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Proper Connection Number of Graphs

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Thesis

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submitted by **M.Sc. Trung Duy Doan**

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Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such.
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Contents

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Glossary of Notation

Non-alphabetic notation

$\binom{n}{2}$	2 choose n	4
$c : E(G) \to [k]$	assigning all the edges of G with colours from $[k]$	7
$\lvert\mathbb{S}\rvert$	cardinality of set $\mathbb{S}$.	4
$G \Box H$	Cartesian product of graphs	7
$\mathbb{A} \times \mathbb{B}$	Cartesian product of sets.	4
$\overline{G}$	complement of G.	6
$v_1 v_2 \dots v_n v_1$	cycle on n vertices	5
$C_n = v_1 v_2 \dots v_n v_1$	cycle C_n	5
$C = v_1 v_2 \dots v_n v_1$	cycle C	5
$G \times H$	the direct product	23
$\sqcup$	the operation symbol for the disjoint union	4
$G - e$	subgraph obtained by deleting an edge e	6
$G[S]$	subgraph induced by S in G	6
$\mathbb{A} \cap \mathbb{B}$	intersection of two sets $\mathbb{A}, \mathbb{B}$	4
$\cap_{i=1}^{k} \mathbb{U}_i$	intersection of k sets $\mathbb{U}_i$	4
$G \cong H$	G is isomorphic to H	6
$G \vee H$	the join of graphs	7
$[k]$	set $\{1, 2, \dots, k\}$.	4
$L(G)$	the line graph	23
$P = v_1 \dots v_n$	path between v_1, v_n	5
$P = (v_1, v_n)$	path between v_1, v_n	5
$v_i P v_j$	subpath of P from v_i to v_j.	5
$P_{\alpha(G)}$	The permuation graph	23
G^k	The kth power graph of G.	23
$\mathbb{A} \setminus \mathbb{B}$	relative complement	4
$[U_1, U_2]$	the set of edges between U_1 and U_2	7
$\mathbb{A} \subseteq \mathbb{B}$	$\mathbb{A}$ is a subset of $\mathbb{B}$	4
$\mathbb{A} \cup \mathbb{B}$	union of two sets $\mathbb{A}, \mathbb{B}$.	4
$\cup_{i=1}^{k} \mathbb{U}_i$	union of k sets $\mathbb{U}_i$	4
$G_1 \cup \dots \cup G_k$	the union of graphs	7
$G - v$	subgraph obtained by deleting a vertex v	6
$G - U$	subgraph obtained by deleting $U \subseteq V(G)$ or $U \subseteq E(G)$ from G	6

Greek alphabet

Θ-graph	graph consists of a cycle and an ear	7

Roman alphabet

1 Introduction

Nowadays, graph theory is one of the most interesting subjects in discrete mathematics. The problem that is often said to be the birth of graph theory is the Seven Bridges of Königsberg: Is there a walk route that crosses each of the seven bridges of Königsberg exactly once? In 1736, Euler published the paper on solution of this problem which is regarded as the first paper in the history of graph theory [33]. After 200 years of the first paper, the first textbook on graph theory was written. This was done by König in 1936 [53]. Since then many publications have been published about the new problems of graph theory. As well as many textbooks have been written about graph theory. One of them is the Introduction to Graph Theory of West. In [71] the author wrote that graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics. Graph theory is to study graphs which are mathematical structures given to model pairwise relations between objects. A graph in this context consists of the vertices which are connected by the edges. By time to time, graph theory has developed into an extensive and popular branch of mathematics, which are widely used to study and model in the distinct areas of mathematics, computer science, social, natural sciences and other scientific and not-so-scientific areas [40]. They include, study of molecules, construction of bonds in chemistry and the study of atoms. Similarly, graph theory is used in sociology for example to measure actor prestige or to explore diffusion mechanisms [2]. In computer science, graphs are used to represent networks of communication, data organizations, computational devices, etc.

In graph theory, graph colouring, which plays an important role, is the assigment of labels or colours to the edges or vertices of a graph. The most common types of graph colouring are edge-colouring and vertex-colouring. There are many interesting applications using graph colourings. The committe-scheduling example used graph colouring to model avoidance of conflicts [71]. Moreover, graph colouring especially is used in computer science such data mining, image segmentation, clustering, image capturing, networking etc. For example, a data structure can be designed in the form of a tree which in turn utilized vertices and edges [69].

The connectivity which is one of the most fundamental concepts of graph theory plays an important role in a combinatorial and an algorithmic sense. There exist many interesting results on connectivity in graph theory. Moreover, the connectivity plays an important role for security in a communication network which is defined in [20, 58, 4] and for accessibility in a communication network which is defined in [57, 35, 4].

Consider a communication network of wireless signal agencies, one fundamental requirement is that the network is connected. Hence, the information is sent through the network from agency A to agency B by an information transmission path. There are two types of these paths in the network which are called a direct information transmission path and an undirect information transmission path, since the direct information transmission paths for all agencies are expensive. The undirect information transmission path between two agencies A and B is a set of the direct information transmission paths such that there are some intermediary agencies on this path connecting them.

Recently, the security of the communication network is very important after the September 11, 2001 terrorist attacks. Hence, the Department of Homeland Security was created in 2003 in response to weaknesses discovered in the transfer of classified information. Ericksen made the following observation [32]:

> *An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels, from radio systems to databases. The technologies utilized were separate entities and prohibited shared access, meaning that there was no way for officers and agents to cross check information between various organizations.*

The information is sent from agency A to agency B by the information transmission path need to be protected, (since the secure requirements which permit access between appropriate parties). Hence, we must require a large enough number of passwords and firewalls that are able to prevent the attack to the information transmission paths. A natural question appears: What is the minimum number of passwords or firewalls needed that allows at least one secure information transmission path between two agencies A and B such that the passwords along each path are distinct? By modeling on a graph, each agency as a vertex, each direct information transmission path between two agencies as an edge, and each possible password of the information transmission path as different colours on the edges of this path. It can be readily seen that the information transfers through the communication network between two agencies A and B by the secure information transmission path is the connection between two vertices A and B of the edge-colouring graph by a coloured path whose edges receive distinct colours. This concept called *rainbow connection number* was introduced by Chartrand et al. [20] in 2008.

Shall we come back the above practical problem with other conditions to apply the results of graph theory for the accessibility of the communication networks? If there is not any direct information transmission path connecting two agencies A and B, then there must be some intermediary agencies of the information transmission path connecting them. To avoid interference, it would help if the input signal and the output signal of the intermediary agencies can not share the same frequency. The communication network can be represented by the edge-colouring graph as follows: each agency as a vertex, each direct information transmission path between two agencies as an edge, and each frequency as the colour of the edge. Clearly, the information sends from agency A to agency B by the information transmission path whose the input signal and the output signal of the intermediary agencies are different is the connection between two vertices A and B of the edge-colouring graph by a coloured path whose consecutive edges receive distinct colous. This concept called *proper connection number* was introduced independently by Borozan et al. [8] and Andrews et al. [4] only recently.

The main contribution of this dissertation is to study the proper k-connection number $pc_k(G)$ of connected graphs G. For $k = 1$, we characterize some classes of 2-connected graph with proper connection number 2. Besides, we disprove Conjecuture 3 which was posed by the authors in [8] by constructing the classes of graphs with the minimum degree at least 3 have proper connection number 3. For $k = 2$, we prove a new upper bound on the proper 2-connection number $pc_2(G)$ and determine $pc_2(G) = 2$ for several classes of graphs. The detailed chapters of this dissertation are structured as follows:

- In Chapter 2, the theoretical background is described. This includes notation and terminology of graph theory, the introduction of some frequently used graphs, and the definition of problem and its related problems.
- In Chapter 3, we present the existent results, which are examples, propositions, corollaries, lemmas, theorems and, conjectures on the proper connection number. This chapters includes four sections which are the preliminary results, the proper connection number of a connected bridgeless graph, the proper connection number of graphs having bridge and proper connection number 2 of some classes of graphs.
- In Chapter 4, we consider the relation between the proper connection number and minimum degree of a connected graph. In this chapter, Conjecture 3 in [8] is disproved by constructing classes of graphs with minimum degree at least 3 that have proper connection number 3. Further, we study proper connection number 2 in a 2-connected graph with the condition of minimum degree.
- In Chapter 5, we characterize the classes of connected, $S_{i,j,k}$-free graphs whose the proper connection number is 2.
- In Chapter 6, we consider the proper 2-connection number of a connected graph G. We prove a new upper bound for $pc_2(G)$ and study proper 2-connection number 2 of several classes of graphs, among them the Cartesian product of two non-trivial connected graphs.
- Finally, in Chapter 7, we propose some open questions and problems of the proper k-connection number.

2 Notation, terminology and definition

In this chapter, we introduce the definitions, notation and terminologies of graph theory, and the notation of frequently used graphs which are needed to follow the dissertation. Moreover, we also describe the problems and its related problems.

2.1 Notation and terminology

In this section, most of the notation and terminologies of graph theory used throughout in the dissertation are described. All of them can be found in [7, 71].

First of all, we briefly summarize some notation of sets which will be used later on. For simplified notation, let $[k]$ be the set $\{1, 2, \ldots, k\}$ for some positive integers k. Let $\mathbb{A}, \mathbb{B}$ be two sets of elements. We denote by $|\mathbb{A}|$ the *cardinality* of the set $\mathbb{A}$ which is its number of elements. $\mathbb{A}$ is a *subset* of $\mathbb{B}$, denoted by $\mathbb{A} \subseteq \mathbb{B}$ if all elements of $\mathbb{A}$ are also elements of $\mathbb{B}$. The *relative complement* of $\mathbb{B}$ in $\mathbb{A}$ denoted by $\mathbb{A} \setminus \mathbb{B}$ is set of elements in $\mathbb{A}$ but not in $\mathbb{B}$. A *Cartesian product* of two sets $\mathbb{A}, \mathbb{B}$ denoted by $\mathbb{A} \times \mathbb{B}$ is the set of all ordered pairs (a, b) where $a \in \mathbb{A}$ and $b \in \mathbb{B}$. That is $\mathbb{A} \times \mathbb{B} = \{(a, b) | a \in \mathbb{A}, b \in \mathbb{B}\}$. The *intersection* or the *union* of two sets $\mathbb{A}, \mathbb{B}$ are written by $\mathbb{A} \cap \mathbb{B}$ or $\mathbb{A} \cup \mathbb{B}$, respectively. For an arbitrary integer $k \geq 3$, we denote by $\cup_{i=1}^{k} \mathbb{U}_i$ or $\cap_{i=1}^{k} \mathbb{U}_i$ the intersection or the union of k sets $\mathbb{U}_i$, where $i \in [k]$, respectively. Let $\sqcup$ be *the operation symbol for the disjoint union* of sets.

All graphs considered in the dissertation are finite, undirected, simple graphs i.e. without multiple edges and loopless. A *graph* is an ordered pair $G = (V, E)$ with a *vertex set* $V(G)$ and an *edge set* $E(G)$. The edge set $E(G)$ is a 2-elements subset of $V(G)$ or an empty set. Since we consider only the finite graph G, it means that both $|V(G)|$ and $|E(G)|$ are finite. Moreover, $|E(G)|$ is bounded by 0 and $\binom{|V(G)|}{2}$ since G is loopless and without multiple edges. Unless stated otherwise, let us denote by $n(G) = |V(G)|$ and $m(G) = |E(G)|$ the *number of the vertices* and the *number of the edges* of G, respectively. Some times, we denote $n(G)$ as the *order* of G and $m(G)$ as the *size* of G. If G is obviously defined , then we can write V, E, n, m, instead of $V(G), E(G), n(G), m(G)$ for short, respectively.

Let $u, v \in V(G)$ be two distinct vertices in G. By the definition above, if $(u, v) \in E(G)$, then u, v are *adjacent* in G. Otherwise, u, v are *non-adjacent* in G. To simplify notation, we can write uv, instead of (u, v). Vice versa, if e is an *edge* of $E(G)$, then there exist two distinct vertices $u, v \in V(G)$ such that $e = uv$. Hence, u, v are called the *end-vertices* of e and u, e or v, e are *incident*. Moreover, v is a *neighbour* of u and we denote by $v \in N_G(u)$ where $N_G(u)$ is called the *neighbour set* of u.

Clearly, for every vertex $u \in V(G)$, we denote by $N_G(u) = \{v|uv \in E(G)\}$. The *close neighbour set* of u is denoted by $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of a vertex u in G which is the cardinality of $N_G(u)$ is denoted by $d_G(u) = |N_G(u)|$. It is clear that $\Delta(G) = \max\{d_G(u)|u \in V(G)\}$ and $\delta(G) = \min\{d_G(u)|u \in V(G)\}$ are the *maximum degree* and the *minimum degree* of G, respectively. We can write $\Delta, \delta, d(u)$, instead of $\Delta(G), \delta(G), d_G(u)$ for short, respectively.

For an integer k, a *$k-$regular graph* is a graph whose maximum degree and the minimum degree are number k, i.e. its all vertices are of degree k.

A *path* is a simple graph on two or more vertices which can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are non-adjacent otherwise. We denote by $P = v_1v_2\dots v_n$ or $P(v_1, v_n)$ the path between v_1, v_n, where n is integer. Hence, $d_P(v_i) = 1$ if and only if $v_i \in \{v_1, v_n\}$ and $d_P(v_i) = 2$ if and only if $v_i \in V(P)\backslash\{v_1, v_n\}$. Moreover, v_1, v_n are end-vertices of P. Clearly, $|V(P)| = n$ and $|E(P)| = n - 1$. For a path P and two vertices $v_i, v_j \in V(P)$, we denote by v_iPv_j the subpath of P from v_i to v_j.

Likewise, a *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are non-adjacent otherwise. Let $n \geq 3$ be an integer. Hence, we denote by $C = v_1v_2\dots v_nv_1$ or $C_n = v_1v_2\dots v_nv_1$ the cycle on n vertices. All vertices of C have degree number 2. Clearly, $|V(C)| = |E(C)| = n$.

A *chord* of a cycle C is an edge not in C whose end-vertices lie in C. A *chordless cycle* in G is a cycle of length at least 4 in G that has no chord (that is, the cycle is an induced subgraph). A graph G is *chordal* if it is simple and has no chordless cycle.

Let P, Q be two paths connecting u, v in G. Two paths P, Q are called *internally vertex-disjoint paths* if they have no common internal vertices, i.e. $V(P) \cap V(G) = \{u, v\}$.

A graph G is said to be *connected* if, for every two distinct vertices $v_i, v_j \in V(G)$, there exists at least one path $P = v_iv_{i+1}\dots v_{j-1}v_j$ connecting them in G. That is a *$(v_i, v_j)-path$*. Otherwise, G is *disconnected*. If G has a v_i, v_j-path, then v_i is *connected to* v_j in G. The *length* of a v_i, v_j-path denoted by $L_G(v_i, v_j)$ is its number of edges. The least length of a v_i, v_j-path written by $d_G(v_i, v_j)$ is said to be the *distance* from v_i to v_j. If G has no such path, then $L_G(v_i, v_j) = d_G(v_i, v_j) = \infty$. If G is known from the context, we can write $L(v_i, v_j), d(v_i, v_j)$, instead of $L_G(v_i, v_j), d_G(v_i, v_j)$, for simplicity. The *diameter* of G denoted by $diam(G)$ is $\max\{d(v_i, v_j)|v_i, v_j \in V(G)\}$.

A graph H is a *subgraph* of G if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *maximal* connected subgraph of G is a connected subgraph which is not contained in an other connected subgraph of G. The *components* of a graph G are its maximal connected subgraphs of G. The number of components of G is written by $\omega'(G)$. A component (or graph) is said to be *trivial* if it has no edges; otherwise it is *non-trivial*. A *forest* or an *acyclic* graph is a graph that does not contain any cycle as a subgraph. Moreover, a connected forest written by T is said to be a *tree*. A *leaf* or *pendant vertex* is a vertex of degree 1.

An operation which is to delete a vertex v from G with together all the edges incident with v or an edge e from G but keeping all the vertices and the remaining edges intact is called *vertex deletion* or *edge deletion*, respectively. A *cut-vertex* or a *cut-edge* of a graph G is a vertex or an edge, respectively, whose deletion increases the number of components. We write $G - v$ or $G - S$ for the subgraph that is obtained by deleting a vertex v or a vertex subset $S \subseteq V(G)$ from G. If $G - S$ has more than one componenent, then we say that S is a *separating set* or a *vertex-cut set* of G. The *connectivity* of G written $\kappa(G)$ is the minimum size of a vertex subset $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. A graph G is *k-connected* if its connectivity is at least k. Likewise, we write $G - e$ or $G - M$ for the subgraph that is obtained by deleting an edge e or an edge subset $M \subseteq E(G)$ from G. If $G - M$ has more than one component, then M is said to be a *disconnecting set* or an *edge-cut set* of G. The *edge-connectivity* of G written $\kappa'(G)$ is the minimum size of an edge subset $M \subseteq E(G)$ such that $G - M$ is disconnected. A graph G is *k-edge-connected* if its edge-connectivity is at least k.

A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. Moreover, if G itself is connected and has no cut-vertex, then G is a block.

An edge in a graph G is said to be a *bridge* if it is a cut-edge of G. Vice versa, a *bridgeless graph* is a graph without cut-edges. Note that if G is a 2-edge-connected graph, then G is bridgeless.

We say that a subgraph H is an *induced subgraph* of G or G induces H if H can be obtained from G by deleting some (possibly none) vertices together with all incident edges. For a vertex subset $S \subseteq V(G)$, the subgraph obtained from G by deleting all the vertices from $V(G) \setminus S$ is induced by S and denoted by $G[S]$. A *spanning subgraph* H of G is a subgraph which is obtained by deleting some (possibly none) edges, i.e. $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A *spanning tree* of G is a spanning subgraph of G that is a tree.

A graph G is said to be *F-free* ($\mathcal{F}$-free) if G contains no induced subgraph F (all graphs of $\mathcal{F}$) which is isomorphic to F (all graphs of $\mathcal{F}$).

Let G and H be two graphs. An *isomorphism* from G to H is a bijection function $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say that G is *isomorphic* to H denoted by $G \cong H$ if there exists a isomorphism from G to H.

The *complement* of a simple graph G, denoted by $\overline{G}$, is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A vertex subset $S \subseteq V(G)$ is said to be an *independent set* of G or a *clique* of G if every two vertices $u, v \in S$ are non-adjacent or adjacent, respectively. The *independence number* of G denoted by $\alpha(G)$ is the maximum size of an independent set in G. The *clique number* of G written by $\omega(G)$ is the cardinality of a maximum clique in G.

A *Hamiltonian path* is a path that visits each vertex exactly once such a graph is also called *traceable*. Morveover, a *Hamiltonian graph* is a graph with a spanning cycle, also called a *Hamiltonian cycle*.

For two simple graphs G and H, the *Cartesian product* G and H, denoted by $G \square H$, is the simple graph with vertex set $V(G) \times V(H)$ specified by putting (u_1, v_1) adjacent to (u_2, v_2) if and only if $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. It means that

$$E(G \square H) = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \text{ and } v_1v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E(G)\}.$$

If G is a graph and $U_1, U_2 \subseteq V(G)$ are two disjoint vertex sets, then $[U_1, U_2]$ denotes the *set of edges between vertices of U_1 and vertices of U_2*.

The *union* of graphs $G_1 \cdots G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\cup_{i=1}^{k} V(G_i)$ and edge set $\cup_{i=1}^{k} E(G_i)$. The graph obtained by taking the union of graphs G and H with disjoint vertex sets is the *disjoint union*, written $G + H$. The *join* of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{xy : x \in V(G), y \in E(G)\}$.

Let us denote F be a subgraph of G. An *ear* of F in G is a nontrivial path whose only two end-vertices are in F and its others vertices are not in F. An *ear decomposition* of G is a decomposition $P_0, P_1, \ldots, P_k$ such that P_0 is a cycle and P_i is an ear of $P_0 \cup \ldots \cup P_{i-1}$.

Let G be a graph, $u, v \in V(G)$ be two distinct vertices, and $P = w_1 w_2 \ldots w_k$ be a path, vertex disjoint from G. We say, we *add the ear* P to G by adding P and the edges uw_1 and vw_k. Hence, for a Θ*-graph* G, there exist a cycle C and a path P such that G is obtained by adding the ear P to C. We define a *2-ear-cycle* and a *3-ear-cycle* to be a graph obtained by adding an ear to a Θ-graph or 2-ear cycle, respectively.

In a graph G, the *subdivision* of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.

A *k-edge-colouring* is a labeling $c : E(G) \to [k]$ that uses exactly k different colours to all the edges of G such that one colour to each edge of G and two adjacent edges can be assigned the same colour. Once the edges of G are assigned by c, an *(edge-)coloured graph* G is given. For an edge $e = uv$ of G, we write by $c(uv)$ or $c(e)$ the colour of the edge uv or e. A path P in an edge-coloured graph G is called a *coloured path*. We denote by *start*(P) and *end*(P), respectively, *colour of the first edge* and *colour of the last edge* of a coloured path P. If P is just an edge uv, then *start*(P) = *end*$(P) = c(uv)$.

2.2 Frequently used graphs

In this section, we give the introduction and denotation of some commonly used graphs that are used several times in this dissertation. Let n be an integer. We denote by $P_n = u_1 \ldots u_n$, where $n \geq 2$ be a path of order n (see Figure 2.1). By the definition of a path, it can be readily seen that the size of P_n is $n - 1$. Likewise, we write by $C_n = u_1 \ldots u_n u_1$, where $n \geq 3$ be a cycle of order n (see Figure 2.2). A cycle is a graph whose the number of vertices and the number of edges are the same, i.e. $|V(C_n)| = |E(C_n)|$. A *complete graph* denoted by K_n (see Figure 2.3), where $n \geq 1$ is a simple graph whose vertices are pairwise adjacent.

A graph G is *bipartite* if its vertex set can be partitioned into two vertex subsets X, Y, i.e $V(G) = X \cup Y$ and $X \cap Y = \{\emptyset\}$ such that every edge has one end-vertex in X and one end-vertex in Y. A bipartite graph is written by $G[X, Y]$. A *complete bipartite*, see Figure 2.4, written by $K_{m,n}$ is a simple biparte graph and every vertex in X is adjacent to every vertex in Y. If $m = 1$, then $K_{1,n}$, see Figure 2.5, is said to be a *star* of order $n + 1$ and size n. Note that a tree whose the maximum degree is its size is a star. A *claw* written by $K_{1,3}$ is a star of order 4 where $m = 1, n = 3$. Let $k \geq 3$ be integer. Likewise, a graph is called *multipartite* or *k-partite* if its vertex set can be partitioned

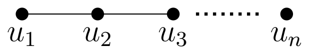

Fig. 2.1: P_n

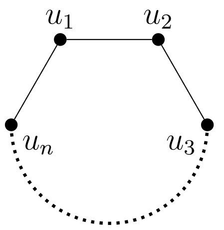

Fig. 2.2: C_n

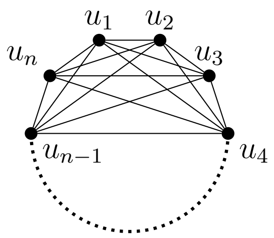

Fig. 2.3: K_n

into k vertex subsets $X_1 \cdots X_k$, i.e $V(G) = \cup_{i=1}^{k} X_i$ and $X_i \cap X_j = \{\emptyset\}$ such that every edge has one end-vertex in X_i and one end-vertex in X_j, , where $i, j \in [k]$ and $i \neq j$.

Let i, j, k be three integers such that $i \geq j \geq k \geq 0$. For generality of two graphs P_n and $K_{1,3}$, we denote by $S_{i,j,k}$ a graph that contains three induced paths P_i, P_j, P_k of orders i, j, k, respectively, such that they have a common initial vertex v, i.e. $P_i \cap P_j \cap P_k = \{v\}$ and $P_x \cap P_y = \{v\}$, where $x, y \in \{i, j, k\}$, see Figure 2.6.

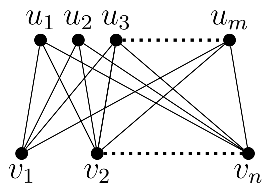

Fig. 2.4: $K_{m,n}$

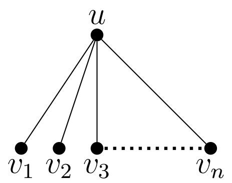

Fig. 2.5: $K_{1,n}$

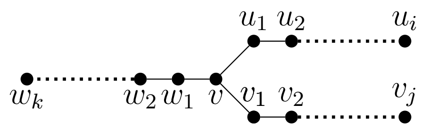

Fig. 2.6: $S_{i,j,k}$

2.3 Definition of problem and its related problems

Let G be an edge-coloured graph by c. If adjacent edges of G receive different colours, then c is *proper (edge-)colouring*. Moreover, G is called a *proper (edge-)coloured* graph. The minimum number of colours needed to colour all the edges of G to make it proper coloured is called *chromatic index number*. That is denoted by $\chi'(G)$.

The edge colouring problem is to determine the chromatic index number $\chi'(G)$ of a nontrivial, connected and edge-coloured graph G, that is, the minimum number of colours needed to colour all the edges of G such that no two adjacent edges have the same colours. This problem, which was first written in 1880 by Tait et al. [68] in relation with four colour problem, is an interesting problem in graph theory. It has many applications in scheduling, for example, the sport timetabling. In [13], the authors show that the edge colouring problem of a complete graph can be used to schedule a round-robin tournament into a few rounds as possible so that each team plays against each other in one of the rounds. In this application, the vertices of G correspond to the teams, the edges of G correspond to the games and the coloured edges correspond to the rounds in which the games are played.

Connectivity which is very important in graph theory has many application in Computer Science and Biology. There are many interesting results which are related to the connectivity in graph theory e.g, independent set in a connected graph, Hamiltonian cycle in 2-connected claw-free graphs, pancyclic of 3-connected graphs,… Recently, many problems of the information security and the information transmission in a communication network are solved by using a connected, coloured graph G. These problems may be used by the graph theoretic model as follows:

Let us denote by G a nontrivial, connected and edge-coloured graph of order n and size m. A path in G is said to be a *$\mathcal{P}$-coloured path*, or more simply a *$\mathcal{P}$-path* if its edges receive colours with the property $\mathcal{P}$. An edge-coloured graph G is called *$\mathcal{P}$-connected* if every two vertices are connected by at least one $\mathcal{P}$-path. The *$\mathcal{P}$-connection number* of a connected graph G is the smallest number of colours that are needed in order to make it $\mathcal{P}$-connected.

The concept of the $\mathcal{P}$-connection number which is said to be the rainbow connection number $rc(G)$ was first introduced by Chartrand, Johns, McKeon, and Zhang [20] in 2008. For a *rainbow connected* graph, every two distinct vertices are connected by a *rainbow path* whose no two edges are assigned the same colours. The rainbow connection number $rc(G)$ of a connected graph G is the minimum number of colours that are needed in order to make it raibow connected graph. Recently, many interesting results of the rainbow connection number $rc(G)$ are published, for example the rainbow connection number 2 for several classes of graphs by Kemnitz et al. [50, 51], the rainbow connection number and forbidden subgraphs by Holub et al. [41, 42, 43], the rainbow connection number and minimum degree by Caro et al. [14] and Schiermeyer et al. [66, 67]. More results in this topic are referred to Li et al. [58] for a survey. Furthermore, the NP-hardness of determining $rc(G)$ and the NP-complete of deciding whether $rc(G) = 2$ were proved by Chakraborty et al. [15].

Inspired of the rainbow connection number and conflict-free colouring of graphs and hypergraphs in [22, 23, 34, 64], the *conflict-free connection number* denoted by $cfc(G)$ was introduced by Czap, S. Jendrol', and J. Valiska [26] in 2016. An edge-coloured graph G is *conflict-free connected* graph if every two distinct vertices are connected by a path, which contains a colour used on exactly one of its edges. The conflict-free connection number $cfc(G)$ of a connected graph G is the smallest number of colours that are needed in order to make it conflict-free connected. After the definition of the conflict-free connection number, there are many nice results which are immediately published in [16, 18, 27].

The newest concept of the $\mathcal{P}-$connection number which is the *odd connection number* is recently introduced by Brause, Jendrol', and Schiermeyer [12] during the C5 workshop 2017 in Rathen. A path in an *odd connected graph G* is an *odd coloured path* if each colour is either used an odd or zero number of times for the edges. The odd connection number of a connected graph G, denoted by $oc(G)$, is the minimum number of colour that are needed in order to make it odd connected graph.

The proper connection number denoted by $pc(G)$ is one of the most interesting concept of the $\mathcal{P}-$connection numbers. Motivated by the rainbow connection number and proper colouring, Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero, and Tuza [8] and Andrews, Lumduamhom, Laforge, and Zhang [4] have independently introduced the concept of $pc(G)$. A path P in an coloured graph G is called a *proper coloured path*, or more simply *proper path* if two its consecutive edges receive different colours. An edge-coloured graph G is called a *properly connected graph G* if every pair of vertices is connected by a proper path. The proper connection number $pc(G)$ of a connected graph G is the smallest number of colours that are needed in order to make G properly connected. There exist many interesting results of proper connection number which are recently studied by the Mathematicians and Researchers, for example the proper connection number 2 and minimum degree by Borozan et al. [8], by Brause et al. [11, 9], by Huang et al. [44, 45], the proper connection number 2 and forbidden subgraph by Brause et al. [10], the proper connection number and size of graphs by Aardt et al. [1], the proper connection of products of graphs by Mao et al. [62], by Hammack et al. [38], the large proper connection number in several graphs by Lumduanhom et al. [60] and some other interesting results in [48, 17, 37, 31, 47, 46, 56, 59, 36]. More detailed results can be seen in the dynamic survey of the proper connection number by Li et al. [57].

After the concepts of the $\mathcal{P}$-connection number are defined, the natural question about its existence in a connected graph G appeared. Since P_2 is the smallest nontrivial connected graph, the $\mathcal{P}$-connection number of G is at least 1. Moreover, if we colour all the edges of G of size m such that each edge is assigned by a different colour from $[m]$, then G is a $\mathcal{P}$-connected graph. Hence, the $\mathcal{P}$-connection number of G is at most m. Therefore, the $\mathcal{P}$-connection number which always exists in a connected graph is bounded by 1 and m.

By the definition above, the connectivity of graph theory is the minimum number of vertices or edges which are removed to disconnect the graphs. Nowadays, the connectivity has many applications, especially in computer science. In [71, 6], the authors described the important role of the connectivity of graph theory in a communication network. They said that a good communication network is hard to disrupt. It means that if a communication network is represented by a graph, then the graph is still connected even when some vertices or edges are removed from it. Let k, l be two integers, where $1 \leq k \leq l$. Suppose that G is an l-connected graph. Hence, it follows from a well-known theorem of Whitney in [72] that every two distinct vertices of G are connected by k internally vertex-disjoint paths. Motivated by this concept, the *$\mathcal{P}$-k-connection number* of a nontrivial, l-connected and coloured graph G is defined as follows:

A graph G is said to be a *$\mathcal{P}$-k-connected graph* if every two vertices of G are connected by at least k internally vertex disjoint $\mathcal{P}$-coloured paths, more simply *k disjoint $\mathcal{P}$-paths*. The $\mathcal{P}$-k-connection number of G is the smallest number of colours that are needed in order to make it the $\mathcal{P}$-k-connected graph.

The *rainbow k-connection number* of the l-connected graph G which was also introduced by Chartrand, Johns, McKeon, and Zhang [21] in 2009 is denoted by $rc_k(G)$. Graph G is called a rainbow k-connected graph if every pair of vertices are connected by k disjoint rainbow path. The rainbow k-connection number is the minimum number of colours that are needed in order to make G rainbow k-connected graph. When $k = 1$, we denote by $rc(G)$, instead of $rc_1(G)$.

Inspired of the rainbow connectivity of the l-connected graph G, Borozan et al. [8] also introduced the concept of *proper $k-$connection number* denoted by $pc_k(G)$. A coloured-graph G is said to be a *proper $k-$connected graph* if there exist at least k *internally vertex disjoint proper paths*, more simply *k disjoint proper paths* connecting two distinct vertices $u, v \in V(G)$. When $k = 1$, we denote by $pc(G)$, instead of $pc_1(G)$.

Recently, there are only some results for $pc_2(G)$ proved by Huang et al. [48], Doan et al. [30] and not many results for $pc_k(G)$, for generality of integer k, of several special classes of graphs proved by Borozan et al. [8], Laforge et al. [56].

Similarly to the $\mathcal{P}$-connection number of a nontrivial connected graph, there always exists the $\mathcal{P}$-k-connection number of a l-connected graph G of size m, where $2 \leq k \leq l$. If we assign every edge $e \in E(G)$ by a different colour from $[m]$, then G is a $\mathcal{P}$-k-connected graph. Hence, there always exists the $\mathcal{P}$-k-connection number of a graph G. Moreover, the upper bound of the $\mathcal{P}$-k-connection number is at most m. By the definition of the P-k-connection number above, it can be readily seen that there are at least two edges of the k-th $\mathcal{P}$-path. It follows that the $\mathcal{P}$-k-connection number of G is at least 2.

3 The proper connection number $pc(G)$

In this chapter, we study the proper connection number $pc(G)$ of a connected graph G. This chapter is organized as follows: the first section is written some fundamental results on the proper conection number, the second section contains the results on the proper connection number of bridgeless connected graphs, the third section consists of the results on the proper connection number of connected graphs having bridges and in the last section some results on the proper connection number 2 of connected graphs are presented. Let G be a nontrivial connected graph of order n and size m. From Section 2.3, we know that the proper connection number $pc(G)$ is bounded by:

$$1 \leq pc(G) \leq m$$

Moreover, the proper connection number $pc(G)$ is related to the rainbow connection number $rc(G)$ and the chromatic index number $\chi'(G)$. If we colour all the edges of G by $\chi'(G)$ colours to make G a properly coloured graph, then G is a proper connected graph by the definition of the proper connected graph in Section 2.3. Hence, $pc(G) \leq \chi'(G)$. By the definitions of a rainbow connected graph and a proper connected graph, it follows that if G is a rainbow connected graph, then G is a proper connected graph, too. Hence, $pc(G) \leq rc(G)$. Therefore, the authors in [4, 8] immediately deduce that the proper connection number $pc(G)$ is bounded by:

$$1 \leq pc(G) \leq min\{\chi'(G), rc(G)\} \leq m$$

To clearly understand the concept of the proper connection number, we consider the following example, the 3-regular graph H_1 in Figure 3.1, that is given by Andrews et al. [4]. This graph consists of three bridges which must receive distinct colours. It can be readily observed that $pc(H_1) \geq 3$. On the other hand, three colours which are shown in Figure 3.1 are enough to make H_1 a proper connected graph. By the definition of the proper connection number, the number of colours is minimum so $pc(H_1) \leq 3$. Hence, $pc(H_1) = 3$. Note that each uncoloured edge can be assigned an arbitrary colour from [3]. Moreover, this assignment has no effect on the result of the proper connection number of H_1.

For general cases, the proper connection number $pc(G)$ is bounded by the maximum of the rainbow connection number $rc(G)$ and the chromatic index number $\chi'(G)$. Hence, the following results illustrate that there exist infinitely many connected graphs whose pairs of $(pc(G), rc(G))$ or $(pc(G), \chi'(G))$ can receive arbitrary values.

Proposition 3.1 (Andrews et al. [4]). *Let a, b be two integers, where $2 \leq a \leq b$.*

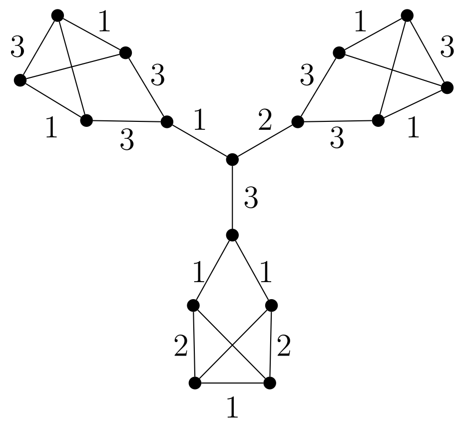

Fig. 3.1: $pc(H_1) = 3$ (Andrews et al. [4])

1. *There is a connected graph G such that $pc(G) = a$ and $rc(G) = b$,*
2. *There exists a connected graph G such that $pc(G) = a$ and $\chi'(G) = b$.*

The most fundamental results on the proper connection number which were claimed by the authors in [4, 8] are listed as follows.

Fact 3.2 (Andrews et al. [4], Borozan et al. [8])**.** *If G is a nontrivial, connected graph of order n and size m, then*

1. *$pc(G) = 1$ if and only if $G \cong K_n$, where $n \geq 2$,*
2. *$pc(G) = m$ if and only if $G \cong K_{1,m}$, where $m \geq 1$,*
3. *$pc(P_n) = 2$, where $n \geq 3$,*
4. *$pc(C_n) = 2$, where $n \geq 4$.*

By Fact 3.2, there are infinitely many connected graphs whose proper connection number obtains the lower bound 1 or the upper bound m, where m is size of a connected graph. Now, we continue to present some known and interesting results of the proper connection number in the next sections.

3.1 Preliminary results

At the first section of Chapter 3, we state some preliminary results on the proper connection number $pc(G)$. Since some uncoloured edges of the graph H_1 in Figure 3.1 can be assigned any colours from [3], it follows that the proper connection number of a new graph which is obtained by removing these edges is not changed. Generally, if every two vertices are connected by a proper path P in a connected spanning subgraph of G, then this proper path P may be still a proper path connecting them in G. Therefore, the following proposition which is the relationship between the proper connection number of G and the proper connection number of a connected spanning subgraph of G is proved by Andrews et al. [4].

Proposition 3.3 (Andrews et al. [4]). *Let G be a nontrivial connected graph. If H is a connected spanning subgraph of G, then $pc(G) \leq pc(H)$. In particular, $pc(G) \leq pc(T)$ for every spanning tree T of G.*

Recently, many interesting results on the proper connection number are proved by using the result of Proposition 3.3, for instance, in [8, 9, 10, 11] the authors proved their results by taking a 2-connected spanning subgraph, in [48] the authors proved that there exists a 2-connected bipartite spanning subgraph of G on the condition of its size. Moreover, by Fact 3.2 and Proposition 3.3, the following result was immediately deduced by Andrews et al. [4] and Borozan et al. [8].

Corollary 3.4 (Andrews et al. [4] & Borozan et al. [8]). *If G is a traceable graph that is not a complete graph, then* $\mathrm{pc}(G) = 2$.

By Proposition 3.3 and Corollary 3.4, it can be readily seen that the proper connection number of a Hamiltonian graph is determined by the following corollary.

Corollary 3.5. *Let G be non-complete, connected graph of order $n \geq 4$. If G is Hamiltonian, then $pc(G) = 2$.*

By Fact 3.2, $pc(G) = 1$ if and only if $G \simeq K_n$, where $n \geq 2$, so many nice results which are published recently study classes of connected graphs whose proper connection number is 2, for more details, see [8, 45, 17, 48, 1, 11, 44, 9, 10]. Moreover, deciding whether a connected graph has proper connection number $pc(G) = 2$ is still an open question by Ducoffe et al. [31]. The main results of this dissertation which are presented in next the chapters are also to determine several classes of connected graphs having proper connection number 2. One of the most beautiful results on the proper connection number 2, which is proved by Borozan et al. [8], is very useful to study the proper connection number 2 of a connected graph. This result is as follows.

Lemma 3.6 (Borozan et al. [8]). *Let G be a nontrivial connected graph and H be a connected subgraph of G such that $pc(H) \leq 2$. If $u \in V(G) \setminus V(H)$ and $N_H(u) \geq 2$, then $pc(H \cup u) \leq 2$.*

As the general case of Lemma 3.6, Yue et al. [73] gave the following proposition.

Proposition 3.7 (Yue et al. [73]). *Let G be a nontrivial connected graph and H be a connected subgraph of G such that $pc(H) = k$, where $k \geq 2$. If $u \in V(G) \setminus V(H)$ and $N_H(u) \geq 2$, then $pc(H \cup u) \leq k$.*

The result of Lemma 3.6 is a process which creates a subgraph of G consisting of a connected graph H of proper connection number at most 2 and a new vertex u that is not in H and has at least two neighbours in H. Then $H \cup \{u\}$ has the proper connection number at most 2. Motivated by Lemma 3.6, we introduce the new extension result as follows which is already published in [11].

Lemma 3.8 ([11]). *Let G be a graph and $H \subset G$ be a subgraph of G such that $pc(H) \leq 2$. If there is a cycle C in G of even length such that $V(C) \cap V(H) \neq \emptyset$ and $V(C) \setminus V(H) \neq \emptyset$, and the colouring of H admits a proper colouring of $C[V(H)]$, then $pc(G[V(H) \cup V(C)]) \leq 2$.*

Proof. [11] For simplicity, let us denote $C = w_1 \cdots w_k v_1 \cdots v_l$ be a cycle of even length, where $w_1 \cdots w_k \in V(H)$ and $v_1 \cdots v_l \in V(G) \setminus V(H)$. Note that possibly $k = 1$ or $l = 1$ but $k + l \geq 4$ since C is the even cycle. By the condition of Lemma 3.8, the edge-colouring of H restricted to the edges $w_1w_2, w_2w_3 \cdots w_{k-1}w_k$ of C admits a proper colouring for $C[V(H)]$. By using this edge-colouring we continue to colour all the remaing edges of C by alternatingly. Hence, a proper colouring of C is obtained, i.e every two consecutive edges of C receive distinct colours. Since C is not necessarily induced in G, we colour all noncoloured remaining edges of $G[V(H) \cup V(C)]$ by some arbitrarily colour.

It can be readily seen that C and H are properly connected by themselves. Hence, it remains to show that there exists a proper path between all pairs of vertices $v_i \in V(C)$ and $w \in V(H) \setminus \{w_1 \cdots w_k\}$. Since H has proper connection number at most 2, there is a shortest proper path, say Q, between w and a vertex $w_j \in V(C)$ in H, i.e. w_j is the first common verter of $V(Q)$ and $V(C)$. Moreover, since C is not only the even cycle but also the properly connected graph, every pair vertices of C are connected by two proper paths P_1, P_2 such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$. So we can choose one of them connecting v_i and w_j on C, say P_1, such that $wQw_jP_1v_i$ is a proper path. Hence, $G[V(H) \cup V(C)]$ has proper connection number 2.

This finishes the proof of our Lemma. □

Clearly, Lemma 3.8 is an extensive version of Lemma 3.6. Furthermore, that lemma can be used as a basic tool to study our several results which will be showed in the next chapters.

Motivated by Proposition 3.7 and Lemma 3.8, we immediately obtain the following result.

Proposition 3.9. *Let G be a nontrivial connected graph and H be a connected subgraph of G such that $pc(H) = k$, where $k \geq 2$. If there is a cycle C in G of even length such that $V(C) \cap V(H) \neq \emptyset$ and $V(C) \setminus V(H) \neq \emptyset$, and the colouring of H admits a proper colouring of $C[V(H)]$, then $pc(G[V(H) \cup V(C)]) \leq k$.*

3.2 The proper connection number of a connected bridgeless graph

In this section, we introduce some well-known results of the proper connection number of graphs having no bridges. First of all, the concept of the *strong property* which was suggested by Borozan et al. [8] is written as follows.

Definition 1 (Borozan et al. [8]). *Let G be a properly connected graph. G is said to have the strong property if for any pair of vertices u, v, there always exist two proper paths (not necessary internally vertex-disjoint proper paths), say P_1 and P_2, such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

The authors in [8] claimed several results of the proper connection number depending on the connectivity of a connected graph. Moreover, the following result improves upon the upper bound of the proper connection number to the best possible.

Theorem 3.10 (Borozan et al. [8]). *If G is a 2-connected graph, then $pc(G) \leq 3$. Furthermore, there exists an edge-colouring $c : E(G) \to [3]$ having the strong property.*

By Theorem 3.10, the proper connection number of a 2-connected graph is bounded by 3. Moreover, the authors in [8] introduced a construction of 2-connected graphs having the proper connection number 3. It means that the upper bound of Theorem 3.10 is reached by the following proposition.

Proposition 3.11 (Borozan et al. [8]). *Given an interger $k \geq 12$. Let $C = u_1 \cdots u_k$ be an even cycle, $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \leq k$ be six integers such that $i_2 - i_1, i_4 - i_3, i_6 - i_5 \geq 3$, and $P_1 = v_1^1 \cdots v_{k_1}^1$, $P_2 = v_1^2 \cdots v_{k_1}^2$, $P_3 = v_1^3 \cdots v_{k_1}^3$ be three vertex disjoint paths such that $k_1 + i_2 - i_1$, $k_2 + i_4 - i_3$, $k_3 + i_6 - i_5$ are even. Then the graph obtained by adding the edges $u_{i_1}v_1^1, u_{i_2}v_{k_1}^1, u_{i_3}v_1^2, u_{i_4}v_{k_2}^2, u_{i_5}v_1^3, u_{i_6}v_{k_3}^3$ has proper connection number 3.*

The smallest 2-connected graph having proper connection number 3 is depicted in Figure 3.2. It follows that B_2 having proper connection number 3 is also an example of Proposition 3.11.

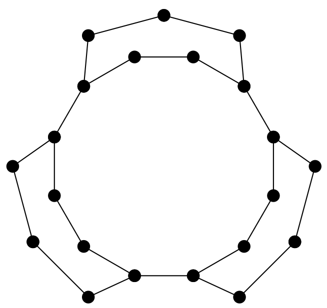

Fig. 3.2: Graph B_1 with $pc(B_1) = 3$ (Borozan et al. [8])

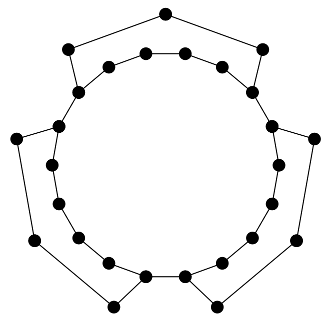

Fig. 3.3: Graph B_2 with $pc(B_2) = 3$

It can be readily seen that all 2-connected graphs with the proper connection number 3 constructed by Propostion 3.11 contain some odd cycles, for example B_1, which contains three odd cycles. A question immediately arises about the proper connection number of a 2-connected graph that has no odd cycle. The answer for this question was studied by the authors in [8] as follows.

Theorem 3.12 (Borozan et al. [8]). *If G is a 2-connected bipartite graph, then $pc(G) = 2$. Furthermore, there exist an edge-colouring $c : E(G) \to [2]$ having the strong property.*

The following result which was proven by Paulraja et al. [65] is very important to determine the proper connection number $pc(G)$ of a connected graph with high connectivity.

Theorem 3.13 (Paulraja et al . [65]). *If G is a 3-connected graph, then G has a 2-connected bipartite spanning subgraph.*

By using Theorem 3.12 and Theorem 3.13 the following result was readily deduced by the authors in [8].

Theorem 3.14 (Borozan et al. [8]). *If G is a 3-connected and non-complete graph, then $pc(G) = 2$. Furthermore, there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

Based on Fact 3.2 and Theorem 3.14, it follows that every non-complete graph G with $\kappa(G) \geq 3$ has proper connection number 2.

The well-known result of the relationship between the (vertex-) connectivity $\kappa(G)$ and the edge-connectivity $\kappa'(G)$ of a graph was published many years ago by Whitney et al. [72].

Theorem 3.15 (Whitney et al. [72]). *If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.*

By Theorem 3.15, every k-connected graph is a k-edge-connected graph. So it is quite natural to consider the proper connection number of a 2-edge-connected graph. In [8] the authors presented theirs proofs of Theorem 3.10 and Theorem 3.12 by using an induction on the number of ears in an ear decomposition. Moreover, using the same argument, the authors in [8] claimed that the results still hold if one replaces 2-connectivity by 2-edge-connectivity. But they did not give the detailed proofs of the proper connection for a 2-edge-connected graph. On the other hand, by using an induction on the number of blocks, the detailed proofs of the proper connection number for a 2-edge-connected graph and a 2-edge-connected bipartite graph were reproven by Huang et al. [48]. Since a graph is said to be 2-edge-connected if it is bridgelese, the results are listed as follows.

Theorem 3.16 (Borozan et al. [8] & Huang et al. [47]). *If G is a bridgeless graph, then $pc(G) \leq 3$. Furthermore, there exists an edge-colouring $c : E(G) \to [3]$ having the strong property.*

Theorem 3.17 (Borozan et al. [8] & Huang et al. [47]). *If G is a bridgeless bipartite graph, then $pc(G) = 2$. Furthermore, there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

Note that any graph which is constructed by Proposition 3.11 is not only 2-connected but also 2-edge-connected, for example B_1, B_2 are depicted in Figure 3.2 and Figure 3.3. Hence, there exist many 2-edge-connected graphs whose proper connection number reaches the upper bound of Theorem 3.16. Further, there are 2-edge-connected graphs which are not 2-connected have proper connection number 3. For example, graph B_3, see Figure 3.4, is 2-edge-connected with a cut-vertex. By a simple case to case analysis, one can readily observe that $pc(B_3) = 3$.

Since all graphs having 3-cut-edges or more and motivated by results of Theorem 3.14, we study the proper connection number of 3-edge-connected graphs. The following result, which is published in [10] determines the proper connection number of graphs having high edge-connectivity.

Theorem 3.18 ([10]). *If G is a 3-edge-connected non-complete graph, then $pc(G) = 2$. Furthermore, there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

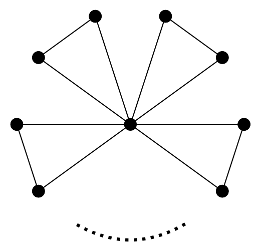

Fig. 3.4: $B_3 = kK_2 \times K_1$, where $k \geq 4$

It can be readily observed that Theorem 3.18 closes the gap in transforming Theorems 3.12 and 3.14 to their edge-connected version.

Before we start proving Theorem 3.18, let us mention some well-known results, which are very important in our proofs.

Theorem 3.19 (Menger [63]). *Let G be a graph. If $u, v \in V(G)$ are two distinct, non-adjacent vertices, then the size of a minimum vertex-cut for u and v equals the maximum number of internally pairwise vertex disjoint $u - v$ paths.*

In our proofs in the following chapters, we use sometimes different versions of Theorem 3.19, which are well-known, too. The following results are described below.

Corollary 3.20. *Let G be a graph. If $u \in V(G)$ is a vertex and $A \in V(G) \setminus \{u\}$ is a vertex set, then the size of a minimum vertex-cut for u and A equals the maximum number of, besides u, pairwise vertex disjoint $u - A$ paths.*

Corollary 3.21. *Let G be a graph. If $U_1, U_2 \subseteq V(G)$ are two disjoint vertex sets, then the size of a minimum vertex-cut for U_1 and U_2 equals the maximum number of pairwise vertex disjoint $U_1 - U_2$ paths.*

Corollary 3.22. *Let G be a graph. If $U_1, U_2 \subseteq V(G)$ are two disjoint vertex sets, then the size of a minimum edge-cut for U_1 and U_2 equals the maximum number of pairwise edge disjoint $U_1 - U_2$ paths.*

The following result, which can be considered as the edge version of Paulraja's result (see Theorem 3.13), is immediately obtained by using Corollay 3.22.

Lemma 3.23 ([10]). *Let G be a 3-edge-connected graph. If H is a 2-edge-connected bipartite graph in G, then there exists a 2-edge-connected bipartite spanning subgraph of G containing H.*

Proof. This lemma is proved by the recursive construction using the following claims.

Claim 3.23.1. *Let H be a bipartite subgraph of G with $n(H) \geq 1$, A and B be the partite sets of H, and $v \in V(G) \setminus V(H)$ be a vertex. If there are three, besides v, vertex-disjoint paths connecting v and $V(H)$, then there exists a bipartite subgraph H' of G such that $V(H) \subset V(H')$. Furthermore, if $n(H) = 1$ or H is 2-edge-connected, then H' is 2-edge-connected.*

Proof. Let P_1, P_2 and P_3 be three vertex-disjonit paths connecting v and $V(H)$. It can be readily observed that all graph H_i, which are obtained by adding P_i to H, are bipartite graph for $i \in [3]$. Now we define A_i, B_i to be the partite sets of H_i such that $A \subset A_i$ or $B \subset B_i$ for $i \in [3]$.

One can easily see that there always exist two vertex-disjoint paths, besides v, from $\{P_1, P_2, P_3\}$ such that $v \in A_i \cap A_j$ or $v \in B_i \cap B_j$. Therefore, H', obtained by adding P_i and P_j to H is bipartite. Moreover, any edge of P_i and P_j is not a bridge since H is connected. Thus, H' is 2-edge-connected if $n(H) = 1$ or H is 2-edge-connected.

This finishes the proof. (□)

Claim 3.23.2. *Let H be a bipartite subgraph of G with $n(H) \geq 1$, A and B be the partite sets of H, and $v \in V(G) \setminus V(H)$ be a vertex. If there are three edge-disjoint paths connecting v and $V(H)$, then there exists a bipartite subgraph H' of G such that $V(H) \subset V(H')$. Furthermore, if $n(H) = 1$ or H is 2-edge-connected, then H' is 2-edge-connected.*

Proof. Let P_1, P_2, P_3 be three edge-disjoint paths connecting v and $V(H)$. Let u_1, u_2, u_3 be three not necessarily distinct end-vertices of P_1, P_2, P_3 in H. It can be readily observed that all graph H_i, which are obtained by adding P_i to H, are bipartite graph for $i \in [3]$. Now we define A_i, B_i to be the partite sets of H_i such that $A \subset A_i$ or $B \subset B_i$ for $i \in [3]$. Furthermore, for $i \in [3]$, let $w_i \in V(P_i)$ be the shortest distance from u_i to w_i in H_i such that $w_i \in V(P_j) \cup V(P_k)$, where $i \notin \{j, k\}$.

It can be readily observed that, if all three vertices w_1, w_2, w_3 are the same, then a 2-edge-connected bipartite subgraph H' is found by Claim 3.23.1. Moreover, by the fact of Claim 3.23.2, we deduce that there always exist two vertices from $\{w_1, w_2, w_3\}$ are the same. Renaming three paths if necessary, we may assume that $w_1 = w_2$ and $w_3 \in V(P_3) \cap V(P_2)$. Now, the three paths $u_1P_1w_1$,$u_2P_2w_1$,$u_3P_3w_3P_2w_1$ are, besides w_1, are vertex-disjoint. We obtain the desired result by Claim 3.23.2.

This finishes our proof.

(□)

Now, we are able to prove our lemma by a recursive constrution: Let $u \in V(G)$ be a vertex and $H = G[u]$ be a subgraph. By Claim 3.23.2, we always construct a 2-edge-connected bipartite subgraph H of G. By the recursive use of Claim 3.23.2, we extend the subgraph H until $V(H) = V(G)$ which is a 2-edge-connected bipartie spanning subgraph of G.

We obtain the result. □

Now we prove Theorem 3.18 which determines the proper connection number of a 3-edge-connected graph. Recall its statement.

Theorem 3.18 *If G is a 3-edge-connected non-complete graph, then $pc(G) = 2$. Furthermore, there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

Proof. Let G be a 3-edge-connected graph. By Lemma 3.23, there is a 2-edge-connected bipartite spanning subgraph of G, say H. Hence, by Theorem 3.17, H has the strong property with two colours. It follows that G has the strong property with two colours, too.

This finishes our proof. □

The proper connection number of a non-complete, bridgeless graph is determined by the results above. The proper connection number of a connected, non-complete graph with high edge-connectivity, $\kappa'(G) \geq 3$, equals 2 by Thereom 3.18. Furthermore, a bipartite graph having edge-connectivity $\kappa'(G) = 2$ has proper connection number 2, too, by Theorem 3.17. For general case, the proper connection number of a bridgeless graph is at most 3 by Theorem 3.16. In the next section, we introduce the results of the proper connection number of a connected graph having bridges.

3.3 The proper connection number of a connected graph having bridges

In this section, we present some existent results of the proper connection number of a graph having bridges which improve bounds of the proper connection number. Recall that the proper connection number of a bridgeless graph is bounded by 3 as shown in the previous section. Unlike the previous results, there does not exist a constant C such that the proper connection number of every graph having bridges is at most C. Note that every edge of a star is a bridge, i.e the number of bridges of a star is its size. By Fact 3.2, the proper connection number of a star equals its number of bridges. Hence, there is a relationship between the number of bridges and the proper connection number of a connected graph. The following result which was studied by Andrews et al. [4] determines a lower bound of the proper connection number in a connected graph having bridges incident to a single vertex.

Proposition 3.24 (Andrews et al. [4]). *Let G be a nontrivial connected graph that contains bridges. If b is the maximum number of bridges incident with a single vertex in G, then $pc(G) \geq b$.*

The proper connection number of a star which is presented in Fact 3.2 is one of the results that can be easily computed by using Propostion 3.24. Further, recall the graph H_1, see Figure 3.1, having three bridges incident with a single vertex, so $pc(H_1) \geq 3$. Now we consider the proper connection number of a nontrivial tree T whose all edges are bridges. Note that the chromatic index number $\chi'(G)$ of a bipartite graph was determined by König's Theorem a long time ago. Further, T is a bipartite graph, since T is acyclic. Hence, $\chi'(T) = \Delta(T)$.

Theorem 3.25 (König et al. [52]). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

On the other hand, the rainbow connection number of T was determined by Chartrand et al. [20].

Proposition 3.26 (Chartrand et al. [20]). *If T is nontrivial tree of size m, then $rc(T) = m$.*

Since $pc(T) \leq min\{\chi'(T), rc(T)\}$, it can be readily obtained that $pc(T) \leq \Delta(T)$. By Proposition 3.24, the authors in [4] immediately deduced the proper connection number of a tree as follows.

Proposition 3.27 (Andrews et al. [4]). *If T is a nontrivial tree, then $pc(T) = \chi'(T) = \Delta(T)$.*

By Proposition 3.3 and Proposition 3.27, the upper bound of a nontrivial connected graph can be improved by the following proposition.

Proposition 3.28 (Andrews et al. [4]). *If G is a nontrivial connected graph, then $pc(G) \leq min\{\Delta(T) : T$ is a spanning tree of $G\}$.*

Just like a lower bound of the proper connection number, bridges also play an important role in its upper bound. In [47], the authors considered the upper bound of the proper connection number of a nontrivial connected graph having bridges. First of all, they proved that the proper connection number is bounded by the cardinality of the set of the pendant vertices which is the set of all vertices of degree 1.

Lemma 3.29 (Huang et al. [47]). *Let G be a graph and $H = G - PV(G)$, where $PV(G)$ denotes the set of the pendant vetices of G. If H is bridgeless, then $pc(G) \leq max\{3, |PV(G)|\}$.*

Furthermore, Huang et al. [47] denote by $B \subseteq E(G)$ the set of cut-edges of a nontrivial connected graph G and denote by $\mathcal{C}$ the set of connected components of $G' = (V(G), E(G) \setminus B)$. Contracting each element of $\mathcal{C}$, which is not a singleton to a new single vertex, a new graph G^*, which is said to be the well-known *bridge-block tree* of G, is obtained. Hence, the following result which is stronger than Lemma 3.29 was proved by Huang et al. [47].

Theorem 3.30 (Huang et al. [47]). *If G is a nontrivial connected graph, then $pc(G) \leq max\{3, \Delta(G^*)\}$.*

Adding the condition of a bipartite graph to the Proposition 3.24, the authors in [73], recently, proved the following result which is the upper bound of a connected, bipartite graphs containing bridges.

Theorem 3.31 (Yue et al. [73]). *Let G be a connected bipartite graph containing bridges. If b is the maximum number of bridges incident with a single vertex in G, then $pc(G) \leq b + 2$ and this upper bound is sharp.*

Together with the results of Proposition 3.24 and Theorem 3.31, Yue et al. [73] directly obtained the following corollary.

Corollary 3.32 (Yue et al. [73]). *Let G be a connected bipartite graph containing bridges. If b is the maximum number of bridges incident with a single vertex in G, then $pc(G) \in \{b, b + 1, b + 2\}$.*

3.4 Proper connection number $pc(G) = 2$ of a connected graph

As already mentioned above, the proper connection number of a nontrivial connected graph G equals 1 if and only if G is a complete graph by Fact 3.2. It can be readily seen that the proper connection number of a non-complete connected graph is at least 2, for example the proper connection number of a path of order at least 2, a cycle of order at least 4, or 3-edge-connected graphs equals 2. The results in Section 3.2 in this chapter have shown that the proper connection number of a non-complete bridgeless graph is 2 or 3. Further, there are infinitely many bridgeless graphs having proper connection number 3 by Proposition 3.11, more details see Figures 3.2, 3.3, 3.4. On the other hand, deciding whether the proper connection number of a non-complete, bridgeless graph equals 2 or 3 is still an open question by Ducoffe et al. [31]. Therefore, many researchers in graph theory study proper connection number 2 of a connected graph. It follows that from time to time, after the concept of the proper connection number is introduced in [8, 4], many beautiful results of connected graphs having proper connection number 2 are published. The section of this chapter is devoted to list several existent classes of connected graphs having proper connection number 2.

Recall that by Theorem 3.17, the proper connection number of a bipartite graph having no bridges equals 2. For generality, there are many bipartite (or multipartite) graphs whose proper connection number is greater than 2, for example, the proper connection number of a tree T with $\Delta(T) \geq 3$ by Proposition 3.27, or a star $K_{1,m}$ with $m \geq 3$ by Fact 3.2 is at least 3. By adding some other conditions for a multipartite graph, the authors in [4] studied the proper connection number of a complete multipartite graph as follows.

Theorem 3.33 (Andrews et al. [4])**.** *If G is a complete multipartite graph that is neither a complete graph nor a tree, then $pc(G) = 2$.*

Now, the results of the proper connection number of the specials classes of graphs obtained from well-known graph operations including the join of graphs, Cartesian product of graphs, direct product, permutation graphs, line graphs and power graphs are presented below.

The authors in [4] studied the proper connection number of the joins of two connected graphs.

Theorem 3.34 (Andrews et al. [4])**.** *Let G, H be two connected graphs. If $G \vee H$ is non-complete, then $pc(G \vee H) = 2$.*

A similar result to Theorem 3.34 was also studied by Andrews et al. [4] for the Cartesian product $G \Box H$ of two nontrivial connected graphs G, H as follows.

Theorem 3.35 (Andrews et al. [4])**.** *If G, H are nontrivial connected graphs, then $pc(G \Box H) = 2$.*

The *direct product* of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edges $\{(g, h)(g', h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}$. The proper connection number of the direct product of two connected non-bipartite graphs, and one of them is 2-connected, were proved by Hammack et al. [38].

Theorem 3.36 (Hammack et al. [38]). *Let G, H be two connected non-bipartite graphs. If one of them is (vertex) 2-connected, then $pc(G \times H) = 2$.*

The concept of *permuation graphs* was first introduced by Chartrand et al. [19]. Let G be a graph with vertex set $V(G) = \{v_1 \cdots v_n\}$ and α be a permutation of the set $[n]$. The permutaion graph $P_\alpha(G)$ of G is the graph of order $2n$ obtained from two copies of G, where the second copy of G is denoted by G' and the vertex corresponding to $v_i \in V(G)$ is denoted by $u_i \in V(G')$. The vertex v_i is joined to the vertex $u_{\alpha(i)} \in V(G')$. The edges $v_i u_{\alpha(i)}$ are called the permutation edges of $P_{\alpha(G)}$. In [4], the authors showed that every permutation graph of a Hamiltonian path has proper connection number 2 by the following result.

Theorem 3.37 (Andrews et al. [4]). *If G is a nontrivial traceable graph of order n, then $pc(P_\alpha(G)) = 2$ for each permutation α of $[n]$.*

Note that Theorem 3.37 holds when G is a traceable graph. However, the proper connection number is also 2 for several classes of permutation graphs of others connected graphs which are not traceable graphs. The result is obtained as follows.

Proposition 3.38 (Andrews et al. [4]). *Every permutation graph of a star of order at least 4 has proper connection number 2.*

The last two results of the proper connection number of line graphs and powers of graphs were presented in the Phd Thesis of Laforge [55]. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G.

Theorem 3.39 (Laforge et al. [55]). *If G is a connected graph of order at least 3 that is neither a star nor K_3, then $pc(L(G)) = 2$.*

Let G be a connected graph and k be a positive integer, the *kth power* of G, written by G^k, is the simple graph G^k with the vertex set $V(G)$ and the edge set $E(G) = \{uv : d_G(u, v) \leq k\}$. The graph G^2 is called the *square* of G

Theorem 3.40 (Laforge et al. [55]). *If G be a connected graph of order at least 3, then $pc(G^2) = 2$.*

By Fact 3.2, the proper connection number of a complete graph equals 1. Note that the diamater of a complete graph is 1. The authors in [8] claimed that if G has small diameter, then its proper connection number is also small. More detailed, the following theorem was proved for 2-connected graphs with small diameter.

Theorem 3.41 (Borozan et al. [8]). *Let G be a nontrivial 2-connected graph. If $diam(G) = 2$, then $pc(G) = 2$.*

After Theorem 3.41 was proved, Li and Magnant in [57] posed the following conjecture of 2-connected with diamater 3.

Conjecture 3.42 (Li et al. [57]). *Let G be a nontrivial 2-connected graph. If $diam(G) = 3$, then $pc(G) = 2$.*

Conjecture 3.42 was proved by Huang et al. [45].

Theorem 3.43 (Huang et al. [45]). *If G is a 2-connected graph with $diam(G) = 3$, then $pc(G) = 2$.*

By Fact 3.2, Theorem 3.41 and Theorem 3.43, it can be readily seen that the proper connection number can decrease if we add edges to a graph. It means that if a graph has many edges or high minimum degree, then its proper connection number is small. Hence, there are several beautiful results of dense connected graphs of proper connection number 2 which are published recently. For a graph with high minimum degree, the famous result of Hamiltonian path was proved by Dirac a long time ago.

Theorem 3.44 (Dirac et al. [28]). *Let G be a graph of order n. If $\delta(G) \geq \frac{n-1}{2}$, then G is traceable. Moreover, if $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

By Theorem 3.44, Proposition 3.3 and Fact 3.2, it can be readily obtained that every connected graph of order n having $\delta(G) \geq \frac{n-1}{2}$ has proper connection number 2. In [8], the authors gave the much better result of a connected graph with the minimum degree that has proper connection number 2. The following result is sharp.

Theorem 3.45 (Borozan et al. [8]). *Let G a connected non-complete graph of order $n \geq 68$. If $\delta(G) \geq \frac{n}{4}$, then $pc(G) = 2$.*

Furthermore, the authors in [8] also confirmed that the minimum degree condition of Theorem 3.45 is best possible by the following counterexample. Let G_i be a complete graph of order $\frac{n}{4}$ and take a vertex v_i of G_i, where $i \in [4]$. Let B_4, see Figure 3.5, be a graph obtained from G_1, G_2, G_3, G_4 by adding the edges v_1v_2, v_1v_3, v_1v_4. Note that $\delta(B_4) = \frac{n}{4} - 1$ and B_4 has three bridges. By Proposition 3.24, it can be readily observed that $pc(B_4) \geq 3$.

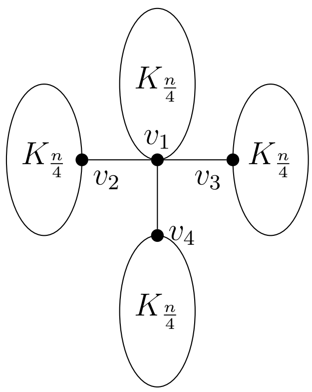

Fig. 3.5: Graph B_4 with $\delta(B_4) = \frac{n}{4} - 1$ has $pc(B_4) = 3$ (Borozan et al. [8])

Although, the lower bound on the minimum degree is sharp, but the authors in [8] did not consider that the lower bound on the order number $n \geq 68$ is best possible. After that Li and Magnant [57] proposed the following conjecture.

Conjecture 3.46 (Li et al. [57]). *Let G be a connected non-complete graph of order $n \geq 5$. If $\delta(G) \geq \frac{n}{4}$, then $pc(G) = 2$.*

The authors in [57] thought that the lower bound on the order could be $n \geq 5$ since the proper connection number of a claw $K_{1,3}$ equals 3. After finding some gaps of the proof of Theorem 3.45, Huang et al. [44] proved Conjecture 3.46 without two counterexamples B_5, B_6, see Figure 3.6 & 3.7, that $pc(B_5) = pc(B_6) = 3$. Note that B_5 is one example of B_3, see Figure 3.4, where $k = 3$.

Theorem 3.47 (Huang et al. [44]). *Let G be a connected non-complete graph of order $n \geq 5$. If $G \notin \{B_5, B_6\}$ and $\delta(G) \geq \frac{n}{4}$, then $pc(G) = 2$.*

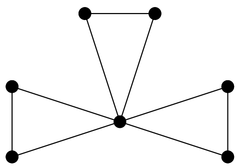

Fig. 3.6: Graph B_5 with $pc(B_5) = 3$ (Huang et al. [44] & Chang et al. [17])

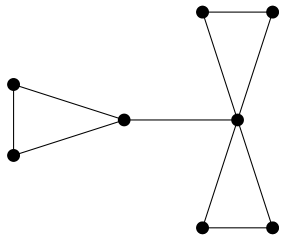

Fig. 3.7: Graph B_6 with $pc(B_6) = 3$ (Huang et al. [44] & Chang et al. [17])

If the minimum degree condition of Theorem 3.47 is changed to the degree sum condition, then the following results were proved by Chang et al. [17].

Theorem 3.48 (Chang et al. [17]). *Let G be a connected non-complete graph of order $n \geq 5$ with $G \notin \{B_5, B_6, B_7\}$, see Figure 3.6 & 3.7 & 3.8. If $d(x) + d(y) \geq \frac{n}{2}$ for every $xy \notin E(G)$, then $pc(G) = 2$.*

The sum degree condition of two non-adjacent vertices of Theorem 3.48 can be improved for a connected, bipartite graph. So the following result was confirmed by the authors in [17].

Theorem 3.49 (Chang et al. [17]). *Let G be a connected bipartite graph of order $n \geq 4$. If $d(x) + d(y) \geq \frac{n+6}{4}$ for every $xy \notin E(G)$, then $pc(G) = 2$.*

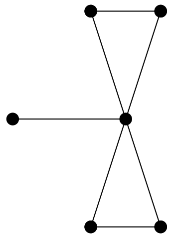

Fig. 3.8: Graph B_7 with $pc(B_7) = 3$ (Aardt et al. [1] & Chang et al. [17])

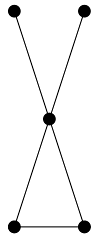

Fig. 3.9: Graph B_8 with $pc(B_8) = 3$ (Aardt et al. [1])

In [1], the authors considered proper connection number 2 of a connected graph with its given number of edges without two counterexamples which are depicted in Figure 3.8 & 3.9.

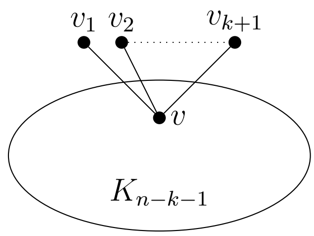

Fig. 3.10: Graph G_k with $|E(G_k)| = \binom{n-k-1}{2} + k + 1$ has $pc(G_k) > k$ (Aardt et al. [1])

Theorem 3.50 (Aardt et al. [1]). *Let G be a connected graph of order n with $G \notin \{B_7, B_8\}$, see Figure 3.8 & 3.9. If G is non-complete and $|E(G)| \geq \binom{n-3}{2} + 4$, then $pc(G) = 2$.*

The authors in [1] improved Theorem 3.50 in general case to obtain a new upper bound of the proper connection number. That is as follows.

Theorem 3.51 (Aardt et al. [1]). *Let $k \geq 3$ be an integer and G be a connected graph of order n. If $|E(G)| \geq \binom{n-k-1}{2} + k + 2$, then $pc(G) \leq k$.*

Furthermore, they deduced that the lower bound on the size of a connected graph G_k is best possible by the following counterexample. Let H be a complete graph of order $n - k - 1$ and take a vertex $v \in V(H)$, $v_i \notin V(H)$ be $k + 1$ others vertices, where $i \in [k + 1]$. Let G_k, see Figure 3.10, be a graph obtained from H and $k + 1$ vertices $v_1, \ldots, v_{k+1}$ by adding edges $v_1v, \ldots, v_{k+1}v$. Note that $|E(G_k)| = \binom{n-k-1}{2} + k + 1$ but $pc(G_k) > k$.

Recently, many results for graph classes with proper connection number 2 have been published. But the problem to determine the proper connection number of an arbitrary connected graph is still difficult. Further, the authors in [31] claimed that the complexity of computing the proper conneciton number of a given graph as an interesting open question. In the next chapters of the thesis, we study several classes of connected graphs having proper connection number 2.

4 The proper connection number and minimum degree condition of graphs

In this chapter, we study sufficient conditions in terms of the ratio between minimum degree and order of a 2-connected graph G implying that G has proper connection number 2. We note that an extended abstract containing some results in this chapter but no detailed proof is already published in [9] in 2016. Moreover, the detailed proof of the results in this chapter can be also found in [11].

4.1 The proper connection number and minimum degree

Recall that by Theorem 3.10, the proper connection number of a 2-connected graph is at most 3. By Proposition 3.11, there are many 2-connected graphs of minimum degree number 2 which have proper connection number 3, see Figure 3.2 & 3.3. Furthermore, from Theorem 3.45 & 3.47, it can be readily seen that every 2-connected graph of order n and minimum degree at least $\frac{n}{4}$ has proper connection number 2. The authors in [8] believed that this condition of minimum degree can be improved in the 2-connected graph. Therefore, they posed the following conjecture.

Conjecture 4.1 (Borozan et al. [8]). *Let G be a 2-connected graph. If $\delta(G) \geq 3$, then $pc(G) = 2$.*

Motivated by Conjecture 4.1, in this section we study the proper connection number of a connected graph with the minimum degree condition. First of all, we disprove Conjecture 4.1 by constructing a series of 2-connected graphs G_i such that $\delta(G_i) = i$, $n(G_i) = 42i$ and $pc(G_i) \geq 3$.

Theorem 4.2 ([9, 11]). *For every integer $d \geq 3$, there exists a 2-connected graph G of minimum degree d and order $n = 42d$ such that $pc(G) \geq 3$.*

For the proof of Theorem 4.2, we will use the graph B_1 which is depicted in Figure 3.2 as a basic tool in our construction. As a further tool for our theorem, we need the following lemma.

Lemma 4.3 ([11]). *Let $k \geq 3$ be an integer, $K_{k,k}$ be a complete bipartite graph on $2k$ vertices, G be a 2-connected graph of proper connection number at least 3, which is vertex disjoint from $K_{k,k}$, $v \in V(G)$ be one of its vertices of degree at most 3,*

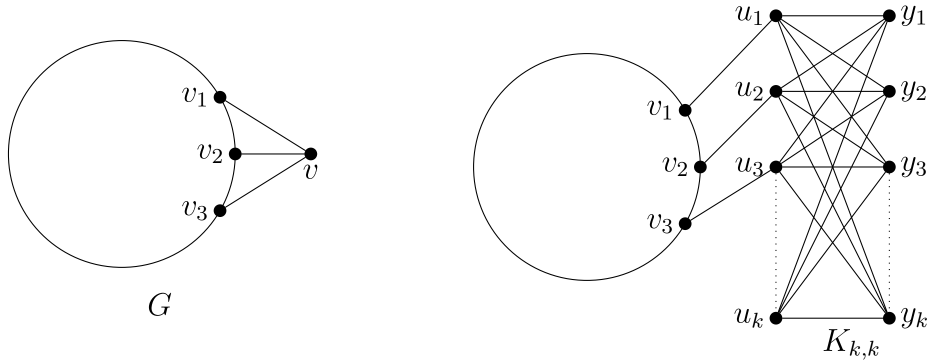

Fig. 4.1: Graph G' is obtained from G and $K_{k,k}$ [11]

$\{v_i, i \in [d_G(v)]\}$ be its neighbours, u_1, u_2, u_3 be three vertices of the same partite set in $K_{k,k}$. If G' is the graph obtained from G by removing v and adding the graph $K_{k,k}$ and the edges u_iv_i for $i \in [d_G(v)]$, then $pc(G') \geq 3$ and G' is 2-connected.

A 2-connected graph G' obtained from a 2-connected graph G and a complete bipartite graph $K_{k,k}$, where $k \geq 3$ by removing a vertex $v \in V(G)$ of degree at most 2 and adding the edges u_iv_i for $i \in [d_G(v)]$ is depicted in Figure 4.1.

Proof. ([11]) Suppose, to the contrary, that $pc(G') \leq 2$. One can readily observe that G' is non-complete. Hence, $pc(G') = 2$ since $pc(G') = 1$ if and only if G' is complete by Fact 3.2. Let us denote w_1, w_2 be two arbitrary distinct vertices of G. We now define two vertices $x_1, x_2 \in V(G')$ which depend on w_1, w_2 as follows: for $i \in [2]$, if w_i is different from v, then $x_i = w_i$, otherwise let $x_i = u_1$. Let us assign 2-edge-colouring c' by labeling $c' : E(G') \to [2]$ to make G' properly connected. Now we define an edge-colouring c to all the edges of G as follows: $c(e) = c'(e)$, for $e \in E(G) \cap E(G')$ and $c(vv_i) = c'(u_iv_i)$ for $i \in [d_G(v)]$.

By the definition of c', there always exists at least one proper path connecting x_1 and x_2 in G', say P. Note that $x_1, x_2 \in (V(G) \setminus \{v\}) \cup \{u_1\}$, implying that no vertex of the added complete bipartite graph $K_{k,k}$ beside u_1 is an end-vertex of P. If P does not contain any edge of $\{u_iv_i : i \in [d_G(v)]\}$, then $x_i \neq u_1$ and $w_i \neq v$ for $i \in [2]$. It can be readily seen that P is a proper path connecting w_1, w_2 by c in G. If P contains only one edge of $\{u_iv_i : i \in [d_G(v)]\}$, say u_iv_i for $i \in [d_G(v)]$, then $v = w_1$ or $v = w_2$. Hence, either $w_2 = x_2Pv_iv = w_1$ or $w_1 = x_1Pv_iv = w_2$ is a proper path connecting w_1, w_2 by c in G since $c(vv_i) = c'(u_iv_i)$. If P contains exactly two edges of $\{u_iv_i : i \in [d_G(v)]\}$, without lost of generality, we may assume that two edges are u_1v_1 and u_2v_2. Now, since at most one vertex of x_1, x_2 is a vertex of $K_{k,k}$, all the internal vertices of $V(v_1Pv_2)$ are vertices of $K_{k,k}$. Furthermore, the length of v_1Pv_2 is even. By our supposition $pc(G') \leq 2$, we deduce that $c'(u_1v_1) \neq c'(u_2v_2)$. It can be readily observed that $c(vv_1) \neq c(vv_2)$. Hence, either $w_1 = x_1Pv_1vv_2Px_2 = w_2$ or $w_1 = x_1Pv_2vv_1Px_2 = w_2$ is a proper path connecting w_1 and w_2 by c in G. It remains to consider that $d_G(v) = 3$ and three edges $u_1v_1, u_2v_2, u_3v_3 \in E(P)$. By the construction of G' the set $\{u_iv_i : i \in [d_G(v)]\}$ is a cut-edge set. Hence, renaming vertices if necessary, we may assume that $x_1 = u_1$ and v_3 is the shortest distance to x_2 on P. Note that $w_1 = v$. Therefore, $w_1 = vv_3Px_2 = w_2$ is a proper path connecting w_1 and w_2 by c in G.

By the observation above, there always exists a proper path connecting any two vertices $w_1, w_2 \in V(G)$ by c. We deduce that G is the properly connected graph by c. Furthermore, the number of colours used by c is at most the number of colours used by c', implying $pc(G) \leq pc(G') \leq 2$, a contradiction. Therefore, $pc(G') \geq 3$.

It remains to prove that G' is the 2-connected graph. Suppose, to the contrary, that G' has a cut-vertex, say $x \in V(G')$. If $x \in V(G) \cap V(G')$, then $G - x$ is disconnected, a contradiction. Furthermore, if $x \in V(G') \setminus V(G)$, then it can be readily observed that $G - v$ is disconnected, a contradiction. Hence, G' is 2-connected.

We complete our proof.

□

By using the graph B_1, see Figure 3.2, and Lemma 4.3, we are able to prove Theorem 4.2. Recall its statement here.

Theorem 4.2 *For every integer $d \geq 3$, there exists a 2-connected graph G of minimum degree d and order $n = 42d$ such that $pc(G) \geq 3$.*

Proof. ([11]) Let B be the graph B_1 which is depicted in Figure 3.2 of proper connection number 3. Let us label all the vertices of $V(B)$ by v_i, where $i \in [n(B)]$. Note that $n(B) = 21$. By choosing $k = d$, an iterative use of the construction described in Lemma 4.3 for every vertex $v_i \in V(B)$ (with the replacement each v_i by a complete bipartite graph $K_{k,k}$ of order $2k$) constructs a new graph B_d which is 2-connected and has proper connection number at least 3. It can be readily seen that $\delta(B_d) = k = d$ and $n(B_d) = 42k = 42d$ since $n(B) = 21$.

We obtain the result. □

By using Theorem 4.2, there are many 2-connected graph of minimum degree 3 that has proper conneciton number 3. Hence, Conjecture 4.1 is disproved. In particular, if we take $d = 3$, then one can be readily obtained that there is a 2-connected graph G of order $n(B) = 126$ with the minimum degree $\delta(G) = 3$ that has proper connection number 3. But G is not the smallest 2-connected graph of $\delta(G) = 3$ with proper connection number 3 by our construction. The following corollary is showed the smallest 2-connected graph with minimum degree at least 3 by our technique that has proper connection number 3.

Corollary 4.4. *There exists a 2-connected graph G of order n with the minimum degree $\delta(G) \geq min\{\frac{n}{32}, 3\}$ that has proper connection number 3.*

Proof. Similarly to the proof of Theorem 4.2, let B be the graph B_1 which is depicted in Figure 3.2 of proper connection number 3. Let us denote $V^*(B)$ the subset vertices of degree 2 of $V(B)$. It can be readily seen that $|V(B)| = 6$ and $|V^*(B)| = 15$. Labeling all the vertices of $V^*(B)$ by v_i, where $i \in [|V(^*(B))|]$. By choosing $k = 3$, an iterative use of the construction described in Lemma 4.3 for every vertex $v_i \in V^*(B)$ (with the replacement each v_i by a complete bipartite graph $K_{3,3}$) constructs a new graph B^* which is 2-connected and has proper connection number 3. One can easily observe that $n(B^*) = 15 * 6 + 6 = 96$ and $\delta(B^*) = 3 = \frac{96}{32}$.

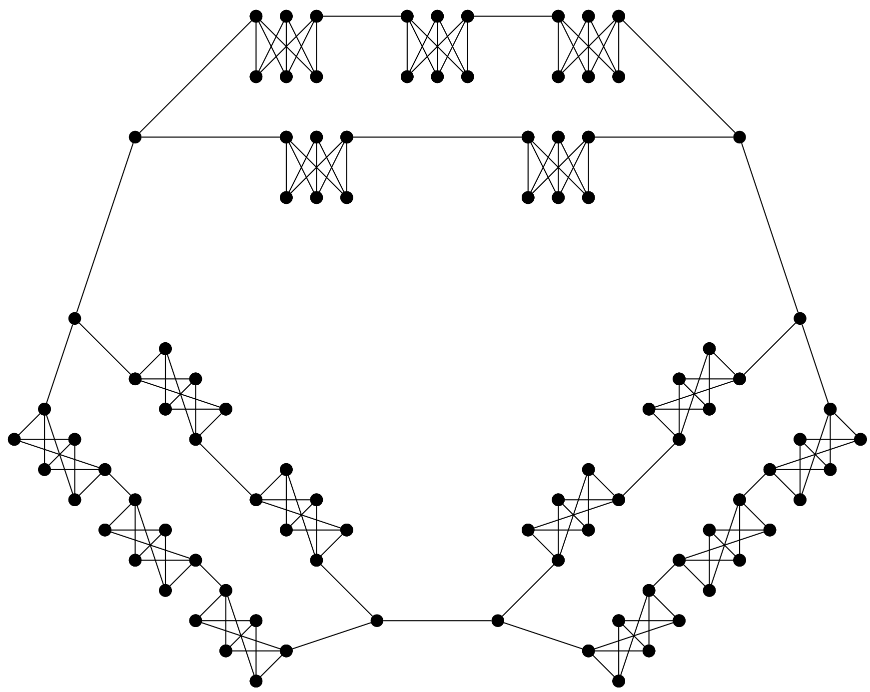

Fig. 4.2: Graph B_{10} is obtained from B_1 and $K_{3,3}$ has $pc(B_{10}) = 3$

The proof is obtained. $\square$

At the same time, when we submited our result in [11], Conjecture 4.1 also is disproved by Huang et al. [45]. But our technique is different from their technique. Moreover, by their technique, the smallest 2-connected graph with minimum degree 3 of proper connection number 3 has order 114. By Corollary 4.4, we have a smaller 2-connected graph of order 96 with minimum degree 3, see Figure 4.2, that has proper connection number 3.

By Theorem 4.2, there are many 2-connected graphs with the arbitrary minimum degree at least 3 having proper connection number 3. Note that, the main idea of the proof of Theorem 4.2 is to replace a vertex of a 2-connected graph by a complete bipartite graph $K_{k,k}$, where $k \geq 3$. Our construction technique still holds for a connected graph in a slightly different way, i.e the result of Theorem 4.2 can be extended by replacing a vertex of a connected graph by a complete graph K_k. More general, there exist many connected graphs with the large minimum degree having the large proper connection number. Hence, we obtain the next result.

Theorem 4.5 ([9, 11]). *Let $d, k \geq 2$ be two integers. There exists a connected graph G of minimum degree d and order $n = (d+1)(k+1)$ such that $pc(G) = k$.*

Proof. ([11]) If $k = 2$, then let G be the graph consisting of three pairwise disjoint vertex sets U_1, U_2, U_3 of cardinality $d+1$ such that $G[U_1, U_2]$ is a complete bipartite graph, $G[U_2, U_3]$ is a complete bipartite graph minus a perfect matching, and $G[U_1, U_3]$ contains no edges. One can easily observe that G is connected, non-complete and a bipartite graph, $\delta(G) = d$, $n(G) = 3(d+1)$. Moreover, G is 2-connected since $d \geq 2$.

By Fact 3.2, $pc(G) = 1$ if and only if G is complete graph. By Theorem 3.12, we deduce that $pc(G) = 2$. Therefore, we may assume that $k \geq 3$.

Let G be the graph obtained from $k+1$ cliques $C_1, C_2, \ldots C_{k+1}$ of size $d+1$, each containing a labeled vertex $v(C_i)$, by adding the edges between $v(C_i)$ such that the graph induced by $\{v(C_i) : i \in [k+1]\}$ is a star $K_{1,k}$. Renaming cliques if necessary, we may assume that $c(C_{k+1})$ is the center of the star. Hence, G is connected and $n(G) = (d+1)(k+1)$. Note that, any colouring c making G properly connected makes $G[\{v(C_i) : i \in [k+1]\}]$ properly connected. Moreover, it can be readily seen that there exists exactly one internally vertex disjoint path between $v(C_{i_1})$ and $v(C_{i_2})$ for any two distinct $i_1, i_2 \in [k+1]$. Therefore, $pc(G) \geq pc(K_{1,k}) = k$ since $pc(K_{1,k}) = k$ by Fact 3.2. For our considerations, we take an k-edge colouring making $G[\{v(C_i) : i \in [k+1]\}]$ properly connected. Moreover, for $i \in [k]$, we colour all the edges of the cliques C_i by a colour different from the colour on the edge connecting C_i to $v(C_{k+1})$. Let w be a vertex in C_{k+1} distinct from $v(C_{k+1})$. Now, we colour the edge $v(C_{k+1})w$ by colour 1, for all vertices z in the non-empty set $V(C_{k+1}) \setminus \{v(C_{k+1}), w\}$, we colour the edge $v(C_{k+1})z$ by colour 2, and colour all the remaining edges by colour 3. Thus, one can easily check by a simple case to case analysis, G is properly connected using k colours and, since $pc(G) \geq k$, we deduce that $pc(G) = k$.

This completes the proof. □

After disproving Conjecture 4.1, we note that there are infinitely many 2-connected graphs with the condition of the minimum degree which have proper connection number 3. Hence, we study the condition of the minimum degree of a 2-connected graph which has proper connection number 2 in the next section.

4.2 The 2-connected graphs with a sufficient minimum degree condition implying proper connection number 2

In Section 4.1, we have already disproved Conjecture 4.1 by constructing a series of 2-connected graphs with minimum degree at least 3 having proper connection number 3. By Theorem 4.2, one cannot bound the minimum degree of a 2-connected graph G from below by a constant such that $pc(G) \leq 2$ follows. Moreover, by Theorem 3.45 and Theorem 3.47, every connected non-complete graph of order $n \geq 5$ with minimum degree at least $\frac{n}{4}$ has proper connection number 2. One can readily observe that this result still holds for every 2-connected graph. Therefore, it is natural to ask for a ratio between minimum degree and order of a 2-connected graph, implying $pc(G) \leq 2$. Motivated by this question, in this section we study the sufficient condition in term of the ratio between minimum degree and order of a 2-connected graph G implying that G has proper conneciton number 2. The following result answeres this question.

Theorem 4.6 ([9, 11])**.** *Let G be a 2-connected graph of order $n = n(G)$ and minimum degree $\delta(G)$. If $\delta(G) > \max\{2, \frac{n+8}{20}\}$, then $pc(G) \leq 2$.*

Before starting to prove Theorem 4.6, we state some useful results which are related to the vertex disjoint paths between two vertex subsets of a graph. Morovere, some results of the proper connection number which are listed in the previous chapters are also used in our proof of this theorem. The first result which is well-known as Menger's theorem is a basic tool at several points throughout the proof. Recalls its statement.

Corollary 3.21 *Let G be a graph. If $U_1, U_2 \subseteq V(G)$ are two disjoint vertex sets, then the size of a minimum vertex-cut for U_1 and U_2 equals the maximum number of pairwise vertex disjoint $U_1 - U_2$ paths.*

By the concept of vertex-cut set and connectivity of a graph, we note that the size of a minium vertex-cut set for two disjoint vertex subsets U_1 and U_2 is at least the connectivity of the graph. Furthermore, we will not only use the existence of the vertex disjoint paths in our proof, but we also need a minimum length of it. A helpful result which was proved by Jackson [49] plays an important role in our proof to determine the ratio between minimum degree and order of 2-connected graph G.

Theorem 4.7 (Jackson [49]). *If S is a 2-connected bipartite graph with bipartition (S_1, S_2) and $u, v \in V(G)$ are two of its vertices, then S contains an $u - v$ path of length at least $2\delta' - 2$, where $\delta' = \min\{d_G(z) : z \in V(S) \setminus \{u, v\}$.*

For simplifying our proof, we introduce graph families. Let G be a multigraph shown in one of the Figure 4.3−4.7. We say that a graph G' belongs to the family $\mathcal{S}(G)$ if and only if it can be obtained from G by subdividing edges. We note that the thick edges which belong to every graph in Figure 4.4−4.7 can be seen as the last added ear in an ear decomposition. Moreover, these edges play an special role to compute the number vertices of a subgraph in our proof.

Coming back to the concepts of 2-ear-cycle and 3-ear-cycle defined in Chapter 2 and by a simple case to case analysis, we immediately obtain the following two facts.

Fact 4.8 ([11]). *If $\mathcal{G}$ is the set of all 2-ear-cycles, then $\mathcal{G} = \sqcup_{i\in[4]}\mathcal{S}(S_i)$.*

Fact 4.9 ([11]). *If $\mathcal{G}$ is the set of all 3-ear cycles, then*

$$\mathcal{G} = \left(\bigcup_{j\in[4]} \mathcal{S}(S_1^j)\right) \cup \left(\bigcup_{j\in[12]} \mathcal{S}(S_2^j)\right) \cup \left(\bigcup_{j\in[13]} \mathcal{S}(S_3^j)\right) \cup \left(\bigcup_{j\in[6]} \mathcal{S}(S_4^j)\right)$$

and the thick edges represent the last added ear.

Now we study a basic result of the proper connection number which we use later on in our proof. The proper connection number of traceable graphs such as a cycle and Θ-graph as well as in 2-ear-cycles is proved as follows.

Lemma 4.10 ([9, 11]). *Each cycle, Θ-graph, and 2-ear-cycles has proper connection number at most 2.*

Fig. 4.3: Graphs S_1, S_2, S_3, S_4

Fig. 4.4: Graphs $S_1^1, \ldots, S_1^4$

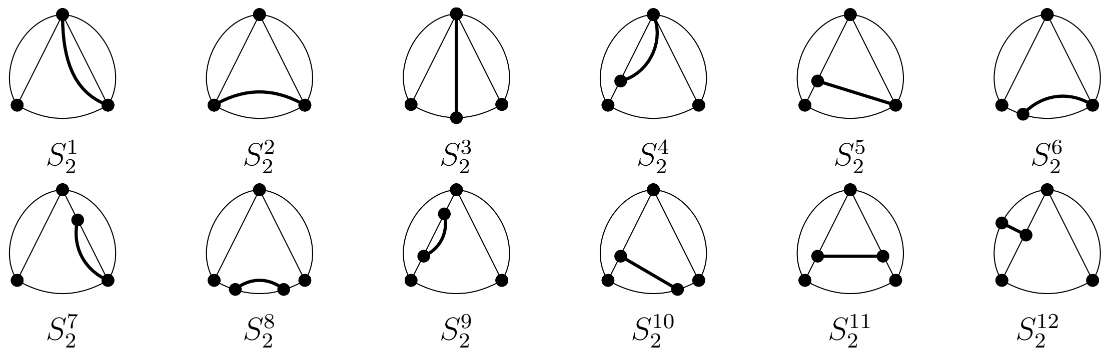

Fig. 4.5: Graphs $S_2^1, \ldots, S_2^{12}$

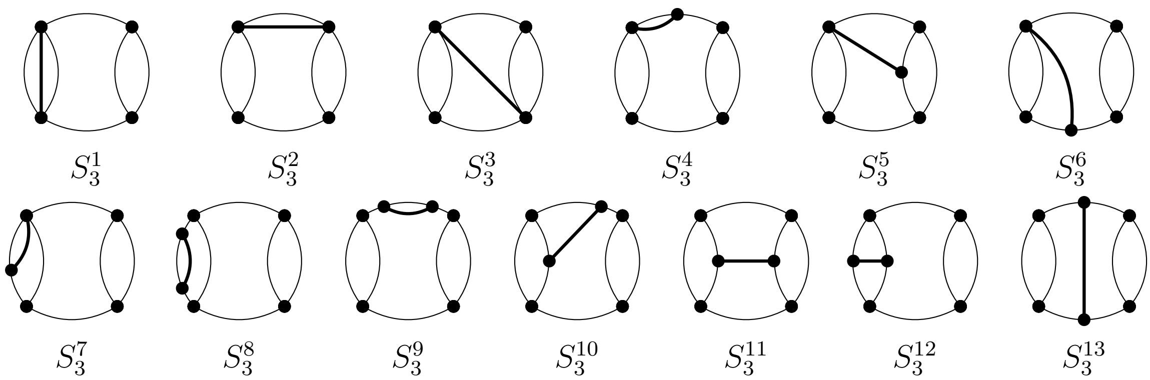

Fig. 4.6: Graphs $S_3^1, \ldots, S_3^{13}$

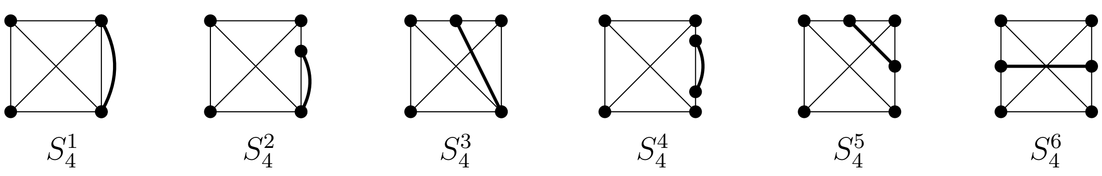

Fig. 4.7: Graphs $S_4^1, \ldots, S_4^6$

Proof. ([11]) By Corollary 3.4, it remains to determine the proper connection number of 2-ear-cycles since cycles and Θ-graphs are traceable.

Let G be a 2-ear-cycle. One can be readily observe that G contains an even cycle, say $C : u_1u_2 \dots u_{2l}u_1$, and two added ears, say R_1 and R_2, since it is constructed from a Θ-graph. Renaming vertices or ears if necessary, we may assume that u_i and u_j, where $1 \leq i < j \leq 2l$, are the end-vertices of R_1. If an end-vertex of R_2 is on C, say u_k, where $k \in [2l]$ and $k = i$ or $k = j$ is possible, then we colour all the edges of C alternatingly by two colours from [2], the edges of R_1 and R_2 such that $u_{i-1}u_iR_1u_j$ and $u_{k+1}u_kR_2$ are proper paths by two colours from [2]. By some simple case to case analysis, one can be readily check that this colouring makes G properly connected. If no end-vertex of R_2 is on C, then the end-vertices of R_2 are two distinct vertices of $R_1 \setminus \{u_i, u_j\}$, say u_k and u_l. Without lost of generality, we may assume that u_k has the smaller distance to u_i on R_1. Now we colour all the edges of C alternatingly by two colours from [2], and the edges of R_1 and R_2 such that $u_{i-1}u_iR_1u_kR_2u_l$ and $u_{j+1}u_jR_1u_k$ are proper paths by two colours from [2]. Again, by some simple case to case analysis, one can be reaidily check that this colouring makes G properly connected.

This finishes our proof. □

Lemma 4.11 ([11])**.** *Let H be a 2-connected graph. If u_1, u_2 are two distinct vertices of H and $P : v_1v_2 \dots v_k$ is a path, vertex disjoint from H, of order $k \geq 1$, then the graph H' obtained by adding edges u_1v_1 and u_2v_k is 2-connected.*

Proof. [11] Suppose, to the contrary, that H' is not 2-connected. Hence, there is a cut-vertex in H', say $z \in V(H')$. If $z \in V(H)$, then z is a cut-vertex in H, contracting the assumption on H. It follows that $z \in V(P)$. But one can be easily seen that $H' - z$ is connected, a contradiction. Therefore, H' is 2-connected.

The result is obtained. □

Now we are able to prove our theorem. Recall its statement.

Theorem 4.6 *Let G be a 2-connected graph of order $n = n(G)$ and minimum degree $\delta(G)$. If $\delta(G) > \max\{2, \frac{n+8}{20}\}$, then $pc(G) \leq 2$.*

Proof. [11] Suppose, to the contrary, that G is a 2-connected graph of order n, minimum degree $\delta(G) > \max\{2, \frac{n+8}{20}\}$, and proper connection number at least 3. Trivially, any 2-connected graph has a cycle as a subgraph. Furthermore, by ear decomposition, any 2-connected graph which is not a cycle or a Θ-graph has a 2-ear-cycle as a subgraph. By our supposition and Lemma 4.10, one can be readily observe that G contain 2-ear-cycles as subgraphs. Now let us take one of largest order, say Q. Note that Q is 2-connected and $pc(Q) \leq 2$ by Lemma 4.10. Now we take a subgraph H of G such that

(i) Q is a subgraph of H, H is 2-connected, $pc(H) \leq 2$, and

(ii) subject to (i), $n(H)$ is maximum.

Since the existent of Q in G, we always find such a 2-connected subgraph H. Moreover, requiring two conditions above, (i) and (ii), we cannot find any 2-connected subgraph of G, say H', such that $n(H') > n(H)$ and $pc(H') \leq 2$. Now, we follow a series of claims to conclude with the implication of the non-existence of G.

First of all, we show that there is no vertex in $V(G) \setminus V(H)$ having two neighbours in H by the following claim. Otherwise, we have a contradiction to the maximal order of H.

Claim 4.11.1. *There exists no vertex in $V(G) \setminus V(H)$ having two neighbours in H.*

Proof. Suppose, to the contrary, that there exists vertex $u \in V(G) \setminus V(H)$ such that u has two neighbours in H. Let us denote $H' = G[H \cup u]$. Hence, $n(H') > n(H)$. By Lemma 4.11, H' is 2-connected. Moreover, by Lemma 3.6, $pc(H') \leq 2$ since $pc(H) \leq 2$, contradicting the maximality of H.

It finishes the proof. (□)

Claim 4.11.2. *There exists no cycle C of even length such that $V(H) \cap V(C) \neq \emptyset$, $(V(G) \setminus V(H)) \cap V(C) \neq \emptyset$, and a colouring of H, using two colours and making H properly connected, restricted to the edges of C makes $C[V(H)]$ a proper path.*

Proof. We use the same technique in the proof of Claim 4.11.1 to prove this claim. Suppose, to that contrary, that there exists such a cycle C. Let us denote $H' = G[H \cup C]$. Hence, $n(H') > n(H)$ since $(V(G) \setminus V(H)) \cap V(C) \neq \emptyset$. By Lemma 4.11, H' is 2-connected. Moreover, by Lemma 3.8, $pc(H') \leq 2$ since $pc(H) \leq 2$, contradicting the maximality of H.

This finishes the proof. (□)

Claim 4.11.3. $G - V(H)$ *is bipartite*

Proof. Suppose, to the contrary, that there exists an odd cycle in $G - V(H)$, say C'. By the well-known different version of Menger's Theorem (i.e. Corollary 3.21), there are two vertex disjoint paths, say P_1, P_2, between $V(H)$ and C' since G is 2-connected graph. We note that these lengths can be one. Let us denote by x_1 and x_2 the end-vertices of P_1 and P_2 in H, respectively, as well as by z_1 and z_2 the end-vertices of P_1 and P_2, respectively. Since H has proper connection number at most 2, there exists a proper path between x_1 and x_2 in H, say P^{x_1,x_2}. Furthermore, let us denote by R_1 and R_2 two disjoint vertex paths connecting z_1 and z_2 in C'. Since C' is the cycle of odd length, hence, one of two cycles $C : x_1P_1z_1R_1z_2P_2x_2P^{x_1,x_2}x_1$ or $C : x_1P_1z_1R_2z_2P_2x_2P^{x_1,x_2}x_1$ is an even cycle such that $C[V(H)]$ is a proper path, contradicting Claim 4.11.2.

This finishes the proof. (□)

Claim 4.11.4. *No vertex of H is adjacent to two vertices of the same component S of $G - V(H)$.*

Proof. Suppose, to the contrary, that there exists a vertex $u \in V(H)$ such that u has two neighbours v_1 and v_2 in one component S of $G - V(H)$. Now, let us denote by R the shortest path between v_1 and v_2 in $G - V(H)$. By Claim 4.11.2, $n(uv_1Rv_2u)$

is odd. By the well-known different version of Menger's (i.e. Corollary 3.21), there exists a path, say R' between a vertex of $V(R)$, say v_r, and a vertex of $V(H)$, say u', which does not contain u or any vertex of $V(R) \setminus \{v_r\}$. Since H has proper connection number at most 2, let us denote by P a proper path between u and u' in H. Now, either $C : uv_1Rv_rR'u'Pu$ or $C : uv_2Rv_rR'u'Pu$ is an even cycle such that $C[V(H)] = uPu'$ is a proper path, contradicting Claim 4.11.2.

This finishes our proof. (□)

Claim 4.11.5. *There exists no 2-edge-connected subgraph S of $G - V(H)$ such that $|[V(H), V(S)]| \geq 2$.*

Proof. Suppose, to the contrary, that there exists such a subgraph. Let us take a 2-edge-connected subgraph S of $G-V(H)$ such that $|[V(H), V(S)]| \geq 2$ and, with respect to this condition, $n(S)$ is minimum. Now, we may assume that there exist two edges u_1v_1 and u_2v_2 of $[V(H), V(S)]$ such that $u_1, u_2 \in V(H)$ and $v_1, v_2 \in V(S)$. Since $n(H)$ is maximal, by Claim 4.11.1 and Lemma 4.11, $v_1 \neq v_2$, and by Claim 4.11.4, $u_1 \neq u_2$. Furthermore, by the minimality of S, $G[V(H) \cup V(S)]$ is 2-connected. Otherwise, there is a cut-vertex, say $x \in V(H) \cup V(S)$. One can be readily seen that $x \in V(S)$ since H is 2-connected. Hence, $G[V(H) \cup V(S)] - x$ consists of k components $S_1, \ldots, S_k$, where $k \geq 2$. It implies that there exists $i \in [k]$ such that $V(S_i) \subseteq V(S)$. Moreover, $S - V(S_i)$ is 2-edge-connected, and $|[V(H), V(S) - V(S_i)]| \geq 2$, contradicting the maximality of S. We obtain the result.

We note that, now, S is bridgeless and bipartite, by Claim 4.11.3. Hence, by Theorem 3.17, there exists an edge-colouring for S using two colours from [2] having strong property. Let us use such a colouring and an edge-colouring of H making H properly connected by using the same two colours. It can be readily seen that there exists a proper path, say R, between u_1 and u_2 in H. We note that the length of this path at least 1 since $u_1 \neq u_2$. We extend the colouring of two edges u_1v_1 and u_2v_2 such that $v_1u_1Ru_2v_2$ is a proper path by two colours from [2]. Clearly, H and $G[S]$ are properly connected. Hence, we show that there exists a proper path between a vertex in H and a vertex in S. If $w \in V(H)$ and $z \in V(S)$, then let us take by R' a shortest proper path in H from w to a vertex in R, say w'. We note that the length of R' can be 0, i.e. $w = w'$, but $|V(R) \cap V(R')| = 1$. One can be readily seen that either $wR'w'Ru_1v_1$ or $wR'w'Ru_2v_2$ is a proper path. Renaming vertices and paths if necessary, we may assume the first case. By the strong property of the edge-colouring used for S, we can extend the proper path $wR'w'Ru_1v_1$ to a proper path from w to z. Hence, between any two vertices $w \in V(H)$ and $z \in V(S)$ there is a proper path connecting them. We deduce that $G[V(H) \cup V(S)]$ has proper connection number 2. Moreover, $G[V(H) \cup V(S)]$ is 2-connected, contradicting the maximality of H.

This finishes our proof. (□)

By Theorem 3.15, any 2-connected graph is 2-edge-connected graph. Therefore, Claim 4.11.5 remains true if we replace the condition of 2-edge-connectivity by the condition of 2-connectivity.

Clearly, since G is 2-connected, there are at least two edges between any component

S of $G - V(H)$ and $V(H)$. Furthermore, Claim 4.11.5, it implies that S contains a bridge. Let T be the block-cut-vertex-tree of S, i.e. the vertices of T represent all maximal 2-connected graphs or cut-edges (also known as blocks) in S, and there is an edge between two vertices of T if and only if the corresponding blocks are connected by a cut-vertex (more detail see [39]). Trivially, T is a graph (more precisely, it is a tree) and contains at least two leaves, say t_1 and t_2. Furthermore, let us denote by T_1, T_2 the two 2-connected graphs which correspond t_1, t_2, respectively.

Claim 4.11.6. *For $i \in [2]$, $|V(T_i)| \geq 3$.*

Proof. Suppose, to the contrary, that there exists an $i \in [2]$ such that $|V(T_i)| \leq 2$. Clearly, T_i is a K_2 since T_i is a block of the block-cut-vertex-tree T. Furthermore, for $i \in [2]$, $V(T_i)$ contains one cut-vertex, say t_i^S, in S. Now we denote by t_i^H the second vertex of T_i. By the 2-connectivity of G, t_i^H has at least and, by Claim 4.11.1, at most one neighbour in $V(H)$. On the other hand, t_i^H is no cut-vertex of S, implying $d_G(t_i^H) = 2$, a contradition to the condition of Theorem 4.6 that $\delta(G) > 2$.

This finishes our proof. (□)

By Claim 4.11.6, it implies that $V(T_i) \geq 3$ for $i \in [2]$. Moreover, for $i \in [2]$, $V(T_i)$ consists of exactly one cut-vertex, say t_i^S, in S. Clearly, G is 2-connected, it implies that $|[V(T_i), V(H)]| \geq 1$. Moreover, by Claim 4.11.5, we deduce the equality, i.e. $|[V(T_i), V(H)]| = 1$. Clearly, again by the 2-connectivity of G, t_i^S is distinct from the vertex in $V(T_i)$ incident to the egde in $[V(T_i), V(H)]$, say t_i^H, for $i \in [2]$. Furthermore, since $|V(T_i)| \geq 3$ and $|[V(T_i), V(H)]| = 1$ implying $\min\{d_G(v) : v \in V(T_i) \setminus \{t_i^H, t_i^S\} \geq \delta(G)$, for $i \in [2]$. By Claim 4.11.5, it can easily deduce that $t_1^S \neq t_2^S$.

By Theorem 4.7, there exists a path, say $P_i \in T_i$ between t_i^S and t_i^H of length at least $2\delta' - 2$, where $\delta' = \min\{d_G(t) : t \in V(T_i) \setminus \{t_i^S, t_i^H\}\} \geq \delta(G)$, for $i \in [2]$. Now, let R be a path connecting t_1^S and t_2^S in S. Let u_1 and u_2 be the neighbours of t_1^H and t_2^H in $V(H)$, respectively. By Claim 4.11.4, u_1 and u_2 are distinct. Moreover, by the 2-connectivity of H and the well-known different version of Menger's Theorem (i.e. Corollary 3.21), there always exist two vertex disjoint paths, say Q_1 and Q_2, connecting $\{u_1, u_2\}$ and $V(G)$. Let u_1, q_1 be two end-vertices of Q_1 and u_2, q_2 be two end-vertices of Q_2. One can reaidly see that $q_1 \neq q_2$. We note that the lengths of Q_1 and Q_2 are possibly 0, i.e. $m(Q_i) = 0$ if and only if $u_i \in V(Q)$, for $i \in [2]$. Therefore, $P : q_1 Q_1 u_1 t_1^H P_1 t_1^S R t_2^S P_2 t_2^H u_2 q_2$ is a path of length $4\delta(G) - 1$ connecting q_1 and q_2.

Let Q' be the graph obtained by adding ear $P \setminus \{q_1, q_2\}$ to Q. Now, we continue with a fact which can be observed by a small case to case analysis. Recall, the thick edges in Figure 4.4−4.7 represent the last added ear.

Fact 4.12. *Any multigraph in $\{S_1^1, \ldots, S_1^4, S_2^1, \ldots, S_2^{12}, S_3^1, \ldots, S_3^{13}, S_4^1, \ldots, S_4^6\}$ has 4 non-thicks edges, say e_1, e_2, e_3, e_4, such that $G - e_k$ is a multigraph which can be obtained by subdividing edges, if necessary, of a multigraph S_1, S_2, S_3, S_4, for $k \in [4]$.*

By using the maximality of Q we consider our last claim as follows.

Claim 4.12.1. $n(Q) \geq 16\delta(Q) - 6$

Proof. By Fact 4.9, there are some i and j such that $Q' \in \mathcal{S}(S_i^j)$ and P correspond to the subdivision of the thick edge. Furthermore, let e_1, e_2, e_3, e_4 be the 4 edges given by Fact 4.12 and $P_{e_1}, P_{e_2}, P_{e_3}, P_{e_4}$ be their correponding paths in Q'. Since, by Fact 4.12, $S_i^j - e_k$ is a multigraph which can be obtained by subdividing edges, if necessary, of multigraph S_1, S_2, S_3 or S_4. It can be readily deduce that $\mathcal{S}(S_i^j - e_k) \subseteq \mathcal{G} = \sqcup_{i\in[4]}\mathcal{S}(S_i)$ for $k \in [4]$. On the other hand, by Fact 4.8, $Q' - \{v \in V(P_{e_k}) : d_{P_{e_k}} = 2\}$ is a 2-ear-cycle for all $k \in [4]$. Furthermore, by choosing the maximality of Q, the lengths of $P_{e_1}, P_{e_2}, P_{e_3}$ and P_{e_4} are at least the length of P. Hence, counting vertices, we obtain the desired result $n(Q) \geq 4(m(P) - 1) + 2 = 16\delta(G) - 6$.

This finish the proof. (□)

From the definition of P and Q it follows that $V(P) \cap V(Q) = \{q_1, q_2\}$, implying $n(G) \geq n(P) + n(Q) - 2 = 20\delta(G) - 8 > n(G)$, a contradiction.

Therefore, the proof is obtained.

□

Theorem 4.6 also shows, that if we require the graph G to be 2-connected, then the minimum degree bound $\delta(G) \geq \frac{n}{4}$ of Theorem 3.47 can be significantly lowered down to $\delta(G) > \max\{2, \frac{n+8}{20}\}$.

5 The proper connection number and forbidden induced subgraphs

In graph theory, a branch of mathematics, the term of *forbidden induced subgraphs*, that is to describe a finite set of individual graphs which do not contain any of these graphs as induced subgraphs or minors, is studied since a long time. There are a lot of interesting problems that have been studied by using this term. One of the most well-known examples of this term is Kuratowski's theorem, see Kuratowski [54], which states that a finite graph is planar if and only if it does not contain either of two forbidden subgraphs, say the complete graph K_5 and the complete bipartite graph $K_{3,3}$.

From time to time, graphs characterized in terms of forbidden induced subgraphs and the connectivity of a graph are known to have many interesting properties. On the one hand, it is well-known that the hereditary graph properties can be described by forbidden induced subgraph characterizations and so it is quite natural to consider them in connection with forbidden induced subgraphs. For example, researchers in graph theory study forbidden induced sugbraphs implying a polynomial-time complexity for computing maximum independent sets or a k-colouring of a graph. On the other hand, there are a lot of properties which are not hereditary, but forbidden subgraphs give nice and simple charaterizations, for example the property of a graph to be hamiltonian or pancyclic.

Moreover, forbidden induced subgraphs play an important role to determine the rainbow connection number, say $rc(G)$, of connected graphs. There are many nice results of the rainbow connection number using this term, see [41, 42, 43]. Since the proper connection number was motivated by the rainbow connection number and proper colouring, one is the starting point for our work to study the proper connection number in connected graphs with forbidden induced subgraphs. Furthermore, in [5], Bedrossian characterized pairs of forbidden induced subgraphs for 2-connected graphs implying hamiltonicity. Thus, since every non-complete hamiltonian graph has proper connection number 2 by Corollary 3.5, this characterization is motivated for us to find sufficient conditions in terms of connectivity and forbidden induced subgraphs such that $pc(G) \leq 2$ holds for all graphs of the corresponding graph classes.

In this chapter, we consider proper connection number at most 2 of connected graphs in the terms of connectivity and forbidden induced subgraphs. We note that an extended abstract containing some results of this chapter is already published in [10].

5.1 The proper connection number of connected, F-free graphs

There are some results of the proper connection in connected graphs with forbidden induced subgraphs that are published, recently. Huang et. al. [46] showed proper connection number 2 of connected graphs by using the condition of triangle-free in its complement.

Theorem 5.1 (Huang et. al. [46]). *Let G be a connected and non-complete graph. If $\overline{G}$ is triangle-free, then $pc(G) = 2$.*

The following result which was studied by the authors in [36] provides an upper bound of the proper connection number in connected graphs with a forbidden induced star.

Theorem 5.2 (Gerek et. al. [36]). *Let $s \geq 2$ be an integer. If G is a connected, $K_{1,s}$-free graph, then $pc(G) \leq s-1$.*

Moreover, the authors in [36] claimed that the result of Theorem 5.2 is sharp by considering any subdivision of a star $K_{1,s-1}$. When we study the proper connection number of connected graphs with the condition of forbidden induced subgraphs and connectivity, the first consideraion is claw-free, as well as a case $s = 3$ of Theorem 5.2. But our technique used to prove this result is different from their technique in [36]. Moreover, this technique will be used several times in our results of later sections. Therefore, we introduce our technique as follows.

Theorem 5.3 ([10]). *If G is a non-complete, connected, claw-free graph, then $pc(G) = 2$.*

Proof. We note that $pc(G) = 1$ if and only if G is complete by Fact 3.2. Since G is connected, non-complete, one can readily deduce that $pc(G) \geq 2$. Suppose, to the contrary, that there exists a connected, non-complete graph of proper connection number at least 3. Let G be a counterexample of minimum order, i.e. G is connected, non-complete, claw-free, but $pc(G) \geq 3$, but for all non-complete properly connected induced subgraphs G' of G, it holds $pc(G') = 2$. Now, let H be a connected induced subgraph in G such that

(i) H has proper connection number 2, and

(ii) $n(H)$ is maximum.

Clearly, there always exists such subgraph H since G is connected and a path P has proper connection number 2 by Fact 3.2. Moreover, by our supposition above, it can be readily observed that $|V(G) \setminus V(H)| \geq 1$. Hence, there is a vertex $v \in V(G) \setminus V(H)$ which is adjacent to a vertex in $V(H)$, say u. If there is at least two distinct neighbours of v in $V(H)$, i.e. v has another neighbour in $V(H)$ that is different from u, then by Lemma 3.6, the connected subgraph $G[H'] = G[H \cup u]$ has proper connection number 2, a contradiction to the maximality of H. Hence, u is only one neighbour of v in $V(H)$. If u has only one neighbour in $V(H)$, say w, then we colour $c(uv) \neq c(uw)$. It can be easily observed that the subgraph $G[H'] = G[H \cup u]$ is properly connected with two colours, a contradiction to the maximality of H. Hence, u has at least two distinct neighbouts in $V(H)$, say w_1 and w_2. Since G, and thus H, is claw-free, one can readily deduce that $N_H(u)$ is complete and of cardinality at least 2. Hence, $H-u$ is connected. By the minimality of G, $pc(H-u) \leq 2$ and therefore, the 2-edge-colouring on $H-u$ can be extended as follows: colour uv by the colour of w_1w_2, and uw_1, uw_2 differently. Clearly, $H-u$ is properly connected with two colours. Let $x \in V(H-u)$ be a vertex. By the concept of the properly connected graph, there is a proper path, say P, from x to w_1 or w_2. Without loss of generality, we may take a shortest one. Moreover, renaming vertices if necessary, we may assume that it contains w_1 but not w_2. Hence, either xPw_1uv or xPw_2w_1uv is proper path between x and v, depending on the colour of the edge incident to w_1 on P. By similar arguments above, $G[H \cup u]$ is properly connected subgraph. This conclusion contradicts the maximality of H.

This completes our proof. □

Now remember the graphs B_1, B_2, B_3, see Figures 3.2, 3.3, 3.4. The graphs B_1, B_2 have proper connection number 3. Moreover, let $k \geq 4$ be integer. By the simple case to case analysis, one can be readily deduced that the proper connection number of the windmill graph $B_3 = kK_2 \times K_1$ is 3, too. Hence, these example graphs show that the boundary class in terms of a single connected forbidden induced subgraph (without adding any further condition) is already reached. But all the results mentioned by Borozan et. al. [8], see Proposition 3.11 and Theorem 3.10, motivated us to find necessary conditions on forbidden induced subgraphs of 2-connected graphs having proper connection number 2.

5.2 The proper connection number of 2-connected, $F-$free graphs

Since the boundary class in terms of single connected forbidden induced subgraph (without adding any further condtion) is already reached by some graphs, for example B_1, B_2 and $B_3 = kK_2 \times K_1$, where $k \geq 4$, in this section, we study proper connection number 2 of 2-connected graphs which are H-free. Let $\mathcal{S}$ be the set of graphs whose every component is of the form $S_{i,j,k}$ for some $0 \leq i \leq j \leq k$. We denote by $\partial(S)$ the subset of vertices of $S \subseteq V(G)$ which have neighbours in $V(G) \setminus S$. First of all, the detailed structure of a finite set of graphs $\mathcal{H}$ is defined as follows

Proposition 5.4 ([10]). *(i) Let $\mathcal{F}$ be a finite set of graphs. If $\mathcal{F} \cap \mathcal{S} = \emptyset$, then there exists a 2-connected, $\mathcal{F}$-free graph G such that $pc(G) = 3$.*

(ii) Let i, j, k be three integers. If $i \geq 3$, or $i = 0$ and $j + k \geq 15$, then B_1 is a 2-connected, $S_{i,j,k}$-free graph such that $pc(B_1) = 3$.

Proof. Let $\mathcal{F}$ be a finite set of graphs such that $\mathcal{F} \cap \mathcal{S} = \emptyset$. Further, let F be a graph of order $n(F)$ of $\mathcal{F}$ and p be an integer such that $p \geq n(F)$ for all $F \in \mathcal{F}$. Hence, by Proposition 3.11, there exists a 2-connected graph G such that G has proper connection number 3, $\Delta(G) = 3$,$d_G(u, v) \geq p + 1$ for all pairs of vertices $u, v \in V(G)$ of degree 3, and G is of girth at least $p + 1$. Therefore, G contains no induced copy of all graphs in $\mathcal{F}$ which have a cycle as a subgraph, a vertex of degree at least 4, or two vertices of degree 3. Thus, G is $\mathcal{F}$-free and (i) is obtained.

We note that (ii) is immediately obtained from the construction of B_1.

This finishes our proof. □

The characterization of Proposition 5.4 is the motivation for us to study the structure of a subgraph F that is forbidden in a 2-connected graph with proper connection number at most 2. Since the structure is described by the characterization, it is the starting point forbid $S_{i,j,k}$ with small i, j, and k, for example P_5 ($\cong S_{0,0,4}$), $S_{1,1,2}$ in a 2-connected graph. Adding the condition of minimum degree at least 3 gives us a further result which is proper connection number 2 in a non-complete, 2-connected, $S_{1,1,6}$-free graphs. Let us recall some useful results in the previous sections that will be used several times throuhout our later proofs. First, the different version of Menger's Theorem are listed.

Corollary 3.20 *Let G be a graph. If $u \in V(G)$ is a vertex and $A \in V(G) \setminus \{u\}$ is a vertex set, then the size of a minimum vertex-cut for u and A equals the maximum number of, besides u, pairwise vertex disjoint $u - A$ paths.*

Corollary 3.21 *Let G be a graph. If $U_1, U_2 \subseteq V(G)$ are two disjoint vertex sets, then the size of a minimum vertex-cut for U_1 and U_2 equals the maximum number of pairwise vertex disjoint $U_1 - U_2$ paths.*

Now, the useful results on the proper connection number are listed as follows.

Lemma 3.6 *Let G be a nontrivial connected graph and H be a connected subgraph of G such that $pc(H) = 2$. If $u \in V(G) \setminus V(H)$ and $N_H(u) \geq 2$, then $pc(H \cup u) = 2$.*

Lemma 3.8 *Let G be a graph and $H \subset G$ be a subgraph of G such that $pc(H) \leq 2$. If there is a cycle C in G of even length such that $V(C) \cap V(H) \neq \emptyset$ and $V(C) \setminus V(H) \neq \emptyset$, and the colouring of H admits a proper colouring of $C[V(H)]$, then $pc(G[V(H) \cup V(C)]) \leq 2$.*

The last result which is constructed from a 2-connected graph and a path is stated below.

Lemma 4.11 *Let H be a 2-connected graph. If u_1, u_2 are two distinct vertices of H and $P : v_1v_2 \ldots v_k$ is a path, vertex disjoint from H, of order $k \geq 1$, then the graph H' obtained by adding edges u_1v_1 and u_2v_k is 2-connected.*

Lemma 5.5. *Let G be a 2-connected graph, H be a subgraph of G with $pc(H) \leq 2$ of maximum order, and S be a component in $G - V(H)$. Every vertex of H has at most one neighbour in $V(S)$ and every vertex in S has at most one neighbour in $V(H)$, i.e. $\Delta(G[V(H), V(S)]) \leq 1$.*

Proof. Let $u \in V(S)$ be a vertex. Suppose, to the contrary, that u has two neighbours in H. Let us denote by $H' = H \cup u$. By Lemma 3.6, $pc(H') \leq 2$. Moreover, one can be readily seen that $n(H') > n(H)$, a contradiction to the maximality of H.

By Claim 4.11.4, one can be readily observed that there is no vertex of H that has two neighbours in the same component S of $G - V(H)$.

This completes our proof. □

In Theorem 5.3, we study proper conneciton number 2 of a connected, claw-free graph (claw $\cong S_{1,1,1}$). We note that the claw is a graph of order 4. Hence, the later results are dealing with proper connection number 2 of a non-complete, 2-connected, $S_{i,j,k}$-free graph, where $S_{i,j,k}$ is of order at least 5. The next two results are considered the cases of $S_{i,j,k}$ of order 5. The first result of $S_{i,j,k}$ of order 5 is obtained as follows.

Theorem 5.6 ([10])**.** *If G is a non-complete, 2-connected, P_5-free graph, then $pc(G) = 2$.*

Proof. By Fact 3.2, G has proper connection number 1 if and only if G is complete. In our proof, we only consider that $\kappa(G) = 2$ by Theorem 3.14. Now, we suppose, to the contrary, that G is non-complete, 2-connected, P_5-free graph of minimum order number that has proper connecion number at least 3. Therefore, all properly induced subgraph of G have proper connection number at most 2. Let us denote by H a 2-connected properly induced subgraph of G of maximum order. It follows that $pc(H) \leq 2$ and that there exists a vertex $v \in V(G) \setminus V(H)$. By Corollary 3.20, there are two, besides v, vertex-disjoint paths, say P_1 and P_2, between v and $V(H)$. Clearly, we may assume that P_1 and P_2 are two induced paths. Let u_1 and u_2 be two end-vertices of P_1 and P_2 in H, respectively. One can readily observe that there is a shortest path, say R, in H connecting u_1 and u_2 by the connectivity of H. Let $C = G[V(R) \cup V(P_1) \cup V(P_2)]$ be a cycle. Trivially, we may assume that C is induced. Otherwise, we can redefine P_1, P_2 and v such that the corresponding cycle is induced. If the length of C is at least 6, then there exists a path P_5 contained in C, a contradiction. Hence, $n(C) \leq 5$. On the other hand, by Lemma 5.5 no vertex in $V(C) \cap V(H)$ has two neighbours in one component of $G - V(H)$ and no vertex in $V(C) \cap (V(G) \setminus V(H))$ has two neighbours in H. One can be readily observed that $|V(C) \cap V(H)| \geq 2$ and $|(V(G) \setminus V(H)) \cap V(C)| \geq 2$, implying $n(C) \geq 4$. If $n(C) = 4$, then $|V(C) \cap V(H)| = 2$ and $|(V(G) \setminus V(H)) \cap V(C)| = 2$. By Lemma 4.11, $G[V(H) \cup V(C)]$ is 2-connected and $n(G[V(H) \cup V(C)]) > n(H)$. Moreover, $R = u_1u_2$ is a proper path of H and the length of C is even. By Lemma 3.8, $G[V(H) \cup V(C)]$ has proper connection number at most 2, contradicting the maximality of H. Hence, $n(C) = 5$. If $|(V(G) \setminus V(H)) \cup V(C)| = 3$, then let us denote by $C : u_1v_1v_2v_3u_2u_1$ a cycle of length 5, where $v_1, v_2, v_3 \in (V(G) \setminus V(H)) \cup V(C)$. Since H is 2-connected, there is another neighbour of u_1 in $V(H)$, say w, different from u_2. By Lemma 3.6, one can be readily seen that w is not adjacent to v_1 and v_3. Moreover, by Lemma 3.8, w is not adjacent to v_2. Hence, $\{w, u_1, v_1, v_2, v_3\}$ induces P_5, a contradiction. Therefore, $|(V(G) \setminus V(H)) \cup V(C)| = 2$, implying that $C : u_1v_1v_2u_2wu_1$ is a cycle of order 5, where $v_1, v_2 \in (V(G) \setminus V(H)) \cup V(C)$ and $w \in V(H)$. Note that C is induced. Let us denote $H' = G[V(H) \cup V(C)]$. By Lemma 4.11, H' is 2-connected. If $H' - w$ is 2-connected, then $n(H' - w) \geq n(H)$, a contradicting the choice of H. Hence, there is a vertex, say x, different from w, such that $\{w, x\}$ is a cut-set in H'.

Since H is 2-connected, w is not a cut-vertex in H, implying $x \notin \{v_1, v_2\}$. But $x = u_i$, where $i \in [2]$, is possible. Since u_1 and u_2 are connected by the path u_1, v_1, v_2, u_2, there exists a vertex, say w', being adjacent to w that is not in the same component with u_1 and u_2 in $H' \setminus \{w, x\}$. Renaming vertices if necessary, we may assume that $x \neq u_1$. Therefore, $\{w', w, u_1, v_1, v_2\}$ induces a path P_5, a contradiction.

This completes our proof. □

Clearly, P_5 is one of the graphs $S_{i,j,k}$ of order 5, where $i = j = 0$ and $k = 4$. The following results is considered the graphs $S_{i,j,k}$, where $1 \leq i \leq j \leq k$. Although the proof of the next theorem is the same as the proof of Theorem 5.6 in their constructs, but their details are distinct.

Theorem 5.7 ([10]). *If G is a non-complete, 2-connected, $S_{1,1,2}$-free graph, then $pc(G) = 2$.*

Proof. By Fact 3.2, the proper connection number of a graph is 1 if and only if this graph is complete. Hence, from the condition of theorem, we note that the proper connection number is at least 2 since the graph is non-complete. Suppose, to the contrary, that there is a 2-connected $S_{1,1,2}$-free graph of proper connection number at least 3. In all the such 2-connected graph, let G be a counterexample of minimum order number, i.e. G is 2-connected, $S_{1,1,2}$-free graph, but $pc(G) \geq 3$ and for all non-complete but 2-connected properly induced subgraph, say G', of G, it holds that $pc(G') \leq 2$. Since G is 2-connected, there is a cycle, as a subgraph of G, that has proper connection number at most 2 by Fact 3.2. Now, let H be a 2-connected properly induced subgraph in G such that $n(H)$ is maximum. It can be readily seen that there always such induced subgraph H. Moreover, by our supposition, $pc(H) \leq 2$ since G is of minimum order.

Hence, there exists a vertex of $V(G) \setminus V(H)$, say v, that is adjacent to at least one vertex in H, say u. By Lemma 5.5, one can easily observe that v has only one neighbour in H that is u. Furthermore, by Corollary 3.20, renaming vertices if necessary, we may assume that there is a path, say $vz_1 \ldots z_k w$, such that $z_1, \ldots, z_k \in V(G) \setminus V(H)$, $w \in V(H) \setminus \{u\}$ since G is 2-connected. Clearly, $k \geq 1$. On the other hand, u has at least two neighbours in H since H is 2-connected. Now, the following claims are obtained.

Claim 5.7.1. *u is not adjacent to z_k.*

Proof. By Lemma 5.5, we immediately obtain Claim 5.7.1. Otherwise, $n(H)$ is not maximum, a contradiction. (□)

Claim 5.7.2. *No neighbour of u in H is adjacent to z_1.*

Proof. Let $u_1 \in N_H(u)$ be a neighbour of u in H. Suppose, to the contrary, that z_1 is adjacent to u_1. Hence, $C = uvz_1u_1u$ is the cycle of even length. By Lemma 3.8, $pc(G[V(H) \cup V(C)]) \leq 2$ since uu_1 is the proper path of H. Moreover, by Lemma 4.11, $G[V(H) \cup V(C)]$ is 2-connected. Therefore, we get the contradiction of the maximality of H.

The proof is obtained. (□)

Recall that, u has at least two neighbours in H. By Claim 5.7.2 and Claim 5.7.1, one can readily deduce that $N_H(u)$ is a clique. Otherwise, $\{u, u_1, u_2, z_1, v\}$ induces $S_{1,1,2}$, a contradiction, if $u_1, u_2 \in N_H(u)$ are independent. The following claim is considered the subgraph $H - u$.

Claim 5.7.3. *$H - u$ is 2-connected.*

Proof. Clearly, $H-u$ is connected since, our supposition, H is 2-connected. Suppose, to the contrary, that $H-u$ is not 2-connected. Hence, there exists a vertex of $V(H-u)$, say x, such that $H-u-x$ is disconnected, i.e. x is a cut-vertex of $H-u$. Therefore, $\{u, x\}$ is cut-set in H. Without lost of generality, let us denote by H_1, H_2 two componnents of $H - \{u, x\}$. Since H is 2-connected, both components H_1, H_2 contain at least one neighbour of u, say x_1 and x_2. On the other hand, by Claim 5.7.2, x_1 is adjacent to x_2, contradicting the fact that H_1 and H_2 are two components of $H - \{u, x\}$.

The proof is obtained. (□)

Now, $H-u$ is 2-connected. Moreover, by the minimality of G, implying $pc(H-u) \leq 2$. We consider the 2-edge-colouring of $H-u$ in order to make it properly connected. Let $u_1, u_2 \in N_H(u)$ be two neighbours of u in H. Now, we colour the two edges uu_1, uu_2 by the colour different from the colour of u_1u_2 and all the edges of the path $uvz_1 \ldots z_kw$ such that it is proper path. Clearly, $H-u$ and $G[u, u_1, u_2, v, z_1, \ldots, z_k]$ are properly connected by theirselves. Now, let $x \in V(H-u)$ and $y \in \{u, u_1, u_2, v, z_1 \ldots z_k\}$. By the definition of the properly connected graph $H-u$, there always exists a proper path, say P, from x to u_1 or u_2. Without lost of generality, we get the shortest one. Renaming vertices if necessary, we may assume that P contains u_1 but not u_2. Now either $xPu_1uvz_1 \ldots y$ or $xPu_1u_2uvz_1 \ldots y$ is a proper path between x and y, depending on the colour of the edge of P incident to u_1. Hence, $G[V(H) \cup \{v, z_1, \ldots z_k\}]$ is properly connected graph. Moreover, one can easily see that, by Lemma 4.11, it is 2-connected. That contradicts the maximality of H.

This completes our proof.

□

Adding the further condition of minimum degree at least 3, we improve the order of $S_{i,j,k}$-free of 2-connected graph having proper connection number 2 as follows.

Theorem 5.8. *If G is a non-complete, 2-connected, $S_{1,1,6}$-free graph of minimum degree δ at least 3, then $pc(G) = 2$.*

Before starting to prove Theorem 5.8, let us denote by $\partial(S)$ the subset of vertices of $S \subseteq V(G)$ which have neighbours in $V(G) \setminus S$.

Proof. Suppose, to the contrary, that the statement is false. Hence, there exists a graph G which is non-complete, 2-connected, $S_{1,1,6}$-free of minimum degree $\delta(G) \geq 3$ with proper connection number 1 or at least 3. On the other hand, by Fact 3.2, $pc(G) = 1$ if and only if G is complete graph. It can readily deduce that $pc(G) \geq 3$.

Moreover, by Corollary 3.5, G is not Hamiltonian since the proper connection number of non-complete, Hamiltonian is 2.

Note that every 2-connected graph with minimum degree $\delta(G)$ at least 3 is either Hamiltonian or contains a cycle of order at least $2\delta(G)$. Hence, G contains a cycle, as a subgraph in G, that has proper connection number at most 2 by Fact 3.2. Now let us denote by H a largest subgraph of proper connection number at most 2 in G. Cleary, such a subgraph always exists, it is induced in G and $n(H) \geq 3$, and $|V(G)\setminus V(H)| \geq 1$. It can easily deduce that every vertex of $V(G) \setminus V(H)$ has at most one neighbour in H. Otherwise, by Lemma 3.6, an induced connected subgraph containing H and the vertex which has two neighbours in H has proper connection number at most 2, a contradiction to the maximality of H. Furthermore, there are at least two vertices in $V(G) \setminus V(H)$ that have only one neighbour in H. On the other hand, since the minimum degree $\delta(G)$ at least 3, implying $|V(G) \setminus V(H)| \geq 3$

Let $\mathcal{B}$ be the set of non-trivial blocks in $G - V(H)$, i.e. the set of 2-connected induced subgraph of order at least 3 in $G - V(H)$. For all $x \in V(G) \setminus V(H)$, we denote $V(x) := V(B)$ if $x \in B$ for some $B \in \mathcal{B}$ and x is not a cut-vertex in $G \setminus V(H)$. Otherwise, we define $V(x) := \{x\}$. Hence, for any two vertices $v_1, v_2 \in \partial(V(G)\setminus V(H))$ since G is 2-connected, connected by a path P in $G - V(H)$, $Q(v_1, v_2)$ denotes the graph $G[\cup_{w\in V(P)} V(w)]$.

Note that we use the same technique of the proof of Claim 4.11.3 and Claim 4.11.5 of Theorem 4.6 to prove two following lemmas although the construction of H of this theorem is different from the construction of H of Theorem 4.6.

Claim 5.8.1. *For any two vertices $v_1, v_2 \in \partial(V(G) \setminus V(H))$, $Q(v_1, v_2)$ is bipartite.*

Proof. Suppose, to the contrary, that $Q(v_1, v_2)$ is not bipartite, i.e. there exists an odd cycle C in $Q(v_1, v_2)$. Let u_1 and u_2 be two neighbour of v_1 and v_2 in H, respectively. Note that u_1 and u_2 are not necessarily distinct. Since H has proper connection number at most 2, there always exists a proper path, say R, connecting u_1 and u_2 in H. By the definition and construction above, there is non-trivial block $B \in \mathcal{B}$ such that $V(C) \subset V(B)$ and there is a path, say P, connecting v_1 and v_2 in $V(G) \setminus V(H)$ such that P contains at least two vertices of $V(B)$, i.e. $|V(P) \cap V(B)| \geq 2$. Let $x_1, x_2 \in V(P) \cap V(B)$ be two vertices having the shortest distance to v_1, v_2 in P, repsectively. One can readily observe that x_1 and x_2 are distinct. Since B is non-trivial block, i.e. B is 2-connected, by Corollary 3.21, there are two vertex-disjoint paths, say P_1 and P_2, between x_1 to C, and x_2 to C, respectively. Without lost of generality, we may assume that $x_1 \in V(P_1)$ and $x_2 \in V(P_2)$. We note that the length of P_1 or P_2 is able to be 0 if $x_1 \in V(C)$ and $x_2 \in V(C)$. Let x_3 and x_4 be the end-vertices of P_1 and P_2 in C, respectively. We denote the orientation of C. Hence, either $x_3\overleftarrow{C}x_4$ or $x_3\overrightarrow{C}x_4$ is odd length since C is odd cycle. Therefore, either $u_1v_1Px_1P_1x_3\overleftarrow{C}x_4P_2x_2Pv_2u_2Ru_1$ or $u_1v_1Px_1P_1x_3\overrightarrow{C}x_4P_2x_2Pv_2u_2Ru_1$ is even cyle. Moreover, R is the proper path of H. By Lemma 3.8, a new induced subgraph in G which contains subgraph H has proper connection number at most 2. Hence, a contradiction to the maximality of H.

The proof is obtained. (□)

Claim 5.8.2. *For any two vertices $v_1, v_2 \in \partial(V(G)\setminus V(H))$, $Q(v_1, v_2)$ contains bridges.*

Proof. Suppose, to the contrary, that $Q(v_1, v_2)$ is bridgeless. Moreover, by Claim 5.8.1, $Q(v_1, v_2)$ is bipartite. Hence, by Theorem 3.17, $Q(v_1, v_2)$ has strong property with two colours, i.e. every two distinct vertices of $V(Q(v_1, v_2))$ are connected by two not necessarily internally vertex-disjoint proper paths such that the colours of the first egde and the last edge of their paths are distinct. We colour all the edges of $Q(v_1, v_2)$ with two colours from [2] in order to make it properly connected with strong property.

Let u_1 and u_2 be two not necessarily distinct neighbours of v_1 and v_2 in H, respectively. By our supposition above, $pc(H) \leq 2$, there exist a proper path, say R, between u_1 and u_2 in H. Now we colour the edges u_1v_1 and u_2v_2 with two colours from [2] such that $u_1v_1Rv_2u_2$ is a new proper path.

Clearly, H and $Q(v_1, v_2)$ are properly connected by themselfves with two colours from [2]. Let $x \in V(H)$ and $y \in V(Q(v_1, v_2))$ be arbitrary vertices, respectively. By definition, there is a shortest proper path, say P, between x and a vertex of $V(R)$ that contains only one vertex in R. Depending on the colour of the last edge of P, either $xPwRw_1$ or $xPwRw_2$ is a proper path. Renaming vertices if necessary, we may assume that $xPwRw_1$ is such proper path. Since $Q(v_1, v_2)$ has strong property, there always exists a proper path, say Q, between y and v_1 such that w_1v_1Qy is a proper path. Hence, $xPwRw_1v_1Qy$ is the proper path between x and y. Therefore, $G[V(H) \cup V(Q(v_1, v_2))]$ is properly connected with two colours, a contradiction to the maximality of H.

The proof is obtained. (□)

By Claim 5.8.2, we deduce that there is a bridge in $Q(v_1, v_2)$. Now we denote by T the graph obtained by removing all bridges in $G - V(H)$, contracting each component, which could possibly consist of one vertex, to a super-vertex, and adding an edge between the super-vertices t_1 and t_2 of the components T_1 and T_2 if a vetex of $V(T_1)$ is adjacent to a vertex of $V(T_2)$ in $G - V(H)$. Clearly, T is a tree, more detail, T is the superblock-cut-vertex tree, and $n(T) \geq 2$.

Claim 5.8.3. *Every component contracted to a leaf of T has exactly one vertex in $\partial(V(G) \setminus V(H)$. Further, that vertex has exactly one neighbour in $V(H)$.*

Proof. Let T' be a component contracted to a leaf t' of T. Since G is 2-connected, i.e. G is bridgeless, there is a vertex, say x, in T_1 such that $x \in \partial(V(G) \setminus V(H))$. Obviously, x has exactly one neighbour in $V(H)$ by Lemma 3.6.

Suppose, to the contrary that, T' has at least two vertices in $\partial(V(G) \setminus V(H)$, say x_1, x_2. Clearly, T' is the non-trivial block. Hence, by Claim 5.8.2, T' contains a bridge, a contradiction.

The proof is obtained. (□)

Claim 5.8.4. *G does not exist.*

Proof. By Claim 5.8.2, there exists bridge in $G - V(H)$ implying T has at least two vertices, i.e. $n(T) \geq 2$. Hence, there are at least two leaves, say t_1 and t_2, in T. Now let T_1 and T_2 be two components contracted to two leaves t_1 and t_2, respectively. We note that there is at most one edge in $G - V(H)$ between vertices of $V(T_1)$ and vertices $V(T_2)$.

Further, we denote $v_1 \in \partial(V(G) \setminus V(H)) \cap V(T_1)$ and $v_2 \in \partial(V(G) \setminus V(H)) \cap V(T_2)$. By Claim 5.8.3, v_i has exactly one neigbour in H, say u_i, for all $i \in [2]$. By Lemma 5.5, we note that u_1 and u_2 are different. Since the minimum degree of G is at least 3, v_i has at least two neighbours in $G - V(H)$, for all $i \in [2]$. Moreover, it can be readily deduced that $n(T_1) \geq 3$ and $n(T_2) \geq 3$ since t_1 and t_2 are leaves of T. Hence, $d_{T_i}(v_i) \geq 2$, for all $i \in [2]$. Let z_1, z_2 and z_3, z_4 be two neighbours of v_1 and v_2 in T_1 and T_2, respectively. Clearly, z_1, z_2, z_3, z_4 are distinct. Renaming vertices if necessary, we may assume that, z_1 and z_3 are non-adjacent. Hence, each vertex of $\{z_1, z_3\}$ has no neighbour in $V(H)$. Otherwise, by Claim 5.8.2, T_i contains a bridge, a contradiction, for all $i \in [2]$. By the minimum degree condition of G and T_i is contracted to a leave t_i, each of both vertices $\{z_1, z_3\}$ has two neighbours in T_i which are different from z_1, z_2, z_3, z_4, for all $i \in [2]$. To be more precise, let z_5 and z_6 be neighbours of z_1, and, z_7 and z_8 be neighbours of z_3. Renaiming vertices if necessary, we may assume that z_5, z_6, z_7, z_8 are distinct, and that z_5 and z_6 are non-adjacent to v_2, z_3, and z_7 and z_8 are non-adjacent to v_1, z_1. Since the bipartiteness of $G[T_1]$ and $G[T_2]$, which follows from Claim 5.8.1, z_5 is non-adjacent to v_1 and z_6, and z_6 is non-adjacent to v_1, and z_7 is non-adjacent to v_2. Moreover, z_5, z_6, z_7 have no neighbour in $V(H)$. Otherwise, by Claim 5.8.2, T_i has a bridge, for all $i \in [2]$. Let $P = u_1 = w_1 \dots w_k = u_2, (k \geq 2)$ be the shortest path between u_1 and u_2 in H. Hence, $G[z_5, z_6, z_1, v_1 w_1, \dots w_k, v_2, z_3, z_7]$ has an induced subgraph $S_{1,1,6}$, a contradiction.

(□)

This completes our proof.

□

The proper connection number of a 3-edge-connected graph, in Therem 3.18, is already proved in Chapter 3 by using the results of Lemma 3.23 and Theorem 3.17. However, the first part of Theorem 3.18 can be proved without major changes from the proof of Theorem 5.8. Recall the first part of Theorem 3.18.

Theorem 3.18 *If G is a 3-edge-connected non-complete graph, then $pc(G) = 2$.*

Proof. To prove the first part of Theorem 3.18, we again use the techniques, notations, claimes of the proof of Theorem 5.8. Suppose, to the contrary, there exists a graph, say G, which is non-complete, 3-edge-connected with proper connection number 1 or at least 3. Since G is non-complete, by Fact 3.2, $pc(G) \geq 2$. Moreover, by Corollary 3.5, G is not Hamiltonian. Thus, $pc(G) \geq 3$.

Clealy, in every 3-edge-connected, there always exists a cycle which has proper connection number at most 2 by Fact 3.2, as a subgraph in G. Now let H be a largest subgraph of proper connection number at most 2 in G. By our supposition above, we note that such a subgraph H always exists, it is induced in G and $n(H) \geq 3$, $|V(G) \setminus V(H)| \geq 1$. On the other hand, by Lemma 3.6, every vertex in $V(G) \setminus V(H)$ has at most one neighbour in H. By the condition of a 3-edge-connected graph, it can be readily observed that $|V(G) \setminus V(H)| \geq 2$. The following notations and claims, which are already mentioned in the proof of Theorem 5.8, are stated here.

Let $\mathcal{B}$ be the set of non-trivial blocks in $G - V(H)$, i.e. the set of 2-connected induced subgraph of order at least 3 in $G - V(H)$. For all $x \in V(G) \setminus V(H)$, we denote $V(x) := V(B)$ if $x \in B$ for some $B \in \mathcal{B}$ and x is not a cut-vertex in $G \setminus V(H)$. Otherwise, we define $V(x) := \{x\}$. Hence, for any two vertices $v_1, v_2 \in \partial(V(G) \setminus V(H))$ since G is 2-connected, connected by a path P in $G - V(H)$, $Q(v_1, v_2)$ denotes the graph $G[\cup_{w \in V(P)} V(w)]$.

Claim 5.8.1 *For any two vertices $v_1, v_2 \in \partial(V(G) \setminus V(H))$, $Q(v_1, v_2)$ is bipartite.*

Claim 5.8.2 *For any two vertices $v_1, v_2 \in \partial(V(G) \setminus V(H))$, $Q(v_1, v_2)$ contains a bridge.*

Claim 5.8.3 *Every component contracted to a leaf of T has exactly one vertex in $\partial(V(G) \setminus V(H)$. Further, that vertex has exactly one neighbour in $V(H)$.*

In the following claim, we verify that there does not exist a 3-edge-connected graph with proper connection number at least 3.

Claim 5.8.5. *G does not exists.*

Proof. By Claim 5.8.2, there is a bridge in $G - V(H)$ implying $n(T) \geq 2$ and T has at least two leaves. Hence, there always exits a least t^* in T. Let T^* be the compoent contracted to t^* in T. By Claim 5.8.3, clearly, there is exact one edge between a vertex of $V(T)$ and a vertex of $V(H)$ in G, say e_1. Furthermore, since t^* is one of all leaves of T, there exists an edge between a vertex of $V(T^*)$ and a vertex of $V(G) \setminus (V(H) \cup V(T^*))$, say e_2. Therefore, $\{e_1, e_2\}$ is 2-edge-cut, contradicting the 3-egde-connectivity of G. Hence, G does not exist.

(□)

This completes our proof.

□

We finish Chapter 5 here.

6 The proper 2-connection number $pc_2(G)$

In graph theory, connectivity is one of the basic concepts and plays an important role. The concepts and results which are related to the connectivity are presented in Chapter 4 of the book 'Introduction to Graph Theory' by West [71]. At the beginning of Chapter 4, the author introduced the important role of the connectivity as follows:

> *A good communication network is hard to disrupt. We want the graph of possible transmissions to remain connected even when some vertices or edges fail. When communication links are expensive, we want to achieve these goals with few edges.*

Let $l \geq k \geq 2$ be two positive integers. The concept of proper k-connection number was recently introduced by Borozan et al. [8]. By the definition above, we note that there always exists the proper k-connection number in every l-connected graph. Moreover, by Fact 3.2, it can be readily seen that $pc_k(G) \geq 2$. On the other hand, as well as the proper connection number $pc(G)$, if G is l-connected graph and a proper colouring, then G is properly k-connected graph, too. Hence, the proper k-connection number $pc_k(G)$ is bounded by $\chi'(G)$. Therefore, we get

$$2 \leq pc_k(G) \leq \chi'(G)$$

The well-known result of the chromatic index number which is called Vizing's Theorem was studied in [70] as follows.

Theorem 6.1 (Vizing's Theorem [70])**.** *If G is a simple graph, then $\chi'(G) \leq \Delta(G)+1$.*

By Vizing's Theorem, as well as Theorem 6.1, the proper k-connection number is bounded by the following result

Fact 6.2 (Borozan et al. [8])**.** *Let k, l be two positive integers such that $k \leq l$. If G is l-connected of maximum degree $\Delta(G)$, then $2 \leq pc_k(G) \leq \Delta(G) + 1$.*

In this chapter, we consider the proper 2-connection number of several classes of connected graphs.

6.1 Results for the proper k-connection number

In this section, we state some existent results of the proper k-connection number $pc_k(G)$. Since the proper connection number of a connected graph is at most the proper connection number of its connected spanning subgraph, see Proposition 3.3, it can be readily deduced that this result still holds for the proper k-connection number, where $k \geq 2$ is an integer. Hence, the following result is immediately obtained.

Lemma 6.3. *Let $l \geq k \geq 2$ be two positive integers and G be l-connected graph. If H is a l-connected spanning subgraph of G, then $pc_k(G) \leq pc_k(H)$.*

Proof. The proof of this lemma is similar to the proof of Proposition 3.3 proved by Andrews et al. [4]. Since $l \geq k \geq 2$, and G, H are two l-connected graphs, there always exist $pc_k(G)$ and $pc_k(H)$. Let us define by $c_H : E(H) \to [pc_k(H)]$ a colouring c_H in order to make H proper k-connected graph.

By the concept of the proper k-connected graph, hence, every two distinct vertices, say $u, v \in V(H)$, are connected by k disjoint proper paths. Now, we define a colouring c of G as follows: $c(e) = c_H(e)$ if $e \in E(G) \cap E(H)$ and $c(e) = 1$ if $e \in E(G) \setminus E(H)$. Clearly, every two vertices, say $u, v \in V(G)$, are connected by k disjoint proper paths since H is l-connected spanning subgraph of G. Hence, by the minimum of the proper k-connection number, it can be readily deduced that $pc_k(G) \leq pc_k(H)$.

This completes our proof.

□

Similar as for the proper connection number $pc(G)$, the results of the proper k-connection number of bipartite graphs are considered. First of all, Borozan et al. [8] posed a general conjecture for $pc_k(G)$ where G is bipartite graph with the specific connectivity that depends on k.

Conjecture 6.4 (Borozan et al. [8])**.** *If G is a $2k$-connected bipartite graph with $k \geq 1$, then $pc_k(G) = 2$.*

Note that, the conjecture holds when $k = 1$, see the first part of Theorem 3.12. If $k \geq 2$, then Conjecture 6.4 is still open. The authors in [8] showed that there is a family of bipartite graphs which are $(2k-1)$-connected with the property that $pc_k(G) > 2$ by the following proposition.

Proposition 6.5 (Borozan et al. [8])**.** *Let $k \geq 1$, p and q be three positive integers. If $p = 2k - 1$ and $q > 2^p$, then $pc_k(K_{p,q}) > 2$.*

Adding further condition, the authors in [8] proved Conjecture 6.4 for complete bipartite graphs.

Theorem 6.6 (Borozan et al. [8])**.** *If $G = K_{m,n}$, where $m \geq n \geq 2k$ for $k \geq 1$, then $pc_k(G) = 2$.*

In thesis [55], which is published recently, the author improved lower bounds for the proper k-connection number of certain complete bipartite graphs $K_{p,q}$ in terms of p and q.

Theorem 6.7 (Laforge [55])**.** *Let p, q and k be three positive integers.*

(i) If $p \leq q$ and $\frac{p}{2} < k \leq p$, then $pc_k(K_{p,q}) \geq \lceil \sqrt[p]{q} \rceil$.

(ii) If $k \geq 2$, $p = 2k - 1$ and $q \leq 3^p$, then $pc_k(K_{p,q}) = \lceil \sqrt[p]{q} \rceil$.

(iii) If $2 \leq p \leq q$ and $2 \leq k \leq p < 2k$, then $pc_k(K_{p,q}) \geq \lceil \sqrt[2p-2k+1]{q} \rceil$.

(iv) If $k \geq 3$, $p = 2k - 2$ and $q \geq 3^{p-1}$, then $pc_k(K_{p,q}) = \lceil \sqrt[p-1]{q} \rceil$.

The following results, which are illustrated Theorem 6.7 were presented in [55].

Proposition 6.8 (Laforge [55]). *(i) $pc_3(K_{4,27}) = 3$.*

(ii) $pc_3(K_{4,28}) = 4$.

After that, the author in [55] suggested some questions as follows.

Question 6.9 (Laforge [55]). *(i) Determine $pc_3(K_{4,q})$ for $q \geq 5$.*

(ii) Determine $pc_{p-1}(K_{p,q})$ for $4 \leq p < q$.

(iii) Determine $pc_{\frac{p-1}{2}}(K_{p,q})$ for $7 \leq p < q$ and p is odd.

For general graphs, Borozan et al. [8] extended Conjecture 6.4 as follows.

Conjecture 6.10 (Borozan et al. [8]). *Let $k \geq 1$ be a positive integer. If G is $2k$-connected graph, then $pc_k(G) \leq 3$.*

Conjecture 6.10 was proved for $k = 1$, see Theorem 3.10. Similar as for Conjecture 6.4, this conjecture is still open for $k \geq 2$. A stronger result for complete graphs was studied by Borozan et al. [8].

Theorem 6.11 (Borozan et al. [8]). *Let $n \geq 4$, $k \geq 2$ be two positive integers. If $n \geq 2k$, then $pc_k(K_n) = 2$.*

Next we state some results of the proper 2-connection number. Since a cycle is one of the most simple 2-connected graphs, the proper 2-connection number is immediately obtained as follows.

Fact 6.12. *If G is a cycle of order n, then $pc_2(G) = 2 + (n \mod 2)$.*

Hence, the proper 2-connection number of a cycle is either 2 or 3 depending on its order. Different to the proper connection number of Hamiltonian graph which is 2, the proper 2-connection number of Hamiltonian graph is not a constant. By Lemma 6.3 and Fact 6.12, we obtain the proper 2-connection number of a Hamiltonian graph.

Corollary 6.13. *If G is Hamiltonian, then $2 \leq pc_2(G) \leq 3$. Moreover, $pc_2(G) = 2$ if $|V(G)|$ is even.*

Although the result of Corollary 6.13 was already mentioned in [48], but the authors did not give the details of a proof. Hence, the proof of this corollary is as follows.

Proof. Clearly, G is a 2-connected graph since G is Hamiltonian. There exists the proper 2-connection number in G. Moreover, by Fact 6.2, $pc_2(G) \geq 2$, where $k = 2$. Let C be a Hamiltonian cycle of G. It can be easily observed that C is a spanning

2-connected subgraph of G. By Lemma 6.3, $pc_2(G) \leq pc_2(C)$, where $k = 2$. On the other hand, by Fact 6.12, $pc_2(C) \leq 3$. Hence, $pc_2(G) \leq 3$. Clearly, if $|V(G)|$ is even, then $pc_2(G) = 2$.

This completes our proof.

□

Hence, the proper 2-connection number of a Hamiltonian graph is either 2 or 3 by Corollary 6.13. In [48], the authors proved the proper 2-connection number 2 of Hamiltonian with some specific conditions. The following lemma is as a basic tool that is used several times in our next results.

Lemma 6.14 (Huang et al. [48]). *Let $C_n = v_1v_2 \dots v_nv_1$ be a cycle of order n. If $G = C_n + v_{n-1}v_1$, then $pc_2(G) = 2$.*

Since a complete graph of order at least 4 has proper 2-connection number 2, by Theorem 6.11, and Hamiltonian has proper 2-connection number 2 or 3, Huang et al. [48] studied the proper 2-connection number of a connected graph which is Hamiltonian and has fewer edges than a complete graph. In particular, they proved proper 2-connection number 2 of a connected graph having specific condition of minimum degree as follows.

Theorem 6.15 (Huang et al. [48]). *Let $n \geq 4$ be a positive integer, and G be a connected graph of order n and minimum degree $\delta(G)$. If $\delta(G) \geq \frac{n}{2}$, then $pc_2(G) = 2$.*

They showed that $\delta(G) \geq \frac{n}{2}$ is best possible. They denote by $B_{11} = K_1 \vee (2K_k)$ a connected graph, where k is a positive integer. Clearly, $\delta(B_{11}) = k < \frac{|V(G)|}{2}$. Since B_{11} is not 2-connected, there does not exist the proper 2-connection number in B_{11}. Hence, the condition of minimum degree cannot be improved. Next, they considered the following theorem having a weaker condition than the condition of Theorem 6.15.

Theorem 6.16 (Huang et al. [48]). *Let $n \geq 4$ be a positive integer and G be a connected graph of order n. If the degree sum of two any non-adjacent vertices is at least n, then $pc_2(G) = 2$.*

Note that there exists connected graphs, where the degree sum of two any non-adjacent vertices is less than their order that the proper 2-connection number neither exist nor is greater than 2. For example, $pc_2(B_{11})$ does not exist, where $B_{11} = K_1 \vee (2K_k)$ or $pc_2(C_5) = 3$, where C_5 is a cycle of order 5. Hence, the condition of the degree sum of Thereom 6.16 cannot be improved.

By using the concept of colour coding which was first introduced by Chartrand et al. [20], the authors in [8] studied the proper 2-connection number for complete bipartite graphs as follows.

Theorem 6.17 (Borozan et al. [8]). *Let $n \geq 3$ be an integer. If $G = K_{n,3}$ is a complete bipartite graph, then*

$$pc_2(G) = \begin{cases} 2 & \text{if} \quad 3 \leq n \leq 6 \\ 3 & \text{if} \quad 7 \leq n \leq 8 \\ \lceil \sqrt[3]{n} \rceil & \text{if} \quad n \geq 9 \end{cases}$$

Recently, many beautiful results of the proper connection number which are already mentioned above are published. For general positive integer $k \geq 2$, the results of the proper k-connection number are studied in the special classes of graphs, for example complete bipartite graphs with specific conditions or complete graphs. Moreover, there are still many open conjectures and questions for the proper k-connection number of general classes of graphs. In the next sections, we consider the proper 2-connection number for several classes of graphs.

6.2 The bounds of $pc_2(G)$

Clearly, it makes only sense to consider the proper 2-connection number in graphs G of connectivity 2 or larger. By Theorem 3.10, if G is a 2-connected graph, then Borozan et al. [8] proved that the upper bound of proper connection number $pc(G)$ is at most 3. One is different from proper connection number $pc(G)$ of connected, in this section we prove that there does not exist a constant C such that the proper 2-connection number is at most C for all 2-connected graphs. Moreover, when $k = 2$, Fact 6.2 implies that $2 \leq pc_2(G) \leq \Delta(G) + 1$. Now we improve the new upper bound for $pc_2(G)$ of 2-connected graphs and characterize 2-connected graphs achieving equality $pc_(G) = \Delta(G) + 1$. The following two results which is about the property of chordless graphs are very important to prove the upper bound of the proper 2-connection number.

Theorem 6.18 (Dirac [29]). *If G is a minimally spanning 2-connected graph, then G is chordless.*

Theorem 6.19 (Machado et al. [61]). *If G is a chordless graph of maximum degree at least 3, then G is $\Delta(G)$-egde-colourable and $(\Delta(G) + 1)$-total-colourable.*

Now we present our result and are able to prove it.

Theorem 6.20 ([30]). *Let G be a 2-connected graph.*

(i) If G is different from an odd cycle, then $2 \leq pc_2(G) \leq \Delta(G)$.

(ii) G has proper 2-connection number $pc_2(G) = \Delta(G) + 1$ if and only if G is an odd cycle.

Proof. For *(i)*, trivially, by Fact 6.12, $pc_2(G) = \Delta(G) = 2$ if G is an even cycle.

Now let G be a 2-connected graph different from a cycle. Hence, $\Delta(G) \geq 3$. Let H be a minimally spanning 2-connected subgraph of G (meaning that if we remove any edge of H then H is 1-connected). One can readily observe that $\Delta(H) \leq \Delta(G)$ and $pc_2(H) \leq \chi'(H)$. By Lemma 6.3, for $k = 2$ implies $pc_2(G) \leq pc_2(H)$. We immediately deduce that $pc_2(G) \leq \chi'(H)$. If H is a cycle, then $pc_2(G) \leq \chi'(H) \leq 3 \leq \Delta(G)$. If H is not a cycle, then $\Delta(H) \geq 3$. Moreover, since H is the minimally spanning 2-connected graph, by Theorem 6.18, H is a chordless graph. By Theorem 6.19, $\chi'(H) \leq \Delta(H)$. Hence $pc_2(G) \leq \chi'(H) \leq \Delta(H) \leq \Delta(G)$ and we obtain the result.

For *(ii)*, we verify that $pc_2(G) = \Delta(G) + 1$ if and only if G is an odd cycle. If G is an odd cycle then $pc_2(G) = 3$ by Fact 6.12. We suppose, now, G is not an odd cycle. Hence G is an even cycle or a 2-connected graph different from a cycle. By our proof above, $pc_2(G) \leq \Delta(G) < \Delta(G)+1$, a contradiction to $pc_2(G) = \Delta(G)+1$. Hence G is an odd cycle.

This completes our proof. $\square$

By Theorem 6.20, we note that $pc_2(G) \geq 2$ for every general 2-connected graph. The lower bound of the proper 2-connection number can be improved for some special classes of graphs having vertices with degree number 2 as in the following lemma.

Lemma 6.21 ([30]). *Let G be a 2-connected graph and w be a vertex of G. If S is a vertex subset of $V(G)$ such that $S = \{v_i \in V(G) \mid d_G(v_i) = 2, w \in N_G(v_i) \cap N_G(v_j), \forall i \neq j\}$, then $pc_2(G) \geq max\{|S|, 2\}$.*

Proof. Note that, we consider only $|S| > 2$. Otherwise, by Theorem 6.20, $pc_2(G) \geq 2 \geq |S|$. Since G is 2-connected, there always exist the proper 2-connection number in G. Now, we may assume, that G is the proper 2-connected graph with $pc_2(G)$ colours by assigning: $c : E(G) \to [pc_2(G)]$, one colour to each edge. Hence, by the concept above, there always exist two disjoint proper paths between two arbitrary distinct vertices, say v_i and v_j, where $v_i, v_j \in S$. Now we verify that $c(v_iw) \neq c(v_jw)$. Since $d_G(v_i) = d_G(v_j) = 2$ and $w \in N_G(v_i) \cap N_G(v_j)$, we denote by $N_G(v_i) = \{u_i, w\}$ and $N_G(v_j) = \{u_j, w\}$ the neighbour set of v_i and v_j in G, respectively (note that, u_i and u_j can be the same).

Suppose, to the contrary, that $c(v_iw) = c(v_jw)$. Hence, $P = v_iwv_j$ is not a proper path from v_i to v_j in G. Let us denote by $P_1 = v_iu_iP_1^*v_j$ and $P_2 = v_iwP_2^*v_j$ two disjoint proper paths between v_i and v_j. One can be readily observed that $|V(P_2^*)| > 0$. Otherwise, $P_2 = v_iwv_j$ is a proper path from v_i to v_j in G, a contradiction to $P = v_iwv_j$ that is not a proper path in G. Therefore, $u_j \in P_2^*$ since $N_G(v_j) = \{u_j, w\}$. On the other hand, we note that $u_j \notin P_1$ and $w \in P_1$. So P_1 and P_2 are not pairwise internally vertex-disjoint paths, a contradiction. It can be immediately deduced that $c(v_iw) \neq c(v_jw)$. By the pigeon hold principal, $pc_2(G) \geq |S|$.

This completes our proof. $\square$

By using Lemma 6.21, we show that there are infinitely many 2-connected graphs whose proper 2-connection number achieves the upper bound of Theorem 6.20.

Proposition 6.22. *Let $k \geq 2$ be an integer. There exist infinitely many 2-connected graphs G with $\Delta(G) = k$ and $pc_2(G) = k$.*

Proof. Trivially, for $k = 2$, let G be a cycle of even order. Hence, $\Delta(G) = 2$ and by Fact 6.12, $pc_2(G) = 2$. So we obtain the result.

Now, we consider, that $k \geq 3$. Let $C = wv_{k-1}u_1 \dots u_nv_kw$ be an even cycle of order at least 4, where n is an integer such that $n \geq 1$. We denote $P_i = v_i^1v_i^2 \dots v_i^m$ as a path of order $m \geq 3$, where i is an integer such that $i \in [k-2]$. Let us construct graph G from C and $k-2$ paths P_i by identifying v_i^1 and w, v_i^m and u_t, where $w, u_t \in V(C)$. By the construction of G above, one can be easily seen that G is 2-connected different from a cycle with $\Delta(G) = k$. Applying Theorem 6.20 implies $pc_2(G) \leq k$. Next, we denote $S = \{v_i = v_i^2, \forall i \in [k-2]\} \cup \{v_{k-1}, v_k\}$ as a vertex subset of $V(G)$. One can readily observe that $|S| = k$ and $d_G(v_i) = 2$, for all $v_i \in S$. Moreover, $w \in N_G(v_i) \cup N_G(v_j)$, where $\forall v_i, v_j \in S$ and $i \neq j$. By Lemma 6.21, $pc_2(G) \geq k$. Therefore, $pc_2(G) = k = \Delta(G)$. The result is obtained.

This completes our proof. □

For a small $k = 4$, the graph depicted in Figure 6.1 with $pc_2(B_{12}) = 4 = \Delta(B_{12})$ is one of the examples of Proposition 6.22.

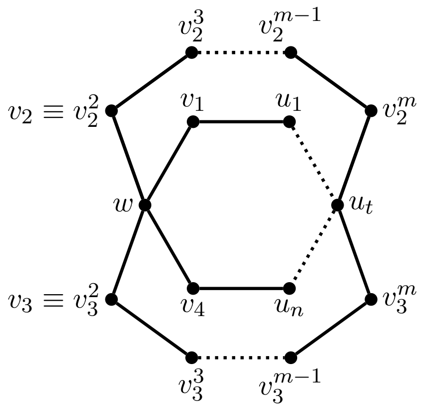

Fig. 6.1: Graph with $pc_2(G_1) = \Delta(G) = 4$

Theorem 6.23 (Whitney [72]). *A graph G is 2-connected if and only if it has an ear decomposition. Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.*

Note that every 2-connected graph G with minimum degree $\delta(G) \geq 2$ is either Hamiltonian or always contains a cycle C with at least $2\delta(G)$ vertices by Dirac's Theorem in [28]. Hence, there always exists at least a cycle in G. Further, among all cycles of G we are able to find a cycle of maximum order. By using Theorem 6.20 and Corollary 6.13, the following result is immediately obtained.

Proposition 6.24. *Let G be a 2-connected graph of order n.*

(i) If G is Hamiltonian, then $2 \leq pc_2(G) \leq 3$.

(ii) If G is not Hamiltonian, and C is the longest cycle of order k in G, i.e. C is of maximum order k, then $2 \leq pc_2(G) \leq n - k + 2$.

Proof. Cleary, we consider only *(ii)*.

Let G be a 2-connected graph different from Hamiltonian. Hence, there is the largest cycle in G, say C of order k. Now, let H be a spanning minimally 2-connected subgraph of G, meaning that the removal of any edge would leave G 1-connected. By Theorem 6.18, H is chordless. It can be readily observed that H is not Hamiltonian and C is the longest cycle of order k in H. Hence, $k \leq n - 1$.

Now we prove that $\Delta(H) \leq n - k + 2$ by the induction on an ear decomposition in H. By Theorem 6.23, let us consider the ear decomposition in H such that C is the initial cycle and $H = C \cup P_1 \cup \ldots P_l$. Since $k \leq n - 1$, it can be readily observed that $l \geq 1$.

The base case of this induction is when $l = 1$. Hence, $\Delta(H) = 3 \leq n - k + 2$, since $k \leq n - 1$

Let P_l be the last ear added to H' and there is at least one internal vertex of P_l that is in $H \setminus H'$. Hence, $n(H') \leq n(H) - 1$ Since H is chordless, $\Delta(H) \leq \Delta(H') + 1$. By our induction on the number of ears, we obtain $\Delta(H') \leq n(H') - k + 2$. After some manipulations, we get $\Delta(H) \leq n - k + 2$. By Theorem 6.20, $pc_2(H) \leq n - k + 2$ since $pc_2(H) \leq \Delta(H)$. Applying Lemma 6.3 implies $pc_2(G) \leq n - k + 2$ since $pc_2(G) \leq pc_2(H)$, where $k = 2$.

This completes our proof. □

Although, the result of Proposition 6.24 is weaker than the result of Theorem 6.20, this proposition is very helpful to determine the relation between the proper 2-connection number and the order of graphs.

Corollary 6.25. *Let $n \geq 3$ be an integer. If G is 2-connected of order n, then*

(i) $pc_2(G) = n$ if and only if $G \cong K_3$.

(ii) There is no 2-connected graph such that $pc_2(G) = n - 1$.

(iii) $pc_2(G) = n - 2$ if and only if $G \in \{K_4, K_4 - e, C_4, C_5, G_5, G_6\}$, where e is an edge of G, see Figure 6.2.

Proof. Let C be the largest cycle of order k in G. Note that if G is Hamiltonian, then $k = n$. Otherwise, G is not Hamiltonian, it can be immediately obtained that $k \geq 4$.

By Proposition 6.24, hence, if G is not Hamiltonian, then $pc_2(G) \leq n - k + 2 \leq n - 2$. Therefore, for *(i)*&*(ii)*, we only consider that G is Hamiltonian.

For *(i)*. One can be readily deduced that G is Hamiltonian since $pc_2(G) = n$. By Corollary 6.13, $2 \leq pc_2(G) \leq 3$. It follows that $n \leq 3$. One can readily observed that $G \cong K_3$, since G is 2-connected of order at most 3. The first case is obtained.

For *(ii)*. Since G is Hamiltonian, by Corollary 6.13, $pc_2(G) = n - 1 \leq 3$. Hence, $n = 3$ or $n = 4$. By simple case to case analysis, we deduce that there is no Hamiltonian G of order 3 or 4 such that $pc_2(G) = n - 1$. The second case is obtained.

For *(iii)*. If $G \in \{K_4, K_4 - e, C_4\}$, then $pc_2(G) = 2 = n - 2$. If $G \in \{G_5, G_6\}$, then applying Theorem 6.20 implies that $pc_2(G) \leq \Delta(G_i) = n - 2$, where $i \in \{5, 6\}$. Let $S = V(G) \setminus \{v_2, v_4\}$ be a vertices subset of G. Clearly, S fulfills conditions of Lemma 6.21 and $|S| = n - 2$. Hence, $pc_2(G) \geq n - 2$. Therefore, $pc_2(G) = n - 2$.

Now we verify that there is no 2-connected graph G, where $G \notin \{K_4, K_4 - e, C_4, C_5, G_5, G_6\}$, such that $pc_2(G) = n - 2$. If G is Hamiltonian, then $n \in \{4, 5\}$ since $2 \leq pc_2(G) = n - 2 \leq 3$ by Corollary 6.13. Clearly, $n = 4$, one can be readily observed that $G \in \{K_4, K_4 - e, C_4\}$, a contradiction. Hence, we consider that $n = 5$ and $G \neq C_5$. Note that G is Hamiltonian. Hence, there exists a spanning 2-connected subgraph, say $H = C_5 + v_i v_{i+2}$, where $C_5 = v_1 \ldots v_5 v_1$ and $v_i \in V(C)$, in G. By Lemma 6.3 and Lemma 6.14, $pc_2(G) \leq pc_2(H) = 2$, a contradiction to $pc_2(G) = n - 2 = 3$.

Now, G is not Hamiltonian and $n \geq 5$. Hence, by Corollary 6.25, $pc_2(G) = n - 2 \leq n - k + 2$. It follows that $k = 4$. Let us denote by $C = v_1 v_2 v_3 v_4 v_1$ the largest cycle

Fig. 6.2: $pc_2(G_i) = n - 2$, where $i \in \{5, 6\}$

of G. Hence, there is at least one vertex $v \in G \setminus C$ since G is not Hamiltonian and C is largest cycle in G. By Corollary 3.20, there are two internally vertex-disjoint paths between v and C, say P_1 and P_2. Renaming vertices if necessary, one can be readily seen that u_1 and u_3 are two end-vertices of P_1 and P_2 in C. Otherwise, there is a larger cycle than C, a contradiction. Moreover, we note that path $P = v_1P_1vP_2v_3$ has exactly three vertices since C is the largest cycle in G. By simple case to case analysis, we deduce that $G \in \{G_5, G_6\}$, a contradiction.

This completes our proof.

□

6.3 The proper 2-connection number 2 of several graphs

By Proposition 6.22, there always exist 2-connected graphs having equal proper 2-connection number $pc_2(G)$ and maximum degree $\Delta(G)$, but the difference $\Delta(G) - pc_2(G)$ can be arbitrarily large. Recently, there are some results of proper 2-connection number 2 for the several graphs proved by Borozan et al. [8] and Huang et al. [48], see results in Section 6.1. In this section, we consider some classes of graphs having $pc_2(G) = 2$. Before starting our results in this section, recall the bounds of the proper 2-connection number of Hamiltonian graphs here.

Corollary 6.13 *If G is Hamiltonian, then $2 \leq pc_2(G) \leq 3$. Moreover, $pc_2(G) = 2$ if $|V(G)|$ is even.*

Hamiltonian graph form one of the most interesting problems of graph theory and has many applications in graph theory and in the real world. Many Researchers and Mathematicians in graph theory still try to find necessary and sufficient conditions of an arbitrary graph being Hamiltonian. Nowadays, there exist some results of the proper k-connection number, where $k \geq 1$ by using the properties of Hamiltonian graphs. For example, $pc(G) = 2$ if G is non-complete traceable of order at least 3 by Andrews et al. [4], or $pc_2(G) = 2$ if G is Hamiltonian with the special conditions by Huang et al. [48], see Theorem 6.15 & Theorem 6.16. It is the starting point for us to consider the Chvátal and Erdős condtion ($\alpha(G) \leq \kappa(G)$ with two exceptions) which is one of the fundamental conditions for Hamiltonian graphs and prove that the proper 2-connection number is 2. Furthermore, we show that our result is sharp. Before starting to prove our result, we mention two well-known results which are very important in our proof.

Theorem 6.26 (Chvátal and Erdős [25])**.** *Let G be a connected graph with $|V(G)| \geq 3$. If $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.*

Theorem 6.27 (Amar et al. [3])**.** *Let G be a simple, k-connected graph of order n with $\alpha(G) \leq \kappa(G)$. If G is different from $K_{k,k}$ and C_5, then G has a C_{n-1}.*

Our result is as follows.

Theorem 6.28 ([30])**.** *Let G be a connected graph with $|V(G)| \geq 3$ which is different from C_3 and C_5. If $\alpha(G) \leq \kappa(G)$, then $pc_2(G) = 2$.*

Now we are able to prove Theorem 6.28 by using Theorem 6.27 & Theorem 6.26 as basic tools.

Proof. By Theorem 6.26, G is Hamiltonian since G is a connected graph with $|V(G)| \geq 3$ and $\alpha(G) \leq \kappa(G)$. We denote, now, $C_n = v_1v_2 \dots v_nv_1$, where $n = |V(G)|$, as a Hamiltonian cycle of G. Applying Corollary 6.13 implies that $2 \leq pc_2(G) \leq 3$ if $|V(G)|$ is odd, and $pc_2(G) = 2$ if $|V(G)|$ is even. Hence, we only consider that $|V(G)|$ is odd. It can be readily deduced that $|V(G)| \geq 5$ since $G \neq C_3$ and $G \neq C_5$. If $|V(G)| = 5$ and $G \neq C_5$, then G has a spanning 2-connected subgraph H, where $H = C_n + v_{i-1}v_{i+1}$. By Lemma 6.14, $pc_2(H) = 2$. By Lemma 6.3, $pc_2(G) \leq pc_2(H) = 2$. Hence, $pc_2(G) = 2$. Thus, $|V(G)| \geq 7$. In order to complete our proof, we follow the series of claims by constructing H as a spanning 2-connected subgraph of G such that $pc_2(H) = 2$.

Claim 6.28.1. *If $\alpha(G) = 2$, then there exists a spanning 2-connected subgraph H of G such that $pc_2(H) = 2$.*

Proof. Suppose, to the contrary, that there does not exist any spanning 2-connected subgraph H of G such that $pc_2(H) = 2$. If either $v_{n-1}v_1$ or $v_{n-1}v_{n-2}$ is an edge of $E(G)$, then we denote H as a spanning 2-connected subgraph of G such that $H = C_n + v_{n-1}v_1$ or $H = C_n + v_{n-1}v_{n-3}$. By Lemma 6.14, $pc_2(H) = 2$, a contradiction. Hence, neither $v_{n-1}v_1$ nor $v_{n-1}v_{n-3}$ does belong to $E(G)$, i.e. $v_{n-1}v_1, v_{n-1}v_{n-3} \notin E(G)$. Since $\alpha(G) = 2$, it can be readily seen that $v_{n-1}v_1 \in E(G)$. Otherwise, $S = \{v_{n-1}, v_{n-3}, v_1\}$ is an independent set of G, a contradiction to $\alpha(G) = 2$. In the same way, one can easily observe that both $v_{n-1}v_3$ and $v_{n-2}v_2$ are edges of G.

We construct, now, $C_{n-1} = v_{n-1}v_3 \dots v_{n-3}v_1v_2v_{n-2}v_{n-1}$ as an even cycle of order $n-1$. Let us denote $H = C_{n-1} + v_nv_{n-1} + v_nv_1 + v_2v_3 + v_{n-2}v_{n-3}$. One can readily observe that H is a spanning 2-connected subgraph of G. We colour all the edges of C_{n-1} alternatingly with two colours from [2]. Thus, every two vertices $x, y \in V(C_{n-1})$ are connected by at least two disjoint proper paths. Renaming colours if necessary, we may assume that $c(v_3v_{n-1}) = 1$. It implies that $c(v_1v_{n-3}) = 1$. Next, we colour all the remaining edges of H by colour 2. One can easily observe that there are two disjoint proper paths between v_n and v_i, where $v_i \in V(C_{n-1}) \setminus \{v_2, v_{n-2}\}$. From v_n to v_{n-2} or from v_n to v_2 we choose two disjoint proper paths as follows: $P_1 = v_nv_1v_{n-3}v_{n-2}$ and $P_2 = v_nv_{n-1}v_3v_2v_{n-2}$, or $P_1 = v_nv_1v_{n-3}v_{n-2}v_2$ and $P_2 = v_nv_{n-1}v_3v_2$, respectively.

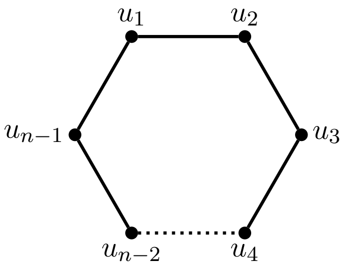

Fig. 6.3: Cycle C_{n-1}.

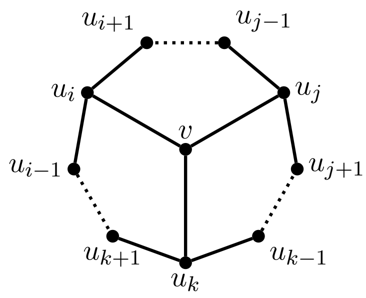

Fig. 6.4: $\deg_G(v) = 3$

Hence, H is a proper 2-connected graph with two colours from [2]. It follows that $pc_2(H) = 2$, a contradiction.

This claim is obtained. $(\square)$

Since $|V(G)|$ is odd, it can be readily seen that $G \neq K_{k,k}$. Moreover, $G \neq C_5$. Applying Theorem 6.27 implies that G has a C_{n-1} being an even cycle of order at least 6. We denote by $C_{n-1} = u_1u_2 \ldots u_{n-1}u_1$ a cycle of order $n-1$ such that direction of movement from u_t to u_{t+1} is clockwise and $v \in V(G) \setminus V(C_{n-1})$. The direction of C_{n-1} by labeling its vertices is depicted in Figure 6.3. Note that taking indices module $n-1$ thus $u_n \equiv u_1$ and $u_0 \equiv u_{n-1}$ in some special cases. By Claim 6.28.1, we have $d_G(v) \geq \delta(G) \geq \kappa(G) \geq \alpha(G) \geq 3$. Let us denote $u_i, u_j \in N_G(v)$ such that $1 \leq i < j < t \leq n-1$, where $u_t \in N_G(v) \setminus \{u_i, u_j\}$ and $P_{ij} = u_i\overrightarrow{C_{n-1}}u_j$ be a path between u_i and u_j along C_{n-1} by clockwise direction. We write $u_i\overrightarrow{C_{n-1}}u_j$ $(u_i\overleftarrow{C_{n-1}}u_j)$ for the path from u_i to u_j along C_{n-1} in clockwise (in counterclockwise) direction. We define $S_{ij} = V(P_{ij})$ as a vertex subset of $V(C_{n-1})$ and $d_{ij} = |E(P_{ij})|$ as a number of edges of P_{ij}. Hence, $u_t \notin S_{ij}$ if $u_t \in N_G(v) \setminus \{u_i, u_j\}$.

First of all, we colour all the edges of C_{n-1} alternatingly with two colours from [2] and some edges as follows: $c(vu_i) = c(u_iu_{i+1})$, $c(vu_j) = c(u_ju_{j-1})$. Hence, there always exist two disjoint proper paths between x, y, where $x, y \in V(C_{n-1})$, and between v, u_t, where $u_t \in V(C_{n-1}) \setminus S_{ij}$ or $u_t \in \{u_i, u_j\}$. Clearly, we only consider that $d_{ij} \geq 2$. Next, we prove that there always exist two disjoint proper paths in G between v, u_t such that $u_t \in S_{ij} \setminus \{u_i, u_j\}$ by the following claims.

Claim 6.28.2. *If $\deg_G(v) \geq 4$, then there are two disjoint proper paths in G between v and u_t, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$.*

Proof. Without lost of generality, we may assume that $u_i, u_j, u_k, u_l \in N_G(v)$ such that $1 \leq i < j < k < l \leq n-1$. Let us colour $c(vu_k) = c(u_ku_{k+1})$ and $c(vu_l) = c(u_lu_{l-1})$. From v to u_t, where $u_t \in S_{ij}$, we take two disjoint proper paths as follows: $P_1 = vu_k\overleftarrow{C_{n-1}}u_j\overleftarrow{C_{n-1}}u_t$ and $P_2 = vu_l\overrightarrow{C_{n-1}}u_i\overrightarrow{C_{n-1}}u_t$. Hence, there always exist two disjoint proper paths between v and u_t.

This claim is obtained. $(\square)$

Applying Claim 6.28.2 implies that if $\alpha(G) \geq 4$, or $\alpha(G) = 3$ and $\delta(G) \geq 4$, then there are two disjoint proper paths in G between any two vertices of G. Thererfore, we consider the last claim such that $\alpha(G) = \deg_G(v) = 3$.

Claim 6.28.3. *If $\alpha(G) = \deg_G(v) = 3$, then there are two disjoint proper paths between v and u_t in G, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$.*

Proof. Since $\deg_G(v) = 3$, let us denote $N_G(v) = \{u_i, u_j, u_k\}$ such that $1 \leq i < j < k \leq n-1$. As in the above definition, we define d_{jk} and d_{ki} as number of edges between v_j, v_k and between v_k, v_i, respectively. Clearly, we only consider that $1 < d_{jk}, d_{ki} < n-1$. An example of $C_{n-1} \cup v$ is depicted in Figure 6.4.

Suppose, to the contrary that, there do not exist two disjoint proper paths between v and u_t in G, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$ if we use two colours from [2]. Thus, $u_{x-1}u_{x+1} \notin E(G)$ such that $x \in \{i, j, k\}$. Otherwise, if $u_{x-1}u_{x+1} \in E(G)$ such that $x \in \{i, j\}$, then, without lost of generality, we may assume that $u_{i-1}u_{i+1} \in E(G)$. Hence, we colour $c(u_{i-1}u_{i+1}) = c(vu_i)$ and $c(vu_k) = c(u_ku_{k+1})$. One can readily see that $P_1 = vu_iu_{i-1}u_{i+1}\overrightarrow{C_{n-1}}u_t$ and $P_2 = vu_k\overleftarrow{C_{n-1}}u_j\overleftarrow{C_{n-1}}u_t$ are two disjoint proper paths between v and u_t, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$, a contradiction. Note that if $u_{k-1}u_{k+1} \in E(G)$, then we change labels between u_i, u_j, u_k, a contradiction. To complete Claim 6.28.3, we consider the follwing two cases depending on the parity of d_{ij}, d_{jk}, d_{ki}. Without lost of generality, we may assume that $c(u_iu_{i+1}) = 2$.

Case 6.28.3.1. *If d_{ij}, d_{jk}, d_{ki} are even numbers, then there exist two disjoint proper paths between v and u_t, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$.*

Proof. Since $c(u_iu_{i+1}) = 2$ and d_{ij}, d_{jk}, d_{ki} are even numbers, $c(u_ju_{j+1}) = 2$ and $c(u_ku_{k+1}) = 2$. Suppose, to the contrary that, there do not exist two disjoint proper paths between v and u_t in G, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$ if we use two colours from [2]. Hence, $u_{j-1}u_{k-1} \notin E(G)$. Otherwise, we colour $c(vu_k) = c(u_{j-1}u_{k-1}) = 1$. From v to u_t, we can choose two disjoint proper paths as follows: $P_1 = vu_ku_{k+1}\overrightarrow{C_{n-1}}u_i\overrightarrow{C_{n-1}}u_t$ and $P_2 = vu_ju_{j+1}\overrightarrow{C_{n-1}}u_{k-1}u_{j-1}\overleftarrow{C_{n-1}}u_t$, a contradiction. By the symmetry of C_{n-1}, one can easily deduce that $u_{i+1}u_{k+1} \notin E(G)$.

Next we consider only, that $d_{ij}, d_{jk}, d_{ki} \geq 4$. Otherwise, without lost of generality, we may assume that $d_{ij} = 2$. Hence, $u_{i+1} \equiv u_{j-1}$. Since neither $u_{i+1}u_{k+1} \notin E(G)$ nor $u_{j-1}u_{k-1} \notin E(G)$, it follows that $S = \{v, u_{i+1}, u_{k-1}, u_{k+1}\}$ is an independent set, a contradiction to $\alpha(G) = 3$.

Now $u_{i+1}u_{k-1} \in E(G)$ and $u_{j-1}u_{k+1} \in E(G)$ since $S = \{u_x, v, u_{k-1}, u_{k+1}\}$ is an independent set of G with $|S| > \alpha(G)$, where $u_x \equiv u_{i+1}$ or $u_x \equiv u_{j-1}$, respectively. Morveover, $u_{j+1}u_{k-1} \in E(G)$ since $S = \{u_{j-1}, u_{j+1}, v, u_{k-1}\}$ is an independent set with $|S| > \alpha(G)$. We colour $c(u_{i+1}u_{k-1}) = 2$ and all three edges $vu_k, u_{j-1}u_{k+1}, u_{j+1}u_{k-1}$ by colour 1. Hence, $P_1 = vu_ku_{k+1}u_{j-1}\overleftarrow{C_{n-1}}u_t$ and $P_2 = vu_ju_{j+1}u_{k-1}u_{i+1}\overrightarrow{C_{n-1}}u_t$ are two disjoint proper paths between v and u_t, a contradiction.

This case is proved. (□)

Case 6.28.3.2. *If exactly two integers among d_{ij}, d_{jk}, d_{ki} are odd, then there exist two disjoint proper paths between v and u_t, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$.*

Proof. Without loss of generality, we may assume that d_{ki}, d_{ij} are odd and d_{jk} is

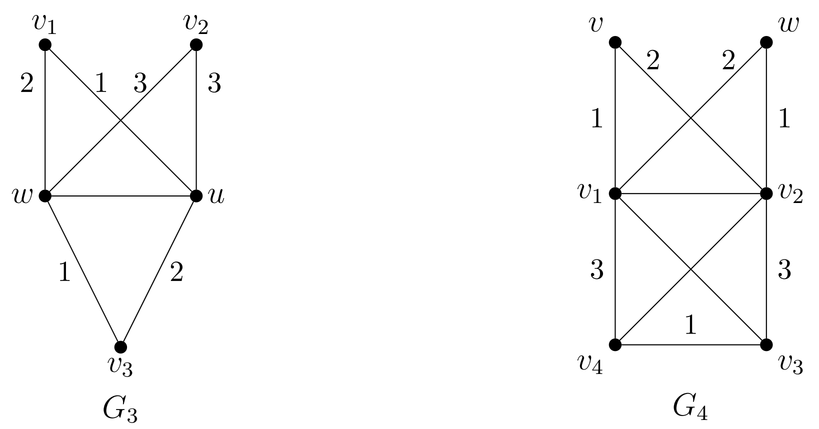

Fig. 6.5: $pc_2(G_i) = 3, i \in \{3, 4\}$

even. Note that $d_{ki}, d_{ij} \geq 3$ and $d_{jk} \geq 4$. Suppose, to the contrary that, there do not exist two disjoint proper paths between v and u_t in G, where $u_t \in S_{ij} \setminus \{u_i, u_j\}$ if we use two colours from [2]. Now, we colour $c(vu_k) = 2$. Hence, $u_{j-1}u_{k-1} \notin E(G)$. Otherwise, we colour $c(u_{j-1}u_{k-1}) = 2$. Thus, $P_1 = vu_ju_{j+1}\overrightarrow{C_{n-1}}u_{k-1}u_{j-1}\overleftarrow{C_{n-1}}u_t$ and $P_2 = vu_ku_{k+1}\overrightarrow{C_{n-1}}u_t$ are two disjoint proper paths between v, u_t, a contradiction. Therefore, $u_{j-1}u_{k+1} \in E(G)$ since $S = \{v, u_{j-1}, u_{k-1}, u_{k+1}\}$ is an independent set, a contradiction to $\alpha(G) = 3$.

If $u_{i+1}u_{k-1} \in E(G)$, then we colour $c(u_{j-1}u_{k+1}) = c(u_{i+1}u_{k-1}) = 2$. Hence, $P_1 = vu_ku_{k+1}u_{j-1}\overleftarrow{C_{n-1}}u_t$ and $P_2 = vu_ju_{j+1}\overrightarrow{C_{n-1}}u_{k-1}u_{i+1}\overleftarrow{C_{n-1}}u_t$ are two disjoint proper paths between v, u_t, a contradiction. Therefore, $u_{i+1}u_{k+1} \in E(G)$. We colour $c(u_{i+1}u_{k+1}) = 2$. We choose $P_1 = vu_ku_{k+1}u_{i+1}\overrightarrow{C_{n-1}}u_t$ as the first proper path between v and u_t.

Hence, either $u_{i-1}u_{j-1}$ or $u_{i-1}u_{j+1}$ is an edge of G since $S = \{v, u_{i-1}, u_{j-1}, u_{j+1}\}$ is an independent set of G, a contradiction to $\alpha(G) = 3$. If $u_{i-1}u_{j-1} \in E(G)$, then $c(u_{i-1}u_{j-1}) = 2$. Hence, it can be readily observed that $P_2 = vu_iu_{i-1}u_{j-1}\overleftarrow{C_{n-1}}u_t$ is the second proper path between v, u_t, a contradiction. If $u_{i-1}u_{j+1} \in E(G)$, then $c(u_{i-1}u_{j+1}) = 2$. Hence, $P_2 = vu_iu_{i-1}u_{j-1}u_j\overleftarrow{C_{n-1}}u_t$ is the second proper path between v, u_t, a contradiction.

This case is proved. (□)

Now Claim 6.28.3 is obtained. (□)

This completes our proof. □

The sharpness examples for Theorem 6.28 are given by the graphs G_i, where $i \in \{2, 3, 4\}$ as follows.

Let G_2 be a connected graph of order $n \geq 3$ consisting of a complete graph K_{n-1} and another vertex v joined to one vertex of K_{n-1}. It can be readily seen that $\alpha(G_2) = 2$ and $\kappa(G_2) = 1$. Clearly, $pc_2(G_2)$ does not exist since G_2 is not a 2-connected graph.

Let $G_3 = 3K_1 \vee K_2$ be the graph depicted in Figure 6.5. There does exist $pc_2(G_3)$ since $\kappa(G_3) = 2$. We denote $S = \{v_1, v_2, v_3\}$ as the unique maximum independent set of G_3

with $\alpha(G_3) = 3$. Moreover, $\deg_{G_3}(v_i) = 2$ for any $v_i \in S$ and $w \in N_G(v_i) \cap N_G(v_j)$ for any two vertices $v_i, v_j \in S$. By Lemma 6.21 , $pc_2(G_3) \geq 3$. On the other hand, we use three colours from [3] to colour all the edges of G_3 in order to make it proper 2-connected graph. Hence, $pc_2(G_3) = 3$.

Let $C_4 = u_1vwu_2u_1$ be a cycle and $K_4 = v_1v_2v_3v_4$ be a complete graph of order 4. We construct G_4 from C_4 and K_4 by identifying u_i and v_i, where $i \in [2]$. It can be easily seen that $pc_2(G_4)$ exists since $\kappa(G_4) = 2$. Moreover, $\alpha(G_4) = 3$. We use three colours to make G_3 a proper 2-connected graph (cf. Figure 6.5). Hence, $2 \leq pc_2(G_4) \leq 3$. Next, we verify that $pc_2(G_4) = 3$. Suppose, to the contrary, that $pc_2(G_4) = 2$. One can readily see that $\deg_{G_4}(v) = \deg_{G_4}(w) = 2$ and $N_{G_4}(v) \cap N_{G_4}(w) = \{v_1, v_2\}$. In the similar way of the proof of Lemma 6.21 implies that $c(vv_1) \neq c(wv_1)$ and $c(vv_2) \neq c(wv_2)$. If $c(vv_1) = 1$, then $c(wv_1) = 2$. By symmetry of G_4, renaming colours if necessary, we may assume that $c(v_4v_1) = 1$. We immediately deduce that $P = v_4v_3v_1v$ has to be one of two disjoint proper paths between v_4 and v. It can be easily seen that $c(v_4v_3) = 1$ and $c(v_3v_1) = 2$ since $c(vv_1) = 1$. Hence, there do not exist two disjoint proper paths connecting w and v_3, a contradiction to the definition of a proper 2-connected graph. Therefore, $pc_2(G_4) = 3$.

The authors in [8] proved $pc_k(K_n) = 2$, where K_n is a complete graph. Next we show that $pc_2(G) = 2$ of 2-connected graphs having $\omega(G) \in \{n, n-1, n-2\}$, where $n = |V(G)|$ by the following result.

Theorem 6.29. *Let G be a 2-connected graph of order n and $\omega(G)$ be the cardinality of a largest clique in G. If:*

(i) $\omega(G) = n$ and $n \geq 4$, or

(ii) $\omega(G) = n - 1$, or

(iii) $\omega(G) = n - 2$ and $G \notin \{G_2, G_3\}$,

then $pc_2(G) = 2$.

Proof. Trivially, we verify *(i)*. If $\omega(G) = n$ and $n \geq 4$, then by Theorem 6.6, $pc_2(G) = 2$, where $k = 2$. Note that $n \geq 4$ is sharp since $pc_2(K_3) = 3$ or $pc_2(K_2)$ does not exist.

To verify *(ii)*. It can be easily seen that $n \geq 4$ since G is a 2-connected graph and $\omega(G) = n-1$. We take two vertices $v_1, v_{n-1} \in N_G(v_n) \cap V(K_{n-1})$, where $v_n \notin V(K_{n-1})$. We choose $C_n = v_1v_nv_{n-1}...v_1$ as a Hamiltonian cycle of G. Now, let us denote by $H = C_n + v_1v_{n-1}$ a spanning 2-connected graph of G. Applying Lemma 6.14 implies that $pc_2(H) = 2$. The result holds.

Finally, we verify *(iii)*. If $n = 4$, then $G \cong C_4$ and $pc_2(C_4) = 2$. Hence we may assume that $n \geq 5$. Suppose, to the contrary that, there does not exist any 2-connected graph G with $\omega(G) = n - 2$ such that $pc_2(G) = 2$. Let G be a 2-connected graph with $\omega(G) = n-2$ and G^* be a maximum complete subgraph of G. Hence $|V(G^*)| = n-2$ and $|V(G^*)| \geq 3$. We denote $v_1, v_2 \in V(G) \setminus V(G^*)$. Since G is the 2-connected graph with $\omega(G) = n - 2$, let $C_{n-2} = v_3v_4v_5..v_nv_3$ be a cycle of G^* of order $n - 2$ such that $v_3 \in N_G(v_1)$, $v_4 \in N_G(v_2)$. If $v_1v_2 \in E(G)$ or $|N_{G^*}(v_1) \cap N_{G^*}(v_2)| \leq 1$ or $|N_{G^*}(v_1) \cap N_{G^*}(v_2)| = 2$ and $N_{G^*}(v_1) \neq N_{G^*}(v_2)$ or $|N_{G^*}(v_1) \cap N_{G^*}(v_2)| \geq 3$, then there always exists a Hamiltonian cycle C_n of G. Hence, $H = C_n + v_3v_{n-1}$ is a spanning

2-connected graph of G. By Lemma 6.3 and Lemma 6.14, $pc_2(G) = 2$, where $k = 2$, a contradiction.

Now we consider that $N_{G^*}(v_1) = N_{G^*}(v_2) = \{v_3, v_4\}$ and $v_1v_2 \notin E(G)$. Furthermore, $|V(G^*)| \geq 5$. Let $C_{n-1} = v_1v_4v_5..v_nv_3v_1$ be a cycle of order $n-1$. To complete our proof we consider the following two cases depending on the parity of $|V(G)|$.

Case 1. If $|V(G)|$ is odd, then C_{n-1} is an even cycle. Hence $H = C_{n-1} + v_2v_3 + v_2v_4 + v_3v_5 + v_4v_n$ is a spanning 2-connected subgraph of G. We colour all the edges of C_{n-1} alternatingly with two colours from [2]. Without lost of generality, we may assume that $c(v_1v_3) = 1$ and $c(v_1v_4) = 2$. Now we colour $c(v_2v_3) = c(v_4v_n) = 2$ and $c(v_2v_4) = c(v_3v_5) = 1$. We show that there are two disjoint proper paths between v_s and v_t, where $v_s, v_t \in V(G)$. We only consider that $v_s = v_2$. If $v_t \in \{v_1, v_3, v_4\}$, then $C' = v_2v_3v_1v_4v_2$ is a 4-cycle with alternating colours. If $v_t \notin \{v_1, v_3, v_4\}$, then $P_1 = v_2v_3v_5...v_t$ and $P_2 = v_2v_4v_n...v_t$ are two disjoint proper paths connecting them. Hence, $pc_2(H) = 2$, a contradiction.

Case 2. If $|V(G)|$ is even, then $|V(G)| \geq 8$ since $G \neq G_3$. We denote $H = C_{n-1} + v_5v_7 + v_3v_5 + v_4v_n + v_2v_3 + v_2v_4$ as a spanning 2-connected subgraph of G. It can be readily seen that $C_{n-2} = C_{n-1} \setminus \{v_6\} + v_5v_7$ is an even cycle. We colour all the edges of C_{n-2} alternatingly with two colours from [2]. Without lost of generality, we may assume that $c(v_5v_7) = 1$. Hence we colour $c(v_5v_6) = c(v_6v_7) = 1$. For any two vertices $v_s, v_t \in V(G) \setminus \{v_2\}$, there are two disjoint proper paths connecting them. Next, we colour $c(v_3v_5) = c(v_2v_4) = 2$, $c(v_4v_n) = c(v_2v_3) = 1$. Hence, $C' = v_2v_4v_1v_3v_2$ is a 4-cycle with alternating colour. If $v_t \in V(H) \setminus \{v_1, v_3, v_4\}$, then $P_1 = v_2v_4v_n..v_t$ is the first proper path between v_2 and v_t. We choose $P_2 = v_2v_3v_5v_6$ if $v_t = v_6$. Otherwise, $P_2 = v_2v_3v_5v_7...v_t$. There are always two disjoint proper paths between v_s, v_t, where $v_s, v_t \in V(H)$. Hence, $pc_2(H) = 2$, a contradiction.

This completes our proof. □

Now we show that the condition of the largest clique number $\omega(G) \in \{n, n-1, n-2\}$ of a 2-connected graph G of order n having $pc_2(G) = 2$ is sharp by the following proposition.

Proposition 6.30. *Let $k \geq 3$ and $n \geq k+3$ be integers. Then there exists a 2-connected graph G of order n with $\omega(G) = n - k$ having $pc_2(G) \geq k$.*

Proof. Since $k \geq 3$, let $P_i = v_i^1 v_i v_i^2$ be $i-$th path of order 3, where $i \in [k]$. We denote K_{n-k} as a complete graph with $V(K_{n-k}) = \{u_1, u_2, ..., u_{n-k}\}$, where $n - k \geq 3$. Construct G by identifying v_i^1 with u_1 and v_i^2 with u_2, where $u_1, u_2 \in V(K_{n-k})$ and $i \in [k]$. Let $S = \{v_1, v_2, ..., v_k\}$ be a vertex subset of $V(G)$. Hence, $|S| = k$ and $\deg_G(v_i) = 2$, where $v_i \in S$. Moreover, $u_1 \in N_G(v_i) \cap N_G(v_j)$, where $v_i, v_j \in S$. By Lemma 6.21, $pc_2(G) \geq k \geq 3$.

This completes our proof. □

6.4 The proper 2-connection number of Cartesian products

Cartesian products are common in Graph Theory. Recently, many interesting problems of Graph Theory are published on classes of these products. Take, for example, Hamiltonian, connectivity, rainbow connection, etc. On the other hand, these products have numerous applications in Biology, Computer Science and Mathematical Chemistry. Let G and H be two simple graphs of vertex set $V(G) = \{u_1 \dots u_s\}$ and $V(H) = \{v_1 \dots v_t\}$, respectively. We denote by $[u_i, v_k]$ a vertex of Cartesian product, say $G\Box H$, of two graphs G and H. Further, we call $P^i_{k,l}$ as a path from $[u_i, v_k]$ to $[u_j, v_l]$, and $P^{i,j}_k$ as a path from $[u_i, v_k]$ to $[u_j, v_k]$ of $G\Box H$. Note that, if $P = P^i_{k,l}P^{i,j}_l$, then $P = [u_i, v_k] \dots [u_i, v_l] \dots [u_j, v_l]$.

The result of the proper connection number $pc(G\Box H)$ of two nontrivial connected graphs G, H is proved by Andrews et al. [4], see Theorem 3.35. Recall its statement.

Theorem 3.35 *If G, H are nontrivial connected graphs, then $pc(G\Box H) = 2$.*

The connectivity of the Cartesian product of two graphs were studied by Chiue et al. [24].

Lemma 6.31 (Chiue et al. [24]). *If G, H are simple graphs, then $\kappa(G\Box H) \geq \kappa(G) + \kappa(H)$.*

In this section, we study the proper 2-connection number of the Cartesian product of two nontrivial connected graphs G, H denoted as $G\Box H$. Applying Lemma 6.31 implies that $G\Box H$ is a 2-connected graph. Moreover, it can be readily observed that $\Delta(G\Box H) = \Delta(G) + \Delta(H)$. By Theorem 6.20, $2 \leq pc_2(G\Box H) \leq \Delta(G) + \Delta(H)$. The following result shows that there is no analogue of Theorem 3.35 for the proper 2-connection number, i.e. there does not exist a constant C such that $pc_2(G\Box H) = C$ for any Cartesian product of two arbitrary nontrivial connected graphs G, H. First of all, we prove that $pc_2(G\Box H) = 2$ for two nontrivial traceable graphs G, H.

Theorem 6.32 ([30]). *If G, H are two nontrivial traceable graphs, then $pc_2(G\Box H) = 2$.*

Proof. Suppose, to the contrary that, $pc_2(G\Box H) > 2$. Let us denote P_m, P_n as two Hamitolnian paths of order at least 2 of G, H, respectively, since G, H are two nontrivial traceable graphs. One can readily observe that $P_m\Box P_n$ is a spanning 2-connected subgraph of $G\Box H$. By Lemma 6.3, $pc_2(G\Box H) \leq pc_2(P_m\Box P_n)$, where $k = 2$. We consider only that both m, n are odd. Otherwise, without lost of generality, we may assume that m is even. Hence $P_m\Box P_n$ is a Hamiltonian of even order with cycle C_{mn} as follows:

$$C_{mn} = P^{1,2}_1 P^2_{1,n-1} P^{2,3}_{n-1} P^3_{n-1,1} P^{3,4}_1 ... P^{m-1,m}_1 P^m_{1,n} P^{m,1}_n P^1_{n,1}$$

For example, Hamiltonian cycle of $P_6\Box P_5$ depicted in Figure 6.6 is an example of C_{mn}. By Corollary 6.13, $pc_2(P_m\Box P_n) = 2$, a contradiction.

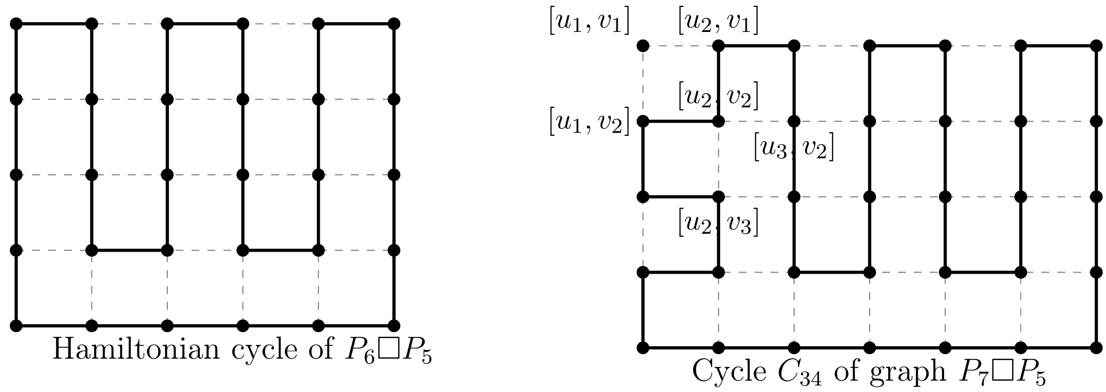

Fig. 6.6: Even cycle of $P_m \Box P_n$

Now let us denote $k = mn - 1$. Hence k is even since m, n are odd. Take C_k as a cycle as follows:

$$C_k = P_n^{1,m} P_{n,1}^{m} P_1^{m,m-1} P_{1,n-1}^{m-1} P_{n-1}^{m-1,m-2} P_{n-1,1}^{m-2} \ldots P_1^{3,2} P_{1,2}^{2} P_2^{2,1} P_{2,3}^{1} \ldots P_{n-1}^{2,1} P_{n-1,n}^{1}.$$

The cycle C_{34} of $P_7 \Box P_5$ depicted in Figure 6.6 is an example of C_k. We colour all the edges of C_k alternatingly with two colours from [2]. Hence, any two vertices of C_k are connected by two disjoint proper paths. Let us denote $H^* = C_k + [u_1, v_1][u_2, v_1] + [u_1, v_1][u_1, v_2] + [u_2, v_2][u_2, v_3] + [u_2, v_2][u_3, v_2]$. Hence, H^* is a spanning 2-connected subgraph of $G \Box H$. By Lemma 6.3, $pc_2(G \Box H) \leq pc_2(H^*)$, where $k = 2$. Without lost of generality, we may assume that $c((u_2, v_2)(u_2, v_1)) = 1$ and $c((u_2, v_2)(u_1, v_2)) = 2$. Colour

$$c([u_1, v_1][u_2, v_1]) = c([u_2, v_2][u_2, v_3]) = 1$$

and

$$c([u_2, v_2][u_3, v_2]) = c([u_1, v_1][u_1, v_2]) = 2$$

Hence, there are two disjoint proper paths between $[u_1, v_1]$ and $[u_i, v_l]$ such that $[u_i, v_l] \in V(C_k) \setminus \{[u_2, v_2]\}$. From $[u_1, v_1]$ to $[u_2, v_2]$, let us take $P_1 = P_1^{1,3} P_{1,2}^{3} P_2^{3,2}$ and $P_2 = P_{1,3}^{1} P_3^{1,2} P_{3,2}^{2}$ as two disjoint proper paths connecting them. Hence, H^* is a proper 2-connected graph with two colours from [2]. It follows that $pc_2(H^*) = 2$, a contradiction.

This completes our proof.

$\square$

The following proposition shows that $pc_2(G \Box H)$ can be arbitrarily large if one of the two graphs G, H is no longer a path.

Proposition 6.33. *Let m, n be two integers such that $m \geq 3$.*

(i) If $K_{1,m}$ is a star and P_n is a path such that $n \geq 2$, then $pc_2(K_{1,m} \Box P_n) = m$.

(ii) If $K_{1,m}, K_{1,n}$ are two stars such that $n \geq 3$, then $pc_2(K_{1,m} \Box K_{1,n}) = \max\{m, n\}$.

For our proof of Proposition 6.33, we use the following result.

Theorem 6.34 (Whitney et al. [72]). *A graph G of order $n \geq 3$ is 2-connected if and only if for any two vertices of G, there is a cycle containing both.*

Proof. Let us denote $V(K_{1,m}) = \{u_0, u_1, ..., u_m\}$ as the vertex set of $K_{1,m}$, where u_0 is the center vertex of $K_{1,m}$. To verify case *(ii)*, without lost of generality, we may assume that $\max\{m, n\} = m$. Hence, we prove that $pc_2(K_{1,m} \square K_{1,n}) = m$. Let us denote $H = P_n$, where $P_n = v_1...v_n$ for case *i)* or $H = K_{1,n}$, where $V(K_{1,n}) = \{v_0, v_1..., v_n\}$ and v_0 is the center vertex of $K_{1,n}$ for case *(ii)*.

First, we prove that $pc_2(K_{1,m} \square H) \geq m$. Let $S = \{[u_i, v_1], \forall i \in [m]\}$ be a vertex subset of $V(K_{1,m} \square H)$. It can be readily seen that $|S| = m$. Furthermore, $\deg_{K_{1,m} \square H}([u_i, v_1]) = 2$, where $[u_i, v_1] \in S$ and $N_{K_{1,m} \square H}([u_i, v_1]) \cap N_{K_{1,m} \square H}([u_j, v_1]) = \{[u_0, v_k]\}$, $\forall [u_i, v_1], [u_j, v_1] \in S$. Applying Lemma 6.21 implies that $pc_2(K_{1,m} \square H) \geq m$.

To complete our proof of Proposition 6.33, we show that $pc_2(K_{1,m} \square H) \leq m$. Now for case *(ii)*, we may assume that $m \geq n + 1$. Let H' be a subgraph of $K_{1,m} \square H$ depending on case *(i)* or case *(ii)* as follows:

(a) If $H = P_n$, then $H' = P^0_{1,n}$. Hence H' is a path of $K_{1,m} \square P_n$.

(b) If $H = K_{1,m}$, then $V(H') = \{[u_0, v_k], k \in \{0\} \cup [n]\}$ and

$$E(H') = \{[u_0, v_0][u_0, v_k], \forall k \in [n]\}$$

Hence, H' is induced as a star in $K_{1,m} \square K_{1,n}$ whose the center vertex is $[u_0, v_0]$.

Let H^* be a spanning subgraph of $K_{1,m} \square H$ such that $E(H^*) = E(K_{1,m} \square H) \setminus E(H')$ and $V(H^*) = V(K_{1,m} \square H)$. Observe that for any two vertices $[u_i, v_k], [u_j, v_l] \in V(H^*)$, there is a cycle connecting them. By Theorem 6.34, H^* is a 2-connected graph. Hence, there does exist $pc_2(H^*)$. By Lemma 6.3, $pc_2(K_{1,m} \square H) \leq pc_2(H^*)$, where $k = 2$. Moreover, one can be readily observed that H^* being different from an odd cycle has $\Delta(H^*) = \deg_{H^*}[u_0, v_k] = m$, where $k \in [n]$ for case *(i)*, or $k \in \{0\} \cup [k]$ for case *(ii)*. Hence, by Theorem 6.20, $pc_2(H^*) \leq \Delta(H^*) \leq m$. One can be readily deduced that $pc_2(K_{1,m} \square H) \leq m$. Therefore, $pc_2(K_{1,m} \square H) = m$.

Now, for case *(ii)*, we consider that $m = n$. We verify that $pc_2(K_{1,m} \square K_{1,n}) \leq m$. Let us colour all the edges of $K_{1,m} \square K_{1,m}$ by the following algorithms, see algorithm 1.

Algorithm 1

for $k = 1$ to m **do**
  for $i = 1$ to m **do**
    $c([u_0, v_k][u_i, v_k]) = ((i + k) \mod m) + 1$
  end for
end for
for $i = 1$ to m **do**
  for $k = 1$ to m **do**
    $c([u_i, v_0][u_i, v_k]) = ((((i + k) \mod m) + 1) \mod m) + 1$
  end for
end for

Now, we consider graph $G[K_{1,m} \square K_{1,m} \setminus [u_0, v_0]]$ with the colouring above. By simple cases to cases analysis, there always exist at least two disjoint proper paths connecting any two distinct vertices $x, y \in V(K_{1,m} \square K_{1,m}) \setminus [u_0, v_0]$. Next, we colour all the remaining edges of $K_{1,m} \square K_{1,m}$ by the following algorithm, see 2.

Algorithm 2

```
for i = 1 to m do
    c([u_0, v_0][u_i, v_0]) = c([u_i, v_0][u_i, v_1])
end for
```

One can observe that there are at least two disjoint proper paths between $[u_0, v_0]$ and $[u_i, v_k]$ such that $i \in \{0\} \cup [m]$ and $k \in \{0\} \cup [n] \setminus \{1\}$. Next, we verify that there are two disjoint proper paths between $[u_0, v_0]$ and $[u_i, v_1]$, for $i \in \{0\} \cup [m]$.

If $m \geq 4$, then for every arbitrary integer i, there are three distinct integers $j, k, l \in [m]$ such that $i \notin \{j, k, l\}$. Now, let us take two disjoint proper paths between $[u_0, v_0]$ and $[u_i, v_1]$ as follows

$$P_1 = P_0^{0,j} P_{0,j}^j P_j^{j,0} P_j^{0,i} P_{j,0}^i P_{0,1}^i$$

and

$$P_2 = P_0^{0,k} P_{0,k}^k P_k^{k,0} P_k^{0,l} P_{k,0}^l P_{0,1}^l P_1^{l,0} P_1^{0,i}.$$

If $m = 3$, then we colour $c([u_0, v_0][u_0, v_1]) = 1$. Now we take two disjoint proper paths between $[u_0, v_0]$ and $[u_i, v_1]$ such that $i \in \{1, 3\}$ as follows

$$P_1 = P_{0,1}^0 P_1^{0,i}$$

and

$$P_2 = P_0^{0,2} P_{0,2}^2 P_2^{2,0} P_2^{0,i} P_{2,0}^i P_{0,1}^i.$$

Two disjoint proper paths between $[u_0, v_0]$ and $[u_0, v_1]$ are

$$P_1 = P_{0,1}^0$$

and

$$P_2 = P_0^{0,2} P_{0,2}^2 P_2^{2,0} P_2^{0,i} P_{2,0}^i P_{0,1}^i P_1^{i,0}.$$

We colour $c([u_0, v_0][u_0, v_2]) = 1$. Hence two disjoint proper paths between $[u_0, v_0]$ and $[u_2, v_1]$ are

$$P_1 = P_{0,2}^0 P_2^{0,2} P_{2,0}^2 P_{0,1}^2$$

and

$$P_2 = P_0^{0,3} P_{0,3}^3 P_3^{3,0} P_3^{0,1} P_{3,0}^1 P_{0,1}^1 P_1^{1,0} P_1^{0,2}.$$

Clearly, $K_{1,m} \square K_{1,m}$ is a proper 2-connected graph with m colours. Hence $pc_2(K_{1,m} \square K_{1,m}) \leq m$.

We deduce that $pc_2(K_{1,m} \square H) = m$.

This completes our proof. $\square$

The graph H^* with $pc_2(H^*) = 3$ depicted in Figure 6.7 is a spanning 2-connected subgraph of $K_{1,3} \square P_6$.

Now, we study the relation between the proper 2-connection number and the proper connection number. We improve the upper bound of the proper 2-connection number of the Cartesian product of two nontrivial connected graphs by the following theorem.

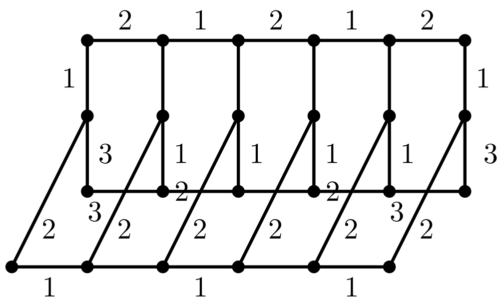

Fig. 6.7: Graph H^* with $pc_2(H^*) = 3$.

Theorem 6.35. *[30] Let G, H be nontrivial connected graphs such that $|V(G)| \geq 4$ and $|V(H)| \geq 2$. If $\delta(G) \geq 2$, then $pc_2(G\Box H) \leq pc(G) + 1$.*

Proof. Since G, H are nontrivial connected graphs, by Lemma 6.31, $G\Box H$ is a 2-connected graph. Hence there does exist $pc_2(G\Box H)$. Let $V(G) = \{u_1, ..., u_m\}$ and $V(H) = \{v_1, ..., v_n\}$, respectively, be the vertex set of G and H. Note that G is a properly connected graph with $pc(G)$ colours since G is a connected graph. Hence we colour all the edges of G in order to make it properly connected graph such that $c[u_i u_j] \in [pc(G)]$. Now we colour all the edges of $G\Box H$ with colour from $[pc(G) + 1]$ as follows:

1 If $u_i u_j \in E(G)$, then $c([u_i, v_k][u_j, v_k]) = c(u_i u_j)$ with $\forall k \in [n]$.

2 If $v_k v_l \in E(H)$, then $c([u_i, v_k][u_i, v_l]) = pc(G) + 1$ with $\forall i \in [m]$.

Since any two vertices $u_i, u_j \in V(G)$ are connected by at least one proper path in G, one can readily observe that there always exists at least one proper path between two vertices $[u_i, v_k], [u_j, v_k] \in V(G\Box H)$. Let us call $P_k^{i,j}$ be the proper path connecting $[u_i, v_k]$ and $[u_j, v_k]$ such that $[u_i, v_k], [u_j, v_k] \in V(G\Box H)$. Moreover, if $v_k v_l \in E(H)$, then $start(P_k^{i,j}) \neq c([u_i, v_k][u_i, v_l])$ and $end(P_k^{i,j}) \neq c([u_j, v_k][u_j, v_l])$. In order to complete our proof, we follow the series of claims by showing two disjoint proper path between any two vertices $[u_i, v_k], [u_j, v_l] \in G\Box H$. Let us denote P_1, P_2 be two disjoint proper paths connecting them.

Claim 6.35.1. *There always exist at least two disjoint proper paths in $G\Box H$ between $[u_i, v_k]$ and $[u_j, v_k]$ such that $u_i, u_j \in V(G)$ and $v_k \in V(H)$.*

Proof. Suppose, to the contrary that, there do not exist two disjoint proper paths in $G\Box H$ between $[u_i, v_k]$ and $[u_j, v_k]$. One can readily observe that $P_1 = P_k^{i,j}$. Since H is the nontrivial connected graph with $|V(H)| \geq 2$. Hence $|N_H(v_k)| \geq 1$. Without lost of generality, we may assume that $v_l \in N_H(v_k)$. Thus $[u_i, v_l] \in N_{G\Box H}([u_i, v_k])$ and $[u_j, v_l] \in N_{G\Box H}([u_j, v_k])$. We immediately deduce that

$$P_2 = P_{k,l}^i P_l^{i,j} P_{l,k}^j,$$

a contradiction.

The proof is obtained. $(\Box)$

Claim 6.35.2. *If $v_k v_l \in E(H)$, then there always exist at least two disjoint proper paths in $G\Box H$ between $[u_i, v_k]$ and $[u_j, v_l]$*

Proof. Suppose, to the contrary that, there do not exist two disjoint proper paths in $G\Box H$ between $[u_i, v_k]$ and $[u_j, v_l]$. Firstly, if $i \equiv j$, then one can readily choose

$$P_1 = P^i_{k,l}.$$

Since G is a nontrivial connected graph, hence $|N_G(u_i)| \geq 1$. Let $u_t \in N_G(u_i)$. Hence $[u_t, v_x] \in N_{G\Box H}([u_i, v_x])$ such that $x \in \{k, l\}$. We choose

$$P_2 = P^{i,t}_k P^t_{k,l} P^{t,i}_l,$$

a contradiction.

Secondly, if $i \neq j$, then we choose P_1, P_2 as follows:

$$P_1 = P^{i,j}_k P^j_{k,l}$$

and

$$P_2 = P^i_{k,l} P^{i,j}_l,$$

a contradiction.

The proof is obtained. (□)

Since H is a nontrivial connected graph, let $P_H = v_k v_{k_1} ... v_{l_1} v_l$ be a path between two vertices v_k, v_l. Now we consider two disjoint proper paths between $[u_i, v_k]$ and $[u_j, v_l]$ in $G\Box H$ such that $d_H(P_H) \geq 2$, for any $P_H = v_k ... v_l$ of H. Otherwise, by Claim 6.35.2 and Claim 6.35.1, there are two disjoint proper paths connecting them.

Claim 6.35.3. *There always exist two disjoint proper paths between $[u_i, v_k]$ and $[u_i, v_l]$ in $G\Box H$ such that $d_H(P_H) \geq 2$, for any $P_H = v_k ... v_l$ of H.*

Proof. Suppose, to the contrary that, there do not exist two disjoint proper paths between $[u_i, v_k]$ and $[u_i, v_l]$ in $G\Box H$ such that $d_H(P_H) \geq 2$. Clearly, $|N_G(u_i)| \geq 2$, since G a nontrivial connected graph with $\delta(G) \geq 2$. Let $u_1, u_2 \in N_G(u_i)$ be two neighbours of u_i. Firstly, if $u_1 u_2 \notin E(G)$ or $|N_G(u_1)| \geq 3$ or $|N_G(u_2)| \geq 3$, then there is another vertex $u_3 \in N_G(u_1)$ (or $u_3 \in N_G(u_2)$) such that $u_3 \notin \{u_i, u_2\}$ (or $u_3 \notin \{u_i, u_1\}$) since $\delta(G) \geq 2$. Without lost of generality, we may assume that $u_3 \in N_G(u_1)$. Hence $(u_3, v_x) \in N_{G\Box H}((u_1, v_x))$, where $v_x \in V(P_H)$. If $d_H(P_H)$ is odd, then $d_H(P_H) \geq 3$. Now

$$P_1 = P^{i,1}_k P^1_{k,k_1} P^{1,3}_{k_1} ... P^{3,1}_{l_1} P^1_{l_1,l} P^{1,i}_l$$

and

$$P_2 = P^{i,2}_k P^2_{k,k_1} P^{2,i}_{k_1} ... P^{i,2}_{l_1} P^2_{l_1,l} P^{2,i}_l,$$

a contradiction. If $d_H(P_H)$ is even, then $d_H(P_H) \geq 2$. Note that $P^{3,i}_l$ is a properly coloured path between $[u_3, v_l]$ and $[u_i, v_l]$. Now

$$P_1 = P^{i,1}_k P^1_{k,k_1} P^{1,3}_{k_1} ... P^{1,3}_{l_1} P^3_{l_1,l} P^{3,i}_l$$

and

$$P_2 = P_k^{i,2} P_{k,k_1}^2 P_{k_1}^{2,i} ... P_{l_1}^{2,i} P_{l_1,l}^i,$$

since P_2 and $P_l^{3,i}$ are two internally vertex disjoint paths, a contradiction.

Secondly, if $u_1u_2 \in E(G)$ and $|N_G(u_1)| = |N_G(u_2)| = 2$, then there exists another vertex $u_3 \in N_G(u_i)$ since $|V(G)| \geq 4$. If $d_H(P_H)$ is odd, then $d_H(P_H) \geq 3$. Now

$$P_1 = P_k^{i,3} P_{k,k_1}^3 P_{k_1}^{3,i} ... P_{l_1}^{i,3} P_{l_1,l}^3 P_l^{3,i}$$

and

$$P_2 = P_k^{i,1} P_{k,k_1}^1 P_{k_1}^{1,2} P_{k_1,k_2}^2 P_{k_2}^{2,1} ... P_{l_1}^{2,1} P_{l_1,l}^1 P_l^{1,i},$$

a contradiction. If $d_H(P_H)$ is even, then

$$P_1 = P_{k,k_1}^i P_{k_1}^{i,3} P_{k_1,k_2}^3 ... P_{l_1}^{i,3} P_{l_1,l}^3 P_l^{3,i}$$

and

$$P_2 = P_{k,k_1}^1 P_{k_1}^{1,2} P_{k_1,k_2}^2 P_{k_2}^{2,1} ... P_{l_1}^{1,2} P_{l_1,l}^2 P_l^{2,i},$$

a contradiction.

The result is obtained. (□)

Claim 6.35.4. *There always exists two disjoint proper paths between $[u_i, v_k]$ and $[u_j, v_l]$ in $G \square H$ such that $d_H(P_H) \geq 2$ and $i \neq j$.*

Proof. Since $i \neq j$ and G is the properly connected graph with $pc(G)$ colours, without lost of generality, we may assume that $P_G = u_i u_{i_1} ... u_{j_1} u_j$ is a shortest proper path connecting u_i, u_j of G. Hence we immediately deduce that $P_x^{i,j} = [u_i, v_x][u_{i_1}, v_x]...[u_j, v_x]$ is the proper path between (u_i, v_x) and (u_j, v_x), where $v_x \in V(P_H)$. Now we consider the following some cases depending on the $d_G(P_G)$

Case 6.35.4.1. *If $d_G(P_G) \geq 3$, then there always exist two disjoint proper paths between $[u_i, v_k]$ and $[u_j, v_l]$.*

Proof. If $L_G(P_G) \geq 3$, then we take P_1 and P_2 depending on the parity of $L_H(P_H)$.

If $L_H(P_H)$ is even, then

$$P_1 = P_k^{i,j_1} P_{k,k_1}^{j_1} P_{k_1}^{j_1,j} P_{k_1,k_2}^j P_{k_2}^{j,j_1} ... P_{l_1}^{j_1,j} P_{l_1,l}^j$$

and

$$P_2 = P_{k,k_1}^i P_{k_1}^{i,i_1} P_{k_1,k_2}^{i_1} P_{k_2}^{i_1,i} ... P_{l_1}^{i,i_1} P_{l_1,l}^{i_1} P_l^{i_1,j}.$$

If $L_H(P_H)$ is odd, then

$$P_1 = P_k^{i,j} P_{k,k_1}^j P_{k_1}^{j,j_1} P_{k_1,k_2}^{j_1} P_{k_2}^{j_1,j} ... P_{l_1}^{j_1,j} P_{l_1,l}^j$$

and

$$P_2 = P_{k,k_1}^i P_{k_1}^{i,i_1} P_{k_1,k_2}^{i_1} P_{k_2}^{i_1,i} ... P_{l_1}^{i_1,i} P_{l_1,l}^i P_l^{i,j}.$$

This case is proved. (□)

Case 6.35.4.2. *If $L_G(P_G) = 2$, then there always exist two disjoint proper paths between $[u_i, v_k]$ and $[u_j, v_l]$.*

Proof. Since P_G is the shortest proper path in G between u_i, u_j and $d_G(P_G) = 2$, it can be readily seen that u_i, u_j are not adjacent. Hence without lost of generality, we may assume that $u_1 \in N_G(u_i)$ such that $u_1 \notin \{u_i, u_{i_1}, u_j\}$, since $\delta(G) \geq 2$. Moreover, there is a proper path $P_l^{1,j}$ in $G\Box H$ between $[u_1, v_l]$ and $[u_j, v_l]$. Taking P_1, P_2 as follows:

If $L_H(P_H)$ is odd, then

$$P_1 = P_k^{i,1} P_{k,k_1}^1 P_{k_1}^{1,i} ... P_{l_1}^{i,1} P_{l_1,l}^1 P_l^{1,j}$$

and

$$P_2 = P_k^{i,j} P_{k,k_1}^j P_{k_1}^{j,i_1} ... P_{l_1}^{i_1,j} P_{l_1,l}^j$$

If $L_H(P_H)$ is even, then

$$P_1 = P_{k,k_1}^i P_{k_1}^{i,1} P_{k_1,k_2}^1 ... P_{l_1}^{i,1} P_{l_1,l}^1 P_l^{1,j}$$

and

$$P_2 = P_k^{i,i_1} P_{k,k_1}^{i_1} P_{k_1}^{i_1,j} P_{k_1,k_2}^j ... P_{l_1}^{i_1,j} P_{l_1,l}^j.$$

One can easily obseve that P_2 and $P_l^{1,j}$ are two internally vertex disjoint path.

This case is proved. (□)

Case 6.35.4.3. *If $L_G(P_G) = 1$, then there always exist two disjoint proper paths between $[u_i, v_k]$ and $[u_j, v_l]$.*

Proof. If $N_G(u_i) \cap N_G(u_j) = \{\emptyset\}$ or $|N_G(u_i) \geq 3|$ or $|N_G(u_j)| \geq 3$, then since $|V(G)| \geq 4$, there are two different vertices u_1, u_2 such that $u_1 \in N_G(u_i)$ and $u_2 \in N_G(u_j)$. Now we choose P_1, P_2 as follows:

Note that $P_l^{1,j}$ is the proper path between $[u_1, v_l]$ and $[u_j, v_l]$. If $L_H(v_k, v_l)$ is odd, then

$$P_1 = P_k^{i,1} P_{k,k_1}^1 P_{k_1}^{1,i} P_{k_1,k_2}^i ... P_{l_1,l}^1 P_l^{1,j}$$

and

$$P_2 = P_k^{i,j} P_{k,k_1}^j P_{k_1}^{j,2} P_{k_1,k_2}^2 P_{k_2}^{2,j} ... P_{l_1}^{2,j} P_{l_1,l}^j.$$

It can be readily seen that P_2 and $P_l^{1,j}$ are two internally vertex disjoint paths.

If $L_H(P_H)$ is even, then

$$P_1 = P_k^{i,1} P^1 k, k_1 P_{k_1}^{1,i} P_{k_1,k_2}^i ... P_{l_1,l}^i P_l^{i,j}$$

and

$$P_2 = P_k^{i,j} P_{k,k_1}^j P_{k_1}^{j,2} P_{k_1,k_2}^2 ... P_{l_1,l}^2 P_l^{2,j}$$

If $u_1 \in N_G(u_i) \cap N_G(u_j)$ and $|N_G(u_i)| = |N_G(u_j)| = 2$, then since $|V(G) \geq 4|$, there is another vertex u_2 such that $u_2 \notin \{u_1, u_i, u_j\}$. Furthermore, $u_2 \in N_G(u_1)$ since G is a connected graph.

If $L_H(P_H)$ is odd, then

$$P_1 = P_k^{i,j} P_{k,k_1}^j P_{k_1}^{j,i} ... P_{l_1}^{i,j} P_{l_1,l}^j$$

and

$$P_2 = P_k^{i,1} P_{k,k_1}^1 P_{k_1}^{1,2} P_{k_1,k_2}^2 ... P_{l_1,l}^1 P_l^{1,j}.$$

Note that $P_l^{2,j}$ is a proper path between $[u_2, v_l]$ and $[u_j, v_l]$. If $d_H(P_H)$ is even, then

$$P_1 = P_k^{i,1} P_{k,k_1}^1 P_{k_1}^{1,2} P_{k_1,k_2}^2 ... P_{l_1}^{1,2} P_{l_1,l}^2 P_l^{2,j}$$

and

$$P_2 = P_{k,k_1}^i P_{k_1}^{i,j} P_{k_1,k_2}^j ... P_{l_1}^{i,j} P_{l_1,l}^j.$$

One can readily see that P_2 and $P_l^{2,j}$ are two internally vertex disjoint paths.

This case is proved. (□)

The result is obtained.

(□)

Therefore, there always exist two disjoint proper paths in $G \square H$ connecting any two vertices $[u_i, v_k]$ and $[u_j, v_l]$ with $pc(G) + 1$ colours.

This completes our proof. □

Remark:

1. The condition of minimum degree $\delta(G) \geq 2$ in Theorem 6.35 is sharp. By Proposition 6.33, if $G \simeq P_m$ and $H \simeq K_{1,q}$ (with $q \geq 4$), then $pc_2(G \square H) = q > pc(G) + 1 = 3$.

2. If $G \simeq K_m$ with $m \geq 4$ and H is an arbitrary nontrivial connected graph, then $pc_2(G \square H) \leq pc(G)+1 = 2$. On the other hand, $pc_2(G \square H) \geq 2$. Hence $pc_2(G \square H) = 2$. We immediately deduce that $pc_2(K_m \square H) = pc(G) + 1$ which is the upper bound of Theorem 6.35.

We finish Chapter 6 here.

7 Conclusions and Perspectives

In this dissertation, we have presented some results of the proper $k-$connection number of a connected graph. In particular, we have studied 2-connected graphs with a given minimum degree, which have proper connection number 2 or 3, and $S_{i,j,k}$-free graphs with given connectivity and proper connection number 2. Finally, we have studied the proper 2-connection number of connected graphs. Now, we summarize the results we have obtained in the present dissertation.

7.1 Contribution summary

In Chapter 4, we study the proper connection number of connected graphs with the condition of minimum degree. In particular, we disprove Conjecture 4.1 of the authors in [8] by constructing a class of connected graphs with minimum degree d, where $d \geq 3$ and order $n = 42d$ such that its proper connection number equals 3, see Theorem 4.2. Furthermore, if the condition of connectivity is not considered, then we prove that there exists a connected graph of minimum degree d and order $n = (d+1)(n+1)$ such that $pc(G) = k$, see Theorem 4.5. Motivated by these results, we study the proper connection number of 2-connected graphs with the condition in term of ratio between its minimum degree and order, see Theorem 4.6. By Theorem 4.6 and Corollary 4.4, we propose the following problem.

Problem 7.1. *Let G be a 2-connected graph of order n and minimum degree $\delta(G) \geq 3$. Characterize the proper connection number of graphs G if $\frac{n}{32} < \delta(G) < \frac{n+8}{20}$.*

In Chapter 5, we consider proper connection number 2 of connected, $S_{i,j,k}$-free graphs, where all i, j, k are small, and the condition of connectivity and minimum degree. In particular, we prove that if G is 2-connected, $S_{1,1,6}$-free graph of minimum degree at least 3, then $pc(G) = 2$, see Theorem 5.8. By Proposition 5.4, we believe that we can improve $S_{i,j,k}$-freeness of a connected graph that has proper connection number 2. Hence, we pose the following problem.

Problem 7.2. *Let G be 2-connected, $S_{i,j,k}$-free graph and minimum degree $\delta(G) \geq 3$, where $min\{i, j, k\} \geq 2$, $max\{i, j, k\} \geq 7$, or $i + j + k \geq 9$. Characterize the proper connection number of G.*

In Chapter 6, we study the proper 2-connection number of connected graphs. Note that, there are still not many results in this area in the literature. We obtain the new upper bound of the proper 2-connection number. In particular, we characterize that a connected graph G has proper 2-connection number $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G if and only if G is an odd cycle, see Theorem 6.20. Furthermore, we study several classes of connected graphs which have proper 2-connection number 2. Motivated by the result of the proper connection number and size of graphs by authors in [1], and the proper 2-connection number of 2-connected graph with the large clique number, see Theorem 6.29, we pose the following conjecture.

Conjecture 7.3. *Let G be a 2-connected graph of order n. If $|E(G)| \geq \binom{n-3}{2} + 7$, then $pc_2(G) = 2$.*

If Conjecture 7.3 is true, then the condition of the size of the graph is sharp. By the proof of Proposition 6.30, if we choose $k = 3$, then $|E(G)| = \binom{n-3}{2} + 6$. It follows that $pc_2(G) \geq 3$. Moreover, the condition of 2-connectivity of Conjecture 7.3 is necessary since there is a connected graph that has no the proper 2-connection number.

In particular, we study the relation between the proper 2-connection number and the proper connection number of the Cartesian product of two arbitrary connected graphs with specified conditions, see Theorem 6.35. By the proof of this theorem, the following conjecture is proposed.

Conjecture 7.4. *Let H be an arbitray nontrivial connected graph. If K_3 is a complete graph of order 3, then $pc_2(K_3 \square H) = 2$.*

As the open question of computing the proper connection number which is given by the authors in [31], and the upper bound of the proper connection number of 2-connected graph is at most 3, see Theorem 3.10, recently, determining the proper connection number of arbitrary 2-connected graphs which is 2 or 3 is still open. Therefore, it might be of interest for further research to study the proper connection number of arbitrary 2-connected graphs that is 2 or 3. Moreover, computing the proper k-connection number of a connected graph is interesting, too.

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