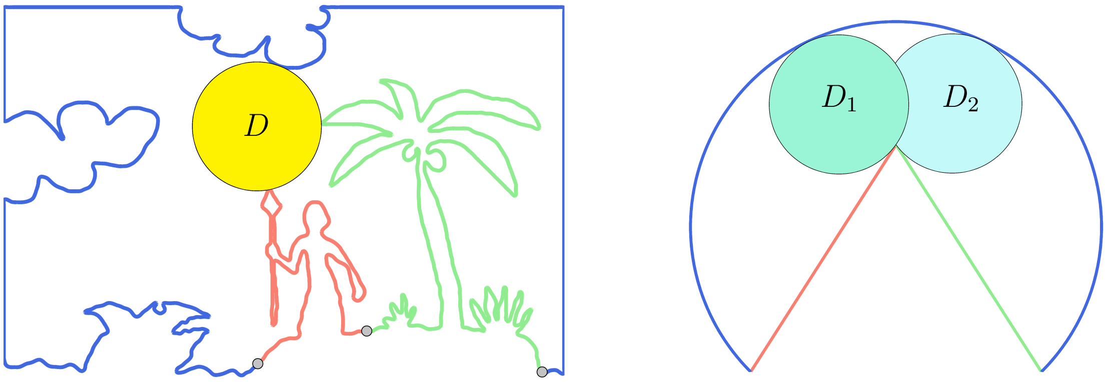


Fig. 3.14.: Two trilaterals with incircles D, D_1 and D_2

3.3.1. Uniqueness of Incircles

Let $G(\alpha, \beta, \gamma)$ be a trilateral. Let $\{T\} = \mathcal{T}_1^*$ be the unique tri-complex T that contains exactly one interior vertex. Let $\mathcal{P} = \{D\}$ be a circle agglomeration that contains exactly one disk D. If $\mathcal{P}$ fills G, then we denote D an *incircle* of G, i.e., $D \subset G$ is a disk in G that touches all three arcs α, β and γ (see Figure 3.14).

Note that despite its name the incircle is a disk. Of course it would be more appropriate to speak of the *indisk*, but since this sounds (in some sense) awkward, we stay with the traditional term and identify D with the actual incircle ∂D.

To guarantee uniqueness of the incircle we need G to be tame (see also [37] Theorem 2).

Lemma 3.16 (Uniqueness of incircles)**.** *Every tame trilateral has at most one incircle.*

Proof. In order to prove uniqueness, we assume that a tame trilateral $G(\alpha, \beta, \gamma)$ has two incircles D_1 and D_2. The idea is to consider appropriate chords connecting the contact points of D_1 and D_2 with the three edges of the trilateral, and to show that two of them intersect each other at some interior point of G. Then we can use Lemma 2.8 to get $D_1 = D_2$.

Let $X_1, X_2 \in \alpha$, $Y_1, Y_2 \in \beta$ and $Z_1, Z_2 \in \gamma$ be prime ends of ∂G^* touched by D_1 and D_2, respectively. Since D_1 and D_2 are incircles of G, all these points exist. Maybe some of them are equal.

First of all, we observe that whenever both disks touch ∂G^* at the same vertex $\alpha \cap \beta$, $\beta \cap \gamma$ or $\gamma \cap \alpha$ of G, then they must be equal. Assuming the contrary, say $X_1 = X_2 = Y_1 = Y_2$, one of the two disks D_1 and D_2 must contain the other, say $D_1 \subset D_2$, since G is tame. Then either $D_1 = D_2$ or $\partial D_1 \cap \partial G$ consists of exactly one boundary point d. In the latter case D_1 must touch X_1, Y_1 and Z_1 in d, hence $\alpha \cap \beta \cap \gamma \neq \emptyset$, which is a contradiction. This solely leaves $D_1 = D_2$.

For $k = 1, 2$ let $p_k, q_k, r_k \in \partial G$ be the contact points of X_k, Y_k and Z_k with D_k, respectively. Assume for the moment that $X_1 = X_2 =: X$, $Y_1 = Y_2 =: Y$ and $Z_1 = Z_2 =: Z$. Then $p_1 = p_2 =: p$, $q_1 = q_2 =: q$ and $r_1 = r_2 =: r$. Furthermore, there are two cases: Either the prime ends X, Y and Z (thus the points p, q and r) are all different, in which case $D_1 = D_2$, or two of this prime ends coincide, in which case we have the situation considered above where D_1 and D_2 touch a common vertex of G, i.e., again $D_1 = D_2$. Therefore, the assumption $D_1 \neq D_2$ implies without loss of generality that $X_1 \neq X_2$.

Assume that $D_1 \neq D_2$, i.e., $X_1 \neq X_2$. Using the notation introduced in Section 2.2.1 and labeling the disks appropriately, we get

$$X_1 \prec X_2 \preceq Y_1 \preceq Z_2 \preceq X_1.$$

Recall, this means that the prime ends X_1, X_2, Y_1 and Z_2 are cyclically ordered along (the positive oriented) ∂G^* in this ordering. To be precise, using the canonical embedding $g : G \to \mathbb{D}$ of G, or rather its extension $g^* : \partial G^* \to \mathbb{T}$, then $g^*(X_1)$, $g^*(X_2)$, $g^*(Y_1)$ and $g^*(Z_2)$ are cyclically ordered along (the positive oriented) $\partial\mathbb{D}$ in this ordering.

Since $X_2 = Y_1$, $Y_1 = Z_2$ and $Z_2 = X_1$ can be excluded by the results from above, we even have the stronger relation

$$X_1 \prec X_2 \prec Y_1 \prec Z_2 \prec X_1.$$

This tells us that the four prime ends X_1, X_2, Y_1 and Z_2 are all different, and that the pairs X_1, Y_1 and X_2, Z_2 are in interlacing position. Let $\sigma_1 = (p_1, q_1)$ be the open chord from p_1 to q_1 in D_1. Let $\sigma_2 = (p_2, r_2)$ be the open chord from p_2 to r_2 in D_2.

We map σ_1 and σ_2 onto two crosscuts $g(\sigma_1)$ and $g(\sigma_2)$ of $\mathbb{D}$, respectively. Since $g^*(X_1), g^*(Y_1)$ and $g^*(X_2), g^*(Z_2)$ are interlacing on $\partial\mathbb{D}$, the two open Jordan arcs $g(\sigma_1)$ and $g(\sigma_2)$ must intersect each other at some point in $\mathbb{D}$. Hence, back to G, the two chords σ_1 and σ_2 intersect each other at some point in G, i.e., $g(\sigma_1) \cap g(\sigma_2) \neq \emptyset$. From the Two-Disk-Lemma (Lemma 2.8, p. 30) we finally infer that $D_1 = D_2$. $\square$

A simple counterexample shows that the tameness of G is crucial: If one of the vertex prime ends of G is an inward corner, then an incircle possibly "rotate" around this corner without loosing its touch to the third edge of G (as shown in Figure 3.14, right). Note that also in the case of generalized circle agglomerations such inward vertex corners are the one and only obstacles that can prevent uniqueness.

3.3.2. Existence of Incircles

In this section we show that every trilateral has an incircle. The proof is based on Sperner's Lemma, which we stated at Section 2.4. Using also Lemma 3.16 we get the following result (see also [37] Theorem 1 and 2).

Theorem 3.17 (Incircle Theorem)**.** *Every trilateral G has an incircle D. If G is tame, then D is unique.*

Proof. Let $f : \mathbb{D} \to G$ be the canonical parameterization of G and let f^* be it's extension. Let $g : \Delta \to \mathbb{D}$ be the canonical embedding of an open equilateral triangle $\Delta \subset \mathbb{C}$ (with side length equal to one). Let g^* be it's extension. The concatenation $h := g \circ f$ is a conformal map $h : \Delta \to G$ from Δ onto G. By the Prime End Theorem (and since Δ is a Jordan domain) the extension $h^* : \overline{\Delta} \to G^*$ is a bijection that maps $\partial\Delta$ onto ∂G^*. Moreover, let a, b and c be the three vertices of Δ. Then h can be normalized in such a way, that $h^*(a) =: A$, $h^*(b) =: B$ and $h^*(c) =: C$ are different prime ends of the arcs α, β and γ, respectively. We choose A, B and C to be different to the vertices prime ends of G (see Figure 3.15).

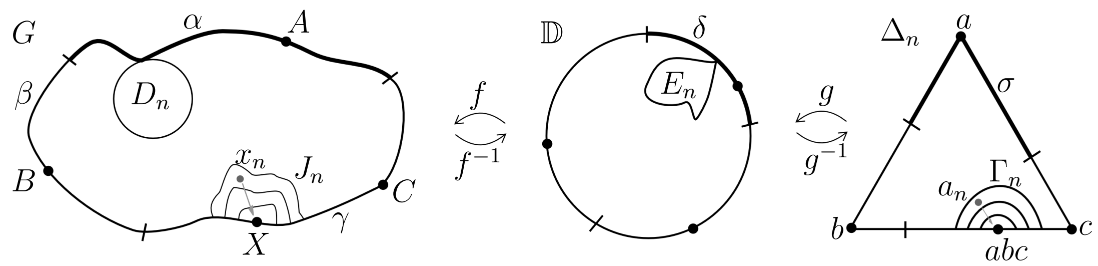


Fig. 3.15.: Constructions for the proof of Theorem 3.17

Uniform subdivision of the sides of Δ into n intervals with equal lengths $1/n$ generates an admissible complex K_n within Δ. In order to define the *color* of a vertex v of K_n, we map v onto a point $z \in G$ via $z = h(v)$ if v is an interior vertex, or onto a prime end $Z \in \partial G^*$ via $Z = h^*(v)$ if v is a boundary vertex. In the former case let $D(z) \subset G$ be that disk in G that has it's center in z and that has maximal radius; the *maximal disk* for z. Then we color v as follows:

- Red: The interior vertex v is colored *red* if $D(z)$ touches α, otherwise
- Green: The interior vertex v is colored *green* if $D(z)$ touches β, otherwise
- Blue: The interior vertex v is colored *blue* if $D(z)$ touches γ.

In the case of a boundary vertex v we set:

- Red: The boundary vertex v is colored *red* if $Z \in \alpha$, otherwise
- Green: The boundary vertex v is colored *green* if $Z \in \beta$, otherwise
- Blue: The boundary vertex v is colored *blue* if $Z \in \gamma$.

Figure 3.16 (left) shows the coloring of $\overline{G}$ according to the distance to the edges of the trilateral in Figure 3.14, and (right) an example of the admissible complex K_n for $n = 4$.

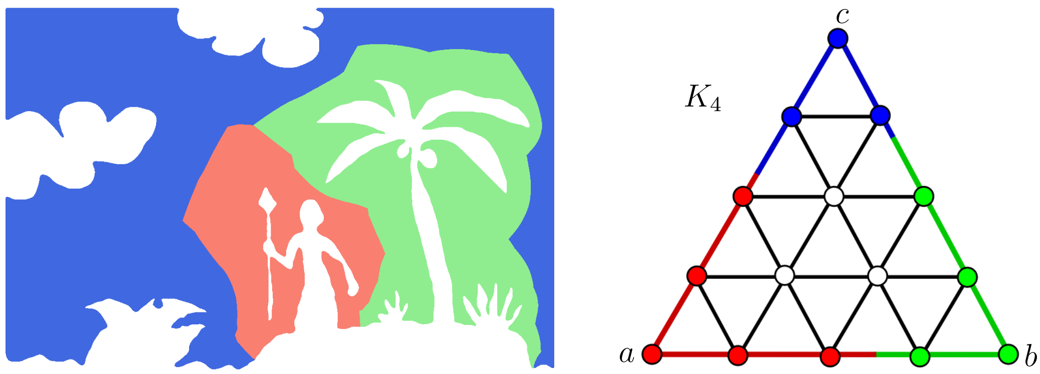


Fig. 3.16.: The coloring of all points within a trilateral (left) and an example for K_4 (right)

By Sperner's Lemma, every K_n (for $n = 1, 2, \ldots$) must contain a face $f_n(a_n, b_n, c_n)$ whose vertices a_n, b_n and c_n are colored red, green and blue, respectively. The sequence (a_n) contains a subsequence (a_{n_k}) that converges to a point $a_0 \in \overline{\Delta}$ (Bolzano-Weierstraß Theorem). Furthermore, the sequence (b_{n_k}) contains a subsequence $(b_{n_{k_l}})$ that converges to a point $b_0 \in \overline{\Delta}$. Since the length of each edge of f_n is $1/n$, the distance between a_0 and b_0 must be 0, i.e., $b_0 = a_0$. Finally, the sequence $(c_{n_{k_l}})$ contains a subsequence that converges to $c_0 \in \overline{\Delta}$, and clearly $c_0 = a_0$. For the sake of simplicity, we write for this last subsquences again a_n, b_n and c_n, and we set $a_0 = b_0 = c_0 = abc \in \overline{\Delta}$. There are two possible cases: Either $abc \in \partial\Delta$ or $abc \in \Delta$.

Assume $abc \in \partial\Delta$. We prove that this case cannot happen. Let (w.l.o.g.) abc lie on the side $[b, c]$ of Δ. Let σ be the pre-image of α under h^*, i.e., $h^*(\sigma) = \alpha$. According to the normalization of h, the edge $[b, c]$ is disjoint to σ. Therefore, and since both sets are closed, no points of any sufficiently small neighborhood of abc within $\overline{\Delta}$ is mapped by h^* onto a prime end of γ. Thus, in order to become red, the vertex a_n of the face $f_n(a_n, b_n, c_n)$ must be an interior vertex for all sufficiently large n. For simplicity we say $a_n \in \Delta$ for every $n \in \mathbb{N}$.

Let (Γ_m) be a null-chain representing abc in Δ and let W_m be it's tails (see Figure 3.15). For all sufficiently large n, since a_n converges to abc, there is a tail W_m so that $a_n \in W_m$. For every such n let m_n be the largest index m granting $a_n \in W_m$. Using again the convergence of a_n to abc we see that m_n goes to infinity as n goes to infinity, i.e., $n \to \infty$ implies $m_n \to \infty$. Therefore, and to simplify things, we may and will assume w.l.o.g. $a_n \in W_n$ for all $n \in \mathbb{N}$.

Now we map everything into G. Let $X := h^*(abc)$, let $J_n := h(\Gamma_n)$ and let $U_n := h(W_n)$. By construction we have $X \notin \alpha$. By Theorem 2.15 of [21] (J_n) is a null-chain for X and U_n are its tails. Since $h(a_n) = x_n \in U_n$ for every $n \in \mathbb{N}$, we can use Lemma 2.17, which states that $D(x_n)$ cannot touch α for all sufficiently large n. This is a contradiction for a_n to be colored red. Hence, $abc \in \partial\Delta$ is impossible.

Assume $abc \in \Delta$. Since h is a conformal map, the sequences $x_n = h(a_n)$, $y_n = h(b_n)$ and $z_n = h(c_n)$ converge to the point $xyz = h(abc) \in G$. We first look at (x_n).

For sufficiently large n we have $x_n \in G$. Let $D_n := D(x_n)$ be the maximal disk for x_n. Using Bolzano-Weierstraß (once for the centers of (D_n), once for their radii) we see that the sequence (D_n) has a convergent sub-sequence, whose limit disk shall be denoted D_x. The center of D_x must be in xyz. Its radius is positive since D_x touches ∂G, as we show next.

Let $\delta \subset \partial\mathbb{D}$ be the pre-image of α under f^*, i.e., $f^*(\delta) = \alpha$. Let $E_n \subset \mathbb{D}$ be the pre-image of D_n under f, i.e., $f(E_n) = D_n$ (see Figure 3.15). Then E_n converges to the pre-image E of D_x under f, i.e., $f(E) = D_x$. Since D_n touches α, we have $\overline{E_n} \cap \delta \neq \emptyset$ for every $n \in \mathbb{N}$. Since $\overline{E_n}$ and δ are closed sets, we infer $\overline{E} \cap \delta \neq \emptyset$. So D_x touches α.

A similar argumentation for y_n and z_n yields the existence of two disks D_y and D_z, respectively, with centers in xyz so that D_y touches β while D_z touches γ. Since all three disks D_x, D_y and D_z lie in G, have the center xyz and touch ∂G, they are the unique maximal disk $D(xyz)$, and hence an incircle of G. □

3.4. Incompressibility

As mentioned before, we will prove the uniqueness of generalized circle agglomerations under the alpha-beta-gamma normalization by complete induction on the number of its disks. The previous section covers the induction base. This section is the core of the induction step although it looks like a totally different setting: We are going to investigate quad-complexes instead of tri-complexes. Moreover, we only consider circle agglomerations, i.e., no degeneration is allowed.

Note that, up to some technical details and minor improvements, this section corresponds almost one-to-one to [20] Chapter 4 and Section 3.7.

3.4.1. Incompressibility Theorem

To start with, we define compressions of arbitrary quadrilaterals. Recall that the subordination of prime ends and inclusions like $\beta' \subset \beta$ are explained in Definition 2.8. Some non-trivial examples are depicted in Figure 3.17 and 3.18. Going one step further, we extend the concept of (in)compressibility to complexes.

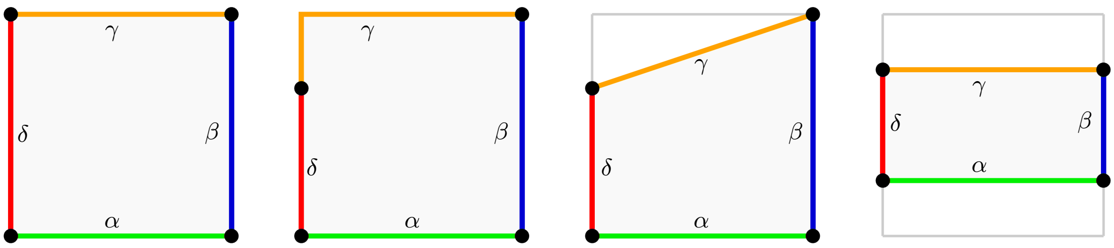


Fig. 3.17.: Subsequent compressions of Jordan quadrilaterals

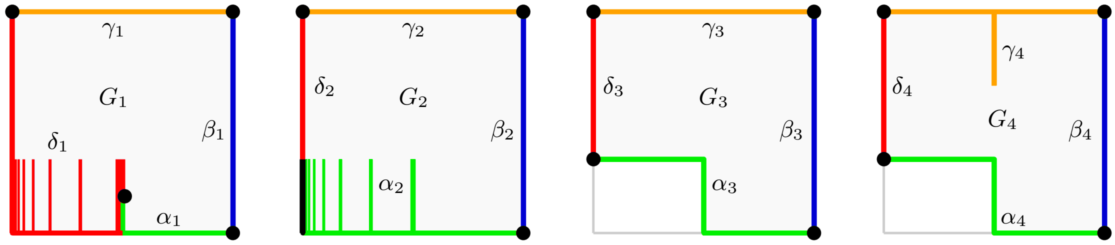


Fig. 3.18.: Subsequent compressions of tame quadrilaterals $G_1 \supset G_2 \supset G_3 \supset G_4$

Definition 3.14. A quadrilateral $G'(\alpha', \beta', \gamma', \delta')$ is said to be a *compression* of the quadrilateral $G(\alpha, \beta, \gamma, \delta)$ if $G' \subset G$, $\beta' \subset \beta$, and $\delta' \subset \delta$.

Definition 3.15. Let $\mathcal{P}$ be a generalized circle agglomeration associated with a quad-complex Q in $\mathcal{Q}^*$ that fills a tame quadrilateral G. A generalized circle agglomeration $\mathcal{P}'$ with quad-complex Q is said to be a *compression* of $\mathcal{P}$ in G if $\mathcal{P}'$ fills a tame quadrilateral G' that is a compression of G. A compression with $\mathcal{P}' \neq \mathcal{P}$ is said to be *nontrivial.*

A quad-complex $Q \in \mathcal{Q}^*$ is called *compressible* if there exist a tame quadrilateral G and a circle agglomeration $\mathcal{P}$ for Q filling G that admits a nontrivial compression in G. Otherwise we call Q *incompressible.* A subset of $\mathcal{Q}^*$ is said to be *incompressible* if all its members are incompressible.

The following theorem is the main result of this section. We state an even more general version at the beginning of Section 3.8.

Theorem 3.18 (Incompressibility Theorem)**.** *All irreducible quad-complexes $Q \in Q^*$ are incompressible.*

The theorem will be proved by induction on the number of vertices of Q. Before we go into the details we give an outline of the proof. Note that irreducibility of Q together with the tameness of the involved quadrilaterals prevents degeneration (Lemma 3.14).

As we will see, the only complex Q in $\mathcal{Q}_1^*$ is irreducible and incompressible. So suppose that $\mathcal{Q}_n^* \cap \mathcal{K}^i$ is incompressible for some $n \geq 1$, and let $\mathcal{P}$ and $\mathcal{P}'$ be circle agglomerations associated with some quad-complex Q in $\mathcal{Q}_{n+1}^* \cap \mathcal{K}^i$ so that $\mathcal{P}$ fills a quadrilateral G and $\mathcal{P}'$ fills a compression G' of G, respectively.

In order to prove that $\mathcal{P} = \mathcal{P}'$, we shall identify a vertex $w \in Q$ (called a *loner*) so that the disk $D_w \in \mathcal{P}$ has "weak interaction" with $\mathcal{P}'$. After removing the loner w from Q, and the disks D_w and D'_w from $\mathcal{P}$ and $\mathcal{P}'$, respectively, we get a quad-complex Q_s in $\mathcal{Q}_{n-1}^* \cap \mathcal{K}^i$ and circle agglomerations $\mathcal{P}_s$ and $\mathcal{P}'_s$ associated with Q_s. Modifying the quadrilaterals G and G' appropriately, we achieve that the resulting quadrilaterals G_s and its compression G'_s are filled by $\mathcal{P}_s$ and $\mathcal{P}'_s$, respectively. Since the underlying quad-complex Q_s in $\mathcal{Q}_{n-1}^* \cap \mathcal{K}^i$ is incompressible, we get $\mathcal{P}'_s = \mathcal{P}_s$, which finally implies $\mathcal{P}' = \mathcal{P}$.

The main steps consist in (i) verifying that a loner always exists, (ii) constructing an irreducible quad-complex Q_s and (iii) modifying the trilaterals G and G' appropriately.

3.4.2. Loners

In this section we assume that $\mathcal{P}$ and $\mathcal{P}'$ are circle agglomerations associated with a quad-complex $Q(a, b, c, d)$ that fill a quadrilateral $G(\alpha, \beta, \gamma, \delta)$ and its compression $G'(\alpha', \beta', \gamma', \delta')$, respectively. Our aim is to study the interplay of the disks $D_1, \ldots, D_k$ in $\mathcal{P}$ that are associated with the vertices in the set $N := \{v_1, \ldots, v_k\}$ of all neighbors of a, and the corresponding disks $D'_1, \ldots, D'_k$ in $\mathcal{P}'$.

Definition 3.16. A vertex $v_i \in N$ with $i \in \{1, 2, \ldots, k\}$ is called a *loner* (with respect to a) if $D_i \cap D'_j = \emptyset$ for all $j \in \{1, 2, \ldots, k\}$ with $i \neq j$.

The concept of loners was introduced by Oded Schramm [25]. It is clear that a similar definition can be made for loners with respect to the "opposite" vertex c of Q. The following result will be crucial for the proof of the main theorem.

Lemma 3.19 (Existence of loners)**.** *If Q is irreducible and G and G' are not spiky, then there exists a loner.*

The proof of Lemma 3.19 needs some preparations. Since the existence of a loner is trivial for $k = 1$, let $k \geq 2$.

For $i = 1, \ldots, k-1$ let c_i be the contact point of D'_i and D'_{i+1}. We choose the contact point c_0 of D'_1 with δ' so that $\widehat{\omega}_1[c_1, c_0] \subset \partial D'_1$ is the shortest negatively oriented (topologically closed) circular arc that connects c_1 with δ', i.e., c_0 is the one and only contact point of D'_1 with a prime end of δ' in $\widehat{\omega}_1[c_1, c_0]$. Note that we do not (yet) exclude the case $c_0 = c_1$ (but see Lemma 3.20). Similarly, we let c_k be the contact point of D'_k with β' so that the positively oriented circular arc $\omega_k[c_{k-1}, c_k] \subset \partial D'_k$ from c_{k-1} to c_k is the minimal closed arc connecting c_{k-1} with β'.

Lemma 3.20 (Distinct contact points)**.** *If Q is irreducible and G' is not spiky, then $c_i = c_j$ for $0 \leq i < j \leq k$ implies that $i = 0$ and $j = k$.*

Proof. It is clear that $c_i \neq c_j$ for $1 \leq i < j \leq k-1$ since three or more disjoint disk cannot touch a common point.

So assume that $c_0 = c_j$ for some $j \in \{1, \ldots, k-1\}$. Since D'_1, D'_j and D'_{j+1} all touch c_0, we must have $j = 1$, i.e., $c_0 = c_1$. According to Definition 2.10 ($\mathcal{P}'$ is contained in G'), both disks D'_1 and D'_2 touch the same prime end $X \in \delta'$ at c_1. Using Lemma 3.8 we see that $\triangle(v_1, a, d)$ and $\triangle(v_2, a, d)$ are triangles in Q. Now, only one of them can be a face since the boundary edge $e(a, d)$ is contained in exactly one face of Q. So one of the triangles is reducible, which is in conflict with Q being irreducible.

The proof of $c_i \neq c_k$ for $i \in \{1, \ldots, k-1\}$ is similar, hence the only case left is $c_0 = c_k$. Figure 3.19 shows an example where this happens. □

For $i = 1, \ldots, k$ let ω_i denote the positively oriented topologically closed subarcs of $\partial D'_i$ from c_{i-1} to c_i. By Lemma 3.20 every such arc as well as the concatenation ω of these arcs is a Jordan arc contained in $\overline{G'}$ and hence in $\overline{G}$.

In the case of $c_0 = c_k$ (with $k > 1$) the Jordan arc ω is closed. However, c_0 is associated with a prime end $X \in \delta'$ by D'_1 while c_k is associated with a prime end $Y \in \beta'$ by D'_k. Eventually, $\beta' \cap \delta' = \emptyset$ implies $X \neq Y$ (see again Figure 3.19).

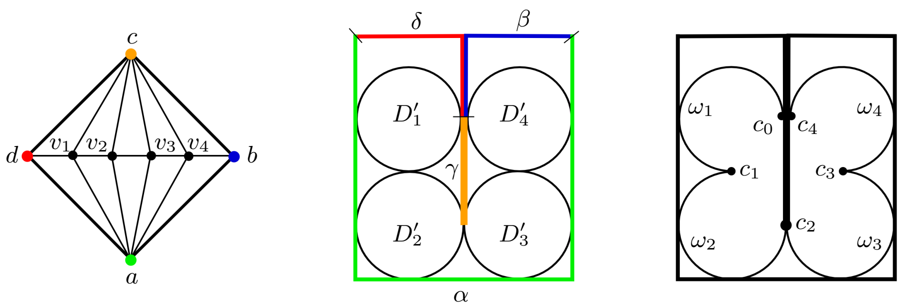


Fig. 3.19.: Structure of the contact points c_i and the arcs ω_i

Now comes the crucial part: With any $i \in \{1, \dots, k\}$ we associate two numbers $l(i)$ and $r(i)$, which describe the interaction of D_i with the disks D'_j for $j = 1, \dots, k$.

In the trivial case, where D_i does not intersect ω, we set $l(i) = r(i) := 0$. Otherwise, we pick an arbitrary contact point s_i of D_i with α. Depending on the location of s_i with respect to ω we distinguish several cases.

Case 1. If $s_i \notin \omega$, then we denote by π_i the maximal positively oriented open subarc of ∂D_i that is contained in $\overline{G} \setminus \omega$, has initial point s_i and terminal point $p_i \in \omega$. Similarly, we denote by ν_i the maximal negatively oriented open subarc of ∂D_i in $\overline{G} \setminus \omega$ with initial point s_i and terminal point $q_i \in \omega$ (see Figure 3.20, left).

The point p_i belongs to ω. Thus, it either lies on exactly one arc ω_j or it is a contact point $c_j = \omega_j \cap \omega_{j+1}$ of two neighboring arcs. In both cases we set $r(i) := j$. Likewise, if q_i lies on exactly one arc ω_j, then we set $l(i) = j$, while $l(i) := j+1$ if $q_i = c_j = \omega_j \cap \omega_{j+1}$ (note the slight difference in the definitions of $l(i)$ and $r(i)$).

Case 2. If $s_i \in \omega$, then the definitions of π_i, ν_i, $r(i)$ and $l(i)$ are more subtle and depend on the behavior of ∂D_i in one-sided neighborhoods of s_i. In what follows we denote by σ_i^- and σ_i^+ (sufficiently small) positively oriented open subarcs of ∂D_i so that s_i is the terminal point of σ_i^- and the initial point of σ_i^+, respectively.

Case 2.1. Let $s_i \in \omega_j \setminus \{c_{j-1}, c_j\}$ for some $j \in \{1, \dots, k\}$. If there exists an arc σ_i^+ contained in $\overline{D'_j}$, then we set $\pi_i := \{s_i\}$, $p_i := s_i$ and $r(i) := j$. Otherwise there exists an arc σ_i^+ contained in $\overline{G} \setminus \omega$ and we define π_i as the maximal such arc. The terminal point p_i of π_i lies either on exactly one semi-closed arc $\omega_l \setminus \{c_{l-1}\}$ $(2 \le l \le k)$ or on ω_l with $l = 1$, and we set $r(i) := l$ (see Figure 3.20, middle).

Analogously, if there exists an arc σ_i^- contained in $\overline{D'_j}$, then we set $\nu_i := \{s_i\}$, $q_i := s_i$ and $l(i) := j$. Otherwise there exists an arc σ_i^- contained in $\overline{G} \setminus \omega$ and we define ν_i as the maximal such arc. The initial point q_i of ν_i lies on exactly one semi-closed arc $\omega_l \setminus \{c_l\}$ $(1 \le l \le k-1)$ or on ω_l with $l = k$, and we set $l(i) := l$.

Case 2.2. Let $s_i = c_j$ for some $j \in \{1, \dots, k-1\}$. We denote by τ_j the common tangent to D'_j and D'_{j+1} at c_j. Let H_j^- and H_j^+ be the open half-planes that are bounded by τ_j so that H_j^- contains D'_j, and H_j^+ contains D'_{j+1}, respectively.

If there exists an arc σ_i^+ contained in H_j^-, then let $\pi_i := \{s_i\}$, $p_i := s_i$ and $r(i) := j$. If there exists an arc σ_i^+ contained in $\overline{D'_{j+1}}$, then we set $\pi_i := \{s_i\}$, $p_i := s_i$ and $r(i) := j+1$.

Otherwise we define π_i as the maximal arc σ_i^+ contained in $\overline{G} \setminus \omega$. In the last case the terminal point p_i of π_i is different from s_i and lies either on exactly one semi-closed arc $\omega_l \setminus \{c_{l-1}\}$ $(2 \leq l \leq k)$ or on ω_l with $l = 1$, and we set $r(i) := l$ (see Figure 3.20, right).

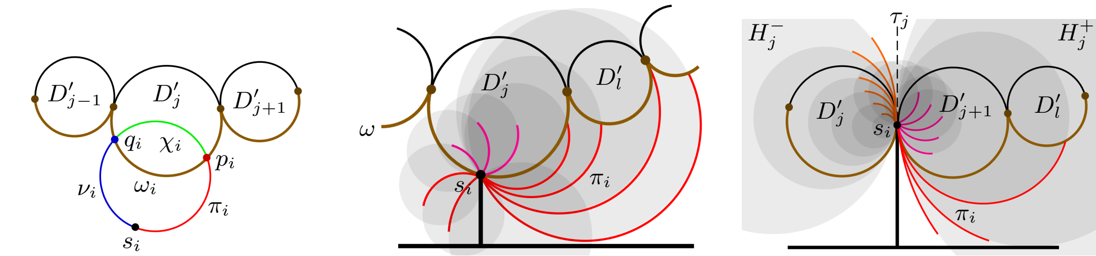


Fig. 3.20.: A typical situation of Case 1, and definition of π_i in Case 2.1 and Case 2.2

Similarly, if there exists an arc σ_i^- contained in H_j^+, then let $\nu_i := \{s_i\}$, $q_i := s_i$ and $l(i) := j + 1$. If there exists an arc σ_i^- contained in $\overline{D'_j}$ $(j = 1, \ldots k)$, then we set $\nu_i := \{s_i\}$, $q_i := s_i$ and $l(i) := j$. Otherwise we define ν_i as the maximal arc σ_i^- contained in $\overline{G} \setminus \omega$. The end point q_i of ν_i that is different from s_i lies on exactly one semi-closed arc $\omega_l \setminus \{c_l\}$ $(1 \leq l \leq k - 1)$ or on ω_l with $l = k$, and we set $l(i) := l$.

Case 2.3. Let $s_i = c_0$. Then D_i touches α and δ' at s_i, and since $\delta' \subset \delta$, we have $s_i \in I(X)$ with $X := \alpha \cap \delta$. Since G is tame, X is a regular prime end, from which we conclude that either $D_i \subset D'_1$ or D'_1 is a proper subset of D_i. In the first case we set $\pi_i := \nu_i := \{s_i\}$ and $r(i) := l(i) := 1$. In the second case, we define π_i as the maximal arc σ_i^+ contained in $\overline{G} \setminus \omega$. The terminal point p_i of π_i lies on exactly one arc $\omega_l \setminus \{c_{l-1}\}$ $(1 \leq l \leq k)$, and we set $r(i) := l$. Further, $\nu_i := \{s_i\}$ and $l(i) := 1$.

Case 2.4. If $s_i = c_k$, then we either have $D_i \subset D'_k$ or D'_k is a proper subset of D_i. In both cases, we set $\pi_i := \{s_i\}$ and $r(i) := k$. If $D_i \subset D'_k$, then we define $\nu_i := \{s_i\}$ and $l(i) := k$, otherwise let ν_i be the maximal arc σ_i^- contained in $\overline{G} \setminus \omega$. If ν_i ends at $q_i \in \omega_l \setminus \{c_l\}$ with $1 \leq l \leq k$, then we set $l(i) := l$.

Lemma 3.21 (Sufficient condition)**.** *If $l(i) = r(i) = i$ for some $i \in \{1, \ldots, k\}$, then v_i is a loner.*

Proof. It follows from the definition of p_i and q_i above that $p_i = q_i$ implies $D_i \subset D'_j$ for some $j \in \{1, \ldots k\}$. If $l(i) = r(i) = i$, then this is only possible if $j = i$, i.e., v_i is a loner. So we suppose $p_i \neq q_i$ in what follows.

By definition of $l(i) = i$ and $r(i) = i$, the (different) points p_i and q_i lie on $\partial D_i \cap \partial D'_i$. Let χ_i be the topologically closed positively oriented subarc of ∂D_i from p_i to q_i. Since $p_i, q_i \in \partial D_i \cap \partial D'_i$, and since π_i or ν_i are disjoint to D'_i, we have $\chi_i \subset \overline{D'_i}$, which implies $\chi_i \cap D'_j = \emptyset$ for all $j \neq i$. Now $\partial D_i = \overline{\nu_i} \cup \overline{\pi_i} \cup \chi_i$, and the construction of the arcs (or points) ν_i and π_i ensures that $\overline{\nu_i} \cap D'_j = \emptyset$, $\overline{\pi_i} \cap D'_j = \emptyset$ and hence $\partial D_i \cap D'_j = \emptyset$. So either $D_i \cap D'_j = \emptyset$ or $D_i \subset D'_j$ or $D'_j \subset D_i$. The second case is impossible otherwise p_i or q_i would not lie on ω_i, i.e., $l(i) \neq i$ or $r(i) \neq i$. The last case can be excluded since then the definitions of $l(i)$ and $r(i)$ would have given $l(i) = r(i) = j$. $\square$

The second main ingredient for proving the existence of a loner is the following lemma.

Lemma 3.22 (Interlacing sequences). *If G' is not spiky and all $l(i)$ and $r(i)$ are positive, then the numbers $l(j)$ and $r(j)$ form two interlacing monotone sequences,*

$$1 \le l(1) \le r(1) \le l(2) \le r(2) \le \ldots \le l(k) \le r(k) \le k.$$

Proof. Using the canonical embedding $g : G \to R$ and its extension $g^* : G^* \to \overline{R}$, we transplant the arcs ω_i, π_i and ν_i to the closed rectangle $\overline{R}$. Since (parts of) these arcs may lie on the boundary of G, this requires some care, but applying Lemma 2.12 we can easily verify the upcoming facts.

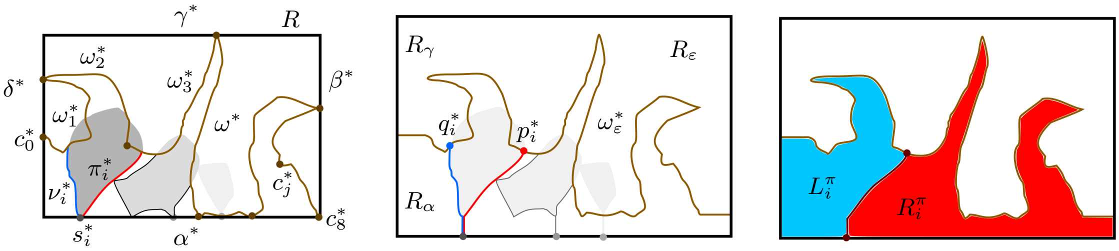


Fig. 3.21.: Illustrations for the proof of Lemma 3.22

The (positively oriented) circular arcs ω_i are mapped to (topologically closed) Jordan arcs ω_i^*. Since every contact point c_i for $i = 1, \ldots, k-1$ is either an inner point of G or associated with the same prime end by both D_i' and D_{i+1}' (see Definition 2.10), the two consecutive arcs ω_i^* and ω_{i+1}^* share exactly one point $c_i^* := \omega_i^* \cap \omega_{i+1}^*$. So the concatenation ω^* of the arcs ω_i^* is an oriented path in the closed rectangle $\overline{R}$ with initial point c_0^* on the (left) edge δ^* of R and terminal point c_k^* on the (right) edge β^* of R, respectively. Moreover, due to Lemma 3.20, ω^* is even a Jordan arc (see for example Figure 3.21, left). The orientation of ω induces a natural ordering of points on ω^*, which we write as $p \prec q$ or $p \preceq q$ (if $p = q$ is admitted). With respect to this relation the images c_i^* of the contact points c_i satisfy

$$c_0^* \prec c_1^* \prec \ldots \prec c_{k-1}^* \prec c_k^*.$$

The conditions $l(i) \neq 0$ and $r(i) \neq 0$ guarantee that π_i and ν_i are well defined. So π_i^* and ν_i^* are either points (on $\alpha^* \cap \omega^*$) or (open) Jordan arcs connecting the (lower) edge α^* of R with ω^*. Apart from their common initial point s_i^* the arcs π_i^* and ν_i^* are disjoint. Also the arcs π_i^* and ν_{i+1}^* can have at most one point r_i of intersection (where $r_i = s_i^* = s_{i+1}^*$ is possible).

Unfortunately, ω^* does not have to be a crosscut in R (it may intersect ∂R not only at its endpoints). Therefore, we embed R in a rectangle R_ε that is a central dilation of R. Prolongating the arcs ω_0^* and ω_k^* by "horizontal" segments (perpendicular to δ^* and β^*) up to the boundary ∂R_ε we get a crosscut ω_ε^* of R_ε. We denote by R_α and R_γ the "lower" and the "upper" component of $R_\varepsilon \setminus \omega_\varepsilon^*$, respectively (see Figure 3.21, middle).

Similarly, we prolongate the arcs ν_i^*, π_i^* by "vertical" segments down to the lower boundary of R_ε, getting crosscuts of R_α. Each of these crosscuts splits R_α into two connected components, which we denote by L_i^ν, R_i^ν and L_i^π, R_i^π, respectively (see Figure 3.21, right). Clearly, π_i^* lies in $\overline{R_i^\nu}$ while ν_i^* is contained in $\overline{L_i^\pi}$, respectively, and hence $q_i^* \preceq p_i^*$.

By Lemma 3.10 the points s_i^* are naturally ordered along α^*, i.e. $s_i^* \preceq s_{i+1}^*$ for $i = 1, \ldots, k-1$. For $s_i^* \prec s_{i+1}^*$ this implies that ν_{i+1}^* lies in $\overline{R_i^\pi}$, and a little thought shows that this inclusion remains valid for $s_i^* = s_{i+1}^*$. In particular, the terminal points p_i^* of π_i^* and q_{i+1}^* of ν_{i+1}^* satisfy $p_i^* \preceq q_{i+1}^*$.

Pulling the relation $q_i^* \preceq p_i^* \preceq q_{i+1}^*$ back from R to G we get (with respect to the ordering on ω) $q_i \preceq p_i \preceq q_{i+1}$, which finally implies the desired relation $l(i) \le r(i) \le l(i+1)$. The reader is invited to convince herself that this also holds in the cases where $s_i \in \omega$. $\square$

In order to complete the proof of Lemma 3.19, we observe that the existence of a loner is trivial if there is some i with $l(i) = r(i) = 0$. So assume that all $l(i)$ and $r(i)$ are positive. Then it follows from Lemma 3.22 and a simple combinatorial argument that there exists an integer $i \in \{1, \ldots, k\}$ so that $l(i) = r(i) = i$. Otherwise $r(1) = 1$ would be impossible, thus $2 \le l(1) \le l(2)$, what makes $r(2) = 2$ impossible, thus $3 \le r(2) \le l(3)$, implying $4 \le r(3) \le l(4)$, and so on, until we eventually arrive at the contradiction $k < r(k)$. Knowing $l(i) = r(i) = i$, the Lemma 3.21 tells us that v_i is a loner.

Some relevant properties of loners are summarized in the next lemma.

Lemma 3.23 (Properties of loners)**.** *Let w be a loner (with respect to a). Then the associated disks $D_w \in \mathcal{P}$ and $D'_w \in \mathcal{P}'$ satisfy the following conditions:*

(i) D_w does not intersect any disk D'_v in $\mathcal{P}' \setminus \{D'_w\}$.

(ii) If D_w touches γ or γ', then $D_w = D'_w$.

(iii) If v is an interior neighbor of w but not of a, and $D_v = D'_v$, then also $D_w = D'_w$.

Proof. Recall from the proof of Lemma 3.22 the definitions of the rectangle R_ε, the Jordan arcs ω_i^*, ω^* and ω_ε^* (which extends ω^* to a crosscut of R_ε). The set $R_\varepsilon \setminus \omega_\varepsilon^*$ consists of exactly two components R_α and R_γ, and the images $g_R(D'_v)$ of all disks $D'_v \in \mathcal{P}'$ are contained in R_γ. It follows directly from the definition of the points s_i that $s_i^* \notin R_\gamma$ for all $i = 1, \ldots, k$.

In order to prove (i), we assume that $w = v_i$ is a loner and that its associated disk D_i intersects a disk $D'_v \ne D'_i$. If p is a point in $D_i \cap D'_v$, then the segment $\sigma := [s_i, p]$ is contained in $\overline{D}_i$. Hence $g_R(\sigma)$ must intersect ω_ε^* at some point on ω_i^*, i.e., $\sigma \cap \partial D'_i \ne \emptyset$.

Obviously, the segment $\sigma' := [c_{i-1}, c_i]$ intersects σ. Moreover, we have $s_i, p \notin D'_i$ and $c_{i-1}, c_i \notin D_i$ otherwise D_i would intersect D'_{i-1} or D'_{i+1}, or, for $i = 1$ or $i = k$, $c_0, c_k \in \partial G$ would be an inner point of D_1, D_k, respectively. So the segments σ and σ' satisfy the assumptions of the Two-Disk-Lemma (Lemma 2.8). Since p is an interior point of D_i, the case $\sigma = \sigma'$ can be excluded. This implies $D_i = D'_i$, hence $D_i \cap D'_v = \emptyset$, which is a contradiction to the assumption made above.

The proof of (ii) runs analogously. If D_i touches γ or $D_i \subset G'$ touches γ', then let $p \in \partial G$ be a contact point of D_i with γ or γ', respectively. Otherwise there is a prime end $X \in \gamma'$ and a point p in its impression so that $p \in D_i$. From $\alpha \cap \gamma = \emptyset$, $\alpha \cap \gamma' = \emptyset$ and the fact that D_i is a disk we conclude that in every case $p \neq s_i$. We set $\sigma := [s_i, p]$ and $\sigma' := [c_{i-1}, c_i]$ as before. Using the arguments from above, we see that σ and σ' satisfy the assumptions of the Two-Disk-Lemma, hence $D_i = D'_i$ or $\sigma = \sigma'$. In the latter case (with $D_i \neq D'_i$) assertion (i) of this lemma implies $c_{i-1} = c_0$ and $c_i = c_k = c_1$, otherwise D_i would intersect D'_{i-1} or D'_{i+1}. Let $Y \in \alpha$ be the prime end that is associated with s_i by D_i. From $s_i = c_0$ or $s_i = c_k$ it follows that $Y \in \delta$ or $Y \in \beta$. So Y is a regular prime end touched by both disks D_i and D'_i, which eliminates the case $\sigma = \sigma'$, i.e., $D_i = D'_i$.

We prove (iii) by using Lemma 2.8 again. Let v be a neighbor of $w = v_i$ but not of a. Let p be the contact point between D_{v_i} and D_v. Assume that $D_v = D'_v$. Then we have $p \neq c_{i-1}$ and $p \neq c_i$, otherwise $v = v_{i-1}$ or $v = v_{i+1}$ would make v a neighbor of a. Furthermore, we have $p \neq s_i$, otherwise p must coincide with c_i, which was excluded. So $\sigma := [s_i, p]$ and $\sigma' := [c_{i-1}, c_i]$ are well defined and $\sigma \neq \sigma'$, hence $D_i = D'_i$ by the Two-Disk-Lemma. □

3.4.3. Cutting and Merging (geometrically)

Once we have identified a loner w we can remove w from $Q \in \mathcal{Q}_{n+1}$ by merging it with a. This generates a *smaller* quad-complex $Q_s := \mu(Q; w, a)$ in $\mathcal{Q}_n$. By omitting the associated disks D_w and D'_w in $\mathcal{P}$ and $\mathcal{P}'$, respectively, we obtain two circle agglomerations $\mathcal{P}_s$ and $\mathcal{P}'_s$ for Q_s in $\mathcal{Q}_n$. It remains to construct two tame quadrilaterals G_s and G'_s that are filled by $\mathcal{P}_s$ and $\mathcal{P}'_s$, respectively, so that G'_s is a compression of G_s. Roughly speaking, this is done by *cutting out* the disks D_w and D'_w from G and G', respectively. The following conclusion from Lemma 3.7 is the first step.

Corollary 3.24 (Disk removal). *Let $G(\alpha, \beta, \gamma, \delta)$ be a quadrilateral. If a disk D touches ∂G^* but not $\gamma \subset \partial G^*$, then there is a simply connected component G_κ of $G \setminus \overline{D}$ so that $\gamma \subset \partial G^*_\kappa$.*

Proof. Since D touches ∂G^*, we have $\overline{D} \cap \partial G \neq \emptyset$. Since D does not touch γ, each prime end $X \in \gamma$ has a tail U that is disjoint to D. Observing that the construction in the proof of Lemma 3.7 yields the same domain G_X for all $X \in \gamma$, we derive the assertion from Lemma 3.7. □

Under the assumptions of Corollary 3.24 we split ∂G^*_κ into four arcs α_κ, β_κ, γ_κ and δ_κ, thus converting the domain G_κ into a quadrilateral $\kappa(G; D, \gamma) = G_\kappa(\alpha_\kappa, \beta_\kappa, \gamma_\kappa, \delta_\kappa)$.

To begin with, we define $\gamma_\kappa := \gamma$. Since D does not touch γ, G_κ contains a tail of $\gamma \cap \beta$, from which we conclude that there is an arc of prime ends in ∂G^*_κ that lies in β (see Definition 2.8) and has terminal point $\gamma \cap \beta$. We define β_κ as the maximal (closed) arc of ∂G^*_κ with these properties. Similarly, we define δ_κ as the maximal (closed) arc of prime ends in ∂G^*_κ with initial point $\gamma \cap \delta$ that is contained in δ.

Finally, let $X \in \partial G^*_\kappa$ be the initial point of β_κ, while $Y \in \partial G^*_\kappa$ is the terminal point of δ_κ. Since X lies in β and Y lies in δ, they must be different. Let α_κ be the arc of ∂G^*_κ from Y to X. Figure 3.22 illustrates the construction.

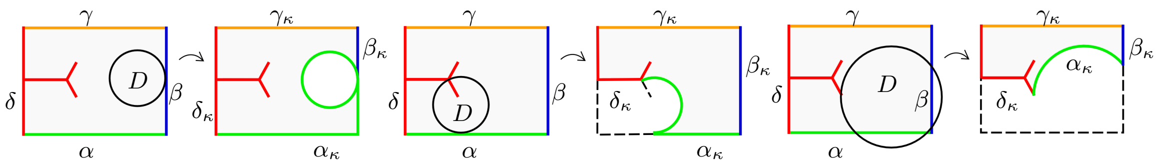


Fig. 3.22.: Three examples illustrating the action of the cut-out-operator κ

Lemma 3.25 (Cutting disks from quadrilaterals). *Let $G(\alpha, \beta, \gamma, \delta)$ be a (tame) quadrilateral. If a disk D touches ∂G^* but not $\gamma \subset \partial G^*$, then $\kappa(G; D, \gamma) := G_\kappa(\alpha_\kappa, \beta_\kappa, \gamma, \delta_\kappa)$ is a (tame) quadrilateral satisfying $G_\kappa \subset G$, $\beta_\kappa \subset \beta$ and $\delta_\kappa \subset \delta$. (So G_κ is a compression of G.)*

Proof. It only remains to show that $\kappa(G; D, \gamma)$ is tame if G is tame. Since two vertices of this quadrilateral coincide with (regular) vertices of G, we need only prove that $X := \alpha_\kappa \cap \beta_\kappa$ and $Y := \delta_\kappa \cap \alpha_\kappa$ are regular.

So assume that $D_1, D_2 \subset G_\kappa$ touch X at p. It follows from the definition of X as endpoint of a maximal arc (of prime ends) contained in β that $X = \beta \cap \gamma$ or $p \in \overline{D}$. In the first case X is regular by assumption, in the second case regularity of X follows from Lemma 3.6 since $D_j \cap D = \emptyset$ for $j = 1, 2$. Regularity of Y is proved the same way. ☐

At some point, we will encounter (trivial) situations where $D \cap G = \emptyset$. In such a case we define $\kappa(G; D, \gamma) := G(\alpha, \beta, \gamma, \delta)$, i.e., κ leaves the quadrilateral unchanged. Furthermore, we introduce cutting operations like $\kappa(G; D, \delta)$that result from $\kappa(G; D, \gamma)$ by a cyclic permutation $\alpha \to \beta \to \gamma \to \delta \to \alpha$ of the edges.

Now, let $M \subset \{a, b, c, d\}$ denote the set of neighbors of the loner w that are boundary vertices of Q. The following constructions summarize (in some sense) the results from above. They are schematically illustrated for typical configurations in Figure 3.23 and Figure 3.24.

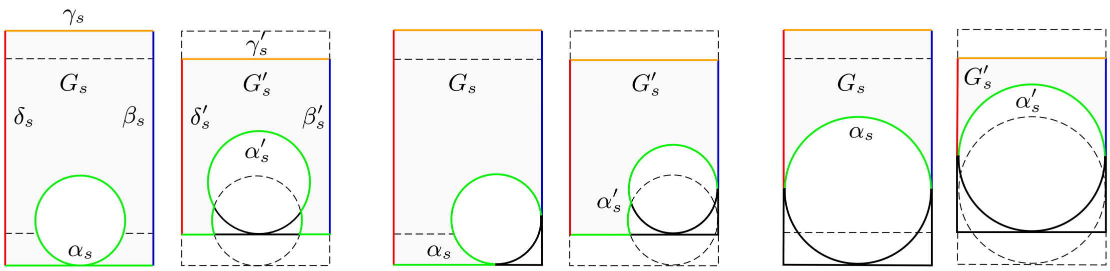


Fig. 3.23.: Schematic construction of $G_s(\alpha_s, \beta_s, \gamma_s, \delta_s)$ and $G_s'(\alpha_s', \beta_s', \gamma_s', \delta_s')$ in Case 1

Case 1. Let $c \notin M$. We define the quad-complex Q_s by merging w with a, i.e., $Q_s := \mu(Q; w, a)$. Since D_w touches α but not γ, the (tame) quadrilateral

$$G_s := \kappa(G; D_w, \gamma)$$

is well defined according to Lemma 3.25, and removing D_w from $\mathcal{P}$ yields a circle agglomeration $\mathcal{P}_s$ for Q_s filling G_s.

In order to construct G'_s, we first observe that D_w does not touch γ'. Assume the contrary, then assertion (ii) of Lemma 3.23 implies $D_w = D'_w$, so D'_w touches γ', and Lemma 3.8 tells us that w must be a neighbor of c, which is a contradiction. Therefore, and because D_w is disjoint to all disks in $P'_s := \mathcal{P}' \backslash \{D'_w\}$ (by assertion (i) of Lemma 3.23), the (tame) quadrilateral

$$G'_s := \kappa\big(\kappa(G'; D_w, \gamma'); D'_w, \gamma'\big),$$

resulting from G' by cutting out first D_w and then D'_w, is well defined according to Lemma 3.25, and filled by $\mathcal{P}'_s$ for Q_s. Moreover, G'_s is a compression of G_s since G' is a compression of G and D_w is cut out of both.

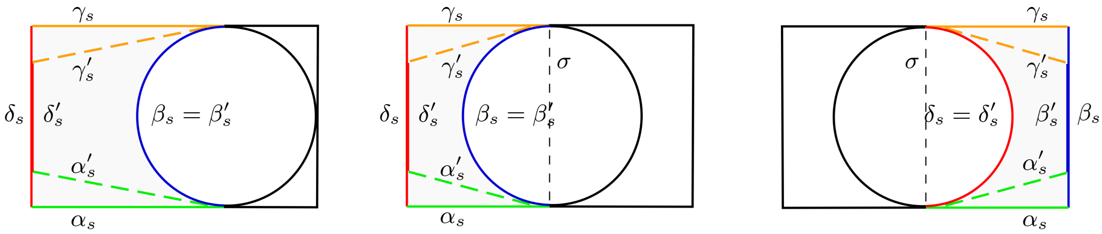


Fig. 3.24.: Schematic construction of $G_s(\alpha_s, \beta_s, \gamma_s, \delta_s)$ and $G'_s(\alpha'_s, \beta'_s, \gamma'_s, \delta'_s)$ in Case 2

Case 2. Let $c \in M$. In this case, merging w with a would result in a complex having an edge between a and c, which is not allowed. Fortunately, there are also some good news: Assertion (ii) of Lemma 3.23 tells us that $D_w = D'_w$.

Case 2.1. Assume that $M = \{a, b, c, d\}$. Then $n = 1$ (since Q is irreducible) and we are done: $\mathcal{P} = \{D_w\} = \{D'_w\} = \mathcal{P}'$.

Case 2.2. Assume that $M = \{a, b, c\}$. The idea here is to exchange the roles of a, α and γ with b, β and δ, respectively. Since Q is irreducible, $f(w, a, b)$ and $f(w, b, c)$ are faces of Q. Merging w with b yields a quad-complex $Q_s := \mu(Q; w, b)$. Let $\mathcal{P}_s$ and $\mathcal{P}'_s$ arise from $\mathcal{P}$ and $\mathcal{P}'$, respectively, by removing D_w. Since D_w touches β but neither δ nor $\delta' \subset \delta$, the (tame) quadrilaterals

$$G_s := \kappa(G; D_w, \delta), \qquad G'_s := \kappa(G'; D_w, \delta')$$

are well defined according to Lemma 3.25. Clearly, $\mathcal{P}_s$ and $\mathcal{P}'_s$ fill G_s and G'_s for Q_s, respectively, and G'_s is a compression of G_s. The case of $M = \{a, c, d\}$ is treated analogously, replacing a, α and γ by d, δ and β, respectively.

Case 2.3. Assume that $M = \{a, c\}$. Instead of merging w with a we split Q into two sub-complexes Q_β and Q_δ, where w plays the role of d or b, respectively.

Let $Q_\beta(a, b, c, w)$ be the minimal admissible sub-complex of Q containing the edges $e(a, b)$, $e(b, c)$, $e(c, w)$ and $e(w, a)$, and let $Q_\delta(a, w, c, d)$ be the minimal admissible sub-complex of Q containing the edges $e(a, w)$, $e(w, c)$, $e(c, d)$ and $e(d, a)$. Then Q_β and Q_δ

are irreducible quad-complexes and each vertex $v \notin \{a, b, c, d, w\}$ of Q is either contained in Q_β or Q_δ.

In order to define suitable quadrilaterals, we first split G into two parts. Let p and q be contact points of D_w with α and γ, respectively. Since $\alpha \cap \gamma = \emptyset$, we have $p \neq q$. Hence, the chord $\sigma := [p, q]$ of D_w is a crosscut in G. The two components G_b and G_d of $G \setminus \sigma$ define (tame) quadrilaterals $G_b(\alpha_b, \beta, \gamma_b, \sigma)$ and $G_d(\alpha_d, \sigma, \gamma_d, \delta)$, respectively. Note that none of their edges degenerates to a single prime end since D_w touches neither β nor δ.

Lemma 3.23 (ii) implies $D'_w = D_w$. Thus, we conclude that D'_w must not only touch α but also α' at p and γ' at q. Therefore, $\sigma := [p, q]$ is also a chord of D'_w, which allows us to define the (tame) quadrilaterals $G'_b(\alpha'_b, \beta', \gamma'_b, \sigma)$ and $G'_d(\alpha'_d, \sigma, \gamma'_d, \delta')$. Since G' is a compression of G, and since $D_w = D'_w$, it is clear that G'_b and G'_d are compressions of G_b and G_d, respectively.

Cutting D_w out of G_b, G'_b, G_d and G'_d we get (tame) quadrilaterals

$$G_\beta := \kappa(G_b; D_w, \beta), \quad G'_\beta := \kappa(G'_b; D_w, \beta'),$$
$$G_\delta := \kappa(G_d; D_w, \delta), \quad G'_\delta := \kappa(G'_d; D_w, \delta'),$$

respectively (see Lemma 3.25). Let $\mathcal{P}_\beta$, $\mathcal{P}'_\beta$ and $\mathcal{P}_\delta$, $\mathcal{P}'_\delta$ be the corresponding sub-agglomerations of $\mathcal{P}$ and $\mathcal{P}'$ that consist of those disks that are associated with interior vertices of Q_β and Q_δ, respectively. Using s as a shorthand for either β or δ, the construction guarantees that $\mathcal{P}_s$ and $\mathcal{P}'_s$ fill G_s and G'_s for Q_s, respectively, and that G'_s is a compression of G_s.

The known properties of Q_s, $\mathcal{P}_s$, $\mathcal{P}'_s$, G_s and G'_s are summarized in the next lemma.

Lemma 3.26 (Merging with loners)**.** *Let w be a loner of the irreducible quad-complex $Q \in \mathcal{Q}^*_{n+1}$, and assume that the quadrilateral G and its compression G' are tame. Then we have:*

(i) The quad-complex Q_s belongs to $\mathcal{Q}^*_n$.

(ii) The quadrilaterals G_s and G'_s are tame, and G'_s is a compression of G_s.

(iii) The circle agglomerations $\mathcal{P}_s$ and $\mathcal{P}'_s$ for Q_s fill G_s and G'_s, respectively.

3.4.4. Proof of the Incompressibility Theorem

After these preparations, we finally prove Theorem 3.18. For the convenience of readers who are only interested in the result, we restate it in an explicit form.

Theorem 3.27 (Incompressibility of circle agglomerations filling quadrilaterals)**.** *Let $\mathcal{P}$ and $\mathcal{P}'$ be circle agglomerations associated with an irreducible quad-complex $Q \in \mathcal{Q}^*$. If $\mathcal{P}$ fills a tame quadrilateral G and $\mathcal{P}'$ fills a tame compression G' of G, then $\mathcal{P}' = \mathcal{P}$.*

Proof. We follow the strategy explained at the beginning of this section. Assume that there are two circle agglomerations $\mathcal{P}$ and $\mathcal{P}'$ for Q filling a tame quadrilateral G and its tame compression G', respectively.

The induction with respect to the number n of inner vertices of Q starts with $n = 1$. The single quad-complex Q in $\mathcal{Q}_1^*$ has only one interior vertex v, which is adjacent to all four boundary vertices of Q. Clearly, v is a loner, and since D_v touches α and γ, Lemma 3.23 (ii) immediately tells us that $D_v = D'_v$. Thus, $\mathcal{Q}_1^*$ is incompressible.

Suppose now that all irreducible quad-complexes in $\mathcal{Q}_n^*$ are incompressible. Let $Q(a, b, c, d)$ be an irreducible quad-complex in $\mathcal{Q}_{n+1}^*$. Let $\mathcal{P}$ and $\mathcal{P}'$ be circle agglomerations for Q filling the tame quadrilateral $G(\alpha, \beta, \gamma, \delta)$ and its tame compression $G'(\alpha', \beta', \gamma', \delta')$, respectively.

Since $Q(a, b, c, d)$ was supposed to be irreducible, Lemma 3.19 guarantees the existence of a loner w. Using Lemma 3.26, we get a quad-complex $Q_s \in \mathcal{Q}_n^*$ and two circle agglomerations $\mathcal{P}_s$ and $\mathcal{P}'_s$ for Q_s filling the tame quadrilateral G_s and its tame compression G'_s, respectively. In Case 2.3 we consider both quad-complexes $Q_s := Q_\beta$ and $Q_s := Q_\delta$ separately.

One might hope that we are (almost) done, but there is still a final obstacle: The quad-complex Q_s may be reducible. Fortunately, complete reduction of Q_s yields the irreducible skeleton $\sigma(Q_s)$ of Q_s (as defined in Section 2.1.1). The corresponding circle agglomerations $\mathcal{P}_\sigma$ and $\mathcal{P}'_\sigma$ still fill G_s and G'_s, respectively. According to our assumption $\sigma(Q_s) \in \mathcal{Q}_n^*$ is incompressible, i.e., $D_v = D'_v$ for all vertices v in the vertex set V_σ of $\sigma(Q_s)$.

In order to prove this relation for all vertices in the vertex set V_s of Q_s, we observe that any u in $V_s \setminus V_\sigma$ lies in the interior of a (reducible) triangle $\triangle(a, v_1, v_2)$ or $\triangle(b, v_1, v_2)$ with $a, b, v_1, v_2 \in V_\sigma$ since only in Case 1 or Case 2.2 the quad-complex Q_s can become reducible. Now Q_s is the result of merging a loner w with the boundary vertex a or b, say a, so Lemma 3.4 tells us that there is an irreducible sub-complex Q_i of Q_s containing u. Depending on the orientation of $\triangle(a, v_1, v_2)$ we either have $Q_i = Q(a, w, v_1, v_2)$ or $Q_i = Q(a, v_2, v_1, w)$. Let $\mathcal{P}_i$ and $\mathcal{P}'_i$ be the sub-agglomerations of $\mathcal{P}$ and $\mathcal{P}'$, respectively, associated with Q_i. In the following we construct two tame quadrilaterals G_i and G'_i so that $\mathcal{P}_i$ fills G_i, $\mathcal{P}'_i$ fills G'_i, and G'_i is a compression of G_i. Then $\mathcal{P}_i = \mathcal{P}'_i$ follows from $Q_i \in \mathcal{Q}_n^*$, and in particular we get $D_u = D'_u$.

Without loss of generality we may and will assume that $Q_i = Q(a, w, v_1, v_2)$. The vertex a is associated with the boundary arcs α and α'. Since Lemma 3.3 does not allow that v_1 is a neighbor of a, it is either associated with disks $D \in \mathcal{P}_i$ and $D' \in \mathcal{P}'_i$ (in which case we have $D = D'$) or it corresponds to γ. In both cases Lemma 3.23 tells us that $D_w = D'_w$. The remaining vertex v_2 (a neighbor of a) is associated either with two identical disks in $\mathcal{P}_i$ and $\mathcal{P}'_i$ or with δ ($v_2 = b$ is excluded by the chosen orientation of Q_i). Since w cannot be a neighbor of d otherwise $\triangle(a, w, d)$ would be a reducible triangle containing v_1 and v_2, we conclude from Lemma 3.8 that D_w does not touch δ.

In order to define associated tame quadrilaterals G_i and G'_i filled by $\mathcal{P}$ and $\mathcal{P}'$, respectively, we again distinguish several cases (schematically depicted in Figure 3.25). If $v_1 = c$ and $v_2 = d$, then we just proceed as in Case 2.3 in Section 3.4.3. If $v_1 = c$ but $v_2 \neq d$, then we first use the construction of Case 2.3 and afterwards either proceed as in Case 2.2 or Case 2.3 (with respect to v_2) depending on whether v_2 is a neighbor of d or not.

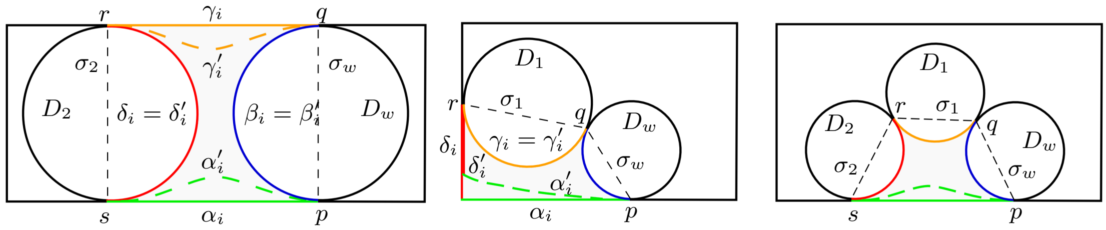


Fig. 3.25.: Schematic construction of $G_i(\alpha_i, \beta_i, \gamma_i, \delta_i)$ and $G_i'(\alpha_i', \beta_i', \gamma_i', \delta_i')$ in different cases

If $v_1 \neq c$, then we adapt the construction of Case 2.3 in the following way. Let p be a contact point of D_w with α, let $q = c(w, v_1)$ be the contact point of D_w and D_1, and let r be a contact point of D_1 either with δ or with D_2 depending on whether $v_2 = d$ or $v_2 \neq d$. In the latter case let s be a contact point of D_2 with α. Let $\sigma_w := [p, q]$, $\sigma_1 := [q, r]$ and (if $v_2 \neq d$) let $\sigma_2 := [r, s]$ be chords of D_w and D_1 (and D_2), respectively. We have $q \neq p, q \neq s$, and $r \neq p, r \neq s$ since D_1 does not touch α. Moreover, $r \neq q$ since D_w does not touch δ or since the three disks do not share a common point. Also, we have $s \neq p$ since otherwise the disk D_u (surrounded by D_w, D_1 and D_2) could not touch α.

If $v_2 = d$, then let $\sigma := \sigma_w \cup \sigma_1$. If $v_2 \neq d$, then let $\sigma := \sigma_w \cup \sigma_1 \cup \sigma_2$. Then σ is a crosscut (or a concatenation of crosscuts) of G and G' so that $G \setminus \sigma$ and $G' \setminus \sigma$ have well-defined connected components G_σ and G_σ' containing D_u and D_u', respectively.

Depending on whether $v_2 = d$ or $v_2 \neq d$ we introduce the quadrilaterals G_σ and G_σ' as $G_\sigma(\alpha_\sigma, \sigma_w, \sigma_1, \delta_\sigma)$ and $G_\sigma'(\alpha_\sigma', \sigma_w, \sigma_1, \delta_\sigma')$, or $G_\sigma(\alpha_\sigma, \sigma_w, \sigma_1, \sigma_2)$ and $G_\sigma'(\alpha_\sigma', \sigma_w, \sigma_1, \sigma_2)$, respectively. Finally, we define the quadrilaterals G_i and G_i' by applying the cut-out operator κ of Section 3.4.3 two or even three times to each of the quadrilaterals G_σ and G_σ', respectively. First we cut out D_w (keeping the actual δ-edges) then D_1 (keeping the actual α-edges) and if $v_2 \neq d$, then we eventually cut out D_2 (keeping the actual β-edges).

The resulting tame quadrilaterals G_i and G_i' are filled by $\mathcal{P}_i$ and $\mathcal{P}_i'$, respectively. Moreover, G_i' is a compression of G_i, since $D_w = D_w'$ and $\delta' \subset \delta$ or $D_2 = D_2'$. This completes the inductive step and the proof of the theorem. □

Replacing the irreducibility constrain by boundary-irreducibility we get a more general version of the Theorem 3.27. Since we need the results of the upcoming section to prove this, we state the result later in Section 3.8 together with another interesting topic: The discrete conformal modulus of quad-complexes.

3.5. Uniqueness

As its heading says, this section is all about uniqueness statements for (generalized) circle agglomerations filling tri- or quadrilaterals with respect to tri- or quad-complexes. The central term is the following.

Definition 3.17. We say that a sub-class of $\mathcal{T}^*$ (or $\mathcal{Q}^*$) has the *uniqueness property* if the following condition is satisfied for all its members T (or Q): If $\mathcal{P}$ and $\mathcal{P}'$ are two generalized circle agglomerations for T (or Q) and both fill the same tame trilateral (or quadrilateral) G, then $\mathcal{P} = \mathcal{P}'$.

The reason why we restrict G to be tame is shown in Figure 3.26. Since we can fill the unit disk for every (proper) tri-complex with a packing $\mathcal{P}$, a slightly rotated packing $\mathcal{P}'$ give rise to a non-tame (Jordan) trilateral G filled by both $\mathcal{P}$ and $\mathcal{P}'$.

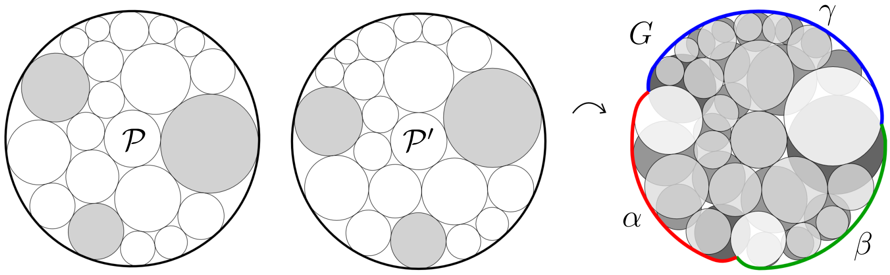


Fig. 3.26.: The packing $\mathcal{P}$ and its rotation $\mathcal{P}'$ both fill $G(\alpha, \beta, \gamma)$

In order to prove uniqueness for the whole class $\mathcal{T}^*$ we do an induction on the sub-classes $\mathcal{T}^*_n$. The Incircle Theorem of Section 3.3 (Theorem 3.17) will be the induction base for $n = 1$. Since touching four edges means especially touching three edges, we get the following corollary.

Corollary 3.28 (Induction base)**.** *The classes $\mathcal{T}^*_1$ and $\mathcal{Q}^*_1$ have the uniqueness property.*

More afford was done in Section 3.4. The following statement is an adopted version of the Incompressibility Theorem. We use the fact that every bounded, simply connected domain is trivially a compression of itself. Note that due to Lemma 3.14 there is no problem that Theorem 3.18 only covers (non-degenerate) circle agglomerations.

Corollary 3.29 (Irreducible uniqueness)**.** *The irreducible subclasses of $\mathcal{Q}^*$ have the uniqueness property.*

In order to extend this result to all classes of $\mathcal{Q}^*$ and $\mathcal{T}^*$, we first prove a more general version of Corollary 3.29 that also includes tri-complexes (see also [20] Theorem 4).

Lemma 3.30 (Irreducible uniqueness)**.** *The irreducible subclasses of $\mathcal{T}^*$ and $\mathcal{Q}^*$ have the uniqueness property.*

Proof. As already mentioned, if $T \in \mathcal{T}_1^*$, i.e., the tri-complex has only one interior vertex, then the assertion of the lemma follows from Theorem 3.17. So we may and will assume that $T \in (\mathcal{T}_n^* \setminus \mathcal{T}_1^*)$ with $n \geq 2$. By Lemma 3.2 we know that $T \in \mathcal{T}$ is proper, i.e., the interior $K := \mathrm{int}T$ of T is an admissible complex.

For technical reasons, we denote the given trilateral by $G(\alpha, \mu, \nu)$. Let v_α, v_μ and v_ν be the boundary vertices of $T(v_\alpha, v_\mu, v_\nu)$ associated with α, μ and ν, respectively. Then the leading vertex v of T that is a neighbor of v_μ and v_ν cannot be a neighbor of v_α otherwise T would be reducible.

Let P and P' be two generalized circle packings for T filling $G(\alpha, \mu, \nu)$. By Lemma 3.14 they are non-degenerate. Let $D_v \in \mathcal{P}$ and $D_v' \in \mathcal{P}'$ be the leading disks associated with v, respectively. By Lemma 3.8 D_v and D_v' cannot touch α. Let $\delta[p, q] \subset \partial D_v$ be the maximal closed positively oriented subarc of ∂D_v so that D_v touches μ at p and ν at q, respectively (note that $\delta[p, q]$ may degenerate to a point). Let $X \in \mu$ and $Y \in \nu$ be the prime ends touched by D_v at p and q, respectively. Define p', q', X' and Y' analogously with D_v replaced by D_v'.

By Lemma 3.7 (and Corollary 3.24) the open set $G \setminus \overline{D_v}$ contains a simply connected component G_α so that $\alpha \subset \partial G_\alpha^*$. The prime ends ∂G_α^* of G_α are decomposed into four arcs α, β, γ and δ, where β runs from the terminal point of α to X, γ from X to Y (corresponding to the negatively oriented circular arc of ∂D_v from p to q), and δ extends from Y to the initial point of α. These arcs define a quadrilateral $G_\alpha(\alpha, \beta, \gamma, \delta)$. A similar definition is made for the quadrilateral $G_\alpha'(\alpha, \beta', \gamma', \delta')$, where G_α' is the connected component of $G \setminus \overline{D_v'}$ with $\alpha \subset \partial G_\alpha'^*$ (see the first three examples in Figure 3.27, illustrating, in this order, the cases 1-3 below).

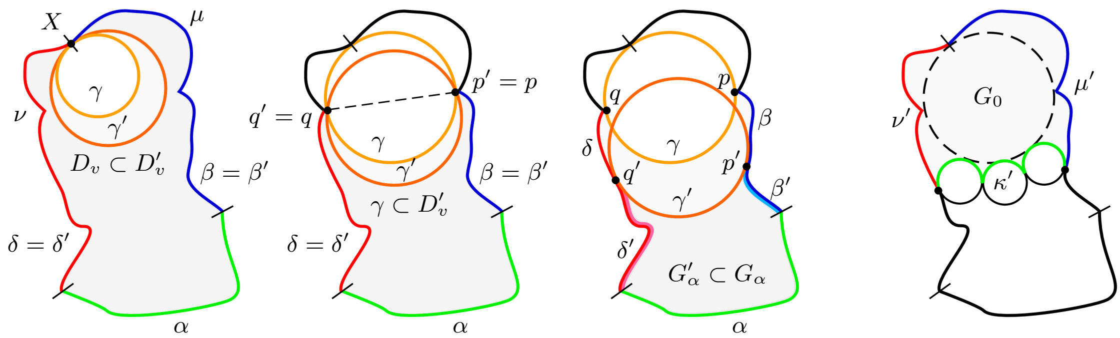


Fig. 3.27.: Constructions for the proof of Lemma 3.30

In order to prove that one quadrilateral G_α or G_α' is a compression of the other we distinguish three cases.

Case 1. Let $p = p' = q = q'$. Then p is the contact point of D_v and D_v' with the prime end $X = \mu \cap \nu$. Since G is tame, X is regular, i.e., $D_v \subset D_v'$ or $D_v' \subset D_v$. This implies $\beta = \beta'$, $\delta = \delta'$ and (w.l.o.g.) $G_\alpha' \subset G_\alpha$ as desired.

Case 2. Let $p = p'$ and $q = q'$ but $p \neq q$. Then $\beta = \beta'$ and $\delta = \delta'$. The chord $\sigma := [p, q]$ is a crosscut dividing G into the two components G_1 and G_2 so that $G_\alpha \subset G_1$

and $G'_\alpha \subset G_1$. Clearly $(D_v \cap G_1) \subset D'_v$ or $(D'_v \cap G_1) \subset D_v$, which implies $G'_\alpha \subset G_\alpha$ or $G_\alpha \subset G'_\alpha$, respectively.

Case 3. Let $p \neq p'$ or $q \neq q'$. We have w.l.o.g. $p \neq p'$ and $\beta' \subset \beta$. Then the prime end X associated with p by D_v is not subordinate to any prime end of G'_α. We prove $D_v \cap G'_\alpha = \emptyset$, from which we get $\delta' \subset \delta$ as well as $G'_\alpha \subset G_\alpha$. Assuming contrarily that there is some $r \in D_v \cap G'_\alpha$, we define the two chords $\sigma := [p, r]$ and $\sigma' := [p', q']$. Then σ is a Jordan arc ending at the prime end X, and σ' is a crosscut of G separating r from X, hence $\sigma \cap \sigma' \neq \emptyset$. Invoking the Two-Disk-Lemma (Lemma 2.8) we get $r = p'$ or $r = q'$, which is a contradiction to $r \notin \partial G$.

Finally, Lemma 3.6 guarantees that the "new" prime ends $\beta \cap \gamma$, $\gamma \cap \delta$, $\beta' \cap \gamma'$ and $\gamma' \cap \delta'$ of $G_\alpha(\alpha, \beta, \gamma, \delta)$ and $G'_\alpha(\alpha, \beta', \gamma', \delta')$, respectively, are regular. Hence, both quadrilaterals are tame. The quad-complex Q that results from T by omitting the edge $e(v_\mu, v_\nu)$ is irreducible. Moreover, v is a boundary vertex of Q and the reduced packings $\mathcal{P} \setminus \{D_v\}$ and $\mathcal{P}' \setminus \{D'_v\}$ are associated with Q and fill G_α and G'_α, respectively. From Theorem 3.27 we get $D_w = D'_w$ for all $w \in V \setminus \{v\}$. Thus, it remains to show that $D_v = D'_v$.

Both disks are incircles of a tame trilateral $G_0(\mu', \nu', \kappa')$, where $\mu' \subset \mu$, $\nu' \subset \nu$ and κ is the union of a collection of subarcs of circles $\partial D_w = \partial D'_w$ with $w \neq v$ (see Figure 3.27, right). Hence, the desired result follows from Theorem 3.17. □

With Lemma 3.30 at hand, the proof of our main uniqueness statement for the alpha-beta-gamma normalization is quite a simple task.

Theorem 3.31 (Alpha-Beta-Gamma Uniqueness Theorem)**.** *The classes $\mathcal{T}^*$ and $\mathcal{Q}^*$ have the uniqueness property.*

Proof. We look at the sub-classes $\mathcal{T}^*_n$ and $\mathcal{Q}^*_n$ and do an induction on n. The Corollary 3.28 covers the induction base. Our induction hypothesis is that $\mathcal{T}^*_n$ and $\mathcal{Q}^*_n$ have the uniqueness property for some $n \geq 1$. Now, let $T \in \mathcal{T}^*_{n+1}$ and $Q \in \mathcal{Q}^*_{n+1}$. Since the remaining proof runs exactly the same for Q as for T, we will only consider tri-complexes (and thus trilaterals). Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two generalized circle agglomerations for T filling the tame trilateral $G(\alpha, \beta, \gamma)$.

Let $T_\sigma := \sigma(T)$ be the skeleton of T. Let $\mathcal{P}^1_\sigma$ and $\mathcal{P}^2_\sigma$ be the sub-packings of $\mathcal{P}^1$ and $\mathcal{P}^2$ associated with T_σ. Since T_σ is irreducible and since $\mathcal{P}^1_\sigma$ and $\mathcal{P}^2_\sigma$ fill the tame G, we obtain $\mathcal{P}^1_\sigma = \mathcal{P}^2_\sigma$ from Lemma 3.30. There is nothing to prove if $T_\sigma = T$ so assume there is a vertex u in $T \setminus T_\sigma$. Let P^1_u and P^2_u be the associated sets (disks or dots) of $\mathcal{P}^1$ and $\mathcal{P}^2$, respectively. Showing $P^1_u = P^2_u$ completes the proof.

By the definition of the skeleton there are three vertices $e, f, g \in T_\sigma$ so that $\triangle(e, f, g)$ is a face in T_σ but a reducible triangle in T containing u. Let $T' = \varrho_-(T, \triangle)$ be the outer reduction of T with respect to $\triangle(e, f, g)$, so $T' \in \mathcal{T}^*_n$. Let $\mathcal{P}'$ and $\mathcal{P}''$ be the sub-packings of $\mathcal{P}^1$ and $\mathcal{P}^2$ associated with T'. We define a tame trilateral G' so that $\mathcal{P}'$ and $\mathcal{P}''$ fill G' for T', i.e., the induction hypothesis provides $\mathcal{P}' = \mathcal{P}''$ and especially $P^1_u = P^2_u$.

To do so, we first note again that none of the sets associated with e, f and g is a dot (Lemma 3.14). Let w.l.o.g. $\triangle(e, f, g)$ be positively oriented. We distinguish between the number of disks associated with e, f and g. There is at least one due to the definition of reducibility for tri-complexes.

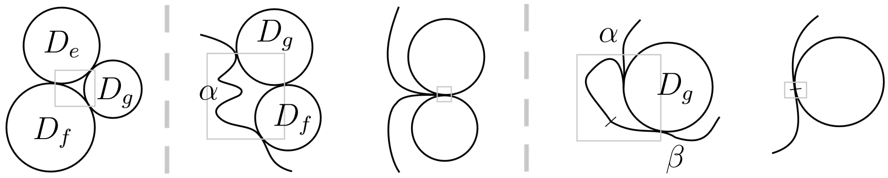


Fig. 3.28.: Combinations for disks associated with $\triangle(e,f,g)$

Case 1. Assume that all three vertices e, f and g are associated with disks $D_e = D_e^1 = D_e^2$, $D_f = D_f^1 = D_f^2$ and $D_g = D_g^1 = D_g^2$, respectively (see Figure 3.28, left). Let the interstice they form be denoted G'. Then we can interpret G' in a natural way as (Jordan) trilateral $G'(\alpha', \beta', \gamma')$ so that the edges α', β' and γ' correspond to circular arcs of D_e, D_f and D_g, respectively. Clearly G' is tame. So we are done using the arguments from above if all disks associated with vertices of $\mathrm{int}T'$ actually lie in G'. Now, maybe this is obvious, maybe it is not. The reader is invited to see a (rather technical and longish) proof in Appendix A.1, which also covers the upcoming Case 2 and Case 3.

Case 2. Assume only f and g are associated with disks D_f and D_g, respectively, while e is associated with an edge of G, say with α (see Figure 3.28, middle).

If the contact point $p = c(f,g)$ between D_f and D_g is also a contact point of D_f (and D_g) with α, then all sets of $\mathcal{P}'$ and $\mathcal{P}''$ must be dots $S = \{p\}$ (see Appendix A.1). So trivially $P_u^1 = P_u^2 = S$ and we are done.

Assume that p is not a contact point of D_f (and D_g) with α. Let G' be the boundary interstice induced by D_f and D_g as defined in Section 2.1.2. We can interpret G' in a natural way as a trilateral $G'(\alpha', \beta', \gamma')$ so that β' and γ' are associated with the corresponding arcs of ∂D_f and ∂D_g, respectively, and $\alpha' \subset \alpha$ is the remaining edge. By Lemma 3.6 the trilateral G' is tame. Since all disks and dots associated with vertices of $\mathrm{int}T'$ must lie in G' and $\overline{G'}$, respectively (see Appendix A.1), we are done.

Case 3. Assume only g is associated with a disk D_g while e and f are associated with edges of G, say α and β, respectively (see Figure 3.28, right). Let $X = \alpha \cap \beta$ be a vertex prime end of G.

If D_g touches X in a point $p \in \partial G$, then all sets of $\mathcal{P}'$ and $\mathcal{P}''$ must be dots $S = \{p\}$ since X is regular (see Appendix A.1). Again, we get $P_u^1 = P_u^2 = S$ and we are done.

Assume that D_g does not touch X. Let $\alpha' \subset \alpha$ be the minimal closed sub-arc of α so that the starting point of α' is a contact point of D_g with α while the ending point of α' is X. Analogously, let $\beta' \subset \beta$ be the minimal closed sub-arc of β so that it starts in X and ends in a touching point of D_g with β. Let δ be the closed, negatively oriented circular arc of ∂D_g so that $\delta \subset G$ starts at the ending point of β' and ends at the starting point of α'. Note that δ is a crosscut of G. Let G' be that component of $G \setminus \delta$ that does not contain D_g. Then we can interpret G' as a trilateral $G'(\alpha', \beta', \gamma')$, where γ' is the prime end interpretation of δ.

We are done since G' is tame (Lemma 3.6) and contains all disks associated with $\mathrm{int}T'$ (Appendix A.1) . $\square$

3.6. Continuity

In this section we mainly do two things: We show the usability of the alpha-beta-gamma normalization and we prepare some statements for its existence proof. In particular we introduce a very useful special case of the *alpha-gamma normalization*, which can be pulled back directly onto the alpha-beta-gamma case. Furthermore, for specific situations, we show that generalized circle agglomerations depend *continuously* on some normalization parameters.

3.6.1. A Special Case of the Alpha-Gamma Normalization

Roughly speaking, the alpha-gamma normalization fixes the center of an alpha disk of a circle packing, and some boundary beta disk is assumed to meet a fixed boundary point (prime end). While Chapter 4 investigates this normalization to its full extend, we are here only interested in that special case of the alpha disk being a boundary disk. The following definition makes this more explicit.

Definition 3.18. Let K be an admissible complex with distinguished boundary vertices $a, b \in \partial V$. In short we write $K(a, c)$. Let G be a bounded, simply connected domain, and let A be a fixed interior point while C shall be a fixed prime end of G, i.e., $A \in G$ and $C \in \partial G^*$. In short we write $G(A, C)$ and we call it a *pinned domain.* Let $\mathcal{P}$ be a generalized circle packing with complex K filling G. Then $\mathcal{P}$ fills $G(A, C)$ for $K(a, c)$ *under the boundary alpha-gamma normalization* if (1) the center of $D_a \in \mathcal{P}$ (the disk associated with a) is A and (2) the disk or dot $P_c \in \mathcal{P}$ (associated with c) meets C.

Note that D_a has to be a disk since it touches ∂G while its center lies in G. Note further that P_c *meeting* the prime end C means that P_c touches C or that there is a crosscut $\Gamma \subset P_c$ separating C from all disks of $\mathcal{P} \setminus \{P_v\}$ (in the sense of Definition 3.13). If P_c is a dot, then meeting C means touching it.

If C is regular, then it turns out that the boundary alpha-gamma normalization uniquely determines $\mathcal{P}$. The following statement is a simple application of our previous results. Later on we sometimes refer to the constructions made within its proof.

Lemma 3.32 (Boundary alpha-gamma uniqueness)**.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ fill a pinned, bounded, simply connected domain $G(A, C)$ for an admissible complex $K(a, c)$ under the boundary alpha-gamma normalization. If C is regular, then $\mathcal{P}_1 = \mathcal{P}_2$.*

Proof. We first note that the alpha disks of $\mathcal{P}_1$ and $\mathcal{P}_2$ must be equal since they are the uniquely determined maximal disks in G with center in A, i.e., $D_a^1 = D_a^2 = D_a$.

If D_a touches C, say in a contact point $s \in \partial G$, then neither can any other disk of $\mathcal{P}$ touch C since it is assumed to be regular, nor can P_c separate C from D_a in any case. We conclude that P_c must be a dot $P_c = S_c = \{s\}$. Since K stays connected when removing a together with its edges and faces, there is a chain of vertices $(v, \dots, c)$ in K that does not contain a, but connects any vertex $v \neq a$ with c. Thus, all sets in $\mathcal{P}_1 \setminus \{D_a\}$ and in $\mathcal{P}_2 \setminus \{D_a\}$ must be dots $S = S_c$, too, since they are neighbors or neighbors-neighbors of

S_c. Hence, we have $\mathcal{P}_1 = \mathcal{P}_2$ in a trivial sense. This only leaves the case that D_a does not touch C.

Let G_c denote the unique simply connected component of $G \setminus \overline{D_a}$ that contains the tails of C (Lemma 3.7). Since the sets P_c^1 and P_c^1 (associated with c) meets C, at least one disk of $\mathcal{P}_1$ and $\mathcal{P}_2$ must be contained in G_C. We show that every disk of $\mathcal{P}_1 \setminus \{D_a\}$ and $\mathcal{P}_2 \setminus \{D_a\}$ must be contained in G_C.

Assume the contrary. Then there is a vertex $v \neq a$ in K so that its associated disk $D_v \in \mathcal{P}_1$ lies not in G_C. As explained above, there is a chain of vertices $(v, \ldots, c)$ in K that does not contain a. Dropping all vertices that are associated with dots, let $D_1, \ldots, D_n$ be the chain of all associated disks in $\mathcal{P}_1$. By construction we have $D_1 = D_v$ and $D_n \subset G_C$. Clearly, this implies that at some point two consecutive disks D_i and D_{i+1} must touch D_a in a common point in order to enter G_C. This is impossible. Hence, $\mathcal{P}_1 \setminus \{D_a\}$ and $\mathcal{P}_2 \setminus \{D_a\}$ are generalized circle agglomerations lying in, thus filling, the domain G_C.

We show that both agglomerations fill a common tame trilateral $G_C(\alpha, \beta, \gamma)$ for some tri-complex $T(a, v_1, v_2)$. Thus, they must be equal by the Uniqueness Theorem 3.31 and we are done.

In order to construct $G_C(\alpha, \beta, \gamma)$, we define X_1 to be that prime end of G touched by D_a that we reach first when we walk along ∂G^* from C with negative orientation. Let β be the set of prime ends we walked through (see Figure 3.29, right). Analogously, let X_2 be the first prime end of G touched by D_a that we reach by walking along ∂G^* from C with positive orientation. Let γ be the set of prime ends we walked through. Eventually, defining α' as the negatively oriented sub-arc of ∂D_a from X_1 to X_2 (if $X_1 = X_2$, then we set $\alpha' := \partial D_a$), we can associated it with an arc α of prime ends in G_C. Note that even if $X_1 = X_2$ are equal in G, then the endpoints of α are different (subordinate) prime ends of G_C.

By construction we can interpret G_C as the trilateral $G_C(\alpha, \beta, \gamma)$. By Lemma 3.7 the prime ends $\alpha \cap \beta$ and $\alpha \cap \gamma$ are regular. Thus, the trilateral $G_C(\alpha, \beta, \gamma)$ is tame since $C = \beta \cap \gamma$ is regular by assumption.

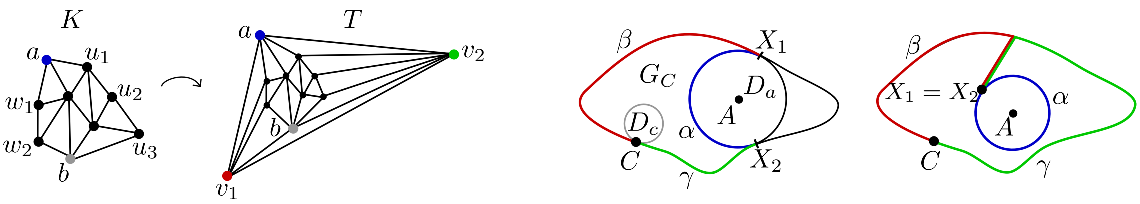


Fig. 3.29.: Construction of $T(a, v_1, v_2)$ and $G_C(\alpha, \beta, \gamma)$ out of $K(a, b)$ and $G(A, C)$, respectively

In order to construct $T(a, v_1, v_2)$, we first follow the boundary chain of K with positive orientation from a to b. Let $a = w_0, w_1, \ldots, w_k, w_{k+1} = b$ be the corresponding boundary vertices we meet (in order of appearance; see Figure 3.29, left). Following the boundary chain with negative orientation from a to b, we get $a = u_0, u_1, \ldots, u_l, u_{l+1} = b$. Maybe we have $k = 0$ or $l = 0$.

We add two vertices v_1 and v_2 together with its edge $e(v_1, v_2)$ to K, and also the edges and faces described next. For $i = 0, \dots, k+1$ and $j = 1, \dots, l+1$ we add the edges $e(v_1, w_i)$ and $e(v_2, u_j)$. For $i = 0, \dots, k$ and $j = 0, \dots, l$ we add the faces $f(v_1, w_i, w_{i+1})$ and $f(v_2, u_j, u_{j+1})$. Finally, we add the face $f(v_1, v_2, b)$. Let T denote the resulting complex. Since every boundary vertex of K appears exactly once in the boundary chain, our construction yields a tri-complex $T(a, v_1, v_2)$ with the boundary vertices a, v_1 and v_2.

Now we put everything together. Let a, v_1 and v_2 be associated with α, β and γ, respectively. Then $\mathcal{P}_1 \setminus \{D_a\}$ and $\mathcal{P}_2 \setminus \{D_a\}$ fill $G_C(\alpha, \beta, \gamma)$ for $T(a, v_1, v_2)$. By the Uniqueness Theorem 3.31 we have $P_v^1 = P_v^2$ for every v in int$T$, i.e., for every $v \neq a$ in K. Hence, $\mathcal{P}_1$ equals $\mathcal{P}_2$ since we already know $D_a^1 = D_a^2 = D_a$. □

The case of non-regular C is discussed in Chapter 4. In this chapter (especially in Section 3.7.1) we apply Lemma 3.32 only to regular domains G, i.e., every prime end of G, in particular C, is regular by assumption. Beside uniqueness, this also ensures that the involved packings are non-degenerate as soon as D_a does not touch C.

Lemma 3.33. *Let the generalized circle packing $\mathcal{P} = \mathcal{D} \cup \mathcal{S}$ fulfill the boundary alpha-gamma normalization for an admissible complex $K(a, c)$ and a pinned, regular, bounded, simply connected domain $G(A, C)$. If the alpha disk D_a does not touch C, then $\mathcal{S} = \emptyset$. Otherwise, if D_a touches C, say in a contact point s, then $\mathcal{D} = \{D_a\}$ and $S_v = \{s\}$ for all $v \neq a$ in K.*

Proof. We use the constructions made within the proof of Lemma 3.32. There, the latter statement of this lemma was shown at the very beginning. So assume that D_a does not touch C. Since $\mathcal{P} \setminus \{D_a\}$ fills the trilateral $G_C(\alpha, \beta, \gamma)$ for the tri-complex $T(a, v_1, v_2)$, there must be at least one disk D contained in it. Thus, the generalized circle packing $\mathcal{P}$ contains at least the two disk D_a and D. By Lemma 2.18 every dot of $\mathcal{P}$ is a pseudo contact point p of two disks of $\mathcal{P}$, and $p \in \partial G$ must be a boundary point of G. Since there are no inward spikes, this is impossible for regular domains. Hence, $\mathcal{P}$ is non-degenerate. □

As conclusion of this little excursion to the alpha-gamma normalization we provide the following existence statement.

Lemma 3.34. *Let $K(a, c) \in \mathcal{K}_n$ be an admissible complex and let $G(A, C)$ be a pinned, regular, bounded, simply connected domain. Let the maximal disk in G with center in A do not touch C. Assume that for every tri-complex in $\mathcal{T}_{n-1}$ there is a generalized circle agglomeration filling a given regular trilateral. Then there is a unique circle packing $\mathcal{P}$ fulfilling the boundary alpha-gamma normalization for $K(a, c)$ and $G(A, C)$.*

Proof. Let D_a be the maximal disk in G with center in A. By assumption D_a does not touch C. We once more use the constructions made within the proof of Lemma 3.32. There, the tri-complex $T(a, v_1, v_2)$ is associated with the trilateral $G_C(\alpha, \beta, \gamma)$. By definition the complex K contains exactly one vertex more than the interior of T. Thus, we have $T \in \mathcal{T}_{n-1}$. By Lemma 3.7 the trilateral $G_C(\alpha, \beta, \gamma)$ is regular. So the assumption of this lemma yields a generalized circle agglomeration $\mathcal{P}'$ for T filling G_C. By construction the packing $\mathcal{P} := \mathcal{P}' \cup \{D_a\}$ is a generalized circle packing fulfilling the boundary alpha-gamma normalization for $K(a, c)$ and $G(A, C)$. Eventually, by Lemma 3.32 and 3.33, $\mathcal{P}$ is uniquely determined and non-degenerate. $\square$

Note that, although Lemma 3.34 depends on the alpha-beta-gamma existence, we actually use it to prove the alpha-beta-gamma existence. This is possible since we do an induction on n.

3.6.2. Continuous Dependence on Parameters

Another application of the alpha-beta-gamma normalization, and yet another preparation for its existence proof, is the continuous dependence of packings on normalization parameters. To begin with, we define what it means for a sequence of prime ends, disks, dots and generalized circle agglomerations to converge.

Let G be a bounded, simply connected domain. Let $g : G \to \mathbb{D}$ and g^* be a canonical embedding of G and its extension to G^*, respectively. A sequence (X_k) of prime ends $X_k \in \partial G^*$ *converges to a prime end* X of G if the points $g^*(X_k)$ on $\partial\mathbb{D}$ converge to $g^*(X)$.

Let (X_k) and (Y_k) be two sequences of prime ends of G. Let δ_k be a (positive oriented) closed arc of prime ends so that X_k and Y_k are its ending points. The sequence (δ_k) *converges to the (positive oriented) closed arc* δ with ending points X and Y if (X_k) and (Y_k) converge to X and Y, respectively.

A sequence of trilaterals $G(\alpha_k, \beta_k, \gamma_k)$ *converges to the trilateral* $G(\alpha, \beta, \gamma)$ if (α_k), (β_k) and (γ_k) converge to α, β and γ, respectively. This definition shall also apply to the case of an arc, say α_k, that converges to a singleton $\alpha = \{X\}$, although here $G(\alpha, \beta, \gamma)$ is not a trilateral anymore.

For each $k \in \mathbb{N}$ let D^k be a disk or dot. We say that the sequence (D^k) *converges to a disk* D if the centers c_k of D^k converge to the center of D, and the radii r_k of D^k converge to the radius of D. The *limit of* (D^k) *is a dot* $S = \{s\}$ if r_k tends to zero and c_k converges to s.

A sequence $(\mathcal{P}_k)$ of generalized circle agglomerations with acceptable complex K is said to *converge to a generalized circle agglomeration* $\mathcal{P}$ for the same complex if for each v in the vertex set of K the sequence (P_v^k) of associated disks or dots in $\mathcal{P}_k$ converges to the corresponding set P_v in $\mathcal{P}$.

Clearly, if a sequence (D^k) of disks lies in G and converges to a disk D, then D is contained in G as well. If the limit is a dot S, then this need not be so; for example S can lie on the impression of an inaccessible prime end. Nevertheless, by Lemma 3.5 every dot S of a non-collapsed generalized circle agglomeration $\mathcal{P}$ is attached to some disk D. So the behavior of S can be traced back to the behavior of D.

Lemma 3.35. *Let G be a bounded, simply connected domain. Let (δ_k) be a sequence of closed arcs of prime ends δ_k of G that converges to a closed arc $\delta \subset \partial G^*$ (which may consist of a single element). Let $(\mathcal{P}_k)$ be a sequence of generalized circle agglomerations contained in G that converges to the generalized circle agglomeration $\mathcal{P}$. For each $k \in \mathbb{N}$ let $(P_1^k, \ldots, P_m^k)$ be a chain of sets in $\mathcal{P}_k$ so that every P_j^k with $j \in \{1, \ldots, m-1\}$ is a disk or a dot, P_1^k touches δ_k, and P_m^k is definitely a disk (maybe $m = 1$). For $j = 1, \ldots, m-1$ the limit of every sequence (P_j^k) shall be a dot S_j.*

Then there is a prime end $X \in \delta$ so that for any tail U of X and sufficiently large k every disk within $\{P_1^k, \ldots, P_{m-1}^k\}$ is contained in U. If (P_m^k) converges to a dot, then also (P_m^k) lies in U for sufficiently large k. If (P_m^k) converges to a disk D, then D touches X at a contact point $s \in \partial G$, and for all $j \in \{1, \ldots, m-1\}$ the dot $S_j = \{s\}$ touches X via D.

Proof. Let $\{D_1^k, \ldots, D_l^k\}$ be the (analogously ordered) subset of all sets in $\{P_1^k, \ldots, P_m^k\}$ that are disks for almost all, say for all $k \in \mathbb{N}$. We have $l \geq 1$ since by assumption $D_l^k = P_m^k$. If $D_1^k = P_1^k$, then D_1^k touches some prime end X_k of δ_k. If $D_1^k = P_i^k$ for some $i > 1$, then all the sets P_j^k with $j \in \{1, \ldots, i-1\}$ are dots attached to D_1^k, and the dot P_1^k touches X_k via D_1^k. So in both cases D_1^k touches $X_k \in \delta_k$.

Let $g : G \to \mathbb{D}$ and g^* be a canonical embedding of G and its extension to G^*, respectively. Let $f : \mathbb{D} \to G$ (and f^*) be the corresponding (extended) canonical parameterization of G, i.e., the inverse mapping of g (or rather g^*). Since D_1^k touches X_k, the closure of $g(D_1^k)$ contains some point $t_k := g^*(X_k)$ on the arc $d_k = g^*(\delta_k) \subset \partial\mathbb{D}$. Since δ_k and hence d_k converge to δ and $d = g^*(\delta)$, respectively, which are assumed to be closed, the t_k have an accumulation point $t \in d$. Let $X := f^*(t)$ be the corresponding prime end in δ.

Let (J_n) be a sequence of crosscuts defining X, and denote by U_n the tail of J_n. For two consecutive crosscuts J_n and J_{n+1} of this sequence the corresponding tails U_n and U_{n+1} satisfy $U_{n+1} \subset U_n$. According to the definition of prime ends, we have $\overline{J_n} \cap \overline{J_{n+1}} = \emptyset$. So the distance of J_n and J_{n+1} is positive (see Figure 3.30, left). Let U be any tail of X. By dropping the first tails of the sequence if necessary, we may and will assume w.l.o.g. that $U_1 \subset U$. Moreover, for some upcoming reasons, let $n \geq 2l - 1$.

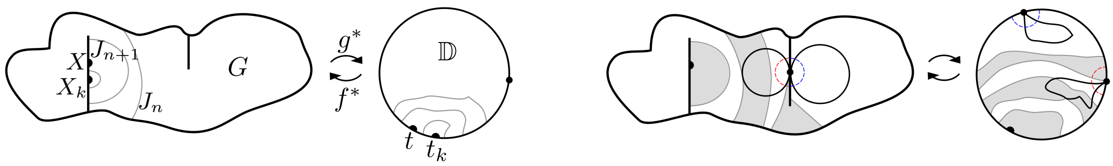


Fig. 3.30.: Constructions for the proof of Lemma 3.35

For sufficiently large k, not only t_k is contained in $\overline{g(U_{n+1})}$ but also the image $g(W_k)$ of some tail W_k of $X_k = f^*(t_k)$, i.e., we have $W_k \subset U_{n+1} \subset U_n$. Since D_1^k touches X_k, there is a point $x_1^k \in (D_1^k \cap W_k) \subset U_{n+1}$. We prove that for $l > 1$ and sufficiently large k the whole disk D_1^k must be contained in U_n.

In order to do so, assume contrary that for $l > 1$ there is some point $y_1^k \in D_1^k$ with $y_1^k \notin U_n$ for almost all, say for all $k \in \mathbb{N}$. Then the straight line $\sigma_1^k = [x_1^k, y_1^k] \subset D_1^k$ lies in G, and it intersects both crosscuts J_n and J_{n+1}. Therefore, on the one hand the length of σ_1^k is bounded below by the distance of J_n and J_{n+1}, but on the other hand it is bounded above be the diameter of D_1^k. The former is a positive constant (depending on n) while the latter tends to zero for $k \to \infty$, a contradiction. Hence, we have $D_1^k \subset U_n$.

Now, we look at the (maybe pseudo) contact point c_1^k between D_1^k and D_2^k. Let $B_\varepsilon(c_1^k)$ be a sufficiently small ε-ball around it, and let $B := B_\varepsilon(c_1^k) \cap G$. If $c_1^k \in G$, then clearly $B \subset G$. If $c_1^k \in \partial G$, then, since $\mathcal{P}$ is contained in G, both disks D_1^k and D_2^k associate c_1^k with the same prime end of ∂G^*, i.e. $g(B)$ is connected in $\mathbb{D}$ and a situation as depicted in Figure 3.30 (right) cannot happen. Therefore, we have $g(B) \subset g(U_{n-1})$ for sufficiently small ε. So $g(B) \cap g(D_2^k) \neq \emptyset$ implies $g(D_2^k) \cap g(U_{n-1}) \neq \emptyset$, thus $D_2^k \cap U_{n-1} \neq \emptyset$.

We pick some point $x_2^k \in (D_2^k \cap U_{n-1})$. If $l > 2$, then we apply the same argumentation as above in order to show $D_2^k \subset U_{n-2}$. Moreover, we get one after another the points $x_3^k, \ldots, x_l^k$ with $x_j^k \in (D_j^k \cap U_{n-2j+3})$ for $j = 3, \ldots, l$ and $D_j^k \subset U_{n-2j+2}$ for $j = 3, \ldots, l-1$ (see Figure 3.31).

If (P_m^k) converges to a dot, then also the radius of $P_m^k = D_l^k$ goes to zero, thus exactly the same argumentation as above yields $D_l^k \subset U_{n-2l+2}$. Since we assumed $n \geq 2l - 1$ and since the tails U_n are nested, we have $D_j^k \subset U_1 \subset U$ for all $j = 1, \ldots, l$, what proves the first two statements of the lemma.

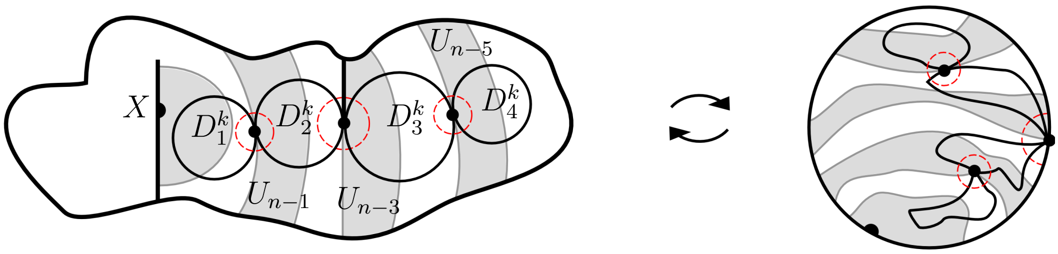


Fig. 3.31.: Constructions for the proof of Lemma 3.35

Assume now that (P_m^k), i.e., (D_l^k) converges to a disk D. We first simplify our notation. Up to changes of indexes, we showed so far that for every $n \in \mathbb{N}$ and $j \in \{1, \ldots, l-1\}$ we have $D_j^n \subset U_n$ and $D_l^n \cap U_n \neq \emptyset$. We prove that D touches X.

In order to do so, let c be the center of D. For sufficiently large, say for every n we have $c \in D_l^n$ since D is the limit of (D_l^n). Moreover, we may and will assume w.l.o.g. that $c \notin U_1$; just drop the first tails if necessary. For every $n \in \mathbb{N}$ we choose some point $x_n \in (D_l^n \cap U_n)$ and define the straight line $\sigma_n := [c, x_n] \subset D_l^n$. By construction there is an intersection point $z_n^i \in (J_i \cap \sigma_n)$ for every $i = 1, \ldots, n-1$ and the distance $|z_n^i - z_n^{i+1}|$ is bounded below by some constant depending on i but not on n.

By the Bolzano-Weierstraß Theorem, a sub-sequence of (σ_n) converges to a straight line $\sigma = [c, x]$. Clearly, up to the endpoint x, we have $[c, x) \subset D$. The limits $z^1, z^2, \ldots$ of $z_n^1, z_n^2, \ldots$ on σ, respectively, are distinct and the fact that $z^{i+1} \in (D \cap U_i)$ implies

$(D \cap U_i) \neq \emptyset$ for every $i = 1, 2, \ldots$. Hence, by definition, D touches X.

Now we look at the remaining sets of $\{D_1^n, \ldots, D_{l-1}^n\}$. Since $\overline{D_j^n} \subset \overline{U_n}$, the limit dot $S_j = \{s_j\}$ of (D_j^n) lies in the impression of X, i.e., $s_j \in I(X)$ for every $j = 1, \ldots, l-1$. Clearly, all dots S_j are equal, i.e., $s_j = s_i = s \in \partial G$ for every $i, j \in \{1, \ldots, l-1\}$. Moreover, we have $s \in \partial D$, i.e., s is the contact point of D with X.

Finally, all dots within $\{P_1^k, \ldots, P_m^k\}$, which we put aside at the very beginning, trivially lie in s, too, what completes the proof. □

Due to the presence of dots the following compactness result for generalized circle agglomerations is not obvious. In particular condition (ii) of Definition 3.8 needs to be treated very carefully.

Lemma 3.36. *Any bounded sequence $(\mathcal{P}_k)$ of generalized circle agglomerations with acceptable complex K contains a sub-sequence that converges to a generalized circle agglomeration $\mathcal{P}$ for K.*

Assume that $\mathcal{P}$ is not collapsed, and let G be a bounded, simply connected domain. If $\mathcal{P}_k$ lies in G for every $k \in \mathbb{N}$, then also $\mathcal{P}$ lies in G. If $\mathcal{P}_k$ fills G for every $k \in \mathbb{N}$, then also $\mathcal{P}$ fills G.

Proof. 1. By the Bolzano-Weierstraß Theorem, there is a sub-sequence of $(\mathcal{P}_k)$ (which we assume to be $(\mathcal{P}_k)$ itself) with radii $r_v^k \geq 0$ converging to $r_v \geq 0$ and centers c_v^k converging to $c_v \in \mathbb{C}$. If $e(v, w)$ is an edge of K, then D_v^k and D_w^k touch each other for every k, which implies that (the disk or dot) P_v touches (the disk or dot) P_w in the limit. Moreover, if $f(u, v, w)$ is a face of K, then the orientation of the triangle $\triangle(c_u^k, c_v^k, c_w^k)$ carries over to the orientation of $\triangle(c_u, c_v, c_w)$, provided that D_u^k, D_v^k, D_w^k and D_u, D_v, D_w are disks. Hence, the ensemble $\mathcal{P}$ of disks or dots P_v with radii r_v and centers c_v is a generalized circle agglomeration for K, what proves the first part of the lemma.

2. Assume that $\mathcal{P}$ is not collapsed and that $\mathcal{P}_k$ lies in a bounded, simply connected domain G. We verify that $\mathcal{P}$ fulfills the conditions (i) and (ii) of Definition 3.8, i.e., $\mathcal{P}$ is contained in G.

Verifying (i) is almost trivial: If the limit D_v of $P_v^k \subset G$ is a disk, then we clearly have $D_v \subset G$, and if the limit is a dot, then there is nothing to prove.

In order to verify (ii), let p be a contact point or pseudo contact point between two disks $D_v, D_w \in \mathcal{P}$ so that p is also a contact point of D_v with a prime end X, and of D_w with a prime end Y. Since D_v and D_w are disks, we may and will assume w.l.o.g. that $P_v^k = D_v^k$ and $P_w^k = D_w^k$ are disks for every $k \in \mathbb{N}$. If we show $X = Y$, then we are done. In order to do so, we distinguish two cases.

2.1. Assume D_v^k and D_w^k touch each other for almost all, say for all k. Then there is a sequence (p_k) of points $p_k = \partial D_v^k \cap \partial D_w^k$ and p_k converges to p. Let $g : G \to \mathbb{D}$ and g^* be a canonical embedding of G and its extension to G^*, respectively. Since $\mathcal{P}_k$ lies in G, we have $\overline{g(D_v^k)} \cap \overline{g(D_w^k)} = \{s_k\}$ with $s_k \in \overline{\mathbb{D}}$. Clearly, the pre-image of the limit s of (s_k) is touched by D_v and D_w, i.e. depending whether $s \in \mathbb{D}$ or $s \in \partial\mathbb{D}$ the two disks D_v and D_w touch the interior point $q \in G$ with $g(q) = s$, or the prime end $Z \in \partial G^*$ with $g^*(Z) = s$, respectively. By assumption, the (maybe pseudo) contact point of D_v and D_w lies on the boundary of G, i.e., the latter case implies $X = Y = Z$.

2.2. Assume D_v^k and D_w^k do not touch each other for almost all, say for all k. Then v is not a neighbor of w in K, thus p is not a contact point of D_v and D_w. In order to make p a pseudo contact point, there must be a chain $(v, u_1, \ldots, u_n, w)$ of vertices in K so that $u_1, \ldots, u_n$ are associated with dots in $\mathcal{P}$. By Lemma 3.5 we may and will assume w.l.o.g. that for some $j \in \{1, \ldots, n\}$ the vertex u_j is a boundary vertex of K. Let $i \geq j$ be the lowest index so that u_i is associated with a disk D_j^k for almost all, say for all k (maybe $i = j$). If such an index does not exist, then let $i < j$ be the largest index so that u_i is associated with a disk D_j^k for (say) all k. By assumption, u_i is well defined. We set $u := u_i$ and $\delta := \partial G^*$. By construction, the disks D_u^k associated with u touch δ.

Let $C = (D_v^k, P_1^k, \ldots, D_u^k)$ be the chain of sets in $\mathcal{P}_k$ associated with $(v, u_1, \ldots, u_i = u)$, and let $C' = (D_u^k, \ldots, P_n^k, D_w^k)$ be associated with $(u = u_i, \ldots, u_n, w)$. Evoking Lemma 3.35 for the reversed C, we get that the disk D_v and the dot S_u touch a prime end Z_1 in $\{s\} = S_u$. Using Lemma 3.35 for C', we see that D_w touches a prime end Z_2 in the same contact point s. We conclude $X = Z_1$, $Y = Z_2$ and $p = s$. Assuming $X \neq Y$ implies tails U_X and U_Y of X and Y, respectively, with $U_X \cap U_Y = \emptyset$. Then, again by Lemma 3.35 and for sufficiently large k, all disks within $\{P_1^k, \ldots, D_u^k\}$ are contained in U_X while all disks within $\{D_u^k, \ldots, P_n^k\}$ are contained in U_Y, which is a contradiction. Hence, we have $X = Y$, what proves the second statement of the lemma.

3. Assume that $\mathcal{P}$ is not collapsed and that $\mathcal{P}_k$ fills G. We verify that $\mathcal{P}$ fulfills the conditions (i) and (ii) of Definition 3.10, i.e., $\mathcal{P}$ fills G.

Verifying (i) is easily done: Since $\mathcal{P}_k$ fills G, it is in particular contained in G. So part 2 of this proof yields that $\mathcal{P}$ is contained in G.

In order to verify (ii), let w be a vertex in K so that its associated set in $\mathcal{P}$ is a disk D_w. Since $\mathcal{P}$ is assumed to be non-collapsed, such a vertex exists. Let D_w^k be the associated disk in $\mathcal{P}_k$. Now, let v be any boundary vertex of K (maybe $v = w$), and let P_v^k be its associated disk or dot in $\mathcal{P}_k$. Since $\mathcal{P}_k$ fills G, the boundary disk or dot P_v^k touches $\partial G^* =: \delta$. Since K is connected, there is a chain of vertices $(v, \ldots, w)$ connecting v with w. Let u denote the first vertex in this chain that is associated with a disk in $\mathcal{P}$. By construction u exists, and to simplify things we assume w.l.o.g. that $u = w$. Let $(P_v^k, \ldots, D_w^k)$ be the associated chain of disks or dots in $\mathcal{P}_k$. Then Lemma 3.35 yields that also the boundary disk or dot $P_v \in \mathcal{P}$ touches a prime end of G, what concludes the proof of the lemma. $\square$

Next we consider sequences of circle agglomerations $\mathcal{P}_k$ filling trilaterals $G(\alpha_k, \beta_k, \gamma_k)$. Of special interest is the case when these trilaterals converge to a limit trilateral $G(\alpha, \beta, \gamma)$ that is tame. The same results are achieved for quad-complexes and quadrilaterals.

Lemma 3.37. *Assume that for each $k \in \mathbb{N}$ the generalized circle agglomeration $\mathcal{P}_k$ fills the trilateral $G(\alpha_k, \beta_k, \gamma_k)$ for the tri-complex T. If $G(\alpha_k, \beta_k, \gamma_k)$ converges to a trilateral $G(\alpha, \beta, \gamma)$, then $\mathcal{P}_k$ contains a sub-sequence converging to a generalized circle agglomeration $\mathcal{P}$ filling $G(\alpha, \beta, \gamma)$. If $G(\alpha, \beta, \gamma)$ is tame, then the complete sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}$.*

Furthermore, one can replace the terms trilateral and tri-complex by quadrilateral and quad-complex, respectively.

Proof. Since the two different cases of trilaterals and quadrilaterals can be proven exactly the same, we assume (for the sake of a shorter notation) that $\mathcal{P}_k$ fills a trilateral. The existence of a convergent sub-sequence $\mathcal{P}_n$ follows from Lemma 3.36. Let $\mathcal{P}$ be its limit. In order for $\mathcal{P}$ to fill $G(\alpha, \beta, \gamma)$ for T, it must fulfill the three conditions (i)-(iii) of Definition 3.11, which we show it does.

If $\mathcal{P}$ is non-collapsed, then this is an easy task using again Lemma 3.36 or rather some arguments of its proof. We directly obtain (i) and (ii) from the statements of Lemma 3.36, and part 2 of its proof yields (iii); we just have to replace δ_n by the desired arc α_n, β_n or γ_n. Therefore, all we have to show is that $\mathcal{P}$ cannot be collapsed.

Since $\mathcal{P}_n$ fills $G(\alpha_n, \beta_n, \gamma_n)$ for each $n = n(k)$, it is not collapsed. Furthermore, there is a leading disk or dot P_1^n meeting $\alpha_n \cap \beta_n$, and a leading disk or dot P_2^n meeting $\beta_n \cap \gamma_n$ (maybe $P_1^n = P_2^n$). By definition, P_1^n touches α_n and β_n while P_2^n touches in particular γ_n.

Let v_1 and v_2 (maybe $v_1 = v_2$) be the associated leading vertices of T, respectively. Since $\mathrm{int}T$ is connected, there is a chain of vertices $(v_1 = u_1, \dots, u_m = v_2)$ from v_1 to v_2 within the interior of T. Let $P_1^n, \dots, P_m^n$ be the associated disks or dots of $\mathcal{P}_n$.

There is a vertex $u = u_i$ for some $i \in \{1, \dots, m\}$ so that its associated set P_i^n is a disk $P_i^n = D_i^n$ for almost all, say for all n. Otherwise, all the sets associated with $u_1, \dots, u_m$ would be dots in the same point $s \in \overline{G}$, i.e., $P_1^n = \ldots = P_m^n = S = \{s\}$, and by Lemma 3.5 the dot S would touch a prime end $X \in \partial G^*$ via a disk D^n that touches X in s. Since P_1^n and P_m^n must touch their associated edges of G, we would have $X \in (\alpha \cap \beta \cap \gamma) = \emptyset$, which is a contradiction.

Assume that $\mathcal{P}$ is collapsed. Let C be the chain $C = (P_1^n, \dots, D_i^n)$, and let C' be the chain $C' = (D_i^n, \dots, P_m^n)$. By Lemma 3.35, the reversed C together with the arc α yields a prime end $X \in \alpha$ so that in particular D_i^n is contained in a tail U_X of X for sufficiently large n. Using instead the arc β, we get a prime end $Y \in \beta$ so that D_i^n lies in a tail U_Y of Y. For $X \neq Y$ we could choose U_X and U_Y to be disjoint, what would be a contradiction to $D_i^n \subset (U_X \cap U_Y)$, i.e., we have $X = Y$. Again by Lemma 3.35, the chain C' together with the arc γ yields a prime end $Z \in \gamma$ so that D_i^n lies in a tail U_Z of Z. For the same reason as before, we conclude $X = Y = Z$, which is a contradiction to $\alpha \cap \beta \cap \gamma = \emptyset$. Hence, $\mathcal{P}$ cannot be collapsed, what proves the first part of the lemma.

If the trilateral $G(\alpha, \beta, \gamma)$ is tame, then it is filled by at most one generalized circle agglomeration $\mathcal{P}$ for T (see Uniqueness Theorem 3.31). Consequently, every sub-sequence of $\mathcal{P}_k$ contains a convergent sub-sequence, and all those convergent sub-sequences must have the same limit $\mathcal{P}$. This implies that the whole sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}$. □

The final part of the proof of the existence theorem is based on an approximation procedure which *exhausts* an arbitrary trilateral $G(\alpha, \beta, \gamma)$ by a sequence of smooth trilaterals. In order to define this approximation, let $f : \mathbb{D} \to G$ be the canonical parameterization of G normalized so that α, β and γ are the images of circular arcs $a, b, c \subset \partial\mathbb{D}$ under the extended mapping $f^* : \overline{\mathbb{D}} \to G^*$.

Definition 3.19. We choose an increasing sequence of positive numbers r_k converging to 1, and we define the *exhausting trilateral* $G_k(\alpha_k, \beta_k, \gamma_k)$ of $G(\alpha, \beta, \gamma)$ by

$$G_k := f(r_k\mathbb{D}), \quad \alpha_k := f(r_k a), \quad \beta_k := f(r_k b), \quad \gamma_k := f(r_k c).$$

Up to some minor (technical) changes within the Lemmas 3.35 to 3.37, the following result can be proven exactly the same way as Lemma 3.37.

Corollary 3.38. *Let $T \in \mathcal{T}^*$ be a tri-complex and let G be a bounded, simply connected domain. Let $G(\alpha, \beta, \gamma)$ be a trilateral and let $G_k(\alpha_k, \beta_k, \gamma_k)$ be exhausting trilaterals as defined above. For each $k \in \mathbb{N}$ let $\mathcal{P}_k$ be a generalized circle agglomeration for T filling $G_k(\alpha_k, \beta_k, \gamma_k)$. Then there exists a sub-sequence of $\mathcal{P}_k$ converging to a generalized circle agglomeration $\mathcal{P}$ that fills $G(\alpha, \beta, \gamma)$ for T. If $G(\alpha, \beta, \gamma)$ is tame, then the whole sequence $\mathcal{P}_k$ converges to $\mathcal{P}$.*

We conclude this section by some thoughts about continuity.We apply Lemma 3.37 to the alpha-beta-gamma normalization and then Lemma 3.36 to the boundary alpha-gamma normalization. In particular the latter result is very useful in several existence proofs.

Let $T \in \mathcal{T}^*$ be a tri-complex. Let G be a bounded, simply connected domain. Let $f : \mathbb{D} \to G$ and f^* be a canonical parameterization of G and its extension to G^*, respectively. Let $I \subset \mathbb{R}$ be a closed, finite interval of $\mathbb{R}$ and let $t \in I$. Let (x_t), (y_t) and (z_t) be three families of points on the boundary of $\mathbb{D}$ depending continuously on t so that for all $t \in I$ the points $x_t, y_t, z_t \in \partial\mathbb{D}$ are pairwise different and positively ordered along $\partial\mathbb{D}$. Let $(\mathbb{D}(a_t, b_t, c_t))$ be the associated family of trilaterals for $\mathbb{D}$.

Definition 3.20. We call $(G(\alpha_t, \beta_t, \gamma_t))$ with $\alpha_t = f^*(a_t)$, $\beta_t = f^*(b_t)$ and $\gamma_t = f^*(c_t)$ a *continuous family of trilaterals* for G. Let $(\mathcal{P}_t)$ be a family of generalized circle agglomerations so that $\mathcal{P}_t$ fills $G(\alpha_t, \beta_t, \gamma_t)$ for T. Then $(\mathcal{P}_t)$ *depends continuously on t at t_0* if for every sequence (t_k) of numbers $t_k \in I$ with $t_k \to t_0$ the associated sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}_0$. If $(\mathcal{P}_t)$ depends continuously on t for all $t_0 \in I$, then it is said to be a *continuous family of generalized circle agglomerations* for T filling $(G(\alpha_t, \beta_t, \gamma_t))$.

Lemma 3.39. *Let $(\mathcal{P}_t)$ fill the continuous family $(G(\alpha_t, \beta_t, \gamma_t))$ for T. If the trilateral $G(\alpha_0, \beta_0, \gamma_0)$ is tame for some $t_0 \in I$, then $(\mathcal{P}_t)$ depends continuously on t at t_0. If G is regular, then $(\mathcal{P}_t)$ is a continuous family.*

Proof. Let (t_k) be an arbitrary sequence of numbers in I with $t_k \to t_0$. By Lemma 3.37, the associated sequence $(\mathcal{P}_k)$ that fills $G(\alpha_k, \beta_k, \gamma_k)$ converges to a generalized circle agglomeration $\mathcal{P}$ for T filling $G(\alpha_0, \beta_0, \gamma_0)$. By Theorem 3.1 there is exactly one such generalized circle agglomeration, i.e. $\mathcal{P} = \mathcal{P}_0$. Hence, $\mathcal{P}_t$ depends continuously on t at t_0. If G is regular, then $G(\alpha_t, \beta_t, \gamma_t)$ is tame for all $t \in I$, i.e., $(\mathcal{P}_t)$ depends continuously on t for all $t_0 \in I$, which makes it a continuous family. □

In order to use the boundary alpha-gamma normalization, let $K \in \mathcal{K}$ be an admissible complex with two distinguished (and different) boundary vertices a and c. Let once more

G be a bounded, simply connected domain and let $f : \mathbb{D} \to G$ and f^* be a canonical parameterization of G and its extension to G^*, respectively. Let $t \in I$. Let (r_t) and (s_t) be two families of points in $\overline{\mathbb{D}}$ depending continuously on t so that for every $t \in I$ the points $r_t \in \partial\mathbb{D}$ are boundary points while $s_t \in \mathbb{D}$ are interior points of $\mathbb{D}$. Let $A_t = f(s_t)$ and $C_t = f^*(r_t)$.

Definition 3.21. We call $(G(A_t, C_t))$ a *continuous family of pinned domains* (associated with the alpha-gamma normalization). Let $(\mathcal{P}_t)$ be a family of generalized circle packings so that $\mathcal{P}_t$ fulfills the boundary alpha-gamma normalization for $G(A_t, C_t)$ and $K(a, c)$. Then we say that $(\mathcal{P}_t)$ *depends continuously on* t *at* t_0 if for every sequence (t_k) of numbers $t_k \in I$ with $t_k \to t_0$ the associated sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}_0$. If $(\mathcal{P}_t)$ depends continuously on t for all $t_0 \in I$, then it is said to be a *continuous family of generalized circle packings* for $K(a, c)$ filling $(G(A_t, C_t))$.

Lemma 3.40. *Let $(\mathcal{P}_t)$ fill the continuous family $(G(A_t, C_t))$ for $K(a, c)$ under the boundary alpha-gamma normalization. If the prime end C_0 is regular for some $t_0 \in I$, then $(\mathcal{P}_t)$ depends continuously on t at t_0. If G is regular, then $(\mathcal{P}_t)$ is a continuous family.*

Proof. Let (t_k) be an arbitrary sequence of numbers in I with $t_k \to t_0$. By Lemma 3.36, the associated sequence $(\mathcal{P}_k)$ contains a sub-sequence $(\mathcal{P}_n)$ that converges to a generalized circle agglomeration $\mathcal{P}$ for K. Clearly, every disk of $\mathcal{P}$ is contained in G.

Every alpha disk D_a^n touches $\delta := \partial G^*$. So Lemma 3.35 implies that the limit alpha disk $D_a \in \mathcal{P}$ also touches the boundary of G. By assumption, the center A_t of D_a^n depends continuously on t, i.e., the sequence (A_n) converges to A_0. Thus, the disk D_a is the unique maximal disk in G with center in A_0, which shows that $\mathcal{P}$ is non-collapsed. Hence, Lemma 3.36 states that $\mathcal{P}$ fills G.

Since the family of prime ends C_t depends continuously on t, the sequence (C_t) converges to C_0. By Lemma 3.35, the limit gamma disk or gamma dot of $\mathcal{P}$ touches C_0. So $\mathcal{P}$ does not only fill G but it also fulfills the boundary alpha-gamma normalization for $K(a, c)$ and $G(A_0, C_0)$. For regular C_0, the Theorem 3.32 states $\mathcal{P} = \mathcal{P}_0$. Thus, every sub-sequence of $\mathcal{P}_k$ contains a convergent sub-sequence, and all those convergent sub-sequences must have the same limit $\mathcal{P}_0$. This implies that the whole sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}_0$, hence $\mathcal{P}_t$ depends continuously on t at t_0.

If G is regular, then C_t is regular for all $t \in I$. So $(\mathcal{P}_t)$ depends continuously on t for all $t_0 \in I$, which makes it a continuous family. □

3.7. Existence

In this section we prove that for each trilateral $G = G(\alpha, \beta, \gamma)$ and each tri-complex $T \in \mathcal{T}^*$ there is a generalized circle agglomeration $\mathcal{P}$ for T filling G. The result will be verified first for regular domains and then extended to general domains by an exhaustion method.

Whether $\mathcal{P}$ is degenerate or not won't be investigated here, please see Section 3.2 instead.

3.7.1. Regular Domains

The existence proof for regular domains proceeds by induction on the number n of disks. For reducible complexes T this turns out to be quite natural. For irreducible T the basic idea is to use a point $z \in G$ together with a fixed vertex $Z \in \partial G^*$ of the given trilateral in order to use the boundary alpha-gamma normalization. Since G is regular and since the interior of T turns out to be admissible, we get a circle packing $\mathcal{P}(z)$ that fills the domain G, and by adjusting z (using Sperner's Lemma 2.20) it even fills the trilateral $G(\alpha, \beta, \gamma)$.

Note that the uniqueness part directly follows from the Uniqueness Theorem 3.31.

Lemma 3.41 (Existence for regular domains)**.** *Let $G(\alpha, \beta, \gamma)$ be a trilateral for a regular, bounded, simply connected domain G. Then for any tri-complex $T \in \mathcal{T}^*$ there is a unique generalized circle agglomeration $\mathcal{P}$ for T filling G.*

The inductive proof of this lemma will occupy the rest of this subsection. For $n = 1$ there is only one tri-complex T in $\mathcal{T}_1^* = \{T\}$, and the assertion of Lemma 3.41 for this complex was verified in Section 3.3 by the Incircle Theorem 3.17. So let us assume that for some $n \geq 2$ the assertion holds for all $T \in \mathcal{T}_{n-1}^*$, and let $T = T(a, b, c) \in (\mathcal{T}_n^* \setminus \mathcal{T}_{n-1}^*)$. In the following construction, we distinguish whether T is reducible or not. We first consider the former case, which can be treated quite easily.

Case 1. Let $T(a, b, c)$ be reducible. Then the number of vertices in its skeleton $\sigma(T) =: T_\sigma$ is strictly less than in T, i.e., $T_\sigma \in \mathcal{T}_{n-1}^*$. According to our induction hypothesis, there is a generalized circle agglomeration $\mathcal{P}_\sigma$ for T_σ filling $G(\alpha, \beta, \gamma)$. Since G is regular and T_σ is (boundary) irreducible, Lemma 3.14 ensures that $\mathcal{P}_\sigma$ is not degenerate.

Every vertex u in $T \setminus T_\sigma$ lies in the interior of some reducible triangle $\triangle(e, f, g)$ of T, which is a face of T_σ but not of T. Additionally, one of the vertices e, f and g is an interior vertex of T and of T_σ as well. Let $T_\triangle := \varrho_-(T, \triangle) \in \mathcal{T}_{n-1}^*$ be the outer reduction of T with respect to $\triangle(e, f, g)$. We explain how to extend $\mathcal{P}_\sigma$ to a generalized (domain filling) circle agglomeration for the union of T_σ and $T_\triangle$.

Since $\mathcal{P}_\sigma$ is not degenerate, the associated sets for the vertices e, f and g of $T_\triangle$ within $\mathcal{P}_\sigma$ are either disks or edges of G (with at least one disk involved). Moreover, those sets either bound a regular trilateral $G_\triangle$ or they touch each other at a single boundary point p (see the proof of Theorem 3.31 and Figure 3.28, p. 92). In the former case, the induction hypothesis guarantees the existence of a generalized circle agglomeration $\mathcal{P}_\triangle$ for $T_\triangle$ filling $G_\triangle$. In the latter case, we define $\mathcal{P}_\triangle$ by letting the associated sets of all vertices within $\mathrm{int} T_\triangle$ be dots at p, which then fulfill all touching conditions in a trivial sense. In both cases, the union of $\mathcal{P}_\sigma$ with all elements of $\mathcal{P}_\triangle$ fills $G(\alpha, \beta, \gamma)$.

By repeating this procedure for all faces of T_σ that are reducible triangles in T we complete the induction step in the first case.

Case 2. Let $T(a, b, c)$ be irreducible. Then we follow the recipe given at the beginning. The details are described in the following steps. Let $X := \gamma \cap \alpha$, $Y := \alpha \cap \beta$ and $Z := \beta \cap \gamma$ denote the three vertices of $G(\alpha, \beta, \gamma)$.

2.1. Construction of the packing $\mathcal{P}(z)$. Since $T \in \mathcal{T}_n^* \setminus \mathcal{T}_{n-1}^*$ with $n \geq 2$, the Lemma 3.2 tells us that $K := \text{int}T$ is an admissible (irreducible) complex, and that $\deg T = 3$. So there are three distinguished leading vertices v_1, v_2 and v_3 of T, and in particular we have $n \geq 3$. Let v_2 be associated with the vertex Z of G, i.e., $f(b, c, v_2)$ is a face in T.

Let $z \in G$ be a point in G so that the maximal disk in G with center in z touches Z. Since G is a regular domain, all these points z form a set λ that is either void or a line segment connecting Z with an interior point of G. In what follows, the set λ often plays an exceptional role.

By Lemma 3.34, for every point $z \in (G \setminus \lambda)$ there is a unique circle packing $\mathcal{P}(z)$ for K fulfilling the boundary alpha-gamma normalization for $K(v_1, v_2)$ and $G(z, Z)$. Moreover, $\mathcal{P}(z)$ depends continuously on z by Lemma 3.40. To be precise: If a sequence (z_k) of points $z_k \in (G \setminus \lambda)$ converges to $z_0 \in (G \setminus \lambda)$, then $(\mathcal{P}(z_k))$ converges to $(\mathcal{P}(z_0))$.

Note that although $\mathcal{P}(z)$ fills the domain G does not need to fill the trilateral $G(\alpha, \beta, \gamma)$ since the boundary disks of $\mathcal{P}(z)$ do not have to touch the necessary edge(s) of $\{\alpha, \beta, \gamma\}$.

2.2. Classification of control points. In order to understand for which $z \in (G \setminus \lambda)$ the circle packing $\mathcal{P}(z)$ fills the trilateral $G(\alpha, \beta, \gamma)$ for T, we consider the leading disks $D_1(z)$, $D_2(z)$ and $D_3(z)$ of $\mathcal{P}(z)$ associated with v_1, v_2 and v_3, respectively. It turns out that the touching properties of $D_1(z)$ and $D_3(z)$ are crucial for the behavior of the whole packing. Thus, we build the following subsets of $G \setminus \lambda$,

$$\begin{aligned}
\mathcal{R} &= \{z \in G : \quad D_3(z) \,\text{touches}\, \alpha \cup \beta,\ D_1(z) \,\text{touches}\, \alpha\} \\
\mathcal{G} &= \{z \in G : \quad D_3(z) \,\text{touches}\, \alpha \cup \beta,\ D_1(z) \,\text{touches}\, \beta\} \\
\mathcal{B} &= \{z \in G : \quad D_3(z) \,\text{touches}\, \gamma\}.
\end{aligned}$$

We prove two useful statements concerning the sets $\mathcal{R}, \mathcal{G}, \mathcal{B}$.

Lemma 3.42. *Every point $z \in (G \setminus \lambda)$ belongs to $\mathcal{R}$, $\mathcal{G}$ or $\mathcal{B}$.*

Proof. Fixing z, we omit it in the notations of this proof. Logically, there is only one possibility that is not covered, namely that D_3 touches $\alpha \cup \beta$ while D_1 touches γ. In order to prove that this cannot happen, we use the cyclic ordering of prime ends (with base point Z).

By assumption D_1 shall touch a prime end $X_1 \in \gamma$, and D_3 touches $X_3 \in (\alpha \cup \beta)$. So with respect to the two vertex prime ends $Z = \beta \cap \gamma$ and $X = \gamma \cap \alpha$ of G, we have

$$Z \preceq X_1 \preceq X \prec Z, \qquad Z \prec X \preceq X_3 \preceq Z.$$

By construction of $\mathcal{P}$ the disk D_2 meets Z. So either D_2 touches Z or it separates it from D_1 and D_3. In every case, the first relation can be complemented by a prime end X_2 touched by D_2 so that we have

$$Z \preceq X_2 \prec X_1 \preceq X \prec Z, \qquad Z \prec X \preceq X_3 \preceq Z.$$

Note that $X_2 \neq X_1$ follows from the fact that G is regular. By Lemma 2.16 the first relation can be further complemented by X_3, which brings

$$Z \preceq X_2 \prec X_3 \prec X_1 \preceq X \prec Z, \qquad Z \prec X \preceq X_3 \preceq Z.$$

Combining this two relations, we arrive at the contradiction

$$Z \prec X_3 \prec X \preceq X_3 \preceq Z,$$

which proves the lemma. □

Lemma 3.43. *The following implications hold true.*

(i) If $z \in \mathcal{R}$, then both disks $D_3(z)$ and $D_1(z)$ touch α.

(ii) If $z \in \mathcal{G}$, then both disks $D_1(z)$ and $D_2(z)$ touch β.

(iii) If $z \in \mathcal{B}$, then both disks $D_2(z)$ and $D_3(z)$ touch γ.

(iv) If $z \in (\mathcal{R} \cap \mathcal{G} \cap \mathcal{B})$, then $\mathcal{P}(z)$ fills $G(\alpha, \beta, \gamma)$ for $T(a, b, c)$.

Proof. In order to verify the facts that are not part of the definition of the sets $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$, we denote by X_1, X_2 and X_3 prime ends touched by D_1, D_2 and D_3, respectively, and we study their ordering relations with respect to $X = \gamma \cap \alpha$, $Y = \alpha \cap \beta$ and $Z = \beta \cap \gamma$. Note that X_1, X_2 and X_3 are pairwise different, since G has no inward spikes.

Proof of (i). Choosing $X_1 \in \alpha$ and $X_3 \in (\alpha \cup \beta)$, we have

$$Z \prec X \preceq X_1 \preceq Y \prec Z, \qquad Z \prec X \preceq X_3 \preceq Z.$$

Since D_2 meets Z, we can choose X_2 in between of Z and X_1, and moreover, by Lemma 2.16, the prime end X_3 lies in between of X_2 and X_1. So we have

$$Z \preceq X_2 \prec X_3 \prec X_1 \prec Z.$$

Combing all three relations yields

$$Z \prec X \preceq X_3 \prec X_1 \preceq Y \prec Z.$$

So D_3 touches α.

Proof of (ii). Let $X_1 \in \beta$. Since D_2 meets Z, we instantly get a prime end X_2 with

$$Z \prec Y \preceq X_1 \prec X_2 \prec Z.$$

So D_2 touches β.

Proof of (iii). Let $X_3 \in \gamma$. Since D_2 meets Z, we instantly get a prime end X_2 with

$$Z \preceq X_2 \prec X_3 \prec X \prec Z.$$

So D_2 touches γ.

Proof of (iv). By the implications (i)-(iii), the disk D_1 touches α and β, the disk D_2 touches β and γ, and the disk D_3 touches γ and α. Let $w_1, \ldots, w_m$ be all boundary vertices of $K = \mathrm{int} T$ that lie on the boundary chain of K between v_1 and v_2 (with positive orientation). In T, they are clearly neighbors of b.

Assume some w_j would also be a neighbor of a or c, say a. Then $\triangle(a, b, w_j)$ would be a triangle in T but not a face. So T would be reducible, which is a contradiction to the assumption of Case 2. Hence, none of the vertices $w_1, \ldots, w_m$ is a neighbor of a or c.

By Lemma 2.16 the associated boundary disks of $\mathcal{P}$ for $w_1, \ldots, w_m$ touch prime ends $W_1, \ldots, W_m \in \partial G^*$ with the ordering

$$Z \prec Y \preceq X_1 \prec W_1 \prec \ldots \prec W_m \prec X_2 \preceq Z.$$

So all the disks associated with $w_1, \ldots, w_m$ touch β.

By reasons of symmetry all boundary vertices of $K = \mathrm{int} T$ that lie on the boundary chain of K between v_2 and v_3 (between v_3 and v_1) are neighbors of c but not of a or b (of a but not of b or c), and their associated disks touch γ (touch α). Hence, $\mathcal{P}$ fills $G(\alpha, \beta, \gamma)$ for $T(a, b, c)$, which completes the proof. □

Our goal is clearly to find a point $z \in (\mathcal{R} \cap \mathcal{G} \cap \mathcal{B})$. We achieve it by using Sperner's Lemma. The next steps are preparations for that.

2.3. Conformal transplantation to a triangle. By Carathodory's version of the Riemann mapping theorem ([21], Cor. 2.7.) there exists a conformal mapping $f : \Delta \to G$ of a triangular domain Δ to G that can be normalized so that the extension $f^* : \overline{\Delta} \to G^*$ to the closure of Δ maps the sides A, B and C of Δ to the arcs (of prime ends) α, β and γ, respectively. As usual, we denote the inverse of f and f^* by g and g^*, respectively. Depending on the location of $f(t)$ (see Figure 3.32), a point t in Δ is colored

red	if $f(t) \in \mathcal{R}$,
green	if $f(t) \in \mathcal{G} \setminus \mathcal{R}$,
blue	if $f(t) \in \mathcal{B} \setminus (\mathcal{R} \cup \mathcal{G})$ or $f(t) \in \lambda$.

For the sake of simplicity, we speak of *red points*, *red sequences*, *red neighborhoods* etc. An explanation for coloring $L := g(\lambda)$ blue will be given in the next step. Note that L is either void or a Jordan arc that connects the vertex $B \cap C$ of Δ with an inner point of Δ (see Lemma 2.11).

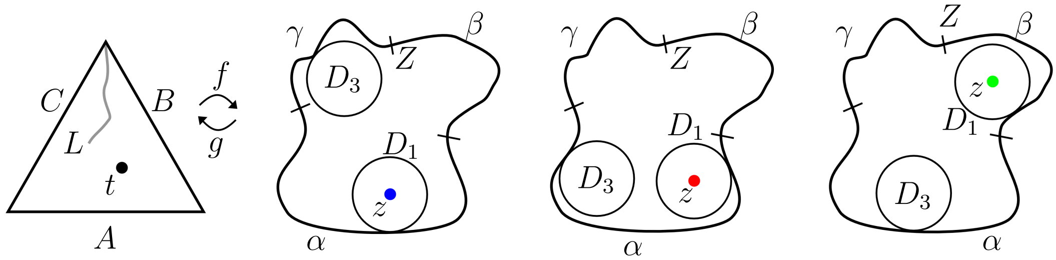


Fig. 3.32.: Examples of the colors blue, red and green for $t \in \Delta$

2.4. Extending the coloring to $\overline{\Delta}$. We examine the color of points in Δ near the boundary and close to the image L of the segment λ. Though not all limiting situations can be classified precisely, it is crucial that we can exclude some cases. Recall that A, B and C are the closed segments that form the sides of the triangle Δ so that $f^*(A) = \alpha$, $f^*(B) = \beta$ and $f^*(C) = \gamma$.

Lemma 3.44. *Let (t_k) be a sequence of points in Δ converging to $t_0 \in \overline{\Delta}$.*

(i) If $t_0 \in ((B \cup C) \setminus A)$, then (t_k) cannot be red.

(ii) If $t_0 \in ((C \cup A) \setminus B)$, then (t_k) cannot be green.

(iii) If $t_0 \in ((A \cup B) \setminus C)$, then (t_k) cannot be blue.

(iv) If $t_0 \in L$, then (t_k) cannot be red.

Proof. For $t_0 \in \partial\Delta$ let $X_0 \in \partial G^*$ be the prime end of G associated with t_0 via $f^*(t_0) = X_0$. Let (J_n) be a null-chain representing X_0, and let (U_n) be its tails. Let $\Gamma_n := g(J_n)$ and $W_n := g(U_n)$ denote the associated null-chain (Γ_n) of t_0 in Δ and its tails (W_n). Since t_k converges to t_0, we may and will assume w.l.o.g. that $t_k \in W_n$ for any fixed n. Thus, the points $z_k := f(t_k)$ lie in U_n, i.e., $z_k \in U_n$.

Proof of (i). Since $t_k \in L$ implies that t_k is blue and not red, we may and will assume $t_k \notin L$. For sufficiently large k, the Lemma 2.17 states that the disk $D_1(z_k)$ cannot touch α, i.e., z_k does not lie in $\mathcal{R}$. Hence, t_k cannot be red.

Proof of (ii). Again, if $t_k \in L$, then t_k is per definition blue so we assume $t_k \notin L$. For sufficiently large k the Lemma 2.17 states that the disk $D_1(z_k)$ cannot touch β, i.e., z_k does not lie in $\mathcal{G}$. Hence, t_k cannot be green.

Proof of (iii). Assume to the contrary that t_k is blue. Since t_0 has a positive distance from L, we may and will assume that $t_k \notin L$. Then $z \in \mathcal{B}$ implies by Lemma 3.43 that both disks $D_2(z_k)$ and $D_3(z_k)$ touch γ. We show that for sufficiently large k this is impossible since Lemma 2.17 states that the disk $D_1(z_k)$ cannot touch γ.

Let $\mathcal{P}$ be the limit of a convergent sub-sequence $(\mathcal{P}(z_m))$ of $(\mathcal{P}(z_k))$ provided by Lemma 3.36. Let $C_1 := (v_3, \dots, v_1)$ be the positively oriented sub-chain of the boundary chain of $K = \mathrm{int}T$ that connects v_3 with v_1. Let $C_2 := (v_2, \dots, v_1)$ be the negatively oriented sub-chain of the boundary chain from v_2 to v_1.

Assume for the moment that all associated sets for C_1 in $\mathcal{P}$ are dots. Then Lemma 3.35 implies that for sufficiently large m the disk $D_3(z_m)$ is contained in the tail of some prime end of G. By Lemma 2.17 this prime end is $X_0 \notin \gamma$. Hence, $D_3(z_m)$ cannot touch γ, which is a contradiction. Analogously, not all of the associated sets for C_2 in $\mathcal{P}$ can be dots.

Since v_1 is associated with a dot in $\mathcal{P}$ by construction while all other vertices of C_1 and C_2 are pairwise different, we conclude that the degenerate circle packing $\mathcal{P}$ contains at least two disks. Therefore, Lemma 2.18 states the existence of a pseudo contact point lying on the boundary of G. This is impossible since G has no inward spikes. Hence, t_k cannot be blue.

Proof of (iv). Assume to the contrary that t_k is red. Then $z_k \in \mathcal{R}$ implies by Lemma 3.43 that both disks $D_1(z_k)$ and $D_3(z_k)$ touch α. We show that this is impossible for sufficiently large k.

Since the case $t_0 \in (\overline{\lambda} \cap \partial\Delta)$ is included in (i), we assume w.l.o.g. that $t_0 \in \Delta$. Let $\mathcal{P}$ be the limit of a convergent sub-sequence $(\mathcal{P}(z_m))$ of $(\mathcal{P}(z_k))$ provided again by Lemma 3.36. The limit of $D_1(z_m)$ is a disk since its center is in $t_0 \in \Delta$. According to construction, the limit of $D_2(z_m)$ is a dot S_2 touching $Z = \beta \cap \gamma$. So $\mathcal{P}$ is a non-collapsed, degenerate circle packing. Since G has no inward spikes, Lemma 2.18 prevents the existence of a second disk in $\mathcal{P}$, i.e., especially $D_3(z_m)$ converges to a dot. Furthermore, all dots of $\mathcal{P}$ equal S_2, and thus they all touch Z.

Let $C_1 := (v_3, \dots, v_2, \dots, v_1)$ be the negatively oriented sub-chain of the boundary chain of $K = \mathrm{int}T$ that connects v_3 with v_1 via v_2. By Lemma 3.35 all dots associated with C_1 touch some prime end in α since $D_3(z_k)$ touches α. Since a dot can touch at most one prime end, we conclude $Z \in (\alpha \cap \beta \cap \gamma) = \emptyset$, what is a contradiction. Hence, t_k cannot be red, what proves the lemma. $\square$

With hindsight to Lemma 3.44, we extend the coloring of Δ to $\overline{\Delta}$ as follows. We simply fix the color of $A \cap B$, $B \cap C$ and $C \cap A$ as well as every remaining point on A, B and C to be red, green and blue, respectively. This guarantees that every point on ∂G, thus on $\partial G \cup \lambda$ has a neighborhood containing not more than two different colors, in particular not the vertices of Δ.

2.5. Application of Sperner's Lemma. Uniform subdivision of the sides of Δ into k intervals with equal lengths generates a regular triangulation T_k of $\overline{\Delta}$. The coloring of the vertices of T_k is a Sperner coloring, and Sperner's Lemma tells us that T_k must contain a triangle Δ_k whose vertices have three different colors (see Section 2.4).

For each $k \in \mathbb{N}$ we denote by t_k^r, t_k^g and t_k^b the red, green and blue vertex of Δ_k, respectively. After replacing the sequence (Δ_k) by an appropriate sub-sequence (Δ_m), we get three sequences (t_m^r), (t_m^g) and (t_m^b) that converge to the same limit $t_0 \in \overline{\Delta}$. Lemma 3.44 tells us that we have $t_0 \in (\Delta \setminus L)$, and hence $z_0 := f(t_0)$ lies in $G \setminus \lambda$.

Since the sequences $z_k^r := f(t_k^r)$, $z_k^g := f(t_k^g)$ and $z_k^b := f(t_k^b)$ converge to $f(t_0) = z_0 \in (G \setminus \lambda)$, we may and will assume w.l.o.g. that we also have $z_k^r, z_k^g, z_k^b \in (G \setminus \lambda)$, i.e., $z_k^r \in \mathcal{R}$, $z_k^g \in \mathcal{G}$ and $z_k^b \in \mathcal{B}$. According to Lemma 3.40, the three sequences of packings $\mathcal{P}(z_k^r)$, $\mathcal{P}(z_k^g)$ and $\mathcal{P}(z_k^b)$ converge to the common limit $\mathcal{P}(z_0)$.

Finally, Lemma 3.35 implies that the touching properties of $\mathcal{P}(z_k^r)$, $\mathcal{P}(z_k^g)$ and $\mathcal{P}(z_k^b)$ carry over to $\mathcal{P}(z_0)$, i.e., $z_0 \in (\mathcal{R} \cap \mathcal{G} \cap \mathcal{B})$. Hence, by Lemma 3.43 $\mathcal{P}(z_0)$ is the desired circle packing filling $G(\alpha, \beta, \gamma)$ for $T(a, b, c)$. This completes the proof of Lemma 3.41.

3.7.2. General Bounded and Simply Connected Domains

After all those preparations, we are ready to state and prove our most general existence statement for the alpha-beta-gamma normalization.

Theorem 3.45 (Alpha-Beta-Gamma Existence Theorem)**.** *Let $T \in \mathcal{T}^*$ be a tri-complex and let $G(\alpha, \beta, \gamma)$ be a trilateral for a bounded, simply connected domain G. Then there exists a generalized circle agglomeration $\mathcal{P}$ for T filling $G(\alpha, \beta, \gamma)$.*

Proof. Let $G_k(\alpha_k, \beta_k, \gamma_k)$ be an exhausting trilateral of $G(\alpha, \beta, \gamma)$ as defined in Section 3.6.2, p. 101. The domains G_k are smooth and hence regular so that Lemma 3.41 guarantees the existence of a generalized circle agglomeration $\mathcal{P}_k$ filling $G_k(\alpha_k, \beta_k, \gamma_k)$ for T. By Lemma 3.38 there is a sub-sequence of $(\mathcal{P}_k)$ that converges to a generalized circle agglomeration $\mathcal{P}$ filling $G(\alpha, \beta, \gamma)$ for T. $\square$

Combining the Uniqueness Theorem 3.31 and the Existence Theorem 3.45 together with Lemma 3.2 and 3.14, we obtain the Alpha-Beta-Gamma Theorem 3.1 as stated at the very beginning of this chapter.

3.8. Discrete Conformal Modulus

Since the Theorem 3.1 provides existence and uniqueness for packings with three boundary constrains (tri-complexes, trilaterals) one cannot expect to get the same result for quad-complexes with respect to quadrilaterals. Nevertheless, it is interesting to investigate what is needed to obtain existence. We first give a slightly more general version of the Incompressibility Theorem 3.18, which then leads to a discrete version of the conformal modulus (see also [20] Theorem 5 and Theorem 7).

Corollary 3.46 (Incompressibility)**.** *All boundary irreducible quad-complexes $Q \in \mathcal{Q}^*$ are incompressible.*

Proof. Assume there are two generalized circle agglomerations $\mathcal{P}$ and $\mathcal{P}'$ for Q filling a tame quadrilateral G and its compression G', respectively. Since Q is assumed to be boundary irreducible, we can use Lemma 3.14 to see that $\mathcal{P}$ and $\mathcal{P}'$ are not degenerate. Complete reduction of Q yields the skeleton $\sigma(Q) = Q_\sigma$, which is an irreducible quad-complex. By the Incompressibility Theorem we have $\mathcal{P}_\sigma = \mathcal{P}'_\sigma$.

As we know, every vertex of Q that does not lie in Q_σ must be contained in a reducible triangle $\Delta = \Delta(a, b, c)$ with vertices a, b and c in Q_σ. Moreover, we have $a, b, c \in K = \text{int}Q$ since Q is boundary irreducible. We denote by $T := \varrho_-(Q, \Delta)$ the outer reduction of Q with respect to Δ (see Definition 2.4). The tri-complex T has the boundary vertices a, b, c and contains all vertices of the interior of $\Delta(a, b, c)$. The interstice formed by the disks $D_a = D'_a$, $D_b = D'_b$ and $D_c = D'_c$ is filled by a circle sub-agglomerations of $\mathcal{P}$ and $\mathcal{P}'$ associated with T. So by Theorem 3.1 we get $D_v = D'_v$ for all v in T. Hence, $D_v = D'_v$ for all v in Q. $\square$

Theorem 3.47 (Conformal modulus)**.** *For every quad-complex $Q \in Q^*$ there is a unique positive number h and a unique circle agglomeration $\mathcal{P}$ for Q filling the rectangle*

$$R = \{z : 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < \mathrm{i}h\}$$

so that its leading disks meet the corners $0, 1, 1 + \mathrm{i}h, \mathrm{i}h$ of R (in a prescribed order).

Proof. 1. We first prove the uniqueness of h, thus of R. To do so, let $h_1, h_2 \in \mathbb{R}^+$ be two positive numbers, say $h_1 \leq h_2$. Assume there are two generalized circle agglomerations $\mathcal{P}_1$ and $\mathcal{P}_2$ for Q filling R_1 and R_2, respectively. Then by Theorem 3.1 and Lemma 3.11 $\mathcal{P}_1$ and $\mathcal{P}_2$ are uniquely determined and not degenerate.

Let $\sigma(Q) = Q_\sigma$ be the skeleton of Q, and let $\mathcal{P}_1^\sigma$ and $\mathcal{P}_2^\sigma$ be the associated sub-packings of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Since $R_1(\alpha_1, \beta_1, \gamma_1, \delta_1)$ is contained in $R_2(\alpha_2, \beta_2, \gamma_2, \delta_2)$ with $\beta_1 \subset \beta_2$ and $\delta_1 \subset \delta_2$, the Incompressibility Theorem 3.27 states $\mathcal{P}_1^\sigma = \mathcal{P}_2^\sigma$. This is impossible for $h_1 < h_2$ since then $\gamma_1 \cap \gamma_2 = \emptyset$. Hence, we have $h_1 = h_2 = h$ and $R_1 = R_2 = R$.

2. We proof the existence of a generalized circle agglomeration $\mathcal{P}$ for $Q \in \mathcal{Q}_n^*$ filling R for some number $h > 0$ by complete induction on the number n of interior vertices of Q.

Our induction base shall be $n = 1$ (since $\mathcal{Q}_0 = \emptyset$). The unique complex Q in $\mathcal{Q}_1 = \{Q(a, b, c, d)\}$ contains exactly one inner vertex, which is therefore adjacent to all four boundary vertices a, b, c and d. So we are searching for a single disk touching every edge of a rectangle. Surely, for a square such a disk exists, but for any other rectangle not. Hence, the theorem is true for $n = 1$ and states in this case $h = 1$.

Our induction hypothesis shall be that Theorem 3.47 is true for every quad-complex in $\mathcal{Q}_{n-1}^*$ for some $n > 1$. Let $Q \in (\mathcal{Q}_n^* \setminus \mathcal{Q}_{n-1}^*)$, i.e., Q has at least two interior vertices. We distinguish whether Q is reducible or not.

2.1. Assume Q is reducible. Since this case is very similar to other reducible situations we encountered before, we keep things very brief. The skeleton $\sigma(Q) = Q_\sigma$ yields a complex with strictly less vertices, i.e., $Q_\sigma \in \mathcal{Q}_{n-1}^*$. By induction hypothesis there is a unique number h and a unique circle agglomeration $\mathcal{P}_\sigma$ filling R_h for Q_σ. For every vertex of Q that does not lie in Q_σ we find a reducible triangle $\Delta = \Delta(a, b, c)$ with vertices a, b and c in Q_σ. As shown for example in the proof of the corollary above, we can associate $\Delta(a, b, c)$ with a tri complex $T(a, b, c) = \varrho_-(Q, \Delta)$ by the outer reduction of Q with respect to Δ, and with the (boundary) interstice $I(a, b, c)$ so that Theorem 3.1 yields a generalized circle agglomeration associated with Δ. In this sense we can construct the desired generalized circle agglomeration $\mathcal{P}$ filling R_h for Q.

2.2. Assume Q is irreducible. By the Lemmas 2.5 and 2.6 we can express all neighbors of d in Q as the ordered set $N(v) = \{a = v_0, v_1, \ldots, v_k, v_{k+1} = c\}$, where v_i and v_j with $i, j \in \{0, \ldots, k + 1\}$ are neighbors if and only if $|i - j| = 1$. To define a new complex T we remove the vertex d and all its edges and faces from Q and add instead the edge $e(a, v_i)$ and the face $f(a, v_{i-1}, v_i)$ for every $i = 2, \ldots, k + 1$ (see Figure 3.33). Since v_1 is the only neighbor of a in $N(v)$, the created T is a tri-complex $T(a, b, c) \in \mathcal{T}_n^*$.

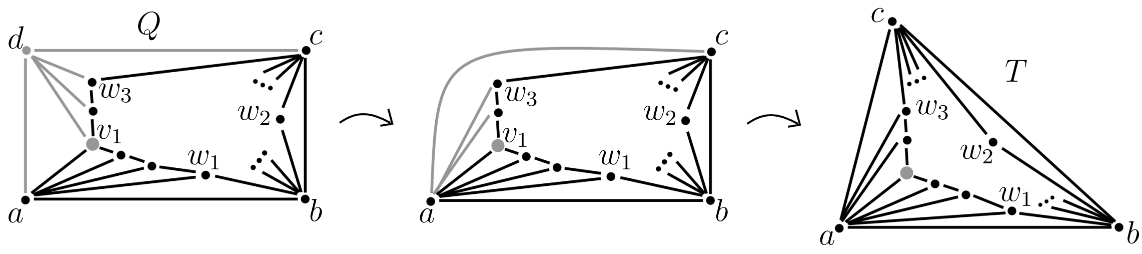


Fig. 3.33.: Construction of T out of Q

Let $w_1, w_2, w_3, v_1 \in Q$ be the (not necessarily different) leading vertices of Q, i.e., $f(w_1, a, b)$, $f(w_2, b, c)$, $f(w_3, c, d)$ and $f(v_1, d, a)$ are faces in Q. Then $f(w_1, a, b)$, $f(w_2, b, c)$ and $f(w_3, c, a)$ are faces in T, i.e., w_1, w_2 and w_3 are also (the) leading vertices of T.

For every $t \in \mathbb{R}^+$ let R_t be the rectangle

$$R_t = \{z : 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < \mathrm{i}t\}.$$

Depending on whether we are using Q or T we interpret R_t as quadrilateral $R_t(\alpha, \beta_t, \gamma_t, \delta_t)$ with the four edges $\alpha = \{x + iy : 0 \le x \le 1, y = 0\}$, $\beta_t = \{x + iy : x = 1, 0 \le y \le t\}$, $\gamma_t = \{x + iy : 1 \ge x \ge 0, y = t\}$ and $\delta_t = \{x + iy : x = 0, t \ge y \ge 0\}$, or as trilateral $R_t(\delta_t \cup \alpha, \beta_t, \gamma_t)$ (see Figure 3.34, left). In both cases R_t is tame for every $t \in \mathbb{R}^+$. Its vertices are even untouchable.

By Theorem 3.1 in combination with Lemma 3.11 there is a non-degenerate circle agglomeration $\mathcal{P}_t$ for T filling the trilateral $R_t(\delta_t \cup \alpha, \beta_t, \gamma_t)$. According to the definition of a leading disk, we know that $D^t_{w_1}$ touches $\alpha \cup \delta_t$ as well as β_t, $D^t_{w_2}$ touches β_t and γ_t, and $D^t_{w_3}$ touches γ_t and $\alpha \cup \delta_t$.

Let $N(a) = \{c, w_3, \ldots, v_1, \ldots, w_1, b\}$ be the ordered set of neighbors of a in T. By construction, the first part of $N(a)$ corresponds to the neighbors $\{c, w_3, \ldots, v_1, a\}$ of d in Q while the second part corresponds to the neighbors $\{d, v_1, \ldots, w_1, b\}$ of a in Q.

Let $D^t \in \mathcal{P}_t$ be associated with a vertex of $\{w_3, \ldots, v_1\}$. By construction the disk D^t touches $\delta_t \cup \alpha$, but if $D^t_{v_1}$ touches δ_t, then Lemma 3.10 implies that also D^t touches δ_t. Analogously, every disk associated with a vertex of $\{v_1, \ldots, w_1\}$ touches α as soon as $D^t_{v_1}$ touches α.

We conclude that $\mathcal{P}_t$ fills not only the trilateral $R_t(\delta_t \cup \alpha, \beta_t, \gamma_t)$ for T, but also the quadrilateral $R_t(\alpha, \beta_t, \gamma_t, \delta_t)$ for Q if $D^t_{v_1}$ touches α and δ_t. Unfortunately, in the general case, $D^t_{v_1}$ only touches α or δ_t. Therefore, we need to adjust t accordingly.

Note that the maximal diameter of a disk in $\mathcal{P}_t$ is $\min\{1, t\}$. So the maximal length l of a disk chain with all n disks of $\mathcal{P}_t$ is n-times this maximal diameter, i.e., $l \le \min\{n, tn\}$. For some sufficiently small $T_1 \in \mathbb{R}^+$ we can achieve $l \le tn < 1$ for every $t \le T_1$. Since at least D_{w_2} touches β, this implies that no disk of $\mathcal{P}_t$ can touch δ_t for $t \le T_1$ (see Figure 3.34, right).

Analogously, for some sufficiently large $T_2 \in \mathbb{R}^+$ we can achieve $l \le n < T_2$ for every $t \ge T_2$. Since at least D_{w_2} touches γ_t, this implies that no disk of $\mathcal{P}_t$ can touch α for $t \ge T_2$. Clearly $T_1 < T_2$. Let R_1 and R_2 be the associated domains for T_1 and T_2, respectively.

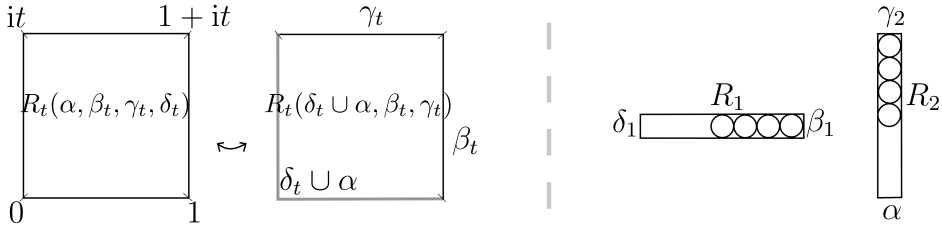


Fig. 3.34.: The quad- or trilateral R_t (left), and R_1 and R_2 for $t = T_1$ and $t = T_2$ (right)

Consequently, there exists a sequence t_k that converges to a positive limit h so that the leading disks $D^k_{v_1}$ of the corresponding circle agglomerations $\mathcal{P}_k$ touch α if k is even while it touches δ_k if k is odd. Let $R_h = R_t$ for $t = h$, and for $\varepsilon > 0$ let $R_\varepsilon = R_t$ for $t = h + \varepsilon$.

Let ε be fix. Then almost all, say all members of $(\mathcal{P}_k)$ are contained in R_ε. By Lemma 3.36, there is a sub-sequence of $(\mathcal{P}_k)$, which we denote again by $(\mathcal{P}_k)$, that converges to a generalized circle agglomeration $\mathcal{P}_h$ lying in R_ε.

In order to see that $\mathcal{P}_h$ is not collapsed, let $N(b) = \{a, w_1, \ldots, w_2, c\}$ be the ordered set of all neighbors of b in T. The disk $D^k_{w_1}$ touches $\delta_k \cup \alpha$, thus δ_k or α for almost all k. In the former case, the diameter of $D^k_{w_1}$ must be 1 in order to reach β_k, i.e., it cannot degenerate. In the latter case, the sum of the diameters of all disks associated with $N(b)$ must be at least $h - \varepsilon$ in order to reach γ_k, i.e., not all of them can degenerate. Hence, $\mathcal{P}$ cannot become collapsed.

By Lemma 3.35 every disk $D^k \in \mathcal{P}_k$ that touches δ_k, α or β_k for (almost) all k converges to a limit disk or dot touching δ_ε, α or β_ε, respectively. Moreover, the distance between γ_ε and any disk $D^k \in \mathcal{P}_k$ touching γ_k is less than 2ε. Hence, $\mathcal{P}_h$ fills the domain R_h, and even the trilateral $R_h(\delta_h \cup \alpha, \beta_h, \gamma_h)$ so that $D^h_{v_1}$ touches α and δ_h. Any assumption to the contrary would easily provide a contradiction; just choose ε sufficiently small.

By our thoughts from above this implies that $\mathcal{P}_h$ fills the quadrilateral $R_h(\alpha, \beta_h, \gamma_h, \delta_h)$ for Q, and both $\mathcal{P}_h$ and R_h are uniquely determined, what concludes this proof. □

Definition 3.22. The (unique) aspect ratio h of the rectangle R in Theorem 3.47 is called the *discrete conformal modulus* of the quad-complex Q.

Note that the discrete conformal modulus as defined in our sense only depends on the used quad-complex Q; Stephenson suggests in [31] (Section 17.3) that it may be associated with the *graph modulus* of Q. It will be interesting to investigate this further.

3.9. Summary

For the convenience of the reader let us summarize the results of this chapter.

Existence. For every bounded, simply connected trilateral $G(\alpha, \beta, \gamma)$ and every tri-complex $T(a, b, c)$ there is a generalized circle agglomeration $\mathcal{P}$ that fills $G(\alpha, \beta, \gamma)$ for $T(a, b, c)$ (Theorem 3.45).

Uniqueness. Let $G(\alpha, \beta, \gamma)$ be a trilateral and let $T(a, b, c)$ be a tri-complex. Let $\mathcal{P}$ be a generalized circle agglomeration for $T(a, b, c)$ filling $G(\alpha, \beta, \gamma)$. If $G(\alpha, \beta, \gamma)$ is tame, i.e., $\alpha \cap \beta$, $\beta \cap \gamma$ and $\gamma \cap \alpha$ are regular, then $\mathcal{P}$ is uniquely determined, independent whether it is degenerate or not (Theorem 3.31).

Continuity. Let I be a closed interval. For $t \in I$, let $(\mathcal{P}_t)$ fill the continuous trilateral family $(G(\alpha_t, \beta_t, \gamma_t))$ for a given tri-complex $T(a, b, c)$. If the trilateral $G(\alpha_0, \beta_0, \gamma_0)$ is tame for some $t_0 \in I$, then $(\mathcal{P}_t)$ depends continuously on t at t_0. If G is regular, i.e., every prime end of G is regular whence ever possible trilateral for G is tame, then $(\mathcal{P}_t)$ is a continuous family (Lemma 3.39).

Guaranteed Non-Degeneration (alpha-beta-gamma)

		Geometric Properties of G			
		No Inward Spikes, vertices are Untouchable	**No Inward Spikes**	**Inward Spikes, but Non-Spiky**	*(no restriction)*
Combinatoric Properties of T	**Boundary Irreducible**	yes (Lemma 3.11)	yes (Lemma 3.14)	yes (Lemma 3.14)	no
	degree=3, Intrinsic Strongly Connected	yes (Lemma 3.11)	yes (Lemma 3.13)	yes (Lemma 3.13)	no
	degree=3, Proper	yes (Lemma 3.11)	yes (Lemma 3.12)	no, but for Skeleton $\mathcal{P}_\sigma$ (Corollary 3.15)	no
	(no restriction)	yes (Lemma 3.11)	no, but for Skeleton $\mathcal{P}_\sigma$ (Corollary 3.15)	no, but for Skeleton $\mathcal{P}_\sigma$ (Corollary 3.15)	no

Non-Degeneration. The table above shows conditions under which a generalized circle agglomeration is guaranteed to be non-degenerate (denoted by *"yes"*). If the table entry says *"no"*, then there are examples of $G(\alpha, \beta, \gamma)$ and $T(a, b, c)$ so that the unique solution $\mathcal{P}$ is degenerate (see Figure 3.35).

If the table entry says *"no, but for Skeleton $\mathcal{P}_\sigma$"*, then one cannot assure (in general) that the complete packing is non-degenerate. Nevertheless, the sub-packing $\mathcal{P}_\sigma$ that is associated with the skeleton T_σ of T cannot become degenerate.

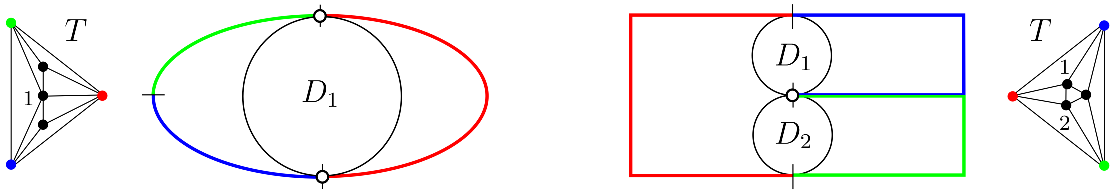


Fig. 3.35.: Two degenerate circle agglomerations; unique under the alpha-beta-gamma normalization

Receipt. The natural setting for the alpha-beta-gamma normalization consists of a tame trilateral $G(\alpha, \beta, \gamma)$ and a boundary irreducible tri-complex $T(a, b, c)$. Then there is a unique circle agglomeration $\mathcal{P}$ for T filling $G(\alpha, \beta, \gamma)$.

If the reader is only interested in circle packings instead of circle agglomerations, then the following connection between boundary irreducible tri-complexes and strongly connected admissible complexes can be helpful.

Lemma 3.48. *Let $T \in \mathcal{T}$ be proper. Let* $\operatorname{degree}(T) = 3$, *and let the critical vertices v_i of T fulfill $v_i \in V^*$ for the kernel K^* of $K = \operatorname{int} T$ (with respect to some interior vertex of K). Then T is boundary irreducible if and only if K is strongly connected.*

The proof for this lemma relies on Lemma 3.2 and some additional thoughts about possible irreducible boundary faces of T. The details are left as an exercise. More important is the following receipt:

Start with a strongly connected admissible complex K and a bounded, simply connected domain G. Choose three different boundary vertices of K and associate them with three regular prime ends of G. Then there is a unique circle packing $\mathcal{P}$ for K filling G under the alpha-beta-gamma normalization.

4. Alpha-Gamma Normalization

The second normalization we are looking at is the *alpha-gamma normalization*. Roughly speaking, we associate an interior and a boundary point of a domain G with two disks of a domain filling packing $\mathcal{P}$ – the alpha and the gamma disk of $\mathcal{P}$, respectively.

More precisely, let G be a bounded, simply connected domain and let X be a prime end of G. Let $\mathcal{P}$ be a (generalized) circle packing filling G, and let D be a boundary disk or boundary dot of $\mathcal{P}$. Then D *meets* X if either D touches X (in the sense of prime ends, see Definition 2.9) or if there is a crosscut $\gamma \subset D$ separating X from $\mathcal{P} \setminus \{D\}$ (in the sense of Definition 3.13; see also Lemma 3.9). Clearly, if D is a boundary dot, then it must touch X.

Let $K(a,c)$ be an admissible complex with a distinguished vertex a, the *alpha vertex*, and a distinguished boundary vertex $c \neq a$, the *gamma vertex*. Let G be a bounded, simply connected domain, let $A \in G$ be a fixed interior point and let $C \in \partial G^*$ be a fixed prime end of G. In short we call $G(A,C)$ a *pinned* domain. Let $\mathcal{P}$ be a generalized circle packing with complex K filling G.

Definition 4.1. We say that $\mathcal{P}$ *fills* the pinned domain $G(A,C)$ for $K(a,c)$ *under the alpha-gamma normalization* if (1) the center of the *alpha disk* or *alpha dot* $P_a \in \mathcal{P}$ is A and if (2) the *gamma disk* or *gamma dot* $P_c \in \mathcal{P}$ meets C.

Whenever we explicitly want a to be a boundary vertex, we speak of the *boundary alpha-gamma normalization* as already introduced in Section 3.6.1. While the boundary case is directly related to the alpha-beta-gamma normalization, the case of interior alpha vertices needs a completely different approach in order to show uniqueness and existence. However, the general idea is very similar to that one of Chapter 3.

In order to prove that two generalized circle packings $\mathcal{P}_1$ and $\mathcal{P}_2$ are unique under the alpha-gamma normalization, we reduce $\mathcal{P}_i$ to its *extended kernel* (analogously to the skeleton of K), show that this must be a non-degenerate circle packing, and use a *rigidity* argument (analogously to incompressibility) in order to see that at least the main parts of $\mathcal{P}_1$ and $\mathcal{P}_2$ must be equal. An application of the alpha-beta-gamma normalization deals with the remaining *detour-packings* in $\mathcal{P}_1$ and $\mathcal{P}_2$. The definitions of these new terms are given in the next section.

Furthermore, we provide again a classification of several types of degeneration that can occur, and eventually we prove existence by another application of Sperner's Lemma (for regular domains) and a well known exhaustion argument (for the general case).

The following statement is the main result, which we prove in Section 4.6.3.

Theorem 4.1 (Alpha-Gamma Theorem)**.** *Let $K(a,c)$ be a strongly connected admissible complex. Let $G(A,C)$ be a pinned, bounded, simply connected domain that is not dubious.*

If C is regular, then there is a unique circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization.

For a pinned domain $G(A, C)$ to be *dubious* means, roughly speaking, that there is a disk $D \subset G$ with center at A and another disk or dot $P \subset (G \setminus D)$ meeting C so that $\partial D \cap \partial P \cap \partial G \neq \emptyset$. A precise definition is given in the text.

We also state some more general results, omitting for example the "strongly connected" and "non-dubious" constraints, but then degeneration effects can occur; in particular within so-called *singular packings*. The final Theorem 4.39 of this chapter yields the existence of a generalized circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization without any additional assumptions to neither K nor G.

4.1. Boundary Alpha-Gamma Normalization

First off all we study the case of boundary alpha vertices. Once we completely understood this special version of the alpha-gamma normalization we will later on often exclude it for the sake of transparency. In Section 3.6.1 we already proved the following result, let Lemma 3.32 be restated here for the convenience of the reader.

Lemma 4.2 (Boundary alpha-gamma uniqueness). *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ fill a pinned, bounded, simply connected domain $G(A, C)$ for an admissible complex $K(a, c)$ under the boundary alpha-gamma normalization. If C is regular, then $\mathcal{P}_1 = \mathcal{P}_2$.*

The reason why we have to restrict C to be regular is shown in Figure 4.1. As proved below, every pinned unit disk $\mathbb{D}(A, C)$ can be filled for every admissible complex $K(a, c)$ with a circle packing $\mathcal{P}$ under the boundary alpha-gamma normalization $\mathbb{D}(A, C)$. By fixing the complex $K(a, c)$ as well as the interior point A while changing C a little bit, we get a slightly shifted circle packing $\mathcal{P}'$ that fulfills the boundary alpha-gamma normalization for $\mathbb{D}(A, C')$. This gives rise to a Jordan domain G with inward corner C'' so that both circle packings $\mathcal{P}$ and $\mathcal{P}'$ fill $G(A, C'')$ for $K(a, c)$ under the boundary alpha-gamma normalization.

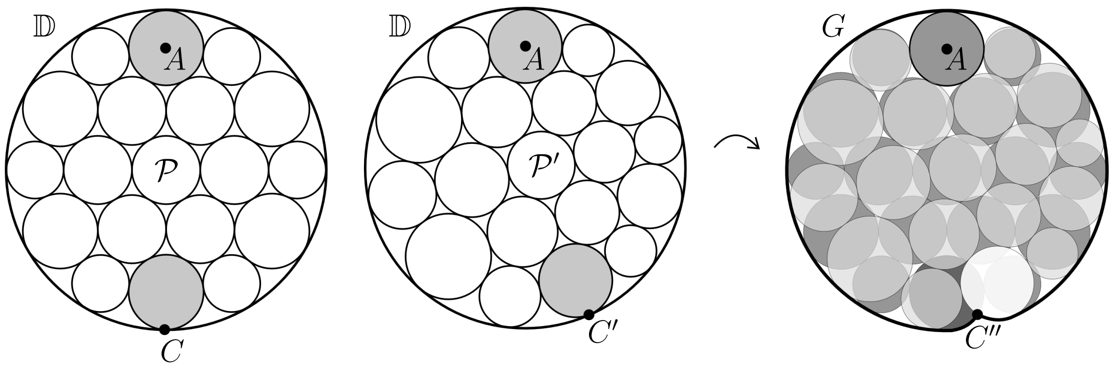


Fig. 4.1.: Two maximal packings $\mathcal{P}$ and $\mathcal{P}'$ with equal alpha disk but slightly shifted gamma disks

Two major ideas of the proof of Lemma 4.2 (or rather Lemma 3.32) are the construction of a tri-complex $T(a, v_1, v_2)$ and a trilateral $G_C(\alpha, \beta, \gamma)$ as depicted again in Figure 4.2 (which is exactly the same as Figure 3.29, p. 94). Note that G_C does not exist if the alpha disk touches C.

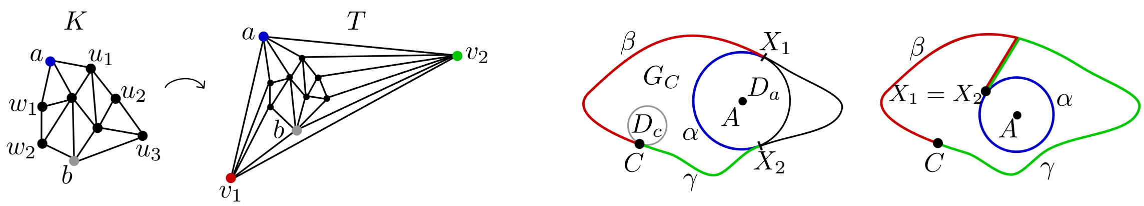


Fig. 4.2.: Construction of $T(a, v_1, v_2)$ and $G_C(\alpha, \beta, \gamma)$ out of $K(a, b)$ and $G(A, C)$, respectively

These constructions together with the Alpha-Beta-Gamma Existence Theorem 3.45 and the idea of the proof of Lemma 3.34 already provides the most general existence statement for the boundary alpha-gamma normalization.

Theorem 4.3 (Boundary Alpha-Gamma Existence Theorem)**.** *Let $K(a, c)$ be an admissible complex and let $G(A, C)$ be a pinned, bounded, simply connected domain. Then there is a (non-collapsed) generalized circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the boundary alpha-gamma normalization.*

Proof. Let D_a be the maximal disk in G with center in A. Assume first that D_a touches C, say in a contact point s. Then we simply define $P_v := \{s\}$ to be dots in s for all $v \in (V \setminus \{a\})$. In a trivial sense, $\mathcal{P} = \{D_a\} \cup \{P_v\}$ is a generalized circle packing for $K(a, c)$ filling $G(A, C)$ under the boundary alpha-gamma normalization.

Assume now that D_a does not touch C. Then the tri-complex $T(a, v_1, v_2)$ can be associated with the trilateral $G_C(\alpha, \beta, \gamma)$ (see the proof of Lemma 3.32 for more details). By Theorem 3.45, there is a generalized circle agglomeration $\mathcal{P}'$ for $T(a, v_1, v_2)$ filling $G_C(\alpha, \beta, \gamma)$. By construction, the packing $\mathcal{P} := \mathcal{P}' \cup \{D_a\}$ is a generalized circle packing for $K(a, c)$ filling $G(A, C)$ under the boundary alpha-gamma normalization. $\square$

Next, we provide some properties which guarantee that $\mathcal{P}$ is non-degenerate. In order to do so, the following is of special interest.

Definition 4.2. Let $G(A, C)$ be a pinned, bounded, simply connected domain with $A \in G$ and $C \in \partial G^*$. Let $D_A^{\max}$ be the maximal disk in G with center at A. If $D_A^{\max}$ does not touch C, then (as above) let $G_C(\alpha, \beta, \gamma)$ be the connected component of $G \setminus \overline{D_A^{\max}}$ that contains C as subordinate prime end. We say that $G(A, C)$ is *dubious*, if $G_C(\alpha, \beta, \gamma)$ has an incircle touching $\alpha \cap \beta$ or $\alpha \cap \gamma$, or if $D_A^{\max}$ touches C.

Since the unit disk $\mathbb{D}$ has no inward spikes, the following lemma together with Lemma 4.3 explains why there is always a maximal packing $\mathcal{P}$ for $K(a, c)$ filling $\mathbb{D}(A, C)$ under the boundary alpha-gamma normalization as long as $A \neq 0$ is not the origin. Recall that $\mathcal{D}$ is the set of all disks within a generalized circle packing $\mathcal{P}$ while $\mathcal{S}$ is the set of all dots.

Lemma 4.4. *Let the generalized circle packing $\mathcal{P} = \mathcal{D} \cup \mathcal{S}$ fill a pinned, bounded, simply connected domain $G(A, C)$ for an admissible complex $K(a, c)$ under the boundary alpha-gamma normalization. Let C be no inward spike. If D_a touches C, say at a contact point s, then $\mathcal{D} = \{D_a\}$ and $S_v = \{s\}$ for all $v \neq a$ in K. If D_a does not touch C, then D_c is a disk. Moreover, we have $\mathcal{S} = \emptyset$ if additionally*

(i) G has no inward spikes, or

(ii) $G(A, C)$ is not dubious and K is strongly connected.

Proof. If D_a touches C in s, then neither can any other disk of $\mathcal{P}$ touch C, because it is assumed to be no inward spike, nor can P_c separate C from D_a. We conclude that P_c must be a dot $P_c = S_c = \{s\}$. Since K stays connected when removing a together with its edges and faces, there is a chain of vertices $(v, \ldots, c)$ in K that does not contain a, but that connects any vertex $v \neq a$ with c. Thus, all sets in $\mathcal{P} \setminus \{D_a\}$ must be dots $S = S_c$, too, since they are neighbors or neighbors-neighbors of S_c.

Let now D_a does not touch C. Since we want to show that c is associated with a disk, we assume contrarily that $P_c = S_c$ is a dot. The normalization states that $S_c = s$ meets C, i.e., it touches C in a boundary point $s \in \partial G$. By Lemma 2.18, there is a disk $D \in \mathcal{P}$ that also touches C in s. By assumption, we have $D \neq D_a$, i.e., $\mathcal{P}$ contains at least two disks. Thus, Lemma 2.18 even states that S_c is a pseudo contact point of two disks of $\mathcal{P}$. So C is an inward spike, what is a contradiction. Hence, $P_c = D_c$ is a disk.

Since $\mathcal{P}$ is not collapsed, with $|\mathcal{D}| \geq 2$, Lemma 2.18 yields the existence of a boundary pseudo contact point as soon as $\mathcal{P}$ is degenerate. Clearly, this cannot happen when G has no inward spikes, what proves (i).

In (ii) there may be inward spikes but not between D_a and D_c since otherwise $G(A, C)$ would be dubious. Thus, there must be a third disk in $\mathcal{P}$. So Lemma 2.19 implies $\mathcal{S} = \emptyset$ for strongly connected K. $\square$

Finally, for the sake of completeness, we restate the continuity result of Lemma 3.40 (with slightly modified phrasing).

Lemma 4.5. *Let $K(a, c)$ be an admissible complex. Let $(G(A_t, C_t))$ be a continuous family of pinned, bounded, simply connected domains. For every $t \in I$ of a compact interval I, let the generalized circle packing $\mathcal{P}_t$ for $K(a, c)$ fill $G(A, C)$ under the boundary alpha-gamma normalization. If the prime end C_0 is regular for some $t_0 \in I$, then $(\mathcal{P}_t)$ depends continuously on t at t_0. If G is regular, then $(\mathcal{P}_t)$ is a continuous family.*

4.2. Additional Notation and Concepts

In what follows, the vertex a of $K(a, c)$ is often assumed to be an interior vertex. Using this fact, we divide K into several parts: The kernel around a, its *extension* with respect to c and several sub-complexes that are only loosely connected to the (extended) kernel. The role of this so called *detour-complexes* within the existence and uniqueness proofs for the alpha-gamma normalization is similar to the role of the reducible triangles for the alpha-beta-gamma normalization.

4.2.1. Detour, Detour-Complex, -Trilateral and -Packing

Let $K(a, c)$ be an admissible complex so that a is an interior vertex of K while c is a boundary vertex of K. Recall that, roughly speaking, the kernel K^* of K is the maximal sub-complex of K that contains only vertices v of K which are accessible from a, i.e. which can be connected with a by a vertex chain $(v, v_1, \ldots, v_n, a)$ using only interior vertices $v_1, \ldots, v_n$ of K (see Definition 2.6, p. 24).

By Lemma 2.3, the kernel $K^*(V^*, E^*, F^*)$ of $K(V, E, F)$ is a strongly connected complex with $\partial V^* = \partial V \cap V^*$, and by Lemma 2.4 we have $K = K^*$ if and only if K itself is strongly connected. We are now interested in the structure of $K \setminus K^*$.

Lemma 4.6. *For $n \geq 1$ let $C = (v_0, \ldots, v_{n+1})$ with $u := v_0$ and $w := v_{n+1}$ be a sub-chain of the boundary chain of K so that u and w are boundary vertices of K^* while $v_1, \ldots, v_n$ do not lie in K^*, i.e., $v_i \in \partial V^*$ if and only if $i \in \{0, n+1\}$. Then u and w are neighbors in K, i.e., $e(u, w) \in E$.*

Proof. Let C^* be the boundary chain of K^*. Let u_1 and u_2 be the predecessor and successor of u in C^*, respectively. By Lemma 2.3, the vertex u has no other boundary vertices of K^* as neighbors but only u_1 and u_2.

Assume $w \neq u_2$. We walk along C^* from w to u. We do not meet u_2 since it is the successor of u in C^*. Arrived at u, we leave C^* and walk through C from u back to w. Again, we do not meet u_2 since $u_2 \in \partial V^*$ with $u_2 \neq u$ and $u_2 \neq w$, i.e., $u_2 \notin C$. Let S be the chain of all those vertices we walked through. According to construction, S is closed and it does not contain u_2. Thus,u_2 lies in the interior of S, what is a contradiction to $u_2 \in \partial V$ (see Figure 4.3, left). Hence, $w = u_2$, what proves the lemma. $\square$

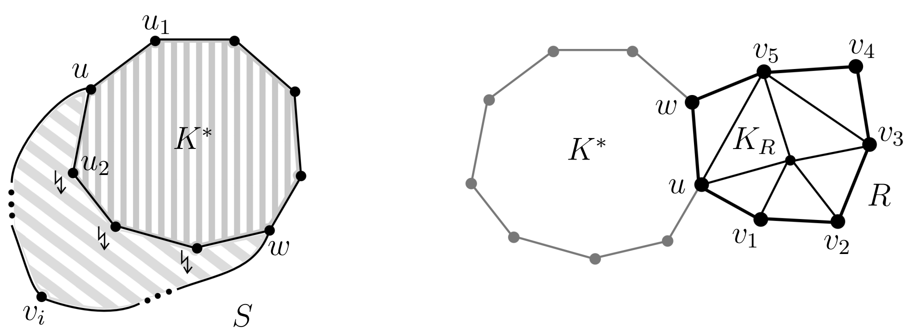


Fig. 4.3.: An impossible detour (left); a detour R and its detour-complex K_R (right)

Definition 4.3. Let $C = (u, v_1, \ldots, v_n, w)$ be again the chain from Lemma 4.6. Let $R = (u, v_1, \ldots, v_n, w, u)$ be the closed version of C. Then we call R a *detour* of K^*. The *detour-complex* K_R shall be the minimal admissible sub-complex of K that contains all vertices of R as well as all the edges $e(u, v_1), e(v_1, v_2), \ldots, e(v_n, w), e(w, u)$ (see Figure 4.3, right).

Let Γ be the edge chain $\Gamma = (e(u, v_1), e(v_1, v_2), \ldots, e(v_n, w), e(w, u))$ associated with R. Since all vertices of R are pairwise different, and there are at least three, we can interpret

Γ as a closed Jordan curve. Let G be the Jordan domain bounded by Γ, i.e., $\partial G = \Gamma$. Then K_R consists exactly of all vertices, edges and faces of K that are contained in $\overline{G}$. Otherwise either additional vertices (edges, faces) would violate the minimal condition of K_R, or missing vertices (edges, faces) would prevent K_R from being admissible. In particular the set of all boundary vertices of K_R is $\{u, v_1, \dots, v_n, w\}$ and the detour R is the boundary chain of its associated detour-complex K_R.

In order to investigate the influence of a detour-complex onto the behavior of the whole complex, the following construction will be helpful.

Definition 4.4. Let $R = (u, v_1, \dots, v_n, w, u)$ be a detour of K^* in K. Let K_R be the corresponding detour-complex. We define $T_R(w, u, d)$ to be the *detour-tri-complex* arising from K_R by adding a new vertex d, the edges $e(d, u), e(d, v_1), \dots, e(d, v_n), e(d, w)$ and the faces $f(d, u, v_1), f(d, v_1, v_2), \dots, f(d, v_n, w)$ (see Figure 4.4).

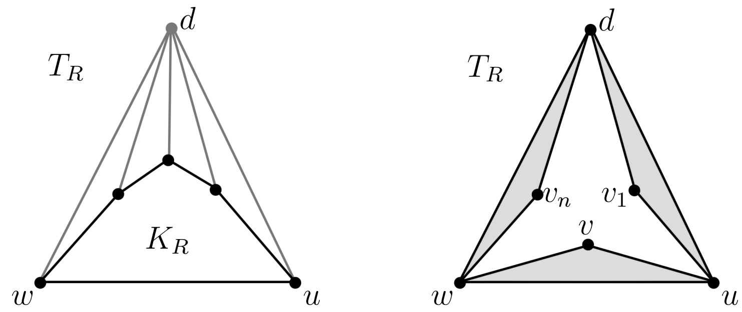


Fig. 4.4.: The definition of the detour-tri-complex $T_R(w, u, d)$

In order to associate the tri-complex T_R with a suitable trilateral, let $\mathcal{P}$ be a generalized circle packing for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization. Let K^* be the kernel of K with respect to a. Let $R = (u, v_1, \dots, v_n, w, u)$ be a detour of K^* in K, let K_R be the corresponding detour-complex and let $T_R(w, u, d)$ be the detour-tri-complex for R.

Assume that the corresponding sets $D_u, D_w \in \mathcal{P}$ for $u, w \in K^*$ are disks, and let $p = c(u, w)$ be the contact point between D_u and D_w. Then either, with respect to $\mathcal{P}^*$, the disks D_u and D_w form a positively oriented boundary interstice I, or p lies on the boundary of G, i.e., we have $\partial D_u \cap \partial D_w \cap \partial G = p$.

In the former case, let G_R denote the boundary interstice I (see Figure 4.5, left and middle). Then at least one vertex of intT_R is associated with a disk and every such disk is contained in G_R. In the latter case, every interior vertex of T_R must be associated with the dot $S = \{p\}$; for the sake of completeness we set $G_R := \emptyset$ (see Figure 4.5, right). Although these assertions are somewhat obvious, the reader is invited to see again Case 2 of Appendix A.1, which is equivalent to this situation here.

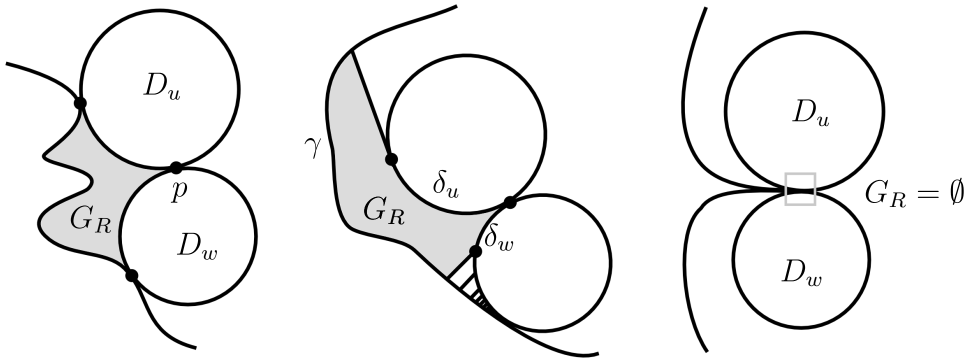


Fig. 4.5.: The boundary interstice of D_u and D_w defines the detour-trilateral $G_R(\delta_w, \delta_u, \gamma)$

Definition 4.5. Let $p \in G$. The boundary interstice G_R formed by D_u and D_w can be interpreted in a natural way as the *detour-trilateral* $G_R(\delta_w, \delta_u, \gamma)$ so that δ_w and δ_u are associated with the corresponding arcs of ∂D_w and ∂D_u, respectively, and $\gamma \subset \partial G^*$ is the remaining edge. Let $\mathcal{P}_R \subset \mathcal{P}$ be the set of all disks and dots of $\mathcal{P}$ that are associated with K_R but neither with u nor w. Then we call $\mathcal{P}_R$ the *detour-packing* for R.

By Lemma 3.6, the detour-trilateral $G_R(\delta_w, \delta_u, \gamma)$ is tame. By Lemma 2.10, the domain G_R is disjoint to the main part of $\mathcal{P}$. Thus, according to construction and since G_R contains all disks associated with $\mathrm{int}T_R$, the detour-packing $\mathcal{P}_R$ is a generalized circle agglomeration for the detour-tri-complex $T_R(w, u, d)$ filling the detour-trilateral $G_R(\delta_w, \delta_u, \gamma)$.

4.2.2. Extended Kernel

Up to now, we only used one of the two special vertices a and c of $K(a, c)$, namely the alpha vertex, in order to define the kernel K^* of K. Whenever c also lies in K^* this is enough, but as soon as it does not we need to extend the idea of the kernel.

If c is contained in K^*, for the sake of completeness, let $\partial\widetilde{E} := \partial E^*$ be the set of all boundary edges of the kernel K^*.

Assume that c is not contained in K^*. Since c is a boundary vertex, it must be contained in a detour $R = (u_m, \dots, u_1, c, w_1, \dots, w_n, u_m)$ so that $u := u_m$ and $w := w_n$ lie in K^*. Let $\widetilde{u}_1$ be that neighbor of c in $\{u_1, \dots, u_m\}$ that has the largest index. Of course, at least u_1 is a neighbor of c, so $\widetilde{u}_1$ exists. Analogously, we define $\widetilde{w}_1$ to be the neighbor of c with largest index in $\{w_1, \dots, w_n\}$. Now let $\widetilde{u}_2$ and $\widetilde{w}_2$ be those neighbors of $\widetilde{u}_1$ and $\widetilde{w}_1$ in $\{u_1, \dots, u_m\}$ and $\{w_1, \dots, w_n\}$, respectively, which have again the largest index. Clearly, for every u_i with $1 \le i < m$ at least u_{i+1} is a neighbor (analogously for w_j with $1 \le j < n$) so we can continue this procedure until we eventually arrive at $u = \widetilde{u}_{\widetilde{m}}$ and $w = \widetilde{w}_{\widetilde{n}}$, respectively. Let $\widetilde{R} = (u, \dots, \widetilde{u}_1, c, \widetilde{w}_1, \dots, w)$ be the resulting chain in K; maybe we have $u = \widetilde{u}_1$ or $w = \widetilde{w}_1$. The Figure 4.6 shows an example of this procedure.

Let's take a walk. We go along $\widetilde{R}$ from u to w, then from w along the boundary chain of the kernel K^* (not of the complex K) in positive direction until we reach again u. The set of edges we walked through shall be denoted $\partial\widetilde{E}$. Note that the edge $e(u,w)$ does not belong to $\partial\widetilde{E}$.

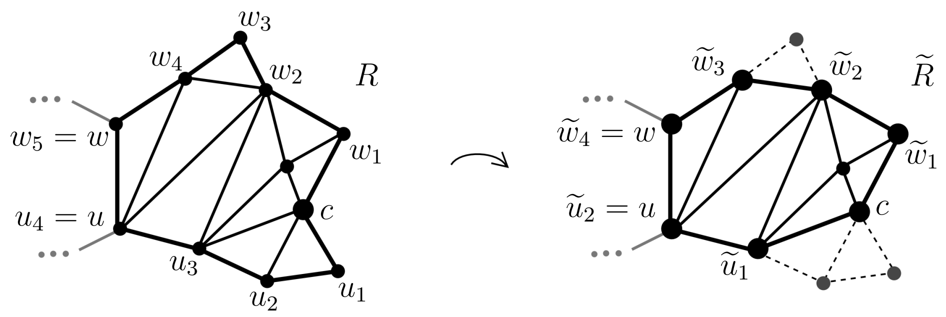


Fig. 4.6.: An example of how to construct $\widetilde{R}$ out of R

Definition 4.6. Let $\widetilde{K} \subset K$ be the smallest admissible sub-complex of K that contains $\partial\widetilde{E}$. Then we call $\widetilde{K}(\widetilde{V},\widetilde{E},\widetilde{F})$ the *extended kernel* of $K(a,c)$ (for examples see Figure 4.7). Let $\mathcal{P}$ be a generalized circle packing with admissible complex $K(a,c)$. Then the *extended main part* $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ denotes that sub-packing of $\mathcal{P}$ that is associated with the extended kernel $\widetilde{K}$ of $K(a,c)$.

All vertices of $\widetilde{R}$ are pairwise different. So we can interpret $\partial\widetilde{E}$ as a closed Jordan curve that defines a Jordan domain G with $\partial G = \Gamma$. Thus, $\widetilde{K}$ consist exactly of all vertices, edges and faces of K that are contained in $\overline{G}$. Hence, the set $\partial\widetilde{E}$ is in fact the set of all boundary edges of $\widetilde{K}$, and moreover $\widetilde{R}$ is the boundary chain of $\widetilde{K}$.

If and only if c lies in K^*, then we have $\widetilde{K} = K^*$. In general, by definition, we have $K^* \subset \widetilde{K}$. So in fact $\widetilde{K}$ is an extension of the kernel K^* of K. The next section explains why we are interested in it.

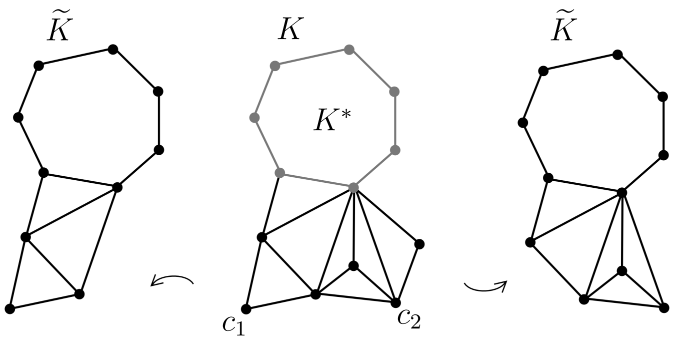


Fig. 4.7.: Two examples of an extended kernel $\widetilde{K}$ for $c = c_1$ (left) and $c = c_2$ (right)

4.3. Characterization of Degeneracy

As we will see, it needs some rather strong restrictions in order to completely avoid degeneracy of $\mathcal{P}$, but it is quite easy to secure non-degeneracy of its extended main part. Therefore, we introduce the following two subsets of generalized circle packings.

Definition 4.7. Let $\mathcal{P}$ be a generalized circle packing for the admissible complex $K(a,c)$. Let $\widetilde{\mathcal{P}}$ be the extended main part of $\mathcal{P}$. If $\widetilde{\mathcal{P}}$ is not degenerate, then we call $\mathcal{P}$ *good-natured*, otherwise it is denoted as *ill-natured*.

The main task of this section is to provide some conditions under which a generalized circle packing $\mathcal{P}$ must be good-natured. First of all, we note that under the alpha-gamma normalization the alpha disk $D_a \in \mathcal{P}$ is never degenerate.

Lemma 4.7. *Let $\mathcal{P}$ be a generalized circle packing for $K(a,c)$ filling $G(A,C)$ under the alpha-gamma normalization. Then a is associated with a disk $D_a \in \mathcal{P}$.*

Proof. Assume contrarily that a is associated with a dot S_a. Then we have $S_a = \{A\}$. By Lemma 2.18, there is a boundary dot $S = \{s\}$ that equals S_a, i.e., $S = S_a$ and $s = A$. Since $\mathcal{P}$ fills G, the point s must be a boundary point. This is a contradiction to $s = A \in G$ and proves the lemma. □

Now, we construct a special ill-natured packing. For some pinned, bounded, simply connected domain $G(A,C)$ let D_1 be the maximal disk in G with center in A. Clearly, D_1 exists, it is unique and it touches ∂G. If D_1 touches C, say at the contact point $s \in \partial G$, then we set $P_2 := S_2 = \{s\}$ to be a dot. If D_1 does not touch C, then let $G_C(\alpha,\beta,\gamma)$ be the connected component of $G \setminus \overline{D_1}$ that contains C as subordinate prime end (see Lemma 3.7). We define $P_2 := D_2$ to be an incircle of $G_C(\alpha,\beta,\gamma)$. Note that if C is regular, then $G_C(\alpha,\beta,\gamma)$ is tame, whence D_2 is uniquely determined by Theorem 3.17.

Definition 4.8. Let $K(a,c)$ be an admissible complex. We set $P_a := D_1$ and $P_c := P_2$. For all $v \in (V \setminus \{a,c\})$ let $P_v := S_v = \{p\}$ be the dot at the contact point p between P_a and P_c, i.e. $S_v = \overline{D_1} \cap \overline{P_2}$. So the set $\mathcal{P}_0 := \{P_v : v \in V\}$ is a degenerate circle packing for $K(a,c)$. If $p \in \partial G$, then $\mathcal{P}_0$ fills $G(A,C)$ for $K(a,c)$ under the alpha-gamma normalization and we call it a *singular packing*.

By definition, a singular packing $\mathcal{P}_0$ only exists for dubious normalizations $G(A,C)$. Moreover, it is ill-natured and independent of the given complex K. If C is regular, then $\mathcal{P}_0$ is unique and (according to the following lemma) it is the one and only possible ill-natured packing at all.

Lemma 4.8. *Let $K(a,c)$ be an admissible complex with interior vertex a. Let $G(A,C)$ be a pinned, bounded, simply connected domain with C being no inward spike. Then there is an ill-natured packing filling $G(A,C)$ for $K(a,c)$ if and only if $G(A,C)$ is dubious. Moreover, every ill-natured packing is a singular packing.*

Proof. If $G(A, C)$ is dubious, then we already constructed an ill-natured packing, the singular packing $\mathcal{P}_0$. So everything left to show is that the existence of an ill-natured packing $\mathcal{P}$ implies that $G(A, C)$ is dubious and $\mathcal{P} = \mathcal{P}_0$.

Assume there is an ill-natured packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization. Let $\widetilde{K}$ be the extended kernel of K and let $\widetilde{V}$ be its vertex set. Then there is a vertex $v_0 \in \widetilde{V}$ that is associated with a dot $S_0 = \{s\}$. In the following we distinguish between three cases.

Case 1. Assume $v_0 = c$, i.e., $S_c = \{s\}$ is a dot meeting C. Then S_c touches C, by definition, and there is a disk $D \in \mathcal{P}$ touching C at the contact point s. Since C is no inward spike, there is exactly one such disk in $\mathcal{P}$. By Lemma 2.18, $\mathcal{P}$ contains exactly one disk that, by Lemma 4.7, must be the alpha disk D_a while all other sets $P_v \in \mathcal{P}$ with $v \in (V \setminus \{a\})$ are dots $S = S_0 = \{s\}$. Thus, the pinned domain $G(A, C)$ is dubious and we have $\mathcal{P} = \mathcal{P}_0$ being the singular packing.

Case 2. Assume that c is associated with a disk D_c while $v_0 \in V^*$ is a vertex of the kernel K^* of K. By Lemma 2.3, the kernel K^* is strongly connected. So Lemma 2.19 assures that the main part $\mathcal{P}^*$ of $\mathcal{P}$ contains at most two disks. Again by Lemma 4.7, one of it must be the alpha disk D_a. By Lemma 2.18, all dots S of $\mathcal{P}^*$ are equal to $S = S_0 = \{s\}$ and are attached to D_a at the boundary point $s \in \partial G$. Thus, we may and will assume w.l.o.g. that $v_0 \in \partial V^*$ is a boundary vertex.

We now walk along the boundary chain of K (so not only of K^*) from v_0 to c. Since at most one other disk of $\mathcal{P} \setminus \{D_a\}$ can touch S, too, and since we can first walk with positive and then with negative orientation, it is clear that $\mathcal{P}$ can have at most one boundary disk, which then must be D_c.

Let K' be any detour-complex of $K(a, c)$. Since K' contains exactly two boundary vertices of K^* while it does not contain a, it stays connected to at least one vertex of ∂V^* even after removing a and c together with all their edges and faces. Furthermore, by Lemma 2.4, the kernel K^* itself is 3-connected. So the whole complex K stays connected without a and c and their edges and faces. Thus, $\mathcal{P}$ contains exactly the two disk D_a and D_c while all other sets $P_v \in \mathcal{P}$ with $v \in (V \setminus \{a, c\})$ must be dots $S = S_0 = \{s\}$. Hence, the normalization $G(A, C)$ is dubious and we have $\mathcal{P} = \mathcal{P}_0$ being a singular packing.

Case 3. Assume that c is associated with a disk D_c while $v_0 \notin V^*$ does not lie in K^*. From $v_0 \in (\widetilde{V} \setminus V^*)$ it follows that $\widetilde{K} \neq K^*$. So c lies in $\widetilde{K} \setminus K^*$. By Lemma 2.18, we may and will assume w.l.o.g. that v_0 is a boundary vertex, i.e., $v_0 \in \partial\widetilde{V}$. If we can show that there is a dot in the main part of $\mathcal{P}$, then we are done by Case 2.

In order to do so, let $R = (u_m, \dots, u_1, c, w_1, \dots, w_n, u_m)$ with $u := u_m$, $w := w_n$ and $u, w \in \partial V^*$ be the (one and only) detour of K^* within $\widetilde{K}$. Due to reasons of symmetry, we can assume w.l.o.g. that $v_0 = w_j \in \{w_1, \dots, w_{n-1}\}$ and $n \geq 2$. Walking through R from v_0 to c, first with negative then with positive orientation, let v_1 and v_2 be the first vertices we meet that are associated with disks $D_1, D_2 \in \mathcal{P}$, respectively. Since D_c is assumed to be a disk, we have $v_1 \in \{c, w_1, \dots, w_{j-1}\}$. If $v_2 \notin \{w_{j+1}, \dots, w_n\}$, then w is associated with a dot and we are done by Case 2. Therefore, assume that we have $v_2 \in \{w_{j+1}, \dots, w_n\}$. This implies $v_1 \neq v_2$, i.e., S_0 is a pseudo contact point of $D_1 \neq D_2$.

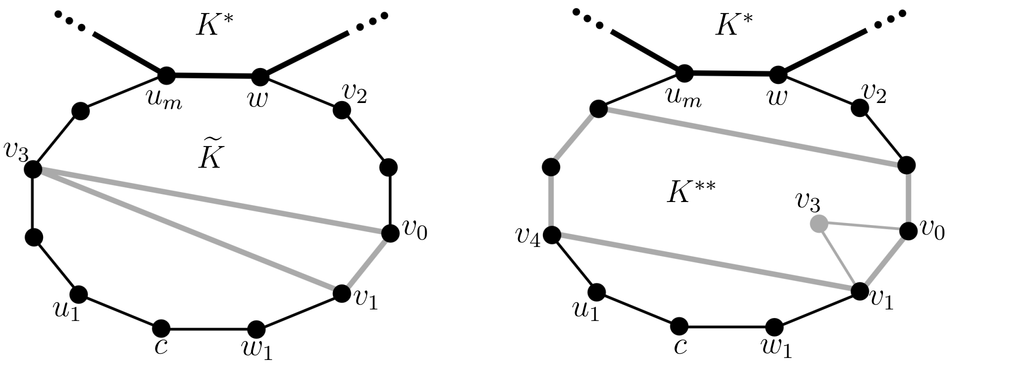


Fig. 4.8.: Constructions for Case 3 of the proof of Lemma 4.8

Now, we look at the successor of v_1 in R, which we may and will assume w.l.o.g. to be v_0. Together with some vertex v_3 they form a face $f(v_0, v_1, v_3)$ in $\widetilde{K}$. We have $v_3 \notin \{w_{j+1}, \ldots, w_n\}$ since otherwise v_1 would have a neighbor in $\{w_1, \ldots, w_n\}$ with a larger index than $v_0 = w_j$, what is a contradiction to the definition of $\widetilde{K}$ (note that this is exactly the reason why the extended kernel is defined the way it is). Therefore, $v_3 \neq v_2$ is not associated with one of the two disks touching S_0. Hence, as a neighbor of v_0, it must also be a dot $S = S_0 = \{s\}$.

If v_3 is a boundary vertex of $\widetilde{K}$, then following R from v_3 to v_2 with positive orientation yields that none of the vertices we walked through, but only v_2, is associated neither with D_1 nor with D_2 (see Figure 4.8, left). Thus, as a neighbor or neighbors-neighbor of v_3, in particular the vertex w is associated with a dot $S = S_0 = \{s\}$ and we are done by Case 2. Therefore, we assume that v_3 is an interior vertex of $\widetilde{K}$.

Let K^{**} be the kernel of K with respect to v_3 instead of a (see Figure 4.8, right). By Lemma 2.3, K^{**} is strongly connected. So Lemma 2.19 states that the associated (degenerated) sub-packing $\mathcal{P}^{**}$ of $\mathcal{P}$ contains at most two disk. One of it is D_1, thus all dots in $\mathcal{P}^{**}$ are attached to it in $S_0 = \{s\}$. If there is a second disk D in $\mathcal{P}^{**}$, then it must touch D_1 at s, what is only possible for $D = D_2$. We conclude that every vertex of K^{**} different to v_1 and v_2 is associated with a dot $S = S_0$.

Let v_4 be the predecessor of v_1 within the boundary chain of K^{**}. Since v_1 has (per definition) no neighbor in $\{w_{j+1}, \ldots, w_n\}$, we have $v_4 \notin \{w_{j+1}, \ldots, w_n\}$, i.e., $v_4 \neq v_2$. Walking once more along the detour R, now from v_4 to v_2 with positive orientation, we see again that none of the vertices we walked through, but only v_2, is associated neither with D_1 nor with D_2. Thus, as a neighbor or neighbors-neighbor of v_4, the vertex w is associated with a dot $S = S_0 = \{s\}$, what allows us to use the arguments of Case 2 and completes the proof of this lemma. □

An alternative version of Lemma 4.8 for strongly connected complexes is stated as Corollary 4.9. There, the property of C to be no inward spike is replaced by the property that the maximal disk with center in A does not touch C. The only difference in the proofs of both statements appears in Case 1, where one has to use Lemma 2.19 instead of Lemma 2.18.

Corollary 4.9. *Let $K(a,c)$ be an admissible complex with interior vertex a. Let $G(A,C)$ be a pinned, bounded, simply connected domain. Let K be strongly connected, and let the maximal disk with center in A does not touch C. Then every degenerate circle packing $\mathcal{P}$ filling $G(A,C)$ for $K(a,c)$ is a singular packing. Moreover, a singular packing exists if and only if $G(A,C)$ is dubious.*

The one-to-one connection between ill-natured packings, the singular packing and dubious pinned domains is very handy since one can easily check a priori whether $G(A,C)$ is dubious or not. But don't be confused: dubiousness has no impact on the existence or non-existence of good-natured packings. Indeed, we prove in Section 4.6 (at least when C is no inward spike) that there is always a good-natured packing, independently of whether $G(A,C)$ is dubious or not.

The following result explains when such a good-natured packing is even non-degenerate.

Lemma 4.10. *Let $G(A,C)$ be a pinned, bounded, simply connected domain and let $K(a,c)$ be an admissible complex with interior alpha vertex. Let $\mathcal{P}$ be a good-natured generalized circle packing for $K(a,c)$ filling $G(A,C)$ under the alpha-gamma normalization. If $K=\widetilde{K}$ or if G has no inward spikes, then $\mathcal{P}$ is not degenerate.*

Proof. If $K=\widetilde{K}$, then trivially $\mathcal{P}$ is not degenerate since we assumed it to be good-natured. For $K\neq\widetilde{K}$ we still have $|\mathcal{D}|\geq|\widetilde{\mathcal{D}}|\geq 2$. So Lemma 2.18 yields the existence of a boundary pseudo contact point as soon as $\mathcal{P}$ is degenerate. This is only possible if G has inward spikes. $\square$

The Figure 4.9 shows an example of a degenerate but yet good-natured circle packing $\mathcal{P}$ for some admissible complex $K(a,c)$ filling $G(A,C)$ under the alpha-gamma normalization. There, all vertices within the detour-complexes K' and K'' are associated with dots in $\mathcal{P}$ except those lying in the extended kernel $\widetilde{K}$ of $K(a,c)$.

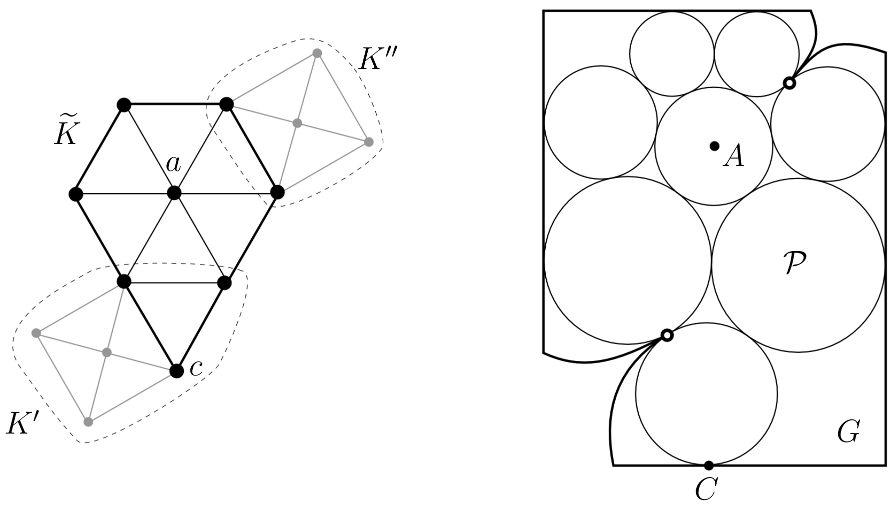


Fig. 4.9.: A good-natured but degenerate packing $\mathcal{P}$ for $K(a,c)$ filling $G(A,C)$

As a final result of this section we show that the limit of a sequence of good-natured circle packings is again good-natured as long as A stays in G and C is no inward spike. Actually, the statement of Lemma 4.11 is even stronger since only the main parts and not the extended main parts are assumed to be non-degenerate.

Lemma 4.11. *For every $k \in \mathbb{N}$ let $\mathcal{P}_k$ be a generalized circle packing with non-degenerate main part and admissible complex K. Assume that $\mathcal{P}_k$ converges to a generalized circle packing $\mathcal{P}$ for $K(a,c)$ filling $G(A,C)$ under the alpha-gamma normalization with interior a. Then the main part of $\mathcal{P}$ is not degenerate. If C is no inward spike, then $\mathcal{P}$ is even good-natured.*

Proof. Since we assumed that $\mathcal{P}$ fills $G(A,C)$ for $K(a,c)$, the Lemma 4.7 states that at least the alpha vertex is associated with a disk D_a in $\mathcal{P}$. Let $N(a) = \{v_1, \dots, v_n\}$ be the ordered set of all neighbors of a.

Since $\mathcal{P}_k$ is assumed to have a non-degenerate main part, the associated sets for $a, v_1, \dots, v_n$ in $\mathcal{P}_k$ are disks $D_a^k, D_1^k, \dots, D_n^k$, respectively. Let U_k and U be the circumferences of D_a^k and D_a, respectively. Then $U_k \to U > 0$ for $k \to \infty$. Let R_k be the sum of the radii of all the disks $D_2^k, \dots, D_n^k$ (note that we excluded the disk D_1^k for reasons explained below). Let R be the limit of R_k, i.e., $R_k \to R$ for $k \to \infty$.

The disks $D_1^k, \dots, D_n^k$ enclose D_a^k for each $k \in \mathbb{N}$. Even if the radius of D_1^k becomes as large as possible, it can never cover more than $\frac{U_k}{2}$. So we have $R_k \geq \frac{2U_k}{2}$, hence $R \geq U > 0$.

Assume now that v_n is associated with a dot in $\mathcal{P}$. Then it is impossible for more than one neighbor of a to be associated with a disk in $\mathcal{P}$. Otherwise, there would be two vertices v_i and v_j with $i < j$ so that for every $i < l < j$ the vertex v_l is associated with a dot in $\mathcal{P}$. Hence, the three disks associated with a, v_i and v_j would have a common (pseudo) contact point, what is impossible. We conclude that w.l.o.g. all vertices $v_2, \dots, v_n$ are associated with dots in $\mathcal{P}$. But this implies $R_k \to 0$, what is a contradiction to $R \geq U > 0$.

So the assumption was wrong and none of the neighbors of a can become degenerate in $\mathcal{P}$. Moreover, we can repeat the thoughts from above but now applied to any interior neighbor of a, in order to see that also the neighbors of $D_1, \dots, D_n$ must be disks in $\mathcal{P}$. And then again the interior neighbors and neighbors-neighbors of $D_1, \dots, D_n$ behave like D_a, too, by the same argumentation.

All in all, we see that every vertex of K is associated with a disk in $\mathcal{P}$ as soon as it can be connected to a by a chain of interior vertices, i.e., the main part of $\mathcal{P}$ is not degenerate. Furthermore, by Lemma 4.8 and since $\mathcal{P}$ is clearly not a singular packing, $\mathcal{P}$ is good-natured if C is no inward spike. $\square$

The example depicted in Figure 4.10 shows a limit packing $\mathcal{P}$ that is ill-natured although the sequence $\mathcal{P}_k$ is good-natured. This is only possible since the limit prime end C is an inward spike. Note that $\mathcal{P}$ together with the singular packing are the two only possible packings for the given complex, domain and normalization.

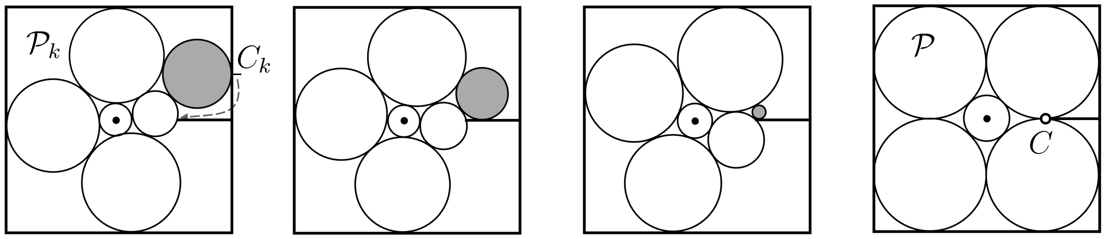


Fig. 4.10.: The limit $\mathcal{P}$ is ill-natured but has a non-degenerate main part

4.4. Rigidity

Similar to the idea of incompressibility of Section 3.4, we need to take a side step, too, before we are able to prove the uniqueness property for the alpha-gamma normalization. Note that, up to some technical details and minor improvements, this section corresponds almost one-to-one to [19] Chapter 3 to Chapter 6. In order to give the reader a flavor of the result, we first state an analogous theorem for analytic functions.

Theorem 4.12 (Identity Theorem for analytic functions)**.** *Let J be a crosscut of a simply connected domain G, with G^- and G^+ denoting the (simply connected) components of $G \setminus J$. If $f : G \to G$ is analytic, $f(z_0) = z_0$ for some $z_0 \in G^+$, and $f(G^-) \subset G^-$, then $f(z) = z$ for all $z \in G$.*

Proof. Let $g : G \to \mathbb{D}$ be the canonical embedding of G onto the unit disk $\mathbb{D}$ with $g(z_0) = 0$. Since g maps the crosscut J of G to a crosscut of $\mathbb{D}$, the composition $g \circ f \circ g^{-1}$ satisfies the assumptions of the lemma with $G := \mathbb{D}$ and $z_0 := 0$. Hence, it suffices to consider this special case.

Let z_1 be a point on J with $|z_1| = \min_{z \in J} |z|$. Since J is a crosscut in $\mathbb{D}$, and $0 = z_0 \in G^+$, we have

$$0 < |z_1| \le \min \left\{ |z| : z \in \overline{G^-} \right\} < 1.$$

By continuity, $f(G^-) \subset G^-$ and $z_1 \in \overline{G^-}$ imply that $f(z_1) \in \overline{G^-}$, and hence $|f(z_1)| \ge |z_1|$. Invoking Schwarz' Lemma, we get $f(z) = cz$ in $\mathbb{D}$, where c is a unimodular constant. Finally, the only rotation of $\mathbb{D}$ which maps G^- into itself is the identity. □

Although Schwarz' Lemma has already been investigated in the framework of circle packing (see [22], or [21] Chap. 13) the following interpretation of Theorem 4.12 is new. Though the definition of a (maximal) crosscut of a circle packing will be deferred to the next section, we hope that Figure 4.11 helps to get an intuitive understanding of the interpretation of Theorem 4.12 in the framework of circle packing. The domain G^- is the one containing the brighter (yellow) disks.

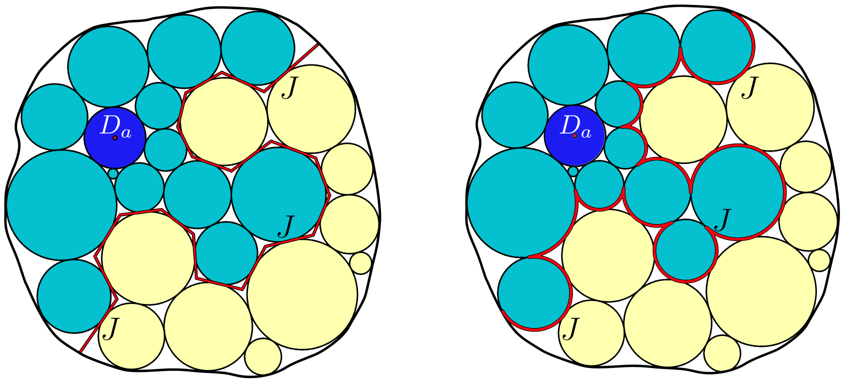


Fig. 4.11.: A domain filling circle packing $\mathcal{P}$ with a crosscut and a maximal crosscut

Assume that a circle packing $\mathcal{P} = \{D_v\}$ for an admissible complex $K \in \mathcal{K}$ fills a bounded, simply connected domain G. Let J be a *(maximal) crosscut of* $\mathcal{P}$ in G so that G^- is a simply connected component of $G \setminus J$, and denote by V^- and V^+ the sets of vertices of K associated with circles in G^- and $G^+ := G \setminus \overline{G^-}$, respectively. Let D_a be an interior disk of $\mathcal{P}$ that is contained in G^+. Let a be the vertex associated with D_a and let K^* be the kernel of K with respect to a (see Definition 2.6, p. 24).

Theorem 4.13 (Rigidity of circle packings with crosscuts)**.** *Let $\mathcal{P}$, G and J be as described above. Let $\mathcal{P}' = \{D'_v\}$ be a second circle packing for K in G so that D_a and D'_a have the same center and $D'_v \subset G^-$ for all $v \in V^-$. Then $D'_v = D_v$ for all vertices $v \in V^*$ in the kernel K^* of K.*

We point out that everything hinges on the assumption about the common center of the two alpha disks. Since we do not assume that $\mathcal{P}'$ fills G, it is solely this condition, which prevents $\mathcal{P}'$ from lying entirely in G^-.

For circle packings with strongly connected admissible complex K the theorem yields *complete rigidity* with respect to crosscuts, i.e., $D'_v = D_v$ for all $v \in V$ since then $K = K^*$.

Figure 4.12 illustrates some effects, which can be observed for packings with general combinatorics. The picture on the left shows an Apollonian packing $\mathcal{P}$ with four generations. The highlighted line is a maximal crosscut, separating the disks in the "lower domain" from the disks in the "upper domain". The disk with the darkest color is the alpha disk with fixed center.

The packing $\mathcal{P}'$, depicted in the middle, satisfies the assumptions of the theorem. In this example only the accessible disks of $\mathcal{P}'$ (shown in darker colors) coincide with their partners in $\mathcal{P}$. The non-accessible disks (shown in lighter colors) differ from the corresponding disks in $\mathcal{P}$.

The example on the right illustrates that the result does not need to hold if the alpha disk is a boundary disk. The depicted packing $\mathcal{P}''$ satisfies all other assumptions (for the same crosscut), but, apart from the alpha disk, it is completely different from the packing $\mathcal{P}$ shown on the left-hand side. This is another reason why we treated the boundary alpha-gamma normalization separately.

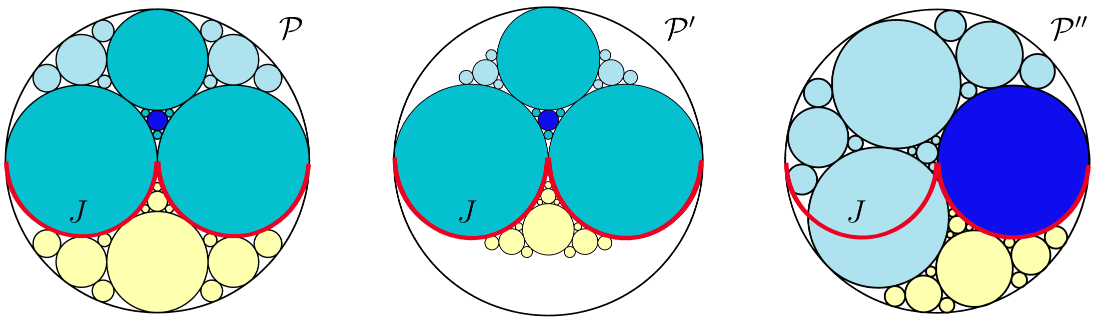


Fig. 4.12.: Some examples illustrating assumptions and assertions of Theorem 4.13

The result has an incompressibility-like interpretation: Suppose that $\mathcal{P}$ fills G and allow its disks to move (change position and size) in such a way that they all remain in G, the center of the alpha disk is fixed in G^+ and the disks in G^- are not allowed to leave G^-. Then only the non-accessible disks can be moved, while the main part of the packing is *rigid*.

4.4.1. Crosscuts of Circle Packings

Before we introduce crosscuts of circle packings, we define crosscuts of its complex.

Definition 4.9. A (combinatoric) *crosscut of an admissible complex* K is a sequence $L = (e_0, e_1, \ldots, e_l)$ of edges in K with the following properties (i)–(iii):

(i) The edges are pairwise different, i.e., if $0 \le j < k \le l$ then $e_j \ne e_k$.

(ii) For $1 \le j \le l$ the edges e_{j-1} and e_j are adjacent to a common face of K.

(iii) Three consecutive edges are not adjacent to the same face of K.

(iv) The edges e_0 and e_l are boundary edges.

It is easy to see that only the first and the last edge of a crosscut can be a boundary edges of K. Since $e_0 \ne e_l$, we have $l \ge 1$. When one edge of a face f belongs to L, then L must contain exactly two edges of f, and these are subsequent members of L. So a crosscut can also be represented by a sequence $(f_1, \ldots, f_l)$ of faces, where e_{j-1} and e_j are adjacent to f_j. Since the three edges of a face are not allowed to be consecutive members of L, all faces f_j must be pairwise different.

After removing the edges and the associated faces of a crosscut L from K, the remaining sub-complex consists of two connected components K_L^- and K_L^+. We assume that K_L^- 'lies to the right' and K_L^+ 'lies to the left', respectively, when we move along the edges $e_0, e_1, \ldots, e_l$ in this order. The vertex sets of K_L^- and K_L^+ are denoted by V_L^- and V_L^+, respectively, and we call them the *lower* and the *upper vertices* of K with respect to L. The set U_L^+ is constituted by all vertices v in V_L^+ that are adjacent to an edge in L. These vertices and the corresponding disks are said to be the *upper neighbors* of L. An analogous definition is made for the set U_L^- of *lower neighbors* of L (see Figure 4.13).

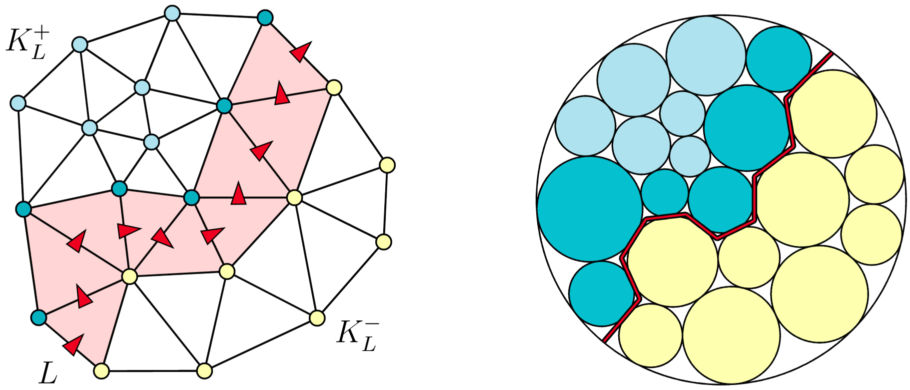


Fig. 4.13.: A crosscut L of K, the vertex sets V_L^-, V_L^+ and U_L^+, and a corresponding packing

Given a (combinatoric) crosscut L of a complex K and a circle packing $\mathcal{P}$ for K filling a bounded, simply connected domain G, we define several related (geometric) crosscuts J of $\mathcal{P}$ in G. To begin with, we associate with every edge $e_j = e(u,v)$ in L the contact point $x_j := \overline{D}_u \cap \overline{D}_v$ of the disks $D_u, D_v \in \mathcal{P}$. The common tangent to D_u and D_v at x_j is denoted τ_j. The set $\mathcal{X} := \{x_0, \ldots, x_l\}$ of all contact points associated with edges of L has a natural ordering, induced by the ordering of the edges in the crosscut. Since the indexing of the elements fits with this ordering, we write $x_j \prec x_k$ if $j < k$.

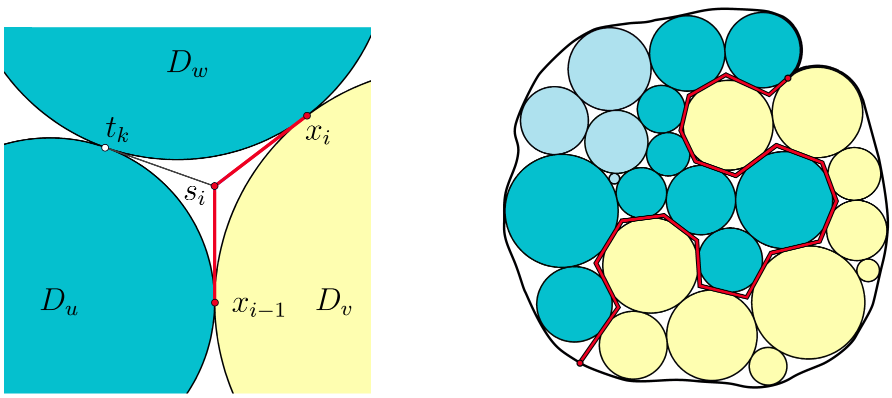


Fig. 4.14.: Local construction and global view of a polygonal crosscut

The *polygonal crosscut* J_L^0 is built from the common tangents τ_i of circles at their contact points x_i as follows. Let $i \in \{1, \ldots, l\}$ and assume that x_{i-1} and x_i are consecutive contact points of the pairs D_u, D_v and D_v, D_w, respectively. Then the three circles ∂D_u,

∂D_v and ∂D_w bound an interstice $I := I(u,v,w)$, which is disjoint to ∂G and any disk of $\mathcal{P}$. The tangents τ_{i-1} and τ_i intersect each other at a point s_i in I, and the union of the closed segments $[s_i, s_{i+1}]$ for $i = 1, \ldots, l-1$ is a Jordan arc in G (see Figure 4.14).

In order to complete this arc to a crosscut in G, we look at the boundary disks D_k and D_{k+1} that touch each other at x_0. If x_0 is not a boundary point of G, then we define s_0 as the endpoint of the largest segment (x_0, s_0) on the tangent τ_0 that is contained in the boundary interstice I_k. Since there is no disk of $\mathcal{P}$ intersecting I_k (Lemma 2.10), we see that $[x_0, s_0) \subset G$ is disjoint from $\mathcal{P}$ and $s_0 \in \partial G$. If x_0 is a boundary point of G, then we simply set $s_0 := x_0$.

A similar construction is made for the point s_{l+1} as ("the first") intersection point of the tangent τ_l with ∂G. Here, $x_0 \neq x_l$ ensures that $[x_0, s_0)$ and $[x_l, s_{l+1})$ live in two different boundary interstices. Although this does not exclude $s_0 = s_{l+1}$, it guarantees that s_0 and s_{l+1} are endpoints of the segments $[s_1, s_0)$ and $[s_l, s_{l+1})$, which belongs to different prime ends s_0^* and s_{l+1}^*, respectively.

Finally, the union of the closed segments $[s_k, s_{k+1}]$ for $k = 0, \ldots, l$ forms the desired polygonal crosscut $J_L^0 := \bigcup_{k=0}^{l} [s_k, s_{k+1}]$. By construction, J_L^0 is a closed Jordan arc with $\mathcal{X} \subset J_L^0 \cap \bigcup_{v \in V} \overline{D_v} \subset \mathcal{X} \cup \{s_0, s_{l+1}\}$, and the open set $G \setminus J_L^0$ has two simply connected components G_0^+ and G_0^- that contain the disks associated with V_L^+ and V_L^-, respectively.

It is clear that, for a fixed combinatorial crosscut L of K, the statement of Theorem 4.13 depends on the choice of the geometric crosscut J: The assertion becomes the stronger, the larger the domain G_J^- is. We therefore define the *maximal crosscut* J_L^+ in $\mathcal{P}$ as follows. Note that J_L^+ does not need to be a Jordan arc.

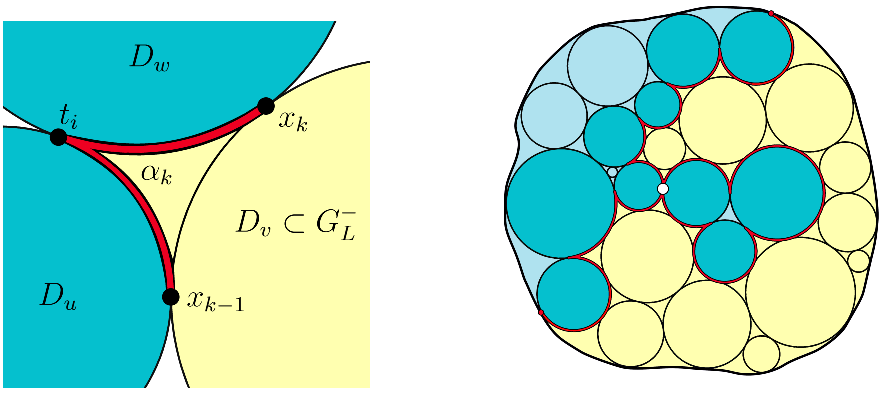


Fig. 4.15.: Construction of a maximal crosscut (which is not a Jordan arc)

Recall that U_L^+ is the vertex set of upper neighbors of L. If x_{k-1} and x_k are contact points of the disks D_u, D_v and D_v, D_w, respectively, then either $v \in U_L^+$ or $u, w \in U_L^+$. The interstice $I(u,v,w)$ is bounded by three (topologically closed) circular arcs α_u, α_v and α_w, respectively. If $v \in U_L^+$, then we connect x_{k-1} with x_k by the arc $a_k := \alpha_v$. In the second case, we connect these points by the concatenation $a_k := \alpha_u \cup \alpha_w$ (see Figure 4.15).

In addition, we connect x_0 and x_l with ∂G by the minimal sub-arcs $a_0 := \delta(g_j^+, x_0) \subset \partial D_j$ and $a_{l+1} := \delta(x_l, g_k^-) \subset \partial D_k$ so that the disks D_j and D_k are upper neighbors of L containing x_0 and x_l, respectively, and $g_j^+, g_k^- \in \partial G$ are its "first" intersection points with the boundary of G. If $g_j^+ = x_0$ or $x_l = g_k^-$, then we set $a_0 := \emptyset$ or rather $a_{l+1} := \emptyset$, respectively. The union $J_L^+ := \bigcup_{k=0}^{l+1} a_k$ of these arcs is a curve, which we call the *maximal crosscut* of $\mathcal{P}$ with respect to L.

The maximal crosscut J_L^+ is composed from a finite number of circular (topologically closed) arcs ω_i that are linked at the *turning points* t_i of J_L^+, and every contact point x_k lies exactly on one arc ω_i (see Figure 4.15). If J_L^+ is not a Jordan arc, $G \setminus J_L^+$ may consist of several connected components (see Figure 4.15, right), one of them containing all disks associated with vertices v in V_L^-. We call this component G_L^- the *maximal lower domain* of G for L with respect to $\mathcal{P}$, and we set $G_L^+ := G \setminus \overline{G_L^-}$. For the sake of brevity we define $\omega := J_L^+$ and $\Omega := G_L^-$.

Since the curve ω can have multiple points (see Figure 4.15, right) there is no natural ordering of the points on ω. However, considering ω as part of the boundary of Ω, we can introduce an ordering of the associated prime ends. In order to describe this procedure we need the following result.

Lemma 4.14. *For any combinatorial crosscut L the maximal lower domain $\Omega = G_L^-$ is simply connected and has a locally connected boundary.*

Proof. Let G_0^- be the lower domain with respect to the polygonal crosscut J_0 in $\mathcal{P}$. Then $G \setminus J_L^0$ consists of two simply connected domains G_0^- and G_0^+, respectively. The maximal lower domain G_L^- is constructed by gluing a finite number of simply connected domains along straight line segments to G_0^-.

Hence the assertion follows from the fact that whenever two simply connected domains with locally connected boundaries G_1 and G_2 touch each other along a Jordan arc J with endpoints a and b, i.e., $G_1 \cap G_2 = \emptyset$ and $\overline{G}_1 \cap \overline{G}_2 = J$, then $\big(G_1 \cup J \cup G_2\big) \setminus \{a, b\}$ is a simply connected domain and its boundary is locally connected. $\square$

The assertion of Lemma 4.14 guarantees that there is a canonical parameterization $f : \mathbb{D} \to \Omega$ of Ω with a continuous extension to $\overline{\mathbb{D}}$, which we again denote by f (see [21] Theorem 2.1). With respect to f, we let $\sigma_i \subset \mathbb{T}$ denote the preimage of the circular arcs ω_i with $i = 1, \ldots, n$. Then $\sigma := \bigcup_{i=1}^{n} \sigma_i$ is the preimage of the maximal crosscut ω.

By the Prime End Theorem, the mapping f induces a bijection f^* between $\mathbb{T}$ and the set of prime ends of Ω. We denote by $\omega^* := f^*(\sigma)$ the set of prime ends associated with Ω, and for $i = 1, \ldots, n$ we let $\omega_i^* := f^*(\sigma_i)$ be the subsets of ω^* corresponding to the arcs σ_i. Note that the preimages σ_i of the circular arcs ω_i are topologically closed subarcs of $\mathbb{T}$, and that the preimage $\mathbb{T} \setminus \sigma$ of $\partial\Omega \setminus \omega$ is not empty. Therefore, σ_i and σ_j, and thus ω_i^* and ω_j^*, are disjoint if $|i - j| > 1$, while their intersection contains exactly one element if $|i - j| = 1$.

Furthermore, we see that the arcs $\sigma_1, \sigma_2, \ldots, \sigma_n$ (in this order) are arranged in clockwise direction on $\mathbb{T}$. It is therefore just natural to order the points on the arc σ (and hence on each subarc σ_i) also in clockwise direction. The mapping f^* transplants this ordering from σ to the set ω^* of prime ends. If $X_1^* = f^*(s_1)$ and $X_2^* = f^*(s_2)$ are two prime ends of ω^*, then the notion $X_1^* \preceq X_2^*$ refers to the ordering $s_1 \preceq s_2$ of the associated points on σ.

Remark. Every ω_i without its endpoints is an open Jordan arc. So the interior points of ω_i and σ_i corresponds one-to-one. Let γ in Ω be an open Jordan arc with terminal point q on ω. Then the associated unique prime end $X \in \omega^*$ must lie in ω_i^* whenever q is an interior point of ω_i. Only if q is an endpoint of ω_i, then there is a chance that the prime end X is not contained in ω_i^* since now X depends on how γ approaches q.

4.4.2. Loners, the Definition

So far we have studied properties of a single circle packing $\mathcal{P}$. In the next step we consider pairs $(\mathcal{P}, \mathcal{P}')$ of packings that are subject to the assumptions of Theorem 4.13.

Definition 4.10. A pair $(\mathcal{P}, \mathcal{P}')$ of circle packings for the complex K is said to be *admissible* (for the crosscut L of K in G with alpha vertex a) if it satisfies the following conditions (i)–(iii):

(i) The packing $\mathcal{P}$ fills the bounded, simply connected domain G, and the packing $\mathcal{P}'$ is contained in G.

(ii) For all vertices $v \in U_L^-$ (the lower neighbors of L) the disks D_v' are contained in G_L^- (the maximal lower domain of G for L with respect to $\mathcal{P}$).

(iii) The centers of the alpha disks of $\mathcal{P}$ and $\mathcal{P}'$ coincide and lie in $G_L^+ = G \setminus \overline{G_L^-}$.

Though it would be more precise to speak of an admissible sixtuple $(K, L, G, \mathcal{P}, \mathcal{P}', a)$, we shall use the term "admissible" generously, for instance saying that "L is an admissible crosscut for $(\mathcal{P}, \mathcal{P}')$".

Recall that U_L^+ denotes the vertex set of those disks in $\mathcal{P}$ that lie in G_L^+ and touch the crosscut ("upper neighbors of L"). In the next step we are going to explore the interplay of the disks D_v' in $\mathcal{P}'$ and D_w in $\mathcal{P}$ for $v, w \in U_L^+$.

Definition 4.11. Let $(\mathcal{P}, \mathcal{P}')$ be an admissible pair of circle packings for the complex K with crosscut L. A vertex v in U_L^+ is called a *loner* if $D_v' \cap D_w = \emptyset$ for all $w \in U_L^+$ with $w \neq v$.

The concept of loners is similar to what we did in Section 3.4.2, but is also somewhat different. The main characteristic of a loner is the following property.

Lemma 4.15. *Let v in U_L^+ be a loner of the admissible pair $(\mathcal{P}, \mathcal{P}')$ with complex K and crosscut L. Then $D_v' \cap (G_L^+ \setminus D_v) = \emptyset$.*

Proof. Let $u \in U_L^-$ and $w \in U_L^+$ be neighbors of v, and let p and q be the contact points of the disks D_v' with D_u' and D_v with D_w, respectively. Clearly $p \neq q$ since otherwise D_u' had to intersect D_v or D_w, what is a contradiction to condition (ii) of the admissible pair $(\mathcal{P}, \mathcal{P}')$.

Assume that p is a boundary point of D_v. Then ∂D_v and $\partial D'_v$ have a common tangent at p since otherwise D'_u had to intersect D_v, what is a contradiction to condition (ii) of the admissible pair $(\mathcal{P}, \mathcal{P}')$. It follows that either $\overline{D'_v} \setminus \{p\} \subset D_v$ or $D'_v = D_v$ or $\overline{D_v} \setminus \{p\} \subset D'_v$. The latter implies that $q \in D'_v$, hence $D'_v \cap D_w \neq \emptyset$, which is impossible since v is a loner. The other two cases imply the statement we want to prove.

Assume that p is not a boundary point of D_v. Suppose that the assertion of Lemma 4.15 were false, i.e., there is some point r in D'_v that is also contained in $G_L^+ \setminus D_v$. Because p lies in the maximal lower domain G_L^-, and r lies in the upper domain G_L^+, the boundary of D'_v must intersect the maximal crosscut J_L^+. Since the vertex v is a loner, every such intersection point must lie in ∂D_v. If $\partial D'_v \cap \partial D_v$ consists of exactly one point r_1, then the boundary of D'_v is the union of $\delta[p, r_1]$ and $\delta[r_2, p]$, hence $D'_v \cap G_L^+ = \emptyset$, what is a contradiction to $r \in D'_v$. If there is a second point $r_2 \in \partial D'_v \cap \partial D_v$ with $r_1 \neq r_2$, then we have $\partial D'_v \cap D_v = \delta(r_2, r_1)$, hence r must be contained in D_v, what is a contradiction to $r \in (G_L^+ \setminus D_v)$. □

In the upcoming Section 4.4.6, the property of loners described in Lemma 4.15 will allow us to move the crosscut L through the packing, reducing in every step the number of circles in G_L^+. The next result is crucial for the applicability of this procedure.

Lemma 4.16 (Existence of loners)**.** *Every admissible pair $(\mathcal{P}, \mathcal{P}')$ of circle packings with crosscut L has a loner.*

The proof is divided into several steps; the first part uses the geometry of disks, then we employ some topology, and finally everything is reduced to pure combinatorics. We start with some preparations.

Recall the definition of the contact points x_k: If $L = (e_0, \ldots, e_l)$ and $e_k = \langle u, v\rangle$ for some $k \in \{0, \ldots, l\}$, then $x_k := \overline{D_u} \cap \overline{D_v}$. Using the same notation, the corresponding contact points of disks in $\mathcal{P}'$ are given by $y_k := \overline{D'_u} \cap \overline{D'_v}$, where $\mathcal{Y} := \{y_0, \ldots, y_l\}$ is the set of all such contact points.

The contact points x_k form an ordered set on the maximal crosscut $\omega = J_L^+$, which is the upper boundary of the maximal lower domain $\Omega = G_L^-$. Since every x_k lies on exactly one arc ω_i, the set $\mathcal{X}$ of contact points splits into classes $X_i := \{x_k \in \mathcal{X} : x_k \in \omega_i\}$ for $i = 1, \ldots, n$. The set $\mathcal{Y}$ of the contact points of $\mathcal{P}'$ is divided accordingly into the classes $Y_i := \{y_k \in \mathcal{Y} : x_k \in \omega_i\}$ (the x_k is no typo here). Like $\mathcal{X}$, the set $\mathcal{Y}$ is endowed with a natural ordering; we write $y_j \prec y_k$ if $j < k$.

Our next aim is to construct a Jordan arc α that is contained in $\overline{\Omega}$ and carries the contact points y_k in their natural order.

Lemma 4.17. *If $(\mathcal{P}, \mathcal{P}')$ is an admissible pair, then there is an oriented Jordan arc α_k from y_{k-1} to y_k such that $\alpha := \cup_{k=1,\ldots l}\alpha_k$ is a Jordan arc in $\overline{\Omega}$ and $\alpha \cap \omega \subset Y$.*

Proof. Let $k \in \{1, \ldots, l\}$. In order to determine the arc α_k of α that connects y_{k-1} with y_k, we remark that both points lie on the boundary of one and the same disk $D'_v \in \mathcal{P}'$. We distinguish two cases:

Case 1. If $v \in V_L^-$, then the disk D'_v is contained in Ω, and we choose the segment $\alpha_k := [y_{k-1}, y_k]$ (see Figure 4.16, left).

Case 2. If $v \in V_L^+$, then e_{k-1}, e_k and a third edge $\langle u, w\rangle$ of K form a face of K and the (neighboring) disks D'_u and D'_w are both contained in Ω. So we let $z_k := \overline{D'_u} \cap \overline{D'_w}$ and connect y_{k-1} with y_k by $[y_{k-1}, z_k] \cup [z_k, y_k] \subset \overline{\Omega}$ (see Figure 4.16, middle).

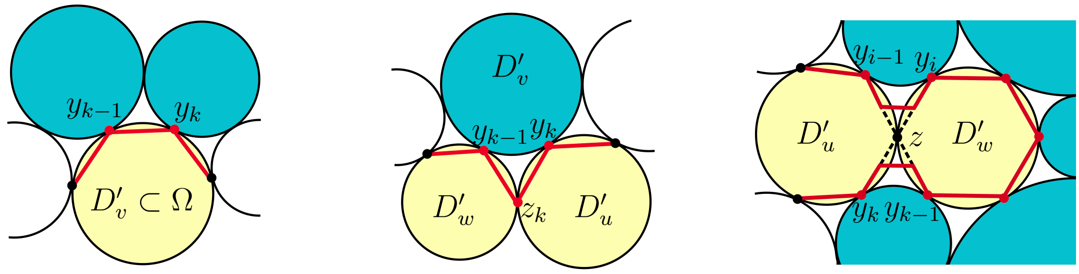


Fig. 4.16.: Construction of the Jordan arc α in Case 1 (left) and Case 2 (middle, right)

It is clear that all open segments (y_{k-1}, y_k), (y_{k-1}, z_k), (z_k, y_k) for $k = 1, \dots, l$ are pairwise disjoint, and that $y_k \neq z_j$. However, it is possible that two endpoints z_k and z_j coincide for $j \neq k$, in which case the concatenation of the arcs α_k is not a Jordan arc.

If this happens, the point $z := z_j = z_k$ is the contact point of two disks D'_u and D'_w with $u, w \in V_L^-$. A little thought shows that then z can neither lie on the boundary of G nor on ω, and hence it must be an interior point of Ω. This allows one to resolve the double point of α at z without destroying its other properties (see Figure 4.16, right.) $\square$

In the next step, we transform the existence of loners to a topological problem. Technically, this is much simpler when α and ω are disjoint. We consider this "regular case" in Section 4.4.3. The "critical case", where intersections of α and ω are admitted, will be treated in Section 4.4.4.

4.4.3. Loners, the Regular Case

Here, we assume that $\alpha \cap \omega = \emptyset$, which implies that all contact points y_k for $k = 0, \dots, l$ lie in the lower domain Ω.

We fix $i \in \{1, \dots, n\}$ and denote by y_i^- and y_i^+ the smallest and the largest member of Y_i with respect to the natural ordering of $\mathcal{Y}$, respectively. Both points (which may coincide), as well as all elements of Y_i, lie on the same circle $\partial D'_v$, associated with a vertex $v = v(i) \in V$.

Let δ'_i be the negatively oriented topologically closed subarc of $\partial D'_v$ from y_i^- to y_i^+. We consider the largest subarcs ν_i and π_i of δ'_i that are contained in $\overline{\Omega} \setminus \omega$ and have initial points y_i^- (for ν_i) and y_i^+ (for π_i), respectively (see Figure 4.17).

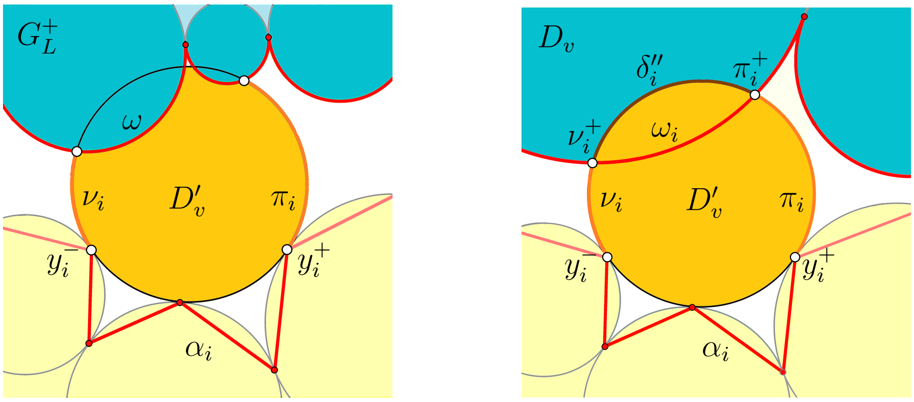


Fig. 4.17.: The arcs ν_i and π_i and their intersection with the boundary of G_L^+

Lemma 4.18. *If there is no loner, then the terminal points ν_i^+ and π_i^+ of ν_i and π_i, respectively, lie on ω for $i = 1, \ldots, n$.*

Proof. If one of the arcs ν_i or π_i does not intersect ω, then both coincide with δ'_i. In this case, the disk $D'_{v(i)}$ is separated from G_L^+ by the union of the arcs α and δ'_i, which implies that $D'_{v(i)}$ cannot intersect any disk D_w with $w \in U_L^+$, so that $v(i)$ is a loner. □

Since (with the exception of their endpoints) the circular arcs ν_i for $i = 2, \ldots, n$ and π_i for $i = 1, \ldots, n-1$ lie in Ω and have terminal points ν_i^+ and π_i^+ on ω, they define prime ends ν_i^* and π_i^* in ω^*. Because the arcs ν_1 and π_n need not lie in Ω, a modified definition is needed for the prime ends ν_1^* and π_n^*. In order to do so, we replace ν_1 and π_n by slightly perturbed circular arcs ν_1^ε and π_n^ε, respectively, that have the same endpoints as ν_1 and π_n, respectively, and lie in Ω (with the exception of their endpoints). Then ν_1^* and π_n^* are defined as the prime ends associated with the terminal points of ν_1^ε and π_n^ε, respectively. Clearly, such arcs ν_1^ε and π_n^ε exist, and for all sufficiently small ε they define the same prime ends $\nu_1^*, \pi_n^* \in \omega^*$, respectively.

Since we endowed the set of prime ends ω^* with a (clockwise) ordering, we can compare the prime ends ν_i^* and π_i^*.

Lemma 4.19. *If $(\mathcal{P}, \mathcal{P}')$ has no loner, the prime ends ν_i^* and π_i^* form an interlacing sequence with respect to the prime end ordering of ω^*, i.e.*

$$\nu_1^* \preceq \pi_1^* \preceq \nu_2^* \preceq \pi_2^* \preceq \ldots \preceq \nu_n^* \preceq \pi_n^*.$$

Proof. Let $y_- := y_0$ and z_- be the initial and terminal points of ν_1, while $y_+ := y_l$ and z_+ are the initial and terminal points of π_n, respectively. We have $z_-, z_+ \in \omega$ due to Lemma 4.18.

Further, let ω_0^* be the set of all prime ends X of ω^* with $\nu_1^* \preceq X \preceq \pi_n^*$, and denote the set of all corresponding points on ω by ω_0. The set ω_0 is a curve or a single point. Together with the Jordan arcs ν_1, α and π_n it forms the boundary of a simply connected domain $\Omega_0 \subset \Omega$ with locally connected boundary. Let Ω_0^* be the set of all prime ends associated with points on $\partial\Omega_0$. Because $\Omega_0 \setminus \omega_0$ is an open Jordan arc, the points y_- and y_+ are associated with uniquely determined prime ends y_-^* and y_+^* of Ω_0, respectively.

Contrary to this, the points z_- and z_+ may be associated with several prime ends of Ω_0. In order to explain which one we choose, let again ν_1^ε and π_n^ε be small perturbations (as explained above) of ν_1 and π_n, respectively, so that both arcs are crosscuts in Ω_0. We define z_-^* and z_+^* as the prime ends in ω^* associated with the terminal points z_- and z_+ of ν_1^ε and π_n^ε, respectively.

We have $n > 1$ since otherwise a loner would exist. It follows that $y_- \neq y_+$, so $y_-^* \neq y_+^*$. From $\alpha \cap \omega = \emptyset$ we get $z_-, z_+ \notin \{y_-, y_+\}$, hence $z_-^*, z_+^* \notin \{y_-^*, y_+^*\}$.

If $z_-^* = z_+^* =: z^*$, then we directly get $(\omega^* \cap \partial\Omega_0^*) = z^*$. This implies $\nu_1^* = \pi_1^* = \nu_2^* = \ldots = \pi_n^* = z^*$, so the lemma holds true. (We consider this case here, though Lemma 4.20 shows that it cannot occur.)

If $z_-^* \neq z_+^*$, then the prime ends y_-^*, y_+^*, z_-^* and z_+^* are pairwise distinct and we have $y_+^* \prec y_-^* \prec z_-^* \prec z_+^* \prec y_+^*$ with respect to the (cyclic, counter-clockwise) ordering of $\partial\Omega_0^*$. Therefore, Ω_0 can be mapped conformally onto a rectangle Q (with appropriately chosen aspect ratio) so that y_+^*, y_-^*, z_-^* and z_+^* correspond to the four vertices of Q (see [21]), which is depicted in Figure 4.18.

Any of the arcs ν_i for $i = 2, \ldots, n$ and π_i for $i = 1, \ldots, n-1$ is mapped onto a crosscut of Q that connects two opposite sides of this rectangle. Since these Jordan arcs cannot cross each other in the interior of Q, the ordering of their initial points on one side of Q is transplanted to the ordering of their terminal points on the opposite side of Q. Translated back to Ω_0, this implies that the ordering of the prime ends ν_i^* and π_i^* is the same as the ordering of the initial points y_i^- and y_i^+ of ν_i and π_i, respectively, along the Jordan curve α. By construction, the latter points form an interlacing sequence. □

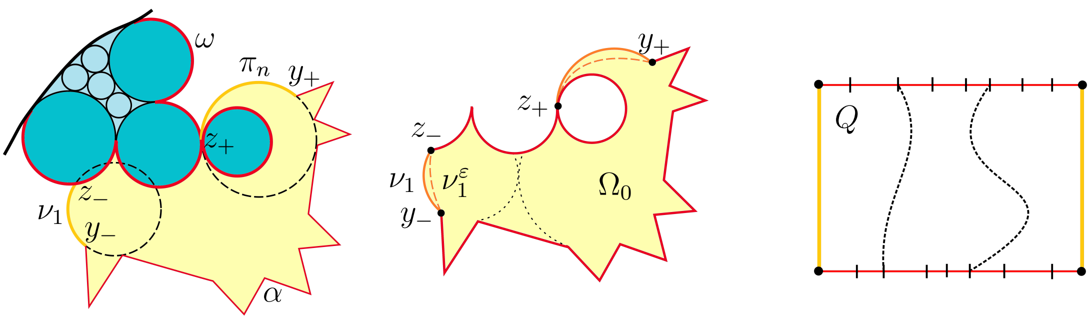


Fig. 4.18.: Construction of Ω_0 and Q from ω, α and ν_1, π_n

Lemma 4.20. *If both prime ends ν_i^* and π_i^* belong to ω_i^*, then the corresponding vertex $v(i)$ is a loner.*

Proof. Let $v := v(i)$. It follows from $\nu_i^*, \pi_i^* \in \omega_i^*$ that $\nu_i^+, \pi_i^+ \in \omega_i \subset \partial D_v$. If $\pi_i^+ \neq \nu_i^+$, then the positively oriented open subarc δ_i'' of P_v' from π_i^+ to ν_i^+ lies in D_v. If $\pi_i^+ = \nu_i^+$, then we set $\delta_i'' := \emptyset$. In both cases, the union of $\alpha_i, \pi_i, \delta_i''$ and ν_i is a Jordan curve that does not intersect the disks D_u with $u \in U_L^+$ and $u \neq v$. So either D_v' is disjoint to all such disks D_u, or one of the disks D_u is contained in D_v'. In the latter case, the prime ends ν_i^* and π_i^* cannot both belong to the same set ω_i^*. □

Proof of Lemma 4.16. After these preparations we are ready to harvest the fruits: Assume that $(\mathcal{P}, \mathcal{P}')$ has no loner. Then, by Lemma 4.18, the endpoint ν_i^+ of the arc ν_i must lie on ω and hence ν_i is associated with a prime end $\nu_i^* \in \omega^*$. If $\nu_i^* \in \omega_k^*$, we choose the smallest such k and set $l(i) := k$. Similarly, we denote by $r(i)$ the smallest number k for which $\pi_i^* \in \omega_k^*$.

Lemma 4.19 tells us that $l(i) \leq r(i) \leq l(i+1)$. In conjunction with Lemma 4.20, we conclude that the first condition implies $r(i) \geq l(i) + 1$. Starting with $l(1) \geq 1$, we get inductively that $r(i) \geq i + 1$ for $i = 1, \ldots, n$, ending up with the contradiction $r(n) \geq n + 1$. This proves Lemma 4.16 in the regular case. □

4.4.4. Loners, the Critical Case

The second case, where we admit that $\alpha \cap \omega \neq \emptyset$, will be reduced to the regular case by an appropriate deformation of the Jordan arc α.

Definition 4.12. A contact point $y \in \mathcal{Y}$ is called *regular* if $y \notin \omega$, otherwise it is said to be *critical.*

If $y \in \mathcal{Y}$ is a critical contact point, then $y \in \alpha \cap \omega \neq \emptyset$, and hence $y \in \omega_j$ for some j. Since $y = \partial D_u' \cap \partial D_v'$ with some $u \in U_L^-$ and $v = v(i) \in U_L^+$, we see that y cannot be an endpoint of ω_j (turning point of ω) since otherwise D_u' would not be contained in Ω. Moreover, the circles $\partial D_u'$, $\partial D_v'$, and ω_j must be mutually tangent at y. The arc ω_j is a subset of the circle ∂D_w (with $w = v(j) \in U_L^+$). Hence, either $D_v' \subset D_w$ (with $D_v' = D_w$ admitted) or D_w is a proper subset of D_v'.

In the next step we modify the Jordan arc α in a neighborhood of y and redefine the arcs ν_i and π_i (connecting y with ω) introduced in the regular case.

Let ε be a sufficiently small positive number. Denote by z the ε-shift of y in the direction of the center of D_u'. Append to D_v' an equilateral open triangular domain T with one vertex at z, two vertices on $\partial D_v'$, and symmetry axis through y and z (see Figure 4.19).

For $y \notin \{y_0, y_l\}$ let ν_i (and π_i) be the largest negatively (positively) oriented subarc of $\partial(D_v' \cup T)$ that has initial point z and is contained in Ω. For $y \in \{y_0, y_l\}$ (and only then) it can happen that y is a boundary point of G. Therefore, we define $\nu_i := [z, y]$ in the case $y = y_0$, and $\pi_i := [y, z]$ in the case $y = y_l$. The case $y_0 = y_l$ can never occur since $l \geq 1$.

Denote by ν_i^+ and π_i^+ the terminal points of ν_i and π_i. Clearly, $\nu_i^+, \pi_i^+ \in \omega$. So let $\nu_i^*, \pi_i^* \in \omega^*$ be their associated prime ends.

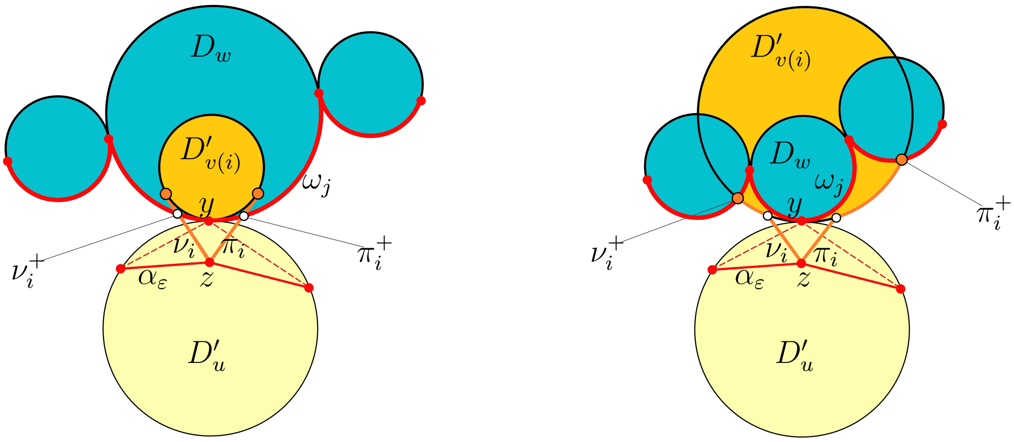


Fig. 4.19.: Modification of α and definition of the arcs ν_i and π_i for critical contact points y

We see that the statement of Lemma 4.18 holds in the critical case, too. Moreover, the Lemma 4.19 can be proved for the critical case in exactly the same way as for the regular case; we just have to apply the adapted definitions of ν_i^* and π_i^*. All what is missing is the following "critical" version of Lemma 4.20.

Lemma 4.21. *Assume that $\partial D'_v$ with $v = v(i) \in U_L^+$ contains a critical contact point $y \in \mathcal{Y} \cap \omega$. Then v is a loner if and only if ν_i^* and π_i^* belong to ω_i^*.*

Proof. We use the notations introduced above with $\varepsilon > 0$ fixed and sufficiently small. We distinguish two cases.

Case 1. Let $D'_v \subset D_w$ (see Figure 4.19, left). Then v is a loner if and only if $w = v$, and this holds true if and only if $j = i$ and $\nu_i^*, \pi_i^* \in \omega_i^*$.

Case 2. Let $D_w \subset D'_v$ and $D_w \neq D'_v$ (see Figure 4.19, right). Then D'_v intersects at least two "upper" disks (namely D_w and one of its neighbors), so that v is not a loner. According to our construction, we have $\nu_i^* \preceq y^* \preceq \pi_i^*$ (where $y^* \in \omega_j^*$ is the prime end corresponding to y and $w = v(j)$), but both equalities are never fulfilled at the same time, and $\nu_i^*, \pi_i^* \notin \omega_j^*$ for $w = v(j)$. Therefore, we have $\nu_i^* \in \omega_m^*$ and $\pi_i^* \in \omega_n^*$ with $m \leq j \leq n$, but $m < n$, so the prime ends ν_i^* and π_i^* cannot both belong to the same class ω_i^*. □

Remark. If D'_v has several critical contact points $y \in \mathcal{Y} \cap \omega_j$ with the same arc ω_j, then D'_v must be tangent to D_w with $w = v(j)$ at two different points. This implies that $D'_v = D_w$, which explains why the criterion is independent of the choice of y.

After replacing all critical contact points y_k by the shifted points z_k and after modifying the construction of the curve α accordingly, the Lemma 4.16 can be proved completely the same way as in the regular case.

In the next section, we need the following generalization of Lemma 4.16. We point out that $v(i) = v(j)$ is allowed in assertion (i).

Lemma 4.22. *Let $D_{v(i)} = D'_{v(i)}$ and $D_{v(j)} = D'_{v(j)}$ with $1 \leq i \leq j \leq n$. Then, in each of the following cases (i)-(iii), there is a loner $v(k)$ that is different from $v(i)$ and $v(j)$ so that k satisfies the following conditions:*

(i) if $1 \leq i < j-1 \leq n-1$, then $i < k < j$,

(ii) if $i > 1$, then $1 \leq k < i$,

(iii) if $j < n$, then $j < k \leq n$.

Proof. The proof differs only slightly from the proof of Lemma 4.16. For example, in order to prove (i), we need only replace the first inequality $l(1) \geq 1$ by $l(i+1) \geq i+1$ (which follows from $D_{v(i)} = D'_{v(i)}$) and, assuming that no loner $v(k)$ with $i < k < j$ exists, proceed inductively for $k = i+1, \ldots, j$ until we arrive at $r(j) \geq j+1$. The last condition contradicts $D_{v(j)} = D'_{v(j)}$.

If $v(k) = v(i)$ or $v(k) = v(j)$, we repeat the procedure, replacing i (in the first case) or j (in the second case) by k, respectively. Iterating this a number of times, if necessary, we eventually find a loner $v(k)$ that is different from $v(i)$ and $v(j)$ since for all $m = 2, 3, \ldots, n-1$ we have $v(m-1) \neq v(m)$ and $v(m) \neq v(m+1)$. $\square$

4.4.5. Structure of Upper Neighbors

In this section, we analyze the structure of the set of upper neighbors U_L^+ and its subset of loners in more detail.

Two consecutive (non-oriented) edges e_{j-1} and e_j of $L = (e_0, \ldots, e_l)$ can be represented as $e_{j-1} = e(u, v)$ and $e_j = e(v, w)$. The third edge of the face $f(u, v, w)$ is considered as oriented from u to w, and we set $e_j^0 := \langle u, w \rangle$. The set of edges e_j^0 splits into two classes. We define E_L^- as the set of those e_j^0, where the face $\langle u, v, w \rangle$ is oriented clockwise, whereas E_L^+ consists of those edges with counter-clockwise orientation of $\langle u, v, w \rangle$. After renumbering the elements of E_L^- and E_L^+ without changing their order, we get two sequences of oriented edges $E_L^- = \{e_1^-, \ldots, e_p^-\}$ and $E_L^+ = \{e_1^+, \ldots, e_q^+\}$ with $p + q = l$, which are called the *sequences of lower* and *upper accompanying edges* of the crosscut L, respectively.

Here are some basic properties of E_L^- and E_L^+, which follow quite easy from the definition of L (proofs are left as exercises). The oriented edges in $E_L^- \cup E_L^+$ are pairwise disjoint. The corresponding non-oriented edges can appear at most twice, and either both in E_L^- or both in E_L^+. Two consecutive edges $e_{j-1}^\pm$, $e_j^\pm$ are linked at a common vertex. The vertex set of all edges in E_L^+ is precisely the set U_L^+ of upper neighbors of L.

Figure 4.20 shows two examples. The involved crosscut on the right models the fourth generation of the Hilbert curve. There, with the exception of boundary edges, all edges in E_L^- (lighter color) and in E_L^+ (darker color) appear with both orientations (not shown in the picture).

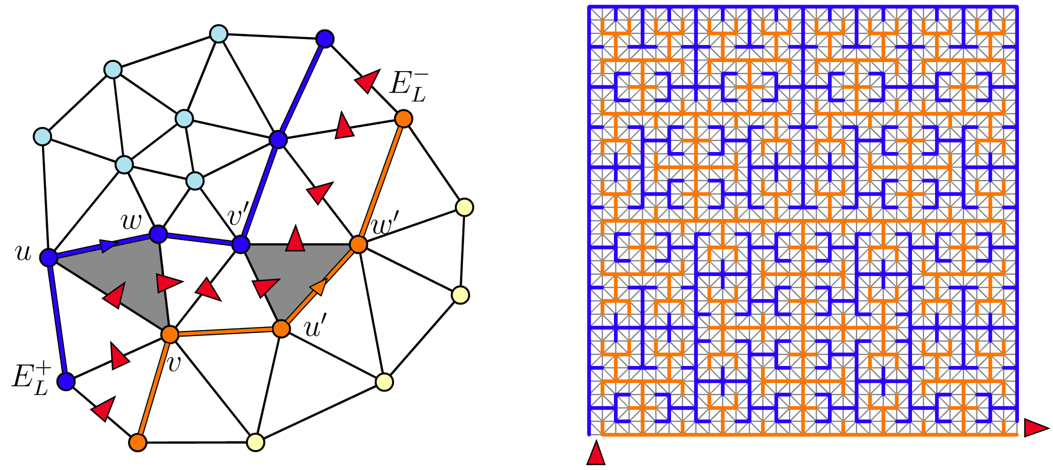


Fig. 4.20.: The upper and the lower accompanying edges of a crosscut

When we arrange the elements of U_L^+ in the order they are met along the edge path E_L^+, we get the *sequence* S_L^+ *of upper accompanying vertices*. A similar definition is made for the *sequence* S_L^- *of lower accompanying vertices*. The geometry of circle packings causes some combinatorial obstructions for these sequences.

Lemma 4.23. *The sequence* S_L^+ *of upper accompanying vertices cannot contain the pattern* $(\ldots, u, \ldots, v, \ldots, u, \ldots, v, \ldots)$ *with* $u \neq v$.

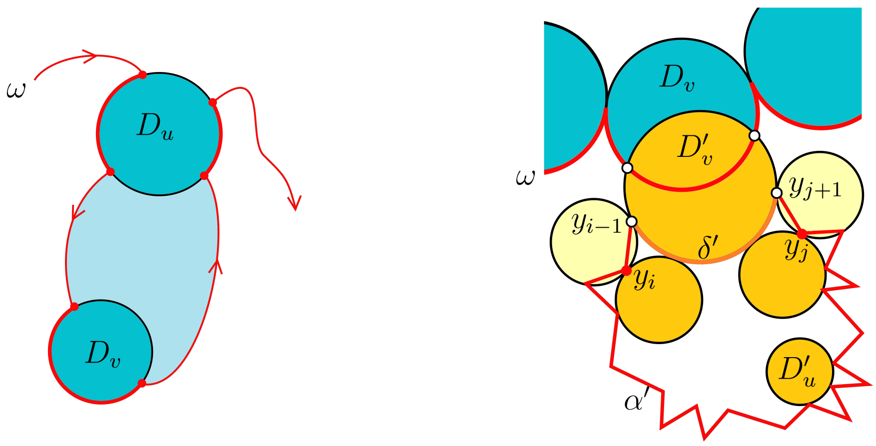


Fig. 4.21.: Illustrations to Lemma 4.23 and Lemma 4.25

Proof. If the sequence S_L^+ contains the pattern $(\ldots, u, \ldots, v, \ldots, u, \ldots)$, the oriented curve ω has three subarcs ω_i, ω_j and ω_k with $i < j < k$ so that $\omega_i, \omega_k \subset \partial D_u$ and $\omega_j \subset \partial D_v$. But then ω cannot contain a subarc of $\partial D_v \setminus \omega_j$ (see Figure 4.21, left), which would be necessary to append another v to the sequence. $\square$

Definition 4.13. A vertex $v \in U_L^+$ that appears only once in the sequence S_L^+ is called *simple*. The other elements in U_L^+ are said to be *multiple* vertices.

If v is a multiple vertex in U_L^+, then there are sequences $M := \{e_i^+, e_{i+1}^+, \dots, e_j^+\} \subset E_L^+$ of accompanying edges so that v is the initial vertex of e_i^+ as well as the terminal vertex of e_j^+ with $i < j$. Any such sequence is called a *loop* for v. We say that a loop M *meets a vertex* $u \in U_L^+$ if u is adjacent to an edge in M and $u \neq v$. The *set of vertices* met by M is denoted by V_M. A loop M also generates a *sequence of vertices* $U_M = \{v, v_1, \dots, v_m, v\}$, when we arrange the elements of V_M in the order they are met along the edge path M.

Lemma 4.24. *Every loop M of a multiple vertex v meets a simple vertex u.*

Proof. We consider the sequence $U_M = \{v, v_1, \dots, v_m, v\}$ of vertices in V_M arranged in the order as they are met by the edge path M. Let w denote the element of this sequence with the earliest second appearance (this does not mean the first element that appears twice). Since w cannot appear twice in direct succession, there is a vertex u in between the first two symbols w.

In order to show that u is a simple vertex, we remark that U_M is a sub-sequence of the sequence S_L^+ of upper accompanying vertices. By definition of w, there cannot be a second u in S_L^+ between the two symbols w next to u, and by Lemma 4.23, the sequence S_L^+ cannot contain a second u outside these two w's. $\square$

Since loners are vertices in U_L^+, it makes sense to speak of simple and multiple loners.

Lemma 4.25. *Let v be a multiple loner with $D'_v \neq D_v$. If $u \neq v$ is a vertex met by a loop of v, then u is a loner and $D'_u \cap D_u = \emptyset$.*

Proof. Let M be a loop of v with $U_M = \{v, v_1, \dots, v_m, v\}$. Let i be the smallest index so that y_i is a contact point of v_1, and let j be the largest index so that y_j is a contact point of v_m. According to the ordering of Y and U_M (as subsequences of S_L^+), y_{i-1} and y_{j+1} are contact points of D'_v. Let $u \in \{v_1, \dots, v_m\}$ with $u \neq v$.

The disk D'_u is enclosed by the union of the subarc $\delta' := \delta[y_{i-1}, y_{j+1}]$ of D'_v and the subarc $\alpha' \subset \alpha$ that connects the points y_{i-1} and y_{j+1} on α (see Figure 4.21). Since v is a loner with $D'_v \neq D_v$, it is clear that $y_{i-1}, y_{j+1} \notin D_v$, and hence either $D'_v \cap D_v = \emptyset$ or $\partial D'_v \cap \partial D_v$ consists of one or two points. In every case, δ' does not intersect D_v. Therefore, the union $\alpha' \cup \delta'$ is contained in $\overline{\Omega}$, hence u is a loner. In particular $D'_u \cap D_u = \emptyset$, which proves the last assertion. $\square$

Combining Lemma 4.16, Lemma 4.22 (applied recursively), Lemma 4.24 and Lemma 4.25 (applied recursively), the essence of the sections 4.4.2 to 4.4.5 can be summarized in the following lemma.

Lemma 4.26. *Let $(\mathcal{P}, \mathcal{P}')$ be an admissible pair of circle packings with crosscut L.*

(i) The pair $(\mathcal{P}, \mathcal{P}')$ contains a simple loner $v \in U_L^+$.

(ii) Every loop of a multiple loner v meets a simple loner u and if $D'_v \neq D_v$ then $D'_u \neq D_u$.

4.4.6. Crosscut Reduction

We are almost in a position to prove Theorem 4.13. The idea is to use the concept of loners and combinatorial surgery to modify the crosscut L. In every step of this procedure, the number of vertices in V_L^+ will be reduced. At the end, we get a special combinatorial structure, which is called a *slit*. Roughly speaking, this is a chain of vertices connecting the alpha vertex with a boundary vertex. We shall prove that the disks of both packings coincide along a slit.

Then a subdivision procedure generates a sequence of slits so that any accessible boundary vertex appears among their end points. So we get $D'_v = D_v$ for all accessible $v \in \partial V$ and finally a well-known theorem tells us that $D'_v = D_v$ for all accessible $v \in V$.

To begin with, we describe how a simple vertex $v \in U_L^+$ can be "shifted" from V_L^+ to V_L^- so that we get a new crosscut L' with $\left|V_{L'}^+\right| < \left|V_L^+\right|$. Depending on the properties of v, we distinguish three cases.

Case 1. Let $v \in U_L^+$ be a simple interior vertex.

Case 2. Let $v \in U_L^+$ be a simple boundary vertex, and assume that neither the initial nor the terminal edge of L are adjacent to v.

Case 3. Let $v \in U_L^+$ be a simple boundary vertex, and assume that either the initial or the terminal edge of L is adjacent to v.

Remark. The case where the initial *and* the terminal edge of L are adjacent to v cannot appear. Indeed, otherwise either v is a multiple vertex (which is not considered) or all edges adjacent to v must belong to L. The latter implies that v is the only vertex in V_L^+, which is not allowed.

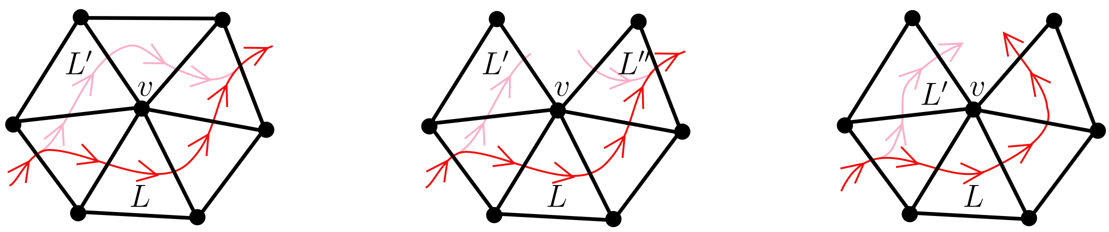


Fig. 4.22.: Modification of the crosscut L in Case 1 (left), Case 2 (middle) and Case 3 (right)

Reduction of Type 1. In order to modify the crosscut $L = (e_0, e_1, \ldots, e_l)$ in Case 1, we consider the flower $B = B(v)$ of v. Since v is simple, the set of edges adjacent to v consists of a sub-sequence $S = (e_i, \ldots, e_j)$ with $0 \le i \le j \le l$ of L and a complementary sequence, which we denote by $S' = (e'_1, \ldots, e'_k)$ with $k \ge 1$. Replacing in L the sequence S by S', we get a new edge sequence

$$L' = (e_0, \ldots, e_{i-1}, e'_1, \ldots, e'_k, e_{j+1}, \ldots, e_l).$$

The reader can easily convince herself (see Figure 4.22, left) that the sequence L' is a (combinatorial) crosscut for K with $|V_{L'}^+| < |V_L^+|$.

Reduction of Type 2. In Case 2 the flower of v is incomplete. Nevertheless, the edges in L adjacent to v form again a sequence of consecutive edges in this incomplete flower since v is simple. However, the local modification of L in a neighborhood of v described above does not result in a crosscut L' since the complementary sequence $S' = S_1' \cup S_2'$ consists of exactly two connected components $S_1' = (e_1', \ldots, e_k')$ and $S_2' = (e_1'', \ldots, e_m'')$ (see Figure 4.22, middle). Replacing in L the sequence S by S_1' or S_2', we get a new edge sequence L' or L'', respectively, with

$$L' = (e_0, \ldots, e_{i-1}, e_1', \ldots, e_k'), \qquad L'' = (e_1'', \ldots, e_m'', e_{j+1}, \ldots, e_l).$$

Both sequences L' and L'' are new crosscuts of K, but only one of it contains a among its upper vertices, say L', so we choose this one as the new crosscut. Clearly $|V_{L'}^+| < |V_L^+|$.

Reduction of Type 3. If either the initial or the terminal edge of L are adjacent to v, then the Type 1 reduction applied to the incomplete flower of v results in an admissible crosscut L', which has one vertex (namely v) less in $V_{L'}^+$ than in V_L^+ (see Figure 4.22, right).

Remark. No matter which type of reduction we used, the sets U_L^- and $U_{L'}^-$ of lower neighbors before and after the reduction, respectively, always fulfill $U_{L'}^- \setminus U_L^- = \{v\}$.

In order to not lose the normalization, we will only reduce vertices different from a. This leads to a situation where none of the above reductions can be applied, namely when a is the one and only simple vertex in U_L^+. This special case will be explored in the next section.

4.4.7. Slits

The next definition and the following lemma describe the situation when all but exactly one vertex of V are multiple.

Definition 4.14. A (combinatoric) *slit* of the complex $K(V, E, F)$ is a sequence $S = (v_1, v_2, \ldots, v_s)$ of vertices in V that satisfies the following conditions (i)–(iv):

(i) The vertices of S are pairwise different, i.e., $v_j \neq v_k$ if $1 \leq j < k \leq s$.

(ii) For $j = 1, \ldots, s-1$, the edges $e_j := e(v_j, v_{j+1})$ belong to E.

(iii) For $j = 1, \ldots, s$, the vertices v_{j-1} and v_{j+1} are the only neighbors of v_j in K that belong to S (where $v_0 := \emptyset$ and $v_{s+1} := \emptyset$).

(iv) The vertex v_1 is a boundary vertex and v_j are interior vertices for $j = 2, \ldots, s$.

The vertices v_1 and v_s are referred to as the *initial vertex* and the *terminal vertex* of S, respectively. The sequence $E_S := (e_1, \ldots, e_{s-1})$ (see (ii)) is said to be the *edge sequence* of S. Note that all e_j are interior edges.

Lemma 4.27. *Assume that the interior vertex v is the only simple vertex in U_L^+. Then the sequence of upper accompanying vertices S_L^+ has the symmetric form $(v_1, \ldots, v_{s-1}, v, v_{s-1}, \ldots, v_1)$ and $S = (v_1, \ldots, v_{s-1}, v)$ is a slit.*

Proof. By definition of a multiple vertex, any vertex in U_L^+ except v must appear at least twice in the sequence S_L^+. If there are vertices that show up twice *at a position left of* v, then we choose one, say u, whose appearances have minimal distance in the sequence $S_L^+ = (\ldots, u, \ldots, u, \ldots, v, \ldots)$. Since neighboring vertices of S_L^+ must be different, there is some $w \neq u$ so that $S_L^+ = (\ldots, u, \ldots, w, \ldots, u, \ldots, v, \ldots)$. Since v is assumed to be simple and w is a multiple vertex, we have $w \neq v$ and w must appear again at another place in S_L^+. By Lemma 4.23, this can only happen in between the two occurrences of u, which is in conflict with the minimal distance property of u.

Similarly, the assumption that there is a vertex that appears in S_L^+ twice at a position right of v, leads to a contradiction. Hence, with the only exception of v, any vertex of U_L appears in S_L^+ exactly once on either side of v. Applying Lemma 4.23 again, we see that the ordering of the vertices left of v must be reverse to the ordering on the right of v. So S_L^+ has the symmetric form stated in the lemma.

Moreover, we have shown that $v_1, \ldots, v_{s-1}, v$ are pairwise different, which is condition (i) of Definition 4.14. The second condition (ii) is trivial.

In order to verify condition (iv), it remains to show that v_j is an interior vertex for $j = 2, \ldots, s-1$ since v_1 is obviously a boundary vertex while $v_s := v$ is an interior vertex, by assumption. Assume v_j is a boundary vertex. The flower of v_j is incomplete and it is clear that v_{j-1} and v_{j+1} are neighbors of v_j. On the one hand, since (v_{j-1}, v_j, v_{j+1}) is a sub-sequence of S_L^+, the crosscut L must look locally like shown in Figure 4.23 left. On the other hand, the sub-sequence (v_{j+1}, v_j, v_{j-1}) of S_L^+ forces L to look locally like depicted in the middle of Figure 4.23, what contradicts our first observation. Hence, v_j must be an interior vertex and its flower must look qualitatively like shown in Figure 4.23 right.

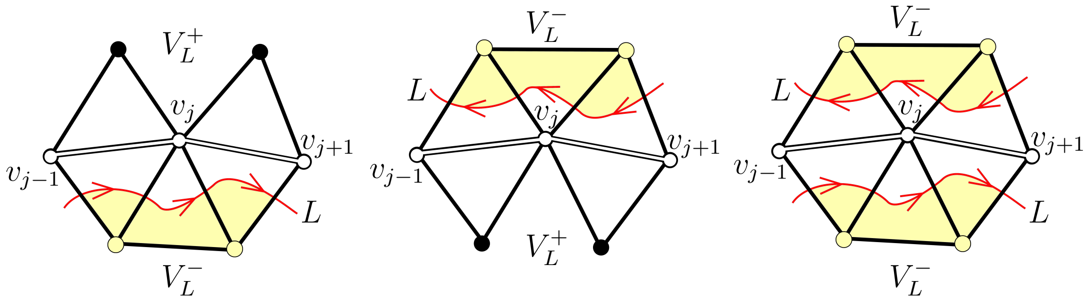


Fig. 4.23.: Illustrations for the proof of Lemma 4.27

In order to verify condition (iii), let $j \in \{2, \ldots, s-1\}$ be fixed. Looking at the behavior of the crosscut L in the flower of v_j it becomes clear that any edge $e(v_{j-1}, v_{j+1})$ (with the convention $v_s := v$) belonging to E must be contained in L twice, what is a contradiction. Furthermore, all other neighbors of v_j belong to V_L^- and hence not to

$V_L^+ \supset S_L^+$. A similar result can be derived by looking at the local behavior of L in the flower of v and the incomplete flower of v_1, now using the sub-sequences (v_{s-1}, v_s, v_{s-1}) and $(v_1, v_2, \ldots, v_2, v_1)$ of S_L^+, respectively. □

The following lemma explains why we are interested in slits.

Lemma 4.28. *Let $(\mathcal{P}, \mathcal{P}')$ be an admissible pair of circle packings for the complex K with crosscut L and alpha vertex a. Then there is a slit $S = (v_1, \ldots, v_s, a) \subset V_L^+$ with terminal vertex a so that $D'_v = D_v$ for all $v \in S$.*

Proof. To begin with, we invoke Lemma 4.26, which tells us that the pair $(\mathcal{P}, \mathcal{P}')$ has a simple loner v_λ. The idea is to use the reduction procedures of the last section to shift v_λ from V_L^+ to V_L^-, which results in a new crosscut L'.

As we remarked at the end of section 4.4.6, the one and only lower neighbor of L' that has not already been a lower neighbor of L is the simple loner v_λ. Therefore, Lemma 4.15 guarantees that L' is admissible for $(\mathcal{P}, \mathcal{P}')$. In order to find the appropriate type of reduction, we distinguish the following cases:

Case 1. There is a simple interior loner v_λ that is different from the alpha vertex a.

Case 2. There is a simple boundary loner v_λ.

Case 3. The only simple loner v_λ is the alpha vertex a.

In Case 1 we apply the reduction of Type 1. In Case 2 either the reduction of Type 2 or Type 3 can be applied, respectively, depending on whether v_λ is adjacent to the initial or the terminal edge of L, or not. In any case, we get a new combinatoric crosscut L' of K. Applying the reduction in Case 1 and Case 2 recursively as long as possible, the number of vertices in V_L^+ decays in every step at least by one. So we eventually arrive at Case 3.

Since the disks D'_a and D_a have the same centers, we either have one of the strict inclusions $D'_a \subset D_a$, $D_a \subset D'_a$ or $D'_a = D_a$. The first case cannot occur since otherwise all neighboring disks of D'_a would intersect D_a, what is a contradiction for those disks associated with a vertex in U_L^-. The second case clearly implies that a is not a loner. So the alpha vertex a is a loner if and only if $D'_a = D_a$. This implies, by Lemma 4.22, that there is another loner v_μ. Since a is the only simple loner, v_μ must be a multiple loner. If $D'_\mu \neq D_\mu$, then according to Lemma 4.26 (i), the vertex set V_M of any loop M of v_μ contains a simple loner, i.e. M meets a. Since we have $D'_a = D_a$, the assertion (ii) of this lemma tells us that $D'_\mu = D_\mu$.

Applying Lemma 4.22 and Lemma 4.26 repeatedly in this manner, we see that all vertices in $U_L^+ \setminus \{a\}$ must be multiple loners, and hence that $D'_v = D_v$ for all $v \in U_L^+$. Furthermore, a is the only simple vertex in U_L^+, so, by Lemma 4.27, we just constructed a slit $S \subset V_L^+$ with terminal vertex a. □

In the next step, we construct crosscuts from slits. To begin with, we introduce some more notations.

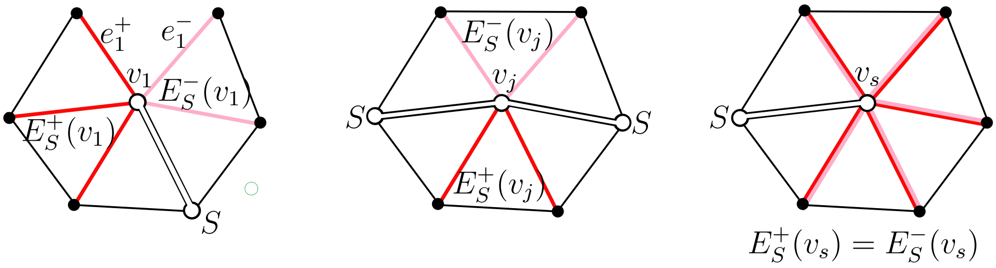


Fig. 4.24.: Left and right neighboring edges of vertices $v = v_1, v_j, v_s$ in a slit S

Let $S = (v_1, \ldots, v_s)$ be a slit. For any vertex v in S we define the subsets $E_S^-(v)$ and $E_S^+(v)$ of $E(v)$ as follows. For $v = v_1$, the (boundary) vertex v_1 has two adjacent boundary edges e_1^- and e_1^+ in $E(v_1)$ so that e_1^- is the predecessor of e_1^+ in the chain of boundary edges. The meaning of the inequalities in the following definitions is explained in Section 2.1.1. For the initial vertex v_1 we set (see Figure 4.24, left)

$$E_S^-(v_1) := \{e \in E(v_1) : e(v_1, v_2) \prec e \preceq e_1^-\},$$
$$E_S^+(v_1) := \{e \in E(v_1) : e_1^+ \preceq e \prec e(v_1, v_2)\}.$$

If $v = v_j$ with $j = 2, \ldots s-1$, then we define (Figure 4.24, middle)

$$E_S^-(v_j) := \{e \in E(v_j) : e(v_j, v_{j+1}) \prec e \prec e(v_{j-1}, v_j)\},$$
$$E_S^+(v_j) := \{e \in E(v_j) : e(v_{j-1}, v_j) \prec e \prec e(v_j, v_{j+1})\},$$

and for the terminal vertex v_s of S we let (see Figure 4.24, right)

$$E_S^-(v_s) = E_S^+(v_s) := \{e \in E(v_s) : e(v_{s-1}, v_s) \prec e \prec e(v_{s-1}, v_s)\}.$$

The edges in

$$E_S^- := \bigcup_{j=1}^{s-1} E_S^-(v_j) \text{ and } E_S^+ := \bigcup_{j=1}^{s-1} E_S^+(v_j)$$

are called the *left* and the *right neighbors of* S, respectively. Note that condition (iii) in Definition 4.14 guarantees that every edge e that is a neighbor of a slit S has exactly one adjacent vertex in S.

Lemma 4.29. *If $S = (v_1, \ldots, v_s, v)$ is a slit in K, then there is a (combinatoric) crosscut L so that $v \in S_L^+$ and that $S_L^- = (v_1, \ldots, v_{s-1}, v_s, v_{s-1}, \ldots, v_1)$ is the sequence of* lower accompanying vertices *of L.*

Proof. Walking along the slit S from v_1 to v_s and back to v_1, we build the crosscut L from the concatenation of the edge sequences

$$E_S^-(v_1),\ \ldots,\ E_S^-(v_s),\ e(v_s, v),\ E_S^+(v_s),\ \ldots,\ E_S^+(v_1).$$

It is easy to see that all edges in L are pairwise different. So L satisfies condition (i) of

Definition 4.9. Condition (ii) can easily be verified and (iv) is obvious. In order to prove (iii), we assume that three edges of L would form a face of K. Since these edges are neighbors of S, exactly one vertex of every edge must belong to S, which is impossible.

The construction also guarantees that the sequence S_L^- of lower accompanying edges of L has the desired form and that v belongs to S_L^+ (see, for example, Figure 4.25, left). $\square$

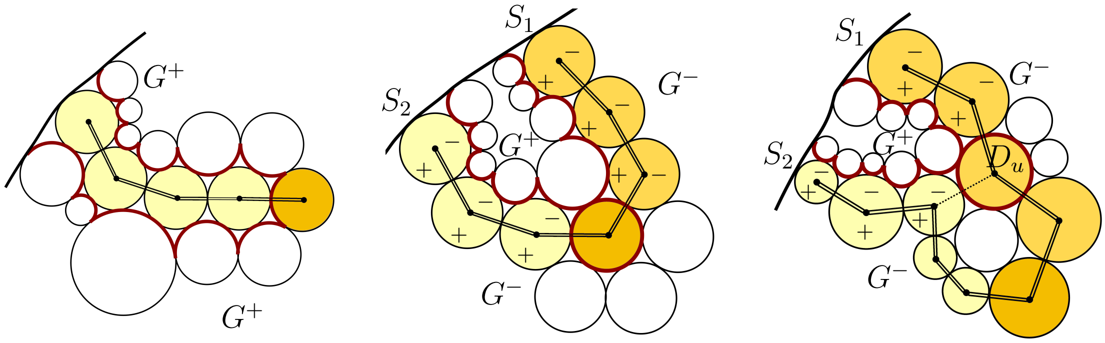


Fig. 4.25.: Constructing crosscuts from one slit (left) and two slits (middle, right)

A crosscut L can also be constructed from joining two slits S_1 and S_2 with a common terminal vertex v. This procedure is somewhat more complicated, in particular when the "right side" of S_1 is close to the "left side" of S_2. In those cases we cannot glue the cuts at their common terminal vertex v since then the resulting edge sequence L would contain some edges more than once. Instead, we modify the procedure by gluing S_1 and S_2 at some appropriately chosen vertex u in S_2 or S_1 that has a neighbor in S_1 or S_2, respectively. Figure 4.25 (middle, right) illustrates the result, showing an associated circle packing and the related maximal crosscuts.

Lemma 4.30. *Let $S_1 = (v_1, \ldots, v_t, v)$ and $S_2 = (w_1, \ldots, w_s, v)$ be slits in K with $S_1 \cap S_2 = \{v\}$. Assume further that $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$. Then there is a combinatoric crosscut L and a vertex $u \in (S_1 \cup S_2) \cap U_L^+$ so that*

$$S_L^- = \big(w_1, w_2, \ldots, w_\sigma, u_1, \ldots, u_k, v_\tau, v_{\tau-1}, \ldots, v_1\big), \qquad 1 \le \tau \le t,\ 1 \le \sigma \le s, \tag{4.1}$$

where $\big(w_\sigma, u_1, \ldots, u_k, v_\tau\big)$ is a (positively oriented) chain of neighbors of u.

Note that the condition $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$ does not exclude that v_1 and w_1 share an edge. Loosely speaking, it means that there is no such edge connecting the "plus side" of S_1 and the "minus side" of S_2.

Proof. We set $v_{t+1} := v$ and $w_{s+1} := v$. Let i be the smallest number in $\{1, \ldots, t+1\}$ for which $E^+_{S_1}(v_i)$ contains an edge $e(v_i, w)$ with $w \in S_2$. Then let j be the smallest number in $\{1, \ldots, s+1\}$ for which $E^-_{S_2}(w_j)$ contains an edge $e(w_j, v_i)$. If $i \ne 1$ and $j \ne s+1$, then we set $\tau := i-1$, $\sigma := j$ and $u := v_i$. If $i \ne 1$ but $j = s+1$, then $i = t$ must hold true (otherwise v would have more then one neighbor in S_1), and we set $\tau := t$, $\sigma := s$ and $u := v$. If $i = 1$, then we set $\tau := 1$, $\sigma := j - 1$ and $u := w_j$. In the last case, we have $j > 1$ since otherwise $i = j = 1$ would contradict the assumption $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$.

In every case, we have $1 \le \tau \le t$ and $1 \le \sigma \le s$ and u is well defined. We now build L as the concatenation of the edge sequences

$$E^-_{S_2}(w_1),\ \ldots,\ E^-_{S_2}(w_\sigma),\quad E^*(u),\quad E^+_{S_1}(v_\tau),\ \ldots,\ E^+_{S_1}(v_1),$$

where $E^*(u) = \big(e(u, w_\sigma), e(u, u_1), \ldots, e(u, u_k), e(u, v_\tau)\big)$ is the chain of edges in the set $\{e' \in E(v) : e(u, w_\sigma) \preceq e' \preceq e(u, v_\tau)\}$.

Since S_1 and S_2 are slits, all edges in the "$E^+_{S_1}$-part" and in the "$E^-_{S_2}$-part" of L are pairwise different. Furthermore, it cannot happen that such an edge is contained in both parts (according to the definition of u), or that it belongs to $E^*(u)$ (by definition of $E^*(u)$). Hence, L satisfies condition (i) of the crosscut definition 4.9.

Condition (ii) can easily be verified and (iv) is trivial. In order to prove (iii), we assume that three edges of L form a face of K. By definition of u, the sequence $(w_1, w_2, \ldots, w_\sigma, u, v_\tau, \ldots, v_2, v_1)$ divides K into two parts K_1 and K_2. All edges of the "$E^+_{S_1}$-part" and of the "$E^-_{S_2}$-part" have exactly one vertex lying in $S^0_1 \cup S^0_2$ and one in K_1. So three of them can never form a face of K. All edges of $E^*(u) \setminus \{e(u, v_\tau), e(u, w_\sigma)\}$ have exactly one vertex lying in $S^0_1 \cup S^0_2$ and one in K_2. So again three of them can never form a face of K. The only remaining edges are $e(u, v_\tau)$ and $e(u, w_\sigma)$, but two edges cannot form a face, and a combination of edges from more than one of the three distinguished edge types can clearly never form a face. Hence, L is a crosscut with $u \in (S_1 \cup S_2) \cap U^+_L$, and S^-_L has the form (4.1). $\square$

The operation described in the proof of Lemma 4.30 is well defined by the slits S_1 and S_2, and it will be referred to as *reflected concatenation* $S_1 \ominus S_2$ of S_1 with S_2. It delivers a crosscut L, a vertex u, and the reduced slits S^0_1 and S^0_2. Note that the reflected concatenation is not commutative.

4.4.8. Subdivision by Disk Chains

Let v_β be an arbitrary accessible boundary vertex. In this final step we describe an approach that allows us to apply Lemma 4.28 recursively until we find a slit S with initial vertex v_β so that $D'_v = D_v$ for all $v \in S$. So especially $D'_{v_\beta} = D_{v_\beta}$. During this procedure, we construct a sequence of crosscuts L_j so that $V^+_{L_j}$ contains v_β and the number of elements in $V^+_{L_j}$ is strictly decreasing for increasing j. This procedure will be crucial for proving the following lemma and finally Theorem 4.13.

Lemma 4.31. *Let $(\mathcal{P}, \mathcal{P}')$ be an admissible pair with complex K, interior alpha vertex a and crosscut L. Then $D'_v = D_v$ for all accessible boundary vertices $v \in \partial V^*$.*

Proof. To begin with, let $S_0 = (v_1, \ldots, v_s, a)$ be a slit according to Lemma 4.28. Let v_β be an accessible boundary vertex. If $v_1 = v_\beta$, then we have $D'_\beta = D_\beta$ and we are done. So let us assume that $v_\beta \notin S_0$.

By Lemma 4.29, there is a crosscut L_1 so that $S^-_{L_1} = (v_1, \ldots, v_{s-1}, v_s, v_{s-1} \ldots, v_1)$ and $a \in S^+_{L_1}$. Applying Lemma 4.28 again, but now with respect to the crosscut L_1, we get another slit $S_1 = (w_1, \ldots, w_t, a) \subset V^+_{L_1}$ so that $D'_v = D_v$ for all $v \in S_1$. If $w_1 = v_\beta$, then we have $D'_\beta = D_\beta$ and we are done. So suppose that $v_\beta \notin S_1$.

The three boundary vertices v_1, w_1 and v_β are pairwise different. We may and will assume w.l.o.g. that they are (positively) oriented along ∂V^*, i.e. $w_1 \prec v_\beta \prec v_1$. This ensures that $E^+_{S_1}(v_1) \cap E^-_{S_0}(w_1) = \emptyset$ since otherwise v_β could be either accessible or a boundary vertex, but not both. Since except a all vertices of S_0 belong to $V^-_{L_1}$, we have $S_0 \cap S_1 = \{a\}$. Consequently, by Lemma 4.30, the reflected concatenation $S_0 \ominus S_1$ of S_0 with S_1 is well defined. It delivers a crosscut L_2, a vertex a_2, and reduced slits $S^-_2 \subset S_0$ and $S^+_2 \subset S_1$ with common terminal vertex a_2. Since $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$ (see above) and by Lemma 4.30, the vertex a_2 belongs to S_1 or S_2 and the set $U^-_{L_2}$ of lower neighbors of L_2 consists solely of elements of $S_0 \cup S_1$ and of (lower) neighbors of a_2. Since we have $D'_v = D_v$ for all $v \in S_0 \cup S_1$, this implies that L_2 is an admissible crosscut for $(\mathcal{P}, \mathcal{P}')$. Moreover, the order of S_0 and S_1 in the reflected concatenation has been chosen in such a way that v_β belongs to $V^+_{L_2}$.

The general step of the procedure is as follows. Assume that we already have an admissible crosscut L_j, the alpha vertex a_j, and the reduced slits S^-_j and S^+_j so that $v_\beta \in V^+_{L_j}$ (see Figure 4.26, left). Denoting by v^-_j and v^+_j the initial vertices of S^-_j and S^+_j, respectively, we may assume that $v^-_j \prec v_\beta \prec v^+_j$ along ∂V^*, which will again be essential to ensure the special condition of Lemma 4.30.

Applying Lemma 4.28, we get a new slit $S_j \subset V^+_{L_j}$ so that S^-_j, S_j and S^+_j are pairwise disjoint, except at their common terminal vertex a_j, and $D'_v = D_v$ for all $v \in S_j$ (see Figure 4.26, middle).

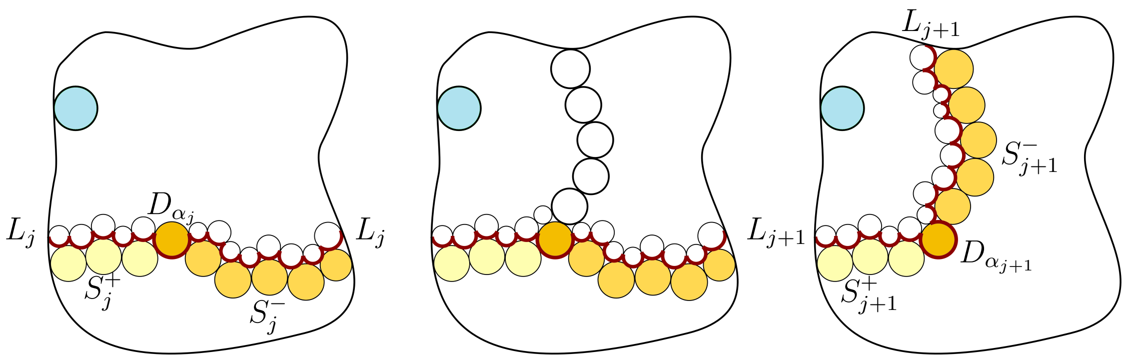


Fig. 4.26.: Construction of the crosscut L_{j+1} from L_j

If $v_\beta \in S_j$, then we are done. Otherwise, we either have $v^-_j \prec v_\beta \prec v_j$ or $v_j \prec v_\beta \prec v^+_j$. In the first case, we build the reflected concatenation $S^-_j \ominus S_j$, in the second case we form $S_j \ominus S^+_j$. The result is a new crosscut L_{j+1}, a corresponding alpha vertex a_{j+1}, and reduced slits S^-_{j+1} and S^+_{j+1} (see Figure 4.26, right)

It follows directly from the construction of the reflected concatenation that $a_{j+1}, v_\beta \in V^+_{L_{j+1}}$. Moreover, we have $a_{j+1} \in S^-_j$, and hence $D'_{\alpha_{j+1}} = D_{\alpha_{j+1}}$. In order to see that L_{j+1} is admissible for the pair $(\mathcal{P}, \mathcal{P}')$, it remains to prove that $D'_v \subset G^-_{L_{j+1}}$ for all $v \in U^-_{L_{j+1}}$.

By Lemma 4.30, the set $U^-_{L_{j+1}}$ of lower neighbors of L_{j+1} consists solely of elements of $S^-_j \cup S^+_j$ and of (lower) neighbors of a_{j+1}. Since $D'_v = D_v$ for all $v \in S^-_j \cup S^+_j \cup \{a_{j+1}\}$, and $D_v \subset G^-_{L_{j+1}}$ for all $v \in U^-_{L_{j+1}}$, the assertion follows.

The number of elements in $V^+_{L_j}$ is strictly decreasing in every step, and hence the procedure must come to an end. This can only happen if $v_\beta \in S_{j^*}$ for some $j^* \in \mathbb{N}$. Since $D'_v = D_v$ for all $v \in S_j$ with $j \leq j^*$, we have shown $D'_{v_\beta} = D_{v_\beta}$. $\square$

4.4.9. Proof of the Rigidity Theorem

Proof of Theorem 4.13. By Lemma 2.3, the kernel K^* is a strongly connected complex with vertex set V^* and $\partial V^* = V^* \cap \partial V$. Since we have shown that $D'_v = D_v$ for all boundary vertices $v \in \partial V^*$ of K^* and since every boundary vertex of K^* is also a boundary vertex of K, the Theorem 11.6 in Stephenson [31] (on the uniqueness of a locally univalent packing with prescribed combinatorics and given radii of boundary circles) tells us that $D'_v = D_v$ for all $v \in V^*$, which is the assertion of Theorem 4.13.

4.4.10. Concluding Remarks

In the general setting of Theorem 4.13, a complete description of which disks are uniquely determined by a crosscut seems not to be known. Figure 4.27 shows some examples. The accessible disks are depicted in darker colors, the alpha disk is the darkest one. By Theorem 4.13 these disks are uniquely determined (rigid) by the crosscut, but the rigid part also comprises the non-accessible disks shown in brighter color.

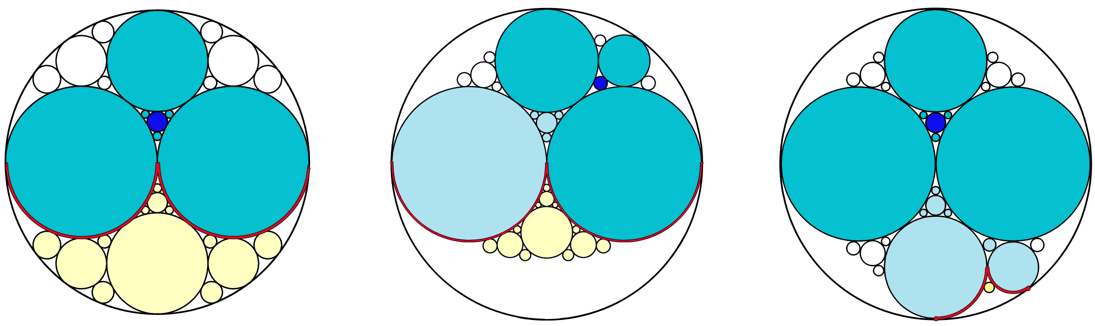


Fig. 4.27.: Rigid configurations of disks in a packing with crosscut

The example on the right is of special interest: A short crosscut separates only one non-accessible disk from the alpha disk. Here, the theorem yields rigidity for the dark (blue) disks – the main part of the packing – while it says nothing about the disks depicted in lighter colors. This is somewhat counterintuitive since the bright disks separate the dark disks from the crosscut so that the latter seem to have no relation to the cut at all. However, a little thought shows that all colored disks in the upper domain are rigid.

A first suggestion would be that the crosscut defines (in some sense) an *extended* kernel of K (maybe similar to that one of Section 4.2.2) and that Theorem 4.13 also applies to the associated *extended* main part of $\mathcal{P}$. But since it is a challenging problem to precisely describe the set of all rigid disks and since we do not need any generalization of Theorem 4.13 in the further approaches of this work, we keep this an open problem.

However, if the second circle packing $\mathcal{P}'$ is assumed to fill the domain G, then the situation becomes much clearer.

Theorem 4.32 (Rigidity of domain filling circle packings with crosscuts)**.** *Let* $(\mathcal{P}, \mathcal{P}')$ *be an admissible pair of two circle packings filling a bounded, simply connected domain* G *with complex* K*, interior alpha vertex* a *and crosscut* L*. Then* $\mathcal{P} = \mathcal{P}'$*.*

Proof. By Theorem 4.13, the main parts of $\mathcal{P}$ and $\mathcal{P}'$ are equal. So all we have to show is $D_v = D'_v$ for every vertex v in K that does not lie in the kernel K^* of K. As discussed in Section 4.2.1, such a vertex $v \in (V \setminus V^*)$ then lies in a detour-complex K_R for a detour $R = (u, v_1, \ldots, v_j, w, u)$ of K^* in K.

Let $T_R(w, u, d)$ be the corresponding detour-tri-complex and let $\mathcal{P}_R$ and $\mathcal{P}'_R$ be the associated detour-packings of $\mathcal{P}$ and $\mathcal{P}'$, respectively. Since $u, w \in K^*$, we have $D_u = D'_u$ and $D_w = D'_w$. So both packings $\mathcal{P}$ and $\mathcal{P}'$ share a common detour-trilateral $G_R(\delta_w, \delta_u, \gamma)$.

By definition, $G_R(\delta_w, \delta_u, \gamma)$ is tame and $\mathcal{P}_R$ and $\mathcal{P}'_R$ fill it for T_R. Hence, Theorem 3.1 states $\mathcal{P}_R = \mathcal{P}'_R$. So eventually $D_v = D'_v$ for all $v \in V$. $\square$

4.5. Uniqueness

In this section, we prove uniqueness for good-natured generalized circle packings under the alpha-gamma normalization. After the efforts of Section 4.4, the proof of the following theorem is refreshingly short.

Note that the general idea of the proof of Theorem 4.33 was already introduced in the concluding remarks of [20].

Theorem 4.33. *Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *fill a pinned, bounded, simply connected domain* $G(A, C)$ *for an admissible complex* $K(a, c)$ *under the alpha-gamma normalization. Let neither* $\mathcal{P}_1$ *nor* $\mathcal{P}_2$ *be a singular packing. If* C *is regular, then* $\mathcal{P}_1 = \mathcal{P}_2$*.*

Proof. The case of a being a boundary vertex is covered by Lemma 4.2. So we may and will assume that a is an interior vertex of K. Since C is regular, the packings $\mathcal{P}_1$ and $\mathcal{P}_2$ are good-natured, otherwise Lemma 4.8 would identify them with singular packings.

Let $\widetilde{K}$ be the extended kernel of $K(a, c)$. Let $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ be the extended main parts of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. By Lemma 4.10, $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ are two non-degenerate circle packings for $\widetilde{K}(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization.

Since the gamma disk $D_c^1 \in \widetilde{\mathcal{P}}_1$ meets the regular prime end C, there is a simply connected component G_1 of $G \setminus \overline{D_c^1}$ that contains all disks of $\widetilde{\mathcal{P}}_1 \setminus \{D_c^1\}$. Analogously, let G_2 be that component of $G \setminus \overline{D_c^2}$ that contains all disks of $\widetilde{\mathcal{P}}_2 \setminus \{D_c^2\}$. In the proof of Lemma 3.30, we showed that $G_1 \subset G_2$ or $G_2 \subset G_1$. Assuming the former one (see Figure 3.27 on p. 90), we have $D_c^2 \subset (G \setminus \overline{G_1})$.

The idea now is to construct a (combinatorial) crosscut L in K so that c is the only lower neighbor of L and that D_c^2 lies in the maximal lower domain of G. In order to do so, let $N = \{v_1, \ldots, v_n\}$ be the ordered set of all neighbors of c within the extended kernel $\widetilde{K}$ of K. Then $L := (e(v_1, c), \ldots, e(v_n, c))$ is the desired combinatorial crosscut of $\widetilde{K}$ (see Figure 4.28; note that in both examples only the single disk D_c^2 of the packing $\mathcal{P}_2$ is shown).

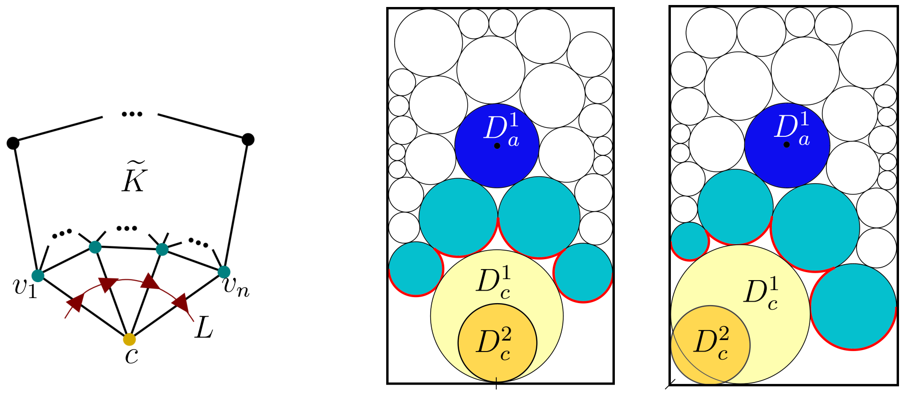


Fig. 4.28.: The construction of L and typical behavior of D_c^2

Let J be the associated maximal crosscut of $\widetilde{\mathcal{P}}_1$ in G. Let G^- and G^+ be the corresponding components of $G \setminus J$ and let $V^- = \{c\}$ and $V^+ = \widetilde{V} \setminus V^-$. Since $D_c^2 \subset G \setminus \overline{G_1} \subset G^-$, the pair $(\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2)$ is admissible for G, K, a and L. Thus Theorem 4.32 yields $\widetilde{\mathcal{P}}_1 = \widetilde{\mathcal{P}}_2$.

After showing uniqueness of the (extended) main parts of $\mathcal{P}_1$ and $\mathcal{P}_2$, the remaining proof runs exactly the same as the proof of Theorem 4.32.

According to the definitions of Section 4.2.1, let $R = (u, v_1, \ldots, v_j, w, u)$ be an arbitrary but fixed detour of K^* in K, let $T_R(w, u, d)$ be the corresponding detour-tricomplex and let $\mathcal{P}_R^1$ and $\mathcal{P}_R^2$ be the associated detour-packings of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Since $u, w \in K^*$, we have $D_u^1 = D_u^2$ and $D_w^1 = D_w^2$. So both packings $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same detour-trilateral $G_R(\delta_w, \delta_u, \gamma)$, which is tame and filled by $\mathcal{P}_R^1$ and $\mathcal{P}_R^2$ for T_R. Thus, Theorem 3.1 states $\mathcal{P}_R^1 = \mathcal{P}_R^2$, what completes the proof. $\square$

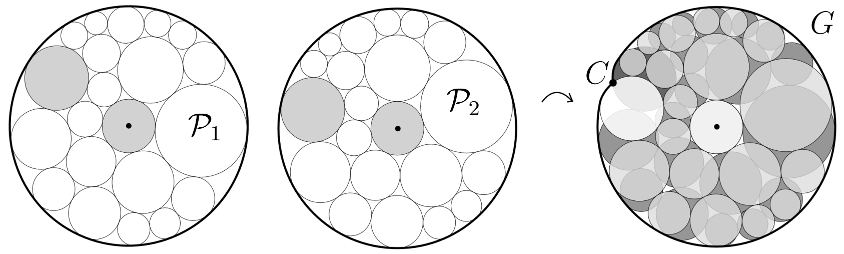


Fig. 4.29.: The packing $\mathcal{P}_1$ and its rotation $\mathcal{P}_2$ both fill $G(0, C)$ under the alpha-gamma normalization

The reason why we restrict C to be regular is shown in Figure 4.29. Since we can fill the unit disk for every admissible complex with a circle packing $\mathcal{P}_1$, a slightly rotated packing $\mathcal{P}_2$ gives rise to a Jordan domain G with inward corner C so that $\mathcal{P}$ and $\mathcal{P}'$ both fill $G(0, C)$ under the alpha-gamma normalization.

4.6. Existence

In this section, we prove that for every admissible complex $K(a, c)$ and every pinned, bounded, simply connected domain $G(A, C)$ there is a generalized circle packing $\mathcal{P}$ filling $G(A, C)$ for $K(a, c)$ under the alpha-gamma normalization. To do this we follow the idea of Section 3.7: At first, we consider regular domains since then degeneracy cannot occur (Lemma 4.10), and then we approximate arbitrary bounded simply connected domains by smooth Jordan domains, which are regular.

4.6.1. Regular Domains

The existence proof for regular domains proceeds almost the same as for the alpha-beta-gamma normalization. The basic idea is to use the gamma vertex c of $K(a, c)$ together with another boundary vertex $b \neq c$ and to associate them with the prime end C of $G(A, C)$ and some interior point $z \in G$, respectively. This allows us to use the boundary alpha-gamma normalization for $K(b, c)$ and $G(z, C)$. Since G is regular, we get a circle packing $\mathcal{P}(z)$ that fills G, and by adjusting z via Sperner's Lemma it eventually fulfills the alpha-gamma normalization for $K(a, c)$ and $G(A, C)$.

Lemma 4.34 (Existence for regular domains)**.** *Let $K(a, c)$ be an admissible complex with interior alpha vertex a. Let $G(A, C)$ be a regular, pinned, bounded, simply connected domain. Then there is a circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization.*

The proof of this lemma will occupy the rest of this section. Different to the proof of Lemma 3.41, we avoid here any induction by relying directly on Lemma 4.3. We follow the recipe given at the beginning.

1. Construction of the packing $\mathcal{P}(z)$. Let $q \in G$ be a point in G so that the maximal disk in G with center in q touches C. Since G is a regular domain, all these points q

form a set λ that is either void or a line segment connecting C with an inner point of G. In what follows, the set λ often plays an exceptional role.

Let $b \in \partial V$ be an arbitrary but fixed boundary vertex of K different from c, i.e., $b \neq c$. By Lemma 4.3, for every point $z \in (G \setminus \lambda)$ there is a generalized circle packing $\mathcal{P}(z)$ for $K(b,c)$ filling $G(z,C)$ under the boundary alpha-gamma normalization. Moreover, $\mathcal{P}(z)$ is unique and non-degenerate by Lemma 4.2 and 4.4, respectively.

By Lemma 4.5, the circle packing $\mathcal{P}(z)$ depends continuously on z in the following sense: If a sequence (z_k) of points $z_k \in (G \setminus \lambda)$ converges to $z_0 \in (G \setminus \lambda)$, then $\mathcal{P}(z_k)$ converges to $\mathcal{P}(z_0)$.

2. Classification of control points. Let $\gamma_1, \gamma_2, \gamma_3 \subset G$ be three pairwise different straight lines that all have their starting point in A and terminal points on ∂G. The angle between any two of them shall be $\frac{2}{3}\pi$. The concatenation of any two of them forms a crosscut of G, thus $G \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)$ is divided into exactly three connected components G_1, G_2 and G_3 so that $\gamma_1 \not\subset \overline{G_3}$, $\gamma_2 \not\subset \overline{G_1}$ and $\gamma_3 \not\subset \overline{G_2}$ (see Figure 4.30, left).

By Lemma 2.11, the terminal points of γ_1, γ_2 and γ_3 are associated with three pairwise different prime ends $X_1, X_2, X_3 \in \partial G^*$, respectively, which are arranged in this order on the positively oriented boundary ∂G^*. We may and will assume w.l.o.g. that $C \notin \{X_1, X_2, X_3\}$ and $C \in \partial G_3^*$ since otherwise we just have to rotate γ_1, γ_2 and γ_3 around A accordingly.

Using G_1, G_2 and G_3 we build the following subsets of $G \setminus \lambda$,

$$\begin{aligned}
\mathcal{R} &= \{z \in G \setminus \lambda : \quad A(z) \in \overline{G_1} \cap G\} \\
\mathcal{G} &= \{z \in G \setminus \lambda : \quad A(z) \in \overline{G_2} \cap G\} \\
\mathcal{B} &= \{z \in G \setminus \lambda : \quad A(z) \in \overline{G_3} \cap G\}.
\end{aligned}$$

Clearly, the following properties of the sets $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ follow directly from the definition of G_1, G_2 and G_3.

Lemma 4.35. *Every point $z \in G \setminus \lambda$ belongs to $\mathcal{R}$, $\mathcal{G}$ or $\mathcal{B}$. If $z \in (\mathcal{R} \cap \mathcal{G} \cap \mathcal{B})$, then $A(z) = A$, i.e. $\mathcal{P}(z)$ fills $G(A,C)$ for $K(a,c)$ under the alpha-gamma normalization.*

Our goal is to find such a point $z \in (\mathcal{R} \cap \mathcal{G} \cap \mathcal{B})$. We achieve it by evoking Sperner's Lemma. The next steps are preparations for that.

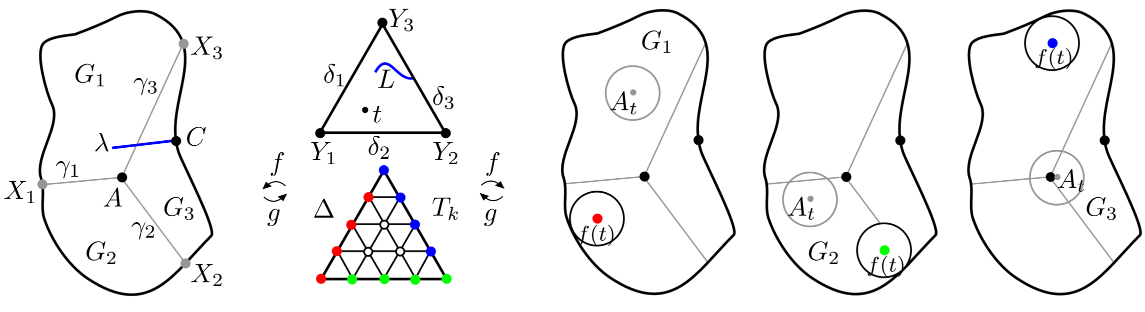


Fig. 4.30.: The subdivision of G, the coloring of $\partial\Delta$ and some red, green and blue points $f(t) \in G$

3. Conformal transplantation to a triangle. By Carathodory's version of the Riemann mapping theorem ([21], Cor. 2.7.), there is a conformal mapping $f : \Delta \to G$ of a triangular domain Δ to G that can be normalized so that the extension $f^* : \overline{\Delta} \to G^*$ to the closure of Δ maps the vertices Y_1, Y_2 and Y_3 of Δ to the prime ends X_1, X_2 and X_3, respectively. As usual, we denote the inverse of f and f^* by g and g^*, respectively.

Depending on its image $z := f(t)$, the location of the center $A_t := A(z)$ of the alpha disk $D_a(z)$ of $\mathcal{P}(z)$ defines the color of a point t in Δ (see Figure 4.30, right). Since $\mathcal{P}(z)$ is only defined for $z \notin \lambda$, we have to treat the case $t \in L$ with $L := g(\lambda)$ separately. All in all, let t be colored

$$\begin{array}{ll} \textit{red} & \text{if } f(t) \in \mathcal{R}, \\ \textit{green} & \text{if } f(t) \in \mathcal{G} \setminus \mathcal{R}, \\ \textit{blue} & \text{if } f(t) \in \mathcal{B} \setminus (\mathcal{R} \cup \mathcal{G}) \text{ or } t \in L. \end{array}$$

For the sake of simplicity, we speak of *red points*, *red sequences*, *red neighborhoods* etc. An explanation for coloring L blue will be given in the next step, where we extend the coloring to $\overline{\Delta}$. The most crucial part is to understand the color of points near the boundary of Δ or close to L. Let $\delta_1 := [Y_3, Y_1]$, $\delta_2 := [Y_1, Y_2]$ and $\delta_3 := [Y_2, Y_3]$ be the three sides of Δ.

Lemma 4.36. *Let (t_k) be a sequence of points in Δ converging to $t_0 \in \overline{\Delta}$.*

(i) If $t_0 \in ((\delta_2 \cup \delta_3) \setminus \delta_1)$, then (t_k) cannot be red.

(ii) If $t_0 \in ((\delta_3 \cup \delta_1) \setminus \delta_2)$, then (t_k) cannot be green.

(iii) If $t_0 \in ((\delta_1 \cup \delta_2) \setminus \delta_3)$, then (t_k) cannot be blue.

(iv) If $t_0 \in L$, then (t_k) cannot be red.

Proof. We start with some preparations for special situations that will occur later on. Given that $t_k \notin L$ for every $k \in \mathbb{N}$, then, by Lemma 3.36, the sequence of circle packings $(\mathcal{P}(z_k))$ contains a convergent sub-sequence, which we again denote $(\mathcal{P}(z_k))$. Let $\mathcal{P}$ be its limit. Assume further that $\mathcal{P}$ is not collapsed. Then Lemma 3.36 tells us that $\mathcal{P}$ fills G. In particular, the limit of the gamma disks $D_c(z_k)$ touches some prime end $X \in \partial G^*$. We show that if the limit of $D_c(z_k)$ is a dot S_c, then $X = C$.

Assume contrarily that S_c is a dot that touches $X \neq C$. Let U_n be the tails of a null-chain defining X. By dropping the first terms if necessary, we may and will assume w.l.o.g. that U_n does not contain C as subordinate prime end for all n. Moreover, Lemma 3.35 implies that for every fixed n and sufficiently large k the disk $D_c(z_k)$ is contained in U_n. Thus, by Lemma 2.17, it cannot touch C. Nevertheless, $D_c(z_k)$ meets C. So we conclude that for all sufficiently large k there is a chord $\sigma_k \subset D_c(z_k)$ that is a crosscut of G separating C from all disks of $\mathcal{P}(z_k) \setminus \{D_c(z_k)\}$.

Let G_k be that component of $G \setminus \sigma_k$ that contains C as subordinate prime end. Since $\mathcal{P}$ is assumed to be non-collapsed, there is a disk D_k in $(\mathcal{P}(z_k))$ with its radius bounded below by some positive constant. Since D_k is disjoint to G_k, it lies in $G \setminus G_k$. Thus, the diameter of $G \setminus G_k$ is also bounded from below by some positive constant.

Recall that we have $D_c(z_k) \subset U_n$ for sufficiently large k and every (fixed) n. This implies either $G_k \subset U_n$ or $(G \setminus \overline{G_k}) \subset U_n$. By definition, the diameter of U_n goes to zero as n goes to infinity, so we can choose n sufficiently large so that D_k, and whence $G \setminus G_k$, cannot be contained in U_n. Thus, there are two numbers n and k so that $G_k \subset U_n$. Since C is a prime end of G_k and since we have $G_k \subset U_n$, C must also be a subordinate prime end of U_n, what was excluded. This contradiction implies $X = C$.

Proof of (i)–(iii). For $t_0 \in \partial\Delta$, let $X_0 \in \partial G^*$ be the prime end of G associated with t_0 via $f^*(t_0) = X_0$. Let (J_n) be a null-chain representing X_0 and let (U_n) be its tails. Let $\Gamma_n := g(J_n)$ and $W_n := g(U_n)$ define the associated null-chain (Γ_n) of t_0 in Δ and its tails (W_n). Since t_k converges to t_0, we may and will assume w.l.o.g. that $t_k \in W_n$ for any fixed n. Thus, the points $z_k := f(t_k)$ lie in U_n, i.e., $z_k \in U_n$.

Since $t_k \in L$ implies that t_k is blue and neither red nor green, we can assume $t_k \notin L$ for (i) and (ii). Since t_0 has a positive distance from L in (iii), we may and will assume that $t_k \notin L$, too. Thus, the very first assumption of this proof holds true and $\mathcal{P}(z_k)$ is well defined for all $k \in \mathbb{N}$.

By construction, the limit of $D_b(z_k)$ is a dot S_b. Let $(b, \ldots, a)$ be a chain of vertices in K that connects b with the alpha vertex a. Let $D_c(z_k), \ldots, D_a(z_k)$ be the associated disks in $\mathcal{P}(z_k)$. If all of them degenerate to dots, then Lemma 3.35 yields for sufficiently large k that the disk $D_a(z_k)$ is contained in any fixed tail U of X_0. Since we can choose U to be disjoint to G_1, G_2 or G_3 in (i), (ii) or (iii), respectively, the center of $D_a(z_k)$ does not lie in the desired set and t_k cannot be red, green or blue, respectively.

Therefore, $\mathcal{P}$ is non-collapsed, what explains the second assumption at the beginning of this proof. But since G is regular, by Lemma 2.18, $\mathcal{P}$ contains only one disk D_v while all other sets are equal dots touching X_0.

Assume $v \neq a$. Since K stays connected after removing v and its adjacent edges and faces, there is a chain $(b, \ldots, a)$ of vertices that connects b with a without containing v. Thus, the same argumentation as above prevents t_k to have its desired color.

This only leaves the case where $v = a$, i.e., $v \neq c$. Let A' be the center of D_a and let S_c be the dot associated with c in $\mathcal{P}$. By our thoughts from the beginning of this proof, S_c touches C, what implies $X_0 = C$. Thus, the degenerate circle packing $\mathcal{P}$ would be the singular packing $\mathcal{P}_0$ for $K(a, c)$ filling $G(A', C)$. By Lemma 4.11, this is impossible. So t_k cannot be red, green or blue for (i), (ii) or (iii), respectively.

Proof of (iv). Since $t_k \in L$ implies that t_k is blue and not red, we may and will assume $t_k \notin L$. Since the case $t_0 \in (\overline{\lambda} \cap \partial\Delta)$ is included in (i), we assume w.l.o.g. that $t_0 \in \Delta$. Thus, the limit D_b of $D_b(z_k)$ is a disk with center in $f(t_0) \in G$. So $\mathcal{P}$ is not collapsed, and both assumptions of the beginning of this proof are fulfilled.

By construction, the limit of $D_c(z_k)$ is a dot S_c. As we have shown above, S_c touches C. Furthermore, as $\mathcal{P}$ is a degenerate circle packing that fills a regular domain, we conclude from Lemma 2.18 that $\mathcal{P}$ contains exactly one disk, namely D_b.

Let $(c, \ldots, a)$ be a chain of vertices in K that connects the gamma and alpha vertex. Clearly, we can choose $(c, \ldots, a)$ so that it does not contain b. Let $D_c(z_k), \ldots, D_a(z_k)$ be the associated disks in $\mathcal{P}(z_k)$. By Lemma 3.35, for sufficiently large k the whole disk $D_a(z_k)$ is contained in a tail U of C. Since we can choose U to be disjoint to G_3, the center of $D_a(z_k)$ does not lie in G_3, hence, t_k cannot be red. $\square$

With hindsight to Lemma 4.36, we extend the coloring of Δ to $\overline{\Delta}$ exactly as shown in Figure 4.30 (middle). We fix the color of Y_1, Y_2 and Y_3 as well as every remaining point on δ_1, δ_2 and δ_3 to be red, green and blue, respectively. This guarantees that every point on $\partial\Delta \cup L$ has a neighborhood containing not more than two different colors, in particular not the points Y_1, Y_2 and Y_3.

4. Application of Sperner's Lemma. Uniform subdivision of the sides of Δ into k intervals with equal lengths generates a regular triangulation T_k of $\overline{\Delta}$. The coloring of the vertices of T_k is a Sperner coloring and Sperner's Lemma tells us that T_k must contain a triangle Δ_k thats vertices have three different colors (see Section 2.4).

For each $k \in \mathbb{N}$ we denote by t_k^r, t_k^g and t_k^b the red, green and blue vertex of Δ_k, respectively. After replacing the sequence (Δ_k) by an appropriate sub-sequence (Δ_m) with $m = m(k)$, we get three sequences (t_m^r), (t_m^g) and (t_m^b) that converge to the same limit $t_0 \in \overline{\Delta}$. Lemma 4.36 tells us that we have $t_0 \in (\Delta \setminus L)$, i.e., $z_0 := f(t_0)$ lies in $G \setminus \lambda$.

Since $z_m^r := f(t_m^r)$, $z_m^g := f(t_m^g)$ and $z_m^b := f(t_m^b)$ converge to $z_0 \in (G \setminus \lambda)$, we may and will assume w.l.o.g. that we also have $z_m^r, z_m^g, z_m^b \in (G \setminus \lambda)$, i.e., $A(z_m^r) \in \overline{G_1} \cap G$, $A(z_m^g) \in \overline{G_2} \cap G$ and $A(z_m^b) \in \overline{G_3} \cap G$. Since the three sets $\overline{G_1} \cap G$, $\overline{G_2} \cap G$ and $\overline{G_3} \cap G$ are closed within G, we get as limit $A(z_0) \in (\overline{G_1} \cap \overline{G_2} \cap \overline{G_3} \cap G)$, i.e., $z_0 \in (\mathcal{R} \cap \mathcal{G} \cap \mathcal{B})$. Furthermore, according to Lemma 3.40, the three sequences of circle packings $\mathcal{P}(z_m^r)$, $\mathcal{P}(z_m^g)$ and $\mathcal{P}(z_m^b)$ converge to the common limit $\mathcal{P}(z_0)$, which then, by Lemma 4.35, is the desired circle packing filling $G(A, C)$ for $K(a, c)$ under the alpha-gamma normalization. This completes the proof of Lemma 4.34.

4.6.2. Convergence and Continuity

Here we restate and extend the convergence and continuity terminology from Section 3.6.2 for the alpha-gamma normalization.

Let $K(a, c) \in \mathcal{K}$ be an admissible complex. Let G be a bounded, simply connected domain and let $f : \mathbb{D} \to G$ and f^* be a canonical parameterization of G and its extension to G^*, respectively. Let $t \in I$ for a compact interval I. Let (r_t) and (s_t) be two families of points in $\overline{\mathbb{D}}$ that depend continuously on t so that for every $t \in I$ the points $r_t \in \partial\mathbb{D}$ are boundary points while $s_t \in \mathbb{D}$ are interior points of $\mathbb{D}$. Let $A_t = f(s_t)$ and $C_t = f^*(r_t)$.

Definition 4.15. We call $(G(A_t, C_t))$ a *continuous family of pinned domains*. Let $(\mathcal{P}_t)$ be a family of generalized circle packings so that $\mathcal{P}_t$ fulfills the alpha-gamma normalization for $G(A_t, C_t)$ and $K(a, c)$. Then we say that $(\mathcal{P}_t)$ *depends continuously on t at t_0* if for every sequence (t_k) of numbers $t_k \in I$ with $t_k \to t_0$ the associated sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}_0$. If $(\mathcal{P}_t)$ depends continuously on t for all $t_0 \in I$, then it is said to be a *continuous family of generalized circle packings* for $K(a, c)$ filling $(G(A_t, C_t))$.

Lemma 4.37 (Alpha-gamma continuity)**.** *Let $(G(A_t, C_t))$ be a continuous family of pinned, bounded, simply connected domains over some compact interval I. For every $t \in I$ let $\mathcal{P}_t$ be a generalized circle packing for an admissible complex $K(a, c)$ that fills*

$G(A_t, C_t)$ under the alpha-gamma normalization. Let (t_k) be a sequence of numbers in I with $t_k \to t_0$. Then the following holds true.

(i) The sequence $(\mathcal{P}_k)$ contains a sub-sequence that converges to a generalized circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A_0, C_0)$ under the alpha-gamma normalization.

(ii) If C_0 is regular, then the complete sequence $(\mathcal{P}_k)$ converges to $\mathcal{P} = \mathcal{P}_0$, i.e., $(\mathcal{P}_t)$ depends continuously on t at t_0.

(iii) If G is regular, then $(\mathcal{P}_t)$ is a continuous family.

Proof. If a is a boundary vertex, then we are done by Lemma 3.40. Note that (i) is not directly mentioned within Lemma 3.40, but it is shown within its proof.

Assume a is an interior vertex. By Lemma 3.36, there is a sub-sequence of $(\mathcal{P}_k)$ that converges to a generalized circle packing $\mathcal{P}$ for K. Let $(\mathcal{P}_n)$ with $n = n(k)$ denote this convergent sub-sequence. Let (D_a^n) be the associated sequence of alpha disks and let D_a be its limit in $\mathcal{P}$. We show that D_a is a disk, i.e., $\mathcal{P}$ is not collapsed.

Assume contrarily that D_a^n converges to a dot $S_a = \{s\}$. By Lemma 3.5, there is a chain of dots $(S, \dots, S_a)$ in $\mathcal{P}$ so that S is a boundary dot. Let $P^n, \dots, D_a^n$ be the associated disks or dots in $\mathcal{P}_n$. Then Lemma 3.35 yields a prime end $X \in \partial G^*$ so that D_a^n is contained in any tail U of X, provided that n is sufficiently large. Since we can choose U so small that it does not contain any of the points A_n, i.e., $A_n \notin U$, we conclude that $\mathcal{P}_n$ cannot fulfill its alpha-gamma normalization, what is a contradiction.

Knowing that $\mathcal{P}$ is not collapsed, we can apply again Lemma 3.36, which guarantees that $\mathcal{P}$ fills G for K. In order to see that $\mathcal{P}$ also fulfills the alpha-gamma normalization for $G(A_0, C_0)$ and $K(a, c)$, we notice that the center of D_a is clearly A_0. So we only have to show that the limit P_c of D_c^n meets C_0.

If (D_c^n) contains a sub-sequence so that all of its disks touch C_n, then Lemma 3.35 directly implies that P_c touches C_0, i.e. P_c meets C_0. Therefore, we assume contrarily that D_c^n does not touch C_n for almost all, say for all n. Since D_c^n meets C_n, there is a chord $\sigma_n \subset D_c^n$ that is a crosscut in G and separates C_n from $\mathcal{P}_n \setminus \{D_c^n\}$.

We transplant everything into the unit disk $\mathbb{D}$ by the canonical embedding $g : G \to \mathbb{D}$ and its extension g^*. Let $r_n := g^*(C_n)$ be the image of C_n, and let $L_n := g(\sigma_n)$ be the image of σ_n. Since σ_n is a crosscut of G, also L_n is a crosscut of $\mathbb{D}$. Let E_n and E_n' be the two components of $\mathbb{D} \setminus L_n$. Then r_n is contained in the boundary of exactly one of them, say $r_n \in \partial E_n$ for all n.

Since D_c^n converges, also σ_n, L_n and eventually the closed sets $\overline{E_n}$ converge. Let E_0 be the limit of E_n. Clearly, the limit $r_0 = g(C_0)$ of r_n is contained in ∂E_0, i.e. $r_n \in \partial E_0$. Thus, either P_c touches C_0, or it still separates it from $\mathcal{P} \setminus \{P_c\}$. So P_c meets C_0. Hence, $\mathcal{P}$ fulfills the alpha-gamma normalization for $G(A_0, C_0)$ and $K(a, c)$, what proves assertion (i) of the lemma.

If C is regular, then, by Theorem 4.33, we know that $\mathcal{P} = \mathcal{P}_0$ is the unique packing for $K(a, c)$ that fills $G(A_0, C_0)$ under the alpha-gamma normalization. Consequently, every sub-sequence of $\mathcal{P}_k$ contains a convergent sub-sequence, and all those convergent sub-sequences must have the same limit $\mathcal{P}_0$. This implies that the whole sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}$, i.e., $(\mathcal{P}_t)$ depends continuously on t at t_0, what proves (ii).

Eventually, let G be regular. Then C_t is regular for all $t \in I$. So $(\mathcal{P}_t)$ depends continuously on t for all $t_0 \in I$, which makes it a continuous family and proves (iii). □

In order to define an exhaustion of $G(A, C)$, let $f : \mathbb{D} \to G$ be the canonical parameterization of G with its extension $f^* : \overline{\mathbb{D}} \to G^*$. Let $p \in \mathbb{D}$ be the pre-image of A under f, i.e., $A = f(p)$, and let $q \in \mathbb{T}$ be the pre-image of C under f^*, i.e., $C = f^*(q)$. We choose an increasing sequence of positive numbers r_k converging to 1 and define the *exhausting pinned domains* $G_k(A_k, C_k)$ of $G(A, C)$ by

$$G_k := f(r_k\mathbb{D}), \quad A_k := f(r_k p), \quad C_k := f(r_k q).$$

Up to some minor technical changes within the Lemmas 3.35 and 3.36, the following result can be proven in exactly the same way as Lemma 4.37.

Corollary 4.38. *Let $G(A, C)$ be a pinned, bounded, simply connected domain and let $G_k(A_k, C_k)$ be exhausting pinned domains of $G(A, C)$. For every $0 < k < 1$ let $\mathcal{P}_k$ be a generalized circle packing for an admissible complex $K(a, c)$ that fills $G_k(A_k, C_k)$ under the alpha-gamma normalization. Then the following holds true.*

(i) The sequence $(\mathcal{P}_k)$ contains a sub-sequence that converges to a generalized circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization.

(ii) If C is regular, then the complete sequence $(\mathcal{P}_k)$ converges to $\mathcal{P}$.

4.6.3. General Bounded and Simply Connected Domains

After all these preparations, we are ready to state and prove our most general existence result for the alpha-gamma normalization. The idea of the proof of Theorem 4.39 is to exhaust a bounded, simply connected domain G by smooth Jordan domains G_k in order to fill G with the limit of the packings for G_k, which we obtain from Lemma 4.34.

Theorem 4.39 (Alpha-Gamma Existence Theorem)**.** *Let $K(a, c)$ be an admissible complex and let $G(A, C)$ be a pinned, bounded, simply connected domain. Then there exists a generalized circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization. Assume additionally that a is an interior vertex. If C is no inward spike, then $\mathcal{P}$ can be assumed to be good-natured. If C is an inward spike, then $\mathcal{P}$ has w.l.o.g. at least a non-degenerate main part.*

Proof. If a is a boundary vertex of K, then the result follows from Theorem 4.3. So assume that a is an interior vertex.

Let $G_k(A_k, C_k)$ exhaust $G(A, C)$. The domain G_k is smooth and hence regular for every $0 < k < 1$. So Lemma 4.34 guarantees the existence of a circle packing $\mathcal{P}_k$ for $K(a, c)$ filling $G_k(A_k, C_k)$ under the alpha-gamma normalization. By Corollary 4.38, there is a sub-sequence of $(\mathcal{P}_k)$ that converges to a generalized circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization.

That the main part of $\mathcal{P}$ is not degenerate or that $\mathcal{P}$ is even good-natured (depending on the behavior of C) follows from Lemma 4.11. □

Recall that there are situations where a pinned domain $G(A, C)$ with an inward spike at C cannot be filled with a good-natured circle packing $\mathcal{P}$ for a given admissible complex $K(a, c)$ with interior a (see again Figure 4.10 on p. 130, right). Nevertheless, we can always achieve a non-degenerate main part. So a restriction to strongly connected complexes leads to the following result.

Corollary 4.40. *Let $K(a, c)$ be a strongly connected complex with interior vertex a. Let $G(A, C)$ be a pinned, bounded, simply connected domain. Then there exists a circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization.*

Finally, by combining Theorem 4.33 and Theorem 4.39 with Lemma 4.4 (if a is a boundary vertex) or with Lemma 4.8 and 4.10 (if a is an interior vertex), we obtain the Alpha-Gamma Theorem 4.1 as stated at the beginning of this chapter.

4.7. Summary

For the convenience of the reader let us summarize the results of this chapter.

Convention. Recall the two possible types of solutions under the alpha-gamma normalization, which we called good-natured and ill-natured circle packings (Definition 4.7). The latter ones are always degenerate, and the so-called singular packing is the best known ill-natured circle packing (Definition 4.8). For the remaining part of this summary we pass the convention that the singular packings are no longer considered to be generalized circle packings. By Lemma 4.8 in combination with Theorem 4.39, this is a valid practice.

Existence. Let $K(a, c)$ be an admissible complex and let $G(A, C)$ be a pinned, bounded, simply connected domain. Then there exists a generalized circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization. Assume additionally that a is an interior vertex. If C is no inward spike, then there is such a $\mathcal{P}$ that is good-natured. If C is an inward spike, then $\mathcal{P}$ has w.l.o.g. at least a non-degenerate main part (Theorem 4.39).

Uniqueness. Let $G(A, C)$ be a a pinned, bounded, simply connected domain and let $K(a, c)$ be an admissible complex. Let $\mathcal{P}$ be a generalized circle packing for $K(a, c)$ filling $G(A, C)$. If C is regular, then $\mathcal{P}$ is uniquely determined, independent whether it is degenerate or not (Lemma 4.2, Theorem 4.33).

Continuity. Let I be a closed interval. For $t \in I$, let $(\mathcal{P}_t)$ fill the continuous pinned family $(G(A_t, C_t))$ for a given admissible complex $K(a, c)$. If the prime end C_0 is regular for some $t_0 \in I$, then $(\mathcal{P}_t)$ depends continuously on t at t_0. If G is regular, i.e., every prime end of G is regular, then $(\mathcal{P}_t)$ is a continuous family (Lemma 4.37, Lemma 4.5).

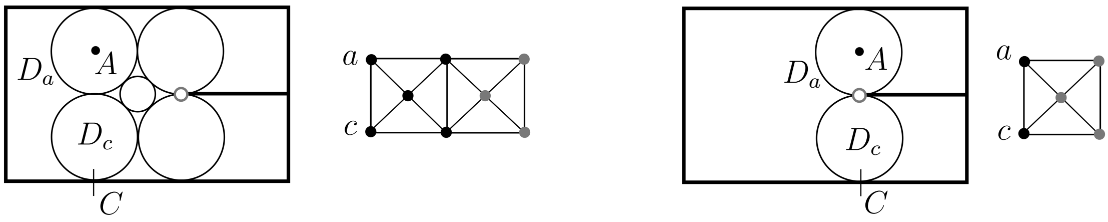


Fig. 4.31.: Unique degenerate packings for non-dubious (left) and dubious (right) alpha-gamma norm.

Non-Degeneration. The following tables show conditions under which a generalized circle packing is guaranteed to be non-degenerate (denoted by *"yes"*). If the table entry says *"no"*, then there are examples of $G(A,C)$ and $K(a,c)$ so that the unique solution $\mathcal{P}$ is degenerate (see Figure 4.31 and Figure 4.32).

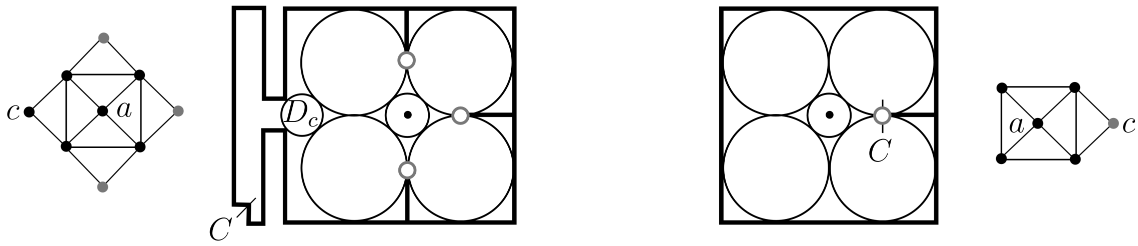


Fig. 4.32.: Two degenerate circle packings that are unique under the alpha-gamma normalization

We first look at the boundary alpha-gamma normalization. Here, the normalization of the pinned domain $G(A,C)$ uniquely defines the alpha disk D_a. If now D_a touches C, and if C is no inward spike, then $\mathcal{P}$ must be degenerate. Thus, we assume that D_a does not touch C.

Guaranteed Non-Degeneration (boundary alpha-gamma)

		Geometric Properties of G		
		No Inward Spikes	**Not dubious**	*(no restriction)*
Combinatoric Properties of K	**Strongly Connected**	yes (Lemma 4.4)	yes (Lemma 4.4)	no
	(no restriction)	yes (Lemma 4.4)	no	no

Now we look at the interior alpha-gamma normalization. If the table entry says *"no, but for $\mathcal{P}^*$"* or *"no, but for $\widetilde{\mathcal{P}}$"*, then one cannot assure (in general) that the complete packing is non-degenerate. Nevertheless, the main part $\mathcal{P}^*$ or the extended main part $\widetilde{\mathcal{P}}$ of $\mathcal{P}$, respectively, cannot become degenerate.

However, the last column is a special case (denoted by *"only existence, no guarantee"*. For the situations shown there all we know is that there exists some $\mathcal{P}$ or $\mathcal{P}^*$, respectively, that is non-degenerate. But we cannot guarantee this property in general, although we already excluded the singular packings by the convention above.

Guaranteed Non-Degeneration (interior alpha-gamma)

		Geometric Properties of G			
		no inward spikes	**inward spikes, but not at C**	**C is inward spike, but $\mathbf{D_A}$ does not touch C**	*(no restriction) CAUTION: only existence, no guarantee*
Combinatorics	**K is strongly connected**	yes (Lemma 4.10)	yes (Lemma 4.10)	yes (Corollary 4.9)	yes (Corollary 4.40)
	$\mathbf{K = \widetilde{K}}$	yes (Lemma 4.10)	yes (Lemma 4.10)	no, but for $\mathcal{P}^*$ (Corollary 4.9)	no, but for $\mathcal{P}^*$ (Corollary 4.40)
	(no restriction)	yes (Lemma 4.10)	no, but for $\widetilde{\mathcal{P}}$ (Lemma 4.10)	no, but for $\mathcal{P}^*$ (Corollary 4.9)	no, but for $\mathcal{P}^*$ (Corollary 4.40)

Receipt. The natural setting for the alpha-gamma normalization is the following:

Start with a bounded, simply connected domain G and a strongly connected admissible complex K. Choose a regular prime C and a point $A \in G$ of G so that $G(A, C)$ is not dubious (Definition 4.2; roughly speaking $D_A^{max} \cap P_C \cap \partial G = \emptyset$). Further choose a boundary vertex c and any other vertex a of K and associate them with C and A, respectively. Then there is a unique circle packing $\mathcal{P}$ for $K(a, c)$ filling $G(A, C)$ under the alpha-gamma normalization.

5. Alpha-Beta Normalization

The last normalization that we want to investigate is the *alpha-beta normalization*. Roughly speaking, we associate an interior point of a domain G with the center of the *alpha disk* of a domain filling circle packing $\mathcal{P}$, and we fix a direction in which the center of a *beta disk* lies.

In order to make this more explicit, let K be an admissible complex. Let the *alpha vertex* a be a distinguished vertex of K and let the *beta vertex* b be another vertex of K with $b \neq a$. Note that, independent of each other, a and b can be boundary vertices. In short we write $K(a, b)$.

Let G be a given bounded, simply connected domain that we want to fill. Let $q \in G$ be a point in G that shall become the center of the alpha disk, and let Γ be a ray with starting point q. We call $G(q, \Gamma)$ again a *pinned domain*, here with respect to the alpha-beta normalization.

Definition 5.1. Let $\mathcal{P}$ be a generalized circle packing for K filling G. Let the alpha disk or alpha dot $P_a \in \mathcal{P}$ have its center at q. Let the beta disk or beta dot P_b have its center on Γ. If a is an interior vertex, then we say that $\mathcal{P}$ fills $G(q, \Gamma)$ for $K(a, b)$ under the *interior alpha-beta normalization*. If a is a boundary vertex and b is a neighbor of a, then we say that $\mathcal{P}$ fills $G(q, \Gamma)$ for $K(a, b)$ under the *boundary alpha-beta normalization*.

That said, this chapter is completely different from its two predecessors. Instead of proving uniqueness with respect to the alpha-beta normalization, we present some examples that destroy every hope for a general uniqueness result.

This is a big surprise since up to this point our presumption was that the alpha-beta normalization could be the best of all those normalizations presented in this work. We looked forward to obtain an uniqueness result free of any additional restrictions to (the boundary of) the chosen bounded, simply connected domain G. However, it turns out that in general we cannot expect uniqueness at all, even if G is assumed to be a smooth Jordan domain.

Nevertheless, at least the boundary alpha-beta normalization works quite well, and also the existence part of the alpha-beta normalization can be proven as expected. The following statements are our boundary and interior main results, which we prove in Section 5.1 and Section 5.3, respectively.

Theorem 5.1 (Boundary Alpha-Beta Theorem)**.** *Let $K(a, b)$ be a strongly connected admissible complex with boundary vertex a and one of its neighbors b. Let $G(q, \Gamma)$ be a pinned, bounded, simply connected domain. If $\Gamma \cap \partial D_a \cap \partial G = \emptyset$, then there is a unique circle packing $\mathcal{P}$ for $K(a, b)$ filling $G(q, \Gamma)$ under the boundary alpha-beta normalization.*

Theorem 5.2 (Interior Alpha-Beta Theorem). *Let $K(a,b)$ be a strongly connected admissible complex. Let $G(q,\Gamma)$ be a pinned, bounded, simply connected domain. Then there is a circle packing $\mathcal{P}$ for $K(a,b)$ filling $G(q,\Gamma)$ under the interior alpha-beta normalization, but $\mathcal{P}$ is in general not unique.*

5.1. Boundary Alpha-Beta Normalization

In this section, we are interested in that special case of the alpha-beta normalization where the alpha vertex a is a boundary vertex and the beta vertex b is a neighbor of a.

Let $K(a,b)$ be an admissible complex with boundary vertex a. Let $(v_1,\dots,v_n)$ be the boundary chain of K so that w.l.o.g. we have $v_1 = a$. Let $N(a) = \{u_1,\dots,u_m\}$ be the ordered set of neighbors of a. Then clearly $v_2 = u_1$, $v_n = u_m$ and $m \geq 2$. Moreover, we have $b = u_j$ for some $1 \leq j \leq m$.

Let $G(q,\Gamma)$ be a pinned, bounded, simply connected domain. The alpha disk D_a is uniquely determined as the maximal disk $D(q) \subset G$ that has its center in q. In particular, $D_a = D(q)$ is always a disk and never a dot. Furthermore, the contact point $c(a,b)$ between D_a and the beta disk or dot P_b is the unique intersection point between ∂D_a and Γ. We set $r := c(a,b)$ and identify $G(q,\Gamma) = G(q,r)$. This allows us to distinguish between the two cases $r \in G$ and $r \in \partial G$.

Our goal is to find a (generalized) circle packing $\mathcal{P}$ for $K(a,b)$ filling $G(q,r)$ under the boundary alpha-beta normalization. Furthermore, we want some criteria under which $\mathcal{P}$ is uniquely determined. We start with an result for $r \in \partial G$.

Lemma 5.3. *Let $K(a,b)$ be an admissible complex. Let $G(q,\Gamma) = G(q,r)$ be a pinned, bounded, simply connected domain with $r \in \partial G$. Let $D_a = D(q)$ and $S_v = \{r\}$ for every $v \neq a$. Then $\mathcal{P} = \{D_a\} \cup \{S_v\}$ is a degenerate circle packing for $K(a,b)$ filling $G(q,\Gamma)$ under the boundary alpha-beta normalization. If the prime end X that is associated with r via D_a is no inward spike, then $\mathcal{P}$ is the one and only solution for the given normalization.*

Proof. That $\mathcal{P}$ fills $G(q,r)$ for $K(a,b)$ under the boundary alpha-beta normalization follows directly by its definition, i.e., here is nothing to prove. Let X be no inward spike.

Assume that there is another generalized circle packing $\mathcal{P}'$ for $K(a,b)$ filling $G(q,\Gamma)$ under the boundary alpha-beta normalization. As we know, the alpha disks are uniquely determined by q, i.e., $D_a = D'_a = D(q)$. This forces $S_b = S'_b = S = \{r\}$ since $r \in \partial G$. By Lemma 2.18, there cannot be a second disk in $\mathcal{P}'$ since otherwise S would become a pseudo contact point of two disks touching X, what makes X an inward spike.

Thus, all sets of $\mathcal{P}' \setminus \{D'_a\}$ are dots, which then must be equal to $S = \{r\}$, i.e., $\mathcal{P} = \mathcal{P}'$. □

In the following we assume that $r \in G$.

Lemma 5.4. *Let $K(a,b)$ be an admissible complex. Let $G(q,\Gamma) = G(q,r)$ be a pinned, bounded, simply connected domain with $r \in G$. Then there is a unique generalized circle packing $\mathcal{P}$ for $K(a,b)$ filling $G(q,\Gamma)$ under the boundary alpha-beta normalization.*

Proof. Once more, we set $D_a := D(q)$. Since $r \in G$, there is exactly one connected component G_q of $G \setminus D(q)$ that contains r on its boundary. Emerging from q, let Γ intersect ∂G the first time in some boundary point $z \in \partial G$. Let $Z \in \partial G^*$ be the associated prime end for z via $[q, z) \subset G$. Then by Lemma 3.7, G_q is simply connected and all accessible prime ends Y of G_q with accessible point $p(Y) \in \overline{D_a}$ are regular.

We want to interpret G_q as a tame trilateral. In order to do so, we walk from r along $\partial D(q)$ with positive orientation (i.e., along ∂G_q^* with negative orientation) until we meet the first boundary point $x \in \partial G$ of G. Let X be its representation on ∂G_q^*. Analogously, we define $Y \in \partial G_q^*$ by walking along $\partial D(q)$ with negative orientation until we first reach the boundary of G in $y \in \partial G$. Finally, let $R \in \partial G_q^*$ be the associated prime end for r in G_q.

Let α be the positively oriented arc of prime ends of G_q connecting X with R. Let β be the positively oriented arc of prime ends of G_q connecting R with Y. Let γ be the positively oriented arc of prime ends of G_q connecting Y with X. Then $G_q(\alpha, \beta, \gamma)$ is a tame trilateral (see Figure 5.1).

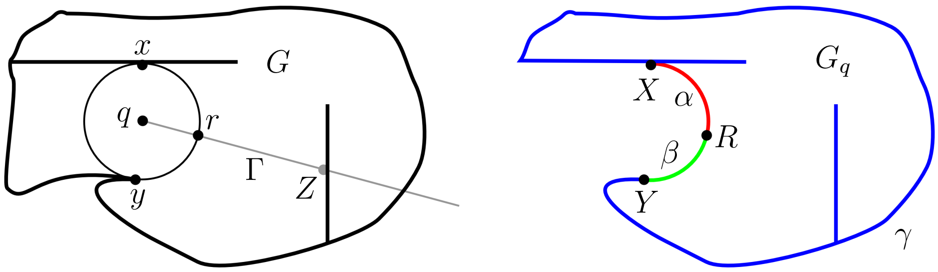


Fig. 5.1.: The definition of the trilateral $G_q(\alpha, \beta, \gamma)$ for the case $r \in G$

In the following step, we transform K into a suitable tri-complex T (see Figure 5.2). In order to do so, let $(v_1, \ldots, v_n)$ denote again the boundary chain of K with $v_1 = a$. Let $N(a) = \{u_1, \ldots, u_m\}$ be the ordered set of all neighbors of a with $v_2 = u_1$, $v_n = u_m$ and $b = u_j$ for some $j \in \{1, \ldots, m\}$.

Now, we remove a as well as all its edges and faces from K, and we add the two vertices a_1 and a_2, the edges $e(a_1, u_1), \ldots, e(a_1, b)$ and $e(a_2, b), \ldots, e(a_2, u_m)$ as well as the faces $f(a_1, u_i, u_{i+1})$ for $i = 1, \ldots, j-1$ (only if $j > 1$) and $f(a_2, u_i, u_{i+1})$ for $i = j, \ldots, m-1$ (only if $j < m$). Then we add the vertex w, the edges $e(w, v_2), \ldots, e(w, v_n)$ and the faces $f(w, v_i, v_{i+1})$ for $i = 2, \ldots, n-1$. Finally, we connect a_1, a_2 and w by edges $e(a_1, w)$, $e(w, a_2)$ and $e(a_2, a_1)$, and we complete the construction by adding the faces $f(a_1, w, v_2)$, $f(w, a_2, v_n)$ and $f(a_2, a_1, b)$.

We denote the arising complex $T = T(a_1, a_2, w)$. Since $n \geq 3$ implies $v_2 \neq v_n$ while $m \geq 2$ implies $u_1 \neq u_2$, the constructed T is a tri-complex.

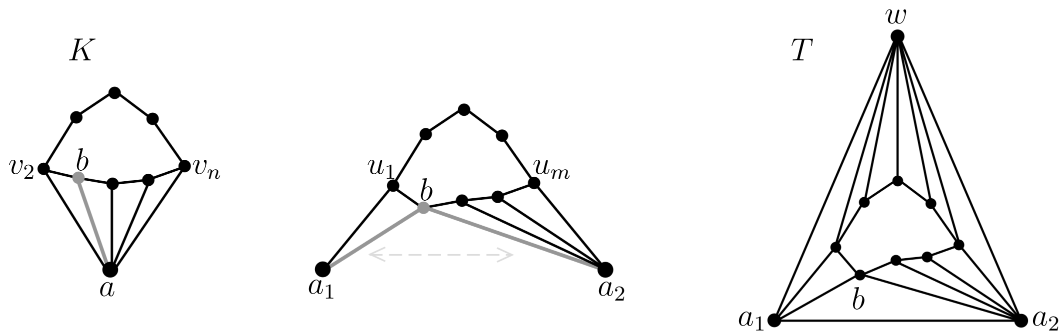


Fig. 5.2.: The construction of the tri-complex $T(a_1, a_2, w)$ out of $K(a, b)$

Eventually, we associate $T(a_1, a_2, w)$ with $G_q(\alpha, \beta, \gamma)$ and apply the Alpha-Beta-Gamma Theorem 3.1. What we get is a (unique) generalized circle agglomeration $\mathcal{P}_T$ for T filling $G_q(\alpha, \beta, \gamma)$. Then the union $\mathcal{P} := \mathcal{P}_T \cup \{D(q)\}$ of $\mathcal{P}_T$ with $D(q)$ is a generalized circle packing for $K(a, b)$ filling $G(q, \Gamma)$ under the alpha-beta normalization.

Assume that there is another generalized circle agglomeration $\mathcal{P}'$ for $K(a, b)$ filling $G(q, \Gamma)$ under the alpha-beta normalization. As we know, the two alpha disks are equal, i.e., $D_a = D'_a = D(q)$. Thus, both packings give rise to the same trilateral $G_q(\alpha, \beta, \gamma)$.

Since the beta disk or dot P'_b touches $r \in G$, it must be a disk $P'_b = D'_b$. Otherwise, by Lemma 2.18 and since $\mathcal{P}'$ fills G, the dot $P'_b = S'_b = \{r\}$ attached to D'_a in r would equal some boundary dot $S'_b = S = \{s\}$ with $s \in \partial G$, which is a contradiction to $s = r \in G$. So at least one disk of $\mathcal{P}'$ lies in G_q.

If all disks of $\mathcal{P}'$ different to D'_a are contained in G_q, then $\mathcal{P}'_T := (\mathcal{P}' \setminus \{D_a\})$ is a generalized circle agglomeration for $T(a_1, a_2, w)$ filling $G_q(\alpha, \beta, \gamma)$. By the Alpha-Beta-Gamma Theorem, we conclude that $\mathcal{P}'_T = \mathcal{P}_T$, i.e., $\mathcal{P}' = \mathcal{P}$.

All that is left to show is that there is no disk $D'_v \in (\mathcal{P}' \setminus \{D'_a\})$ lying outside of G_q. Assume contrarily that such a disk D'_v exists. As explained above, we have $D'_b \subset G_q$. Since K stays connected after removing a and its adjacent edges and faces, there is a chain $(b, \ldots, v)$ of vertices in $V \setminus \{a\}$ that connects b with v and does not contain a. Let $C := (D'_b, \ldots, D'_v)$ be the associated chain of disks and dots in $\mathcal{P}'$. Since $D'_b \subset G_q$ and $D'_v \cap G_q = \emptyset$, there must be two disks D'_1 and D'_2 in C so that D'_1 touches D'_2 at least at a pseudo contact point while $D'_1 \subset G_q$ but $D'_2 \cap G_q = \emptyset$. By definition of G_q, this is only possible if D'_1, D'_2 and D'_a touch each other at a common point, which is impossible. Hence, the assumption was wrong and we are done. □

In order to complete our thoughts about the boundary alpha-beta normalization, the next step is the investigation of the behavior of degeneracy. Recall that $\mathcal{D}$ is the set of all disks within a generalized circle packing $\mathcal{P}$ while $\mathcal{S}$ is the set of all dots.

Lemma 5.5. *Let $\mathcal{P} = \mathcal{D} \cup \mathcal{S}$ be a generalized circle packing for $K(a, b)$ filling $G(q, \Gamma) = G(q, r)$ under the boundary alpha-beta normalization with $r \in G$. If G has no inward spikes or if K is strongly connected, then $\mathcal{P}$ is not degenerate, i.e., $\mathcal{S} = \emptyset$.*

Proof. Since we have $r \in G$, the trilateral $G_q(\alpha, \beta, \gamma)$ exists and $\mathcal{P}_T$ fills it as explained in the proof of Lemma 5.4, which implies that $\mathcal{P}_T$ must contain at least one disk. Thus, $\mathcal{P}$ contains at least the two disks $D_a = D(q)$ and some $D_u \subset G_q$.

Assume that $\mathcal{P}$ is degenerate. By Lemma 2.18, we directly see that G must have an inward spike. This only leaves the case of K being strongly connected. Here, Lemma 2.19 implies that $\mathcal{P}$ contains exactly two disks. By the alpha-beta normalization, the contact point of these two disks must be r. Thus, Lemma 2.18 implies $r \in \partial G$, which is a contradiction. Hence, $\mathcal{P}$ cannot be degenerate. $\square$

A combination of the Lemmas 5.3 to 5.5 proves the Boundary Alpha-Beta Theorem 5.1. Using the fact that the kernel K^* of K is strongly connected, the following corollary directly follows from Lemma 5.5.

Corollary 5.6. *Let $\mathcal{P}$ be a generalized circle packing for $K(a, b)$ filling $G(q, \Gamma) = G(q, r)$ under the boundary alpha-beta normalization with $r \in G$. Let b be an interior vertex and let K^* be the kernel of K with respect to b. Then the main part of $\mathcal{P}$ with respect to K^* is not degenerate.*

We conclude this section by some thoughts about continuity. Let $I \subset \mathbb{R}$ be a compact interval and let $t \in I$. Let G be a bounded, simply connected domain. Let (q_t) be a continuous family of points $q_t \in G$ and let (φ_t) be a continuous family of numbers $\varphi_t \in [-2\pi, 2\pi]$. Let (Γ_t) be a family of rays Γ_t with starting point q_t so that the angle between Γ_t and the positive real axis is φ_t. Let r_t be the intersection point between Γ_t and the boundary of the maximal disk $D(q_t)$ in G with center q_t. Then we call $(G(q_t, \varphi_t)) = (G(q_t, \Gamma_t)) = (G(q_t, r_t))$ a *continuous family of pinned, bounded, simply connected domains.*

Let $K(a, c)$ be an admissible complex. Let $(\mathcal{P}_t)$ be a family of generalized circle packings so that $\mathcal{P}_t$ fills $G(q_t, \Gamma_t)$ for $K(a, c)$ under the boundary alpha-beta normalization. Then we say that $(\mathcal{P}_t)$ *depends continuously on t at t_0* if for every sequence (t_k) of numbers $t_k \in I$ with $t_k \to t_0$ the associated sequence $(\mathcal{P}_{t_k})$ converges to $\mathcal{P}_{t_0}$.

If $(\mathcal{P}_t)$ depends continuously on t for all $t_0 \in I$, then it is said to be a *continuous family of generalized circle packings* for $K(a, c)$ and the continuous family $(G(q_t, \Gamma_t))$ with respect to the boundary alpha-beta normalization.

Lemma 5.7. *For the admissible complex $K(a, b)$ let the family of generalized circle packings $(\mathcal{P}_t)$ fill the continuous family of pinned, bounded, simply connected domains $(G(q_t, \varphi_t)) = (G(q_t, \Gamma_t)) = (G(q_t, r_t))$. If Γ_{t_0} does not associate the maximal disk $D(q_{t_0})$ with an inward spike of ∂G^* for some $t_0 \in I$, then $(\mathcal{P}_t)$ depends continuously on t at t_0. If G is regular or if $r_t \in G$ for all $t \in I$, then $(\mathcal{P}_t)$ is a continuous family.*

Proof. Let (t_k) be an arbitrary sequence of numbers in I with $t_k \to t_0$. By Lemma 3.36, the associated sequence $(\mathcal{P}_{t_k})$ contains a sub-sequence $(\mathcal{P}_{t_n})$ (we assume $k_n = n$) that converges to a generalized circle packing $\mathcal{P}$ for K. Since q_{t_n} converges to $q_{t_0} \in G$, the alpha disk or dot of $\mathcal{P}$ is the maximal disk $D_a = D(q_{t_0})$, i.e., $\mathcal{P}$ is not collapsed. Thus, Lemma 3.36 even states that $\mathcal{P}$ fills G.

In order to prove that $\mathcal{P}$ also fulfills the alpha-beta normalization for $G(q_{t_0}, \Gamma_{t_0}) = G(q_{t_0}, r_{t_0})$ we further need to show that the limit $P_b(t_0)$ of the beta disks or dots $P_b(t_n)$ touches r_{t_0}. But this is clear since $P_b(t_n)$ touches $D_a(t_n)$ in r_{t_n} for every n while r_{t_n} converges to r_{t_0}. So $\mathcal{P}$ is a generalized circle packing for $K(a,b)$ filling $G(q_{t_0}, \Gamma_{t_0})$ under the alpha-beta normalization.

By the assumption of the lemma, the Theorem 5.1 states $\mathcal{P} = \mathcal{P}_{t_0}$. Thus, every sub-sequence of $\mathcal{P}_{t_k}$ contains a convergent sub-sequence, and all those convergent sub-sequences must have the same limit $\mathcal{P}_{t_0}$. This implies that the whole sequence $(\mathcal{P}_{t_k})$ converges to $\mathcal{P}_{t_0}$. Hence, $\mathcal{P}_t$ depends continuously on t at t_0.

If G is regular or if $r_t \in G$ for all $t \in I$, then Γ_t can never associate the maximal disk $D(q_t)$ with an inward spike of ∂G^* for any $t \in I$. Hence, $(\mathcal{P}_t)$ is a continuous family. $\square$

5.2. Interior Alpha-Beta Uniqueness

Compared to the other normalizations that we discussed in this work the boundary alpha-beta normalization has the weakest uniqueness constraints: We only have to avoid inward spikes, i.e., even inward corners are no problem. Moreover, there is no constraint at all if we choose $r \in G$. Now, imagine what this could imply for the interior alpha-beta normalization with an interior alpha vertex! There, the case $r \in \partial G$ might never happen and we could expect to have uniqueness completely independent of the boundary of G.

Sadly, the reality does not behave like this as shown in this section. We present some circle packings filling smooth Jordan domains that are counterexamples for uniqueness under the interior alpha-beta normalization (although we additionally choose b as a neighbor of a). Every used complex $K(a,b)$ is of the following type.

Definition 5.2. Let (K_n) be a sequence of complexes starting with K_1 consisting of exactly one interior vertex a and its $d \geq 3$ boundary neighbors (one of it is b). For $n = 1, 2, \ldots$ we define K_{n+1} by adding at least 3 boundary vertices to K_n so that (1) the interior of K_{n+1} equals K_n and (2) within ∂K_{n+1} every vertex has exactly two neighbors. We call K_n an *onion complex* and all K_i with $0 < i \leq n$ the *layers* of K_n.

By definition, every onion complex K_n, thus also every layer K_i of K_n, is a strongly connected admissible complex. The set of all boundary vertices of K_i is called the *i-th generation* of K_n. We define the alpha vertex a to be the one and only vertex of *generation* 0, and we choose for b one of the neighbors of a, i.e., b is in generation 1.

5.2.1. A Counterexample with one Generation

For our first counterexample let $K(a,b)$ be a *hexagonal* onion complex with exactly one generation, i.e., it contains exactly one interior vertex a together with 6 boundary vertices. Let $N(a) = \{v_1, \ldots, v_6\}$ with $b = v_1$ be the ordered set of the six boundary vertices of $K(a,b)$.

Now look at Figure 5.3. Let $\mathcal{P} = \{P_a, P_1, \ldots, P_6\}$ with $P_b = P_1$ be the brighter packing (say the *white* one) and let $\mathcal{Q} = \{Q_a, Q_1, \ldots, Q_6\}$ with $Q_b = Q_1$ be the darker packing (say the *gray* one) within Figure 5.3. Both circle packings have complex $K(a,b)$, both alpha disks are centered at the origin and both beta disks are centered on the positive real axis. Nevertheless, we have $\mathcal{P} \neq \mathcal{Q}$.

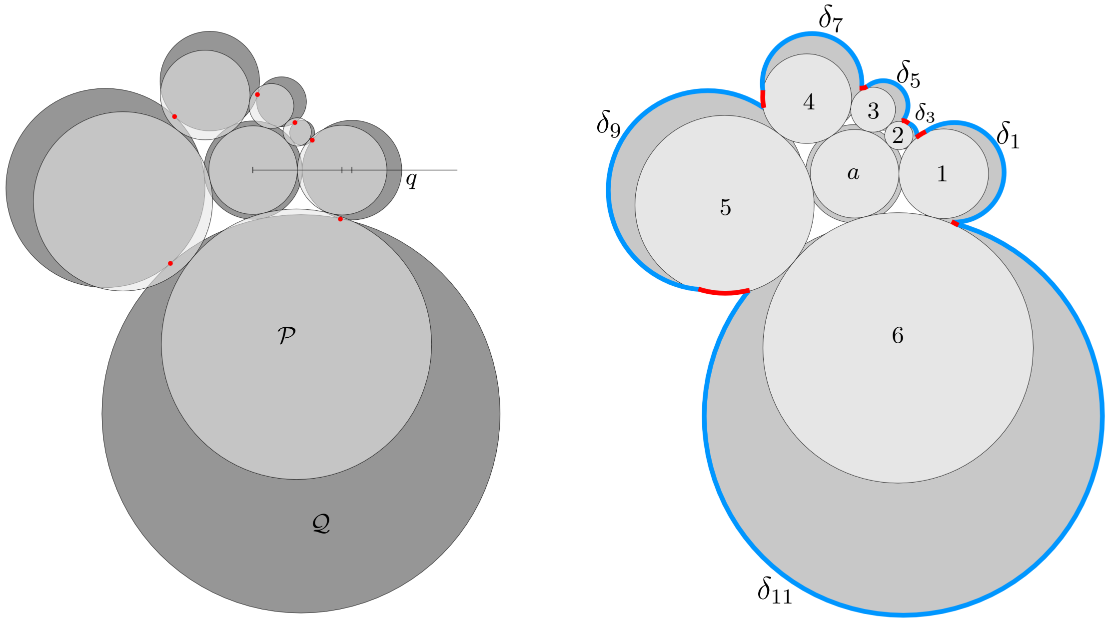


Fig. 5.3.: Two hexagonal circle packings filling a Jordan domain under the alpha-beta normalization

So far this is nothing special since also under our previous normalizations we only obtain uniqueness from packings filling some (bounded, simply connected) domain. But indeed we are able to construct a (Jordan) domain that is filled by $\mathcal{P}$ as well as by $\mathcal{Q}$.

To do so, let us start at the rightmost intersection point q between the boundary of the gray beta disk Q_1 and the real axis. From q we walk along the boundary of Q_1 with positive orientation up to the first intersection point t_1 with a white disk, which is in $\partial Q_1 \cap \partial P_1$. Note that we reach t_1 before we pass the contact point $c_Q(v_1, v_2)$ between Q_1 and Q_2 since $c_Q(v_1, v_2)$ is contained in the interior of the white beta disk P_1.

Now, we walk along the boundary of P_1 with positive orientation up to the next intersection point t_2 with a gray disk, which is in $\partial P_1 \cap \partial Q_2$. Note that we again reach t_2 before we pass the contact point $c_P(v_1, v_2)$ between P_1 and P_2 since $c_P(v_1, v_2)$ is contained in the interior of Q_2.

We keep on walking in this sense along the boundaries of $Q_2, P_2, Q_3, P_3, \ldots, Q_6, P_6, Q_1$ until we are back at our starting point q. Let Γ' denote the set of all points we walked through. Clearly, Γ' is a Jordan curve. Let G' be the Jordan domain bounded by Γ', i.e., $\partial G' = \Gamma'$. According to construction, every disk of $\mathcal{P}$ and $\mathcal{Q}$ is contained in G'. We show that $\mathcal{P}$ and $\mathcal{Q}$ even fill G'.

Every contact point $c_Q(v_i, v_{i+1})$ for $i = 1, \ldots, 6$ and $v_7 := v_1$ is contained in the interior of P_i (highlighted in Figure 5.3, left). Every contact point $c_P(v_i, v_{i+1})$ for $i = 0, \ldots, 5$

and $v_0 := v_6$ is contained in the interior of Q_{i+1} (not highlighted). Therefore, every boundary disk of $\mathcal{P}$ as well as of $\mathcal{Q}$ touches Γ'. Moreover, Γ' is the concatenation of the (closed, non-degenerate) circular arcs $\delta_j = \delta_j[t_{j-1}, t_j]$ for $j = 1, \ldots, 12$ and $t_0 := t_{12}$ with $\delta_{2i} \subset \partial P_i$ and $\delta_{2i-1} \subset \partial Q_i$ (in Figure 5.3, right, the former arcs are colored blue, the latter red).

This observation allows us to smooth Γ' near all points t_j, which are (the only) inward corners of ∂G, in order to obtain a smooth Jordan curve Γ bounding a smooth Jordan domain G that is still filled with $\mathcal{P}$ and $\mathcal{Q}$. According to this construction, we get the following result.

Theorem 5.8 (Interior alpha-beta uniqueness counterexample)**.** *There is a smooth Jordan domain $G(q, \Gamma)$, an admissible complex $K(a, b)$ with interior vertex a, and two circle packings $\mathcal{P} \neq \mathcal{Q}$ so that $\mathcal{P}$ and $\mathcal{Q}$ fill $G(q, \Gamma)$ for $K(a, b)$ under the interior alpha-beta normalization.*

Of course the depicted packings of Figure 5.3 have been carefully calculated. In Section 5.2.3 we discuss numerical stability and in Appendix A.2.1 we list the centers and radii of all involved disks to invite the readers to verify the results. Some more counterexamples are shown in Appendix A.2.2 to A.2.4 as well as their calculated centers and radii.

In the next section we introduce a counterexample with two generations.

5.2.2. A Counterexample with two Generations

The counterexample presented in this section consists of two circle packings for an octagonal onion complex K with 2 generations. We say K is octagonal since every interior vertex has exactly eight neighbors.

The two packings are depicted in Figure 5.4. Let $\mathcal{P}$ be the brighter and let $\mathcal{Q}$ be the darker one. Both circle packings $\mathcal{P} \neq \mathcal{Q}$ have the same complex $K(a, b)$, both alpha disks are centered at the origin and both beta disks are centered on the positive real axis.

Following the same idea as within the previous section, we construct a smooth Jordan domain G that is filled by $\mathcal{P}$ and $\mathcal{Q}$. To do so, let us start at the most right intersection point q between the real axis and $\mathcal{Q}$. From q we walk (with positive orientation) along the boundary of the corresponding disk $Q \in \mathcal{Q}$ up to the first intersection point with a disk $P \in \mathcal{P}$. Then we walk along ∂P (with positive orientation) up to the next intersection point with another disk of $\mathcal{Q}$, and so on, until we are back at our starting point q.

Let Γ' denote the Jordan curve we walked through and let G' be the associated Jordan domain with $\partial G' = \Gamma'$. According to construction, every disk of $\mathcal{P}$ and $\mathcal{Q}$ is contained in G'. Since the contact points of consecutive boundary disks of $\mathcal{Q}$ are contained in disks of $\mathcal{P}$ (highlighted in Figure 5.4) and vice versa, the curve Γ' contains a boundary arc of every boundary disk of $\mathcal{P}$ and $\mathcal{Q}$. Thus, both packings fill G'.

Smoothing ∂G near all those points (inward corners) where a disk of $\mathcal{P}$ intersects a disk of $\mathcal{Q}$, we can construct a smooth Jordan domain G so that $\mathcal{P}$ and $\mathcal{Q}$ fill $G(0, \mathbb{R}_0^+)$ for $K(a, b)$ under the interior alpha-beta normalization.

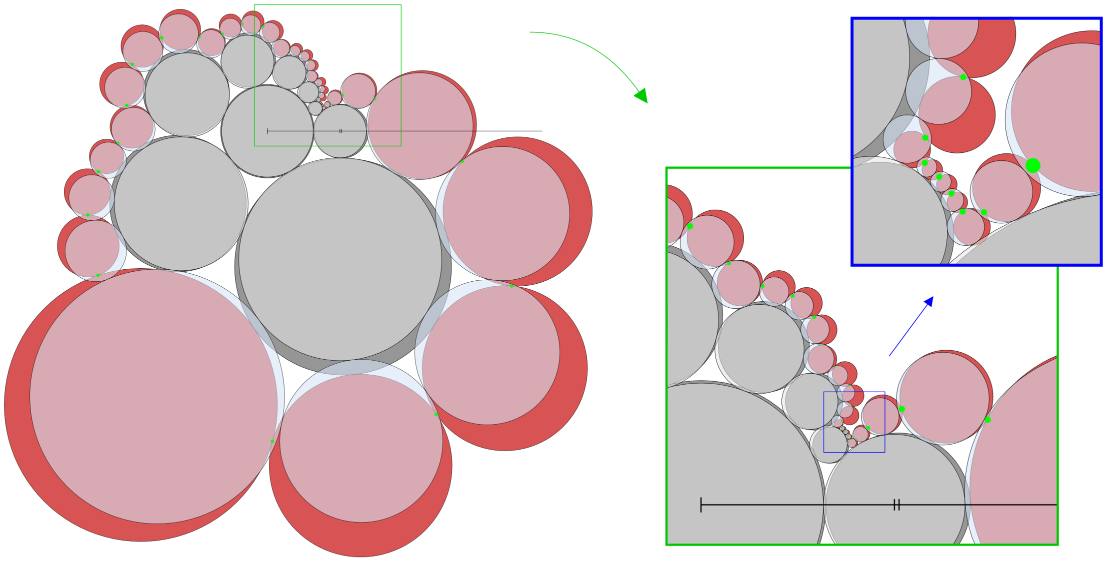


Fig. 5.4.: Two octagonal circle packings fulfilling the interior alpha-beta normalization

In the next section we discuss the computational accuracy of all counterexamples depicted in this chapter and in Appendix A.2. Afterwards, in Section 5.2.4, we explain how to construct counterexamples with n generations.

5.2.3. Computational Accuracy

In this section we provide some facts about the computational accuracy of our counterexamples. At first, we explain how we checked that the packings are really circle packings, i.e., that the disks are pairwise disjoint and that all touching conditions are fulfilled. Then we look at the most critical distances between circles and contact points.

In order to show univalence and to verify the contact conditions between neighboring disks we did the following. Let D_1 and D_2 be any two disks of one of the circle packings shown in this chapter or in Appendix A.2. Let r_1 and r_2 be their radii and let c_1 and c_2 be their (complex valued) centers, respectively.

Analytically, D_1 and D_2 are disjoint if $(r_1+r_2) \leq |c_1-c_2|$. If D_1 and D_2 are neighbors, then we even have $(r_1+r_2) = |c_1-c_2|$. Numerically, we say that D_1 and D_2 are disjoint if

$$(r_1 + r_2) - |c_1 - c_2| < 10^{-12},$$

and that D_1 touches D_2 if

$$-10^{-12} < (r_1 + r_2) - |c_1 - c_2| < 10^{-12}.$$

In this sense, our packings are univalent circle packings up to an error of at most the order 10^{-12}.

Considering two circle packings $\mathcal{P}$ and $\mathcal{Q}$ at the same time (in the sense we did in this chapter), there is an additional obstacle: One boundary disk of $\mathcal{P}$ may touch the "wrong" boundary disk of $\mathcal{Q}$. What we need is the following *co-univalence* condition.

Let $v \neq w$ be two boundary vertices of K. Let $D_v \in \mathcal{P}$ be the associated boundary disk for v in $\mathcal{P}$ and let $D_w \in \mathcal{Q}$ be the associated boundary disk for w in $\mathcal{Q}$. Whenever v and w are no neighbors, then D_v shall be disjoint to D_w.

All pairs of co-univalent disks D_1 and D_2 within any of our examples are disjoint in the sense as described above. Moreover, within our examples with one generation, we even have

$$|c_1 - c_2| - (r_1 + r_2) \geq 10^{-2}.$$

Within the octagonal example, D_1 and D_2 fulfill (in the worst case)

$$|c_1 - c_2| - (r_1 + r_2) \geq 10^{-3}.$$

Finally, we look at the contact points. Analytically, a point p is contained in the closure of a disk D with radius r and center c if $|p - c| \leq r$. Numerically, we say that p lies in $\overline{D}$ if

$$10^{-12} \leq r - |p - c|.$$

Every such contact point p that was assumed to lie in the interior of its associated disk within the examples with one generation even fulfills

$$10^{-2} \leq r - |p - c|.$$

Within the octagonal example we still have (in the worst case)

$$10^{-3} \leq r - |p - c|.$$

The smallest disks of the packings with one generation and with an alpha vertex degree 6 or 7 have radii within the order of 10^{-1}. The smallest disks of the remaining packings, especially of the octagonal ones, still have radii within the order of 10^{-2}. The critical order of co-univalence and the contact points within all our examples is not worse than 10^{-3}. Compared to the accuracy of at least 10^{-12} that we already need to calculate and verify a circle packing without any additional property, we see that those numbers are larger by at least 9 orders.

5.2.4. Counterexamples with n Generations

Here, as a concluding remark of the uniqueness section, we want to explain why it is no problem to construct counterexamples for onion complexes with every order and every number of generations.

The essence of all the counterexamples is the property of the boundary disks of $\mathcal{P}$ and $\mathcal{Q}$ to alternately contain the contact points of the boundary disks from the other packing, respectively. In order to achieve this, the neighbors of the beta disks within the first generation create the initial impulse.

Look again at Figure 5.3. On the one hand, Q_2 is much smaller than Q_1. Thus, the contact point of Q_1 and Q_2 shifts into the interior of P_1. On the other hand, Q_6 is much larger than Q_1. Thus, the contact point of Q_1 and Q_6 stays out of reach of P_1. The remaining disks of $\mathcal{P}$ and $\mathcal{Q}$ just have to transmit this impulse as if $\mathcal{P}$ would be a slight rotation of $\mathcal{Q}$ around the origin.

Of course, there are some natural limits regarding the ratio between P_a, P_1 and Q_a, Q_1, respectively. For example, if the radii of P_a and P_1 are so small that both disks are contained in Q_a, then clearly nothing works. Likewise, if we choose P_1 too large, then Q_1 together with all its contact points becomes contained in P_1. Keeping P_a and Q_a as well as P_1 and Q_1 almost the same size is the preferred choice.

Now look again at Figure 5.4. The idea behind the layout of $\mathcal{P}$ and $\mathcal{Q}$ is basically the same as within the former case. In fact, the first layers are created exactly the same way as explained above. In order to complete the packing we only have to add (one by one) the disks of the second generation so that the initial impulse of Q_2, Q_1, Q_8 and P_2, P_1, P_8, respectively, is further transmitted.

Following this idea even further we are convinced that analogous examples can be constructed for arbitrary onion complexes with any number of generations.

Conjecture 5.9 (Interior alpha-beta onion counterexamples). *For every onion complex $K(a, b)$ there is a smooth Jordan domain $G(q, \Gamma)$ and two circle packings $\mathcal{P} \neq \mathcal{Q}$ so that $\mathcal{P}$ and $\mathcal{Q}$ fill $G(q, \Gamma)$ for $K(a, b)$ under the interior alpha-beta normalization.*

5.3. Interior Alpha-Beta Existence

In this section, we prove that for every admissible complex $K(a, b)$ and every pinned, bounded, simply connected domain $G(q, \Gamma)$ there is a generalized circle packing $\mathcal{P}$ filling $G(q, \Gamma)$ for $K(a, b)$ under the interior alpha-beta normalization. Note that this implies that a is an interior vertex of K, and that b does not need to be a neighbor of a.

We first consider smooth Jordan domains since then degeneracy cannot occur. Then we approximate arbitrary bounded, simply connected domains by an exhausting method.

5.3.1. Smooth Jordan Domains

The existence proof for smooth Jordan domains relies heavily on our results from Chapter 4. The basic idea is to use the alpha vertex a together with an arbitrary (but fixed) boundary vertex c of K in order to apply an alpha-gamma normalization. The associated gamma disk shall meet a boundary point C_t that we move along the boundary of G so that it winds once around the interior point q. Since G is regular and by Lemma 4.34, we get a sequence of circle packings $\mathcal{P}_t$ for $K(a, c)$ filling $G(q, C_t)$. As we show, as soon as C_t winds around q, every disk of $\mathcal{P}_t$ does so, too. Thus, the center of the beta disk must intersect Γ for some number t_0. Hence, $\mathcal{P}_0$ fulfills the interior alpha-beta normalization for $K(a, b)$ and $G(q, \Gamma)$.

Lemma 5.10. *Let $K(a, b)$ be an admissible complex. Let $G(q, \Gamma)$ be a pinned, smooth Jordan domain. Then there is a circle packing $\mathcal{P}$ for $K(a, b)$ filling $G(q, \Gamma)$ under the interior alpha-beta normalization.*

Proof. We change the normalization from alpha-beta to alpha-gamma so that we can use our results from the previous chapter, and we follow the recipe given at the beginning.

To do so we keep the interior point q, but for consistency of notation we rename it as $A := q$. Since G is assumed to be a Jordan domain, we can use the fact that here the prime ends of ∂G^* correspond one-to-one with the boundary points of ∂G. So we replace the former with the latter and speak of the pinned domain $G(A, C)$ for some $C \in \partial G$ with respect to the alpha-gamma normalization.

Let $f : \mathbb{D} \to G$ be the canonical parameterization of G, and let f^* be its extension. We set $C_t := f^*(e^{it})$ for $0 \leq t \leq 2\pi$, i.e., we have $\partial G = \{C_t : t \in [0; 2\pi]\}$. Let c be an arbitrary (but fixed) boundary vertex of K, maybe we have $c = b$. Then, by Lemma 4.34, for every $t \in [0; 2\pi]$ there is a circle packing $\mathcal{P}_t$ for $K(a, c)$ filling $G(A, C_t)$ under the alpha-gamma normalization. By Theorem 4.33, we have $\mathcal{P}_0 = \mathcal{P}_{2\pi}$.

First of all, we see that the radii R_t^a of the alpha disks D_t^a are bounded below by a positive constant $R > 0$, i.e., $R_t^a \geq R > 0$. Since Lemma 4.34 provides the existence of a circle packing $\mathcal{P}_t$ for every t, we clearly have $R_t^a > 0$ for every $t \in [0; 2\pi]$. Assume there is a sequence (t_n) of numbers $t_n \in [0; 2\pi]$ with $t_n \to t_0$ so that $R_{t_n}^a \to 0$ for $n \to \infty$. Then, by Lemma 4.37, we have $\mathcal{P}_{t_n} \to \mathcal{P}_{t_0}$. Thus, $R_{t_n}^a \to R_{t_0}^a > 0$, what is a contradiction to our assumption. Hence, the lower bound $R > 0$ must exist. In fact we can even assume w.l.o.g. that $R > 0$ is a lower bound for the radii of all disks in $\mathcal{P}_t$ for every $t \in [0; 2\pi]$.

Now, let M_t^v be the center of a disk D_t^v in $\mathcal{P}_t$. Let $\delta_v : [0; 2\pi] \to G$, $t \mapsto M_t^v$ be the path whose image contains all the points M_t^v. If $v = a$, then we clearly have $\delta_a(t) \equiv A$. If $v \neq a$, then the image of δ_v is a closed curve, which we also denote as δ_v. Since the radii of the alpha disks are bounded below by R, the point A does not lie on δ_v for every $v \in V \setminus \{a\}$. Thus, it makes sense to speak of the winding number of δ_v around A.

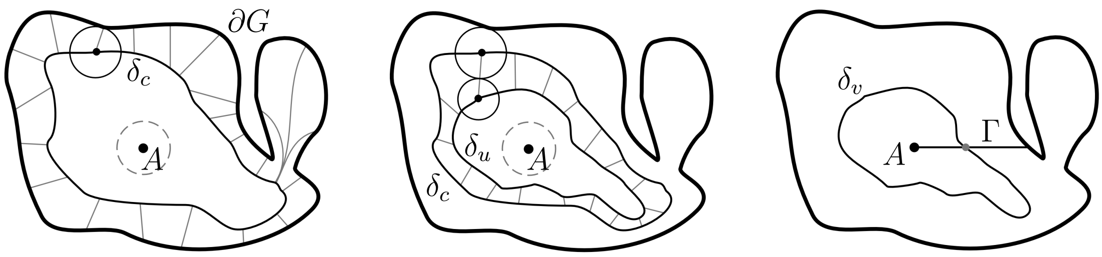


Fig. 5.5.: The closed curves ∂G, δ_c and δ_u are homotopic in $\mathbb{C} \setminus \{A\}$; every curve δ_v intersects Γ

The curve δ_c, associated with the gamma vertex c, has winding number 1 (around A). To see this all we have to apply is the definition of the meeting point C_t between D_t^c and ∂G. Either we have $C_t \in \partial D_t^c$, or there is a chord $\sigma_t \subset D_t^c$ so that $G \setminus \sigma_t$ contains two connected components G_t^1 and G_t^2 with $D_t^a \subset G_t^1$, $C_t \in \partial G_t^2$ but $C_t \notin \partial G_t^1$. So for every $t \in [0; 2\pi]$ there is a Jordan arc $\mu_t \subset G$ with endpoints C_t and M_t^c that is disjoint to D_t^a, i.e., to A. Moreover, μ_t depends (w.l.o.g.) continuously on t since C_t and M_t^c do so. Thus, the two curves ∂G and δ_c are homotopic as closed curves in $\mathbb{C} \setminus \{A\}$ (see [17], and also Figure 5.5, left). And since ∂G has winding number 1 around A, so must have δ_c, too.

A similar construction shows that every curve δ_u with $u \in V \setminus \{a\}$ has winding number 1 if u is a neighbor of c. To see this, let $u \in V \setminus \{a\}$ be a neighbor of c. Then we can connect M_t^c with M_t^u by a straight line $\mu_t \subset \overline{D_t^c} \cup \overline{D_t^u}$ for every $t \in [0; 2\pi]$. Such a straight line μ_t is always disjoint to D_t^a, i.e., to A. Moreover, μ_t depends continuously on t since M_t^c and M_t^u do so. Thus, the two curves δ_c and δ_u are homotopic as closed curves in $\mathbb{C} \setminus \{A\}$ (see Figure 5.5, middle). From this we conclude that δ_u has winding number 1 around A.

Repeating these thoughts also for every neighbor and neighbors-neighbor w of u with $w \in V \setminus \{a\}$, we see that every curve δ_w has winding number 1, too. Eventually, since K stays connected after removing a and its edges and faces, all the curves δ_v with $v \neq a$ are homotopic as closed curves in $\mathbb{C} \setminus \{A\}$.

After these preparations we now look at the beta disks D_t^b. Since the associated curve δ_b winds once around A, it must intersect the straight line Γ at some point M_τ^b for some $\tau \in [0; 2\pi]$ (see Figure 5.5, right). Hence, the associated circle packing $\mathcal{P}_\tau$ not only fills $G(A, C_\tau)$ for $K(a, c)$ under the alpha-gamma normalization, but it also fills $G(q, \Gamma)$ for $K(a, b)$ under the interior alpha-beta normalization, what proves the lemma. □

5.3.2. General Bounded and Simply Connected Domains

After the little excursion to smooth Jordan domains, we are ready to state and prove our most general existence result for the interior alpha-gamma normalization. The idea of the proof of Theorem 5.11 is once again to exhaust a bounded, simply connected domain G by smooth Jordan domains G_k in order to fill G with the limit $\mathcal{P}$ of the packings $\mathcal{P}_k$ for G_k, which we obtain from Lemma 5.10. To keep things short, we re-interpret $\mathcal{P}_k$ with respect to the alpha-gamma normalization so that we can rely again on our results from Chapter 4

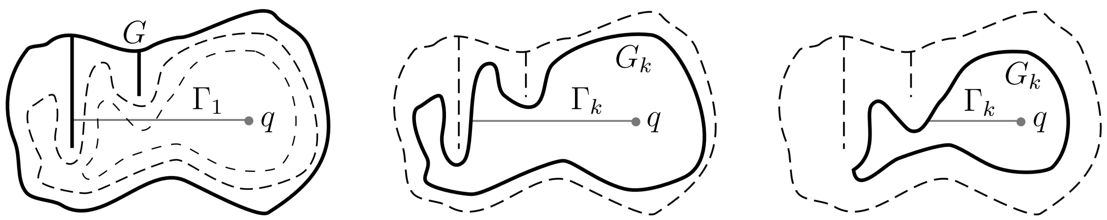


Fig. 5.6.: Exhausting domains $G_k(q, \Gamma_k)$ of $G(q, \Gamma)$ for the interior alpha-beta normalization

In order to define an exhaustion of $G(q, \Gamma)$, let $f : \mathbb{D} \to G$ be the canonical parameterization of G. We normalize f so that the origin is the pre-image of q, i.e., $q = f(0)$. Let (r_k) be an increasing sequence of positive numbers converging to 1. We set $G_k := f(r_k\mathbb{D})$, and we define $\Gamma_k \subset \Gamma$ to be the line segment that starts in q, ends at $p \in \partial G_k$ and fulfills $\Gamma_k \setminus \{q,p\} \subset G_k$. Then $G_k(q,\Gamma_k)$ are the *exhausting pinned domains* of $G(q,\Gamma)$ (see Figure 5.6).

Theorem 5.11 (Interior Alpha-Beta Existence Theorem)**.** *Let $K(a,b)$ be an admissible complex. Let $G(q,\Gamma)$ be a pinned, bounded, simply connected domain. Then there exists a generalized circle packing $\mathcal{P}$ for $K(a,b)$ filling $G(q,\Gamma)$ under the interior alpha-beta normalization so that the main part $\mathcal{P}^*$ of $\mathcal{P}$ is not degenerate. If K is strongly connected or if G has no inward spikes, then there exists such a $\mathcal{P}$ that is a circle packing.*

Proof. Let $G_k(q,\Gamma_k)$ exhaust $G(q,\Gamma)$ as explained above. The domain G_k is a smooth Jordan domain for every $0 < k < 1$. So Lemma 5.10 guarantees the existence of a circle packing $\mathcal{P}_k$ for $K(a,b)$ filling $G_k(q,\Gamma_k)$ under the interior alpha-beta normalization. Let M_k^b be the center of the beta disk $D_k^b \in \mathcal{P}_k$. By definition of Γ_k, we have $M_k^b \in \Gamma_1$ for all $0 < k < 1$. By the Bolzano-Weierstraß-Theorem and since Γ_1 is a compact set, the sequence (M_k^b) contains a convergent sub-sequence $(M_{k_n}^b)$ with limit on Γ_1. To keep things simple we assume $k_n = n$ and denote the associated sub-sequence of circle packings by $(\mathcal{P}_n)$.

Now, we change the normalization from alpha-beta to alpha-gamma. Let c be an arbitrary (but fixed) boundary vertex of K, maybe $c = b$. Let $D_n^c \in \mathcal{P}_n$ be its associated disks. Since D_n^c is a boundary disk for every n, it touches at least one prime end $C_n \in \partial G_n^*$ of G_n. Let f^* be the extension of the canonical parameterization f that we used to define the exhausting pinned domains. For every n let t_n be the pre-image of C_n under f^*, i.e., $C_n = f^*(t_n)$. Since $t_n \in \mathbb{D}$ for every n, since $\overline{\mathbb{D}}$ is a compact set and by the Bolzano-Weierstraß-Theorem, the sequence (t_n) contains a convergent sub-sequence. To keep things simple we denote this sub-sequence again by (t_n). Since the distance between t_n and $\partial\mathbb{D}$ goes to zero as n goes to 1, we conclude that its limit t_0 lies on the boundary of $\mathbb{D}$, i.e. $t_n \to t_0 \in \partial\mathbb{D}$.

Let $C_0 := f^*(t_0)$ and, for consistency of notation, let $A := q$. Then for every n we have a pinned domain $G_n(A, C_n)$, and $\mathcal{P}_n$ fills $G_n(A, C_n)$ for $K(a,c)$ under the alpha-gamma normalization. Following the proof of Lemma 4.37, up to some minor technical changes in Lemma 3.35 and 3.36, we see that the sequence $(\mathcal{P}_n)$ contains a sub-sequence that converges to a generalized circle packing $\mathcal{P}$ for $K(a,c)$ filling $G(A, C_0)$ under the alpha-gamma normalization. By Lemma 4.11, the main part of $\mathcal{P}$ is not degenerate.

Since, by our construction from above, the center of the alpha disk $D_a \in \mathcal{P}$ is $q = A$, and the center M^b of the beta disk or beta dot lies on Γ_1, the packing $\mathcal{P}$ even fills $G(q,\Gamma)$ for $K(a,b)$ under the interior alpha-beta normalization.

Finally, if K is strongly connected, then Lemma 2.4 states $K = K^*$. So $\mathcal{P}$ is not degenerate since we assumed that $\mathcal{P}$ has a non-degenerate main part. If K is not strongly connected, then nevertheless $\mathcal{P}$ contains more than 2 disks (in its main part) so that Lemma 2.18 yields the existence of a boundary pseudo contact point as soon as $\mathcal{P}$ is degenerate. This is only possible if G has inward spikes. □

The Interior Alpha-Beta Theorem 5.2, which we stated at the beginning of this chapter, is a weaker version of Theorem 5.11.

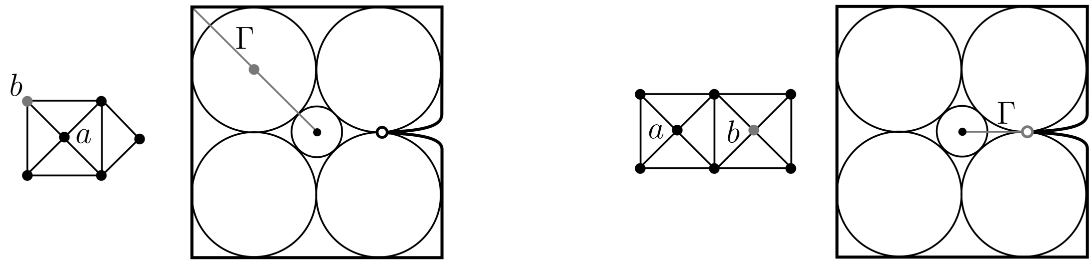


Fig. 5.7.: Two examples of degeneration under the interior alpha-beta normalization

The examples of Figure 5.7 show that some situations only allow degenerate circle packings under the interior alpha-beta normalization if K is not strongly connected while G has inward spikes. The reader is invited to verify that the depicted packings are unique under the given normalization. We are more interested in the reverse: conditions that guarantee non-degeneration. Recall that $\mathcal{D}$ is the set of all disks within a generalized circle packing $\mathcal{P}$ while $\mathcal{S}$ is the set of all dots.

Lemma 5.12. *Let $\mathcal{P} = \mathcal{D} \cup \mathcal{S}$ be a generalized circle packing for $K(a, b)$ filling $G(q, \Gamma)$ under the interior alpha-beta normalization. Let D_q^{max} be the maximal disk in G with center q. Assume $\partial D_q^{max} \cap \Gamma \cap \partial G = \emptyset$. If G has no inward spikes or if K is strongly connected, then $\mathcal{P}$ is not degenerate, i.e., $\mathcal{S} = \emptyset$.*

Proof. Assume D_a is a dot $S_a = \{q\}$. By Lemma 2.18, there is a boundary dot $S = S_a = \{s\}$ with $s \in \partial G$. This contradicts $s = q \in G$, hence D_a is a disk.

Assume D_b is a dot $S_b = \{s\}$. By definition and again by Lemma 2.18, we have $s \in \partial D_a \cap \Gamma \cap \partial G$. This implies $D_a = D_q^{max}$, thus $D_q^{max} \cap \Gamma \cap \partial G \neq \emptyset$, what is a contradiction. Hence, $\mathcal{P}$ contains at lest the two disks D_a and D_b.

Assume that $\mathcal{P}$ is degenerate. By Lemma 2.18, we directly see that G must have an inward spike. This only leaves the case of K being strongly connected. Here, Lemma 2.19 implies that $\mathcal{P}$ contains exactly two disks. By the alpha-beta normalization, the contact point of these two disks must be $\partial D_a \cap \Gamma$. Thus, Lemma 2.18 implies $\partial D_a \cap \Gamma \cap \partial G \neq \emptyset$. This contradicts the assumption $\partial D_q^{max} \cap \Gamma \cap \partial G = \emptyset$ since clearly $D_a = D_q^{max}$. Hence, $\mathcal{P}$ cannot be degenerate. $\square$

Using the fact that the kernel K^* of K is strongly connected, the following corollary directly follows from Lemma 5.12.

Corollary 5.13. *Let $\mathcal{P}$ be a generalized circle packing for $K(a, b)$ filling $G(q, \Gamma)$ under the interior alpha-beta normalization. Let K^* be the kernel of K with respect to a. Let $\partial D_q^{max} \cap \Gamma \cap \partial G = \emptyset$. Then the main part $\mathcal{P}^*$ of $\mathcal{P}$ is not degenerate.*

5.4. Summary

For the convenience of the reader let us summarize the results of this chapter.

Existence. For every pinned, bounded, simply connected domain $G(q, \Gamma)$ and every admissible complex $K(a, b)$ there is a generalized circle packing $\mathcal{P}$ that fills $G(q, \Gamma)$ for $K(a, b)$ under the alpha-beta normalization. (Lemma 5.3, Lemma 5.4 and Theorem 5.11).

Uniqueness. Let $G(q, \Gamma)$ be a pinned, bounded, simply connected domain and let $K(a, b)$ be an admissible complex. Let $\mathcal{P}$ be a generalized circle packing for $K(a, b)$ filling $G(q, \Gamma)$. If a is an interior vertex, then $\mathcal{P}$ is in general not uniquely determined, even if G is smooth (Theorem 5.8). If a is a boundary vertex and b is a neighbor of a, and if $\partial D_a \cap \Gamma \cap \partial G = \emptyset$ for the unique alpha disk D_a, then $\mathcal{P}$ is uniquely determined, independent whether it is degenerate or not (Lemma 5.3 and Lemma 5.4).

Continuity. Let I be a closed interval. For $t \in I$, let $(\mathcal{P}_t)$ fill the continuous family of pinned, bounded, simply connected domains $(G(q_t, \Gamma_t)) = (G(q_t, r_t))$ for a given admissible complex $K(a, b)$. Let X_t be the prime end associated by Γ_t at its first intersection point with ∂G. Since we cannot expect continuity for the interior alpha-beta normalization, let a be a boundary vertex of K and let b be a neighbor of a. If X_{t_0} is no inward spike or if $r_{t_0} \in G$ for some $t_0 \in I$, then $(\mathcal{P}_t)$ depends continuously on t at t_0. If G is regular, i.e., every prime end of G is regular, or if $r_t \in G$ for all $t \in I$, then $(\mathcal{P}_t)$ is a continuous family (Lemma 5.7).

Non-Degeneration. The following table shows conditions under which a generalized circle packing is guaranteed to be non-degenerate (denoted by *"yes"*).

If the table entry says *"no, but for $\mathcal{P}^*$ if a or b is interior"*, then one cannot assure (in general) that the complete packing is non-degenerate. Nevertheless, the main part $\mathcal{P}^*$ of $\mathcal{P}$ cannot become degenerate as long as one of the vertices a and b is an interior vertex. In the former case the kernel K^* (thus $\mathcal{P}^*$) is defined with respect to a, and only if $a \in \partial V$, then K^* is defined with respect to b.

Recall that D_q^{max} is the maximal disk in G with center q (if $a \in \partial V$, then $D_a = D_q^{max}$). Note that we always assume $r \in G$ for $\{r\} = \partial D_q^{max} \cap \Gamma$ since otherwise degenerate circle packings do trivially always exist.

<table>
<tr><th colspan="4">Guaranteed Non-Degeneration (alpha-beta)</th></tr>
<tr><td colspan="2" rowspan="2"></td><td colspan="2">Geometric Properties of G</td></tr>
<tr><td>No Inward Spikes, r ∈ G</td><td>r ∈ G</td></tr>
<tr><td rowspan="2">Combinatoric Properties of K</td><td>Strongly Connected</td><td>yes
(Lemma 5.5, Lemma 5.12)</td><td>yes
(Corollary 5.6, Corollary 5.13)</td></tr>
<tr><td>(no restriction)</td><td>yes
(Lemma 5.5, Lemma 5.12)</td><td>no, but for $\mathcal{P}^*$ if a or b is interior
(Corollary 5.6, Corollary 5.13)</td></tr>
</table>

Receipt. The natural setting for the alpha-gamma normalization is the following:

Start with a bounded, simply connected domain G *and a strongly connected admissible complex* K. *Choose a point* $q \in G$. *Let* D_q^{max} *be the maximal disk in* G *with center* q. *Choose now a boundary point* $r \in \partial D_q^{max}$ *of* D_q^{max} *so that* $r \in G$ *is an interior point of* G. *Let* Γ *be the ray that emerges from* q *and intersects* r. *Further choose any two vertices* $a \neq b$ *of* K, *but if* a *is a boundary vertex, then* b *must be one of its neighbors. Associate* a *with* q, *and* b *with* Γ. *Then there is a unique circle packing* $\mathcal{P}$ *for* $K(a, c)$ *filling* $G(q, \Gamma)$ *under the alpha-beta normalization.*

6. Concluding Remarks

Knowing the existence of domain filling circle packings for smooth Jordan domains, one expects that a limit procedure yields similar results for general simply connected domains – and as we have seen this indeed works. So one may speculate why Schramm did not actually study this further. We conjecture that he has foreseen the technical difficulties which arise in the limiting procedure: since disks may degenerate to dots, one needs a framework to handle these objects. This becomes even more involved if the domain is not Jordan, because then prime ends enter the scene. Reading Schramm's later papers on circle packing [27], [27] and his papers with He [12], [13], [14], [15], [16], we feel that Schramm's main interest was slightly different from ours, and so he probably did not want to waste time working out these "details".

Establishing an appropriate framework and creating the necessary technical tools was one aim of this thesis. As a result we obtained general existence and uniqueness criteria for (generalized) domain-filling circle packings (agglomerations) which are subject to one of five different types of normalizations. For each normalization we also investigated which conditions on the complex and the domain guarantee that solutions of the problem are non-degenerate classical circle packings.

In our investigations we also had to learn that the issue of uniqueness is much more intricate as we expected. In particular the Alpha-Beta normalization does not guarantee uniqueness even for smooth Jordan domains. So it is no surprise that Schramm did not address this question seriously.

Here is yet another observation which may be worth mentioning. As was shown by Bauer, Stephenson and Wegert [4], the set of all circle packings for a fixed complex K forms a smooth manifold of dimension $m + 3$, where m is the number of boundary vertices of K. Since a domain-filling circle packing satisfies m contact conditions with the boundary of the domain, this fits with our expectation that we (basically) need three additional normalization conditions to eliminate the three remaining degrees of freedom. Nevertheless, it is still surprising that this works so nicely (in general), for instance in cases where some boundary disks touch ∂G at several points.

As was already sketched in the introduction, domain-filling circle packing are closely related to discrete conformal mappings. Assume that for a given complex K, the normalized circle packings $\mathcal{P}_K$ and $\mathcal{Q}_K$ fill the domains G and D, respectively, and denote by f_K the induced discrete conformal mapping. What happens with f_K when the number of vertices of K tends to infinity?

In his talk at the Bieberbach conference Thurston conjectured that the discrete conformal mappings f_K generated by the cookie-cutting method converge to a conformal mapping of G onto $\mathbb{D}$. This was first proven by Rodin and Sullivan ([23]). Several generalizations and enhancements of this result were found by Carter, Stephenson, He, Rodin, Schramm, Dubejko and others (see [7],[9], [11],[15],[16],[29] and the references therein). He and Schramm also discovered that Theorem 1.1 provides an independent proof of the classical Riemann Mapping Theorem. To express the impact of this finding with the words of K. Stephenson: *It is hard to imagine more dramatic evidence of the fidelity of the discrete theory to its classical roots* ([31], p.267).

Considering general domain-filling circle packings, several questions arise. Which conditions guarantee that the radii of the disks in $\mathcal{P}_K$ converge to zero? Does this depend only on the sequence of complexes K_n or also on the domain G?

In the classical case of maximal *hexagonal* packings, where the $\mathcal{P}_K$ fill the unit disk and every interior disk has exactly *six* neighbors, the answer is positive. This changes already if the $\mathcal{P}_K$ are maximal *heptagonal* packings, where each interior disk has exactly seven neighbors.

The different behavior of hexagonal and heptagonal complexes is related to the Discrete Uniformization Theorem of Beardon and Stephenson: every (infinite) complex has one of three mutually exclusive intrinsic geometries, either K is *spherical*, *parabolic* or *hyperbolic*, making its circle packings affine for the Riemann sphere, the complex plane or the unit disk, respectively (see [5], [6], [13] and the references therein). While the infinite hexagonal complex is parabolic, the infinite heptagonal complex is hyperbolic.

Unfortunately, the Discrete Uniformization Theorem only states that K corresponds to one of the three types, but not to which one exactly, and it cannot be directly applied to the radii of circle packings filling an arbitrary domain. Some first results indicate that for every bounded, simply connected domain G, and at least one type of normalization, the sequence of hexagonal complexes always yields domain-filling circle packings $\mathcal{P}$ whose radii all go to zero – but these results are preliminary and we do not include them in this thesis.

But the playground for domain-filling circle packings is much larger. Conformal mapping is just a special case of more general boundary value problems, so called *Riemann-Hilbert problems.* Corresponding discrete problems have been introduced and studied by Dubejko [8],[9] (functions with prescribed modulus on the boundary) and in a more general setting by Wegert and Bauer [36]. Special Riemann-Hilbert problems give rise to a *discrete Hilbert transform* (see Volland [35]). Other types of boundary value problems, involving the derivative of the unknown function, have also been translated to the language of circle packing. Wegert, Roth and Kraus [38] have proven existence and uniqueness of solutions to a discrete *Beurling problem* for circle packings. Numerical methods for the solution of these problems have been developed by Frank Martin.

All problems mentioned above were modeled on *maximal* circle packings (filling the unit disk), domain-filling circle packings open the door to study these problems on arbitrary domains.

A. Appendix

A.1. Comments to the Proof of Theorem 3.31

This supplement provides some additional thoughts about a very specific situation within the proof of Theorem 3.31. Without explaining again all the details, we simply use the same notation as introduced there.

We only consider the packing $\mathcal{P} := \mathcal{P}^1$ since for $\mathcal{P}^2$ everything runs analogously. Let T_σ be the skeleton of T and let $\mathcal{P}_\sigma$ be the associated sub-packing of $\mathcal{P}$. Assume w.l.o.g. that $\triangle = \langle e, f, g\rangle$ is a positive oriented face of T_σ. Let V' be the set of all vertices in the interior of $\triangle$ within T and let $\mathcal{P}'$ be the associated subset of $\mathcal{P}$ for V'.

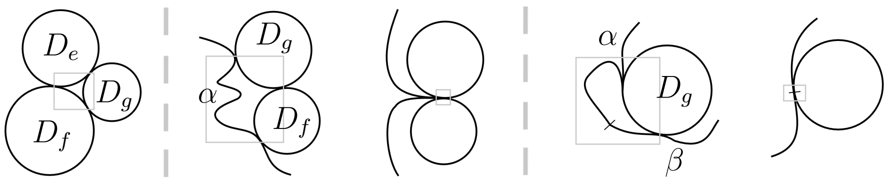


Fig. A.1.: Combinations for disks associated with $\triangle(e, f, g)$

Case 1. Assume that all three vertices e, f and g are associated with disks D_e, D_f and D_g, respectively (see Figure A.1, left). Then D_e, D_f and D_g form a positively oriented interstice $I_0(e, f, g)$ in $\mathcal{P}_\sigma$. Since $\triangle$ is not face of T, there is a vertex $u \in V'$ so that $\langle u, f, g\rangle$ forms a positively oriented face of T. We show that u is associated with a disk in $\mathcal{P}'$.

Assume contrarily that u is associated with a dot $S_u = \{s\}$. Then we have $s = \partial D_f \cap \partial D_g$. Since the interior of $\triangle$ is connected in T and since no third disk of $\mathcal{P}$ can touch D_f and D_g in s, all sets of $\mathcal{P}'$ must be dots $S = \{s\}$. Now, there is at least one vertex in V' that is a neighbor of e. This implies $s \in \partial D_e$. Since this is impossible, our assumption was wrong.

Let $I(u, f, g)$ be the interstice formed by D_u, D_f and D_g. Since I must be positively oriented, we either have $I \subset I_0$ or $I_0 \subset I$. The former implies $D_u \subset I_0$ while the latter implies $D_e \subset I$. We show that the former holds true and that this implies that eventually every disk of $\mathcal{P}'$ is contained in I_0 (see Figure A.2).

Assume contrarily that $D_e \subset I$. By pairwise connecting the centers of D_u, D_f and D_g with a straight line, we get a Jordan curve Γ bounding a domain U, i.e., $\partial U = \Gamma$. By assumption, we have $D_e \subset U$. Since T_σ is 3-connected, there is a chain of vertices $(w_1, \ldots, w_n, w)$ in T_σ with $w_1 = e$ and $w \in \{a, b, c\}$, say $w = a$, so that f and g are not contained in the chain. Let $(D_1, \dots, D_n)$ be the associated chain of disks in $\mathcal{P}_\sigma$. Then, in order to reach ∂G, at least one of the disks D_i must touch two of the disks D_u, D_f, D_g in a single point. This is impossible. Hence, our assumption was wrong and we have $D_u \subset I_0$.

Assume that there is a disk $D_v \in \mathcal{P}'$ with $D_v \cap I_0 = \emptyset$. Since the interior of $\triangle$ is connected, there is a chain $(u, \dots, v)$ of vertices in V' connecting u with v. Let $(D_u, \dots, D_v)$ be the associated chain of disks or dots in $\mathcal{P}'$. Since this chain starts with $D_u \subset I_0$ and ends with $D_v \subset (G \setminus I_0)$, there must be two disks $D_1, D_2 \in \mathcal{P}'$ sharing at least a pseudo contact point with two of the disks D_e, D_f, D_g, what is impossible.

Hence, also our last assumption was wrong, i.e., all the disks of $\mathcal{P}'$ lie in I_0, whence all its dots lie in $\overline{I_0}$.

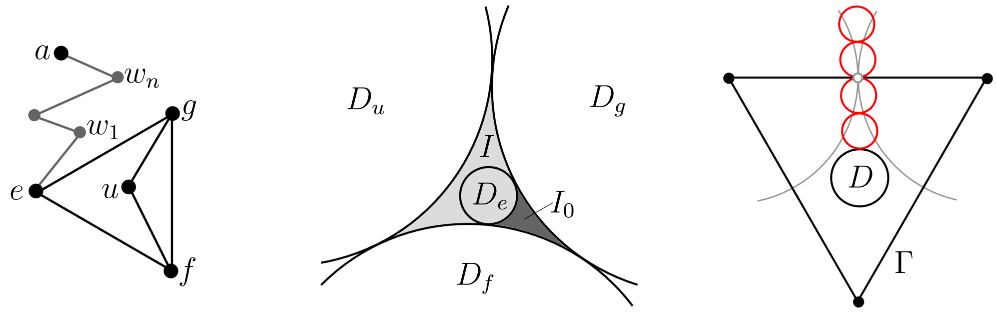


Fig. A.2.: The nested triangles $\langle e, f, g\rangle$ and $\langle u, f, g\rangle$ with associated nested interstices I_0 and I

Case 2. Assume that the vertices f and g are associated with disks D_f and D_g, respectively, while e is associated with an edge of G, say with α (see Figure A.1, middle). Then either D_f and D_g form a positively oriented boundary interstice $I_0(e, f, g)$ in $\mathcal{P}_\sigma$, or the contact point $p := c(f, g)$ between D_f and D_g lies on the boundary of G, i.e., we have $\partial D_f \cap \partial D_g \cap \partial G = p$.

Since $\triangle$ is not a face of T, there is a vertex $u \in V'$ so that $\langle u, f, g\rangle$ forms a positively oriented face of T. We show that I_0 exists if and only if u is associated with a disk in $\mathcal{P}'$.

Assume that u is associated with a dot $S_u = \{s\}$. Then we have $s = \partial D_f \cap \partial D_g$, i.e., $s = p$. Since the interior of $\triangle$ is connected in T and since no third disk of $\mathcal{P}$ can touch D_f and D_g in p, all sets of $\mathcal{P}'$ must be dots $S = S_u = \{p\}$ as neighbors and neighbor-neighbors of S_u. Now, there is at least one vertex in V' that is a neighbor of e, what implies $p \in \partial G$.

Assume that u is associated with a disk D_u. Since the edge between f and g is an interior edge of T_σ, there are two faces $\langle e, f, g\rangle$ and $\langle w, g, f\rangle$ in T_σ. By Lemma 3.2, the interior of T_σ is an admissible complex, i.e., every interior edge of T_σ must be contained in at least one interior face of T_σ. We conclude that $w \notin \{a, b, c\}$, i.e., D_w is a disk in $\mathcal{P}_\sigma$. According to the orientations of the two faces $\langle u, f, g\rangle$ and $\langle w, g, f\rangle$, we have an associated quadruple of disks $D_u, D_f, D_w, D_g \in \mathcal{P}$ as depicted in Figure A.3 (middle). Since $\mathcal{P}$ is contained in G, we have $p \in G$.

We just showed that if $\partial D_f \cap \partial D_g \cap \partial G = p$, then $\mathcal{P}'$ contains only dots $S = \{p\}$, and if $\partial D_f \cap \partial D_g \cap \partial G = \emptyset$, then D_u is a disk forming the interstice $I(u, f, g)$. By showing $I \subset I_0$, we conclude $D_u \subset I_0$, whence the same argumentation as in Case 1 implies that every disk of $\mathcal{P}'$ lies in I_0.

Since $\langle e, f, g\rangle$ and $\langle u, f, g\rangle$ have the same orientation, we have $\overline{I_0} \cap \overline{I} \cap \partial D_f \neq \emptyset$ as well as $\overline{I_0} \cap \overline{I} \cap \partial D_g \neq \emptyset$. Thus, we either have $I \subset I_0$ or $I_0 \subset I$. Since the latter implies $I \cap \partial G \neq \emptyset$, what is impossible, we conclude $I \subset I_0$, whence $D_u \subset I_0$. Assuming that another disk $D_v \in \mathcal{P}'$ does not lie in I_0 leads to a chain of disks or dots of $\mathcal{P}'$ from D_u to D_v, i.e., from I_0 to $G \setminus I_0$ (see Figure A.3, right). Similar to Case 1 this implies that there are two disks D_1 and D_2 within this chain that share at least a pseudo contact point with the disk D_f or D_g, what is impossible.

Hence, all the disks of $\mathcal{P}'$ lie in I_0 and all its dots lie in $\overline{I_0}$ or rather in $p = \partial D_f \cap \partial D_g \cap \partial G$.

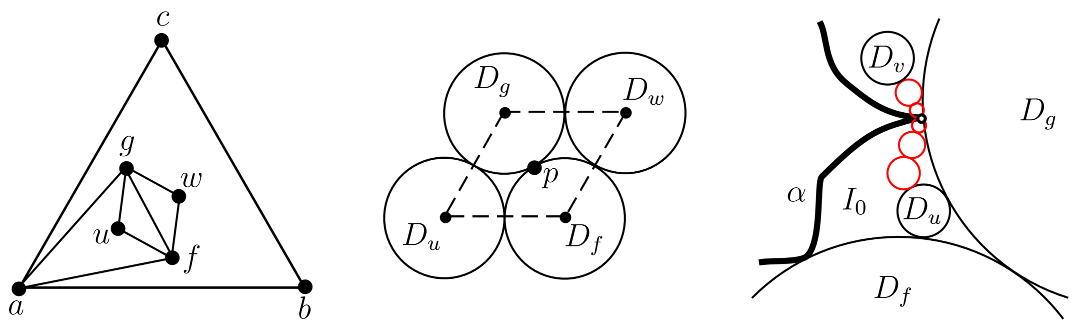


Fig. A.3.: Quadruple D_u, D_f, D_w, D_g encloses p, and $\mathcal{P}'$ cannot leave the boundary interstice I_0

Case 3. Assume that only the vertex g is associated with a disk D_g while e and f are associated with edges of G, say with α and β, respectively (see Figure A.1, right). Since $\triangle = \langle a, b, g\rangle$ is only a face of T_σ but not of T, there is a face $\langle a, b, u\rangle$ in T so that $u \in V'$ is the leading vertex of T. Let $P_u \in \mathcal{P}'$ be the disk or dot associated with u. Let $X := \alpha \cap \beta$. We distinguish whether D_g touches X or not.

Assume that D_g touches X in $p \in \partial G$. Since X is assumed to be regular, no disk of $\mathcal{P} \setminus \{D_g\}$ can touch it. Thus, P_u can meet X if and only if it is the dot $P_u = S_u = \{p\}$. Moreover, since the interior of $\triangle$ is connected, all sets of $\mathcal{P}'$ must be equal dots $S = S_u = \{p\}$ as neighbors and neighbor-neighbors of S_u.

Assume that D_g does not touch X. Then $G \setminus D_g$ contains exactly one component I_0 that has X as subordinate prime end (Lemma 3.7), which can be interpreted it in a natural way as trilateral $I_0(\alpha', \beta', \gamma')$ with $\alpha' \subset \alpha$, $\beta' \subset \beta$ and γ' being associated with an arc on ∂D_g. In order to meet X, the set P_u is either a disk $D_u \subset I_0$ touching α' and β', or it is a dot S_u touching X. In the latter case, S_u must be attached to a disk

$D_w \in \mathcal{P}'$ otherwise, since the interior of $\triangle$ is connected, S_u would be attached to D_g, what is impossible since we assumed that D_g does not touch X.

In every case, if D_g does not touch X, then there is a disk $D_w \subset I_0$ with $w \in V'$. Assuming that another disk $D_v \in \mathcal{P}'$ does not lie in I_0 leads to a well known chain of disks or dots of $\mathcal{P}'$ from D_w to D_v, i.e., from I_0 to $G \setminus I_0$. As above, we eventually arrive at the contradiction that three disks D_1, D_2 and D_g share at least a single pseudo contact point.

Hence, if D_g does not touch X, then all the disks of $\mathcal{P}'$ lie in I_0 and all its dots lie in $\overline{I_0}$, and if D_g touches X, then $\mathcal{P}'$ only contains dots lying in p.

A.2. Counterexamples, their Centers and Radii

A.2.1. Hexagonal Onion Complex with one Generation

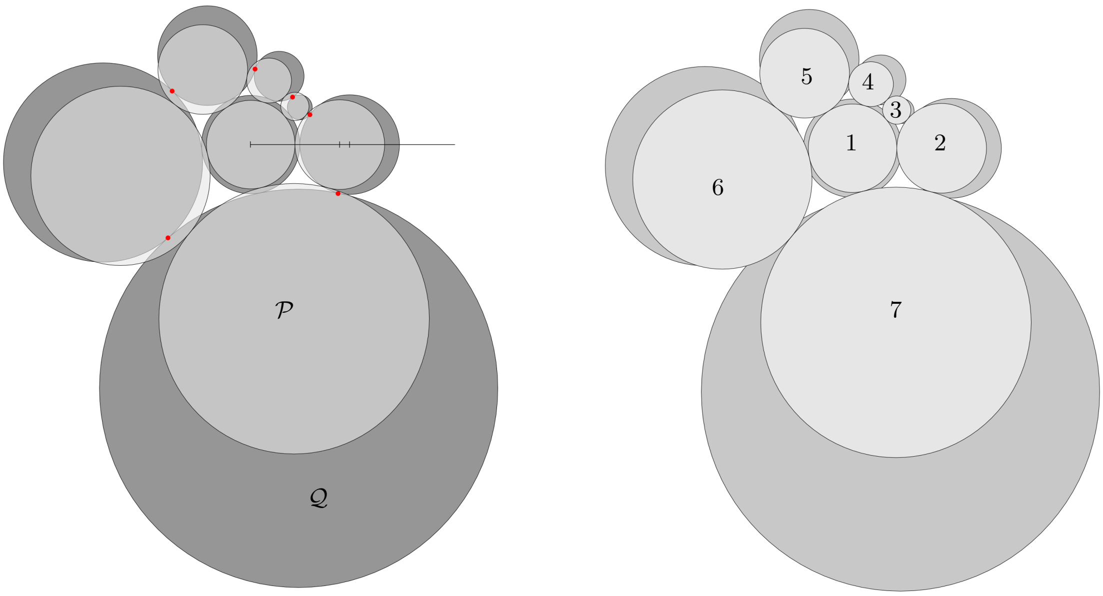


Fig. A.4.: Hexagonal counterexample with 1-generation onion complex

Circle packing $\mathcal{P}$ (brighter color)

Index	Radii	Center
1	0.900000000000	$0.000000000000 + 0.000000000000i$
2	0.911284351257	$1.811284351257 + 0.000000000000i$
3	0.287378773612	$0.898209622347 + 0.776587294737i$
4	0.455642175628	$0.373895268476 + 1.303061025644i$
5	0.911284351257	$-0.974608556384 + 1.526724979468i$
6	1.822568702514	$-2.647317671542 - 0.635680333068i$
7	2.749462562263	$0.882870772720 - 3.541061449910i$

Circle packing $\mathcal{Q}$ (darker color)

Index	Radii	Center
1	1.000000000000	$0.000000000000 + 0.000000000000i$
2	1.012538168063	$2.012538168063 + 0.000000000000i$
3	0.253134542015	$0.998422964850 + 0.757296351273i$
4	0.506269084031	$0.579161866860 + 1.390474050633i$
5	1.012538168063	$-0.880662732377 + 1.809625107505i$
6	2.025076336127	$-3.003168635406 - 0.363407466497i$
7	4.050152672263	$0.974767437609 - 4.955186218069i$

A.2.2. Heptagonal Onion Complex with one Generation

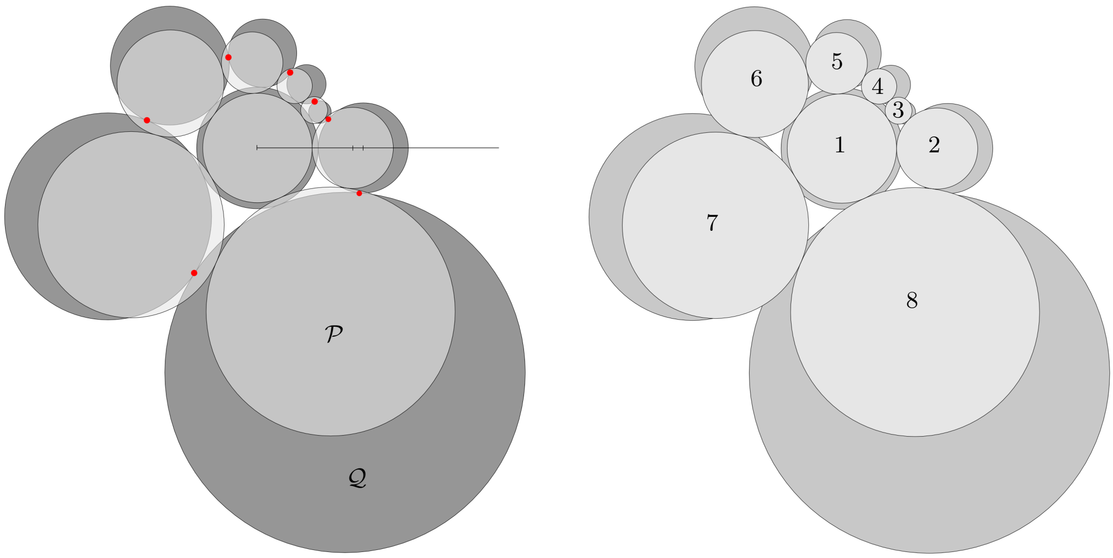


Fig. A.5.: Heptagonal counterexample with 1-generation onion complex

Circle packing $\mathcal{P}$ (brighter color)

Index	Radii	Center
1	0.900000000000	$0.000000000000 + 0.000000000000i$
2	0.667078244356	$1.567078244356 + 0.000000000000i$
3	0.221593297696	$0.932936389822 + 0.622576274831i$
4	0.290362670693	$0.611673367543 + 1.021185085681i$
5	0.505550773065	$-0.086749089851 + 1.402871188339i$
6	0.880215020533	$-1.429633793440 + 1.060807586692i$
7	1.532543363893	$-2.079014553040 - 1.262919516623i$
8	2.047763358783	$1.204368105668 - 2.690502980007i$

Circle packing $\mathcal{Q}$ (darker color)

Index	Radii	Center
1	1.000000000000	$0.000000000000 + 0.000000000000i$
2	0.741198049285	$1.741198049285 + 0.000000000000i$
3	0.185299512321	$1.027541884322 + 0.590840595992i$
4	0.322625189659	$0.806608779760 + 1.048198296475i$
5	0.561723081183	$0.084988624146 + 1.559408835445i$
6	0.978016689481	$-1.441109249359 + 1.354900053611i$
7	1.702825959881	$-2.457133995812 - 1.125948798140i$
8	2.964792197135	$1.440670149152 - 3.693784872973i$

A.2.3. Pentadekagonal Onion Complex with one Generation

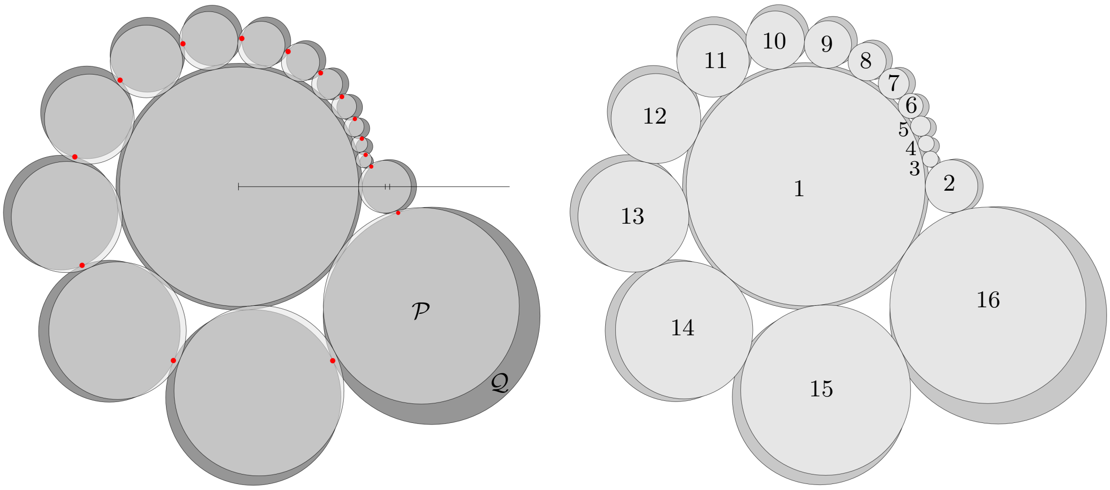


Fig. A.6.: 15s counterexample with 1-generation onion complex

Circle packing $\mathcal{P}$ (brighter color)

Index	Radii	Center
1	0.970000000000	$0.000000000000 + 0.000000000000i$
2	0.213510144827	$1.183510144827 + 0.000000000000i$
3	0.064974764275	$1.011531329690 + 0.219036827364i$
4	0.066066779608	$0.976800243482 + 0.345392032536i$
5	0.081772589706	$0.932521835286 + 0.486444865499i$
6	0.101212083697	$0.851190590406 + 0.650361366520i$
7	0.125272856382	$0.713994291561 + 0.830562930514i$
8	0.155053507179	$0.497754073383 + 1.008953059586i$
9	0.191913801465	$0.179829029340 + 1.147913412345i$
10	0.237536756586	$-0.248287617048 + 1.181735282424i$
11	0.294005487353	$-0.753063684487 + 1.015187154747i$
12	0.363898319721	$-1.216180149595 + 0.547896496688i$
13	0.450406515497	$-1.399009023280 - 0.245618448102i$
14	0.557479983303	$-0.987251960731 - 1.165559378764i$
15	0.690007628866	$0.161427574649 - 1.652139965632i$
16	0.795556772012	$1.478513281336 - 0.964981446510i$

Circle packing $\mathcal{Q}$ (darker color)

Index	Radii	Center
1	1.000000000000	$0.000000000000 + 0.000000000000i$
2	0.220113551368	$1.220113551368 + 0.000000000000i$
3	0.055028387842	$1.035173688481 + 0.203716307221i$
4	0.068110082070	$1.017289020961 + 0.325549067348i$
5	0.084301638872	$0.976046481652 + 0.472274612608i$
6	0.104342354326	$0.897333214065 + 0.643711999651i$
7	0.129147274621	$0.761559505394 + 0.833667012378i$
8	0.159848976474	$0.544277963237 + 1.024212452063i$
9	0.197849279861	$0.220988714054 + 1.177287936540i$
10	0.244883254212	$-0.219116646231 + 1.225447759785i$
11	0.303098440570	$-0.744444795269 + 1.069517411084i$
12	0.375152906929	$-1.236192843350 + 0.602389219263i$
13	0.464336613914	$-1.449256493750 - 0.209612342609i$
14	0.574721632272	$-1.053553536933 - 1.170373087506i$
15	0.711348070996	$0.114988655375 - 1.707480550178i$
16	0.880454205474	$1.562779015707 - 1.045863168369i$

A.2.4. Octagonal Onion Complex with two Generations

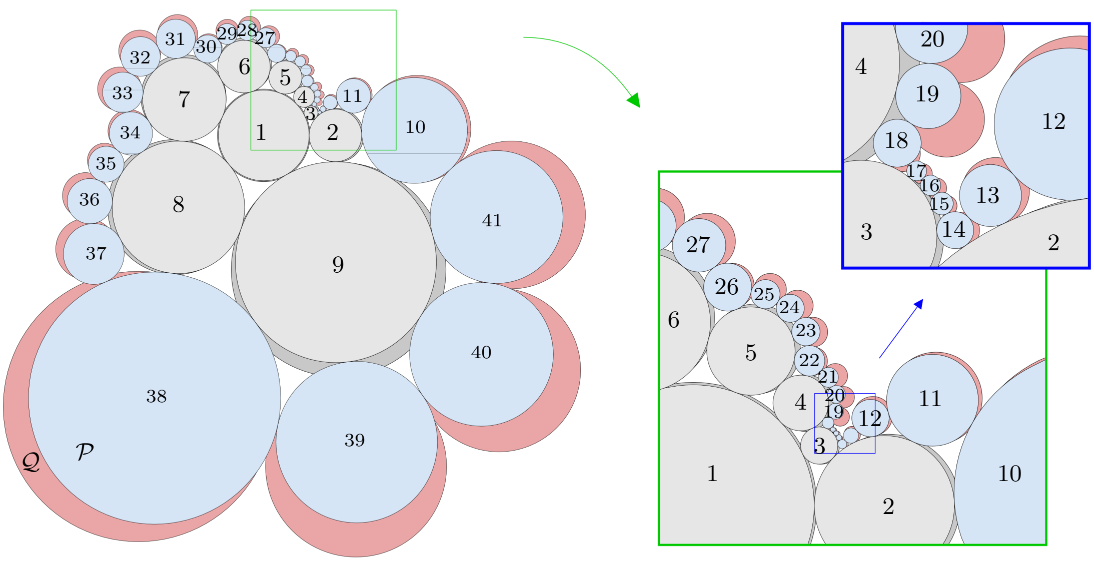


Fig. A.7.: Octagonal counterexample with 2-generations onion complex

Circle packing $\mathcal{P}$ (brighter color)

Index	Radii	Center
1	0.979670000000	$0.000000000000 + 0.000000000000i$
2	0.567799389023	$1.547469389023 + 0.000000000000i$
3	0.150626189358	$1.019760292628 + 0.487502226923i$
4	0.225331336860	$0.870947249299 + 0.832753931707i$
5	0.357691201174	$0.467010202059 + 1.253170560449i$
6	0.567799389023	$-0.428461920677 + 1.486970710032i$
7	0.901325347442	$-1.713748564552 + 0.775376911312i$
8	1.430764804696	$-1.843380817706 - 1.553107565046i$
9	2.170698555026	$1.557417738628 - 2.738479873927i$
10	1.142439582565	$3.254351901225 + 0.107095422313i$
11	0.371640216651	$1.937649541240 + 0.854579558336i$
12	0.151127089103	$1.435240385838 + 0.710112619093i$
13	0.061605922610	$1.276750790889 + 0.568209896901i$
14	0.036680647389	$1.206697112053 + 0.499269836531i$
15	0.022421219830	$1.180282319246 + 0.552140333311i$
16	0.020259845798	$1.157706989261 + 0.588362258153i$
17	0.019656918642	$1.130103905384 + 0.617196575275i$
18	0.047507613418	$1.091530985845 + 0.672180247808i$
19	0.061653909278	$1.149709291956 + 0.764546490211i$
20	0.070214754467	$1.159637840120 + 0.896040856667i$
21	0.080511738664	$1.100612197641 + 1.034729173630i$
22	0.129206048879	$0.948190178703 + 1.178774578121i$
23	0.110820701522	$0.906621562762 + 1.415174432345i$
24	0.111673270827	$0.783044493921 + 1.600194095594i$
25	0.117367380668	$0.584043205034 + 1.713587689145i$
26	0.193945739425	$0.278159214750 + 1.771474157608i$
27	0.216217178554	$0.047047595751 + 2.110326690208i$
28	0.217853850026	$-0.356843883492 + 2.269352885685i$
29	0.225655142475	$-0.793464047532 + 2.191487239963i$
30	0.301818852217	$-1.210136801151 + 1.868048479965i$
31	0.423475048001	$-1.901661672650 + 2.086782550912i$
32	0.427544193173	$-2.662605687260 + 1.705734055996i$
33	0.437452229207	$-3.043660592592 + 0.929193026909i$
34	0.466017447487	$-2.868347129528 + 0.042895872239i$
35	0.386466967335	$-3.403272829213 - 0.620869276990i$
36	0.487898451768	$-3.757441722654 - 1.420293620530i$
37	0.656226444396	$-3.670788415416 - 2.561132339355i$
38	2.723775381244	$-2.357774556188 - 5.675679970524i$
39	1.743580425336	$2.007150223582 - 6.626836934189i$
40	1.551459683101	$4.701736317432 - 4.730394185502i$
41	1.434443270081	$5.027898248299 - 1.762358666731i$

Circle packing $\mathcal{Q}$ (darker color)

Index	Radii	Center
1	1.000000000000	$0.000000000000 + 0.000000000000i$
2	0.579582297124	$1.579582297124 + 0.000000000000i$
3	0.144895574281	$1.038565046346 + 0.481838687234i$
4	0.230007387039	$0.905586171860 + 0.832365218823i$
5	0.365113968146	$0.501796541155 + 1.269541798174i$
6	0.579582297124	$-0.407381967487 + 1.526145525811i$
7	0.920029548156	$-1.733388669442 + 0.825758552147i$
8	1.460455872585	$-1.912485483173 - 1.547980095994i$
9	2.318329188492	$1.617040741537 - 2.897669381314i$
10	1.159164594246	$3.312817726552 + 0.138331841688i$
11	0.373595694908	$1.966556779671 + 0.871090715340i$
12	0.158225267311	$1.454587573450 + 0.727142572809i$
13	0.067011573091	$1.287028652342 + 0.576624659483i$
14	0.036223893570	$1.218860469258 + 0.499096210563i$
15	0.020292665824	$1.190048372148 + 0.547717034764i$
16	0.020292665824	$1.169413132285 + 0.582664939149i$
17	0.020292665824	$1.140879343363 + 0.611526540540i$
18	0.036223893570	$1.101208477906 + 0.651780036285i$
19	0.075376573726	$1.189540873441 + 0.719985990569i$
20	0.086632744827	$1.218721624338 + 0.879345656342i$
21	0.099569827936	$1.150703285335 + 1.052680314339i$
22	0.115003693519	$0.971689529526 + 1.170984461946i$
23	0.120938753547	$0.968037194292 + 1.406898638662i$
24	0.126071952201	$0.844900779795 + 1.621028767742i$
25	0.131423027489	$0.622694060687 + 1.751135795028i$
26	0.182556984073	$0.310298814578 + 1.782642268473i$
27	0.227353536480	$0.113662257816 + 2.142309752972i$
28	0.237003469469	$-0.308062750655 + 2.336668818117i$
29	0.247062990134	$-0.786162910518 + 2.260902239386i$
30	0.289791148562	$-1.189235530847 + 1.906297066022i$
31	0.435111589861	$-1.862608048930 + 2.174724769631i$
32	0.453579733131	$-2.678524472886 + 1.822513672871i$
33	0.472831749605	$-3.114645710988 + 1.005178673605i$
34	0.460014774078	$-2.904845599482 + 0.096230627805i$
35	0.369874642538	$-3.439342584143 - 0.538615351310i$
36	0.494791353375	$-3.851762382600 - 1.298587189163i$
37	0.661895829612	$-3.831215528203 - 2.455091865359i$
38	2.920911745167	$-2.706525275362 - 5.856794667348i$
39	1.954861794710	$1.993451099564 - 7.154249748381i$
40	1.766735196560	$5.076643197483 - 5.069972738776i$
41	1.596713007134	$5.350017542792 - 1.717652603964i$

Glossary

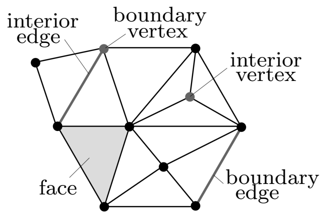
A

Acceptable complex K. *p.51*
Simplicial complex that can be framed by a tri- or quad-complex. The set of acceptable complexes is $\mathcal{K}^*$, allowing at most n vertices yields the class $\mathcal{K}_n^*$.

Accessible. *p.24,35*
Different meanings. A vertex is accessible from an alpha vertex a if it can be connected with a by a chain of interior vertices. The set of all accessible vertices of K defines the kernel K^*. A boundary point p is accessible via a prime end X if every tail of X contains a Jordan arc ending at p. A prime end is accessible if it has an accessible point. Every prime end has at most one accessible point.

Admissible complex K. *p.20*
Simplicial complex with specific properties, finite triangulation of a topological closed disk, sometimes denoted as combinatorial closed disk. The set $\mathcal{K}$ of all admissible complexes comprises the set $\mathcal{K}^0$ of strongly connected complexes and is contained in the set $\mathcal{K}^*$ of acceptable complexes, i.e., $\mathcal{K}^0 \subset \mathcal{K} \subset \mathcal{K}^*$. The class of all K with at most n vertices is $\mathcal{K}_n$.

notation	$K = K(V, E, F)$
set of vertices	$V = \{v_i\}$
set of edges	$E = \{e(v_i, v_j)\}$
set of faces	$F = \{f(v_i, v_j, v_k)\}$



Admissible pair $(\mathcal{P}, \mathcal{P}')$. *p.136*
More precisely the admissible sixtuple $(K, L, G, \mathcal{P}, \mathcal{P}', a)$ so that $\mathcal{P}$ fills G, $\mathcal{P}'$ is contained in G, the lower neighbors of $\mathcal{P}'$ lie in the maximal lower domain and the alpha disks of $\mathcal{P}$ and $\mathcal{P}'$ have the same center in the upper domain.

Alpha-beta-gamma normalization. *p.49*
See normalization.

Alpha-beta normalization. *p.117*
See normalization.

Alpha-gamma normalization. *p.167*
See normalization.

Attached. *p.44*
A dot S is attached to a disk D if it touches D, or if there is a chain of dots connecting S with

D. A dot S attached to two disks $D_1 \neq D_2$ is a pseudo contact point of D_1 and D_2.

B

Blue, red, green. *p.107*
See red, green, blue.

Boundary alpha-beta normalization. *p.168*
See normalization.

Boundary alpha-gamma normalization. *p.93,118*
See normalization.

Boundary chain. *p.24*
Chain of boundary edges of admissible K. Contains every boundary edge exactly once and associates it with a precursor and a successor. Induces a cyclic ordering to the boundary vertices.

Boundary interstice. *p.33*
See interstice.

Boundary irreducible. *p.22*
See reducible.

C

Canonical embedding g, g^*. *p.34*
See canonical parameterization.

Canonical parameterization f, f^*. *p.34*
A conformal mapping $f : \mathbb{D} \to G$. Its extension $f^* : \overline{\mathbb{D}} \to G^*$ is a bijection mapping $\partial\mathbb{D}$ onto ∂G^*. The inverses of f and f^* will be denoted by the canonical embedding g and its extension g^*. Can be normalized in various ways.

Chain. *p.21*
Finite sequence of vertices, edges or faces. Neighboring elements share common edge or vertex.

Circle agglomeration $\mathcal{P}$. *p.57*
See circle packing.

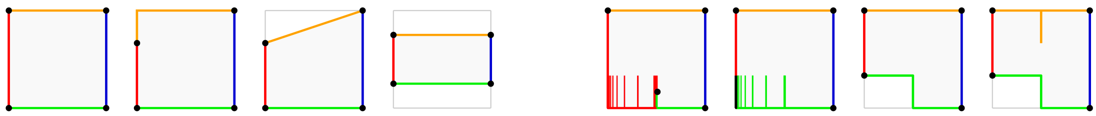
Circle packing $\mathcal{P}$. *p.31*
Collection $\mathcal{P} = \{D_v\}$ of open, non-overlapping disks D_v; univalent. Touching pattern is admissible complex K. Allowing the complex to be acceptable yields circle agglomerations. Allowing disks to be dots yields generalized circle packings or agglomerations.

Collapsed. *p.43,57*
A collapsed generalized circle agglomeration only contains dots and no disks.

Complete reduction $\varrho_+(K)$. *p.23*
See reduction.

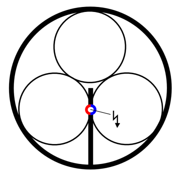
Compressible. *p.77*
See Compression.

Compression. *p.23*
Different Meanings. A quadrilateral $G'(\alpha',\beta',\gamma',\delta')$ is a compression of the quadrilateral $G(\alpha,\beta,\gamma,\delta)$ if $G' \subset G$, $\beta' \subset \beta$ and $\delta' \subset \delta$. If both quadrilaterals are tame, then a generalized circle agglomeration $\mathcal{P}'$ filling $G'(\alpha',\beta',\gamma',\delta')$ is a compression of a generalized circle agglomeration $\mathcal{P}$ filling $G(\alpha,\beta,\gamma,\delta)$. A compression with $\mathcal{P}' \neq \mathcal{P}$ is nontrivial. A quad-complex is called compressible if it admits a nontrivial compression. Otherwise it is incompressible. All boundary irreducible quad-complexes are incompressible (Theorem 3.18, Corollary 3.46).



Contact point. *p.32,43,38*
Different meanings. The contact points $c(u,v)$ of $\mathcal{P}$ are associated with edges $e(u,v)$ and disks or dots $P_u, P_v \in \mathcal{P}$ so that $c(u,v) = \overline{P}_u \cap \overline{P}_v$. The contact point p of a disk or dot touching a prime end X is the accessible point $p \in I(X)$ of X.

Contained. *p.45,58*
Some standard meanings. A generalized circle agglomeration $\mathcal{P}$ is contained in a bounded, simply connected G domain if all its disk lie in G and no contact point or pseudo contact point on ∂G between two disks of $\mathcal{P}$ is accessible via more than one prime end. Not defined if $\mathcal{P}$ is collapsed.



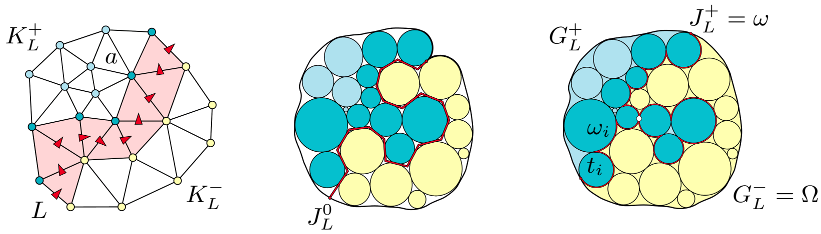
Continuous. *p.102,161,171*
Different Meanings. A family $(G(\alpha_t,\beta_t,\gamma_t))$ of trilaterals is continuous if the via canonical parameterization f associated arcs on $\mathbb{T}$ are so. A family $(G(A_t,C_t))$ of pinned domains with respect to the alpha-gamma normalization is continuous if (r_t) and (s_t) are so for $f(r_t) = A_t$ and $f^*(s_t) = C_t$. A family $G(q_t,\Gamma_t)$ of pinned domains with respect to the boundary alpha-beta normalization is continuous if (q_t) and (Γ_t) are so. A family $(\mathcal{P}_t)$ of generalized circle agglomerations depends continuously on t at t_0 if for every $t_k \to t_0$ the sequence $(\mathcal{P}_{t_k})$ converges to $\mathcal{P}_{t_0}$. If $(\mathcal{P}_t)$ is continuous at all t, then it is a continuous family.

Continuous family. *p.102,161,171*
See continuous.

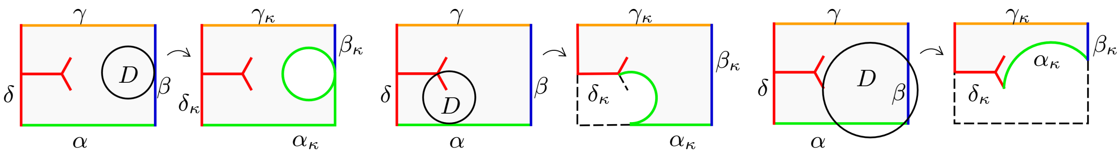
Converge. *p.96*
Different meanings. For prime ends defined via canonical parameterization. Sequence of prime ends converges to limit prime end if corresponding points on $\mathbb{T}$ do so. Analogously for arcs of prime ends. Sequence of trilaterals converges to limit trilateral if all edges converge to limit edges. Sequence of a disk converges if centers and radii converge. Limit can be a dot. Sequence of generalized circle agglomerations converges to limit generalized circle agglomeration if all its disks converge.

Crosscut. *p.34,132*
Different meanings. For bounded, simply connected domains G, it is an open Jordan arc in G

with endpoints on ∂G. Divides G into two simply connected components. For an admissible complex K, it is a special sequence of pairwise different consecutive edges starting and ending on ∂K. Divides K into two connected sub-complexes K_L^+, K_L^- with a in K_L^+. Their vertex sets V_L^+, V_L^- are the upper and lower vertices of K for L. All vertices of V_L^+, V_L^- adjacent to L are the upper and lower neighbors of L forming the sets U_L^+, U_L^-. Also refers to the associated disks. One geometric representation of L is the geometric crosscut J_L^0 formed by the tangents of disks associated with the edges of L. Using instead the corresponding circular arcs of upper neighbors of L defines the maximal crosscut $\omega = J_L^+$. The former is always a crosscut of G. The latter is at least composed from a finite number of circular arcs ω_i linked at the turning points t_i. That component of $G \setminus \omega$, which contains all disks associated with V_L^-, is the maximal lower domain $\Omega = G_L^-$. The upper domain is $G_L^+ := G \setminus \overline{G_L^-}$.



Cutting $\kappa(G; D_v, \gamma)$. *p.83*
Defines a compression $\kappa(G; D_v, \gamma) = G_\kappa(\alpha_\kappa, \beta_\kappa, \gamma_\kappa, \delta_\kappa)$ for $G(\alpha, \beta, \gamma, \delta)$. Is that component of $G \setminus \overline{D_v}$ which contains γ. Not defined if D_v touches γ. Geometrical analog for merging v with a in $Q(a, b, c, d)$.

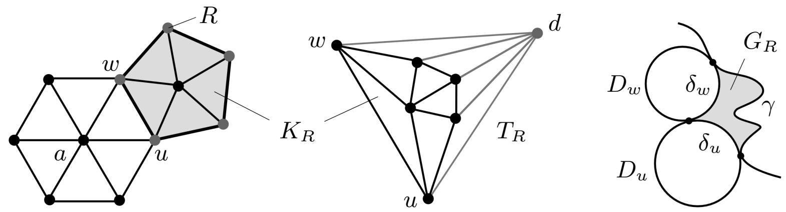


D

Degenerate. *p.43,57*
A degenerate circle agglomeration contains at least one dot.

Degree. *p.52*
Different meanings. The degree of a vertex is the number of its neighbors. For degree of a quad- or tri-complex see leading vertex.

Detour R. *p.121*
Special chain of boundary vertices. Starts at u in the kernel K^* and follows the boundary chain until it is back in K^* as a neighbor w of u. Sub-complex of $K(a, c)$ bounded by R is the detour-complex K_R. Connecting all vertices of R to a new vertex d yields the detour-tri-complex $T_R(w, u, d)$. The sub-packing $\mathcal{P}_R$ of $\mathcal{P}$ for K_R is the detour-packing. The boundary interstice formed by D_u and D_w defines the detour-trilateral $G_R(\delta_w, \delta_u, \gamma)$. Maybe $G_R = \emptyset$. Otherwise, $\mathcal{P}_R$ is a generalized circle agglomeration for $T_R(w, u, d)$ filling $G_R(\delta_w, \delta_u, \gamma)$.



Detour-complex K_R. *p.121*
See detour.

Detour-packing $\mathcal{P}_R$. *p.123*
See detour.

Detour-tri-complex $T_R(w, u, d)$. *p.122*
See detour.

Detour-trilateral $G_R(\delta_w, \delta_u, \gamma)$. *p.123*
See detour.

Discrete conformal modulus. *p.113*
The unique aspect ratio of all standard quadrilaterals for which a quad-complex admits circle agglomerations (Theorem 3.47).

Dot S. *p.43*
A set $S = \{p\}$ consisting of a single point $p \in \mathbb{C}$.

Dubious. *p.119*
Special type of pinned domain $G(A, C)$ with respect to the alpha-gamma normalization. The maximal disk D_a in G with center in A has a contact point s on ∂G with C or with a disjoint disk D_c meeting C. The singular packing $\mathcal{P}$ containing D_a, D_c and any number of dots $S = \{s\}$ trivially fulfills the alpha-gamma normalization for $K(a, c)$ and any dubious $G(A, C)$.

E

Exhaust. *p.101,163,179*
Different meanings. Defined via canonical parameterization f. For a trilateral $G(\alpha, \beta, \gamma)$ with $\alpha = f(a)$, $\beta = f(b)$ and $\gamma = f(c)$ the exhausting trilateral $G_k(\alpha_k, \beta_k, \gamma_k)$ yields $G_k = f(r_k\mathbb{D})$ with $\alpha_k = f(r_k a)$, $\beta_k = f(r_k b)$ and $\gamma_k = f(r_k c)$. For a pinned domain $G(A, C)$ with $A = f(p)$ and $C = f^*(q)$ the exhausting pinned domains $G_k(A_k, C_k)$ are defined via $G_k := f(r_k\mathbb{D})$, $A_k := f(r_k p)$ and $C_k := f(r_k q)$ with respect to the alpha-gamma normalization. For a pinned domain $G(q, \Gamma)$ with $q = f(0)$ and $\Gamma_k \subset \Gamma$ the exhausting pinned domains $G_k(q, \Gamma_k)$ are defined via $G_k := f(r_k\mathbb{D})$ with respect to the alpha-beta normalization.

Extended flower $B^*(v)$. *p.21*
See flower.

Extended kernel $\widetilde{K}$. *p.124*
See kernel.

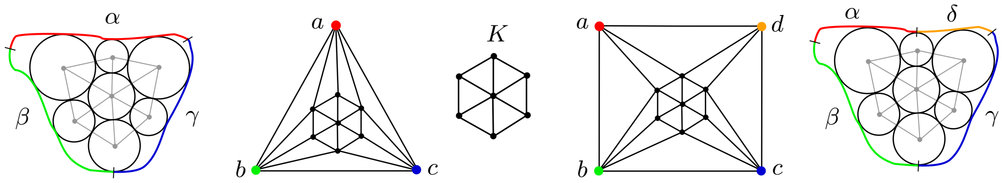
Extended main part $\widetilde{\mathcal{P}}$. *p.124*
See main part.

F

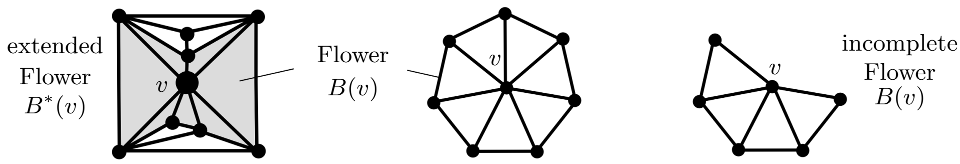
Filling a domain G. *p.58*
A generalized circle agglomeration $\mathcal{P}$ fills a bounded, simply connected domain G if $\mathcal{P}$ is contained in G and all boundary disks or dots touch a prime end of ∂G^*.

Filling a quadrilateral $G(\alpha, \beta, \gamma, \delta)$. *p.58*
See filling a trilateral.

Filling a trilateral $G(\alpha, \beta, \gamma)$. *p.58*
Only defined if K is framed with a tri-complex $T(a, b, c)$ and the vertices a, b, c are associated with α, β, γ. Then a generalized circle agglomeration $\mathcal{P}$ fills a trilateral $G(\alpha, \beta, \gamma)$ if $\mathcal{P}$ is contained in G and every boundary disk or dot P touches its associated edge(s) of ∂G^*. If $G(\alpha, \beta, \gamma)$ is not spiky, then P touches no other edges (Lemma 3.8). Carries the definition of meeting points for leading disks and dots. Applies the alpha-gamma-beta normalization to $\mathcal{P}$. Analogously defined for quadrilaterals associated with a quad-complex.



Flower $B(v)$. *p.21*
Minimal admissible sub-complex of K containing v and all its neighbors. Also known as star of v. Incomplete if v is a boundary vertex. The extended flower is the admissible sub-complex of K containing $B(v)$ and all edges between any of its vertices. For irreducible K and interior v we have $B^*(v) = B(v)$ (Lemma 2.1).



Frames. *p.51*
See quadrilateral or trilateral complex.

G

Generalized circle agglomeration $\mathcal{P}$. *p.57*
See circle packing.

Generalized circle packing $\mathcal{P}$. *p.43*
See circle packing.

Generation, i-th. *p.172*

See onion complex.

Good-natured. *p.125*
Generalized circle packing with non-degenerate extended main part. Otherwise ill-natured. If C is no inward spike, then ill-natured packings for $G(A, C)$ identifies with the singular packing (Lemma 4.8).

Green, blue, red. *p.107*
See red, green, blue.

H

I

Ill-natured. *p.125*
See good-natured.

Impression $I(X)$. *p.35*
The intersection of $\overline{U}_n$ for all tails U_n of a prime end X. At most one point of $I(X)$ is accessible via X. Different prime ends can have the same impression.

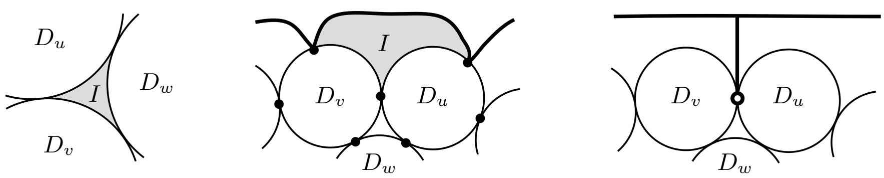
Incircle. *p.72*
A disk touching every edge of a trilateral. Always exists. Is unique for tame trilaterals (Theorem 3.17).

Incompressible. *p.77*
See Compression.

Incomplete flower $B(v)$. *p.21*
See flower.

Inner reduction $\varrho_+(K, \triangle)$. *p.23*
See reduction.

Interstice $I(u, v, w)$. *p.32,40*
Domain bounded by three disks $D_u, D_v, D_w \in \mathcal{P}$ for an oriented face $\langle u, v, w, \rangle$ of K. Not defined for dots. Provided $\mathcal{P}$ fills G, the boundary interstice is the domain bounded by ∂G and two disks $D_u, D_v \in \mathcal{P}$ for an oriented face $\langle u, v, w\rangle$ with boundary edge $\langle u, v\rangle$ of an admissible K. Can be an empty set and is not defined for dots.



Intrinsic strongly connected. *p.52*
See quadrilateral or trilateral complex.

Inward corner. *p.60*
See regular.

Inward spike. *p.60*
See regular.

Irreducible. *p.21*
See reducible.

J

Jordan arc, curve. *p.30*
Homeomorphic images of a segment and a circle, respectively.

Jordan domain G. *p.30*
Bounded component of $\mathbb{C} \setminus J$ for a Jordan curve J. Simply connected domain. Prime ends correspond one to one with boundary points.

K

Kernel. *p.24*
Minimal admissible sub-complex K^* containing all accessible vertices of K. See also strongly connected. Alternatively, minimal admissible sub-complex of K containing a so that $\partial V^* \subset \partial V$. Extended kernel $\widetilde{K}$ is minimal admissible sub-complex of $K(a, c)$ containing a and c so that $\partial\widetilde{V} \subset \partial V$.

L

Layer K_i. *p.172*
See onion complex.

Leading disk, dot. *p.59*
Disk or dot P_v of a generalized circle agglomeration $\mathcal{P}$ associated with a leading vertex in a face $f(v, a, b)$ of a tri- or quad-complex framing K. If $\mathcal{P}$ fills a non-spiky tri- or quadrilateral, then P_v touches α and β, it meets the vertex $\alpha \cap \beta$. If the trilateral is not spiky, then P_v touches $\alpha \cap \beta$ or separates it from $\mathcal{P} \setminus \{P_v\}$ (Lemma 3.9). This property defines meeting for the alpha-gamma normalization.

Leading vertex. *p.52*
Interior vertex v of a quad- or tri-complex forming a face $f(a, b, v)$ with two boundary vertices. The number of pairwise different leading vertices is the degree of the complex. A quad-complex has a degree of 1–4. A tri-complex has degree 1–3. Up to $T \in \mathcal{T}_1^*$, T is reducible if it has degree 1 or 2.

Loner. *p.78,136*
Different meanings. For incompressibility of a quad-complex $Q(a, b, c, d)$ it is a special neighbor v of a. The disk D_v in $\mathcal{P}$ is disjoint to all disks in $\mathcal{P}'$ associated with neighbors $w \neq v$ of a. D_v is even disjoint to $\mathcal{P}' \setminus \{D'_v\}$ (Lemma 3.23). D_v and D'_v are cut out of G and G'. For rigidity it is an upper neighbor v of a crosscut L. The roles of $\mathcal{P}$ and $\mathcal{P}'$ are exchanged, so D'_v is disjoint

to all disks in $\mathcal{P}$ associated with an upper neighbor $w \neq v$. D'_v is even disjoint to $G^+_L \setminus D_v$. Both D_v and D'_v are cut out of G^+_L.

Lower neighbors U^-_L. *p.132*
See crosscut.

Lower vertices V^-_L. *p.132*
See crosscut.

M

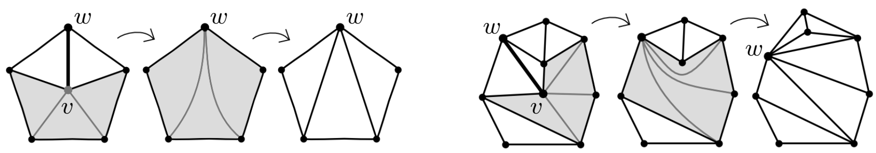
Main part $\mathcal{P}^*$. *p.31,43,124*
For a generalized circle agglomeration $\mathcal{P}$ with complex K, the main part $\mathcal{P}^*$ is the subset of $\mathcal{P}$ associated with the kernel K^* of K. Analogously, the extended main part of $\mathcal{P}$ is associated with the extended kernel of K.

Maximal crosscut $\omega = J^+_L$. *p.134*
See crosscut.

Maximal lower domain $\Omega = G^-_L$. *p.135*
See crosscut.

Meet. *p.59,117*
See leading disk, dot.

Merging $\mu(K; v, w)$. *p.54*
Specific construction for irreducible admissible complex K with neighboring vertices v and w. Removes v and makes its neighbors to be adjacent to w instead. Resulting admissible complex can be reducible. Useful for inductions.



N

Normalization. *p.49,117,167*
Additional constraints in order to obtain uniqueness. Different types. Alpha-beta-gamma normalization lets three different regular prime ends of ∂G^* be met by up to three leading disks. Alpha-gamma normalization fixes the center of the alpha disk and lets the gamma disk meet a regular prime end. Boundary version uses a boundary alpha disk. Alpha-beta normalization fixes the center of the alpha disk and places the center of the beta disk on a fixed ray. Yields uniqueness only for a boundary alpha disk that neighbor the beta disk.

Null-chain (J_n). *p.34*
See prime end.

O

Onion complex K_n. *p.172*
Special type of strongly connected admissible complex. K_1 is a flower around an interior alpha-vertex. K_{n+1} arises from K_n by adding boundary vertices so that (1) the interior of K_{n+1} equals K_n and (2) within ∂K_{n+1} every vertex has exactly two neighbors. The sub-complexes K_i with $0 < i \leq n$ are the *layers* of K_n. The set of all boundary vertices of K_i is called the *i-th generation* of K_n. The alpha vertex is of *generation* 0.

Outer reduction $\varrho_-(K, \triangle)$. *p.23*
See reduction.

P

Pinned domain $G(A, C)$,$G(q, \Gamma)$. *p.117,167*
Different meanings. With respect to the alpha-gamma normalization, a be a bounded, simply connected domain G with fixed interior point $A \in G$ and prime end $C \in \partial G^*$. With respect to the alpha-beta normalization, a bounded, simply connected domain G with fixed interior point $q \in G$ and a ray Γ with starting point q.

Polygonal crosscut J_L^0. *p.133*
See crosscut.

Prime end. *p.34*
A null-chain (J_n) is a nested sequence of crosscuts J_n of G whose diameters shrink to zero. The associated components U_n of $G \setminus J_n$ with $U_{n+1} \subset U_n$ are the tails of J_n. If for every n there exists an m so that $U_m \subset U'_n$ and $U'_m \subset U_n$, then the two null-chains (J_n) and (J'_n) are equivalent. The corresponding equivalence classes are the prime ends of G. There are alternative definitions. The set of all prime ends is denoted ∂G^*, the intrinsic boundary of G. Given a conformal mapping $f : \mathbb{D} \to G$, its extension f^* is a bijection between $\mathbb{T}$ and ∂G^*. This defines an orientation of ∂G^*.

Proper. *p.50*
See quadrilateral or trilateral complex.

Pseudo contact point. *p.44*
See attached.

Q

Quadrilateral $G(\alpha, \beta, \gamma, \delta)$. *p.56*
Bounded, simply connected domain G with ∂G^* decomposed into four arcs $\alpha, \beta, \gamma, \delta$ of prime ends, its edges. The positively oriented vertices of $G(\alpha, \beta, \gamma, \delta)$ are the prime ends $\alpha \cap \beta$, $\beta \cap \gamma$, $\gamma \cap \delta$ and $(\delta \cap \alpha)$. Often associated with a quad-complex. Sometimes conformally mapped onto a standard quadrilateral R by a special embedding $g_R : G \to R$ with extension $g_R^* : G^* \to \overline{R}$.

Quadrilateral complex, quad-complex $Q(a, b, c, d)$. *p.50*
Strongly connected admissible complex with exactly four positively oriented boundary vertices a, b, c, d. Forms w.l.o.g. a square. By definition, there are no edges $e(a, c)$ and $e(b, d)$. The set of all quad-complexes is $\mathcal{Q}^*$, restricting to at most n interior vertices it is $\mathcal{Q}_n^*$. The interior $K = \text{int}Q$ is said to be framed by Q, so Q is a frame for K. Proper quad-complexes frames an

admissible K. The set of all proper quad-complexes is $\mathcal{Q}$. Intrinsic strongly connected Q frame a strongly connected K. The set of all such complexes is $\mathcal{Q}^0$. We have $\mathcal{Q}^0 \subset \mathcal{Q} \subset \mathcal{Q}^*$.

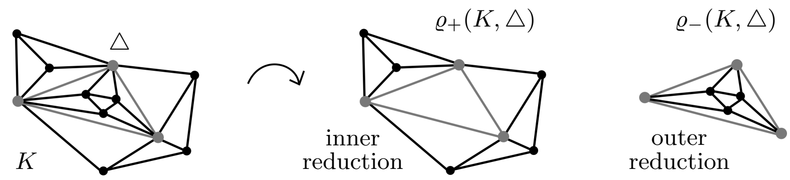
R

Red, green, blue. *p.74,107,159*
Different meanings. Colors of the points of an equilateral triangle $\triangle$. Used to define a Sperner coloring for a triangulation of $\triangle$. Depends on the decomposition of G into the sets $\mathcal{R}, \mathcal{G}, \mathcal{B}$. Depends on the used normalization.

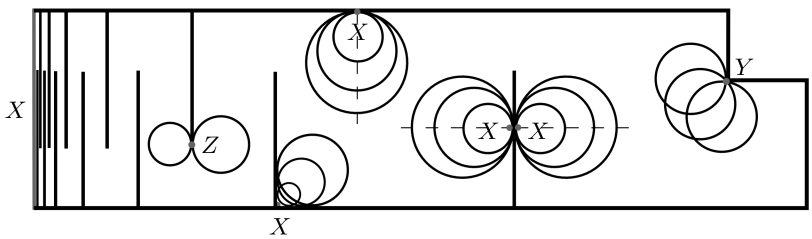
Reducible. *p.21*
Triangle $\triangle(u, v, w)$ is reducible in K if it is no face of K and $K \neq T(u, v, w)$ is no tri-complex. Otherwise $\triangle$ is irreducible; $\mathcal{K}^i$ is the set of all irreducible admissible complexes. For reducible K one can apply inner and outer reductions. If all reducible triangles of K are interior ones, then K is boundary irreducible. Important properties:

- Boundary irreducible quad-complexes are incompressible (Theorem 3.18, Corollary 3.46).
- Up to $\mathcal{T}_1^*$, boundary irreducible tri-complexes are proper (Lemma 3.2).
- Circle agglomerations filling non-spiky tri- or quadrilaterals for boundary irreducible tri- or quad-complexes are not degenerate (Lemma 3.14).

Reduction. *p.23*
Extracts a triangle $\triangle$ from K. Inner reduction $\varrho_+(K, \triangle)$ removes the interior of $\triangle$ making it a face. Outer reduction $K \setminus \varrho_-(K, \triangle)$ removes the exterior of $\triangle$ making it a tri-complex. Complete reduction $\varrho_+(K)$ removes all reducible triangles by inner reduction. Resulting complex of complete reduction is the skeleton $\sigma(K)$ of K.



Regular. *p.59,60*
Different meanings. Characterization of ∂G^*. A prime end X is regular if two disks D_1, D_2 in G touching X imply $D_1 \subset D_2$ or $D_2 \subset D_1$. If X cannot be touched by any disk in G, then it is not only regular but untouchable. A prime end Y is an inward corner if it is not regular. An inward spike Z is an inward corner touched by two disjoint disks in G. Regular prime ends are needed to obtain uniqueness for several normalizations. Inward spikes can lead to degeneration and are often excluded as exceptional cases. A tri- or quadrilateral is spiky if it has an inward spike as vertex. A tri- or quadrilateral is tame if all its vertices are regular. A domain is regular if all its prime ends are regular.



S

Separate. *p.64*
A set $\mathcal{D}$ of disjoint disks in G is separated by one of its members D from a prime end X if X is subordinate to a component of $G \setminus \overline{D}$ that is disjoint to $\mathcal{D} \setminus \{D\}$.

Simplex. *p.19*
Convex hull of a number of affinely independent points.

Simplicial complex. *p.19*
Set of simplexes with specific properties.

Singular packing. *p.125*
See dubious.

Skeleton. *p.23*
See reduction.

Sperner coloring. *p.47*
See Sperner's lemma.

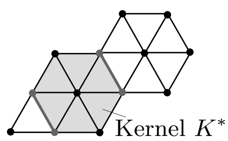
Sperner's lemma. *p.47*
Two dimensional version. Considers triangulation T with vertices colored either 1, 2 or 3 and three distinguished boundary vertices v_1, v_2, v_3. A Sperner coloring for T assumes that v_i is colored i and that all boundary vertices between v_i and v_j are colored i or j. Sperner's Lemma guarantees the existence of a face $f(u, v, w)$ with u, v, w having different colors.

Standard quadrilateral $R(\alpha, \beta, \gamma, \delta)$. *p.56*
Rectangle R with lower, right, upper and left sides α, β, γ and δ parallel to the coordinate-axes. A special quadrilateral.

Spiky. *p.60*
See regular.

Strongly connected. *p.24*
Every boundary vertex of admissible K has exactly two other boundary vertices as neighbors. Is often assumed to avoid degeneration. The class of all strongly connected complexes is $\mathcal{K}^0$. All tri- and quad-complexes are strongly connected. Equivalent definitions (Lemma 2.4):

- Edges between boundary vertices are boundary edges.
- All vertices are accessible; K equals its kernel $K^* = K$.
- K is 3-connected; stays connected after removing 2 vertices.



Subordinate prime end. *p.36*
For $G' \subset G$, a prime end X' of G' is subordinate to a prime end X of G if the tails of X' are contained in the tails of X.

T

Tails (U_n). *p.34*
See prime end.

Tame. *p.59*
See regular.

Triangle. *p.21*
Different meanings. As triangle $\triangle(u,v,w)$ of K it suggests that u,v,w are pairwise adjacent vertices. Can be reducible or a face of K.

Trilateral $G(\alpha,\beta,\gamma)$. *p.56*
Bounded, simply connected domain G with ∂G^* decomposed into three arcs α,β,γ of prime ends, its edges. The positively oriented vertices of $G(\alpha,\beta,\gamma)$ are the prime ends $\alpha\cap\beta$, $\beta\cap\gamma$ and $\gamma\cap\alpha$. Often associated with a tri-complex.

Trilateral complex, tri-complex $T(a,b,c)$. *p.50*
Strongly connected admissible complex with exactly three positively oriented boundary vertices a,b,c. Forms w.l.o.g. an equilateral triangle. By definition, the triangle $\triangle(a,b,c)$ of T is irreducible. The set of all tri-complexes is $\mathcal{T}^*$, restricting to at most n interior vertices it is $\mathcal{T}_n^*$. The interior $K = \mathrm{int}T$ is said to be framed by T, so T is a frame for K. Proper tri-complexes frames an admissible K. The set of all proper tri-complexes is $\mathcal{T}$. Intrinsic strongly connected T frame a strongly connected K. The set of all such complexes is $\mathcal{T}^0$. We have $\mathcal{T}^0 \subset \mathcal{T} \subset \mathcal{T}^*$.

Turning points t_i. *p.135*
See crosscut.

U

Uniqueness property. *p.89*
A class of tri- or quad-complexes has the uniqueness property if all its members yield unique generalized circle agglomerations filling given tri- or quadrilaterals. Every class of tri- or quad-complexes has the uniqueness property (Theorem 3.31).

Untouchable. *p.59*
See regular.

Upper domain G_L^+. *p.135*
See crosscut.

Upper neighbors U_L^+. *p.132*
See crosscut.

upper vertices V_L^+. *p.132*
See crosscut.

V, W, X, Y, Z

List of Symbols

$\ominus$, $S_1 \ominus S_2$	reflected concatenation of slit S_1 with slit S_2; p. 152
$\prec, \preceq, \succ, \succeq$	expresses the ordering of elements within an oriented set; p. 20, p. 35
$\langle u, v\rangle$	oriented edge of K from vertex u to vertex v; p. 20
$\langle u, v, w\rangle$	oriented face of K with vertices u,v and w; p. 20
$\triangle(u, v, w)$	triangle (non-oriented) for pairwise adjacent vertices u, v and w; p. 20
$\mathcal{B}, \mathcal{R}, \mathcal{G}$	decomposition of G, used to define a Sperner coloring; p. 105
$B(v)$	flower, smallest admissible sub-complex of K containing $N(v)$; p. 21
$B^*(v)$	extended flower, a sub-complex of K comprising $B(v)$; p. 21
$c(u, v)$	contact point of the disks or dots P_u, P_v, $c(u, v) = \overline{P}_u \cap \overline{P}_v$; p. 32, p. 43
c_k^-, c_k^+	contact points of boundary disk D_k with D_{k-1} and D_{k+1}, respectively; p. 33
∂	boundary operator, applied to various objects
$\delta(c_k^-, c_k^+)$	exterior boundary arc of D_k; p. 32
$\delta(c_k^+, c_k^-)$	interior boundary arc of D_k; p. 32
$\delta(p, q)$	positively oriented open circular arc from p to q on ∂D; p. 30
$\delta[p, q]$	positively oriented closed circular arc from p to q on ∂D; p. 30
D, D_v	a disk, associated with a vertex v of K; p. 30, p. 31
$\mathcal{D}$	set of all disks within $\mathcal{P}$; p. 43, p. 57
D_q^{max}	maximal disk in G with center $q \in G$; p. 181
E	the set of edges of the complex K; p. 20
e_j^-, e_j^+	lower and upper accompanying edges of the crosscut L, respectively; p. 143
η_k, η	segments connecting the centers of D_k and D_{k+1} and their concatenation; p. 32
$E_L^{\pm}(v)$	sequences of upper and lower accompanying edges of the crosscut L; p. 143
E_S	the edge sequence of the slit S; p. 147
$E_S^{\pm}(v)$	sequences of edges adjacent to a vertex v in a slit S; p. 150
$E_S^{\pm}$	sequences of left and right neighbor edges of slit S, respectively; p. 150
$E(v)$	the (cyclically ordered) set of all edges adjacent to v in K; p. 21
$e(u, v)$	single (non-oriented) edge of K between the vertices u and v; p. 20
f, f^*	the (extended) canonical parameterization of G, or a vertex of K; p. 34
F	set of faces of the complex K; p. 20
$F(v)$	the (cyclically ordered) set of all faces adjacent to v in K; p. 21
$f(u, v, w)$	single (non-oriented) face of K for the vertices u, v and w; p. 20
g, g^*	the (extended) canonical embedding of G, or a vertex of K; p. 34
G	bounded simply connected domain in $\mathbb{C}$, sometimes Jordan domain; p. 30
G^*	compactification $G \cup \partial G^*$ of G using its prime ends ∂G^*; p. 34
$\mathcal{G}, \mathcal{B}, \mathcal{R}$	decomposition of G, used to define a Sperner coloring; p. 105
$G(\alpha, \beta, \gamma)$	trilateral with arcs $\alpha, \beta, \gamma)$; p. 56
$G(\alpha, \beta, \gamma, \delta)$	quadrilateral with arcs $\alpha, \beta, \gamma, \delta)$; p. 56

$G_C(\alpha,\beta,\gamma)$	trilateral of $G \setminus \overline{D_a}$ containing C for $G(A,C)$ and $K(a,c)$; p. 94, p. 119
$G(A,C)$	pinned domain, $A \in G$, $C \in \partial G^*$ for alpha-gamma normaliz.; p. 93, p. 117
G_k	set of contact points of D_k with ∂G, $G_k := \overline{D}_k \cap \partial G$; p. 33
G_k	exhausting domain for G with $G_k = f(r_k\mathbb{D})$; p. 101, p. 163, p. 179
G_L^-, G_L^+	lower and upper domains of G with maximal crosscut J_L^+, $G_L^- = \Omega$; p. 135
$G_R(\delta_w,\delta_u,\gamma)$	detour trilateral of G for $\mathcal{P}$ detour R; p. 123
$G(q,\Gamma)$	domain with $q \in G$, ray Γ and intersection point r ...
$= G(q,r)$	...for alpha-beta normalization; p. 167, p. 168
G_s, G_s'	quadrilaterals after cutting a loner out of G, G'; p. 84
g_k^-, g_k^+	first and the last contact point of D_k with ∂G; p. 33
g_R, g_R^*	conformal mapping onto a standard quadrilateral, its extension; p. 56
$I(u,v,w)$	interstice between the disks D_u, D_v and D_w; p. 32
$I(X)$	the impression of the prime end X; p. 35
I_k	boundary interstice between D_k and D_{k+1}; p. 33, p. 40
J_L^0	polygonal crosscut in G for (combinatoric) crosscut L in K; p. 133
J_L^+	maximal 'crosscut', upper boundary of the lower domain G_L^-, $J_L^+ = \omega$; p. 134
J_n	null-chain with tails U_n defining a prime end X; p. 34
$\kappa(G;D,\gamma)$	cut-out operator, component of $G \setminus \overline{D}$ containing γ; p. 83
K	admissible or acceptable complex, finite triangulation, $K(V,E,F)$; p. 20
K^*	kernel of K, largest sub-complex of K with vertex set V^*; p. 24
$\widetilde{K}$	extended kernel of K with respect to $K(a,c)$; p. 124
$\mathcal{K}, \mathcal{K}_n,$	set of all admissible complexes, with at most n vertices; p. 20
$\mathcal{K}^0$	set of all strongly connected complexes; p. 24
$\mathcal{K}^i$	set of all irreducible admissible complexes; p. 21
$\mathcal{K}^*, \mathcal{K}_n^*$	set of all acceptable complexes, with at most n vertices ; p. 51
$K(a,b)$	complex for alpha-beta normalization; p. 167
$K(a,c)$	complex for alpha-gamma normalization; p. 93, p. 117
K_n	onion complex with n layers around alpha vertex a; p. 172
K_i	i-th layer of onion complex K_n, the i-th generation is ∂K_i; p. 172
K_R	detour-complex of $K(a,c)$ with respect to detour R; p. 121
K_σ	the skeleton $\sigma(K) = K_\sigma$ of K; p. 23
L	combinatorial crosscut, sequence of edges in K; p. 132
$l(i)$	smallest label k of prime end set ω_k^* associated with a loner; p. 79, p. 141
M	loop of a multiple loner v_μ, a sequence of edges; p. 145
$\mu(K;v,w)$	merging of v with w in K; p. 54
$N(v)$	the (cyclically ordered) set of all neighbors of v in K; p. 20
ν_i, π_i	negatively, positively oriented arcs on ∂D from y_i^-, y_i^+ to ω, resp.; p. 138
ν_i^+, π_i^+	terminal points of the arcs ν_i, π_i, respectively; p. 139
ν_i^*, π_i^*	prime ends of Ω associated with ν_i, π_i, respectively; p. 139
Ω	lower sub-domain of G with respect to a maximal crosscut, $\Omega = G_L^-$; p. 135
ω	upper boundary of lower domain Ω, maximal crosscut; p. 134
ω^*	prime ends of Ω associated with ω p. 135
ω_i	circular subarcs of ω in between its turning points; p. 135

ω_i^*	classes of prime ends associated with the arcs ω_i; p. 135
P, P_v	a disk or dot, associated with a vertex v of K; p. 30; p. 43
$\mathcal{P}$	circle packing, generalized circle agglomeration; p. 31, p. 57
$\mathcal{P}^*$	main part of $\mathcal{P}$, associated with kernel K^*; p. 31, p. 43
$\mathcal{P}_0$	minimal packing for dubious $G(A,C)$ and $K(a,c)$; p. 125
$\mathcal{P}_R$	detour packing of $\mathcal{P}$ for detour-complex K_R and detour R; p. 123
$\mathcal{P}_s, \mathcal{P}'_s$	sup-agglomeration of $\mathcal{P}, \mathcal{P}'$ for Q_s without a loner; p. 84
$\mathcal{P}_\sigma$	the sup-packing of $\mathcal{P}$ associated with the skeleton; p. 23
$Q(a,b,c,d)$	quad-complex with boundary vertices a,b,c,d, shortly Q; p. 50
$\mathcal{Q}$	the set of all proper quad-complexes Q; p. 50
$\mathcal{Q}^0$	the set of all intrinsic strongly connected quad-complexes Q; p. 52
$\mathcal{Q}^*$	the set of all quad-complexes Q; p. 50
$\mathcal{Q}_n, \mathcal{Q}_n^*$	class of $\mathcal{Q}, \mathcal{Q}^*$ with at most n interior vertices; p. 50
Q_s	quad-complex after merging with a loner; p. 84
Q_σ	the skeleton $\sigma(Q) = Q_\sigma$ of Q; p. 23
$\varrho_+(K)$	complete (inner) reduction, yields the skeleton $\sigma(K)$ of K; p. 23
$\varrho_+(K, \triangle)$	inner reduction, removes the interior of $\triangle$ from K; p. 23
$\varrho_-(K, \triangle)$	outer reduction, removes the exterior of $\triangle$ from K; p. 23
R	detour of $K(a,c)$ with respect to kernel K^*; p. 121
$R(\alpha,\beta,\gamma,\delta)$	standard quadrilateral, rectangle with sides parallel to coordinate-axes; p. 56
$\mathcal{R}, \mathcal{G}, \mathcal{B}$	decomposition of G, used to define a Sperner coloring; p. 105
$r(i)$	largest label k of prime end set ω_k^* associated with a loner; p. 79, p. 141
S, S_v	a dot, degenerate disk associated with a vertex v of K; p. 43
S	combinatoric slit, a sequence of vertices; p. 147
$\mathcal{S}$	set of all dots within $\mathcal{P}$; p. 43, p. 57
$\sigma(K)$	skeleton of K, result of complete reduction $\varrho_+(K)$; p. 23
S_L^-, S_L^+	sequences of lower and upper accompanying vertices of L, resp.; p. 144
$T(a,b,c)$	tri-complex with boundary vertices a,b,c, shortly T; p. 50
$T(a,v_1,v_2)$	special tri-complex for alpha-gamma normalization $K(a,c)$; p. 94, p. 119
$\mathcal{T}$	the set of all proper tri-complexes T; p. 50
$\mathcal{T}^0$	the set of all intrinsic strongly connected tri-complexes T; p. 52
$\mathcal{T}^*$	the set of all tri-complexes T; p. 50
$\mathcal{T}_n, \mathcal{T}_n^*$	class of $\mathcal{T}, \mathcal{T}^*$ with at most n interior vertices; p. 50
$T_R(w,u,d)$	detour tri-complex for detour-complex K_R and detour R; p. 122
T_σ	the skeleton $\sigma(T) = T_\sigma$ of T; p. 23
t_i	turning points of the upper boundary ω, cusps of Ω; p. 135
U_L^-, U_L^+	sets of lower and upper neighbors of L, $U_L^- \subset V_L^-, U_L^+ \subset V_L^+$; p. 132
U_M	sequence of the vertices in V_M for a loop M; p. 145
U_n	tails of a null-chain J_n defining a prime end X; p. 34
V	vertex set of the complex K; p. 20
V^*	vertex set of the kernel of K; p. 24
$\widetilde{V}$	vertex set of the extended kernel of K; p. 124
$v(i)$	vertex of the disk which contains the circular arc ω_i, $v(i) \in U_L^+$; p. 138

V_L^-, V_L^+	lower and upper vertices of K with crosscut L, subsets of V; p. 132
V_M	set of all vertices met by a loop M; p. 145
$x_k, \mathcal{X}$	contact points of upper with lower disks in $\mathcal{P}$, the set of all x_k; p. 137
X_i	sets of contact points x_k on ω_i, $X_i \subset X$; p. 137
y_-, y_+	initial point and terminal point of α, respectively; p. 139
y_k, Y	contact points of upper with lower disks in $\mathcal{P}'$, the set of all y_k; p. 137
y_i^-, y_i^+	minimal and maximal element of Y_i, respectively; p. 138
Y_i	sets of contact points y_k with $x_k \in \omega_i$, $Y_i \subset Y$; p. 137
z_-, z_+	terminal points of ν_1 and π_n, respectively; p. 139

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