GDP per Capita Differentials between Nations: Patterns and Models

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To my grandma Elfriede, who never ceased to believe in me

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List of Abbreviations

CES	Constant elasticity of substitution
DW	Durbin Watson
EU-15	European Union (15 member states)
GDP	Gross domestic product
GNI	Gross national income
GNP	Gross national product
HPI	Human poverty index
IMF	International Monetary Fund
LDC	Less developed country
OECD	Organization for Economic Cooperation and Development
PPP	Purchasing power parity
SPSS	Statistical Package for the Social Sciences
$	Dollar (international dollars if not mentioned otherwise)
UK	United Kingdom
US	United States

1 Introduction

"The world distribution of income has become more unequal over the 1965-90 period […]. By decomposing world income inequality into between- and within-country components, [there is] strong evidence that between-country inequalities are of significantly greater importance in shaping the trends in question. Overall, while between-country inequality has become more pronounced over the period under consideration, the opposite was the case of within-country inequality. However, the attenuation of income inequality within nations was not nearly sufficient to compensate for the accentuation of between-country inequality. Inequality in the distribution of income between-countries continues to be of essential importance to global stratification" (Korzeniewicz and Moran, 1997, pp. 1029f).

1.1 Problem Statement

Both income disparities between and within are central to economic research. Barro (2000), for example, finds that higher inequality within a country is bad for growth if the country is poor but has a positive impact on growth if it is rich. On the contrary, Galor and Zeira (1993) point out that differences in the income distribution across countries – and hence in wealth – are responsible for differences in the abilities of countries to react to macroeconomic shocks. Even looking at the headlines in newspapers shows that income inequality is an ever prevailing issue.[1]

Korzeniewicz and Moran (1997) work out that income inequality across countries accounts for about 90 percent of the total world income inequality. The authors conclude that "data on the between-country distribution of world income can indeed be used as appropriate indicators of inequality" (Korzeniewicz and Moran, 1997, p. 1017). The authors also point out that including within-country inequality might be more accurate; however, this will probably not change the conclusions to be drawn as the trends are very well represented by between-country inequality. This enables researchers to provide a much more detailed research on the question of the worldwide income inequality, as data between-country inequality are available not only for more years but especially for more countries of the world (Korzeniewicz and Moran, 1997).

In recent decades, polarization of the world income distribution has become a trend. According to Korzeniewicz and Moran (1997), the polarization theory has to be included as a decisive factor when trying to understand the development of the world economy. As they explore, the world income distribution became indeed more unequal over time. While some countries – for example East Asia – managed

[1] In politics, the focus is often on income inequality within countries. Barack Obama, the President of the United States, for example, "declared rising income inequality a "fundamental threat" to the United States" (Sink, 2013, p. 1). However, in economic research both kinds of inequality are relevant.

to move upwards, the worldwide income distribution did not experience such a development. “These results highlight the continuing need for more detailed inquiries into the processes that generate growing inequality in the world distribution of income” (Korzeniewicz and Moran, 1997, p. 1031).

The world income distribution, to be more precise especially the polarization of the world income distribution, has been central to economic research. “Compare labor productivities and incomes (per capita) across countries and ask, Are poorer countries catching up to richer ones? Are they likely to in the future? Or are countries converging only within "clubs"? If so, are these clubs of the very rich and the very poor, or is most of the world becoming only middle class? Answers to these questions – on catch-up and convergence – are basic for thinking about economic growth: they can be viewed either as checks on different growth models or as empirical regularities to be explained by theory” (Quah, 1996c, p. 95).

Convergence always concerns the question of “catching up, forging ahead, and falling behind” as Abramovitz (1986, p. 385) calls it. In 1996, a famous convergence debate was published in the Economic Journal. The participants of this debate are Sala-i-Martin, Bernard and Jones, Quah, and Galor, all of them being famous convergence economists. Following this debate, club convergence became more and more central to discussion. This contravenes the idea that the poor countries would catch up with the rich ones and thereby narrow the gaps in the world income distribution, hence experience convergence. Instead, empirical analyses show that convergence clubs formed in which countries converge towards each other: a club of rich countries and a club of poor countries. Consequently, instead of a Gibrat distribution[2], a bimodal distribution appears where the middle income class seems to decrease sharply (see for example Quah (1993a; 1997), Jones (1997), Pearlman (2003), and Beaudry, Collard and Green (2002), just to name a few).

Also the second focus on the question of stratification as mentioned by Quah (1996c), namely the check on different growth models, was already pursued in the past. One of the most influential economic growth models is the Solow growth model. In his article, Solow (1956) shows graphically how the model may yield two stable steady states and thus explain multimodality.

1.2 Research Objectives

Based on the empirical findings of the past as well as the hypothesis by Solow (1956) that his neoclassical growth model is able to yield multimodality, this doctoral thesis seeks to answer the following two questions:

1. Is there really club convergence in the real per capita income distribution across the countries of the world?

[2] The term Gibrat distribution refers to a distorted normal distribution of income (or any other economic variable) in form of a lognormal distribution, which is skewed to the right (Gabler, 2014).

2. Is the Solow growth model indeed able to explain the polarization phenomenon?

This examination will be done in three ways. First of all, Solow's findings shall be examined graphically and verbally. In excess of this, the hypothesis will also be explored analytically by inserting an endogenous savings rate into the neoclassical growth model. As a third way, an empirically determined version of the Solow growth model will be determined and then checked for the existence of two stable steady states. This will be based on the empirical analyses, which are pursued to examine whether the real per capita income distribution across the countries of the world is indeed polarized.

1.3 Thesis Organization

This doctoral thesis is organized as follows.

Chapter 2 provides an overview of the theoretical background of the polarization of the world income distribution. As a fundamental basis, the main terms underlying this doctorate will be defined. In addition, an overview of the existing literature on the convergence debate and on the theory of bimodality will be given.

In Chapter 3, the basic Solow growth model will be presented. In this chapter, the approach of Solow (1956) shall be followed to show graphically and verbally the ability of his growth model to capture twin peaks. Contrary to Solow (1956), the production function will not be augmented here. Instead, the possibility of an endogenous savings rate and an endogenous population growth rate will be discussed. These possibilities were also mentioned by Solow (1956), yet not very extensively. In the course of the chapter, an augmented Solow growth model including human capital will be presented as well. It will be elaborated on the ability of an endogenous savings rate in human capital together with an endogenous savings rate in physical capital to yield two stable steady states within this model framework.

Chapter 4 looks at the empirical methods to be applied in the analyses of this doctoral thesis. Here, different methods for distribution analysis will be compared. After presenting in detail the kernel density distribution method, the Markov chain will be described. This is a method for analyzing the future significance of the twin peaks phenomenon. Finally, the loess fit curves will be introduced. They enable a judgement on the correlation between real per capita GDP and the different variables included in the basic and the augmented Solow growth model: the savings rate (approximated by the investment rate as will be explained later on), the population growth rate, and human capital.

In Chapter 5, this doctorate empirically explores the polarization hypothesis. After examining whether there are indeed twin peaks in the real income distribution across the countries of the world, the same will be done for the variables likely to yield bimodality in the Solow growth model. In addition, loess fit curves may help to decide on the variable being most likely to lead to bimodality in the real per capita income distribution across the countries of the world.

Chapter 6 provides the second method to investigate the ability of the Solow growth model to capture twin peaks. In an analytical way a savings rate function dependent on income shall be determined and inserted in the Solow growth model. The resulting model shall then be solved analytically in order to find out whether indeed two stable equilibria result.

In Chapter 7, a third procedure to examine the hypothesis that the Solow growth model is able to explain real per capita income differentials between nations will be pursued. Based on the empirical data, an endogenous savings rate and an endogenous population growth rate shall be determined. These functions will be used in the Solow growth model. By calculating the steady states of this empirically determined Solow growth model, it shall be checked whether bimodality results.

Finally, Chapter 8 will conclude this doctoral thesis.

2 Theoretical Background

Economic growth research of the past decades shows that a number of stylized facts turned out to be fundamental to the economic developments of 20th century growth. In 1961, Kaldor formulated his famous stylized facts which dealt as a summary of the conclusions to be drawn from analyzing economic growth in the 20th century. These facts should also build the framework for the future research agenda (Kaldor, 1961). The six stylized facts can be summarized as follows:

"1. Labor productivity has grown at a sustained rate.
2. Capital per worker has also grown at a sustained rate.
3. The real interest rate or return on capital has been stable.
4. The ratio of capital to output has also been stable.
5. Capital and labor have captured stable shares of national income.
6. Among the fast growing countries of the world, there is an appreciable variation in the rate of growth of the order of 2-5 percent"

(Jones and Romer, 2010, pp. 224f). The first five facts can be found in any textbook on economic growth. They are also covered by the neoclassical growth model by Solow. Yet, as Jones and Romer (2010) state, research on economic growth nowadays focuses on the sixth stylized fact. In addition, the authors found a number of new stylized facts which are center to modern economic growth research:

"1. Increases in the extent of the market. Increased flows of goods, ideas, finance, and people – via globalization, as well as urbanization – have increased the extent of the market for all workers and consumers.
2. Accelerating growth. For thousands of years, growth in both population and per capita GDP has accelerated, rising from virtually zero to the relatively rapid rates observed in the last century.
3. Variation in modern growth rates. The variation in the rate of growth of per capita GDP increases with the distance from the technology frontier.
4. Large income and total factor productivity (TFP) differences. Differences in measured inputs explain less than half of the enormous cross country differences in per capita GDP.
5. Increases in human capital per worker. Human capital per worker is rising dramatically throughout the world.
6. Long run stability of relative wages. The rising quantity of human capital, relative to unskilled labor, has not been matched by a sustained decline in its relative price."

(Jones and Romer, 2010, p. 225). Modern growth research of the recent decades tried to extend their models to cover the "new stylized facts". While, at the time Kaldor formulated his facts, it seemed to be sufficient to focus on physical capital as an explanatory factor, additional factors are important from a recent point of view.

A further popular subject of growth literature in the past has been the well-known convergence hypothesis being central to a large discussion.[3] It basically states that countries tend to converge over time.[4] However, this hypothesis has been challenged by several authors during the past forty years. They found that countries are not just member of a single group, all converging towards each other. On the contrary, since the late 1970s rather two convergence clubs have been forming, the so-called twin peaks: a group of developed countries and a (larger) group of developing countries.[5] This feature will be central to this doctoral thesis. Twin peaks in the distribution of real per capita income[6] imply – within the framework of the Solow growth model – that the income distribution polarizes into two groups of countries, each growing towards a separate steady state. This possibility was already mentioned by Solow (1956). He proposes that his model would be able to capture multiple peaks. Before this statement can be examined, in this chapter the main literature on the subject of convergence, especially on twin peaks convergence, will be reviewed.

This chapter is organized as follows: Section 2.1 gives the definitions of the basic terms used in this dissertation. Thereafter, Section 2.2 reviews the convergence debate published in the Economic Journal of 1996. It can be seen as a kind of starting point of the twin peaks discussion.[7] Section 2.3 will then give an overview of the theoretical literature on the subject.[8] Section 2.4 will conclude this chapter.

2.1 Definitions

Any scientific work should start by a clarification of the basic terms. Often scientists differ in their view on the exact meaning of central expressions. The differences may seem negligible, but in fact, often they are not. As they form the basis for the conclusions to be drawn from the analyses in this doctoral dissertation, a clarification is indispensable. Furthermore, an exact definition may also be decisive when empirical analyses are performed, as many features can be measured in different ways. This doctoral thesis is about twin peaks in the Solow growth model. Hence, the terms twin peaks and thus also poverty are crucial. The following fundamental concepts necessary for the discussion of the twin peaks phenomenon

[3] Of course, there is a large class of articles dealing with endogenous growth models. Yet, this dissertation concentrates on the Solow growth model and on convergence.
[4] Converging means that the dispersion among countries decreases. The convergence debate will be reviewed in Section 2.2.
[5] It will be shown later on in this dissertation that countries may also switch groups. Among the examples are the Asian Tiger states (partially having switched from the lower income group to the higher one) and Argentina (having fallen back from Group 2 to Group 1).
[6] From now on, the term "income" refers to real income. In addition, if the context is clear, the term "per capita" will be left out.
[7] It should be kept in mind that there were already contributions, which can be attributed to the twin peaks discussion, earlier than 1996. However, the broad discussion did not really start before that time.
[8] The empirical literature will be dealt with in Chapter 4.

will be explored: economic growth and economic development; income inequality; bimodality; poverty; and poverty traps. Specific terms which are not fundamental to the whole discussion but rather specific to individual sections will be discussed if needed.

2.1.1 Economic Growth and Economic Development

Twin peaks theory is part of economic growth theory. "Economic growth refers to increases in a country's production or income per capita" (Nafziger, 2006, p. 15). It is a decisive factor in poverty reduction. The Gabler Economic Encyclopedia (2014) defines economic growth as the increase of the economic efficiency of an economy. This efficiency can be indicated by gross national product (GNP) or gross domestic product (GDP).[9] Hence, economic growth means an increase of the overall economic production and thus of GNP or GDP. The causes of economic growth will not be discussed here. They are examined by several economic growth models (Gabler, 2014).[10]

Economic growth is not only an economic feature. It might also be measured by other factors such as health, material possessions, and differences between rich and poor countries as well as the level of inequality within a country (Weil, 2005). Nevertheless, in this doctoral thesis only the economic meaning of the term economic growth will be applied in order to be able to make statements about the relative positions of countries. Using only real per capita GDP as a reference is quite restrictive. Nevertheless, this will be done as the central question in this doctoral thesis concerns the income distribution across the countries of the world and not the one within them. Furthermore, some of the data which would otherwise be needed are not easy to gather or their quality is bad. Last but not least, growth models are not able to focus on several variables constituting economic growth. Rather, the left-hand side of these models is made up by one variable, which generally is income.

Economic growth accompanied by changes in output distribution and economic structure is a form of economic development. Economic development can either be understood as a process or as a state. Development strategies principally focus on the process of economic development, in which real per capita GDP is increasing over a longer term without an increase in the number of people living

9 Even though most authors work with GDP data, some authors use gross national product (GNP) or gross national income (GNI) instead. "GNP is the sum of all income earned by the factors of production owned by the residents of a given country" (Weil, 2005, p. 301). The decision on which of the two measures, GDP or GNP, is better to be used is dependent on the question to be answered. GDP data are easier to get as it is easier to measure the amount of output produced within a specific country (which equals GDP) than to find out who owns the respective production factors and hence being able to calculate GNP. For these reasons, also in the doctoral thesis at hand GDP will be used, which is also justified by the fact that GDP data are available for more countries than GNP data, especially when using only one data source.

10 Economic growth models will not be treated here. For a good overview of economic growth models see for example Aghion (2009).

below the subsistence level, without a rise in income inequality within the country, without more environmental damage, or a larger economic reliance on foreign countries. Economic development can be measured in economic terms on the one hand. On the other hand, also social or partial indicators are of importance, such as the literacy rate, birth and death rates, life expectancy, fertility, the use of electricity, the share of international trade in GDP, the intensity of political competition among the parties, and the strength of democratic institutions (Gabler, 2014). Yet, here, the focus is on real per capita GDP data as an indicator.

2.1.2 Income Inequality

A crucial concern of this doctoral thesis is income inequality. Income inequality is an often analyzed aspect in economics. In general, the discussions on this aspect concern income inequality within a nation, for example income inequality in Germany or in South Africa and so on. Income inequality refers to a number of important questions: "how much inequality is there in our society? How many people live in poverty? What problems arise in measuring the amount of inequality? How often do people move among income classes?" (Mankiw and Taylor, 2014, p. 386). There are a number of measures which shall help to examine income inequality: the Lorenz curve, the Gini coefficient and so on. Yet, it is not straightforward to interpret these values even if they were available for all countries. These coefficients just give an insight into how incomes are distributed. What we do not know, however, is the standard of living which accompanies these different incomes. As this is the most interesting aspect when talking about income inequality, it already becomes apparent that even if just looking at one country, the examination of income inequality is not an easy one (Mankiw and Taylor, 2014).

Also the United Nations are concerned with the problem of inequality. In 1994, it was stated at the Cairo International Conference on Population and Development that "despite decades of development efforts, both the gap between rich and poor nations and inequalities within nations have widened"[11] (Todaro and Smith, 2006, pp. 193f). However, inequality is more than just income inequality. Even though economic analysis generally focuses on inequality in the distribution of incomes (and assets), the general problem of inequality additionally comprises inequality in power, prestige, status, gender, job satisfaction, conditions of work, degree of participation, freedom of choice, and many other dimensions of inequality. Of course, these aspects are often influencing each other. Yet, the biggest problem is that such indicators of inequality are difficult to measure. For this reason, even though being the second best option, it just remains to focus on the distribution of incomes (Todaro and Smith, 2006).

Another way to distinguish measures of income distribution is by looking at the personal or size distribution of income on the one hand, and the functional or

[11] In this doctoral thesis the focus is on inequality between rich and poor countries, not within them.

distributive factor share distribution of income on the other hand. The former measure is most widely used and “simply deals with individual persons or households and the total incomes they receive” (Todaro and Smith, 2006, p. 195). In this context, Lorenz curves and Gini coefficients become important. The latter measure, the functional factor share distribution, “attempts to explain the share of the total national income that each of the factors of production (land, labor, and capital) receives” (Todaro and Smith, 2006, p. 201). Especially this latter aspect of inequality is just possible at a national level.

As stated before, factors of production use to be rather mobile at a national level. However, at an international level national borders and also other borders such as languages usually work very well as obstacles to factor mobility. In the past decades, there could be found some examples of trying to eliminate such obstacles. An obviously very famous example is the European Union with its common market concept. The Schengen contract and the EU contract as a whole aimed at easing of the obstacles to factor mobility (Facchini, 2002). The result was, for example, that the EU residents may work and live anywhere in the EU. Nevertheless, one overlooked the problem of the different languages, for example, which are an obstacle to factor mobility that cannot be easily overcome. For this reason, we do not have perfect factor mobility, though it is higher than before (Facchini, 2002). The countries of the EU converged in their economic developments, often they are seen as one country, at least those countries having the Euro as a common currency, hence the Eurozone. However, a look at current economic data shows that the countries tend to have quite different problems. Spain and Greece, for example, faced unemployment rates of above 25 percent in the first half of 2014, while Germany and Austria had rates of 5 percent or even less (Eurostat, 2014). If factor mobility were perfect, then the unemployment rates should be much closer.

From these different aspects one can conclude that an analysis of worldwide income inequality in the framework of an economic growth model will be hardly possible. Instead, a second best solution for the analysis of worldwide income inequality is to look at the distribution of real per capita GDP across the countries of the world irrespective of the income distributions within each of these countries.[12]

Past analyses do not show high correlations of per capita income with any of the widely used measures of inequality such as the Gini coefficient, the income share of the lowest 40 percent of households, and the ratio of the highest 20 percent to the lowest 20 percent, at least for developing countries. This supports the view that it is acceptable to talk about nationwide inequality. Beyond this, income inequality is also quasi-independent of economic growth, and vice versa. Thus, even though a country might face high economic growth, this does not say anything about the income distribution within this country nor about the changes in the standard of living (Todaro and Smith, 2006). These findings can be interpreted such that the

[12] Hence, per year and country there is just one point of observation.

statements made out of real per capita income inequalities are acceptable for worldwide comparisons even though the individual income inequalities are not considered.

2.1.3 Bimodality

Another important term to be clarified is the term bimodality, which stems from the intent to use growth theory to examine international income inequality. Bimodality is a phenomenon in economic growth analysis. It refers to the fact that the real per capita income distribution across the countries in the world is characterized by two peaks instead of being distributed according to a Gibrat distribution. This phenomenon is often called polarization indicating that the world distributes into rich and poor. In this doctoral thesis, the terms bimodality, polarization, twin peaks, and club convergence will be used interchangeably.

The expression twin peaks was found by the “father” of the twin peaks hypothesis, namely Danny Quah. In 1996, he published an article in a famous debate on convergence theory in the Economic Journal. This article can be interpreted as the starting point for the actual polarization discussion. It will be reviewed in Section 2.2 in the context of the whole convergence debate.

2.1.4 Poverty

When analyzing the income inequality between the countries of the world, the term poverty becomes of central importance. Generally speaking, poverty refers to the situation in which plight is no longer temporary but rather permanent instead. Poverty can be differentiated into absolute and relative poverty. Absolute poverty refers to the situation in which a person does not have the means to pay for subsistence consumption, such as food, clothing, and shelter. Contrary to this, relative poverty refers to the lack of a socio-cultural minimum standard of living. Hence, relative poverty is a more subjective definition. While absolute poverty is mainly a problem in developing countries, inhabitants of developed countries may also be deemed poor, but then usually on a relative basis. This makes it difficult to compare countries; and even a comparison of the status of people in the same country at different points in time is hardly possible (Nafziger, 2006).

Poverty is of a multifaceted nature. Ruggeri Laderchi, Saith and Stewart (2003) point out that poverty can be defined as a lack of monetary means, a lack of capabilities, as social exclusion, or rather by participatory approaches. In their studies for the World Bank, Narayan, Patel, Schafft, Rademacher and Koch Schulte (2000) name the following dimensions of poverty:

1. being poor means facing a lack of at least one thing – most important is the lack of food, in other words hunger;

2. poverty has a psychological dimension made up by powerlessness, voicelessness, dependency, shame, and humiliation;
3. poverty often stands for a lack of adequate infrastructure such as roads, transportation, and above all clean water;
4. poverty generally goes hand in hand with bad education and a low level of literacy;
5. poverty, also known as destitution, is often accompanied by poor health and illness;
6. poor people are rarely interested in income but rather in managing assets to cope with their vulnerability (these assets may be physical, human, social, or environmental).

Just as there is a wide variety of dimensions of poverty, measurements of poverty differ as well. While it is often measured in monetary terms only and hence as a particularly low real per capita income level at a certain point in time, the World Bank calculates a human poverty index (HPI-1), which is based on the multidimensionality of poverty as described by Narayan et al (2000). This index comprises the following elements: “[first, the] probability at birth of not surviving to age 40; [second, the] adult illiteracy rate; and [third, the] lack of a decent standard of living, as measured by the average of the percentage of the population without sustainable access to improved water source and the percentage of children underweight under the age of five” (Nafziger, 2006, p. 168).

Poverty in a developed country does not necessarily coincide with the definition of poverty in a less developed country (LDC). Just as the World Bank calculates the HPI-1 to measure poverty in developing countries, it also takes into account different definitions of poverty in developed countries. Consequently, the World Bank calculates the HPI-2 for the developed countries including functional literacy, survival rate to age 60, and a higher poverty line than for the LDCs (Nafziger, 2006).

One part of the HPI, both the one for LDCs and the one for developed countries, is the poverty line. Poverty is multidimensional, but it can also be defined on a monetary base only. For this purpose, often a poverty line is drawn which means that anybody being below this line is considered to be poor. Some economists argue in favor of two poverty lines instead of only one. The lower one, generally set at $1 per day[13], indicates the absolute minimum by international standards. This value “is based on a standard set in India, the country with the most extensive literature on the subject and close to the poverty line of perhaps the poorest country, Somalia” (Nafziger, 2006, p. 171).

As compared to the lower poverty line at $1, there is another one at $2. This second poverty line stands for consumption in excess of the subsistence minimum.

[13] The value here is given in purchasing power parity (PPP) dollars, hence the dollars are corrected for different PPPs in the countries considered. It can be compared to the international dollar used in the Penn World Table (see Chapter 5).

However, it varies from country to country and focuses on the possibility to participate in everyday life of society. Consequently, this is a more subjective measure than the $1 poverty line – it serves as a measure of relative poverty. Just to compare the meaning of these two poverty lines, the more objective subsistence line and the more subjective line addicted to the standard within a country, the World Bank found that poverty in the world was 17.6 percent in the first case but 43.7 percent in the second case in the year 2000 (Nafziger, 2006).

Much of today's discussion is based on the human development index published by the World Bank. However, in this doctoral thesis the focus will be on income comparisons only, as it is not on poverty in general but rather on the income inequality across the countries of the world.[14] For this reason, the other elements on deciding whether a country is to be classified as rich, poor, or in between including the income inequality within a country, hence the income inequality on an individual basis, are not of relevance here, although being aware that this is indeed a rather limited view on poverty.

2.1.5 Poverty Trap

The final term to be defined in this section, namely the term poverty trap, is also crucial to the discussion of the twin peaks phenomenon. "The malady of many underdeveloped economies can be diagnosed as a stable equilibrium level of per capita income at or close to subsistence requirements. Only a small percentage, if any, of the economy's income is directed toward net investment. If the capital stock is accumulating, population is rising at a rate equally fast; thus the amount of capital equipment per worker is not increasing. If economic growth is defined as rising per capita income, these economies are not growing. They are caught in a low-level equilibrium trap" (Nelson, 1956, p. 894).

Mehlum, Moene and Torvik (2006, p. 81) define a poverty trap "as the bad equilibrium in a situation where there also exists a good equilibrium. […T]he poverty trap can be avoided, but once you are in a trap it is difficult to escape". Becoming poor does not automatically mean to be trapped in poverty. It is hard work, but the trap can be overcome at the initial status. The discussion on poverty traps is rather on long run poverty, which should be overcome. There has been a broad discussion on poverty traps in the literature. This aspect will be dealt with in Section 2.3, in which a literature review of the twin peaks discussion will be given. Before, the next section will introduce into the twin peaks subject by reviewing the basic convergence debate published in the Economic Journal in 1996.

[14] Just to remind, the focus is not on income inequality within nations but rather between them.

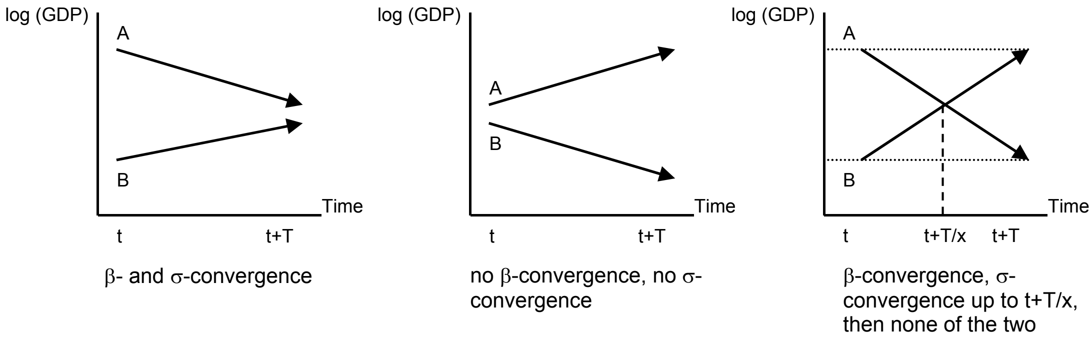
2.2 The Convergence Debate

The discussion on the twin peaks phenomenon in the income distribution across the countries of the world – although it started already earlier – can be seen as being founded on a controversy given in the Economic Journal of 1996: "On the Convergence and Divergence of Growth Rates". The convergence hypothesis has been examined in detail by many authors – among them those economists who participated in the controversy: Sala-i-Martin, Bernard and Jones, Quah, and Galor. There are many opposing views on the convergence hypothesis, especially concerning the definition of convergence and the way it should be examined.

Basically all definitions have in common the notion that countries initially having very different levels of real per capita income will converge towards each other – especially if technologies, preferences, and population growth rates are similar, or even equal. These discussions support the importance of technology as an explanatory factor, and thus contribute to the endogenous growth theory.[15] Next to convergence, some authors also find evidence on divergence, especially among rich and poor countries. This results in polarization – or, to use Quah's words, in twin peaks.

To start the controversy, Sala-i-Martin (1996) first distinguishes two different concepts of convergence: $\beta -$convergence and $\sigma -$convergence. The former refers to the case when "poor countries tend to grow faster than rich ones" (Sala-i-Martin, 1996, p. 1020), also called absolute $\beta -$convergence. The latter exists "if the dispersion of countries' per capita GDP levels tends to decrease over time" (Sala-i-Martin, 1996, p. 1020). To illustrate these two concepts, Sala-i-Martin uses graphs as given in Figure 2.1.

Figure 2.1 The Relation between $\beta -$ and $\sigma -$Convergence[16]



Source: Own representation based on Sala-i-Martin, 1996

[15] Technology spillovers, for example, can be seen as a main factor leading to multiple locally stable steady states (Azariadis and Drazen, 1990).

[16] Country A is initially richer than country B.

The first panel shows $\beta -$ and $\sigma -$convergence at the same time. In the second panel, the opposite is shown, namely the absence of both types of convergence. Finally, the third panel shows a situation in which there is $\beta -$ and $\sigma -$convergence up to point $t + T/x$. Beyond the point of intersection, both types of convergence disappear again, hence there is divergence. Sala-i-Martin gives cross-country evidence on convergence in his article. He distinguishes several groups of countries and finds that convergence takes place at 2 percent per year. Next to the absolute $\beta -$convergence mentioned above, there is also conditional $\beta -$convergence. According to Sala-i-Martin, it describes a situation in which the “growth rate of an economy will be positively related to the distance that separates it from its own steady state” (Sala-i-Martin, 1996, p. 1027). Both absolute and conditional $\beta -$convergence will coincide if the countries involved have the same steady state. This is possible, for example, among OECD[17] countries. Sala-i-Martin concludes that neither absolute $\beta -$convergence nor $\sigma -$convergence exist in our world. However, there is conditional $\beta -$convergence. The speed of convergence is equal to 2 percent per year.

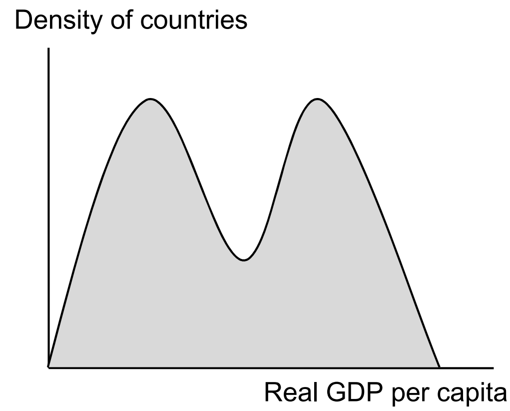
Bernard and Jones (1996) give the second contribution to the convergence debate. In their article, the authors examine the role of technology in the context of convergence. According to them, “technology transfer is [...] a potential force behind convergence” (Bernard and Jones, 1996, p. 1038). Using the neoclassical Solow growth model including technological progress, they show that the result of this model crucially depends on the parameters of the technology transfer equation and on those of the production function. Technology differs across countries. The authors state that endogenous growth theory, which developed over the past decades, deals with technology. However, this does not fully solve the problem. Consequently, further work needs to be done, especially as far as technology is concerned.

Subsequently, Quah (1996c) adds to the convergence debate. First, he describes the traditional approach which is just looking at whether a single country converges towards its own steady state. Yet, Quah points out that it is more important to see what happens to the entire income distribution. And there, he says, contrary to his colleagues of the controversy, the empirics rather show club convergence and polarization into rich and poor. The new approach he proposes does not estimate a cross-section regression of growth rates on income levels and other variables, as has been done so far. Rather, the new approach is based on the dynamics of the cross-section distribution of income across countries. As already specified by several authors, also Quah considers technological progress to be a crucial element of economic growth. However, he strongly emphasizes the existence of poverty traps. This means that there are actually at least two – if not multiple –

[17] OECD stands for Organization for Economic Cooperation and Development.

equilibria: a low-income equilibrium which can be seen as a poverty trap, and a high-income equilibrium which covers the rich countries. Figure 2.2 illustrates these twin peaks in a simplified manner.

Figure 2.2 Bimodality



Source: Quah, 1996b

In each year, a certain density plot can be made for the income distribution.[18] Yet, since about the 1980s, the distribution has not looked the same anymore. As will be seen in Chapter 5, even before 1980, the distribution was not normal. It has only one peak at a low income level and the distribution is skewed to the right.[19] Consequently, the middle income class was rather small already then. Looking at which countries belonged to the middle income class before the 1980s, the data were checked for the year 1975. The middle income class, if defined to comprise incomes between \$5,000 and \$10,000, basically covers the countries of Middle and Latin America, some former communist economies as Hungary and Poland[20], some Asian Tigers as Singapore and Hong Kong, as well as a few European countries, for example Malta and Portugal. Since then, however, the middle class has declined and even more, twin peaks have emerged (Quah, 1996c).

Of course, there are also intra-distribution dynamics. Countries which were rich before can be either rich afterwards, or they may have become poor in the meantime. One often-quoted example is Argentina. Such dynamics can be due to external shocks like natural diseases or wars, but they can also be man-made. However, this is not the subject of the article summarized here. Quah tries to give an econometric explanation of the distribution discussed above. He argues on the basis of stochastic difference equations. He does not show how growth models such as the Solow growth model can explain the existence of multiple equilibria.

[18] A more detailed description of methods to analyze the distribution of real per capita GDP will be given in Chapter 4. Please keep in mind that the term "income distribution" referred to in this doctoral thesis concerns the distribution across the countries of the world rather than within them.

[19] This is in line with the basic assumption of income being distributed according to a Gibrat distribution.

[20] Using data of formerly communist countries has to be done carefully. Among the problems are that the prices were not market determined and hence probably not fair for a number of products, the economies were planned, exchange rates may be incorrect, and data may not be reliable. In the Penn World Table, those countries are covered nevertheless. The data are not provided for all formerly communist countries before 1990. Yet, if so those are countries for which benchmark studies helped to offer reliable data. This benchmarking together with the treatment of China due to its economic importance in the world are described by Kravis, Kenessey, Heston and Summers (1975) as well as Heston (2001). In this doctoral thesis, the data from the former communist countries are used as being assumed to be reliable – hence, this doctoral dissertation follows the majority of economic growth analysts keeping those countries in the dataset.

Quah also points to the fact that the concept of conditional convergence as mentioned by Sala-i-Martin (1996) may be misleading. In his eyes, the existence of twin peaks, thus of convergence clubs, directly influences the factor inputs. These will be endogenously determined by each country's convergence club – at least to a large extent. Hence, not the factor inputs determine a country's position but rather the club membership affects the values of these variables. Summing up, the factors that decide on club membership determine everything. As a result, researchers not taking this into account will never find twin peaks. Instead, they will talk about conditional convergence. The "varying degrees of capital market imperfection [...] lead to twin-peaks dynamics in the model" (Quah, 1996c, p. 1053). Quah addresses this aspect further in other articles and papers, as will become apparent in the next section as well as in Chapter 5.

Finally, Galor (1996) provides the last contribution to the convergence debate in the Economic Journal. In his article, Galor suggests that "the convergence controversy may reflect [...] differences in perception regarding the viable set of competing testable hypotheses generated by existing growth theories" (Galor, 1996, p. 1056). In his eyes, both the conditional convergence hypothesis as well as that of club convergence are supported by the neoclassical growth paradigm. According to him, the emergence of club convergence is crucially influenced by human capital, income distribution, fertility, capital market imperfections, externalities, and convexities. Some of these variables are central to the analyses of other economists as will become apparent later on.

According to Galor, convergence in structural characteristics among countries is necessary for absolute convergence. As already seen before in the other articles of the controversy, countries with similar characteristics but different levels of GDP per capita will tend towards the same steady state. Besides, Galor states that in the case of multiple locally stable steady-state equilibria, "a (conditional) club convergence hypothesis rather than a conditional convergence hypothesis would emerge" (Galor, 1996, p. 1058). The assumption of diminishing marginal productivity of production factors is crucial to the conditional convergence hypothesis. Galor argues that heterogeneity of countries in factor endowments in the Solow growth model can lead to multiple equilibria so that the club convergence hypothesis arises. Other sources of club convergence were already mentioned above. In the last sections of the article, Galor investigates the robustness of the convergence hypothesis as well as that of the club convergence hypothesis. He discusses the individual variables advancing club convergence (see above) and also the influence of perfect international capital mobility and technological progress. Galor concludes that club convergence is consistent with the neoclassical growth model, and it is also a robust result.

To sum it up and make a point, this controversy can be seen as a short introduction to the subject of club convergence and its importance in growth theory in general.

In the following section, an overview of theoretical explanations for the twin peaks phenomenon will be given.[21]

2.3 The Theory of Bimodality

As already stated in the introduction to this chapter, the finding of club convergence needs to be added to the stylized facts of economic growth theory. Yet, also for poverty trap analysis, which is directly linked to the twin peaks theory, six stylized facts exist (Azariadis, 2006):

1. two thirds of global income inequality among households are due to international differences; one third is due to intra-country variations;
2. poorer countries seem to catch up with richer ones only in samples dominated by nations in the OECD, East Asia, and South-East Asia;
3. advanced countries grow slightly faster and much more predictably than the world average;
4. LDCs grow a bit slower and less predictably than the world average;
5. the significance of the explanatory variables in cross-country growth regressions is very sensitive to the variables chosen – the only robust variable is investment;
6. if East and South East Asia are excluded from the sample, the group of LDCs does not catch up to the OECD countries unless one controls for a long and not altogether meaningful list of differences in structural features.

These facts were found by several authors. If the number of countries exceeds that of the underlying steady state paths, then a clustering in the cross section distribution could well arise. This is then called twin peaks or multiple peaks. Basically, the share of the world population living in the richest part of the world decreased over time, while that in the poorest part increased (Quah, 2000). Most authors classify two income groups, one characterized by low real per capita incomes and one by high incomes. Cetorelli (2002) points out that in the poor steady state there is a low capital level. The productivity of this capital is difficult to be increased. Hence, the poor countries do not have good chances to get out of poverty and reach the high income peak according to the theory of poverty traps. The poor peak is seen as a poverty trap which is due to the fact that savings are dependent on growth in physical and human capital rather than being constant, due to technology and influences of human capital via productivity growth (Quah 1992). Besides these two income groups, Kejak (2003) describes three growth stages in his article. The first one covers the rich countries, the second one is characterized by low growth rates, and finally, the third stage faces zero growth and hence represents the poverty trap. Switches between the groups – whether two or three groups are identified – are possible; however, switches occur mainly

[21] This chapter explores only a review of theoretical articles and arguments. Empirical findings will be the subject of Chapter 5.

because of external shocks like, for example, wars or the like (Stiglitz, 1987; Becker, Murphy and Tamura, 1990).

Some authors, among them for example Galor (2007), argue that the twin peaks phenomenon is rather a temporary phenomenon than a long run one. In the long run, the equilibrium known from standard growth theory might still be reached. In the short run and the medium run, instead, there are slow growing economies in the vicinity of a Malthusian regime.[22] The fast growing countries are facing sustained economic growth. Additionally, there are also countries in transition from one regime to another. Also Galor (2007) mentions endogenous forces as reasons for switches between the clubs. Such endogenous forces may, for example, be changes in the rate of technological progress, in the rate of population growth, and in human capital formation.

A problem which should not be obliged is that less developed countries often face a trade-off between lower output in the short run associated with higher unemployment using inappropriate technologies and higher future output (Stiglitz, 1987). Altogether, it is difficult to find a way out of the poverty trap. Many authors point out that the poverty trap might be overcome by a sharp increase in investment by development aid (Ben-David, 1998). Stiglitz (1987) also argues that when looking for an optimal development strategy one ought not to look at the current comparative advantage but rather at the dynamic one. If it could be found out which one it is, then exploiting it will open the door for getting out of the poverty trap. Becker, Murphy and Tamura (1990) state that a way out of poverty is only possible in case of “reasonably prolonged good fortune and policies that favor investment” (Becker, Murphy and Tamura, 1990, p. S36). Such temporarily increasing savings may help a country to get out of the poverty trap characterized by a low initial capital stock (Deardorff, 2001).

In many of the articles reviewed in this chapter, models are presented which yield multiple steady states. The models will not be summarized here, the interested reader is referred to the respective articles cited above. However, what is important to note is that some authors mention that models generating twin peaks in real per capita income should also generate twin peaks in other variables.[23]

2.3.1 Reasons for Bimodality

The reasons for bimodality discussed in the literature are manifold, ranging from elementary factors of the Solow growth model to not yet included factors like trade, for example. Table 2.1 summarizes the main arguments for the existence of twin

[22] Malthus‘ theory states that population tends to outgrow the resource base. For more details refer to Ekelund and Hébert (1997).

[23] This could, though, not be verified by the empirics (Ziesemer, 2004). Nevertheless, the idea will be applied in Chapter 5 when some variables possibly yielding twin peaks will be examined for multimodality as well.

peaks and gives an overview of which authors use which arguments. Some authors, for example Galor and Weil (2000) and Galor (2007), believe that population growth is responsible for the formation of the twin peaks. Different rates of population growth lead to the formation of income clubs. Furthermore, Galor and Weil (2000) point out that demographic transition, hence the evolution of population, alongside with an acceleration of technological progress, and increasing investment in human capital enables the transition from Malthusian stagnation to sustained economic growth. The idea of population growth being responsible for the emergence of twin peaks is very plausible. From the data it is known that poor countries tend to have higher population growth rates (see also Chapter 5). A high population growth yields the need for more income growth if per capita income is to be held constant or shall be even rising. Yet, this will be the subject of later chapters.

Just as Galor and Weil (2000), also other authors see technological progress as a main reason for emerging twin peaks (Quah, 2000). Azariadis and Drazen (1990) work with a Diamond model in which there are technological externalities with a threshold property. Including these yields two steady states: one with low labor quality and no growth of per capita income, which is called underdevelopment trap, and one being characterized by higher labor quality and positive per capita income growth.

Another group of authors discussing the reasons for the twin peaks phenomenon concentrate on the savings rate (for example Dalgaard and Hansen, 2004). The savings rate is, just as the population growth rate, a central element of the Solow growth model, one of the most prominent models in growth theory. Generally speaking, poor countries tend to have high interest rates which make it hard to get out of the poverty trap as investments are very expensive then (Quah, 1992). Countries characterized by lower investment rates tend to have lower levels of development (Ben-David, 1998). Ben-David (1998) uses savings depending on the capital stock, whereby the savings rate is negative for very small capital stock levels; though, this assumption cannot be proven by the empirics. Also Stiglitz (1987) points out that differences in per capita income levels are related to differences in the savings rate. Deardorff (2001) concentrates on savings out of wages specifically. When wages rise with the capital stock at a rate that essentially depends on the elasticity of substitution of that particular sectoral production function, then the wage curve equals the per capita savings curve and two stable steady states arise. By increasing the share of wages to be saved, a country can escape the poverty trap in that the savings curve shifts upwards so that only one steady state results. This, however, only happens if the increase in savings is large enough to eliminate the poverty trap instead of a country just moving away from it (Deardorff, 2001).

Table 2.1 The Main Arguments for the Emergence of Bimodality

Argument	Authors
Population growth	Galor and Weil (2000) Galor (2007)
Technological change	Azariadis and Drazen (1990) Galor and Weil (2000) Quah (2000a)
Savings rate	Ben-David (1998) Dalgaard and Hansen (2004) Deardorff (2001) Quah (1992) Stiglitz (1987)
Imperfect capital markets	Quah (1992) Semmler and Ofori (2007)
Depreciation rate	Dalgaard and Hansen (2004) Redding (1996)
Human capital	Azariadis and Drazen (1990) Becker, Murphy and Tamura (1990) Chakraborty (2004) Deardorff (2001) Eicher and Garcia Peñalosa (2004) Galor and Moav (2004) Galor (2007) Grimalda and Vivarelli (2004) Kejak (2003) Quah (1999) Stiglitz (1987)
Consumption preferences	Ben-David (1998) Galor and Moav (2002) Kejak (2003)
Time preferences	Chakrabarty (2012)
History matters	Azariadis and Drazen (1990) Quah (1992) Stiglitz (1987)

A further reason for the emergence of twin peaks mentioned by some authors is imperfect capital mobility. Quah (1992) points out that imperfect capital mobility is a major reason for why poor countries being sufficiently distant from capital-rich countries remain poor. Semmler and Ofori (2007) state that locally increasing returns to scale and capital market constraints yield twin peaks in per capita income. Only countries with developed capital markets can reach a high-development stage according to him.

Looking at the Solow growth model also allows for another theory yielding multimodality. Dalgaard and Hansen (2004) believe that an endogenous rate of depreciation might yield multiple steady states due to endogenous capital utilization despite of a constant savings rate. Both equilibria can face rising growth rates due to a decrease in the depreciation rate along with an increase in productivity of education parameters (Redding, 1996).

Having basically discussed arguments which might stem from the basic Solow growth model, a large class of articles considers human capital, which might also be included in the Solow growth model as shown in Chapter 3. To begin with, Galor (2007) mentions different rates of human capital formation as a major reason for the formation of twin peaks. According to him, existing research says that the levels of income and human capital yield convergence. Also Grimalda and Vivarelli (2004) are of the opinion that the degree of endowment with a skilled labor force determines in which steady state a country will end up. While in the long run income inequality in the transition phase is possible, in the short run small scale Kuznets curves may arise.[24] According to the authors, if the degree of skill endowment is far too low in the economy, it will be trapped in a low growth equilibrium. Then, decreasing income inequality will result with small scale Kuznets curves in the short run. However, as in both situations there are small scale Kuznets curves in the short run, policy makers have it difficult to figure out in which situation a country is and hence which policy would be needed. This adds to the above mentioned arguments that it is difficult for a country to get out of the poverty trap. Grimalda and Vivarelli (2004) advise to fight the poverty trap by eliminating the scale effects in the dynamic of the income inequality. They propose to reach this by slowing down the migration of the workforce towards the skill-intensive sector.

Another argument for human capital being responsible for the emergence of twin peaks stems from Azariadis and Drazen (1990). The authors point out that the accumulation of human capital can enforce threshold externalities. However, contrary to Grimalda and Vivarelli (2004), they are convinced that human capital alone is not sufficient to yield twin peaks in a growth model unless the initial values of the average level of human capital are appropriate. Human capital usually has to induce increasing returns to scale somewhere in order to be responsible for multiple peaks. The authors state that “multiple, locally stable balanced growth paths will exist in this model economy whenever individual yields on human capital rise with the average quality of labor” (Azariadis and Drazen, 1990, p. 515).

Several authors point to the importance of learning in the process of economic growth. Generally speaking, Stiglitz (1987) states that learning is important: learning by doing, learning by learning, and localized learning with some spillovers will lead to multiple equilibria. According to Becker, Murphy and Tamura (1990), societies with limited human capital choose large families and invest little in each

[24] The Kuznets curve gives the relationship between per capita GDP and the income distribution within a country. According to the Kuznets-U-hypothesis, a transitional process is accompanied by an increasing level of inequality in the beginning. This inequality will decline again in the ongoing process of transition (Gabler, 2014).

member. Hence, two steady states arise. The first one is characterized by small families with high and growing human capital and physical capital and low rates of return to human capital investment as the level is already high. The second steady state, on the contrary, consists of countries with large families, little human capital and also little physical capital. The rate of return to human capital is high due to the low level. Overall, there are increasing returns to human capital because "education and other sectors that produce human capital use educated and other skilled inputs more intensively than sectors that produce consumption goods and physical capital" (Becker, Murphy and Tamura, 1990, p. S13).

According to Kejak (2003), sustained growth can only be reached if both physical and human capital are growing. In his article, Kejak presents a "two-sector endogenous growth model with threshold externalities in the process of human capital accumulation" (Kejak, 2003, p. 795). Within the framework of the model, there may be underdevelopment traps and sustained growth. During the transitional phase, Kejak (2003) distinguishes three stages: the stage of low growth (this is the stage before the productivity miracle occurs – productivity, hence, is low in this stage); the take-off stage (in this stage the miracle occurs so that "the economy switches from low productivity to high productivity in the education sector" (Kejak, 2003, p. 782)); and finally, the stage of high growth (this stage occurs after the productivity miracle – productivity is high in this stage). Within the framework of the model, a temporary underdevelopment trap may arise. The reasoning is that there is no growth in human capital combined with slowly declining growth in physical capital. This trap is only temporary as it is "followed by a sudden transition to a sustained or quasi-sustained growth path" (Kejak, 2003, p. 795). The second phenomenon covered by Kejak's model is seemingly sustainable growth. This is the phase in which "the economy temporarily goes through a transition with positive growth of human capital but is finally trapped in a zero growth stage" (Kejak, 2003, p. 795). There might also be a slowdown in productivity growth. This is the case when there is a temporary decline in growth rates during the transition from low to high growth. If there are increasing returns to education due to an increasing effect of externalities, people are likely to spend more time on skills improvement. This makes total productivity growth decrease. However, over time the higher skills can be used to increase productivity growth again. Summing up, Kejak "provides an explanation for the productivity slowdown as a temporary phenomenon during the transition to a stage of higher growth of an economy facing a new "industrial" revolution" (Kejak, 2003, p. 795).

Another aspect of human capital despite of learning is mentioned by Chakraborty (2004). He states that high mortality may lead to being caught in a poverty trap. Mortality can be seen as destruction of human capital. If it is high, it is a disincentive to investments, it blocks productivity, and it has a negative impact on the level of education in this country and hence again on human capital. If mortality is introduced as an endogenous variable in a growth model, threshold effects may arise in the human capital technology. Thus, the rate of return to human capital will be lower and people will be impatient about the return on capital investment so that the savings rate will be low just as the investment rate. Thus, economic growth will be low if not zero or even negative.

Quah (1999) also adds to the large discussion on human capital as a factor yielding twin peaks in pointing out that clubs remain because ideas (also part of human capital) can freely spread but not across clubs. This shows that the view on human capital is widely spread – many aspects fall into that category and it will be discussed later on in this doctoral thesis how difficult it is to define and then to measure human capital.

After having looked at arguments that might lead to twin peaks in the framework of the Solow growth model as well as including human capital, three other types of arguments shall be considered. First, Eicher and Garcia Peñalosa (2004) point out that introducing endogenous institutions such as imperfect property rights in a growth model might yield multiple equilibria. In the presence of imperfect property rights, profit maximizers have incentives to improve the institutions. This, then, leads to the emergence of a two-camp world. A second class of arguments applies to trade. Deardorff (2001) states that convergence is more likely without trade, even though this will not be the higher steady state. He includes trade in the form of multi-good Heckscher-Ohlin trade[25] in the Solow growth model and then finds multiple peaks. Also Galor and Moav (2004) point out that international trade has widened the gap between the technological level as well as the skill abundance of industrial and non-industrial economies. Consequently, sustained differences in income per capita across countries result. International trade and technological differences may explain differences across countries and trade is found to have an influence on human capital via skill abundance (Galor and Moav, 2004).

Finally, a last class of arguments explaining the emergence of twin peaks are preferences. Here, two subgroups can be formed: the first one covers preferences for consumption and the second time preferences.[26] Galor and Moav (2002) argue in their article that after a stimulation of the “natural selection” due to the long period of economic stagnation, a country may enter in the transition to sustained growth. This “natural selection” the authors view as the basis of the evolution of the human species. Furthermore, they argue that the evolution of the human species can be seen as the impulse for the movement from a period of stagnation to sustained growth. The key element here is demographic transition (Galor and Moav, 2002). Twin peaks are above all the result of individuals’ preferences for consumption and for the quantity and quality of their children. In a country with higher preferences

[25] Heckscher-Ohlin trade describes trade based on international differences. According to this theory, a country which is abound in the production factor labor will specialize in the production of labor-intensive goods. On the contrary, a country being abound in the production factor capital will specialize in the production of capital-intensive goods. By this Heckscher-Ohlin trade, an international equalization of factor prices tends to be reached (Gabler, 2014).

[26] Consumption preferences refer to the composition of the basket of goods consumed, whereas time preferences rather focus of the timing of decisions (on consumption, on having children and so on).

for the quality of children an evolutionary advantage arises. If the number of such individuals increases in a country, there is technological progress and in the end there will be sustained growth. Summing up, preferences determine which steady state can be reached.

Also Ben-David (1998) argues on the basis of consumption for the existence of twin peaks. He states that by including subsistence consumption into the neoclassical growth model with labor-augmenting technological progress, convergence clubs will arise at the bottom and at the top of the distribution. Basically, a poor country, which is sufficiently poorly endowed and whose inhabitants deplete their capital stock to survive, experiences negative growth. Thus, the countries in the poor club (that is in the poverty trap) will survive on subsistence levels alone. Within the poor group of countries, there will be downward convergence due to subsistence consumption (Ben-David, 1998). This is a new argument in the twin peaks theory.

Apart from consumption preferences, also time preferences may yield twin peaks. Chakrabarty (2012) argues that it is more realistic to assume endogenous time preferences which lead to poverty traps, hence multiple peaks. He describes a model which explains “why two economies that have identical production technologies and identical preferences may converge to different levels of income depending on initial conditions” (Chakrabarty, 2000, p. 2) due to different time preferences.

Having reviewed the reasons for the emergence of twin peaks, another argument facing the club membership should be considered. As noted by several authors, the club membership depends to a large extent on history. Quah (1992) states that history and geography can determine the starting position and explores how likely it is to break through the borders determining where a country will end up. According to Azariadis and Drazen (1990), it is decisive where a country starts off. If the country starts off below a critical value of the capital stock, the low income steady state will be reached. Alike, when starting off above it, the high income equilibrium will be reached. Also Stiglitz (1987) points out that the starting position matters, especially for those countries being caught in the poverty trap. These findings again reinforce the above mentioned view that getting out of the poverty trap might be very difficult.

2.4 Conclusion

In this chapter, the theoretical background on which this doctoral thesis is based was given. To begin with, the basic terms important in the twin peaks theory were defined. These terms include economic growth and economic development, income inequality, bimodality, poverty, and poverty traps. In particular, it was worked out that even though it is desirable to analyze worldwide individual income inequality, growth models need the factor mobility level between countries. This is, however, only existent within countries. Between countries there are several obstacles to factor mobility: national borders or different languages, for example. For this reason, it is admissible to analyze income inequality across nations by use of economic growth models.

Thereafter, an overview of the convergence debate was given which can be seen as the official starting point of the twin peaks discussion. Here, it was worked out that the twin peaks hypothesis is indeed an alternative to the convergence hypothesis which was central to convergence discussion before. According to the convergence hypothesis, there is only one steady state towards which countries should converge. In contrast, the discussion on twin peaks brought up the possibility of having two stable steady states instead.

The purpose of Section 2.3 was to give a review of the past literature on bimodality. It summarized a large number of articles which are basic to the twin peaks discussion. In this chapter, the focus was on theoretical articles. The empirical articles will be central to discussion in Chapter 5. First, general aspects were summarized followed by more specific reasons for the emergence of twin peaks mentioned in the literature. It was shown that there are many ways to reach multiple steady states in basic growth models.[27] These arguments are not exclusionary but may be combined.

Generally speaking, it became obvious that human capital is the factor which is treated in a huge amount of studies of the twin peaks phenomenon. Whereas the arguments of human capital yielding twin peaks are very plausible, in this doctorate it shall be checked whether including an endogenous savings rate or an endogenous population growth rate in the basic Solow growth model indeed yields two stable equilibria, just as Solow (1956) claims. These arguments were also treated in the literature, however Solow's claim was not investigated yet. This shall be done this doctoral thesis. Additionally, due to its importance in the literature, human capital will also be considered empirically in Chapter 5.

Before the empirical analyses will be presented in Chapter 5, the next chapter will provide the theoretical framework for them. As a first step, Chapter 3 will help to answer the question of whether the Solow growth model is indeed able to yield bimodality from a graphical point of view.

[27] It should be mentioned that the literature review given in this chapter does not at all claim completeness. As there is a huge mass of contributions to that subject, especially in the recent years, the focus was on the most important authors and on giving a broad overview of potential explanations for the twin peaks phenomenon.

3 Growth Models Capturing Bimodality

In the previous chapter an overview of the twin peaks literature was given. Chapter 2 dealt with the theoretical literature while the empirical studies will be treated in more detail in Chapter 5. The purpose of this chapter is to lay the foundations for the examinations in order to check whether the Solow growth model is indeed able to capture bimodality just as Solow (1956) proposes.[28] Section 3.1 will deal with the Solow growth model and a broad overview of the critique of it working out why it still has such a relevance that it is worth being the center of concern in this doctoral thesis. The Solow growth model is a neoclassical growth model. The term “neoclassical” is due to the production function to be used in the model. This production function allows for factor substitution, hence for producing a certain amount of output by different combinations of input factors (Maußner, 1996).

As outlined above, this doctoral thesis aims to examine Solow’s claim (1956) that his growth model is able to yield multiple equilibria. This hypothesis is supported by Maußner (1996) as well. However, none of the authors goes into detail to really prove this claim. They rather apply verbal analysis for that purpose. This chapter is supposed to give an overview of the Solow growth model as it is described in standard macroeconomic literature. It forms the basis for the analyses presented in the Chapters 6 and 7. In Section 3.1, the Solow growth model as the main framework to be used will be presented. Section 3.2 will show in general how the Solow growth model can yield twin peaks. This will be done by geometrical and verbal analysis. It should be noted, however, that there are many different ways to get bimodality within the Solowian framework. Here, only one alternative for each factor will be considered.[29] Section 3.3 will describe a modified version of the Solow growth model including human capital, while in Section 3.4 the possibilities to capture twin peaks within this framework shall be discussed. Section 3.5 will conclude this chapter.

3.1 The Solow Growth Model

3.1.1 Assumptions

The Solow growth model was developed by Robert Solow in 1956. The model assumes a closed economy without state activity. On the goods market, there is perfect competition and only one homogenous good is produced.

There are two production factors, namely capital and labor. Also their markets are characterized by perfect competition. Technology is assumed to be exogenous in

[28] It is well known that the neoclassical growth model has some disadvantages. For example, the problem of unemployment is not considered in this model world.

[29] Several authors tried to describe models capturing twin peaks. The theoretical literature was already described in the previous chapter; hence, the reader is referred back to Section 2.3.

the basic Solow growth model. Technological progress may be labor-augmenting, that is Harrod-neutral (this means entering as $Y = F(K, AL)$ in the model), capital-augmenting, that is Solow-neutral (entering as $Y = F(AK, L)$ in the model), or Hicks-neutral (entering as $Y = AF(K, L)$ in the model) (Allen, 1967).[30] Here, it is assumed to be labor-augmenting, hence Harrod-neutral. The production function is of the Cobb-Douglas type. In the entire economy, both the consumers and the companies are price takers while the former act to maximize their utility and the latter to maximize profits. All markets are characterized by perfect information. As there is no international trade in the economy, savings have to be equal to investment, which is a basic neoclassical assumption. Additionally, savings are assumed to be a constant fraction of income (Maußner, 1996).

3.1.2 The Steady State

The Solow growth model is based on a constant-returns-to-scale production function with labor-augmenting (Harrod-neutral) technological progress:

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \tag{3.1}$$

where Y is income, K is capital, A is technology, L is labor and α is the capital share $(0 < \alpha < 1)$. The production function can also be written per efficiency unit of labor:

$$y = f(k) = k^{\alpha} \tag{3.2}$$

with $f'(k) > 0, f''(k) < 0, \lim_{k \to \infty} f(k)k^{-1} = 0$, where $y = \frac{Y}{AL}, k = \frac{K}{AL}$. In addition, there are decreasing marginal products of K and $AL.$ Next to the constant-returns-to-scale assumption, there are further assumptions about the evolution of the inputs. Labor and technology grow exponentially:

$$\frac{dL(t)}{dt} = nL(t) \Leftrightarrow L(t) = e^{nt}L(0), \tag{3.3}$$

$$\frac{dA(t)}{dt} = gA(t) \Leftrightarrow A(t) = e^{gt}A(0), \tag{3.4}$$

where g_L and g_A describe the growth rates of labor and technology respectively, n indicates the population growth rate, and g the rate of technological progress. As mentioned in the previous section, the savings rate s is assumed to be exogenous and constant. The same holds for the depreciation rate δ. Central to the Solow growth model is the importance of capital accumulation for economic growth. The capital stock evolves according to Equation (3.5):

$$\dot{K} = sY - \delta K, \tag{3.5}$$

[30] "[Technological progress] can take various forms. Inventions may allow producers to generate the same amount of output with either relatively less capital input or relatively less labor input, cases referred to as capital-savings or labor-saving technological progress, respectively. Inventions that do not save relatively more of either input are called neutral or unbiased" (Barro and Sala-i-Martin, 2004, p. 52).

where $s \in [0,1]$. In efficiency per capita terms, the capital accumulation Equation (3.5) can be written as:

$$\dot{k} = sf(k)(n + g + \delta)k. \tag{3.6}$$

In the steady state, capital accumulation will be equal to zero, so that the following holds:

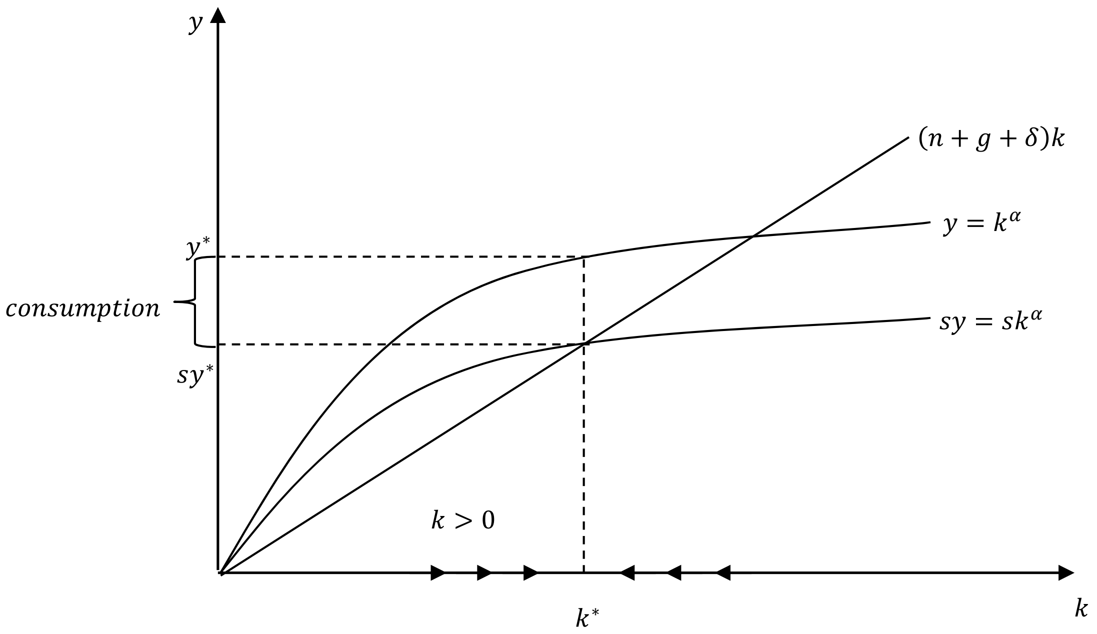
$$\dot{k} = 0. \tag{3.7}$$

Stars as a subscript indicate steady state values so that this implies:

$$sf(k^*) = (n + g + \delta)k^*. \tag{3.8}$$

In the steady state, income per efficient unit of labor, $y^* = f(k^*)$, is constant. Along a balanced growth path, per capita income as well as capital per capita grow at the rate of technological progress, g (Jones, 1998).

Figure 3.1 The Solow Growth Model



Source: Jones, 1998

The Solow growth model can also be analyzed graphically. This is done in Figure 3.1. The shaded area indicates capital deepening, as here capital accumulation is positive so that capital per efficient worker is increasing. If only K grows instead of k, hence the absolute capital stock rather than the one in efficiency terms, there is capital widening.[31]

As indicated by the arrows along the k-axis, starting off to the left of k^*, a country is automatically converging towards k^* due to the dynamics described above.

[31] Capital widening means that k is constant in the steady state at k^*.

Accordingly, when starting off to the right of the steady state, it will converge downwards to k^* in the long run.

In the basic Solow growth model as it appears here, there is one steady state. The aim of this doctoral thesis is to check whether the Solow growth model can indeed be changed such that two stable steady states arise. However, in this section, the Solowian steady state shall be reproduced first.

As stated above, in the steady state Equation (3.8) holds. As $f(k^*) = y^* = k^{*\alpha}$, this can be inserted into Equation (3.8):

$$sk^{*\alpha} = (n + g + \delta)k^*. \tag{3.9}$$

By rearranging the equation, the steady state values k^* and y^* can be determined:

$$k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{3.10}$$

Inserting this result into Equation (3.2) yields the steady state value of y:

$$y^* = k^{*\alpha} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{3.11}$$

The steady state values given by Equations (3.10) and (3.11) indicate those given in Figure 3.1. It can be read off that the Solow growth model predicts an economy to converge towards a stable long run capital stock and income level per efficient unit of labor – both are determined by s, n, g, δ, and α according to Equations (3.10) and (3.11). In the long run, the growth rates of per capita[32] capital and income are given by

$$g_y = g_k = g\ , \tag{3.12}$$

hence the economy grows along a balanced growth path at the rate of technological progress (Jones, 1998).

3.1.3 Implications

As could be seen in the previous subsection, technological progress is the only source of sustained per capita income growth because it can offset the decreasing marginal product of capital. Policy changes have no long run growth effects in the Solow growth model but can have level effects. Different growth rates arise because some countries are further away from their steady states than others. Another important implication of the model is that the further an economy is below its steady state, the faster it should grow, and vice versa (Jones, 1998).

The Solow growth model is based on a number of preconditions. First, in the long run, the economy will approach its steady state independent of its initial conditions. Second, the steady state level of per capita income depends on the savings rate

[32] The term “per capita” refers to per efficient unit of labor in this doctoral thesis.

and the population growth rate: the higher the rate of savings, the higher the steady state level of per capita income will be; conversely, the higher the rate of population growth, the lower the steady state level of per capita income. Next, the steady state growth rate of per capita income depends only on the rate of technological progress, as stated above. It is independent of the savings rate and the rate of population growth. In addition, the capital stock grows at the same rate as income. Thus, the capital-to-income ratio is constant. Finally, in the steady state, the marginal product of capital is constant, whereas the marginal product of labor grows at the rate of technological progress (Mankiw, 1997).

By use of his model, Solow (1956) was able to answer the question crucial to all studies of economic growth: why are they so poor and we so rich? According to him, it is due to a higher savings rate and thus investments, higher capital accumulation (an increase in labor productivity), and low population growth that make up economic growth. The empirics seem to support his model. Mankiw, Romer and Weil (1992), for example, prove the Solow growth model by use of cross-country data provided by the Penn World Table 5.1, which is the predecessor of the dataset being used in this doctoral thesis. However, recent empirical studies showed that there are rather twin peaks in the real per capita GDP data (please refer to Chapter 5). Hence, the question is whether the basic Solow growth model is based on realistic assumptions. This statement is to be examined in this work. To begin with, this will be done graphically and verbally in this chapter.

3.1.4 Discussion and Relevance of the Solow Growth Model in Modern Growth Theory

The Solow growth model stems from the year 1956, in which Robert Solow published his article “A Contribution to the Theory of Economic Growth”. It is one of the most widely used growth models in economic theory. It explains differences in growth by differences in investment rates, population growth rates, and technological growth rates. Due to exogenous differences in technology, per capita incomes may differ across countries. As stated above, the Solow growth model offers answers to the question of why there is a two-camp world. The model states that “we invest more and have lower population growth rates, both of which allow us to accumulate more capital per worker and thus increase labor productivity” (Jones, 1998, p. 39). Furthermore, sustained growth is reached via technological progress. The reason for this is that technological progress may counteract the falling marginal product of capital in the long run (Jones, 1998).

The exogeneity of technology as well as savings and population growth is often criticized. It refers to the fact that these variables are assumed to come from somewhere outside the model instead of being determined within it. McCallum (1996), for example, criticizes that “Solow’s paper [does] not include dynamic optimizing analysis of households’ saving behavior, however, but simply [takes] the fraction of income saved to be a given constant" (McCallum, 1996, p. 49). It is not very realistic to assume homogenous households; hence, an equal exogenous savings rate for all people in all countries of the world is not realistic either. The same accounts also for the assumption of exogenously give population growth. In the following section, it will be discussed why population growth should rather be considered to be endogenous. Razin and Sadka (1995) show, for example, that population growth is indeed dependent on the level of income, among a number of other factors, of course. Another critique concerns the production function underlying the Solow growth model. The production function is Cobb-Douglas and quite restrictive. The CES function might be more unobstructed (Masanjala and Papageorgiou, 2004); Allen (1967), for example, analyzes the Solow growth model also by use of a CES function.

Based on the above mentioned critiques[33] the new growth theory developed. The new growth theory uses endogenous growth models taking into account that people decide on household consumption and hence on savings; in addition (or instead), also technological progress or population growth might be endogenous.

Despite the critiques of the neoclassical growth model and the emergence of new growth models trying to get rid of the above mentioned disadvantages, the Solow growth model is still widely used in economic growth research. One of the most famous articles trying to prove the relevance of the Solow growth model is the one by Mankiw, Romer and Weil (1992), who show that the Solow growth model is able to explain the empirics; at least it gives the right signs. The magnitudes are not correct for the basic model. Yet, if it is augmented by human capital, the Solow growth model is able to describe the cross-country data. Under the assumptions of the model, savings ought to be equal to investment which is a realistic assumption for most of the countries. Exceptions are countries which are financially globalized. Yet, only a few countries belong to this group. Hence, the neoclassical assumption may well be followed. In addition, none of the alternative models reached the same reputation as the Solow growth model did. There is a huge class of endogenous growth models. Nonetheless, the Solow growth model is still extensively used for growth analyses.

A look at the literature, even though this is only a very brief overview of the vast amount of articles which can be found, shows that the Solow growth model is still central to research. Some of the authors use it in order to analyze local growth aspects, for example Richardson (1973) as well as Durlauf, Kourtellos and Minkin (2001), who formulate a "local" Solow growth model. Other authors try to extend the model by further variables. Karras (2010), for example, uses the Solow growth model extended by land.

From these examples, it becomes clear that the model still influences economic growth research and it continues to yield several options to augment the model in

[33] This is just an overview of the most prominent criticism. For a more detailed discussion the interested reader is referred to Romer (1996), for example.

order to overcome the shortcuts accompanying this neoclassical growth model. In addition, there remain a number of “open questions” surrounding the Solow growth model. A statement by Solow himself shall be examined in this doctoral thesis: he states in his article that his basic model is able to yield multiple peaks. He argues that in the framework of his growth model there might be two stable steady states due to a different form of the production function, an endogenous savings rate, or an endogenous population growth rate (Solow, 1956). In the following section, the latter two ideas will be pursued: the Solow growth model will be modified graphically and argumentatively so that it can be explored whether there are indeed twin peaks dynamics.

3.2 Capturing Bimodality in the Basic Solow Growth Model

In the previous section, the Solow growth model and the resulting steady state were summarized. With the basic neoclassical assumptions, a single steady state, hence unimodality will emerge. Nevertheless, as Solow states, all neoclassical growth models are able to explain bimodality, and so is the Solow growth model. This is the subject of this section.[34] In order to capture twin peaks, the underlying assumptions and equations need to be modified. In this section, two alternative modifications will be provided: first, an endogenous savings rate and thereafter, an endogenous population growth rate.[35] It should be kept in mind that here only one possible modification for each of the two variables will be shown.

3.2.1 Savings as a Source of Bimodality

To begin with, the savings rate as a potential source of bimodality will be the center of concern. In the previous chapter, the savings rate was already found to be an explanatory factor for twin peaks when reviewing the literature on the subject. Galor (1996), for example, explained the phenomenon via the savings rate. Basically, most assumptions underlying the Solow growth model can remain unchanged. The production function still exhibits diminishing returns to scale.[36] In the basic model, it was assumed that savings are a constant fraction of income. This is quite unrealistic. As will be shown later on, the savings rate is indeed positively related to income. Hence, it is more realistic to have an endogenous

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[34] However, it should be kept in mind that the examination in this chapter is restricted to a graphical and verbal analysis. Analytical solutions are the subject of the Chapters 6 and 7.

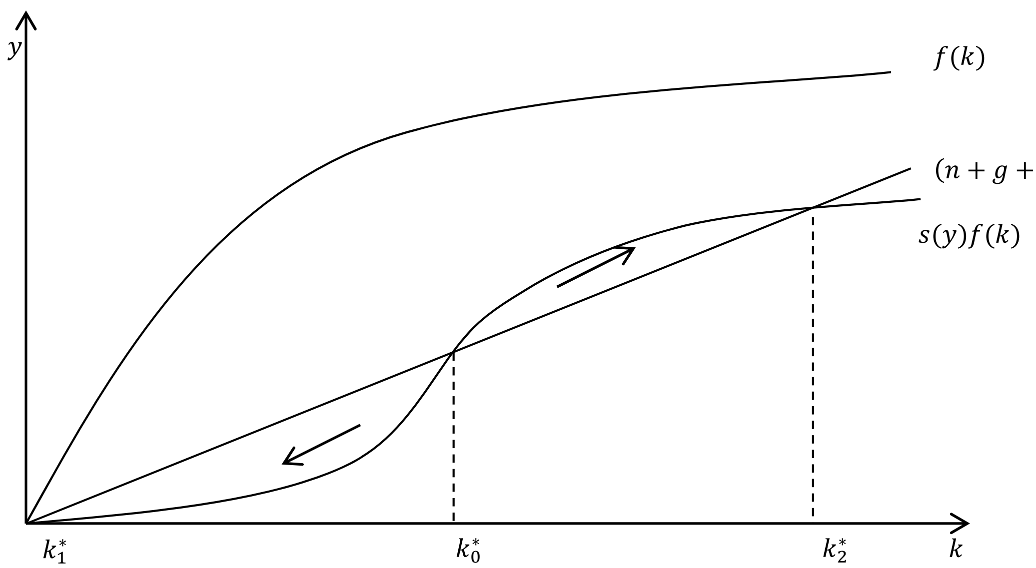
[35] It is also possible to assume an endogenous rate of technological progress. Some studies assume threshold externalities in the rate of technological progress. Here, two other quite plausible alternatives shall be elaborated on, namely endogenous savings and population growth. Already here, it should be noted that the probability of just one factor explaining the emergence of twin peaks is rather low. Instead, it is much more likely that there is a combination of endogenizations which in the end yields twin peaks.

[36] Instead, also a different production function could be assumed. However, this will not be done here and instead an endogenous savings rate is assumed.

savings rate or a threshold effect. In this section, the savings rate is assumed to be dependent on income: $s(y)$. Endogenizing it will alter the shape of the $sf(k)$ curve in Figure 3.1. Figure 3.2 shows the modified Solow growth model capturing twin peaks with s being a logistic function of y.

In Figure 3.2, the savings curve cuts the break-even investment line three times. Stability within the Solowian framework arises if the savings curve is above the $(n+g+\delta)k$-line[37] which indicates capital destruction. If capital accumulation in the form of savings exceeds capital destruction, the economy grows and is pushed towards the right until a steady state is reached (for example k_1^* or k_2^* depending on where the economy starts off). k_1^* capital destruction is higher than the capital accumulation. This implies that the economy shrinks and is pushed towards the left until k_1^* is reached. To the right of k_0^*, the dynamics again push the economy to the right. Figure 3.2 shows these dynamics. It can be concluded that k_1^* and k_2^* are locally stable steady states (a small move to the left or to the right of them always push the economy back to the steady states); k_0^*, on the contrary, is instable. A small move away from the steady state immediately pushes the country towards k_1^* or k_2^*.[38]

Figure 3.2 The Solow Growth Model with Two Stable Steady States



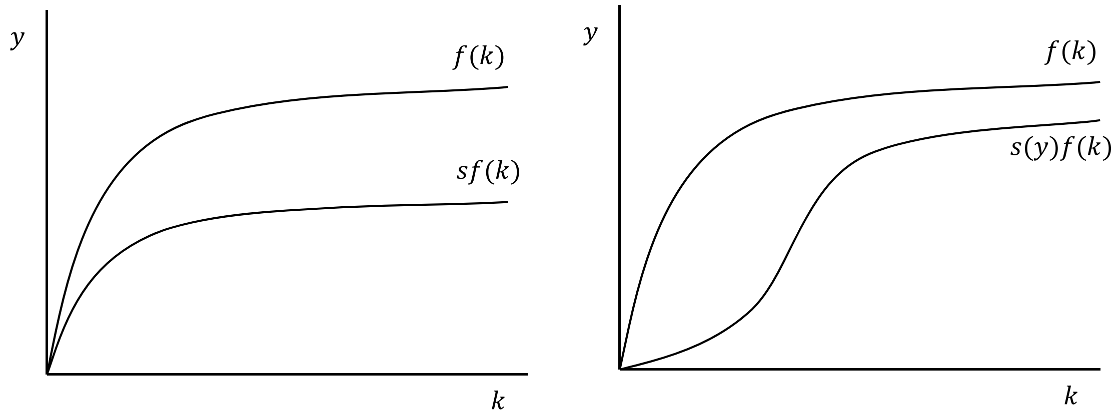
Source: Stiglitz (1987), pp. 136f

[37] This line is also called investment requirement line.

[38] The lowest steady states seems to be at an income of zero. This would mean that the country would die off. As this is not a very realistic assumption, it is also possible to shift the $s(y)f(k)$ curve upward so that the points of intersection do not appear to include the origin. Alternatively, one might assume that the origin does not represent $y = 0$ and $k = 0$ but some small amount of y and k. Whether and where the curves intersect depends, of course, on the exact positions of the curves. The figure here shall be seen as an example.

What accounts for the form of the savings function? The basic assumptions remain the same; however, instead of assuming savings to be a constant fraction of income, s is a function of y. The motivation behind this is that the assumption of homogeneous individuals should be relaxed to the more realistic state of individuals with heterogeneous preferences. Though, it should be noted that the logistic S-function as described below is also possible with homogenous individuals. Nevertheless, heterogeneity is assumed here. People tend to choose different savings rates in accordance with their income and hence also savings rates of countries should differ depending on the income level.[39] In this context, the savings gap should be mentioned, which was a central aspect of development economics in the 1950s. Developing countries are often not able to generate the savings necessary to meet the investment requirements. In order to overcome this savings gap, capital aid by the industrial countries is required (Szirmai, 1997). Another possibility is to have debt agreements with industrial countries. Consequently, the savings function is no longer concave over the whole range but rather convex in some regions despite of the neoclassical characteristics of the production function. This is shown in Figure 3.3.

Figure 3.3 Homogeneous and Heterogeneous Savings



Panel a: Homogeneous Savings | Panel b: Heterogeneous Savings

The savings rate is positively dependent on income over the whole range of per capita income. This finding is supported by a number of authors, among them Steger (2001) as well as Harms and Lutz (2004). Yet, “the largest increase in the savings rate occurs[, however,] with the transition from low-income to lower

[39] Income may be assumed to be total income or one can distinguish between capital income and labor income (Quah, 1996a). This distinction seems plausible but is not of crucial importance in this doctoral dissertation – the conclusions are independent of that, at least in this case. Galor (1996), instead, uses it as an explanation for multiple equilibria: if savings out of labor income are larger than out of capital income and if production technology is CES with a low elasticity of substitution or non-CES, multiple equilibria exist.

middle-income countries" (Steger, 2001, p. 4). A crucial element for explaining why the savings rates tend to be very low at low levels of income refers to the theory of subsistence consumption. Harms and Lutz (2004) assume a zero savings rate until a certain subsistence level of income is reached in order to finance subsistence consumption. Also Christiano (1989) focuses on subsistence consumption as an explanatory factor of low savings rates at low levels of income. According to him, subsistence consumption leads to a time-varying intertemporal elasticity of substitution. This, in turn, is able to explain why savings rates tend to be low in the early phase of the growth transition. Even though the interest rates and hence the rates of return to capital may be high in poor countries, capital accumulation may be low because of the very low intertemporal elasticity of substitution as a consequence of subsistence consumption at low capital / income levels (Chang and Hornstein, 2011).

Ogaki, Ostry and Reinhart (1996) present a table comparing GNP per equivalent adult in dollars of 1985 as the average of the years 1980 to 1987 with the personal savings as a percentage of GDP for four groups of countries: low-income countries, lower middle-income countries, upper middle-income countries, and high-income countries.[40] The data are presented for 58 countries.[41] Yet, here only the averages of these groups shall be considered. The corresponding data points are shown in Figure 3.4. It becomes obvious that, as Steger (2001) states, the largest change in the average savings rate is indeed observable between the low-income countries and the lower-income countries. Thereafter, the changes become only marginal. Connecting the points and using the view of Harms and Lutz (2004) that the savings rate may be even zero[42] at very low subsistence levels of income in combination with the views of Christiano (1989) and Chang and Hornstein (2011) that the savings rate may be low in the early phase of the growth transition, the assumption of a logistic savings function seems to be rather plausible. In Figure 3.4, the points A to D represent the data points obtained from the table by Ogaki, Ostry and Reinhart (1996).[43] The idea of a zero or close to zero savings rate indicated by the extrapolation to the left of point A is based on Harms and Lutz (2004) and Christiano (1989). Beyond point D, the function is also extrapolated based on the curvature determined by the connection of points B to D.

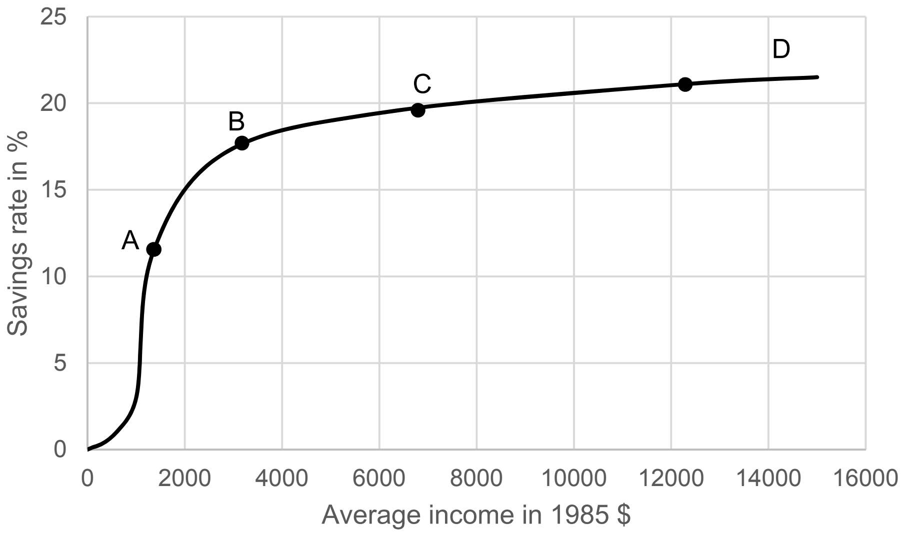
[40] The income groups are defined by the World Bank (1994): the low-income group has an income of $675 or less; the lower-middle-income group has an income of $676 to $2,695; the upper-middle-income group has an income of $2,696 to $8,355; and the high-income group has an income of $8,356 or more. What is important to note is that the authors concentrate on personal savings only. Hence, they totally ignore the corporate savings and the state savings. Especially the former is a decisive source of savings. Hence, the results of Ogaki, Ostry and Reinhart (1996) have to be judged according to this shortcoming of sticking to the personal savings only.

[41] An overview is given in the Appendix (A.1).

[42] This refers to net savings if the population stagnates, or, in the case of gross savings, only for the short run. In the long run, gross savings have to be positive.

[43] Point A is the data point for the low-income group, point B for the lower middle-income group, point C for the upper middle-income group, and point D for the high-income group.

Figure 3.4 The Savings Function



Source: Own representation based on Ogaki, Ostry and Reinhart (1996), Christiano (1989), and Harms and Lutz (2004)

Knowing this relationship, it will be shown that the resulting $s(y)f(k)$ curve will look as in Figure 3.2.[44] For this purpose, one first of all has to look at the production function itself. It is a monotonically growing function where the derivatives are as follows: $f'(k) > 0$ and $f''(k) < 0$. Economically spoken, the production function exhibits diminishing returns to effective labor. If the production function is indeed monotonically growing, then multiplying this curve with the logistic savings function again yields an S-curve. However, it may be stretched.

As stated above, there are three intersection points in Figure 3.2, two of which represent locally stable equilibria – one with a high capital-labor ratio and high income, and one with a low level of income or even zero. It becomes apparent that history is important in this model. Depending on the initial k, a country will either end up in the low-income equilibrium or in the high-income steady state. A country with $k < k_0$ has only one possibility to achieve k_2^*: the “Big Push” strategy. This refers to a comprehensive industrialization plan in which investments in the capital goods sector, the intermediate sector, and various consumer goods industries take place simultaneously (Szirmai, 2005). Big Push is a key concept in development economics. To reach self-sustaining growth, a strong investment incentive is needed which requires a massive capital input in many sectors simultaneously

[44] It should be pointed out that the basic equations of the Solow growth model are still valid. The new feature is that the savings rate is determined endogenously according to a logistic function. A logistic curve is likely to result from introducing a heterogeneous population into a simple growth model (Castanova, 1999). This condition is fulfilled, as the individuals are now assumed to be heterogeneous, especially with respect to the choice of the savings rate. The general form of the logistic function is $\mu_x = c + \frac{ae^{bx}}{1+\alpha e^{bx}}$. Whether the savings function indeed follows a logistic form will be the subject of Chapter 6.

(Gabler, 2014). One of the most important cofounders of the Big Push debate is Gershenkron (1992). Also Chenery (1980) argues that problems of economic development would be caused by a lack of capital. He formulates the two-gap model[45], which determines two main limits to output growth: a trade limit and a saving-investment limit. Capital imports in the form of foreign exchange are needed, according to him, in order to reach sufficient saving for investment in the end (Chenery, 1980).

Such a Big Push investment can "push" a country beyond k_0 so that the process of moving towards k_2^* can start. However, it should be kept in mind that this is a very expensive strategy which has to be carefully implemented. Hence, it is not as easy as it might seem. If it were, why then are more than half of the countries still caught in the low-capital-per-labor equilibrium? In fact, there are countries which succeeded in getting out of the low-income equilibrium, namely those countries experiencing a growth miracle (for example Japan, the Asian Tigers). However, many countries are caught in a poverty trap.[46] What does this mean? It implies that even though a country succeeds in increasing its k in the short run (though not as much as by the Big Push), in the long run the economy is forced back towards the low-income equilibrium. The only chance of escaping this trap is by a large capital investment pushing the country beyond k_0.

It should be mentioned that the idea of the Big Push did not really prove to be successful in the past. Easterly (2002) shows that even though a lot of well-intentioned trials to foster growth by development aid in the form of investments in machines, in human capital in the form of education and health and so on did not prove to be successful. The receiving countries did not reach growth as theory suggests. "The problem was not the failure of economics, but the failure to apply the principles of economics in practical work" (Easterly, 2002, p. xii). According to Landsburg (2012), most of economics can be summarized in four words: ""People respond to incentives." The rest is commentary" (Landsburg, 2012, p. 3). This incentive argument is also used by Lucas (2002, p. 17): "For income growth to occur in a society, a large fraction of people must experience changes in the possible lives they imagine for themselves and their children, and these new visions of possible futures must have enough force to lead them to change the way they behave, the number of children they have, and the hopes they invest in these children: the way they allocate their time." Hence, the role of the capital in economic development has to be seen carefully because capital alone did not prove to be successful in the past.

[45] "The basic argument of the two-gap model is that most developing countries face either a shortage of domestic savings to match investment opportunities or a shortage of foreign exchange to finance needed imports of capital and intermediate goods" (Todaro and Smith, 2006, p. 724).

[46] It should be kept in mind that each country is used as one observation point irrespective of its size. Consequently, this doctoral thesis is about how many countries are in a poverty trap and not about what fraction of the world population is in a poverty trap.

3.2.2 Population Growth as a Source of Bimodality

So far, endogenous savings as an explanation for the emergence of twin peaks in the Solow growth model was considered. Another factor which might lead to bimodality shall be investigated in this section: population growth.[47] From the basic Solow growth model it is well known that lower population growth leads to higher income per head. However, it also seems plausible that the causality goes the other way round: higher income leads to lower population growth. Dornbusch, Fischer and Stertz (2008) already worked out a way to include the idea of endogenous population growth in the Solow growth model. Population growth basically depends on three elements: fertility, mortality, and migration, fertility probably being the most important of the three. Fertility “(parents’ decisions about how many children to have) is the endogenous source of population growth” (Razin and Sadka, 1995, p. 48). Already Becker (1993) pointed out in his Nobel lecture that the more productive a country, the higher is the price of the time spent on child care. Thus, children become “more expensive” which means that the demand for large families decreases. To make a point, Becker (1993, p. 397) states that “the growing value of time and the increased emphasis on schooling and other human capital explain the decline in fertility as countries develop”.

When talking about fertility, two main motives can be distinguished: first, there is a parental altruistic motive. This points to the tradeoff between the quantity and the quality of children. This tradeoff is not only described by Becker (1993) but also by other authors such as De la Croix (2013), for example. The quality of children refers to such aspects as welfare, human capital, health, and providing for the child’s future consumption (Razin and Sadka, 1995). Also the World Bank points to this tradeoff. “All parents everywhere get pleasure from children. But children involve economic costs; parents have to spend time and money bringing them up. Children are also a form of investment – providing short-term benefits if they work during childhood, long-term benefits if they support parents in old age. There are several good reasons why, for poor parents, the economic costs of children are low, the economic (and other) benefits of children are high, and having many children makes economic sense” (World Bank, 1984, p. 51). The other motive mentioned by Razin and Sadka (1995) is an old age security motive which sees children as a capital good in that they take care of their parents in case that they get old.[48]

In consequence of the two motives mentioned above, rich people often decide to invest more in their few children rather than having more children. This is underlined by the higher population growth rates often found in poorer countries. It should be kept in mind that in poor countries the mortality rates are usually higher than in rich countries. This means that the fertility rates have to be even higher in

[47] This idea is, for example, followed by Feyrer (2008). Most literature focusing on population as an explanatory factor for twin peaks emergence uses the fertility rate as explanatory factor. Among those studies are, for example, Barro and Becker (1989).

[48] This argument has to be seen with some care as life expectancy in poorer countries is low so that many people might not get old enough to be reliant on their children.

order to compensate these higher mortality rates and still have higher population growth rates in total. When income per capita rises, fertility declines and life expectancy increases. "At very high income levels, fertility shows a very weak tendency to increase with income, suggesting the very rich could also desire to have many children" (Razin and Sadka, 1995, p. 245).

To sum it up, Razin and Sadka (1995) argue that when income rises, parents tend to decide for improving the quality of their children instead of increasing their quantity. This explains why it is indeed a good idea to assume endogenous population growth depending on income.

Also other authors focus on the fertility factor as a crucial element of the population growth rate. Looking at the daily press shows that especially in the industrial countries low fertility rates are an important topic. De la Croix (2013) examines the relationship between fertility and income. He formulates four stylized facts characterizing this relationship:

"1. In all species, when available resources are more abundant, reproduction increases. This is true for plants, animals, and humans before the Industrial Revolution.
2. before the Industrial Revolution, the rich had more surviving children than the poor.
3. the transition from income stagnation to economic growth is accompanied by a demographic transition from high to low fertility.
4. now, both within and across countries, the rich and educated households have fewer children than poor and unskilled households.
5. most of the literature finds that the income of the father positively affects fertility, while the income of the mother negatively affects fertility" (De la Croix, 2013, pp. 1f).

The idea behind the fifth fact is that if the mother has a higher wage, she faces higher opportunity costs of having children. On the other hand, the higher income of the father brings about an income effect (De la Croix, 2013). When deciding on the number of children, parents have to look at their budget constraint given as follows:

$$\text{income} = \text{number of children} \cdot \text{spending per child} + \text{other spending}. \quad (3.13)$$

The more children a couple has the less can be spent per child and / or the less can be spent on other things (De la Croix, 2013).

There are a number of reasons for having children and for having less children in consequence of the demographic transition. One reason that De la Croix (2013) mentions is that children might have served as capital good for old age. With the introduction of pension systems by the states, however, this was no longer necessary to the same extent. This theory is called the old-age support hypothesis and is advanced by Ehrlich and Lui (1991), for example. It is also supported by Colombo (2010). He shows that "public pension system programs have a negative effect on the fertility rate of a country” (Colombo, 2010, p. 1). Similarly, the level of development of the financial markets also determines the fertility rate. In a study on the United States and Europe, Boldrin, De Nardi and Jones (2005) find that if the size of the social security system rises by 10 percent of GNP, the fertility rate decreases by 0.7 to 1.6 children. Colombo (2010) also points out that in the majority of developing countries, public pension schemes refer to workers in the public sector only. The resting employees in the informal sector are not addressed by the scheme, but as they represent up to 80 percent of the working force, the majority of workers will not get pensions. This might be overcome by micro pension systems aimed at “reducing poverty and the role of children as a natural insurance in order to free familiar resources that are then allowed to be invested in human and physical capital” (Colombo, 2010, p. 2).

Another school of thought argues that when a country develops, mortality, especially child mortality, declines. Consequently, fertility rates decline because “replacement” of the deceased children in order to have the same number of children reaching adolescence is no longer necessary when the mortality rates decline. This school of thought is represented by Bar and Leukhina (2010) and Doepke (2005), just to name a few. It is called the child replacement hypothesis.

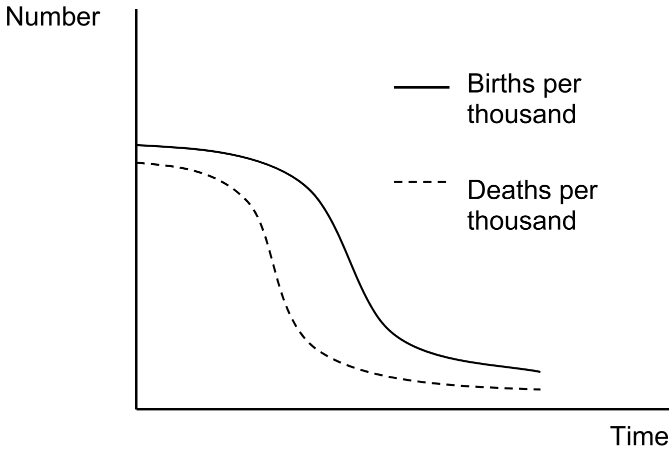
A third school of thought concentrates on the rising income and education of mothers. This raises the opportunity costs of having and raising children. Consequently, a smaller number of children is chosen in combination with higher investments in their quality, hence in their education (De la Croix, 2013). For less educated women these opportunity costs are lower which in turn explains why in poorer countries fertility rates are generally higher – here, women are often less or even not educated at all.

A fourth school of thought points out that increasing skills premiums due to more demand for educated workers from the firms make “the rate of return to quality [rise] relative to the implicit return of quantity” (De la Croix, 2013, p. 4). This school of thought is represented by Galor and Weil (2000), for example, and points once again to the above-mentioned budget constraint: the quality of the children can either be increased by higher incomes shifting upward the budget constraint or by decreasing the quantity of children, hence by substituting quality for quantity.

Summing up, whichever of the different schools of thought is seen as the correct one, all of them come to the same conclusion: as the average income rises, hence as countries start to develop, fertility rates decline. In consequence, even though also mortality rates decrease, the overall effect leads to lower population growth rates. Thus, it is very realistic to assume the population growth rate to be dependent on real per capita GDP.

Poor, developing countries indeed tend to have higher population growth rates than rich, developed countries. Apart from the reasons mentioned above, demographic transition is also an important explanation. This is illustrated in Figure 3.5.

Figure 3.5 Demographic Transition



Source: Szirmai, 1997, p. 94

Prior to demographic transition, both birth rates and death rates are high. As the number of births is somewhat above the number of deaths, there is a slightly positive population growth rate. When transition starts, the death rates begin to decrease due to improved nutrition, better hygienic circumstances, and a higher standard of living. The fall of the death rate causes an acceleration of the population growth rate. When the process of transition goes on, there comes a point at which urbanization, modernization, and increasing living standards lead to a decline of the birth rate.[49] After a long time, there will be a new equilibrium with a low population growth rate. This was reached by the Western World by the 20th century. For developing countries, however, the situation is much worse than it was in the Western World during transition: death rates are declining much faster than they did in Europe.[50] This is partly due to the medical progress achieved in the Western World which increasingly spreads out in the developing countries today.

Another aspect that might be the role arranged marriages. Edlund and Lagerlöf (2004) argue that love marriages lead to a redistribution of resources from old to young and hence tend to encourage human and physical capital accumulation. "[…A]rranged marriage – common in many parts of the Middle East, South Asia, and Africa – may be an institution that hampers development" (Edlund and Lagerlöf, 2004, pp. 23f). Yet, they also find that in countries which face a low level of development often face the problem of low returns to human capital. There, moving away from arranged marriages to love marriages might have only minor effects on economic growth. Often, these effects are hardly visible, so that countries might come to the conclusion that there is no economic need to switch to love marriages (Edlund and Lagerlöf, 2004).

Weinreb and Manglos (2013) find in their study that the effect of arranged marriages on the fertility rate differs across countries. In Turkey, for example, they find that there does not seem to be a difference in the fertility rate when comparing

[49] For a more detailed discussion of the reasons for this decline please refer back to Section 3.2.2.
[50] In Latin America transition is finished. Hence, here the situation was much worse than during the transition phase in the Western World – today these countries are no longer in this group of transition economies.

arranged marriages with love marriages. Hence, it remains ambiguous whether the role of arranged marriages should really be exposed when talking about fighting the high fertility rates in developing countries.

Another aspect which might also have an impact on the level of fertility rates, especially in developing countries, is the role of unintended pregnancy. Looking at the literature shows that this is especially a problem in poor countries where there are few possibilities to avoid pregnancy. Hence, fertility rates tend to be higher in poorer countries. This phenomenon ceases with rising incomes (see Hubacher, Mavranezouli and McGinn, 2008). Also Bongaarts considers the problem of unwanted fertility. He argues that when the countries go through the fertility transition, the wanted fertility rate declines. Yet, the unwanted fertility has an inverted U-shape. “During the first half of the transition, unwanted fertility tends to rise, and it does not decline until near the end of the transition” (Bongaarts, 1997, p. 267). Bongaarts concludes that “efforts to reduce unwanted pregnancies through family planning programs and other measures are needed early in the fertility transition because, in their absence, unwanted fertility and abortion rates are likely to rise to high levels” (Bongaarts, 1997, p. 267).

Summing up, birth rates in developing countries still remain high so that the developing countries are in fact characterized by a tremendously high population growth rate (Szirmai, 1997). This supports the conclusions drawn before concerning the population trap, possible twin peaks in the population growth rate, and the role of endogenous population growth in the Solow growth model.

Consequently, n will be endogenized so that the shape of the break-even investment line rather than that of the savings curve will be changed in the Solow growth model. Before doing so, the influence of population growth on the steady state in which a country ends up will briefly be investigated. This is best done via the so called Neo-Malthusian Trap.

3.2.2.1 The Neo-Malthusian Trap

According to the Neo-Malthusians, population grows unchecked and to its own damage. People do not want to allow the cruel forces of nature to correct population growth; hence, they have to find more humane ways to get back to equilibrium (Gabler, 2014). A high population growth rate has to be avoided for several reasons. It might lead to nutritional shortages and, most importantly, it has a dampening effect on growth of per capita income.[51] The latter is formalized in the Neo-Malthusian Trap. “Developing countries [are] in danger of getting caught in an equilibrium at a low level of economic development. This low-level equilibrium [is]

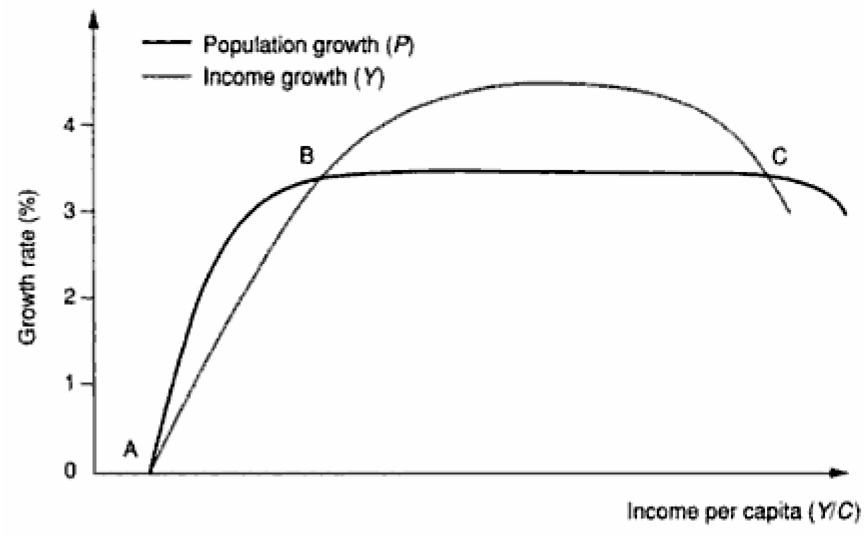
[51] It can also have positive impacts, of course: as a stimulation of production growth, higher labor supply in the presence of labor scarcity, encouragement of large-scale investment in infrastructure and so on (Szirmai, 2005).

known as the Neo-Malthusian Trap" (Szirmai, 2005, p. 152). Figure 3.4 illustrates the Neo-Malthusian Trap.

The horizontal axis measures per capita income (Y/C), where Y stands for income and C for capita.[52] The vertical axis shows the growth rates of population (g_P) and income (g_Y) respectively, where P stands for population. Neo-Malthusian theory states that "as per capita income goes up, […] population growth […] will increase till the biological maximum of around 3 percent growth per year is reached" (Szirmai, 2005, p. 153). Therefore, the population growth line is horizontal between points B and C. If income is sufficiently high, the population growth rate starts to decrease again. The rate of income growth (national income) also depends on per capita income within a country. If people become richer, they can save and hence also invest more so that the growth rate will be positively affected up to a certain level of saturation (Szirmai, 2005). Between points A and B population grows more rapidly than income. This implies that per capita income is declining, which results in the low-level equilibrium A. Between points B and C, on the other hand, income growth exceeds population growth. Consequently, economic growth is self-sustaining as per capita income increases until C, the high-level equilibrium, is reached.

Even if developing countries reach a population growth rate between A and B, population growth is faster than income growth ($g_P > g_Y$) and will force per capita income back towards the subsistence level A. Only by reaching a population growth rate which exceeds B, the high-level equilibrium C becomes feasible.

Figure 3.4 The Neo-Malthusian Trap



Source: Szirmai, 2005, p. 153

[52] This refers to the C on the horizontal axis and should not be mixed up with the point C as shown in the graph.

Usually, developing countries are caught in the low-level equilibrium, hence the term “population trap” or “Neo-Malthusian Trap”. The only opportunity to escape this trap is by tremendous developmental effort. Thus, especially higher development aid is often demanded – this connects to the Big Push (see Section 3.2.1, p.32). Now that the theoretical background is known, the impact of endogenous population growth on real per capita income in the context of the Solow growth model will be discussed.

3.2.2.2 Endogenous Population Growth in the Solow Growth Model

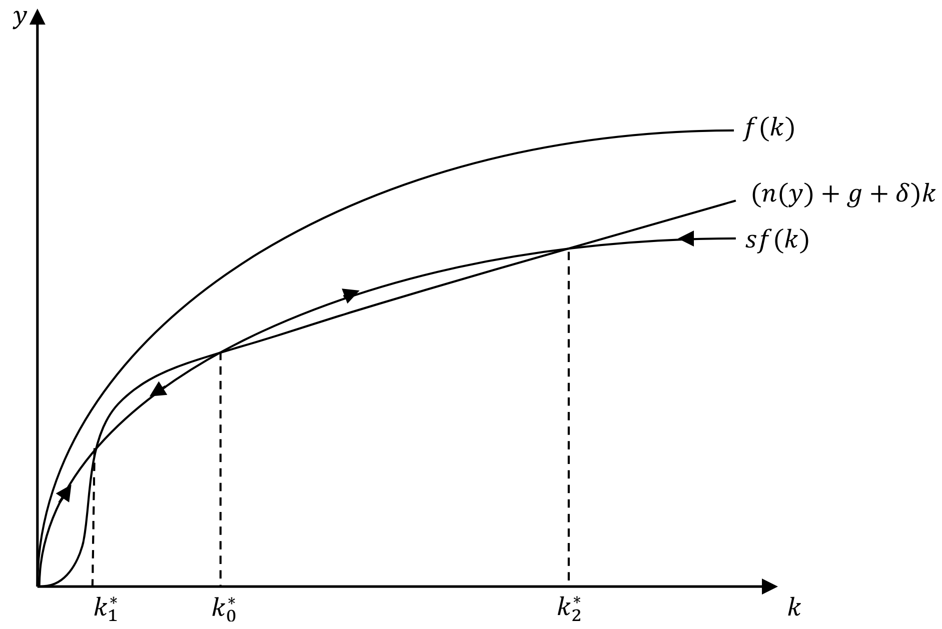
According to the basic Solow growth model a high rate of population growth leads to a lower steady state and thus a lower income per capita (Dornbusch, Fischer and Stertz, 2008). Yet, as already stated in the introduction to Section 3.2.2, population growth is very likely to be dependent on income, too. From the data it is well-known that poor countries tend to have higher population growth rates than rich ones. Dornbusch, Fischer and Stertz (2008) also underline this. Furthermore, they point out that “as incomes rise, death rates fall (especially through reductions in infant mortality) and population growth rises” (Dornbusch, Fischer and Stertz, 2008 p. 86). Yet, the higher incomes rise, the lower birth rates will be so that rich countries often have population growth rates close to zero.

Dornbusch, Fischer and Stertz (2008) graphically show a version of the Solow growth model with an endogenous population growth rate. According to the authors, a curve graphing n against y “would rise, fall, and then level off near zero” (Dornbusch, Fischer and Stertz, 2008, p. 86). Based on this endogenization, the so called investment requirement line $(n(y) + g + \delta)k$ becomes a curve. The result is shown in Figure 3.6. As in Section 3.2.1 (see p. 32), whether the curves indeed intersect depends on the exact form and position of the curves. In the example given here, three steady states arise, two of which are stable, namely k_1^* and k_2^*. Possibilities to escape the poverty trap, as mentioned by Dornbusch, Fischer and Stertz (2008), are either a Big Push policy to push a country beyond k_0^* or moving the savings curve upward by increasing the productivity or the savings rate (or the investment requirement curve downwards by population control policies[53], for example) so that only one steady state remains, namely k_2^*.[54]

[53] Population control policies are used by China, for example, in form of their “one-child policy” (for example Bongaarts and Greenhalgh (1985), or Rosenzweig and Zhang (2009)).

[54] Stability within the Solowian framework arises if the savings curve is above the $(n + g + \delta)k$-line which indicates capital destruction. If capital accumulation in the form of savings exceeds capital destruction, the economy grows and is pushed to the right until a steady state is reached (for example k_1^* or k_2^* depending on where the economy starts). Between k_1^* and k_0^*, the capital destruction is higher than the capital accumulation, hence the economy shrinks and is pushed to the left until k_1^* is reached. To the right of k_0^*, the dynamics again push the economy to the right. Figure 3.2 shows the dynamics. Hence, it can be concluded that k_1^* and k_2^* are locally stable steady states (a small move to the left or to the right of them always pushes the economy back to the steady states). k_0^*, on the contrary, is instable. A small move away from the steady state immediately pushes the country towards k_1^* or k_2^*.

Figure 3.6 The Population Induced Poverty Trap



Source: Dornbusch, Fischer and Stertz (2008), p. 87

Having examined two very plausible factors of the Solow growth model which are likely to lead to the twin peaks observed in the empirical data (see Chapter 5), a third option which is also often mentioned in the literature shall be considered: human capital. This is done by first describing the Solow growth model extended by human capital as the basic model. The section thereafter will then deal with the question of how twin peaks might emerge in this model.

3.3 The Solow Growth Model with Human Capital

In Section 3.2, the basic Solow growth model was graphically manipulated such that twin peaks arise as a plausible result. It was shown that an endogenization of the savings rate or the population growth rate may lead to two stable steady states in the Solow growth model. A further decisive factor discussed in the twin peaks literature is human capital. [55] Thus, in this section another class of models also being able to cover the twin peaks phenomenon will be introduced: neoclassical growth models with human capital. Already Azariadis and Drazen (1990) point out that human capital alone is not sufficient to explain twin peaks in the income distribution across nations.[56] The crucial question is whether it induces increasing

[55] What is exactly human capital is, how it is measured, and further aspects will be treated in more detail in Chapter 5.

[56] Once again, the reader shall be reminded that this doctoral thesis is about income inequality across nations and not within them though being aware that this is an important aspect of inequality as well.

returns somewhere. If it does, then bimodality arises. Galor (1996), for example, investigates social increasing returns to scale from the accumulation of human capital. He specifies that countries being similar in their structural characteristics, such as the initial output level and human capital per capita, but differing in their initial distribution of human capital, will end up in different steady state equilibria. Romer (1996) claims that theories which are based on knowledge accumulation are unlikely to explain cross-country differences in incomes. Hence, economists tried to find new models able to explain these differences. One class of models includes human capital, which consists of abilities, skills, and knowledge of workers, amongst others. Human capital is rival and excludable but faces a number of positive and negative externalities, hence it is an imperfect private good.

3.3.1 The Model

In this section, the basic Solow growth model supplemented by human capital shall be presented. The model shown here is the one by Romer (1996). Just as in the basic Solow growth model, constant returns to scale are assumed. However, including human capital implies that changing the resources devoted to physical and human capital accumulation respectively may lead to large changes in output per worker.

In the basic neoclassical growth model including human capital, output is given by the following Cobb-Douglas production function with constant returns to scale:

$$Y = K^{\alpha} H^{\beta} [AL]^{1-\alpha-\beta} \tag{3.14}$$

with $\alpha > 0, \beta > 0, \alpha + \beta = 1$, where Y is income, K is capital, H is human capital, A is technology, L stands for labor[57], α is the physical capital share, β is the human capital share, and $1 - \alpha - \beta$ is the labor share. There are constant returns to K, H, and L together. K and H are assumed to be symmetric.

$$\dot{K} = s_K Y - \delta K \tag{3.15}$$

$$\dot{L} = nL(t) \tag{3.16}$$

s_K indicates the savings rate in physical capital, δ is the rate of depreciation, and n is the population growth rate. Technological progress is exogenous, as in the basic Solow growth model, and human capital behaves as physical capital.

$$\dot{A} = gA(t) \tag{3.17}$$

$$\dot{H} = s_H Y - \delta H \tag{3.18}$$

Again, g stands for the rate of technological progress and s_H indicates the savings rate in human capital. In the following, the dynamics of the model will be examined.

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57 This means that "a skilled worker supplies both one unit of L and some amount of H" (Romer, 1996, p. 128)

3.3.2 The Dynamics

In this section, the dynamics of physical as well as human capital need to be considered. But first, the production function will be reformulated in terms of efficient units of labor,

$$k = \frac{K}{AL}, h = \frac{H}{AL}, y = \frac{Y}{AL} \tag{3.19}$$

so that the production function becomes:

$$y = k^{\alpha} h^{\beta}. \tag{3.20}$$

In order to be able to calculate the steady state, $\dot{k}$ and $\dot{h}$ have to be determined respectively.

$$\dot{k} = \frac{\dot{K}}{AL} - \frac{K}{(AL)^2}\left[A\dot{L} + L\dot{A}\right] \tag{3.21}$$

$$\Leftrightarrow \dot{k} = \frac{s_K Y - \delta K}{AL} - \frac{K}{AL}\left[\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right] \tag{3.22}$$

$$\Leftrightarrow \dot{k} = s_K y - \delta k - k(n + g) \tag{3.23}$$

$$\Leftrightarrow \dot{k} = s_K k^{\alpha} h^{\beta} - (n + g + \delta)k \tag{3.24}$$

To get the steady state value of k, Equation (3.24) has to be set equal to zero:

$$\dot{k} = 0 \tag{3.25}$$

$$\Leftrightarrow s_K k^{\alpha} h^{\beta} - (n + g + \delta)k = 0 \tag{3.26}$$

$$\Leftrightarrow s_K k^{\alpha} h^{\beta} = (n + g + \delta)k \tag{3.27}$$

$$\Leftrightarrow \frac{s_K h^{\beta}}{n+g+\delta} = k^{1-\alpha} \tag{3.28}$$

$$\Rightarrow k^* = \left(\frac{s_K}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} h^{\frac{\beta}{1-\alpha}}. \tag{3.29}$$

The same has to be done for $\dot{h}$ (the result is symmetric to that for $\dot{k}$ as the two kinds of capital evolve symmetrically). Hence, it can be found:

$$\dot{h} = s_H k^{\alpha} h^{\beta} - (n + g + \delta)h, \tag{3.30}$$

and this results in the following condition for the steady state:

$$\dot{h} = 0 \tag{3.31}$$

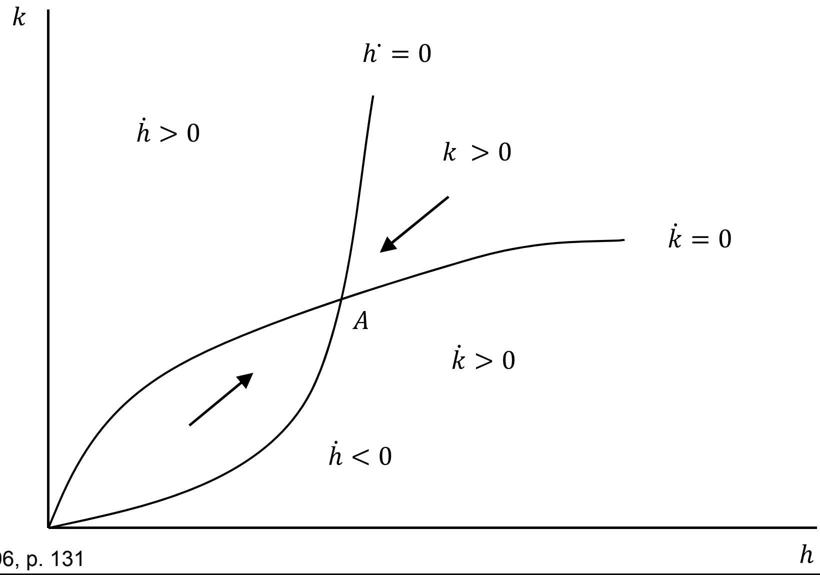
$$\Rightarrow s_H k^{\alpha} h^{\beta} - (n + g + \delta)h = 0 \tag{3.32}$$

$$\Leftrightarrow s_H k^{\alpha} h^{\beta} = (n + g + \delta)h \tag{3.33}$$

$$\Leftrightarrow k^{\alpha} = \frac{n+g+\delta}{s_H} h^{1-\beta} \tag{3.34}$$

$$\Rightarrow k^* = \left(\frac{n+g+\delta}{s_H}\right)^{\frac{1}{\alpha}} h^{\frac{1-\beta}{\alpha}}. \tag{3.35}$$

Figure 3.7 The Dynamics of k and h



Source: Romer, 1996, p. 131

In the steady state, both conditions – (3.25) and (3.31) – have to be fulfilled. Figure 3.7 illustrates the determination of the equilibrium. The forms of the two curves depend on the assumptions given before: $\alpha + \beta < 1$. The derivatives of $\dot{k}$ and $\dot{h}$ with respect to k would indicate the shapes given in Figure 3.7.[58] Point A is the steady state which is globally stable.

In this model, an increase in the savings rate in physical capital leads to an upward shift of the $\dot{k} = 0$ line due to a higher s_K. There will be a gradual transition towards the new balanced growth path, with both higher h and higher k. In principle, this result is equal to the one of an increase in savings in the basic Solow growth model, where $sf(k)$ shifts up and the country moves towards a higher steady state k^*.

Now that the Solow growth model with human capital was explored, the next section will show how this model needs to be modified to yield twin peaks.

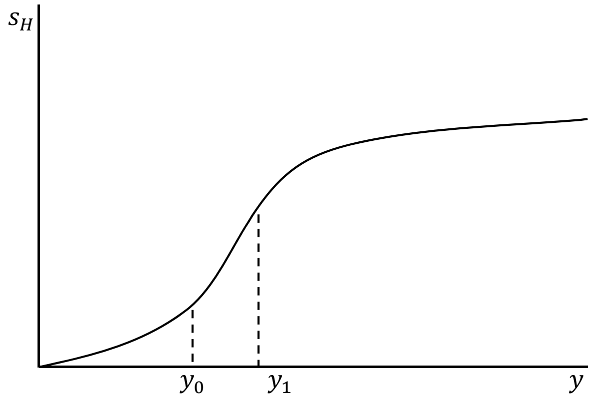
[58] The derivatives will not be calculated here. The interested reader is referred to Romer (1996) for this purpose. What is important here is to understand the dynamics. Above and to the left of the $\dot{k} = 0$ curve, $\dot{k} < 0$, hence k declines. Below it and to the right, $\dot{k} > 0$ and k increases. For the $\dot{h} = 0$ curve, the opposite is true. Above and to the left of the $\dot{h} = 0$ curve, $\dot{h} > 0$, hence h increases. To the right and below the curve $\dot{h} < 0$, hence h declines. What becomes apparent is that as long as the $\dot{k} = 0$ curve is above the $\dot{h} = 0$ curve, the economy is pushed towards the right until the curves intersect. This makes economic sense. If physical capital grows faster than human capital, an economy can grow as production can be increased. By increasing human capital, the economy can reach a higher production level. In the other hand, if human capital grows faster than physical capital, a country is not able to increase production. What happens is either a labor hoarding if the economy is closed or, if it is open, human capital flees to other countries. Then, a country will rather decline. Hence, if the $\dot{k} = 0$ curve is above the $\dot{h} = 0$ curve, the economy is pushed to the right. If it is the other way around, the economy is pushed to the left. This is shown by the arrows in Figure 3.7.

3.4 Human Capital and Bimodality

Some authors treated the question of how human capital yields twin peaks. Among them are, for example, Azariadis and Drazen (1990), Durlauf (1996), Lucas (1988), and Becker, Murphy and Tamura (1990). Human capital alone is not sufficient to capture twin peaks “unless it induces increasing returns somewhere” (Azariadis and Drazen, 1990, p. 513). This could be seen in the previous section. The basic assumptions only led to a single steady state. How does the model need to be modified in order to be able to explain bimodality? An example of such a modification will be shown in this section.

3.4.1 Assumptions

Figure 3.8 Human Capital Savings



The assumptions remain essentially the same as in the basic neoclassical growth model with human capital. Evidently, human capital and physical capital are symmetrical. However, the inclusion of human capital alone is not sufficient to yield twin peaks in the income distribution across nations as outlined above. Galor (1996) proposes to include capital market imperfections in the model along with non-convexities in the production of human capital. The inclusion of capital market imperfections seems plausible. If there is perfect capital mobility, there will not be twin peaks in the distribution even though human capital is included in the model. However, usually, capital is not perfectly mobile among countries.[59] Or, if there are perfect international capital movements, there are domestic capital market imperfections (Galor, 1996). It can be concluded that the assumption of highly imperfect capital markets is both realistic and economically plausible for nearly all developing countries due to nonprotected property rights, for example.

Azariadis and Drazen (1990), on the other hand, point to the importance of increasing returns which have to be induced somewhere. The explanations in this section are slightly connected to Galor (1996) and Azariadis and Drazen (1990). In the augmentations, the main assumptions of the basic model will remain valid. As

[59] Yet, Barro, Mankiw and Sala-i-Martin (1992) show that the results obtained by Mankiw, Romer and Weil (1992) remain valid if credit rationing occurs in a specific form. For details, the interested reader is referred to the article by Barro, Mankiw and Sala-i-Martin (1992).

in Section 3.2, the savings rate will be endogenized in this section – however, this time not only the one in physical capital. There are good reasons to assume that also the savings rate in human capital evolves in the form of a logistic curve. People with very low income have to afford subsistence consumption. Hence, they cannot save physical capital and, instead of accumulating human capital, they are usually forced to work as soon and as much as possible. This explains the rather flat part in Figure 3.8 up to y_0. From a certain point on, people have slightly more money than needed for subsistence consumption. In memory and fear of worse times, they start saving; hence, this part of the savings rate curve $(y_0 - y_1)$ is very steep.

Accordingly, they also accumulate human capital in order to be able to ensure a better income and thus a higher standard of living in the future or to guarantee their children a better life (or both). However, beyond y_1, a high level of human capital is reached. From there on, s_H will increase only slightly. This is especially due to high opportunity costs of accumulating more human capital. Someone, who has studied for four years or more at university, will have to think carefully about whether it is really worth doing a doctorate thereafter. It often means three or four further years at university, perhaps with high fees unless he or she receives a scholarship. But more importantly, it means that at least three years of regular income and of work experience are forgone. Is the return to further education high enough in this case to be worth the additional effort? This example clearly demonstrates the high opportunity costs associated with further human capital accumulation when a high level is already reached.

In conclusion, a logistic curve for both s_K and s_H seems plausible – either when reasoning for each kind of savings separately or when referring back to the fact that s_K and s_H are assumed to be symmetric. Having decided to endogenize both s_K and s_H, the augmented growth model will be discussed in order to explain the existence of twin peaks in the following section.

3.4.2 The Model

In the previous subsection, it was already stated that s_H and s_K will be endogenized to capture twin peaks. The consequences of this (in formal terms) are:

$$\dot{k} = s_K(y)k^{\alpha}h^{\beta} - (n + g + \delta)k = 0 \tag{3.36}$$

$$\dot{h} = s_H(y)k^{\alpha}h^{\beta} - (n + g + \delta)h = 0. \tag{3.37}$$

Figure 3.9 shows the graphical representation of the augmented growth model. As in the models in Section 3.2, there are three equilibria. However, B is unstable as the dynamics indicate. Hence, even if a country manages to reach an $h \in [h_A, h_B[$, it will be forced back towards point A, the low k-low-h-equilibrium. On the other hand, a country with $h \in [h_B, h_C[$ will be forced to point C as will those countries with $h > h_C$. This is the high-k-high-h-equilibrium. Both A and C are locally stable equilibria – history determines in which equilibrium a country ends up.

Figure 3.9 Human Capital and Bimodality[60]

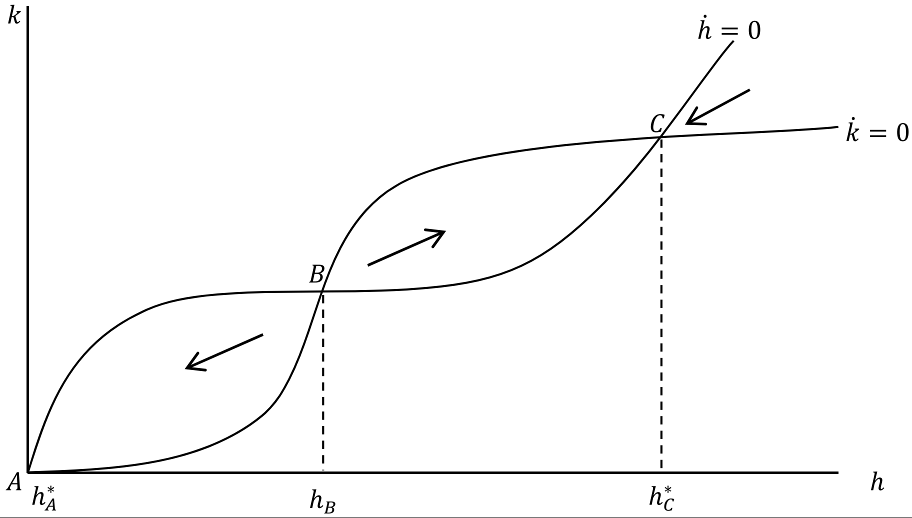


Figure 3.9 illustrates the dilemma of developing countries. They tend to have a very low level of human capital. Hence, they face a sort of "education trap", similar to the population trap explained above. A low initial level of human capital will be reflected in a low level of physical capital and hence low income. Consequently, the country will be caught in the low-income equilibrium. This again points to a possible role of development aid not only financially but also in the fields of human capital, hence education, training on-the-job, and the like. Human capital, thus education, has very important functions and tasks such as promotion of economic growth and development (Szirmai, 2005). This helps to understand why the income distribution is as unequal as shown in Chapter 5.[61] Unless a high level of education is reached, the poor countries will not be able to catch up. Yet, looking at the developments of recent decades it can be seen that developing countries are narrowing the gap to developed countries. In the future, they will probably have universal primary education (Szirmai, 2005). This makes it clear that there still is a large gap towards the developed countries, even though it is about to narrow.[62]

[60] As explained before (see footnote 58), if the $\dot{k} = 0$ curve lies above the $\dot{h} = 0$ curve, the country is pushed to the right. If the $\dot{h} = 0$ is above the $\dot{k} = 0$ curve, the economy is pushed to the left. The result is indicated by the arrows. Consequently, A and C represent stable equilibria: a small movement to the right of C pushes a country back to point C. Equally, if the country moves a bit to the left of C, it is pushed upwards until C is reached again. Contrary, B is an unstable equilibrium: moving a little bit to the left forces a country downwards until A is reached. If, on the contrary, it moves a bit to the right, then the country is forced to move upwards by the dynamics until point C is reached. Hence, A and C are stable and B is unstable. The same argument accounts for k, of course.

[61] Once again, the reader is reminded that income inequality in this doctoral thesis concentrates on the one across nations and not within them.

[62] For a more detailed discussion of the characteristic features and problems of the educational system in developing countries please refer to Szirmai (2005).

However, it clearly underlines the finding of the "education trap" examined before and also explains why these two locally stable steady states exist.[63]

To sum it up and make a point, this section clarified why human capital, by assuming endogenous savings both in human and physical capital, can serve to explain the twin peaks in the distribution of per capita income across countries.

3.5 Conclusion

In this chapter, it was attempted to give economic explanations for the existence of twin peaks in the income distribution. First, a discussion of the relevance of the Solow growth model in modern economic growth theory was given. This was followed by a brief review of the basic Solow growth model. Thereafter, two alternative augmentations of the neoclassical growth model were considered. Both are plausible and realistic and both lead to the desired outcome as proposed by Solow: two locally stable steady states. The first augmentation was the endogenization of the savings rate based on the assumption that the relationship between y and s is best given by an S-shaped logistic curve. As a consequence, this leads to three intersections with the break-even investment line. Two of these intersection points are locally stable, and may hence represent the two peaks shown in Chapter 5.

The second modification of the basic Solow growth model was the introduction of an endogenous population growth rate instead of assuming it to be constant. Also the population growth rate was assumed to be dependent on the income level. The two variables were related via the mirror image of an S-curve. This endogenous population growth led to a transformation of the break-even investment line into the mirror image of an S-curve. Again, there were three intersection points, two of which being stable steady states. In this context, also the Neo-Malthusian Trap was discussed in order to show that high population growth rates are responsible for the poor countries being caught in the low-income equilibrium.

After the basic Solow growth model was used as the framework for capturing twin peaks, an augmentation of the model with human capital was used. First, the basic model was reviewed. Thereafter, some modifications were made in order to be able to explain why there are two stable steady states.

All three alternatives considered in this chapter showed how important history is in determining whether a country will end up in the low-income or the high-income equilibrium. Yet, in any case, each of the augmented growth models was able to explain why twin peaks rather than a unique peak may arise in the distribution of per capita income across countries in the Solowian framework. Hence, from a graphical and verbal point of view it can be concluded that Robert Solow (1956)

[63] It has to be mentioned, however, that especially in Latin America the reduction of this education trap did not really have a remarkable impact.

indeed was right: his model is able to capture multiple steady states. The next step is to examine whether Solow is right from an analytical point of view by use of formulas and mathematics. This will be done in Chapter 6. Before, however, a statistical analysis will follow in order to prove that it is really plausible to have bimodality in the per capita income distribution. Additionally, it will be examined in how far the different variables may deal to explain bimodality within the Solowian framework. Before the empirical analysis can be done, Chapter 4 will present an overview of the empirical methods to be used in this doctoral thesis.

4 Empirical Methods for Distribution Analysis

In Chapter 2, a review of the relevant theoretical literature for the twin peaks phenomenon was given. Then, the previous chapter gave an overview of the basic Solow growth model as well as a version including human capital. It was shown in how far these models are able to capture twin peaks, hence in how far multiple steady states are possible within the framework of the Solow growth model as claimed by Solow (1956). After this more theoretic attempt to check Solow's hypothesis, an analytical examination shall follow. Yet, before this can be done, an empirical analysis should be undertaken in order to find out whether the real per capita GDP data indeed exhibit multimodality and if so how many peaks can be found. Summing up, it will be shown that it is indeed empirically relevant to test Solow's claim of being able to capture multimodality within his growth model.

Prior to the empirical study, the methods to be used for this analysis need to be introduced. This is the task of this chapter. It will give an overview of the existing statistical methods to deal with distribution dynamics. In the first section, a review of the relevant empirical studies published so far will be given with a focus on the methods applied.[64] Then, this doctoral thesis will explore that kernel densities are a good method to be used. For this purpose, a technical overview of the main methods that can be applied will be given. Thereafter, due to its importance the kernel density will be presented in more detail. In Section 4.4, the Markov chain will be examined. It is a discrete tool for the analysis of the distribution of real per capita GDP allowing for a look into the future. Finally, the loess fit method will be described. This method will be used for the determination of the endogenous savings function. The last section will conclude this chapter.

4.1 Literature Review

Before the methods to be used in this doctoral dissertation will be described, a literature review of the basic empirical studies will be given with a focus on the methods used. It becomes obvious that two main methods are relevant for twin peaks analysis: nonparametric density estimation, especially the kernel density, and the Markov chain. This is shown by Table 4.1 which gives an overview of the use of methods in the literature on empirical twin peaks analysis. In the following, the empirical literature will be reviewed with a focus on the methods applied for the twin peaks analysis. To begin with, the kernel density as a nonparametric density estimation will be considered. Thereafter, the Markov chain (as well as Markov transition matrices) will be the center of concern. Finally, other methods used in the empirical literature will briefly be reviewed. For all subsections, it has to be kept in mind that the focus of this chapter is on the application of statistical methods

[64] The literature overview given in this section does not claim completeness.

only. The results of the empirical analyses will be discussed in Chapter 5 and hence will not be mentioned here.

Table 4.1 Literature Review: The Methods Applied

Author	Kernel Density Estimation[65]	Markov Chain	Other Methods
Bandyopadhyay (2001)	Stochastic kernel		
Beaudry, Collard and Green (2002)	Gaussian kernel		Interquartile ranges Mass around the mean
Ben David (1997)			Frequency plots Lorenz curves Annual standard deviations
Bianchi (1997)	Kernel		Bootstrap multimodality tests Box plots Normal quantile plots
Cantner et al (2001)	Kernel		
Chumacero (2006)	Kernel	Markov chain	Contour and surface plots
Feyrer (2008)		Markov chain	
Jones (1997)		Markov chain	Nonparametric density estimation (not clearly specified)
Kremer, Onatski and Stock (2001)		Markov chain	
Paap and van Dijk (1998)		Markov chain	
Pearlman (2003)		Markov chain	
Krüger, Cantner and Pyka (2003)	Kernel		
Quah (1993a)		Markov chain	
Quah (1996c)		Markov chain	
Semmler and Ofori (2007)	Gaussian kernel	Markov chain	
Villaverde (2001)	Gaussian kernel	Markov chain	

[65] If known, the specific kind of kernel function used is indicated.

4.1.1 The Kernel Density

Looking at Table 4.1, in half of the articles considered for this literature review, the kernel density is used to analyze the distribution of real per capita GDP in the world. Often, this is not the only method applied. Hence, several articles be part of this subsection as well as of the next or even the third one. While some authors just use this method without giving reasons for that (Cantner, Ebersberger, Hanusch, Krüger and Pyka, 2001; Chumacero, 2006; Villaverde, 2001; Beaudry, Collard and Green, 2002), others not even describe which nonparametric density estimation method they use (for example Jones, 1997). Here, one can only guess from looking at the graphs presented in the respective article. Most authors give at least some explanations for their methodological choice and discuss it in more detail.[66] Basically, the kernel density is the method mostly applied for the graphical representation of the twin peaks phenomenon. The traditional nonparametric approach for frequency distribution analysis, which was also applied by one of the authors as seen in Section 4.1.3, is the histogram. However, as this method is dependent on the position of the bin edges and as it appears to be rather jagged, it is quite difficult to discriminate between sampling errors and the real structure in the data sample. These disadvantages can be overcome by the use of kernel densities (Krüger, Cantner and Pyka, 2003). They can be used for testing the type of modality in the real per capita income data. Moreover, this method can be used to test the convergence hypothesis: if there were convergence, then unimodality would be a robust result which could not be rejected (Bianchi, 1997). However, it has to be noted that "no formal test of this [(twin peaks)] theory can be provided with this visual evidence" (Chumacero, 2006, p. 6).

Looking at the literature, it can be seen that some authors combine this method with bootstrap modality tests. These are used in order to check whether the result of twin peaks is robust. The bootstrap test is used for testing the null hypothesis that the function $f(x)$ has m modes versus its alternative that $f(x)$ has more than m modes. Using the kernel density as the basis, the bandwidth, which will be described in more detail later, determines whether the null hypothesis can be rejected. "A large value of [the bandwidth] indicates more than m modes, thus rejecting the null" (Bianchi, 1997, p. 396).

Another way to see kernel densities in the field of twin peaks analysis is indicated by Bandyopadhyay (2001). He argues that the stochastic kernel is a continuous version of the Markov chain which then can be used for graphical representation in this respect. This leads to the following subsection in which the use of the Markov chain for income distribution analysis shall be discussed.

[66] Semmler and Ofori (2007) and Krüger, Cantner and Pyka (2003) just argue that other authors use this method and hence they follow their proposal. But at least the theoretical background of the kernel method is described in both articles.

4.1.2 The Markov Chain

The Markov chain is used in the majority of the articles considered in Table 4.1. It becomes clear that this method along with the kernel density really dominates the polarization literature. Again, a number of authors only apply this method without really mentioning reasons except for some other authors doing so (for example Feyrer, 2008[67]; Semmler and Ofori, 2007). The Markov chain is a suitable method to answer the convergence question (Villaverde, 2001). It estimates a country's income tomorrow solely on the basis of its income today (Kremer, Onatski and Stock, 2001). Furthermore, it is a solution to the problem that standard regression analysis just looks at the average or representative behavior instead of the entire distribution (Bandyopadhyay, 2001). Chumacero (2006) defines the Markov chain as a trial to "formalize the twin peaks hypothesis by deriving the ergodic distribution of the transition matrix of relative incomes among countries" (Chumacero, 2006, p. 6). The Markov chain "allows for a more flexible relationship between the level of income and the growth rate of income than the standard convergence approach in which countries' growth rates are assumed to be a linear (or sometimes quadratic) function of their (log) income levels" (Kremer, Onatski and Stock, 2001, p. 6). By use of the Markov chain, it is possible to examine the transitions between the income states (high income and low income) and to determine whether countries convergence towards the mean, if income classes are defined via the mean income in the world, or US income, if this is taken as a reference (Pearlman, 2003).

Already Quah (1993a and 1996c, for example), who can be seen as the predominant economist in this field of research, makes use of the Markov chain in his empirical studies. He argues that the advantage of the Markov chain compared to other methods is that it is "not tied to restrictive assumptions on the nature of long run growth" (Quah, 1993a, p. 429). The law of motion underlying this process is like a standard first-order auto-regression of the form:

$$F_{t+1} = MF_t, \tag{4.1}$$

where M maps one distribution into another, F_t is the original distribution of real per capita GDP at time t and F_{t+1} is the distribution at time $t+1$. Of course, this can also be extended further for a long run analysis or even a prediction of the future distribution.

In addition to its use for addressing the convergence question in general, Paap and van Dijk (1998) point out that the Markov chain allows to analyze the mobility within and between the income groups formed. Hence, it can be found out whether it is likely for poor countries to escape the poverty trap or whether they rather tend to fall behind even more instead of catching up. Furthermore, it can be observed in which of the income groups the largest mobility can be found and whether the

[67] However, Feyrer (2008), contrary to many others, gives an extensive description of the Markov method. This will not be reviewed here; the interested reader is referred to the article.

middle income group really decreases over time as often mentioned in empirical studies (Paap and van Dijk, 1998). Also Bandyopadhyay (2001) underlines that "Markov chains are used to approximate and estimate the laws of motion of the evolving distribution. The intra-distribution dynamics information is encoded in a transition probability matrix, and the ergodic (or long run) distribution associated with this matrix describes the long term behavior of the income distribution" (Bandyopadhyay, 2001, p. 5).

The Markov chain provides the possibility to look into the future and check whether the future income distribution is really twin peaked or rather unimodal instead (Jones, 1997). Yet, Chumacero (2006) also points to a disadvantage of the Markov chain. He states that the "resulting ergodic distribution is sensitive to the choice of thresholds for each category, the number of years used to compute the transition matrix, and the variable used to perform the comparisons" (Chumacero, 2006, p. 6).

4.1.3 Other Methods

Table 4.1 shows the predominance of the kernel density and the Markov chain as methods to search for twin peaks in the real per capita GDP data across the countries of the world. However, it also shows that some other methods are applied as well. These will be reviewed briefly in this subsection. Only in one of the articles considered regression analysis is used; however, Beaudry, Collard and Green (2002) point out that this method is just applied in order to be able to say more about the reasons for the emergence of bimodality. Most of the methods mentioned in Table 4.1 are rather other kinds of nonparametric density estimators used to give a graphical representation of the real per capita GDP data. For example, Bianchi (1997) uses box plots to demonstrate that making log transformations of the GDP data transforms the distribution into a symmetric one without outliers. Histograms are connected to such box plots. They are used, for example, by Paap and van Dijk (1998). The authors use a smoothed version of the histogram as well as histograms with estimated density functions in order to be able to visualize the distribution in each year considered. However, the authors also point to a disadvantage of the histogram (this will be discussed in more detail in the following of this chapter): the histogram is rather sensitive to the choice of the bandwidth. This choice is quite arbitrary and hence, different choices might yield different solutions concerning the finding of the type of modality. Paap and van Dijk (1998) also use a mixture of a Weibull and a truncated normal density as a graphical representation of the data. Yet, these methods are not described in more detail in the article.

As an alternative to histograms, Ben-David (1997) uses frequency plots in his analysis. In these plots, no bars are used but ragged lines instead, where frequencies are put into the diagram as points which then are connected. Another method applied along with kernel densities are normal quantile plots (Bianchi, 1997). They are used to answer the question "how well the distributions of the data in the original scale and the two above-mentioned transformations [(income divided by the sum of all incomes, log transformation)] are approximated by the normal distribution" (Bianchi, 1997, p. 400).

Beaudry, Collard and Green (2002) apply interquartile ranges and the mass around the mean together with the kernel densities for analyzing the income data. "These numbers – when taken together with the percentile differences – indicate a widening process that has taken place around the interquartile range and corresponds to a hollowing out of the middle of the distribution with mass moving towards two modes, but without a fattening of the tails" (Beaudry, Collard and Green, 2002, p. 6).

Finally, Ben-David (1997) also makes use of Lorenz curves to illustrate the changing income inequality in the world. Basically, this tool is known from inequality analysis within a country; however, it can also be applied to the data for all countries in the world. In addition, Ben-David (1997) uses the annual standard deviation of the log per capita incomes in his article. He argues that this measure allows for the determination of the degree of inequality within the income groups defined.

4.2 Distribution Analysis

In the previous section, the key literature on the empirical methods used for twin peaks analysis was reviewed. It became obvious that two methods turn out to be of central concern, namely the kernel density and the Markov chain. In most of the articles it is clear which methods are used. But often, the choice of methods is not well-founded. In this doctoral thesis, the nonparametric density estimation methods will be clearly described and analyzed according to the applicability to the twin peaks phenomenon. Furthermore, the Markov chain method will be presented in more detail in Section 4.4 so that there is a good overview of the statistical methods for the distribution analysis which is the subject of Chapter 5.

In the last section, it was shown that most authors used the kernel density distribution to analyze the per capita income distribution. Some of the authors explained their choice of statistical method by stating that the kernel density is better than the histogram, for example. But they did not really prove that this is indeed the case. This is the task of this section. Here, density analysis will be described in general and the advantages and shortages of the respective methods will be discussed and shown by the data.[68]

[68] At this point, part of the data will be used. An exact description of the data sources as well as the measurement and so on will follow in Chapter 5.

The distribution of per capita income can be statistically approached by density estimation. “Density estimation […] is the construction of an estimate of the density function from the observed data” (Silverman, 1986, p. 1). It can be subdivided into parametric and nonparametric density estimation. In the former case, there have to be assumptions about the type of distribution in the population. For example, the normal distribution with mean μ and variance σ^2 is a parametric density estimation which can be approached by calculating the mean and the variance and inserting these into the formula of the normal density (Silverman, 1986). Nonparametric density estimation, on the contrary, is based on weaker assumptions about the distribution of the dataset. By considering nonparametric density estimation, the analyst lets the data “speak for themselves in determining the estimate of f [(the probability density function)] more than would be the case if f were constrained to fall in a given parametric family” (Silverman, 1986, p. 1).

Table 4.2 Advantages and Shortages of the Methods for Distribution Analysis

Method	Advantages	Disadvantages
Nearest neighbor measure	– respects not only individual points but also their neighbors	– no smooth curve – no probability density – not appropriate for the entire density
Naïve estimator	– histogram where every point is the center of a sampling interval	– ragged character may lead to misinterpretations of the data
Stem-and-leaf plots	– numerical data is not lost	– unclear for large datasets
Box plot	– easy to interpret – symmetry and distribution around the median can be read off	– no type of modality can be retrieved
Histogram	– easy to interpret	– sensitive to the choice of origin and bin width – not smooth
Kernel method	– smooth – rather objective	

Density estimates help checking the data for skewness and multimodality and they are more or less self-explanatory, even to non-statisticians. All important conclusions can be drawn from the dataset by use of density estimation (Silverman, 1986). Density estimators can be formulated for the univariate and the multivariate case. Univariate (just as multivariate) density estimation allows for several alternative density estimators. Among them are the nearest neighbor method, the naïve estimator, stem-and-leaf plots, the box plot, the histogram, the kernel density, the variable kernel method, the orthogonal series estimators, the maximum penalized likelihood estimators, and the general weight function estimators. For reasons of importance, only the first six will be dealt with here.[69] However, they will be presented in the opposite direction starting off with the nearest neighbor method and ending with the two most important methods, namely the histogram and the kernel density. Table 4.2 summarizes the most important aspects of the six methods divided into advantages and disadvantages.

4.2.1 The Nearest Neighbor Measure

The first measure to be discussed is the nearest neighbor method which is an "attempt to adapt the amount of smoothing to the "local" density of data" (Silverman, 1986, p. 19). The k^{th} nearest neighbor measure is given as follows:

$$\hat{f}(t) = \frac{k}{2nd_t(t)}. \tag{4.2}$$

To reach this equation, it is assumed that $f(t)$ gives the density at t. The sample size is given by n. In this case, about $2rnf(t)$ observations will fall in the interval $[t-r, t+r]$ for each $r > 0$. Furthermore, "exactly k observations fall in the interval $[t - dk(t), t + dk(t)]$" (Silverman, 1986, p. 19). Then, setting

$$k = 2d_k(t)n\hat{f}(t), \tag{4.3}$$

the above equation for the k^{th} nearest neighbor estimate can be found. It should be noted that $d(x, y)$ stands for the distance between two points x and y on a line and k is an integer which is smaller than the sample size n. The distance evolves according to

$$d_1(t) \leq d_2(t) \leq \cdots \leq d_n(t). \tag{4.4}$$

The elements $d_n(t)$ indicate the distance from t to the points of the sample. As $k < n$[70], $d_k(t)$ is one element of this rule. The distance here is calculated as the absolute difference between x and y, hence $|x - y|$, which is commonly done. As an example, Figure 4.1 gives the nearest neighbor estimate for the Old Faithful data[71] with $k = 20$ (Silverman, 1986).

The nearest neighbor measure has some disadvantages. In contrast to (at least some kinds of) the kernel density estimator, which will be shown later, it is not a smooth curve. It is a continuous function; however the derivative is discontinuous at some points. It is not, like others, by itself a probability density because it does

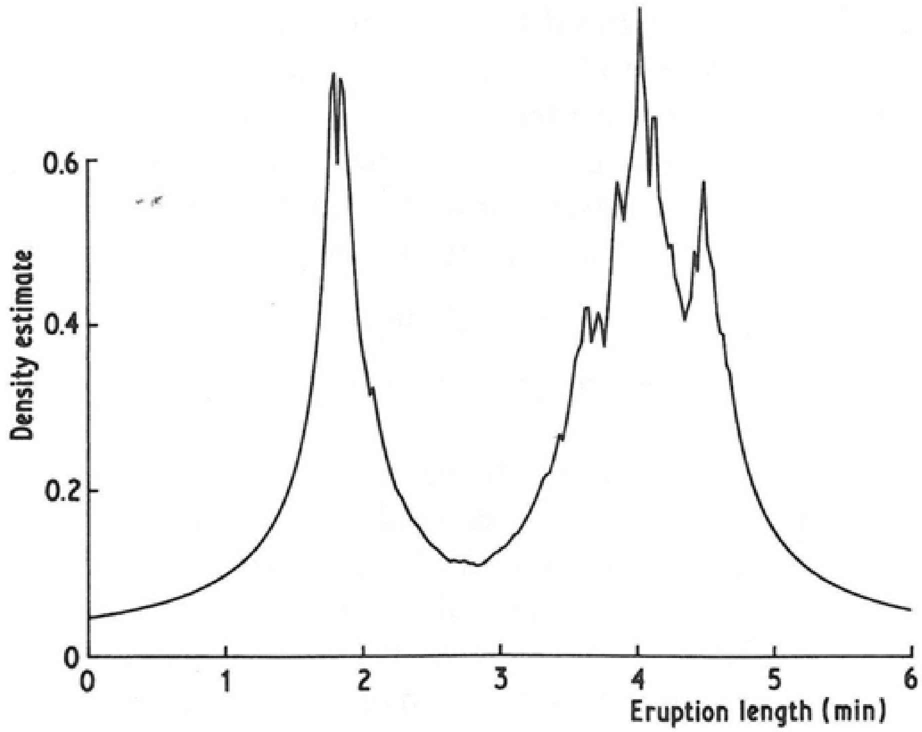
[69] For the remaining four estimators, the interested reader is referred to Silverman (1986).

[70] To be more precise, k is a considerably smaller integer than n, typically $k \approx n^{1/2}$ (Silverman, 1986).

[71] The Old Faithful is a geyser in the Yellowstone National Park in the USA. The dataset comprises eruption data for the Old Faithful Geyser. It is used for density estimations by Silverman (1986). Table A2 in the Appendix (A.2) gives an overview of this dataset.

not integrate to unity. Additionally, it is not appropriate if an estimate of the entire density is needed (Silverman, 1986).

Figure 4.1 The Nearest Neighbor Estimate



Source: Silverman, 1986, p. 20

4.2.2 The Naïve Estimator

Another method for density estimation is the already mentioned naïve estimator, which is defined as follows:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}w\left(\frac{x-X_i}{h}\right) \tag{4.5}$$

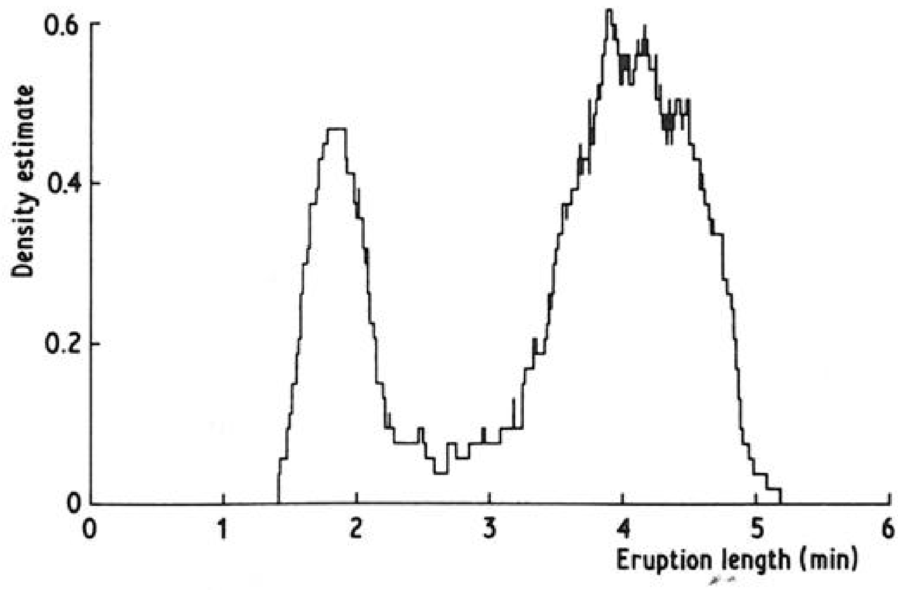
where w is a weight function defined as

$$w(x) = \begin{cases} \frac{1}{2}\, if\ |x| < 1 \\ 0\ otherwise \end{cases}. \tag{4.6}$$

X_i denotes the observations whose underlying density shall be estimated, x is a specific observation, h stands for the box width, and n for the size of the population. "[T]he estimate is constructed by placing a "box" of width $2h$ and a height $(2nh)^{-1}$ on each observation and then summing to obtain the estimate" (Silverman, 1986, p. 12). The naïve measure is the sum of boxes centered at the observations. It is an attempt to construct a histogram where every point is the center of a sampling interval. Figure 4.2 gives an example of such a naïve estimator for the

Old Faithful Geyser data[72] with $h = 0.25$. One disadvantage of the naïve estimate is that it is not continuous. There are several jumps and in the other points the derivative is equal to zero. It gives the measure a ragged character, which is not only aesthetically undesirable but can also lead to misinterpretations of the actual distribution. These disadvantages are overcome by the kernel density which will be the subject of Section 4.3.

Figure 4.2 The Naïve Estimator



Source: Silverman, 1986, p. 13

4.2.3 Stem-and-Leaf Plots

Stem-and-leaf plots are useful for the presentation of statistic characteristics. They give information about the form of the density and the numerical data are not lost, as it is the case with other density estimators such as the histogram (Reinboth, 2009). It looks like a sort of check list: to begin with, the first digit of an observation within a class is noted down for all classes. Then, a vertical line is drawn. To the right of it, the second digit is written according to the size. The following digits will be ignored. In case that the common digits of observations are equal for several classes, the respective interval can be marked by an additional sign (Hartung, 1985). With the output of the stem-and-leaf plot, also the stem width and the information about how many cases are represented by one leaf are given in the output (Reinboth, 2009).

The advantage of stem-and-leaf-plots is that the values of the original dataset are not lost (except for some rounding) and hence they give a good insight into the

[72] The data can be found in The Appendix (A.2).

structure of the data. If for example, the data in the data block 40 to 49 are mainly close to 50, then this information can be read off from the stem-and-leaf plot. The same is true if most numbers in this block lay around 42. The histogram, on the contrary, would not give a different picture in these two cases, even though it has to be noted here that this strongly depends on the class division used for the histogram. This detailed information would be lost. However, stem-and-leaf plots also have shortages. For example, they are unclear for large datasets: they cannot easily be presented on paper or on the screen. For this reason, they are no good alternative for the purpose of this doctoral thesis. Instead, it is better to group the data and hence form a histogram to visualize the structure of the data (Fahrmeir, Hamerle and Tutz, 1996).

Figure 4.3 Stem-and-Leaf Plot of Real per Capita GDP 1990[73]

```
RGDP 1990: Stem-and-Leaf Plot
Frequency   Stem &  Leaf
   37.00       0 .  0000000000000001111111111111111111111
   26.00       0 .  22222222222222333333333333
   17.00       0 .  44444444444555555
    9.00       0 .  666667777
    8.00       0 .  88889999
    2.00       1 .  11
    7.00       1 .  2233333
    2.00       1 .  44
    2.00       1 .  67
    7.00       1 .  8999999
    8.00       2 .  00000011
    2.00       2 .  22
    3.00    Extremes    (>=26078)

Stem width:  10000.00
Each leaf:   1 case(s)
```

Though, before going on with the presentation of the next nonparametric estimation method, the real per capita GDP data[74] for 1990 will be analyzed in the form of a stem-and-leaf plot to demonstrate what it actually looks like. Figure 4.3 shows the results. It can be read off that the stem width is very large with a value of 10,000. This means that the value to be read off the diagram needs to be multiplied by 10,000 (for example $2.2 \cdot 10{,}000 = 22{,}000$). The stem is the number in front of the

[73] The dataset in 1990 used here comprises 130 countries. The stem-and-leaf plot is determined by the computer package SPSS. The underlying data are taken out of the Penn World Table 6.3.

[74] The dataset will be described in detail in Chapter 5. There, the reader can find the information about the countries covered, the data source, the variables to be used and so on. The dataset used in this chapter does not comprise the same countries as the one in Chapter 5. Yet, the purpose here is to show how the conclusions to be drawn from distribution graphs for the same dataset may differ. Hence, it is not necessary to include the same countries.

decimal point. As there are many observations, each number is used several times as the stem value. Each value behind the decimal point stands for a leaf, hence for one observation point. In the stem-and-leaf plot, only the first decimal is shown. The rest is cut off. Hence, an income of $19,768, for example, would be indicated as 1.9 in the plot, where 1 is the stem and 9 is the leaf. The plot in Figure 4.3 leads the analyst to conclude that there are at least two peaks, but also a third one might be read off. Furthermore, there are three extremes with real per capita GDP values of more than $26,078. As stated above, this plot is not easy to handle for large datasets. In addition, a large amount of more detailed information gets lost due to the cutting off of decimals. Hence, this method will not be used in the doctoral thesis at hand.

4.2.4 The Box Plot

Another nonparametric estimation method is the box plot. The box plot is able to answer the questions whether the empirical distribution of data is symmetrical, how the observations scatter around the median, whether there are outliers and so on (Hartung, 1985). It is a clear way to graphically depict a series of observations. The resulting box covers 50 percent of the observed values. Furthermore, a thick line in the box marks the median of the whole data. Finally, there is an inner-fence and an outer-fence which round off the graphical representation. The borders of the inner-fence are given by

$$h_L - 1.5(h - spread) \; and \; h_U + 1.5(h - spread), \quad (4.7)$$

where h indicates hinges, h_L is the lower hinge and h_U is the upper hinge. The borders of the outer-fence are shown by

$$h_L - 3(h - spread) \; and \; h_U + 3(h - spread). \quad (4.8)$$

Values which lay between the inner-fence and the outer-fence are marked by crosses, those outside the outer-fence by circles. In addition, the adjacent values laying inside the inner-fence and differing the least from the upper or the lower border of the inner-fence are marked by dashed lines. Finally, a box plot can also be notched, which means that a confidence interval at a 90 percent level for the median is marked as well. The upper and the lower notches are given by

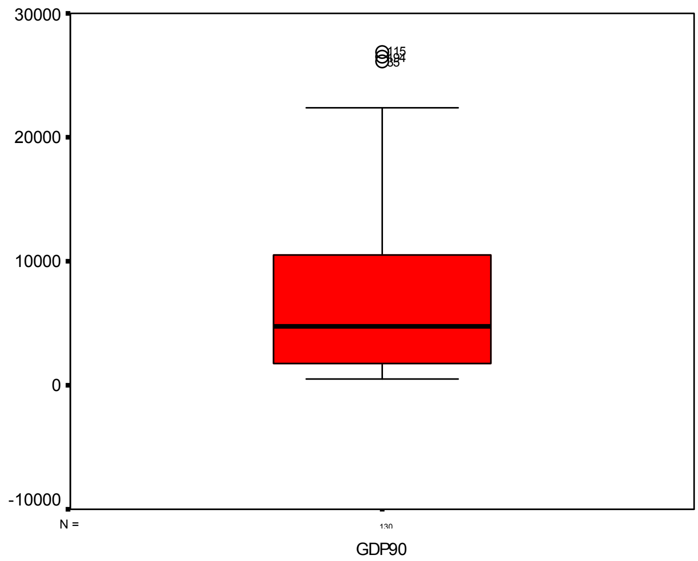
$$m \pm 1.58 \frac{(h - spread)}{\sqrt{n}}. \quad (4.9)$$

Figure 4.4 shows a box plot for real per capita GDP[75] in the year 1990. The box plot is indeed able to show where the median can be found and furthermore, also the skewness can be read off. In this case, the median lies at $4,420.55 and the borders of the box are given by the 25th percentile, namely $1,546.87, and the 75th

[75] Just to remind the reader: the dataset will be described in detail in Chapter 5. There the reader can find the information about the countries covered, the data source, the variables to be used, and so on.

percentile, namely $11,447.83. The outer-fences lie at about $0 and $22,000. It is obvious that the distribution is skewed to the left[76] and there are three outliers, bthe highest one at an income of about $26,892.84. Looking at the data helps identifying these outliers as Switzerland, Luxembourg, and the US. However, by use of the box plot it is not possible to find out whether there are twin peaks or whether the distribution of real per capita GDP is unimodal instead. This shortness of the box plot can be overcome by the histogram, which will be the subject of the next subsection.

Figure 4.4 Box Plot of Real per Capita GDP in 1990[77]



4.2.5 The Histogram

After having described some of the methods for density estimation, now the focus will be on the histogram. It is the most widely used nonparametric density estimator consisting of several bins, each defined by the interval

$$[x_0 + mh, x_0 + (m+1)h[, \qquad (4.10)$$

where x_0 is the origin given, h is the bin width and m represents positive and negative integers. The respective density function is given as

$$\hat{f}(x) = \frac{1}{nh} \cdot (no.\ of\ X_i\ in\ the\ same\ bin\ as\ x), \qquad (4.11)$$

where n stands for the number of observations and X_i indicates a real observation for $i \in [1; n]$ (Silverman, 1986). The choice of the bin width h is of crucial

[76] This underlines the assumption of a Gibrat distribution underlying the real per capita GDP data.
[77] There are again 130 nonoil countries in the dataset. The box plot is determined by the computer package SPSS. The underlying data are taken out of the Penn World Table 6.3.

importance because it controls the amount of smoothing within the procedure. In the graphical representation of the histogram, the area above the intervals is equal or proportional to the absolute or relative frequency (Fahrmeir, Künstler and Tutz, 2008). This characteristic of the histogram is called the principle of equal-areas. The bins of the histogram should be chosen to be of equal size if possible.

From the histogram, the skewness can be read off as well as whether a distribution is unimodal or multimodal. This is its advantage over the box plot. However, it has some shortages as well. The choice of the origin as well as the bin width and hence the number of intervals is decisive for the conclusions to be drawn from the histogram.

The more bins are chosen, the more details can be read off. However, the histogram will be very irregular. The less intervals are chosen, the less jumps will be there, but valuable information will be lost. There are rules of thumb for the choice of the number of bins like, for example, $k = \sqrt{n},\ k = 2\sqrt{n},\ k = 10log_{10}n.$ Nonetheless, the bin widths can also be chosen subjectively. Furthermore, histograms are not continuous, which might be a problem if derivatives are needed.

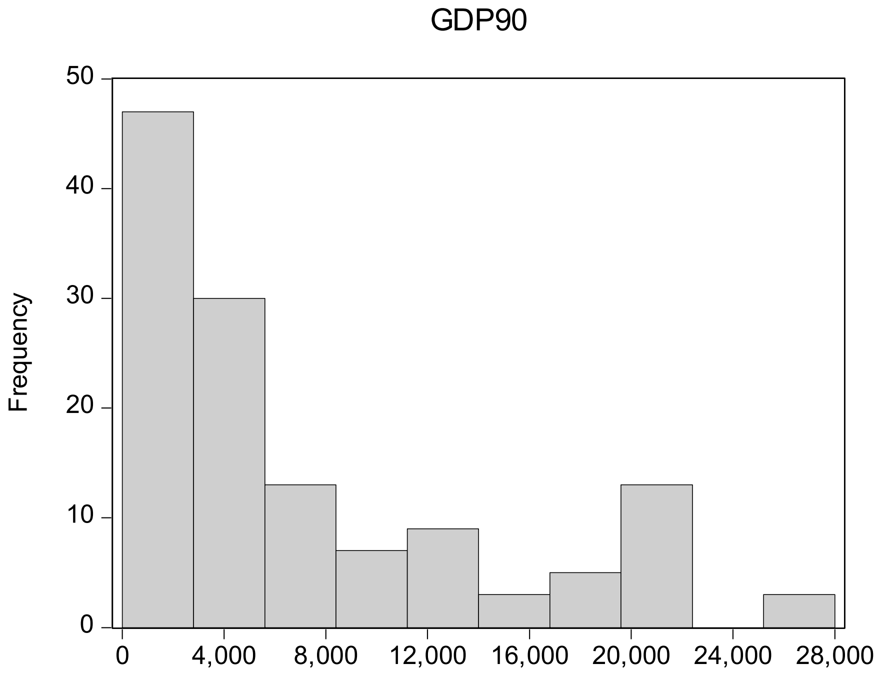
Figure 4.5 shall help to prove that the histogram indeed is a very sensitive way to show distributions. By use of the computer package EViews[78], the data are used with different origins or different bin widths so as to show that these differences might indeed lead to different conclusions. The first two panels present a histogram with a bid width of 2,800 and different origins, namely zero and -500, and the latter two use a bin width of 1,400 with the same origin choices as in Panels (a) and (b). The results are given in Figure 4.5. Here, it becomes apparent that the histogram is indeed sensitive to the choice of the bin width as well as to the choice of origin.

The conclusions to be drawn from Panel (a) are that there are two peaks when the bin width is 2,800 and the origin is zero. The shifting of the origin does not really influence this conclusion. Panel (c), on the contrary, shows that the use of a bin width of 1,400 instead has indeed an influence. Here, the analyst might conclude that there are at least three peaks, if not four.

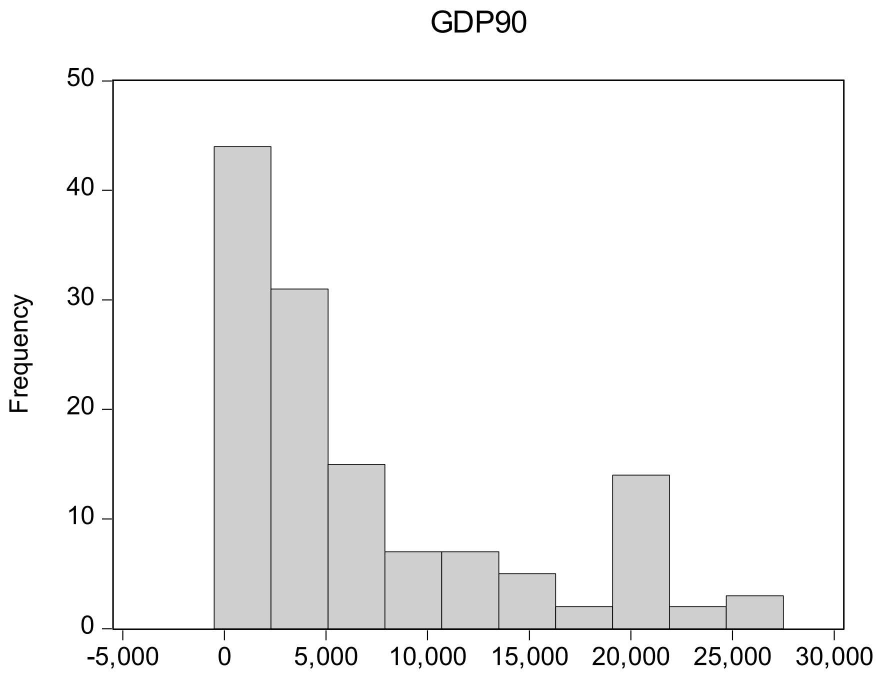
The choice of -500 as origin influences this result as well. There are only two clear peaks but in addition, there are three bins which might as well be interpreted as slight peaks. To sum it up, because of the sensitivity of the histogram for the choice of bin width and origin the histogram is not sufficiently objective for the purpose of finding out whether there are twin peaks in the real GDP data for the respective years. Hence, even though it is still the most widely used method of density estimation, an alternative needs to be found. For the reasons discussed above, the four methods presented before are no good alternative either. The next section will provide the best alternative to be chosen for the purpose of this doctoral thesis, namely the kernel density.

[78] EViews is another econometric computer package which will be used in this doctoral thesis along with SPSS.

Figure 4.5 The Sensitivity of the Histogram[79]

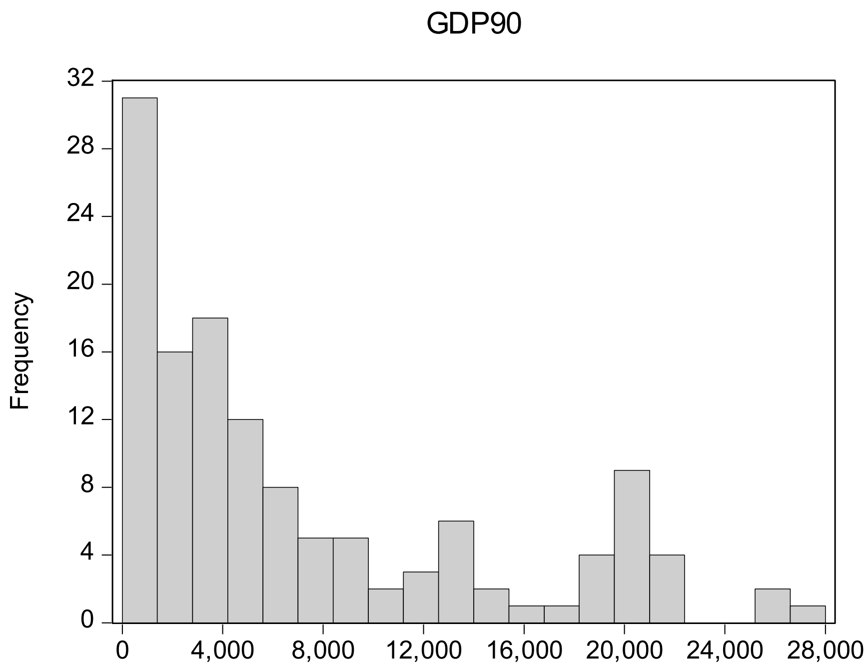


Panel (a): Bin width 2,800, origin 0

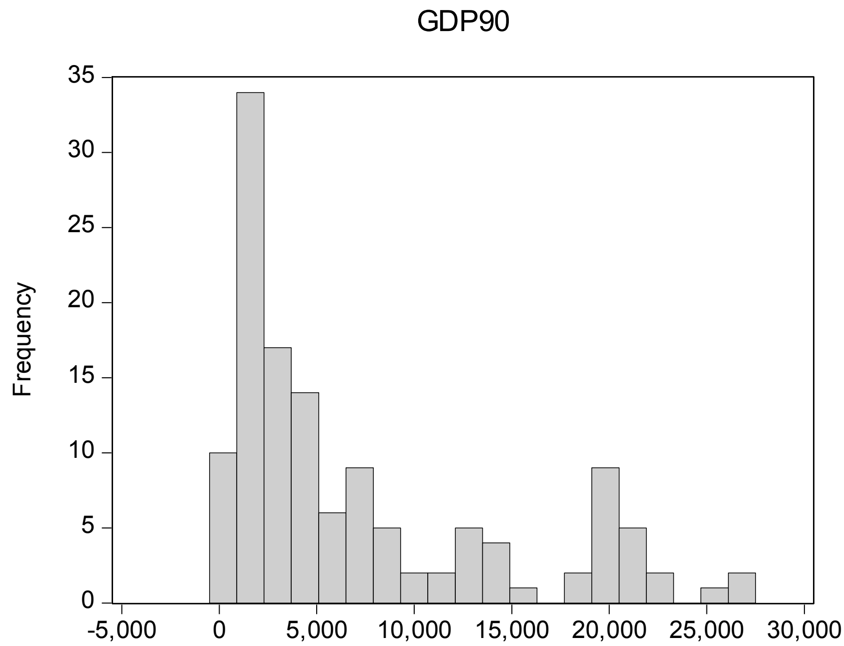


Panel (b): Bin width 2,800, origin -500

79 The dataset comprises 130 nonoil countries in 1990. The histograms are determined by the computer package EViews. The underlying data are taken out of the Penn World Table 6.3.



Panel (c): Bin width 1,400, origin 0



Panel (d): Bin width 1,400, origin -500

4.3 The Kernel Density

"To remove the dependence on the end points of the bins, kernel estimators center a kernel function at each data point. And if we use a smooth kernel function for our building block, then we will have a smooth density estimate" (Mishra, 2007, p. 1). In this way, two of the three main disadvantages of the histogram can be overcome, namely the dependence on the origin and hence the end points of the bins as well as the lack of smoothness of the histogram. The only real disadvantage that seems to remain is the dependence on the choice of the bin width. However, as will be shown later, also this shortage can be overcome by choosing an optimal bandwidth for the kernel density.

The kernel of a function is the main part of the function. It is the part that remains when constants are disregarded (Casella and Berger, 1990). The kernel method is widely used. The kernel density estimator is in fact a generalized naïve estimator which can overcome several of the estimator's problems by replacing the weighting function w by a kernel function K fulfilling the condition

$$\int_{-\infty}^{\infty} K(x)dx = 1. \tag{4.12}$$

Silverman (1986) states that the kernel density estimator replaces the bins or boxes as used in the histogram by so-called smooth bumps, which are placed at the observations. This is done by "putting less weight on observations that are further from the point being evaluated" (EViews, 2009). The density function is given by

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right), \tag{4.13}$$

where n is the number of observations, and is thus known, h is the bin width, K is the kernel function, x is a specific point of a series X, and X_i indicates the $X_1, \dots, X_n$ independent, identically distributed observations of a distribution series X.

In contrast to the histogram and the naïve estimator, the kernel density estimator $(\hat{f})$ is continuous and differentiable, provided that K is nonnegative and satisfies Equation (4.12). The kernel function K determines the shape of the bumps and h determines their width. The choice of h is very decisive, because only a correct – the optimal – h gives the proper presentation of the data, of what the distribution looks like. In the next two subsections the kernel function and the bin width h will be discussed in more detail.

4.3.1 The Kernel Function

The kernel function K can have several forms. Silverman (1986) distinguishes five options which are summarized in Table 4.3. When looking at the efficiencies of these kernels, one can see that they are all close to one and do not differ very much. Thus, the choice of the kernel function does not necessarily have to be based on these efficiencies. The most commonly used kernel function is the Gaussian kernel. However, in this dissertation, the Epanechnikov kernel will be used. The reason for this choice is that according to Table 4.3, it is the most efficient kernel.

Table 4.3 Some Kernels and their Efficiencies

Kernel	K(t)		Efficiency[80]
Epanechnikov	$\frac{3}{4}\left(1-\frac{1}{5}t^2\right)/\sqrt{5}$ 0	$for\ \lvert t\rvert < \sqrt{5}$ otherwise	1
Biweight	$\frac{15}{16}(1-t^2)^2$ 0	$for\ \lvert t\rvert < 1$ otherwise	$\sqrt{\frac{3087}{3125}} \approx 0.9939$
Triangular	$1-\lvert t\rvert$ 0	$for\ \lvert t\rvert < 1$ otherwise	$\sqrt{\frac{243}{150}} \approx 0.9859$
Gaussian	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$		$\sqrt{\frac{36\pi}{125}} \approx 0.9512$
Rectangular	$\frac{1}{2}$ 0	$for\ \lvert t\rvert < 1$ otherwise	$\sqrt{\frac{108}{125}} \approx 0.9295$

Source: Silverman, 1986

4.3.2 The Optimal Bandwidth

In addition to the kernel function, also the appropriate bandwidth needs to be determined. The bandwidth controls the smoothness of the kernel density estimate. A larger bandwidth implies a smoother estimate. The optimal value of the bandwidth h is the solution to Equation (4.14), hence the minimum of the approximate mean integrated square error (MISE; see Silverman, 1986):

$$min\frac{1}{4}h^4k_2^2\int f''(x)^2dx + n^{-1}h^{-1}\int K(t)^2\,dt \tag{4.14}$$

The MISE is the “first and most widely used way of placing a measure on the global accuracy of $\hat{f}$ as an estimator of [the true but unknown density] f”, which needs to be determined (Silverman, 1986, p. 35):

$$MISE(\hat{f}) = E\int\{\hat{f}(x) - f(x)\}^2\,dx. \tag{4.15}$$

The solution to Equation (4.15) is

$$h_{opt} = k_2^{-\frac{2}{5}}\{\int K(t)^2\,dt\}^{\frac{1}{5}}\{\int f''(x^2)dx\}^{-\frac{1}{5}}n^{-\frac{1}{5}}, \tag{4.16}$$

[80] Exact to the fourth decimal place.

where $k_2 = \int t^2 K(t)dt \neq 0$. The ideal bandwidth is expected to converge to zero when the sample size n increases. Nevertheless, this will occur at a slow rate.[81]

The procedure of finding the optimal h is rather complex. Hence, Silverman (1986) gives six alternatives for this procedure. The first is subjective choice. This means that the analyst plots out several curves and then chooses the estimate which is most in accordance with his prior expectations of the density. The second option is called "reference to a standard distribution". It implies the use of a standard family of distributions to assign a value to the term $\int f''(x)^2 dx$ in Equation (4.16). The least-squares cross-validation procedure is completely automatic. A fourth alternative is the likelihood cross-validation in which there is a natural development of the idea of using likelihood to judge the adequacy of fit of a statistical method. Next, Silverman describes the test graph method which "aims to yield estimates that are uniformly close to the true density" (Silverman, 1986, p. 55). Finally, there is the internal estimation of the density roughness. For a more detailed discussion please refer to Silverman (1986).

For the purpose of this doctorate it is sufficient to look at the alternatives given by EViews[82], namely the options "Silverman" and "user specified". The latter allows choosing one of the methods described before and then entering the result into the program. However, the option "Silverman" means that a data based automatic bandwidth is chosen according to the second option, namely reference to a standard distribution. This option will be described in more detail here. Silverman (1986) gives an example of the normal distribution with the variance σ^2 and the standard normal density Φ. Then

$$\int f''(x)^2 dx = \sigma^{-5} \int \phi(x)^2 dx \tag{4.17}$$

$$\int f''(x)^2 dx = \tfrac{3}{8}\pi^{-\frac{1}{2}}\sigma^{-5} \approx 0.212\sigma^{-5}. \tag{4.18}$$

Using a Gaussian kernel and substituting the value of Equation (4.17) into (4.16) yields

$$h_{opt} = (4\pi)^{-\frac{1}{10}} \left(\tfrac{3}{8}\pi^{-\frac{1}{2}}\right)^{-\frac{1}{5}} \sigma n^{-\frac{1}{5}} \tag{4.19}$$

$$h_{opt} = \left(\tfrac{4}{3}\right)^{\frac{1}{5}} \sigma n^{-\frac{1}{5}} = 1.06\sigma n^{-\frac{1}{5}}. \tag{4.20}$$

Hence, "a quick way of choosing the smoothing parameter, therefore, would be to estimate σ from the data and then to substitute the estimate into [Equation (4.19)]. Either the usual sample standard deviation or a more robust estimator of σ could be used" (Silverman, 1986, p. 46). Equation (4.19) is a good estimator for the bandwidth only if the population is normally distributed. Multimodal populations

81 For a more detailed description of the process, the interested reader is referred to Silverman (1986). The calculus to find h_{opt} can be found in Parzen (1962, Lemma 4A).

82 EViews is another econometric computer package which will be used in this doctoral dissertation along with SPSS.

may be oversmoothed (Silverman, 1986). To overcome this risk of oversmoothing, a robust measure of spread should be used.

Writing Equation (4.19) in terms of the interquartile range R of the underlying normal distribution yields

$$h_{opt} = 0.79Rn^{-\frac{1}{5}}. \quad (4.21)$$

Though, this equation is not appropriate for bimodal distributions as it oversmooths even more. To overcome this problem, the adaptive estimate of spread should be used instead:

$$A = min(standard\ deviation,\ interquartile\ range/1.34). \quad (4.22)$$

A shall replace σ in Equation (4.19). Furthermore, to improve the smoothing procedure even further, the factor 1.06 should be reduced. Equation (4.22) gives the bandwidth for a Gaussian kernel which "will yield a mean integrated square error within 10 [percent] of the optimum for all the t-distributions considered, for the log-normal with skewness up to about 1.8, and for the normal mixture with separation up to three standard deviations" (Silverman, 1986, p. 48). The resulting function, also underlying the option "Silverman" in EViews is given by Equation (4.23):[83]

$$h = 0.9An^{-\frac{1}{5}} \quad (4.23)$$

4.3.3 The Kernel Density in 1990

After having defined the kernel function and the bandwidth to be used, this section will show the application of the kernel density to the real per capita GDP data[84] in 1990, so that the results can then be compared with those of the box plot and the histogram. From Figure 4.6, it becomes apparent that in 1990, there were indeed twin peaks in the per capita GDP data. The optimal bandwidth calculated by EViews is 5,123.1 and it can be retrieved from the graph that the first peak lays at about \$2,800 and the second peak at about \$20,000. As stated before, the box plot is not able to answer the question whether there are one or more peaks in the real per capita GDP data.

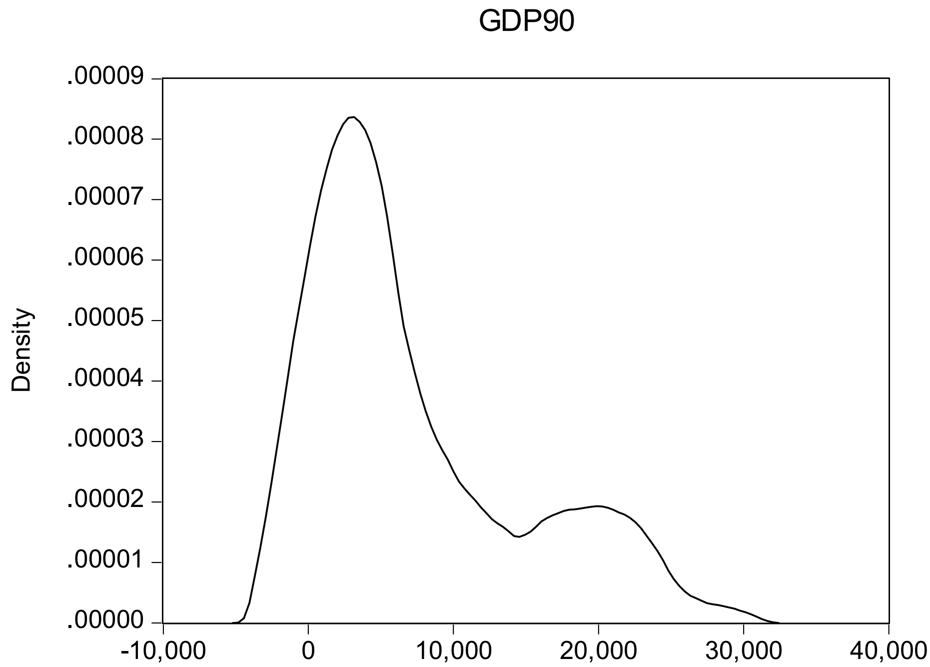
The skewness, which is also shown in the kernel density, however, is the same. Both graphs allow the analyst to conclude that the distribution is skewed to the left. Taking into account the histogram, first with ten classes and an origin of zero, leads to the conclusion that the histogram as well as the kernel density suggest twin peaks. However, the position of the peaks differs. Whereas the histogram in

[83] One further option which can be chosen is "bracket bandwidth" leading to kernel densities with the bandwidths $0.5h$, h, and $2h$. This offers the analyst the opportunity to examine the sensitivity of the estimates to variations in the bandwidth.

[84] The dataset will be described in detail in Chapter 5. There the reader can find the information about the countries covered, the data source, the variables to be used and so on.

Panel (a) of Figure 4.5 (see p. 68f) shows a first peak at $1,375 and a second one at $20,625, the one in Panel (b) shows a first peak at $900 and a second one at $20,500. In contrast, the kernel density which is not sensitive to the choice of the bin width and the origin, shows the peaks at $2,800 and $20,000. Both, the histogram with ten classes as well as the kernel density come to the same conclusion, but especially the lower peak is underestimated by the histogram. Using 20 classes instead, the two methods come to different conclusions. The histogram points to more than two peaks. While the second large peak is still at about $20,000, the first peak is underestimated even further. Additionally, further peaks appear at $13,062.5 and at $26,800 in Panel (d) of Figure 4.5.

Figure 4.6 The Kernel Density in 1990[85]



4.4 The Markov Chain

After having described the several methods that can be used for density analysis in this doctoral thesis, this section will deal with another important method which is widely used in the context of twin peaks analysis: the Markov chain. The Markov chain or more generally spoken the Markov process is “used to measure or estimate movements over time” (Chiang and Wainwright, 2005, p. 78). The Markov chain consists of two elements, namely the Markov transition matrix and a vector

[85] The dataset comprises 130 countries in 1990. The kernel density is determined by the computer package EViews. The underlying data are taken out of the Penn World Table 6.3.

that contains the initial distribution across the various states. This vector is then multiplied repeatedly with the initial vector. The consequence is that the situation at time t only depends on the situation at time $t-1$.[86] In this doctorate, the vector will contain the states of the real per capita GDP income distribution in the starting year of the analysis. In the next two subsections, the elements of the Markov chain, namely the transition matrix and the corresponding vector of the initial distribution will be described in more detail.

4.4.1 The Transition Matrix

The transition matrix gives the transition probabilities. In its simplest form, there are only two income groups: Group 1 and Group 2. Thus, the probabilities that a country moves from one income group to the other are as follows:

p_{11} = probability that a country currently in Group 1 remains in Group 1
p_{12} = probability that a country currently in Group 1 moves to Group 2
p_{22} = probability that a country currently in Group 2 remains in Group 2
p_{21} = probability that a country currently in Group 2 moves to Group 1.

Consequently, the transition matrix has the form

$$M = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix}. \tag{4.24}$$

This matrix is based on several conditions (Chiang, 2005):

$$p_{11}, p_{12}, p_{21}, p_{22} \geq 0 \tag{4.25}$$

$$p_{11} + p_{12} = 1 \tag{4.26}$$

$$p_{21} + p_{22} = 1. \tag{4.27}$$

Of course, this can also be extended to more income groups so that a $5x5$ matrix or a $6x6$ matrix results, depending on the number of income groups defined.

4.4.2 The Vector of the Initial Distribution

As stated above, the vector of the initial distribution gives the distribution of countries across the income groups at time t

$$\overrightarrow{x_t} = \begin{bmatrix} G1_t \\ G2_t \end{bmatrix}, \tag{4.28}$$

where $G1_t$ stands for the size of the income Group 1 at time t and $G2_t$ stands for the size of the income Group 2 at time t respectively.

[86] In the Appendix (A.3) it is explored that under certain conditions the Markov chain already implies a stationary distribution as its result. It will be shown that also in this doctorate, this is the case so that the results shown in Section 5.5.1.3 do not come as a surprise but rather stem to the underlying characteristics of the Markov chain as presented here.

4.4.3 The Distribution across Income Groups in the next Period

In order to determine the distribution of countries across the income groups in the next period, the transition matrix and the distribution vector need to be multiplied.

$$M\overrightarrow{x_t} = \overrightarrow{x_{t+1}} \tag{4.29}$$

$$\begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} G1_t \\ G2_t \end{bmatrix} = \begin{bmatrix} p_{11}G1_t + p_{21}G2_t \\ p_{12}G1_t + p_{22}G2_t \end{bmatrix} = \begin{bmatrix} G1_{t+1} \\ G2_{t+1} \end{bmatrix} \tag{4.30}$$

$$p_{11}G1_t + p_{21}G2_t = G1_{t+1} \tag{4.31}$$

This indicates that the number of countries currently in Group 1 ($G1_t$) has a probability of p_{11} to remain there. Being multiplied, this product needs to be added to those countries which fall behind in the next period. $G2_t$ indicates the number of countries currently being in Group 2. This number is then multiplied by p_{21} which is the probability of a country currently in Group 2 falling back to Group 1 ($G1_{t+1}$) at time $t+1$. For Group 2 ($G2_{t+1}$) the same can be done. Hence, we arrive at $\overrightarrow{x_{t+1}}$. Now, this process can be repeated for the next period:

$$M\overrightarrow{x_{t+1}} = \overrightarrow{x_{t+2}} \tag{4.32}$$

$$\Leftrightarrow MM\overrightarrow{x_t} = \overrightarrow{x_{t+2}} \tag{4.33}$$

$$\Leftrightarrow M^2\overrightarrow{x_t} = \overrightarrow{x_{t+2}} \tag{4.34}$$

$$\Leftrightarrow \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} G1_t \\ G2_t \end{bmatrix} = \begin{bmatrix} G1_{t+2} \\ G2_{t+2} \end{bmatrix} \tag{4.35}$$

$$\Leftrightarrow \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix}^2 \begin{bmatrix} G1_t \\ G2_t \end{bmatrix} = \begin{bmatrix} G1_{t+2} \\ G2_{t+2} \end{bmatrix}. \tag{4.36}$$

This can also be done for n periods:

$$M^n\overrightarrow{x_t} = \overrightarrow{x_{t+n}} \tag{4.37}$$

$$\begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix}^n \begin{bmatrix} G1_t \\ G2_t \end{bmatrix} = \begin{bmatrix} G1_{t+n} \\ G2_{t+n} \end{bmatrix}. \tag{4.38}$$

This is called the Markov process or the Markov chain; if n is exogenous, the Markov chain is said to be finite.[87]

4.5 The Loess Fit Method

After having discussed the Markov chain, this section will present another way to analyze the real per capita income distribution: the loess fit method. The classical estimation methods are linear and nonlinear least squares regressions. Loess

[87] In this chapter, the subject is the theoretical discussion of the methods to be used in this doctoral thesis for the analysis of the real per capita GDP data. The empirical analyses in this chapter were only for demonstrating that different methods might lead to different conclusions. The essential analyses of the data for the empirical study will be undertaken in Chapter 5. Hence, at this point in time no Markov chain will be calculated. The reader is referred to Chapter 5 for this part.

(locally weighted polynomial regression) is one of several modern modeling methods. It was found for those situations in which the classical methods do not work well or are too complicated. The loess method "combines much of the simplicity of linear least squares regression with the flexibility of nonlinear regression […] by fitting simple models to localized subsets of the data to build up a function that describes the deterministic part of the variation in the data, point by point" (Nist/Sematech, 2012, p. 1). In this way, the analyst does not have to specify a model to the whole dataset but only to segments.

The loess method was first found by Cleveland (1979). Together with Devlin he extended this method later on (Cleveland and Devlin 1988). The method works as follows: for each point in the dataset a subset of the data is determined. In this subset, a low-degree polynomial is fit to the data by using weighted least squares. The idea is to give those points being close to the point to be estimated more weight because they are assumed to be highly related to this point. The further away a point of the subset is to this point, the less related it is expected to be. For this reason, the weight given to the points within the subset decreases with the distance to the point to be estimated (Nist/Sematech, 2012).

For finding the subset of data for each of the weighted least squares fits in loess, a nearest neighbors algorithm is applied. The size of the subset depends on the bandwidth given by the analyst. This bandwidth is the smoothing parameter. "The smoothing parameter, q, is a number between $(d+1)/n$ and 1, with d denoting the degree of the local polynomial. The value of q is the proportion of data used in each fit. The subset of data used in each weighted least squares fit is comprised of the nq points (rounded to the next largest integer) whose explanatory variables are close to the point at which the response is being estimated" (Nist/Sematech, 2012, p. 2). The bandwidth q controls the flexibility of the loess regression function: if q is too large, then the loess regression function does not well represent the data. If q is too small, the random error may be captured. Typically, q is chosen out of the interval $[0.25; 0.5]$ (Nist/Sematech, 2012).

The local polynomials fit to each subset of the data tend to be either locally linear or locally quadratic. Other polynomials are possible but usually not very helpful. If the polynomials used are too high, the data might be overfit in each subset. In general, "loess is based on the ideas that any function can be well approximated in a small neighborhood by a low-order polynomial and that simple models can be fit to data easily" (Nist/Sematech, 2012, p. 2).

Another element of loess is the weight function. It usually gives most weight to the closest points, just as outlined above. The further the points are away from the point to be estimated, the less weight they will have, because then they tend to be less related to the point to be estimated. Traditionally, the following weight function is used for loess:

$$w(x) = f(x) = \begin{cases} (1-|x|^3)^3, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}. \tag{4.39}$$

This is the tri-cube weight function.[88] "The weight for a specific point in any localized subset of data is obtained by evaluating the weight function at the distance between that point and the point of estimation, after scaling the distance so that the maximum absolute distance over all of the points in the subset of data is exactly one" (Nist/Sematech, 2012, p. 3).

The advantages of the loess method can be summarized as follows: no specification of a function fitting all of the data in the sample is needed, the method is very flexible, and it is simple. However, there are also some shortcomings of this method. First of all, the use of data is not efficient. Large datasets are needed in order to get good results. In addition, the results cannot easily be presented. There is no regression function that results. Finally, outliers are controlled for by the loess method. But if the outliers are extreme, the result will definitely be affected. Due to the advantages mentioned above, the loess method shall also be applied to the data used in this doctorate.

4.6 Conclusion

This chapter gave an overview of the empirical methods for income distribution analysis. After a short introduction, a literature review of the relevant empirical studies concentrating on the twin peaks phenomenon was given. The focus of this review was on the choice of methods for the empirical analyses. It could be shown that two methods dominate the polarization debate: the kernel density estimate and the Markov chain. In order to work out why these two methods are of importance, first an overview was given of the most common methods available for the purpose of describing the distribution of real per capita GDP.

The advantages and disadvantages of the methods of nonparametric data analysis were discussed in detail. It was demonstrated that neither the stem-and-leaf-plots nor the box plot or even the histogram are suitable for the problem at hand. The box plot proved to be inconvenient for the phenomenon of twin peaks as it is unable to show more than one peak. It only covers the median and shows how the observations are distributed around it. Furthermore, the histogram has a decisive disadvantage: it is highly sensitive to the choice of the bin width as well as to the choice of the origin. Thus, different conclusions might be drawn from the dataset when the histogram is based on different assumptions. This is not desirable. The shortages of the box plot and the histogram are overcome by the kernel density if the optimal bandwidth h can be found. As this is the case, the kernel density is the first-best method for the multimodality analysis of real per capita GDP; hence, it will be used in this doctoral thesis. In addition to this, the method of the Markov chain was described which is the most common discrete method for the twin peaks analysis already used by several authors before and also being used here to

[88] Other weight functions are also possible. For the properties the weight functions have to fulfill, the interested reader is referred to Cleveland (1979).

enable a look into the future. Finally, in this chapter also the loess fit method was presented. It will be used to support the determination of the savings function later on.

The application of the methods described in this chapter will be the subject of the next chapter, which contains the empirical study being the basis for the determination of the models in Chapters 6 and 7.

5 Empirical Analysis of the Polarization Phenomenon

In this chapter, the empirical analysis of the twin peaks phenomenon will be presented. It will be shown that the world income distribution indeed polarizes into rich and poor. At this point, the reader should be reminded that the focus of this doctorate is on the income distribution across countries ignoring population size as well as income inequality within countries, hence between individuals, for the reasons outlined above. Before this analysis can be done, a literature review will be given. This review does not claim completeness. The growth literature is rising on the polarization issue and so there are several studies which empirically elaborate on the problem. In the following section, the main empirical literature will be reviewed in order to show which results other authors found. Afterwards, Section 5.2 will give an insight into the different data sources being used. In Section 5.3, the focus is on the countries covered by the empirical analyses presented in this chapter. Knowing which data to use, Section 5.4 will then present the descriptive statistics which help to understand what the data look like. Thereafter, Section 5.5 will present the distribution analysis for the central variables, namely real per capita GDP, the savings rate, and human capital. Here, also the question of whether there are outliers to be excluded from the dataset will be addressed. In Section 5.6, loess fit curves will then give further insight into the conjunction of the savings rate and GDP on the one hand, and human capital and GDP on the other hand. Section 5.7 will conclude this chapter.

5.1 Literature Review

The twin peaks phenomenon was empirically analyzed in several studies. In this section, the main empirical findings will be reviewed. From growth theory it is well-known that each country tends towards its own steady state; the steady state paths may differ among countries, depending on the determinants of long run growth (Quah, 1993a). Discussing growth is directly associated with the examination of convergence. The convergence debate has engaged economists for decades and will probably do so in the future. In general, there has been a tendency of the countries to move upwards in the income distribution (Jones, 1997). Pritchett (1997) finds that the ratio of per capita income of the rich countries to the poor ones increased by a factor of about five from the year 1870 to 1990. This shows that indeed convergence and especially divergence are important subjects in the context of economic growth. Yet, it is not sufficient to talk about convergence versus divergence only (Quah, 1997). Instead, the term “club convergence” has become more and more important as a third alternative. This means that two or more clubs form endogenously according to their economic structure in the income distribution across the countries of the world. Within these clubs there might be convergence or divergence. The clubs may be institutional, cultural, or geographical (Desdoigts, 1999). The existence of these clubs even became a new stylized fact in economic growth theory.

As already outlined in the previous chapter, the authors use different methods for the examination of the real per capita income data for twin peaks. Because they were already described in more detail before, these methods will only briefly be considered here. The traditional method for analyzing convergence questions was regressing average growth rates on initial levels of income using a cross-section analysis. If the coefficient is smaller than zero, there is convergence. If the coefficient is rather positive instead, there is divergence (Quah, 1993b). Yet, several authors, such as Ben-David (1997) or Quah (1993b), work out that this method is not without problems. Especially Quah (1993b) points to Galton’s classical fallacy of regression towards the mean. It can be defined as follows: “a negative initial condition provides a force for the cross-section distribution to collapse; however, ongoing disturbances provide a force in the opposite direction” (Quah, 1993b, p. 433). Another way of explaining Galton’s fallacy is that the coefficient of the initial condition is never bigger than zero, even if the cross-section distribution remains invariant over time. As the traditional method for convergence analysis is not being used anymore nowadays, alternatives need to be found. What becomes apparent is that one of the most commonly used methods is the kernel density as a method of nonparametric density estimation. This method was used by several authors: for example Quah (1997), Beaudry, Collard and Green (2002), Bianchi (1997), Jones (1997), Krüger, Cantner and Pyka (2003), Chumacero (2006), as well as Margaritis, Färe and Grosskopf (2007).

A further tool, which is also often used in the context of twin peaks analysis, is the Markov chain. Next to the kernel densities, Quah (1993a) also uses this method. Additionally, Pearlman (2003) needs to be mentioned in this context.[89] Yet, he also points out the shortages of this method. He forms five different income groups, which is a common practice: State 1 (0 to 25 percent of the mean of the world income), State 2 (25 to 50 percent of the mean world income), State 3 (50 to 100 percent of the mean world income), State 4 (100 to 200 percent of the mean world income), and State 5 (more than 200 percent of the mean world income). Yet, “[if] all countries end up in the lowest income state, then the lowest income state should actually be the mean, but the mean of the distribution is by definition exactly between states 2 and 3” (Pearlman, 2003, p. 79). Pearlman (2003) also criticizes that many authors use the real per capita GDP of the United States as a reference and not the mean of world income. This means, according to him, that no country will ever have a per capita GDP greater than that of the United States.

[89] Other authors using the Markov chain method are for example Kremer, Onatski and Stock (2001), Chumacero (2006), as well as Feyrer (2008). Another, very rarely used method for convergence analysis is the Lorenz curve. Ben-David (1997) uses it in order to show that the income distribution has indeed become more unequal. Additionally, also convergence tests and panel unit root tests are applied by some authors (see for example Chumacero (2006), Margaritis, Färe and Grosskopf (2007)). However, these additional methods will not be elaborated on here as they are of minor relevance for the twin peaks discussion.

However, he also points out that while he does not find evidence for twin peaks by using the world income mean as a reference, slight evidence for polarization can be found by using the income of the United States as a reference.

Looking at the results from several empirical studies shows that most authors indeed find twin peaks. This gives rise to the importance of examining this phenomenon further. Danny Quah (as the so called father of the twin peaks theory who formed the expression “twin peaks”) comes to the conclusion that the two camp world, as he calls it, is indeed a robust result in the long run (Quah 1993a, 1993b). A country is deemed to be either rich or poor, there is nothing in between. The convergence clubs form endogenously, hence, the income distribution across the countries of the world polarizes over time. Even the existence of multiple peaks, which is also called stratification, is a possible result (Quah, 1996c). Quah finds in his analyses that there is a large group of poor countries and a much smaller group of very rich countries (Quah, 1997). Besides, Jones (1997) shows that there are twin peaks in real per capita GDP across the countries of the world. He examines the income distributions in two years, namely in 1960 and in 1988. While in the former case, there is only one peak, 28 years later twin peaks formed (Jones, 1997). Paap and van Dijk (1998) also find this evidence on the existence of twin peaks. They use a period from 1960 to 1989 as a reference and a dataset covering 120 countries. The twin peaks in the income distribution seem to be a robust result. In addition, Bianchi (1997) reports polarization in the income distribution. The clubs form over time and the middle income class decreases in size. Bianchi (1997) also points to the fact that there are few intra-distributional dynamics.

Other authors who find only a single peak in 1960 but twin peaks later on are Semmler and Ofori (2007) as well as Beaudry, Collard and Green (2002). According to them, the polarization has established in 1998. The authors find that bimodality evolved from about 1978 on. Looking at the results published by Krüger, Cantner and Pyka (2003) confirms that there are indeed twin peaks from 1970 on. They use income per worker relative to US income as a variable. However, they also point to the question of whether twin peaks are indeed a long run phenomenon or rather a transitory fact instead. This option is treated by Kremer, Onatski and Stock (2001) who find that in the end the high income peak gets larger over time and might once dominate the whole income distribution. Yet, “a prolonged transition during which some inequality measures increase” (Kremer, Onatski and Stock, 2001, p. 275) might exist. This finding of a strong high income peak is also supported by Feyrer (2008), even though the twin peaks are found to be a persistent result here.

Table 5.1 Differences in Empirical Analyses

Author	Data Source[90]	Variable Used[91]	Years Covered	Number of Countries[92]	Method Applied
Beaudry, Collard and Green (2002)	Penn World Table 6.0	Real GDP per worker at constant world prices	1960 to 1998	75	Kernel density
Bianchi (1997)	Penn World Table 5.6	Per capita GDP at constant US $	1970, 1980, 1989	119	Kernel density
Chumacero (2006)	Penn World Table	Real GDP per capita relative to average world GDP	1960, 1995	85	Kernel density Markov chain
Feyrer (2008)	Penn World Table	RGDPCH (real per capita GDP as chain index)	1970 to 1989	90	Markov chain
Jones (1997)	Penn World Table	Real GDP per worker relative to US income; weighted by population	1960, 1988	121	Kernel density Markov chain
Kremer, Onatski and Stock (2001)	Kraay (1999)[93]	RGDPCH (real per capita GDP as chain index)	1960 to 1996	128[94]	Markov chain[95]
Paap and van Dijk (1998)	Penn World Table 5.6	Real per capita GDP	1960 to 1989	120	Histogram Markov chain
Pearlman (2003)	Penn World Table	RGDPCH (real per capita GDP as chain index)	1960 to 1984	120	Markov chain
Krüger, Cantner and Pyka (2003)	Penn World Table	Real income per worker relative to US income	1960 to 1990	104	Kernel density
Quah (1993a)	Penn World Table 5.6	RGDPL (real per capita GDP as Laspeyres index)	1962 to 1985	118	Markov chain
Quah (1997)	Penn World Table 5.6	Log of per capita income relative to world average; weighted by population	1961 to 1988	105	Kernel density
Semmler and Ofori (2007)	Penn World Table	RGDPL (real per capita GDP as Laspeyres index)	1960 to 1985	113	Kernel density Markov chain

[90] Where known, the exact version is mentioned.
[91] Where known, the exact name of the variable as mentioned in the data source is indicated.
[92] Many authors (for example Jones (1997), Kremer, Onatski and Stock (2001), Pearlman (2003), and Feyrer (2008)) indicate that they only use countries offering data in all years under consideration. For the others nothing is stated in the respective articles.
[93] Extended version of Penn World Table 5.6
[94] Without countries where oil and natural resources make up for more than 15 percent of GDP.
[95] Twin peaks are only found when using 5-year transitions instead of 1-year transitions.

Contrary to the previously mentioned authors, Chumacero (2006) not only finds evidence for twin peaks from a certain point in time on. He examines the income distributions in 1960 and in 1995, where he finds clear twin peaks in the latter year, but a slight bimodal income distribution appears also for the year 1960. Even though the kernel estimation as well as the Markov chain analysis point to this result, Chumacero (2006) argues that the results in this case are sensitive to the thresholds for each category chosen, for the number of years used for the computation of the transition matrix, and for the variable used to perform the comparisons.

As could be seen, many authors found evidence for twin peaks in the real per capita income distribution. Yet, there are also authors who cannot support this hypothesis. Pearlman (2003) cannot find strong evidence for the existence of twin peaks in the data. This result changes, however, when using log transformations instead of levels. Also Quah (1997) uses logs. He finds that taking logs makes the peaks getting closer together than taking levels. Furthermore, he also tries to weigh the income data by population. While Quah (1997) finds multiple peaks in this case, Jones (1997) states that weighing by population[96] makes the poor countries start to catch up with the rich ones.

The question that arises is why these different findings of twin peaks or unimodality occur. Of course there are several possibilities. The first one is that the authors use different datasets characterized for example by a different source, different time periods, different variables, or even different methods for the analysis. Table 5.1 gives an overview of the main articles treating the twin peaks phenomenon. Here, also information about the data source, the variable used, the years covered, the number of countries considered, and the method applied are summarized.

What becomes apparent is that basically all empirical analyses use the Penn World Table as a data source. Most use the old version 5.6, only in one article (Beaudry, Collard and Green, 2002) a newer version, namely 6.0, is indicated. Concerning the variable used there are some differences. While all are based on real per capita GDP some authors use it relative to US income or to the world average income, some authors use the chain index, others the Laspeyres index. And again others do not indicate any details. Most empirical studies start in about 1960, while only a few start later, namely in 1970. The final year varies between 1984 and 1998, while the majority of studies ends in the late 1980s. Hence, the results are rather old compared to what will be done in this doctoral thesis. The number of countries included in the analyses differs quite sharply. Beaudry, Collard and Green (2002) use the smallest dataset with only 75 countries while the largest set by Kremer, Onatski and Stock (2001) comprises 128 countries (here it should be kept in mind that the dataset is only based on the Penn World Table; beyond, the data are

[96] The unit of observation is then a person instead of a country. The most important fact to note in this regard is that roughly 40 percent of the world's population live in China (23 percent in 1988) and India (17 percent in 1988)." (Jones, 1997, p. 22).

extended also for additional countries). The largest dataset on the basis of the Penn World Table is used by Jones (1997) with 121 countries. The application of different methods for analysis comprises kernel densities and Markov chains, while none of the two is really covered by the majority. As especially Pearlman (2003) finds a different result when using levels of income (namely no evidence for twin peaks); the interesting question is whether this might be due to a different dataset. Comparing the variable used, there is only one other study using a chain indexed income variable, namely Kremer, Onatski and Stock (2001). In both articles, the Markov chain is applied. Pearlman (2003) includes 120 countries, while Kremer, Onatski and Stock (2001) use a different dataset which is based on extrapolations even though having the Penn World Table as a basis as well. Thus, not only more countries are covered (128 instead of 120, even though explicitly mentioning a number of countries being excluded, for example oil-producing countries; otherwise it would be 140 countries) but also more years (1960 to 1984 compared to 1960 to 1995). This might be reasons for the different findings, even though it should be remembered that using logs yields twin peaks also in Pearlman's study (2003).

Club convergence is not only an alternative to convergence and divergence; they may also coexist. The latter terms play a central role when looking at each club individually.[97] Basically, most authors agree that there is evidence for divergence in the club of poor and convergence exists in the group of the rich countries. This means that the poor countries tend to stay poor, while the rich ones cluster closer together and tend to grow richer. Over time, the distribution polarizes more and more, and the middle class decreases (Quah, 1993a). Also other authors find this result, for example Jones (1997), Pritchett (1997), and Ben-David (1997). Furthermore, Paap and van Dijk (1998) point to the increasing gap between rich and poor. They state that "the probability to catch up for the poor countries is smaller than the probability of falling behind [for the rich ones]" (Paap and van Dijk, 1998, p. 1292). Even though most authors seem to agree on the results, Ben-David (1997) argues that it is more likely to find convergence among the poor than among the rich. In this point, he differs from the others who clearly point to divergence at the lower end of the distribution.

A further point which is discussed in the literature is the question of whether countries indeed might switch groups. Quah (1996c) clearly answers this question by stating that there might be intra-distribution dynamics. On the one hand, there might be crisscrossing, which means a switching of the position with another country, or, on the other hand, leapfrogging, which means taking over the position of an initial leader. The number of growth miracles increased over time, while that of growth disasters decreased. The reason for this may be that governments learned what kind of institutions and policies favor economic growth and which

[97] It should be mentioned that the possibility of finding of multiple peaks instead of only two shall not be explicitly excluded. However, the theoretical analyses of this doctorate concentrate on finding two stable steady states within the framework of the Solow growth model.

counteract it (Jones, 1997). In contrast, Paap and van Dijk (1998) point out that the downward movements are larger than the upward movements.[98]

Based on the articles reviewed here, it can be concluded that existing research has a number of weaknesses. The first one is obviously the use of older data. Another weakness which became apparent is that many authors just use certain data series and, even more important, certain econometric tools for their research without explaining the reasons for their choices. This is clearly overcome in the doctoral thesis at hand. Already in Chapter 4, the econometric methods were explained in detail and it was examined which ones ought to be used for the analyses of the world income distribution. Additionally, also the data sources will be explained in detail in the next section. Alternatives will be discussed and the choices made for the analyses will be described in detail. The empirical analyses presented here are much more detailed – concerning the choice of methods, of data sources, and of variables – so that the reader can easily comprehend what stands behind the phenomenon from an empirical point of view.

5.2 The Data Sources

Knowing the empirical results of other authors, in the rest of the chapter, the own empirical analysis will be presented. Here, several questions will be explored. First, do the data support the twin peaks hypothesis? Second, if this is true, is there divergence or convergence in the individual clubs? Third, do the peaks move further apart over time? And fourth, does the middle income group indeed decrease more and more? This section forms the basis for the own empirical analysis. For this purpose, first of all the data sources need to be described. To start with, in the next subsection the possible data sources for GDP, the savings rate and the population growth rate will be discussed. Thereafter, the focus will be on potential data sources for human capital.

5.2.1 Gross Domestic Product, the Savings Rate, and Population Growth

When working with empirical data, the analyst usually has a range of possible data sources. Especially GDP data are provided by several organizations and of course by the officials of the countries themselves as well. For analyses as in this doctorate, it is desirable to have the data out of one source only. In this case, there is a maximum of data comparability as each data series is measured in the same way and there are no breaks due to different bases used when calculating the data.

[98] There are also studies on regional data which concern the question of the existence of twin peaks in certain areas. However, as this doctoral thesis is about the world income distribution, these articles will not be discussed in more detail. For convenience, they shall briefly be named here: López-Rodríguez (2007) works on the EU15, Margaritis, Färe and Grosskopf (2007) work on the OECD countries, and Bandyopadhyay (2001) works on Indian States.

In this context, it should be kept in mind that reliability of the data is very important[99] – analyses are useless if the data contain too many mistakes. For GDP, for example, each country calculates its own values. Yet, measurements might differ sharply, just as the definitions of what is included in GDP, how it is manipulated in order to reach real GDP, and so on. Especially the data of developing countries are to be used with special care. Often, there are large black markets which disturb the GDP data. In addition, data quality tends to be nonsatisfying and sometimes, data are not provided completely. The data on GDP might also be exaggerated in order to manipulate the view of foreigners on that country. In order to avoid these problems, the analyst should use only one single, reliable data source, if possible. A commonly used and decent dataset in this field of research[100] is the Summers and Heston dataset, also called the Penn World Table.[101] In this doctoral thesis, the Penn World Table 6.3 (Heston, Summers and Aten, 2009) will be used. It is a newer version of the dataset; hence, the empirical twin peaks analysis will be based on more recent data. The dataset offers data for 189 countries for the years 1950 to 2007 and it uses the year 2005 as the base year while the former version used the year 2000 as its basis. The Penn World Table provides several income measures, consumption measures, growth rates of GDP, population measures, investment data, price indices, imports and exports, and many more (Summers and Heston, 1991). The variable being used in this doctorate for real per capita GDP is rgdpl (real per capita GDP, deflated by a Laspeyres Index)[102], for which 2005 is used as the base year. The data are measured in international dollars.[103]

[99] Nevertheless it has to be pointed out that no data source will be perfectly reliable. The reasoning is that all data have to be measured by someone. Usually, also data bases as the Penn World Table are to some extent dependent on the data they are offered. Zhang and Zhu (2015), for example, reestimate China's final consumption expenditure and show that it is about 10 percent higher than the official figure. If the consumption expenditure is reported too low, there might also be mistakes in the publication of investment figures and of GDP as a whole. And of course this might also have consequences for the correctness of the savings data. This under- or overestimation of data is likely to be a problem in other countries as well. Consequently, the term reliability refers to "relative reliability" in this doctoral thesis.

[100] For example Cantner et al (2001), Mankiw, Romer and Weil (1992), Pearlman (2003), and Quah (1996b), just to name a few. Hence, even though one should always be a bit skeptical about the data, and hence also about the Penn World Table, this data source will be used in this doctoral thesis, as it is commonly done in growth studies.

[101] Alternative data sources are the IMF Financial Statistical Yearbook (see for example IMF, 2000) and the World Development Indicators published by the World Bank (see for example World Bank, 2009). Though, despite of the deficiencies of the Penn World Table (see for example Johnson, Subramanian, Larson and Papageorgiou (2009) it was decided to stick to the habit of growth analyses and use this dataset.

[102] Summers and Heston constructed the dataset based on the data they had and a number of manipulation methods in order to reach a dataset which covers comparable data. Hence, the data should not be mixed up with the GDP data published by the statistical offices. It is possible that Summers and Heston used different ways to calculate real GDP data, for example they used a Laspeyres index to deflate instead of a chain index. Furthermore, they used a common currency to eliminate exchange rate fluctuations and purchasing power parity (PPP) associated problems with the measurement of real per capita GDP.

[103] These are calculated via the exchange rates of the domestic currencies to the US$ and then adjusted for the PPP in order to get the real exchange rates. The international dollar is defined such that one international dollar is equal to one US$.

The Penn World Table has been the foundation of most empirical growth research since the mid-1980s. “The Penn World Table displays a set of national accounts economic time series covering a large number of countries. Its unique feature is that its expenditure entries are denominated in a common set of prices in a common currency so that real international quantity comparisons can be made both between countries and over time. In addition, it presents data on relative prices, within and between countries, and demographic data and capital stock estimates as well” (Summers and Heston, 1991, p. 327). In conclusion, it can be said that the data as provided by the Penn World Table really satisfy the above mentioned requirements. Thus, they are of sufficient quality to enable reliable analyses and decent conclusions concerning the distribution of real per capita income across the countries of the world.

Next to real per capita GDP data, data on the savings rate are needed. Data on savings are usually difficult to find, especially for such a large number of countries as covered by the Penn World Table. As this dataset, as well as many others, does not provide data on savings, it has to be decided on how to measure the savings rate instead. Mankiw, Romer and Weil (1992), for example, approximate the savings rate by the investment share in GDP – this is a neoclassical assumption and hence, as the Solow growth model is a neoclassical model, this is a common approach in this field of research.[104]

Basic macroeconomic theory (see for example Mankiw, 1997) already points out that in the optimal situation, the interest rates serves to equilibrate savings and investment, hence to clear the market. This assumption will be used in this doctoral thesis though being aware that it is very straight forward and assumes away the openness of a country. There are neither perfect markets nor are the countries considered here closed economies in any manner. Nevertheless, the task of this doctorate is to examine in how far the Solow growth model is indeed able to capture twin peaks. Additionally, as data on net foreign borrowing are not really available either, this pitfall of a closed economy assumption cannot be overcome. Another option to measure savings would be by subtracting final consumption expenditure from GDP. This is done, for example, by Mohan (2006). Yet, also this measurement is not the savings rate which would be needed. Borrowing possibilities are assumed away, the countries are still assumed to be closed. Summing up, there is not really a perfect measure of the savings rate. If savings data are needed for empirical analyses, the economist needs to decide on which measure to take. They all have their advantages and shortcomings. In this doctoral thesis, although being aware of the pitfalls of that choice, the approach of Mankiw, Romer and Weil (1992)

[104] Feldstein and Hoioka (1980) show that normally, savings are equal to investment.

and others will be followed by approximating the savings rate by the investment share in GDP (called ki in the Penn World Table 6.3).[105]

The last variable taken out of the Penn World Table 6.3 is the population growth rate. The Summers and Heston dataset does not directly cover the population growth rate. Yet, absolute population data are provided so that the population growth rate can easily be calculated by simple arithmetics.[106]

5.2.2 Human Capital

Knowing which dataset will be used for the GDP data, for the data on the investment rate (as an approximation of the savings rate), and for those on population growth, two further datasets need to be discussed in this section. The Penn World Table covers the most important GDP-related data. Though, it does not provide any measure of human capital, which is important as a possible explanation for the emergence of bimodality in the real per capita income distribution across the countries of the world.

Human capital comprises the capabilities of the educated and highly qualified workforce of a country. The term human capital stems from the fact that capital refers to durable objects which were "produced" and may be used over a long time, independent of whether it costs money. When examining human capital data, it needs to be decided on how to measure human capital. A view at the literature shows that different variables are used for human capital. A commonly applied procedure is to approximate human capital by the average years of (total) schooling.[107] In recent years, this has been the most popular measure of human capital, used, for example, by Barro and Sala-i-Martin (2004), and Islam (1995).[108] The average years of schooling indicate the educational attainment of the population in a specific age group (for example of the population aged 15 years and over). The advantage is that the total amount of education is considered and not only the absolute minimum. In this way, further aspects crucial for the classification of the level of human capital, which can be used for production, are considered as well. Nevertheless, also this measure has disadvantages. First of all, Wößmann (2003) criticizes that additional years of schooling are treated identically while it makes a difference whether someone has his or her first year of

[105] For this reason, in the following the terms "savings rate" and "investment rate" will both be used interchangeably keeping in mind that the latter is used as an approximation of the former. In general, when talking about the theoretical Solow growth model the term "savings rate" will be preferred while in the empirical part the term "investment rate" will be applied instead.

[106] Of course, population data are the easiest variable to obtain, they are as well provided by the World Bank and the IMF. However, it was decided to stick to one single data source and take the data out of the Penn World Table. This has the advantage that the population size is exactly the one used when calculating GDP per capita data. This implies perfect coherence of the data.

[107] See for example Mankiw, Romer and Weil (1992), Castelló and Doménech (2002), as well as Feyrer (2008), just to name a few.

[108] Other examples are Barro (1997; 2001), Benhabib and Spiegel (1994), Gundlach (1995), Krueger and Lindahl (2001), Miller and Upadhyay (2002), O'Neill (1995), and Temple (1999).

education or the tenth or even twentieth year. The variable does not take into account the findings that wage differentials occur due to decreasing returns to schooling (Psacharopoulos, 1994). Thus, a different weighting of years would be desirable. Furthermore, Wößmann (2003) states that the quality of education is not considered by using the average years of schooling as an indicator. This hints especially to different efficiencies of education systems, different qualities of teaching, of the educational infrastructure in a country, or of the curriculum (Wößmann, 2003). Hence, he proposes to weigh the years of schooling also by the quality of education.

A second variable which might be used as a proxy for human capital is public spending on education. This variable is the only monetary one which will be considered here. It is published by the World Bank in the World Development Indicators (for example World Bank, 2009). Ziesemer (2004), for example, uses this indicator as a human capital variable. This variable measures public spending on public education as a percentage of GDP, plus subsidies to private education at the primary, secondary, and tertiary levels. The advantage of this variable is that it is a monetary measure, which can easily be compared across countries. No differences in the quality of education or the like might be considered here, it is a "hard" variable. Though, the disadvantage is that this variable totally ignores private spending on education. In some countries, private spending might be even more important, so that using the variable "public spending on education" tends to underestimate the real spending on education in those countries. Hence, the level of human capital in a country might be underestimated leading to wrong conclusions.[109]

As a third alternative, authors such as Azariadis and Drazen (1990) and Romer (1990) use the adult literacy rate as human capital variable. The literacy rate measures the number of adults able to read and write, with understanding, a simple statement related to one's daily life as a percentage of the total population aged 15 years and over (Wößmann, 2003). Also this measurement has disadvantages. Most investment expenditures in human capital is ignored, as only the very basic educational level is considered. "Hence using adult literacy rates as a proxy for the stock of human capital implies the assumption that none of these additional investments [(for example in numeracy, logical and analytical reasoning, scientific and technical knowledge)] directly adds to the productivity of the labor force" (Wößmann, 2003, p. 243).

A fourth way to measure human capital, which is also used in the literature, is the school enrolment ratio. This ratio indicates the number of students enrolled in a certain grade level as a percentage of the total population in the age group considered. This measure was used by Barro (1991) and Mankiw, Romer and

[109] In addition, a further disadvantage is that this measure does not say anything about the quality of education nor the value of it. There might be high public spending on education but low rates of return while in other countries public spending is low but the rate of return is high.

Weil (1992), just to name a few. The problem with this measure is that the human capital measure is important for current production, while the enrolment ratios only hint to the human capital available for future production. "The accumulated stock of human capital depends indirectly on lagged values of school enrolment ratios, where the time lag between schooling and future additions to the human capital stock can be very long and also depends on the ultimate length of the education phase" (Wößmann, 2003, p. 244). Hence, the enrolment ratio is a flow variable.

Given the different definitions of possible measures of human capital along with the discussions of the advantages and the discrepancies of them, a decision needs to be taken concerning which measure to use in this doctorate. There is no first-best option. The first two alternatives are not only widely used, they are also covered by common datasets for human capital data. Even though the criticism on the average years of schooling is definitely plausible, in this doctoral thesis it was decided to follow the common approach and use the average years of schooling as a measure of human capital. The reasoning is that, first of all, it is a widely used measure. Second, the data availability is quite good compared to having to find out more about the quality of the different educational systems, for example. Even if this information could be found, there is too much insecurity about changes in these systems over years, as a time frame of 1960 to 2000 at least needs to be considered. Hence, despite of being aware of the deficiencies even of this variable, it will be used here.[110]

Knowing which variable to use the available data sources need to be discussed. As outlined above, an important aspect is reliability of the data. The analyst has to be sure that the data are correct and allow him to draw useful conclusions. One such dataset is the Barro-Lee dataset (Barro and Lee, 2009), which also provides the variable "average years of schooling". The Barro-Lee dataset is a collection of education data for a large number of countries, though not for all countries covered by the Penn World Table 6.3. It offers data for the years 1955, 1960, 1965, 1970, 1975, 1980, 1985, 1990, 1995, and 2000.[111] In addition to the average years of schooling, this dataset provides data on the percentage of the population aged 15 and over with no schooling as well as the percentages in each level of schooling. From this, an index of human capital can be determined. Another expedient source of education data are the World Development Indicators by the World Bank. Here,

[110] In the Appendix (A.4), public spending on education will be used to find out whether this might be an alternative to the variable average years of schooling.

[111] When data are not offered for all years of interest, this is called the missing value problem in statistics. According to Luengo (2011), there are three problems which might occur due to missing values: first, a loss of efficiency; second, complications in handling and hence also in analyzing the data; and third there might be a bias which results from the differences between missing and complete data. Apart from just leaving the data away or treating the missing values as special values, there is also the possibility to use missing values imputation methods. Yet, as Kaiser (2011, p.43) points out: "missing values imputation methods are suitable only for missing values caused by missing completely at random". As this is not the case here, and going back to the original data sources on which the Barro-Lee-Dataset is based, it was decided to follow the first proposal by Kaiser (2011) and use the reduced dataset instead.

data on the second measure of human capital, which is applied for the sensitivity analysis in the Appendix (A.4), are provided, namely public expenditure on education as a percentage of GDP. These data can also be used as a proxy to measure human capital, though this measure totally ignores the private means spent on education, which might play a large role in some countries. Consequently, this variable tends to underestimate the level of human capital. The sensitivity analysis presented in the Appendix (A.4) supports the choice to use the average years of schooling out of the Barro-Lee dataset. The reason is that the analyses based on public spending on education do not show a clear picture of whether human capital is twin peaked.

5.3 The Countries Covered

As already stated in the previous section, the Penn World Table, the Barro-Lee-dataset, and the World Development Indicators 2009 cover different numbers of countries and years. Hence, when discussing which countries are covered, these three datasets need to be distinguished.

To begin with, the Penn World Table 6.3, which is the version used in this doctoral thesis, covers 189 countries. Yet, when looking at the data, it becomes clear that not all countries offer data in each year under consideration.[112] Consequently, the question arises whether only those countries offering data in every year should be included in the analysis. This is the procedure followed by Quah (1992), for example.

The Barro-Lee dataset covers 134 countries. Again, also here not all countries yield data on the average years of schooling in each year.[113] In the dataset, only every fifth year is covered starting in 1955, while in this year there are only very few data points. The last year for which data are provided is 2000.

Finally, the World Development Indicators 2009 cover the largest set of countries, namely 209 providing data from 1960 to 2008. However, also here not all countries offer data in each year.[114] The dataset starts rather late, as already stated above, and only recently every year is covered by the dataset.

Table 5.2 shows the number of countries in the different datasets for every tenth year starting in 1960. As the Barro-Lee dataset and the World Development Indicators 2009 are not used together but each would be combined with the Penn World Table, for the last two sources the number in brackets indicates the number

[112] For a complete list of the countries covered by the Penn World Table 6.3 for the variables real per capita GDP, investment rate, and population growth rate please refer to Table A.4 in the Appendix (A.5).

[113] For a complete list of the countries covered by the Barro-Lee-Dataset for the variable “average years of schooling please refer to Table A.5 in the Appendix (A.5).

[114] For a complete list of the countries covered by the World Development Indicators 2009 for the variable “public spending on education” refer to Table A.6 in the Appendix (A.5).

of countries the two datasets, the Barro-Lee dataset and the Penn World Table, and the World Development Indicators and the Penn World Table respectively, have in common. It becomes obvious that this number differs significantly – not only between the two human capital datasets but also over time. When looking at the different sizes of the datasets, it becomes clear that further decisions need to be taken for the empirical analyses:

1. Should the countries covered by the different datasets be identical or should different sets be used because of more data available?
2. Which should be the starting year for the analyses?
3. Should only those countries be used which provide data in all years under consideration, and if so, over all variables or for each variable individually?
4. Should outliers be excluded, and if so, in all years? According to which rule?

Table 5.2 The Sizes of the Data Sources

Year	Penn World Table 6.3	Barro Lee Dataset	World Development Indicators
1960	110	111 (84)	0
1970	163	110 (103)	61 (60)
1980	163	118 (109)	104 (102)
1990	174	127 (114)	124 (112)
2000	187	113 (109)	121 (119)
2007	186	0	49 (48)

All these questions and those associated will be addressed later on when needed.

One further aspect about the choice of countries is important when looking at the literature, namely the treatment of the oil-producing countries. Many authors working on economic growth in general and on the twin peaks phenomenon in specific have corrected for the oil-producing countries. The incomes of the oil-producing countries tend to be high, even much higher than those of the industrialized countries. For example, Brunei had an income of $34,683.60 in 1975 compared to the maximal income of the non-oil countries of $24,690.37 in Bermuda. Such incomes can also be seen as outliers and should thus be excluded from the dataset. However, the main reasoning behind the exclusion of oil countries is that countries which have a wealth of resources such as Oman would stand on top of the table with regard to productivity even though there is not really production in this country. Some authors, for example Hall and Jones (1996), try to deal with this problem by subtracting the part from total income which stems from the extraction of resources. Yet, this is a rather vague procedure. As there is no good alternative, many authors exclude the oil countries from the dataset in order to avoid this problem.

In this doctorate, the data are corrected for the following countries: Brunei due to the missing diversification of the economy which is focused on just one product; Kuwait due to the immense dependence on oil and gas; Macao due to its special situation with China and its position as a gambling den and hence the absence of a functioning economy; Norway due to its big dependence on oil as the main contributor to GDP; Oman as it is highly dependent on oil which will run out in about 20 years; Qatar due to its huge dependence on gas; Saudi Arabia, the United Arabic Emirates, Libya, Algeria, Iraq, and Venezuela for their dependency on oil; Trinidad and Tobago for their dependency on oil and gas; and finally Botswana for its dependency on diamonds (ADAC, 2004).[115]

Now that the datasets to be used were described and also the countries covered were discussed, the next section will deal with the descriptive statistics on the individual variables to be considered in the empirical analyses of Sections 5.5 and 5.6.

5.4 Descriptive Statistics

When presenting comprehensive datasets as it is the case here, descriptive statistics are very important. They can be used for a descriptive and sometimes also graphical preparation and compression of data. This can either be done by use of graphs or by use of tables which summarize, among others, statistics such as the mean or the standard deviation (Fahrmeir, Hamerle and Tutz, 1996). This is the subject of the section at hand. The descriptive statistics will be provided for all variables considered in this chapter, namely real per capita GDP, the investment rate, the population growth rate, and human capital.[116]

5.4.1 The GDP data

As already stated above, in this doctoral thesis the variable rgdpl out of the Penn World Table 6.3 will be used as a measure for real per capita GDP. In the Penn World Table, rgdpl is defined as real GDP per capita, derived from the growth rates of consumption, government expenditure, and investment, measured in

[115] Additionally, it was considered to eliminate Luxembourg from the dataset. The reasoning behind this is that GDP of Luxembourg tends to be overestimated as GDP is determined by many commuters. Additionally, international service providers contribute to GDP by 65 percent and make the country prone to changes in the economic conditions abroad (ADAC, 2004). As the conclusions to be drawn from the graphs did not change it was decided to keep Luxembourg in the dataset. Appendix (A.6) covers the sensitivity analysis on the elimination of Luxembourg.

[116] Each country equals one observation. GDP is given in per capita terms but the countries themselves are not weighed by population. Hence, the United States are one observation point just as Costa Rica.

international dollars in constant prices of 2005. It is calculated as a Laspeyres index (Heston, Summers and Aten, 2009).

Table 5.3 shows the descriptive statistics of this variable for a selection of years. Data are available from 1950 to 2007, as stated above. Here, the results for the years 1955, 1960, 1965, 1970, 1975, 1980, 1985, 1990, 1995, 2000, and 2005 will be reported. The descriptive statistics of the complete dataset are given in the Appendix (A.7). In this section, the complete dataset of nonoil countries is analyzed irrespective of whether there are outliers. The detailed outlier discussion is presented in Section 5.5.

Figure 5.1 shows the developments of the main descriptive statistics over time: the number of observations (measured on the right-hand scale), the mean, and the standard deviation (both measured in international dollars on the left-hand scale). What becomes apparent is that all of the three variables are increasing over time. Especially the number of observations is an important indicator which helps to decide in which year to start the analysis if only countries offering data in all years shall be considered. This will be done later in this chapter. The mean and the standard deviation are important statistics for the outlier identification which will be undertaken in Section 5.5.

Figure 5.1 Descriptive Statistics for Real per Capita GDP

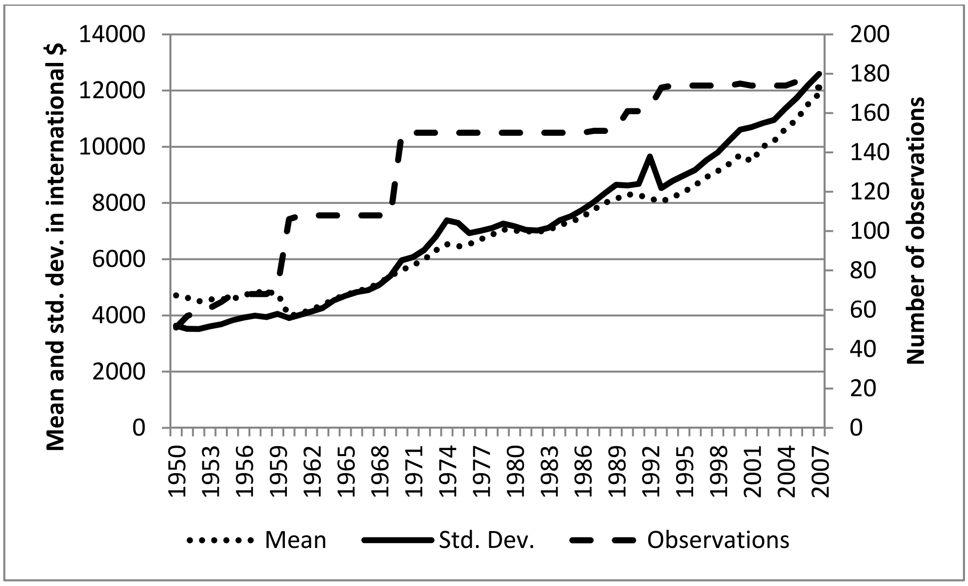


Table 5.3 Descriptive Statistics for Real per Capita GDP

	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Number of observations	68	106	108	150	150	150	150	161	174	174	176
Mean ($)	4589.25	3980.34	4727.20	5619.91	6455.82	7021.14	7327.19	8310.74	8396.58	9713.29	10969.34
Median ($)	3203.40	2380.07	2580.30	2955.06	3841.26	4385.21	4600.71	4807.79	5071.59	5577.72	6281.94
Maximum ($)	15396.52	18102.63	21012.32	28309.82	53713.12	33093.42	33552.81	43562.79	49723.61	63419.39	71209.28
Minimum ($)	406.84	418.64	480.77	336.48	617.27	495.44	595.13	510.60	153.44	312.41	359.85
Standard Deviation	3826.61	3905.58	4700.09	5958.99	7292.57	7172.97	7530.71	8620.55	8979.95	10607.09	11734.17

5.4.2 The Investment Rate

The next variable for which descriptive statistics will be presented is the investment rate as an approximation of the savings rate. Table 5.4 shows the number of observations, the mean, the median, the minimum and maximum values, and the standard deviation for the years 1955, 1960, 1965, 1970, 1975, 1980, 1985, 1990, 1995, 2000, and 2005, respectively. The descriptive statistics of the complete dataset are again given in the Appendix (A.7). Additionally, the complete dataset of nonoil countries is analyzed irrespective of whether there are outliers, which will be discussed later on.

Figure 5.2 shows the developments of the main descriptive statistics over time: the number of observations (measured on the right-hand scale), the mean, and the standard deviation (both measured as percentage of GDP on the left-hand scale). The number of observations again increases over time as the dataset is exactly identical to the one of GDP. The mean increases from values around 20 percent in 1950 to values of about 25 percent in 2007. The curve does not steadily slope upward, but the ups and downs are only very slight. The same accounts for the standard deviation. Also this curve looks rather stable. It increases over time from about 10 percent in 1950 to about 15 percent in 2007. From this it can be concluded that the savings rate is rather stable over time, as neither the mean nor the standard deviation show strong movements.

Figure 5.2 Descriptive Statistics for the Investment Rate

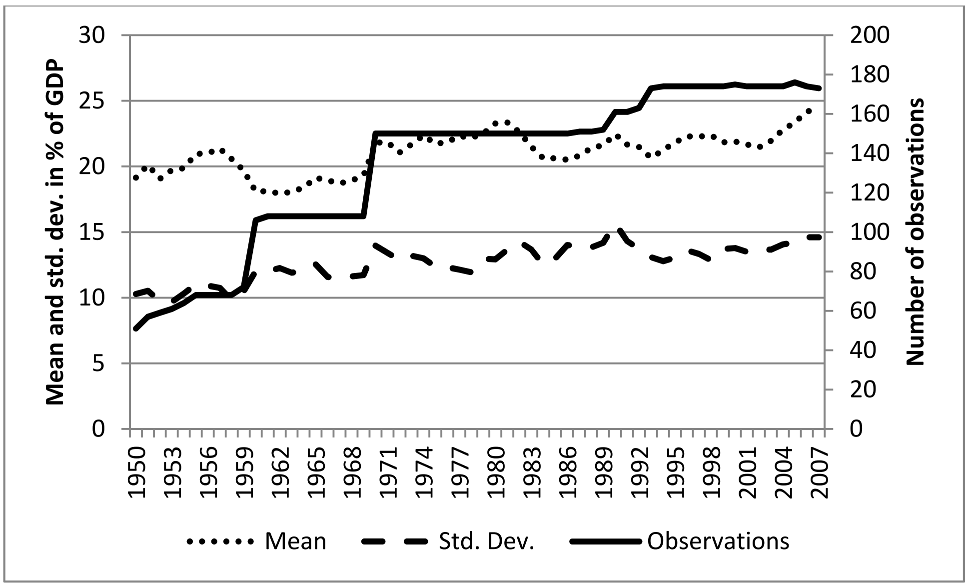


Table 5.4 Descriptive Statistics for the Investment Rate

	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Number of observations	68	106	108	150	150	150	150	161	174	174	176
Mean ($)	20.96	18.09	19.14	21.81	21.72	23.27	20.62	22.46	21.85	21.91	23.37
Median ($)	19.72	17.05	16.24	18.51	21.85	22.80	17.86	20.65	20.65	20.56	22.82
Maximum ($)	62.94	53.36	58.26	63.71	69.94	69.80	72.87	105.68	72.44	79.14	93.60
Minimum ($)	3.60	1.32	2.18	1.84	1.62	1.47	0.98	1.59	1.97	1.00	3.31
Standard Deviation	11.04	12.03	12.55	13.97	12.35	12.92	13.00	15.67	13.06	13.79	14.22

5.4.3 The Population Growth Rate

Another variable which plays a crucial role in the context of the Solow growth model is the population growth rate. In this subsection, the descriptive statistics for the population growth rate shall be presented. Table 5.5 gives an overview of the descriptive statistics for a selection of years. Also the data on the population growth rate are available from 1950 to 2007 and the data source is the Penn World Table. As for GDP and the investment ratio, here the results for the years 1955, 1960, 1965, 1970, 1975, 1980, 1985, 1990, 1995, 2000, and 2005 are reported. The complete dataset is described in the Appendix (A.7). Figure 5.3 gives an overview of the development of the main descriptive statistics over time. It becomes obvious that the number of observations is stable at 189.

The mean is relatively stable with a downwards tendency over the years while the standard deviation shows a sharp increase in the early 1990s. It is obvious from Figure 5.3 that the mean decreased in the very early 1990s while one or two years later there is a sharp increase in the mean again.

Figure 5.3 Descriptive Statistics for the Population Growth Rate

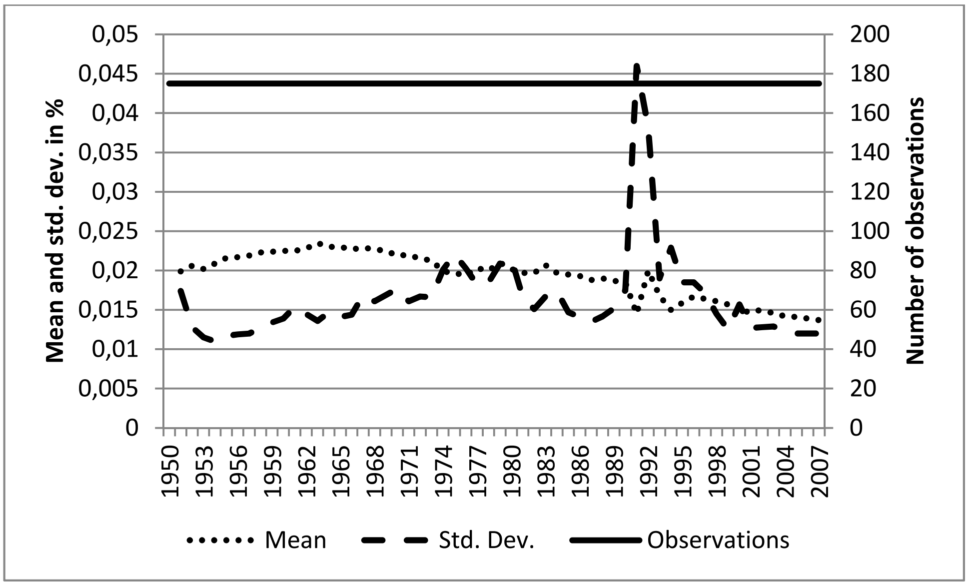


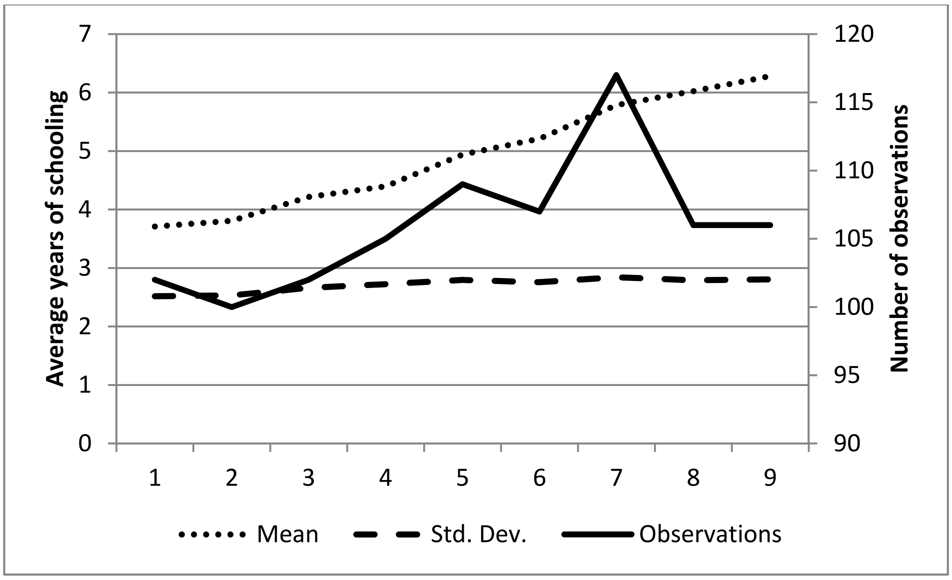
Table 5.5 Descriptive Statistics for the Population Growth Rate

	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Number of observations	175	175	175	175	175	175	175	175	175	175	175
Mean	0.022	0.022	0.023	0.022	0.020	0.020	0.020	0.018	0.016	0.014	0.014
Median	0.021	0.022	0.023	0.022	0.021	0.0196	0.020	0.020	0.017	0.015	0.014
Maximum	0.069	0.115	0.101	0.145	0.153	0.157	0.097	0.081	0.070	0.053	0.050
Minimum	-0.017	-0.022	-0.014	-0.024	-0.147	-0.041	-0.012	-0.122	-0.128	-0.124	-0.010
Standard Deviation	0.012	0.014	0.014	0.018	0.022	0.021	0.015	0.018	0.019	0.016	0.012

5.4.4 Human Capital

The final variable which needs to be described in this section is human capital. As stated above, human capital is the potential of the educated and highly qualified workforce of a country. The variable “average years of schooling” is part of the Barro-Lee dataset, which will be used here. The Barro-Lee dataset provides data for the total population aged 15 and over and for the one aged 25 and older. The same population shares are provided for females and for males only. In this doctorate, the total population aged 15 and over will be used as in many countries, working age starts already with 15 (or even before). Hence, also these younger workers should be covered by human capital data. In the dataset, there are 129 non-oil countries; however, not all provide data for all years considered.[117]

Figure 5.4 Descriptive Statistics for Human Capital



In Table 5.6, the descriptive statistics of the human capital variable “average years of schooling” are given. Again, outliers are not yet identified in the dataset; this will be done later on. Furthermore, Figure 5.4 gives the graphical representation of the main descriptive statistics, namely the mean, the standard deviation, and the number of observations. The number of observations ranges between about 105 countries in 1965 and 122 countries in 1990, while decreasing again after 1990. The mean increased constantly from almost four years in 1960 to roughly above six years in 2000, while the standard deviation remained rather stable over time with values of about 2.5 years.

[117] The analyses of this chapter will be repeated for public spending on education as well. The results can be found in a sensitivity analysis in the Appendix (A.4). As this variable does not yield any measurable advantage over the usage of the average years of schooling, the latter variable is used in this doctoral thesis from now on as the measure of human capital.

Table 5.6 Descriptive Statistics for Human Capital

	1960	1965	1970	1975	1980	1985	1990	1995	2000
Number of observations	107	105	107	110	114	112	122	111	111
Mean ($)	3.7100	3.8098	4.2174	4.3990	4.9425	5.2128	5.7871	6.0265	6.2833
Median ($)	3.5100	3.3900	3.9400	3.9850	4.7650	5.1700	5.5750	6.0000	6.1800
Maximum ($)	9.7300	9.7400	10.2400	11.2700	11.8600	11.5700	11.7400	11.8900	12.0500
Minimum ($)	0.1200	0.1700	0.2000	0.0900	0.2600	0.4900	0.6500	0.7600	0.8400
Standard Deviation	2.5183	2.5338	2.6624	2.7250	2.7964	2.7578	2.8426	2.7925	2.8090

5.5 Distribution Analysis

After having looked at the descriptive statistics for the individual variables of concern in this doctoral thesis, the section at hand shall deal with a closer look at the data. Here, especially the empirical distribution of real per capita GDP will be examined in detail in order to check whether there are indeed twin peaks in the distribution of real per capita income. This will be done by use of kernel densities. The theoretical background was already given in the previous chapter. Furthermore, as it was already outlined above, there are several distinct explanatory factors which might lead to twin peaks; the distributions of these factors[118] will also be examined by kernel densities. The hypothesis behind this is that if there are twin peaks in real per capita GDP, and if the neoclassical growth model is correct, then there should also be twin peaks in the inputs into the model, hence either in the savings rate, the population growth rate, or in human capital (Ziesemer, 2004). This will be checked as well in this section.

5.5.1 Gross Domestic Product

Twin peaks analysis is about the analysis of the real per capita GDP data. By use of nonparametric density estimation, it shall be found out whether the income distribution is unimodal, bimodal, or even multimodal instead. In Chapter 4, an overview of the techniques that might be used for this purpose was given. It was worked out that kernel densities are the best alternative to answer the question of the type of modality and that these results are robust as they are not subject to the choice of the bandwidth or the origin as in the case of the histogram, for example. Hence, the kernel density is the statistical method which will be applied in this chapter.

As stated before, the data source is the Penn World Table 6.3. The years for which the kernel densities will be presented are 1960, 1965, 1970, 1975, 1980, 1985, 1990, 1995, 2000, and 2005.[119] The oil-producing countries and Macao as well as Botswana were eliminated from the dataset for the above-mentioned reasons. Furthermore, it was decided to follow Quah (1992) and use only those countries, which offer data in all years under consideration. In this way, comparability is given.[120] The first year under consideration will be 1960, even though there are already data available from 1950 on. But then, there are only 70 countries offering data. Starting in 1970 would be sensible when just looking at the number of countries.[121] Though, as the twin peaks phenomenon starts to evolve around this

[118] The focus is on the savings rate, the population growth rate, and on human capital.

[119] In the Appendix (A.8), the kernel densities for every year individually are presented. Here, the interested reader can find more information on when exactly the twin peaks appear.

[120] In the Appendix (A.9), a sensitivity analysis on considering all countries offering data in each year is presented to elaborate on whether the conclusions to be drawn differ in this case.

[121] In the Appendix (A.10), a sensitivity analysis on using a different starting year while considering only those countries offering data in all years is presented.

year, it is not a good choice as a starting point. Hence, it was decided, despite of the deficiency of having less countries in the dataset, to use 1960 as the starting year. In the following subsection, it will briefly be discussed whether there are outliers which should be excluded from the dataset.

5.5.1.1 The Outliers

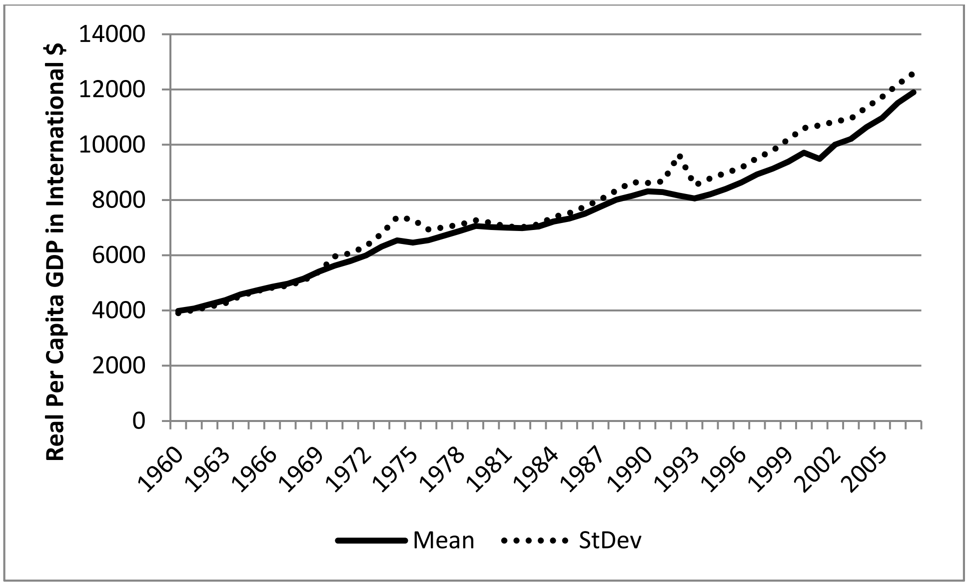
Outlier definition is a very sensitive subject in statistical analysis. It is not really commonly agreed on how to define outliers and when to exclude them. Statistical literature is very vague in this respect. It is usually stated that data should be excluded which disturb the conclusions to be drawn from an analysis. However, when do data disturb an analysis? Again, the literature is not clear on this. Looking at the GDP data, it can be noted that, for example, the oil producing countries are potential outliers in that these countries tend to have high GDP values even though they are not reached through production. Hence, it is quite sensible, to follow the common habit of other economists working on economic growth and exclude them from the dataset, as already outlined above. The exact reasons shall not be replicated here. Another reason for identifying outliers is usually an obvious typing error. This is rather unrealistic in a dataset on real per capita GDP. Here, a brief look at the data can help to scrutinize whether it makes sense, for example, that a country like Switzerland has such a high income as indicated. Switzerland is a country which is found to be an outlier in several years due to its high income. Nevertheless, even if this is formally correct, it does not make sense to exclude this country from the dataset. First of all, it is well known that Switzerland is a rich country. There are no obvious mistakes in the data entry. Hence, excluding it from the dataset might lead to wrong conclusions about the real income distribution across the countries of the world. However, such country might indeed lead to a further peak in the kernel analysis. This "small peak" should then be interpreted with care in order to draw the right conclusions on the degree of peakedness.[122]

A general, commonly used rule to identify outliers is to exclude all observations falling outside of the interval [mean ± 3 · standard deviation] (Fahrmeir, Hamerle and Tutz, 1996). Even though the expectations are already that the real per capita GDP data are not Gibrat distributed (twin peaks are expected), it was decided to follow this outlier identification rule anyway. However, only those countries should be excluded which are outliers quite often, also depending on the size of the country. Furthermore, as outlined above, it needs to be questioned, whether it makes sense to exclude a country like, for example, Switzerland due to its

[122] Another outlier is Luxembourg. It may make sense to exclude Luxembourg from the dataset as here, a large part of GDP is determined by commuters. Yet, it was decided not to do so in this doctorate and rather keep the point of observation in the dataset instead.

importance for the analysis as a rich industrial country. Hence, no countries are excluded in any year.[123]

Figure 5.5 Descriptive Statistics for Real per Capita GDP (105 Countries)



5.5.1.2 The Kernel Densities

As described in the previous subsection, the kernel densities, which will be shown here, are presented for a dataset of the nonoil countries only considering those countries which offer data in all years under consideration. Some outliers are excluded, while others such as Switzerland remain in the dataset because it was found to be too important to be excluded from the dataset. Hence, before the kernels are presented, Figure 5.1 is reproduced for this dataset to see what the mean and the standard deviation look like.[124] What can be seen is that the mean as well as the standard deviation are constantly rising over time. Now, the kernel densities shall be presented for every fifth year. The results are shown in Figure 5.6.[125]

[123] In the Appendix (A.12), there is a table which indicates which countries are outliers in how many years. Additionally, it is shown how often (as a percentage of the total number of years) a country is defined as an outlier. Here, a brief discussion of the countries then being excluded or not excluded will be presented.

[124] The number of observations is constant at 105 and hence it is not reported as a separate line in Figure 5.1.

[125] The kernel densities are based on an Epanechnikov kernel function with the optimal bandwidth chosen according to the rule of Silverman (see Chapter 4).

Figure 5.6 The Distribution of Real per Capita GDP

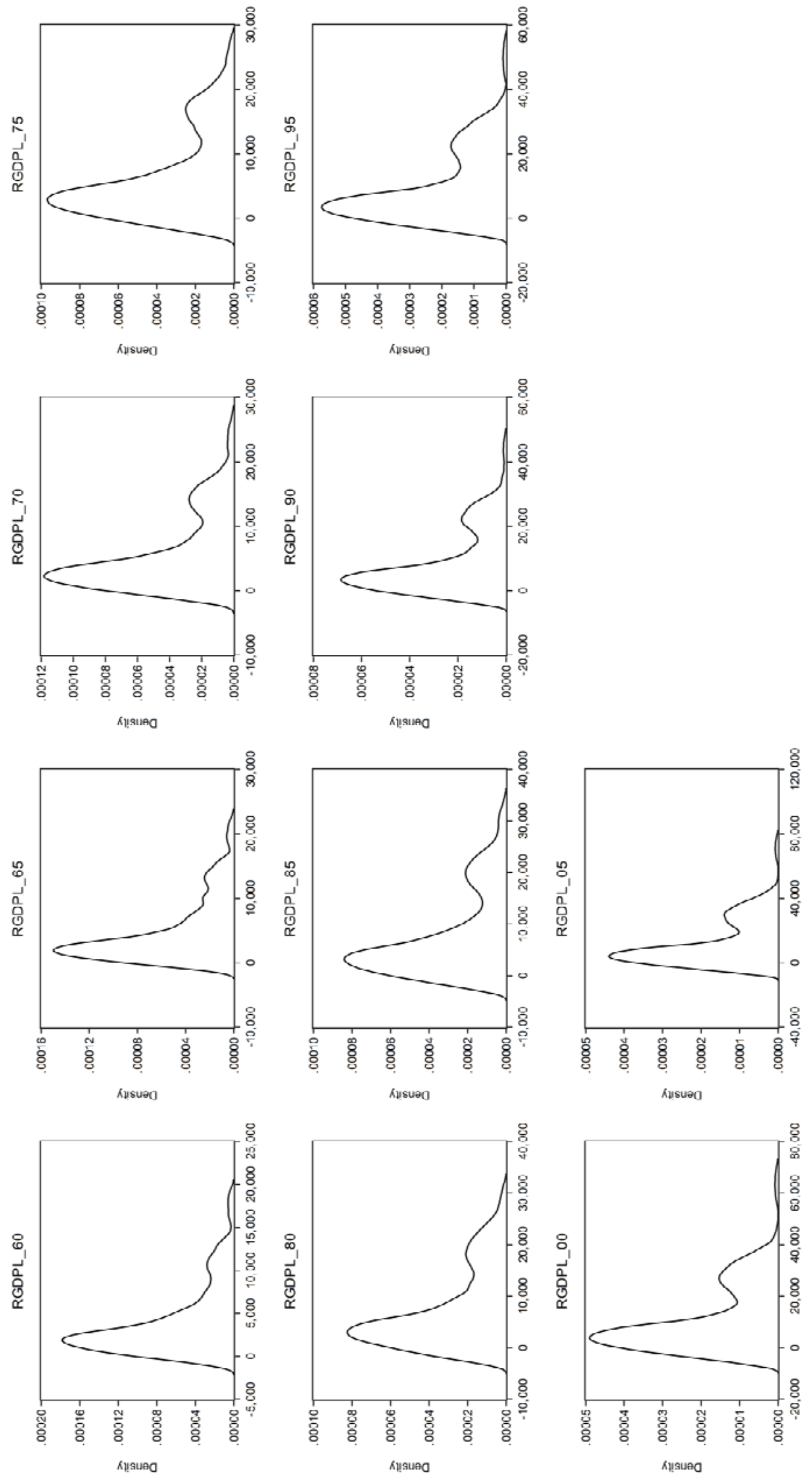


Table 5.7 The Peaks

Year	Low income peak	High income peak	Valley bottom
1960	$1,923		$8,846*
1965	$1,923		$8,846*
1970	$2,500	$13,500	$10,000
1975	$3,000	$16,000	$11,250
1980	$3,125	$17,500	$13,750
1985	$3,333	$20,000	$14,667
1990	$3,636	$21,818	$16,364
1995	$3,636	$21,818	$16,364
2000	$3,750	$26,250	$17,500
2005	$4,615	$30,256	$18,462

* = there is not really a point of inflection here but rather an identified "end of the low-income peak" which is, of course, to an extent subjective

In Table 5.7, the peaks (low-income peak, high-income peak; point of inflection, which is the point which separates rich and poor) are described in more detail. Looking at Figure 5.6 shows that there was only one large peak at a low level of income in 1960 and 1965. The distributions are skewed to the right, hence Gibrat distributed as expected. The peak is at an income of about $1,923.[126] Thereafter, the curve continuously flattens. Hence, there is unimodality in the real per capita GDP data in the first two years given in Table 5.7.

Looking at the whole distribution for all years, which is presented in the Appendix (A.8), it can be found that unimodality is a feature of the real per capita income distribution until 1967. Thereafter, bimodality appears. It is obvious that there are no multiple peaks implying more than two peaks. The little dip which can be seen in some of the kernel density graphs is made up by one country (or at most two countries), namely Switzerland and / or Luxembourg. Hence, it should not really be interpreted as a peak. Table 5.7 gives an overview of the value at which the low-income and the high-income peak can be identified.[127]

In the Appendix (A.12) it is shown which countries are in the low-income peak and which are in the high-income peak. However, in the years 1960 and 1965 there is only one peak so that countries not being identified as being member of the group of poor are indicated by a value of zero. In all other years, a one indicates that a country is poor and a two indicates that a country is in the high-income group. From this, it can be read off that some countries made it to get out of the low peak, namely especially some of the Asian Tigers (Hong Kong, Singapore, South Korea,

[126] These values, which are also given in Table 5.7, result from an analysis of the graphs. They give an indication of how far the two peaks are really separated. Additionally, they are necessary to get an overview of which countries can be found in which peak. This is given in the (see Appendix (A.12)) in form of a table where a one indicates membership in the "club of poor" and a two indicates membership in the "club of rich".

[127] It should be kept in mind that these are approximated values from a graphical analysis.

Taiwan), some European countries (Cyprus, Greece, Ireland, Portugal, Spain) as well as Japan, and Puerto Rico, just to name the most important ones. Yet, there are also some examples for the other direction, namely countries which once were rich but then switched to the club of the poor: Argentina, for example. In the direction of poor to rich, there are, fortunately, more examples. But looking at the data indicates that it is rather difficult to switch groups. If it were easier, there would be more changes in between. But nevertheless, it shows that switches are possible under certain circumstances.

Summing up, twin peaks seem to be a common feature of the world income distribution. Hence, it is worth examining whether the Solow growth model is indeed able to yield bimodality. Furthermore, it is also important to work out how switches between the groups are possible, as they obviously occur, even though this is rarely the case. In the next subsection, the future distribution will be examined in more detail by use of the Markov chain, before further analyses of the three possible explanatory variables, namely the savings rate, the population growth rate, and human capital will be presented.

5.5.1.3 The Future Income Distribution

In the previous subsection, the data on real per capita GDP were analyzed by use of kernel densities in order to get an idea of what exactly the world wide income distribution looks like. It was found that since the 1970s, twin peaks have indeed been a common feature of the world income distribution. As outlined in Section 5.1, for the empirical analyses of the twin peaks subject also the Markov chain is applied in order to get an idea of what the distribution might look like in the future. In this doctorate, the Markov chain method will also be applied using to the dataset described above, namely those 105 countries offering data in all years from 1960 on. In this way, it will be examined what the future income distribution across those countries might look like.[128] Before the method can be applied, several decisions need to be made. The first one concerns the number of income groups. This choice might have an influence on the conclusions to be drawn. Jones (1997) proposes to use domestic income relative to the one of the United States as the basis for this division. The income of the US is often used as a reference income. It is not the highest one; in several years, Switzerland and Luxembourg were found to be on the top of the income distribution. Nevertheless, it was decided to follow Jones' proposal and use the United States data as a reference. Jones distinguishes six income groups as given in Table 5.8, where $\tilde{y}$ refers to the relative income.[129] Here, also the number of countries being in each income group in the source year and in the target year is indicated.

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[128] The Markov chain method was described in more detail in Chapter 4. The reader is referred back to Section 4.4.

[129] $\tilde{y} = \frac{Y_C}{Y_{US}}$, where C = any country

A further decision which needs to be taken is on the source year and the target year. Jones (1997) uses 1960 as the source year. He covers 100 countries in his dataset. The target year is 1988. In this doctoral thesis, his result shall be checked. For this reason, the same source and target year is used. It will be examined whether his result (he finds a single peak in 1960 and twin peaks in 1988 along with a single peak again in the very long run) can be replicated or whether the data show a different distribution. In order to find out more about the sensitivity of this result with respect to the choice of the target year, the same will be done for shorter time periods, namely for every decade starting in 1960 and ending up in 2007.

Table 5.8 Group Division (Jones Distribution)

Income group	Rule for grouping	Number of countries in 1960	Number of countries in 1988
1	$\tilde{y} \leq 0.05$	7	21
2	$0.05 < \tilde{y} \leq 0.1$	25	20
3	$0.1 < \tilde{y} \leq 0.2$	27	21
4	$0.2 < \tilde{y} \leq 0.4$	24	16
5	$0.4 < \tilde{y} \leq 0.8$	14	20
6	$\tilde{y} > 0.8$	8	7

On the basis of Table 5.8, the countries providing data in 1960 and 1988 will now be assigned to these income classes. Table 5.8 summarizes the number of countries[130] in each group in 1960 and 1988. Table 5.9 shows the same in form of a matrix[131] indicating the dynamics of the countries across the income classes. For example, in 1960, only seven countries are in the lowest income group, while in 1988, the group is made up by 21 countries. This underlines the observation mentioned before, namely that in 1960, the number of rich countries increases slightly while the number of poor countries increases much more. Looking at the dynamics shows that only one country succeeds to escape income Group 1 and move upwards to Group 2, while 6 countries stay in Group 1. In addition, 13 countries belonging to Group 2 in 1960 and two countries of Group 3 in 1960 move down to Group 1 in 1988. Hence, in total the number of countries being in the lowest income group increases from 7 to 21.

The only possibility to move up one or more classes as defined above is by growing quicker than the United States, because the income classes are based on domestic income relative to the one of the United States. In this case, a country is called growth miracle. In terms of the Solow growth model, there is a shift of the long run steady state by the country (Jones, 1997). Or there might be a move from the

[130] To see which countries are in which group, the interested reader is referred to the Appendix (A.13).

[131] The interested reader is referred to Chapter 4, Section 4.4, in which the theory on the Markov chain is discussed. Appendix (A.14) shows general form of Table 5.9 using the movement probabilities of Chapter 4.

low-income steady state towards the second, the high-income equilibrium. However, Table 5.9 also shows growth disasters, which are countries with negative growth rates from 1960 to 1988. Following the line of reasoning from above, these movements can be seen as a downward shift of the steady state in the basic Solow growth model or, in the framework of a model capturing bimodality, as a move from the upper steady state towards the lower one. Overall, it should be kept in mind that the Markov chain analysis allows the analyst to make statements about the number of countries being in each group and about how many countries switch groups. However, it is not possible to read off which countries are in the respective groups.[132]

Table 5.9 The Movement among Income Classes (1988)

Target (1988)	Source (1960)						Total
	1	2	3	4	5	6	
1	6	13	2	0	0	0	21
2	1	8	10	1	0	0	20
3	0	4	11	6	0	0	21
4	0	0	4	10	2	0	16
5	0	0	0	6	10	4	20
6	0	0	0	1	2	4	7
Total	7	25	27	24	14	8	105

Table 5.10 Transition Matrix[133] (1988)

Target (1988)	Source (1960)					
	1	2	3	4	5	6
1	0.857	0.520	0.074	0	0	0
2	0.143	0.320	0.370	0.042	0	0
3	0	0.160	0.407	0.250	0	0
4	0	0	0.148	0.417	0.143	0
5	0	0	0	0.250	0.714	0.500
6	0	0	0	0.042	0.143	0.500

Knowing the distribution of the countries in the source and in the target year, the next step of the Markov method is to calculate the transition matrix, as it was

[132] The interested reader is referred to the Appendix (A.13) indicating which countries are in which group and which countries are growth disasters or growth miracles. A famous example of a growth miracle which moved up at least two groups is Taiwan (Group 3 to Group 5). And indicated by a downward move by two groups at least, growth disasters are, for example, Chad, Congo, Nigeria, Zambia (all Group 3 to Group 1), and Guinea (Group 5 to Group 2).

[133] The transition probability indicates, for example, the probability that a country currently in Group 1 (in the source year) will be in Group 1 also in the target year. Hence, the position "Group 1 – Group 1" will be divided by the total number of countries in this group in the source year. This holds for all transition matrices used in this doctoral thesis.

described in the previous chapter. The result is shown in Table 5.10. The values in the cells can be interpreted as follows: there is a probability of 0.857 that a country currently in Group 1 will stay in Group 1 in the next generation. In addition, there is a probability of 0.143 that it manages to move up one group. Alike, there is a chance of 0.520 that a country currently in Group 2 will move to Group 1 in the next 28 years, and so on. The probabilities given above are based on the income distributions in 1960 and 1988 and add up to one.[134] Based on this transition matrix, it can be calculated what the distribution looks like in the long run. The results will be reported for the year in which the distribution seems to stabilize finally.

By use of the transition matrix, the development of the distribution of countries in each income group can be examined in the long run. The values are calculated as follows (here the example of countries in Group 1 in the target year is presented):

$$\begin{aligned} no.\ of\ countries\ 1_{target} &= p_{11} \cdot 1_{source} + p_{21} \cdot 2_{source} + p_{31} \cdot 3_{source} \\ &+ p_{41} \cdot 4_{source} + p_{51} \cdot 5_{source} + p_{61} \cdot 6_{source} \end{aligned} \quad (5.1)$$

In the end, this can be repeated over and over again until the distribution stabilizes. In this way, the long run distribution can be calculated and it can also be shown in which year this distribution is reached. Table 5.11 summarizes these results.[135]

Table 5.11 The Long Run Income Distribution

Income group		1988		1990		2000		2007	
	1960	1988	3920	1990	4210	2000	4800	2007	4780
1	7	21	68	24	69	28	70	24	59
2	25	20	18	17	16	13	6	17	17
3	27	21	6	22	7	22	16	19	15
4	24	16	3	13	3	12	3	11	3
5	14	20	7	21	7	21	7	23	8
6	8	7	2	8	2	9	3	11	4

Using different target years and analyzing the sensitivity of the conclusions to be drawn when using different target years shows that the decision on the target year might have decisive consequences for the conclusions to be drawn. From Table 5.11, it becomes obvious that different target years lead to different results. First of all, as found by Jones (1997), there is unipeakedness in 1960. In 1988, Jones (1997) finds twin peaks. Here, there are rather three peaks, even though the the difference between Group 1, Group 2, and Group 3 is minor. In the long run,

[134] Sometimes, they might not exactly add up to 1, but to 0.999 instead, which is due to rounding errors.

[135] In this table also results for using 1990, 2000, and 2007 as target years respectively instead of 1988 are given. This yields something new beyond what other authors did so far. The corresponding tables containing the group membership as well as the transition probabilities can be found in the Appendix (A.15). Table 5.11 just deals to show that using a different target year and hence a different time frame yields slightly different results.

this target year yields bimodality as a stable result by 3920.[136] Using 1990 instead shows three peaks in the target year and twin peaks as a long run result by 4210. If 2000 is the target year, then there are again three peaks in this year and in the long run, there are also three steady states. Finally, using 2007 as the target year means that there are three peaks in 2007 but only two in the long run. The sensitivity analysis of this subsection shows that the Markov chain is rather sensitive to the choice of the target year. Furthermore, a critical aspect is the group division. Here, the proposal by Jones (1997) was followed for this division.

Another important question that arises is whether the findings of the Markov analysis are stable over time. For this purpose, the time frame is split up into decades and the group divisions as well as the transition probabilities are calculated for each decade separately.[137] The results are presented in Tables 5.12 to 5.21.

The tables show that the number of countries in the poorest income group is definitely increasing. Hence, there is a tendency towards divergence over time as also the number of countries in the highest income group increases, even though only slightly compared to Group 1. What is obvious as well is that the tables indicate a single peak for 1960 and emerging twin peaks thereafter even though sometimes there are also three peaks. More interesting than the absolute number of countries in each group is the question whether the transition probabilities from one group to another change over time. This can be found out by looking at the individual transition matrices. Comparing the figures in these tables leads to the conclusion that at the lower end (Group 1) and at the upper end (especially Group 5, but to a certain degree also Group 6) of the distribution, the transition probabilities seem to be more or less stable (not in absolute terms but the values are very close); this is not the case in the middle income classes, especially in Group 2, Group 3, and Group 4. Based on these findings it can be concluded that the Markov chain yields results which are quite sensitive to the base and the target year. The transition matrix changes over time within decades so that one has to be careful about how to interpret the findings. Even though the twin peaks phenomenon could be replicated using the Jones (1997) distribution as well as his conditions, this finding is not robust to the choice of the time frame. Hence, no really stable result of bimodality appears.

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[136] The calculations yielding the stabilization years are made by use of Excel. The process $M^n\overrightarrow{x_t} = \overrightarrow{x_{t+n}}$ is repeated until stabilization appears up to the third decimal. It has to be kept in kind, however, that the transition probabilities of the transition matrices are also rounded values. Hence, the results are subject to rounding errors. The numbers indicated in the long run income distribution (see Table 5.11) are rounded to whole numbers. The stability of the result can be proved. It was stated before that the Markov chain in this dissertation has the implicit result of stationary results as outlined above. This means that $M\vec{x} = \vec{x}$. This can be proved by using the matrix M and multiplying it by the distribution of the long run target year, for example 3920 in the distribution using 1988 as the target year for the transition matrix. This is shown in the Appendix (A.4).

[137] The decades to be considered are from 1960 to 1970, from 1970 to 1980, from 1980 to 1990, from 1990 to 2000, and from 2000 to 2007.

Table 5.12 The Movement across Income Classes (1960 to 1970)

Target (1970)	Source (1960) 1	2	3	4	5	6	Total
1	6	3	0	0	0	0	9
2	1	21	7	0	0	0	29
3	0	1	20	2	0	0	23
4	0	0	0	17	1	0	18
5	0	0	0	5	12	1	18
6	0	0	0	0	1	7	8
Total	7	26	27	24	14	8	105

Table 5.13 Transition Matrix (1960 to 1970)

Target (1970)	Source (1960) 1	2	3	4	5	6
1	0.857	0.120	0	0	0	0
2	0.143	0.840	0.259	0	0	0
3	0	0.040	0.741	0.083	0	0
4	0	0	0	0.708	0.071	0
5	0	0	0	0.208	0.857	0.125
6	0	0	0	0	0.071	0.875

Table 5.14 The Movement across Income Classes (1970 to 1980)

Target (1980)	Source (1970)						
	1	2	3	4	5	6	Total
1	8	7	0	0	0	0	15
2	1	20	3	0	0	0	24
3	0	2	12	1	0	0	15
4	0	0	8	14	1	0	23
5	0	0	0	3	12	1	16
6	0	0	0	0	5	7	12
Total	9	29	23	18	18	8	105

Table 5.15 Transition Matrix (1970 to 1980)

Target (1980)	Source (1970)					
	1	2	3	4	5	6
1	0.889	0.241	0	0	0	0
2	0.111	0.690	0.130	0	0	0
3	0	0.069	0.522	0.056	0	0
4	0	0	0.348	0.778	0.056	0
5	0	0	0	0.167	0.667	0.125
6	0	0	0	0	0.278	0.875

Table 5.16 The Movement across Income Classes (1980 to 1990)

Target (1990)	Source (1980)						
	1	2	3	4	5	6	Total
1	14	10	0	0	0	0	24
2	1	10	6	0	0	0	17
3	0	4	9	9	0	0	22
4	0	0	0	11	2	0	13
5	0	0	0	3	12	6	24
6	0	0	0	0	2	6	8
Total	15	24	15	23	16	12	105

Table 5.17 Transition Matrix (1980 to 1990)

Target (1990)	Source(1980)					
	1	2	3	4	5	6
1	0.933	0.417	0	0	0	0
2	0.067	0.417	0.400	0	0	0
3	0	0.167	0.600	0.391	0	0
4	0	0	0	0.478	0.125	0
5	0	0	0	0.130	0.750	0.500
6	0	0	0	0	0.125	0.500

Table 5.18 The Movement across Income Classes (1990 to 2000)

Target (2000)	Source (1990)						
	1	2	3	4	5	6	Total
1	23	5	0	0	0	0	28
2	0	11	2	0	0	0	13
3	0	1	19	2	0	0	22
4	1	0	1	10	0	0	12
5	0	0	0	1	19	1	21
6	0	0	0	0	2	7	9
Total	24	17	22	13	21	8	105

Table 5.19 Transition Matrix (1990 to 2000)

Target (2000)	Source (1990)					
	1	2	3	4	5	6
1	0.958	0.294	0	0	0	0
2	0	0.647	0.091	0	0	0
3	0	0.059	0.864	0.154	0	0
4	0.042	0	0.045	0.769	0	0
5	0	0	0	0.077	0.905	0.125
6	0	0	0	0	0.095	0.875

Table 5.20 The Movement across Income Classes (2000 to 2007)

Target (2007)	Source (2000)						
	1	2	3	4	5	6	Total
1	23	1	0	0	0	0	24
2	5	12	0	0	0	0	17
3	0	0	18	1	0	0	19
4	0	0	4	7	0	0	11
5	0	0	0	4	19	0	23
6	0	0	0	0	2	9	11
Total	28	13	22	12	21	9	105

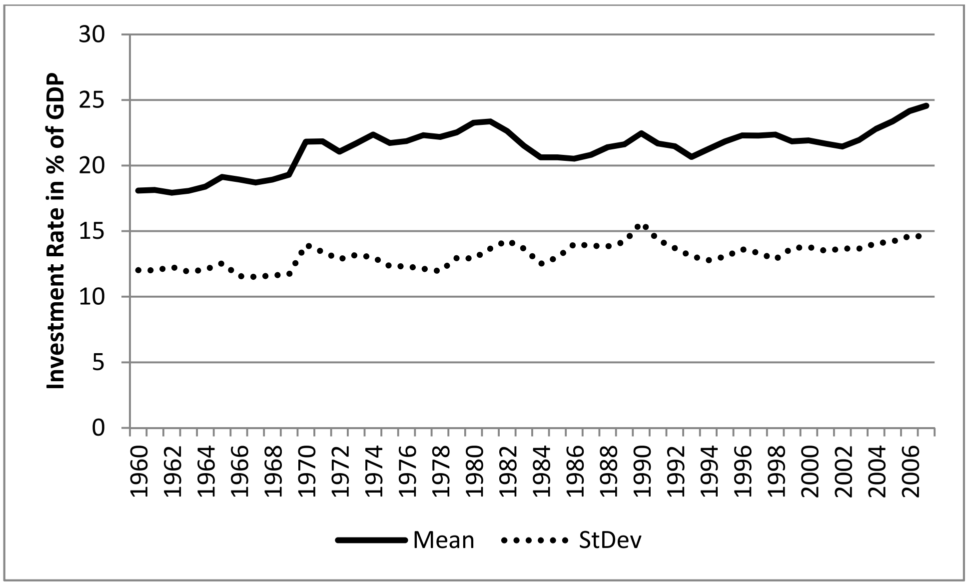
Table 5.21 Transition Matrix (2000 to 2007)

Target (2007)	Source (2000)					
	1	2	3	4	5	6
1	0.821	0.077	0	0	0	0
2	0.179	0.923	0	0	0	0
3	0	0	0.818	0.083	0	0
4	0	0	0.182	0.583	0	0
5	0	0	0	0.333	0.905	0
6	0	0	0	0	0.095	1.000

5.5.2 The Investment Rate

As stated in Chapter 3, the savings rate can be seen as a possible explanation for the emergence of twin peaks in the real per capita income distribution. For this reason, it is worth to find out more about whether the distribution of the savings rate is twin peaked as well. As described before, the savings rate will be approximated by the investment rate. Thus, in the following the terms “savings rate” will be replaced by the term “investment rate”. As done by Ziesemer (2004), it might be argued that the emergence of twin peaks in real per capita GDP should be influenced by twin peaks somewhere else, hence, in any of the explanatory factors in the model. One of these factors which should be considered in this doctorate is the investment rate; the other two are the population growth rate and human capital. The former will be the subject of this section, the population growth rate is the subject of Section 5.5.3, and human capital will be dealt with in Section 5.5.4.

Figure 5.7 Descriptive Statistics for the Investment Rate (105 Countries)



5.5.2.1 The Outliers

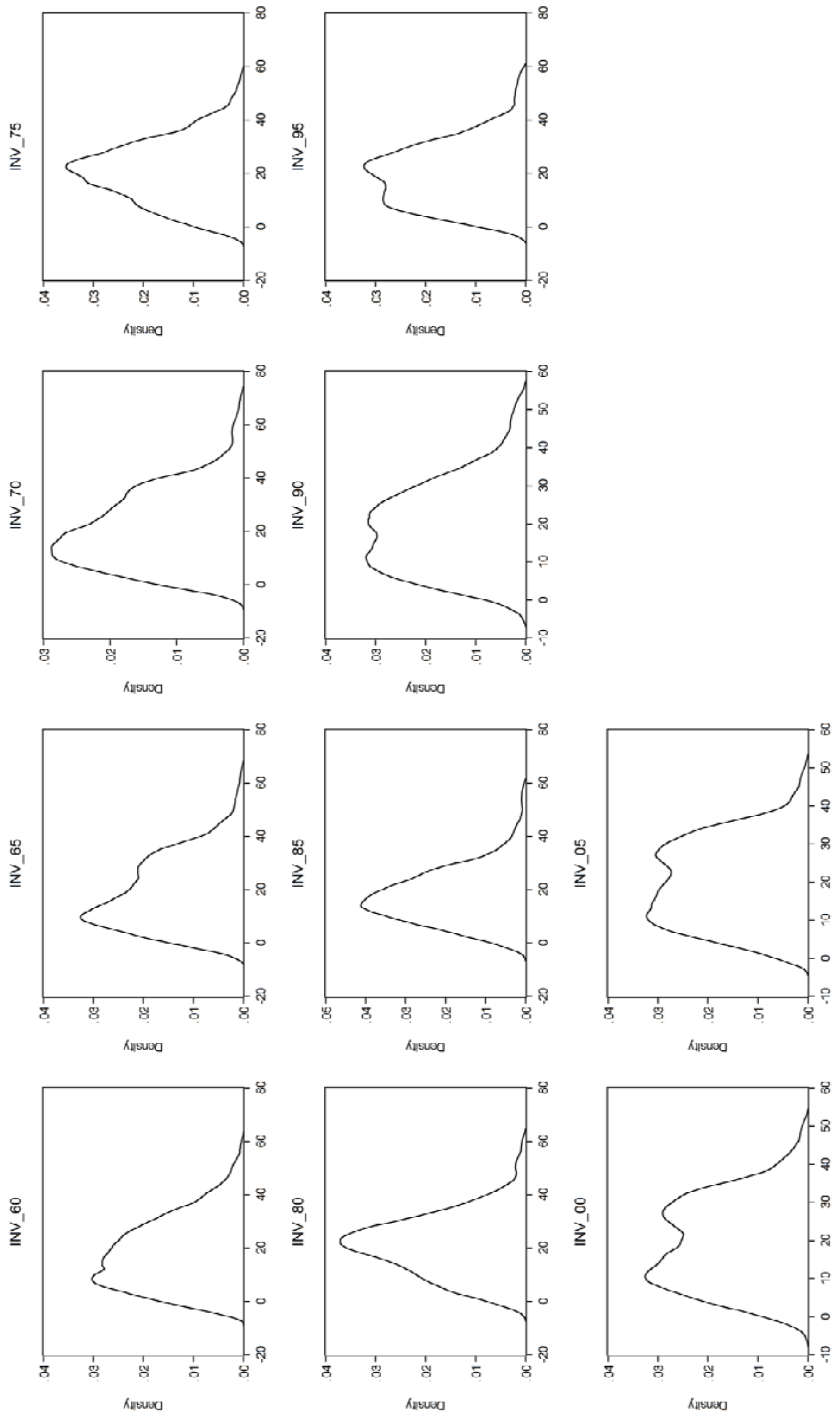
The descriptive statistics were already presented in Section 5.4. Here, the focus is on distribution analysis. Though, before this can be done, also the investment rate needs to be checked for outliers which should be excluded in the whole dataset, hence in all years under consideration. The outliers for the variable investment rate are defined as in the case of GDP. Hence, outliers are countries which have a data point outside of the interval of three standard deviations around the mean of the whole world in a sufficient number of years. Looking at the outliers[138] shows that several countries fall out of the dataset in some years, sometimes only once or twice. However, there is no country which is an outlier in a sufficient number of years so that it should indeed be excluded from the whole dataset in all years. It can be concluded that according to the above mentioned definition no outliers are excluded from the dataset. However, in some kernels they are then responsible for an additional very small peak at the upper end of the distribution.

5.5.2.2 The Kernel Densities

Knowing that there are no countries which need to be excluded from the dataset despite of the already excluded oil-producing countries, the kernel density analysis will be repeated for the investment data. Again, before the kernel densities are shown, Figure 5.2 is reproduced for the reduced dataset of 105 countries. Figure 5.7 shows the results. On average, the mean is lower than in Figure 5.2, but still rather stable. The standard deviation is also a bit lower. Contrary to before, it is very slightly downward sloping rather than upward sloping over time. Figure 5.8 gives the kernel densities for selected years, the complete set of graphs can be found in the Appendix (A.8). Looking at the distribution graphs leads to the conclusion that the investment rate is obviously not twin peaked until 1990. From then on, in the majority of years twin peaks seem to arise, even though it should be noted that these twin peaks are characterized by flat valleys. However, the distribution looks totally different than the one of real per capita GDP. There, a large peak at a low level of income could be found, while the peak at a high level of income was much smaller. For the investment rate, this is different. In the years in which two peaks can be observed, those peaks have a similar height, so that the investment rate obviously does not directly influence the income distribution. However, this does not mean that the investment rate does not influence the emergence of twin peaks. There are more analyses necessary to judge on this issue. In Section 5.6, a further statistical method will be applied to show that the investment rate is indeed an important variable, namely the loess fit method based on a nearest neighbor regression. Looking at Figure 5.8, there seem to be several peaks in some years even before 1990. As these peaks appear to be very small, they should be obeyed as they only stand for one or two countries and hence might be interpreted as outliers instead. At this point it should be remembered that these outliers were deliberately not excluded from the dataset because they are not an outlier in many years. Nevertheless, in order to be interpreted as a peak, there should be more countries in it. The findings of this section fit the findings of other authors on twin peaks in the investment rate. Ziesemer (2004), for example, also finds that there are not really twin peaks in the distribution of the investment rate.

[138] The list of those countries together with the number of years in which they are identified as outliers can be found in the Appendix (A.11).

Figure 5.8 The Distribution of the Investment Rate



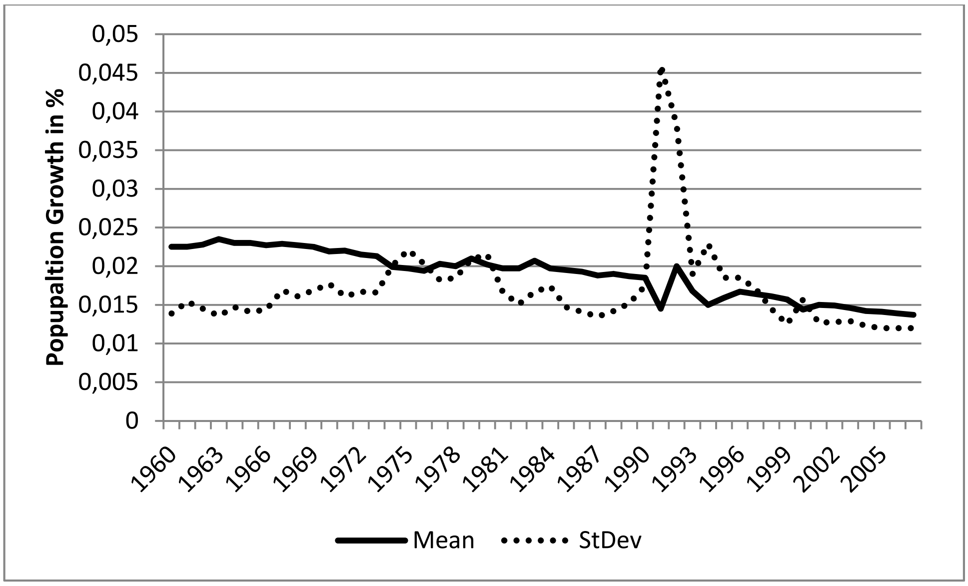
5.5.3 The Population Growth Rate

In Chapter 3, it was already pointed out that also the population growth rate might yield twin peaks in the real per capita income distribution. Hence, the distribution of this variable should be considered further. This is the purpose of this subsection.

5.5.3.1 The Outliers

The descriptive statistics were already presented in Section 5.4. Here, the focus is on distribution analysis. As for the investment rate, also here the dataset needs to be checked for outliers. The outliers for the variable population growth rate are defined as in the case of GDP and the investment rate. Hence, outliers are countries which have a data point outside of the interval of three standard deviations around the mean of the whole world in a sufficient number of years. Looking at the outliers[139] shows that again, several countries fall out of the dataset in some years, sometimes only once or twice. Qatar is the country which is an outlier in 22 of the years under consideration. Yet, it is excluded from the dataset anyhow as it is an oil-producing country.

Figure 5.9 Descriptive Statistics for the Population Growth Rate (105 Countries)



What is important to note is that identifying outliers in the case of the population growth rate is not easy. The reason is that the population growth rate might be "too high" or "too low" for several external reasons. An extremely low population growth

[139] The list of those countries together with the number of years in which they are identified as outliers can be found in the Appendix (A.11).

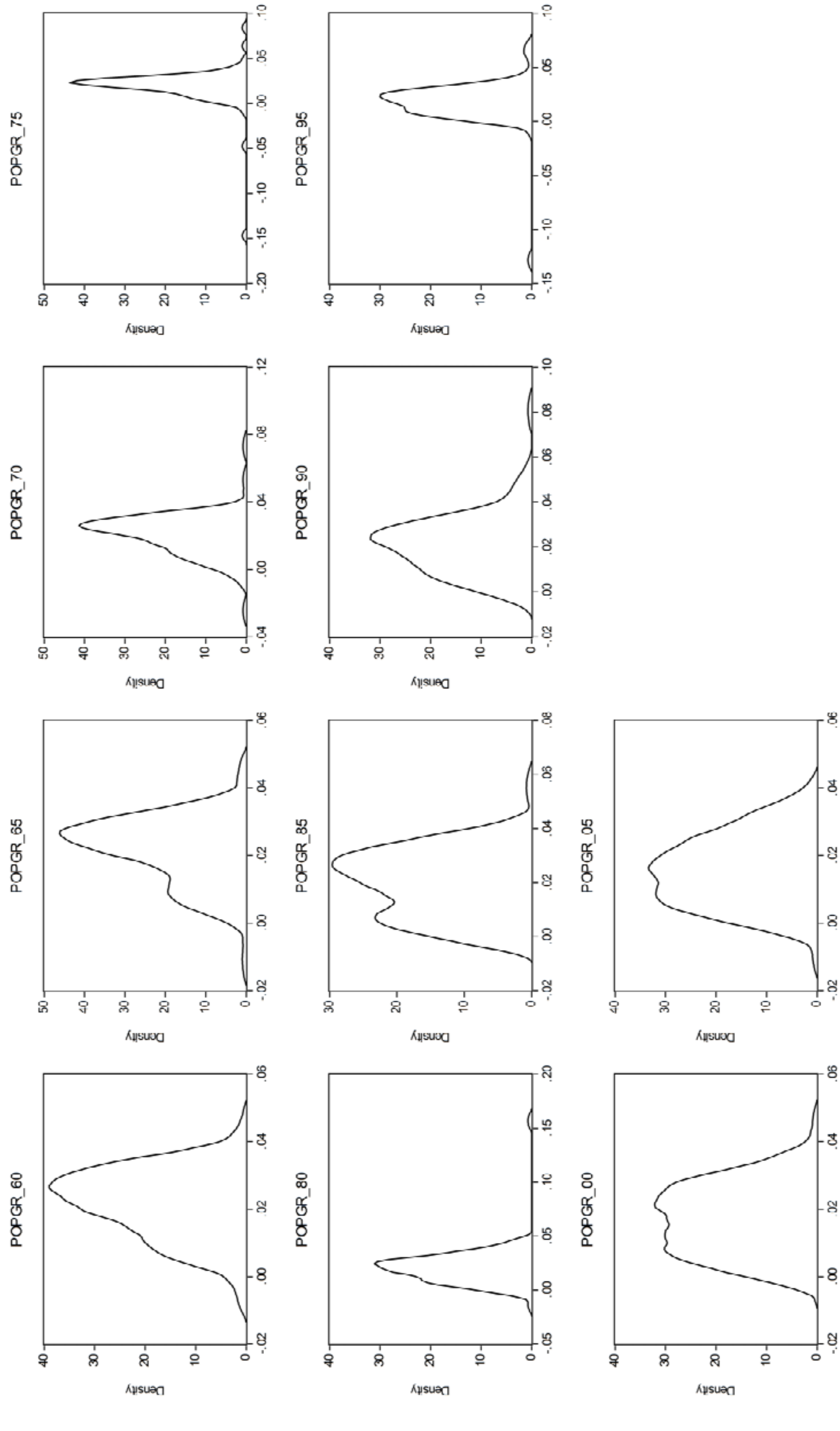
rate may be due to a war, a natural disease, or illnesses. On the other hand, the population growth rate may be unexpectedly high due to wars in neighboring countries, hence immigrations or enlargements of a country, among others. For example, it is not surprising that the population growth rate in Israel looks like an outlier at the upper end of the spectrum in 1951 and 1952, as a large number of immigrants came from Europe to Israel after World War II. Excluding such countries from the dataset would mean losing important data. Such external facts do have an influence on the economic development of a country and hence cannot be denied by just excluding such countries from the dataset. In addition, it is likely that wars, natural diseases, and illnesses on the one hand, and higher population growth rates due to, for example, habits and religion on the other hand are decisive in poor countries and might be a reason for ending up in a poverty trap. Hence, it was decided not to delete any country for the population growth rate from the dataset. Consequently, the dataset includes the same countries as already noted under the section about real per capita GDP.

5.5.3.2 The Kernel Densities

After having decided not to exclude countries as outliers from the dataset on the basis of the population growth rate, this variable will be checked for twin peaks. This is the task of this subsection. Again, before the kernel densities are shown, Figure 5.3 is reproduced for the reduced dataset of 105 countries. Figure 5.9 shows the results. It becomes obvious that the mean is still about the same while the standard deviation is much more volatile than when including all countries offering data.

The density distribution of the population growth rate will be checked as well. Hence, the kernel densities will be presented in Figure 5.10. Again, the kernel densities are only reported for selected years here while in the Appendix (A.8) the complete set of density distributions is shown. The kernel densities show mainly single peaks. Exceptions are slightly 1960 – and more obviously 1985. In some years, there seem to be outliers, for example in 1975, 1980, and 1995. This is definitely due to external factors already described above. As stated in the previous subsection, it was decided to leave those countries in the dataset because external factors do have an influence on the economic development of a country and hence, excluding such countries would mean losing valuable information. As in the case of the investment rate, this again does not mean that twin peaks might not be due to differences in population growth rates. A further analysis to judge this is the loess fit method based on a nearest neighbor regression. This will be done in Section 5.6.

Figure 5.10 The Distribution of the Population Growth Rate



5.5.4 Human Capital

As mentioned before, human capital can be measured in several ways. In this doctoral thesis, it was already argued that human capital will be measured by the average years of schooling provided by the Barro-Lee-dataset.[140] Here, the same procedures as in the previous sections will be repeated. Before the kernel densities can be shown, again an outlier analysis needs to be made. This will be the subject of the following subsection.

5.5.4.1 The Outliers

The data on the average years of schooling are, as outlined above, available for every fifth year only. Hence, the years for which analyses can be shown fit to those shown for real per capita GDP and the investment rate, namely the years 1960, 1965, 1970, 1975, 1980, 1985, 1990, 1995, 2000, and 2005. Concerning the outlier definition, again the same rule is used, hence only countries which are outside of the interval [mean ± 3 · standard deviation] in a sufficient number of years are excluded from the total dataset. Using this rule yields no outliers at all. For this reason, no countries except for the above mentioned oil-producing countries need to be excluded from the dataset.

5.5.4.2 Kernel Densities

Knowing the exact size of the dataset, in this section, the kernel densities for human capital measured by the average years of schooling will be presented. As with income and the investment rate, only those countries offering data in all years considered will be taken into account. Doing this and not taking into account yet whether those countries offer GDP data in all years, yields a reduced dataset covering only 82 countries instead of 129. This is already a much smaller dataset. Reducing this set even further and using only those countries which also yield income data in all of the years covered by the human capital dataset yields only 65 countries.

For the now reduced dataset as compared to Section 5.4, the mean and the standard deviation are determined again. Figure 5.11 shows these statistical values for the larger as well as for the smaller dataset. What becomes apparent from this figure is that excluding also those countries not offering GDP data in all years leads to a slightly higher mean in all years, while the standard deviation decreases slightly overall.

140 Again human capital will also be analyzed for another variable, namely public spending on education, offered by the World Development Indicators from the World Bank. The results of this sensitivity analysis are shown and briefly discussed in the Appendix (A.4).

In Figure 5.12, the kernel densities are shown for the dataset of 82 countries, while Figure 5.13 shows the kernel densities of the further reduced dataset of only 65 countries. Obviously, the conclusions to be drawn are different in both cases. In the former case, with 82 countries used as input, twin peaks in human capital seem to appear, while using only 65 countries yields a distribution which is rather unipeaked. It can be seen that the interpretation of the results of the human capital analysis is indeed dependent on the choice of countries. However, at this point in time, it does not make sense to exclude those countries not offering GDP data in all years. As GDP and human capital shall not be brought together here, it is sufficient to care about the human capital data only. When analyzing the two variables together, those countries should be taken out. This will be done in the following section. Here, the connection of the investment rate and income on the one hand, and human capital and income on the other hand will be examined further by use of the loess fit method.

Figure 5.11 Descriptive Statistics for Human Capital for Different Datasets

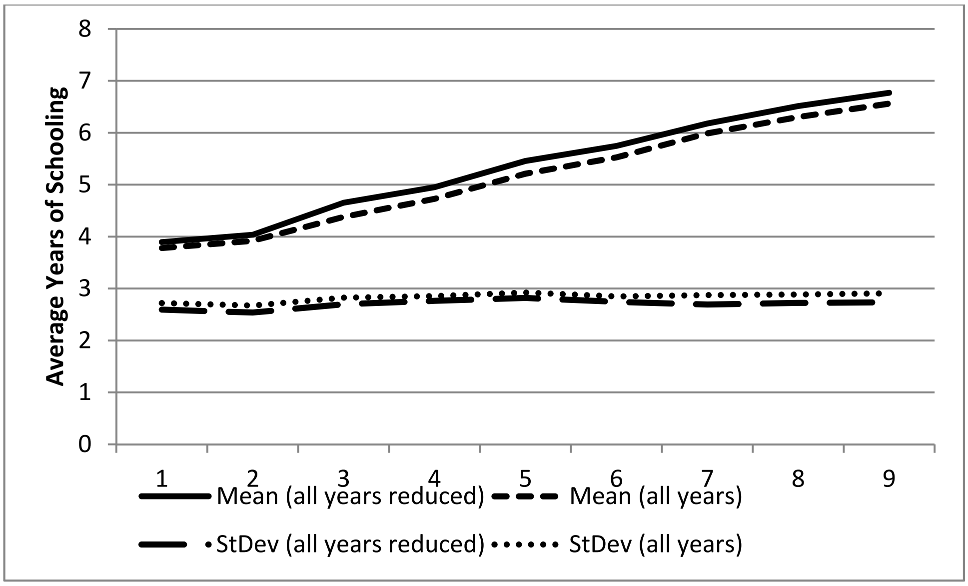


Figure 5.12 The Distribution of the Average Years of Schooling (82 Countries)

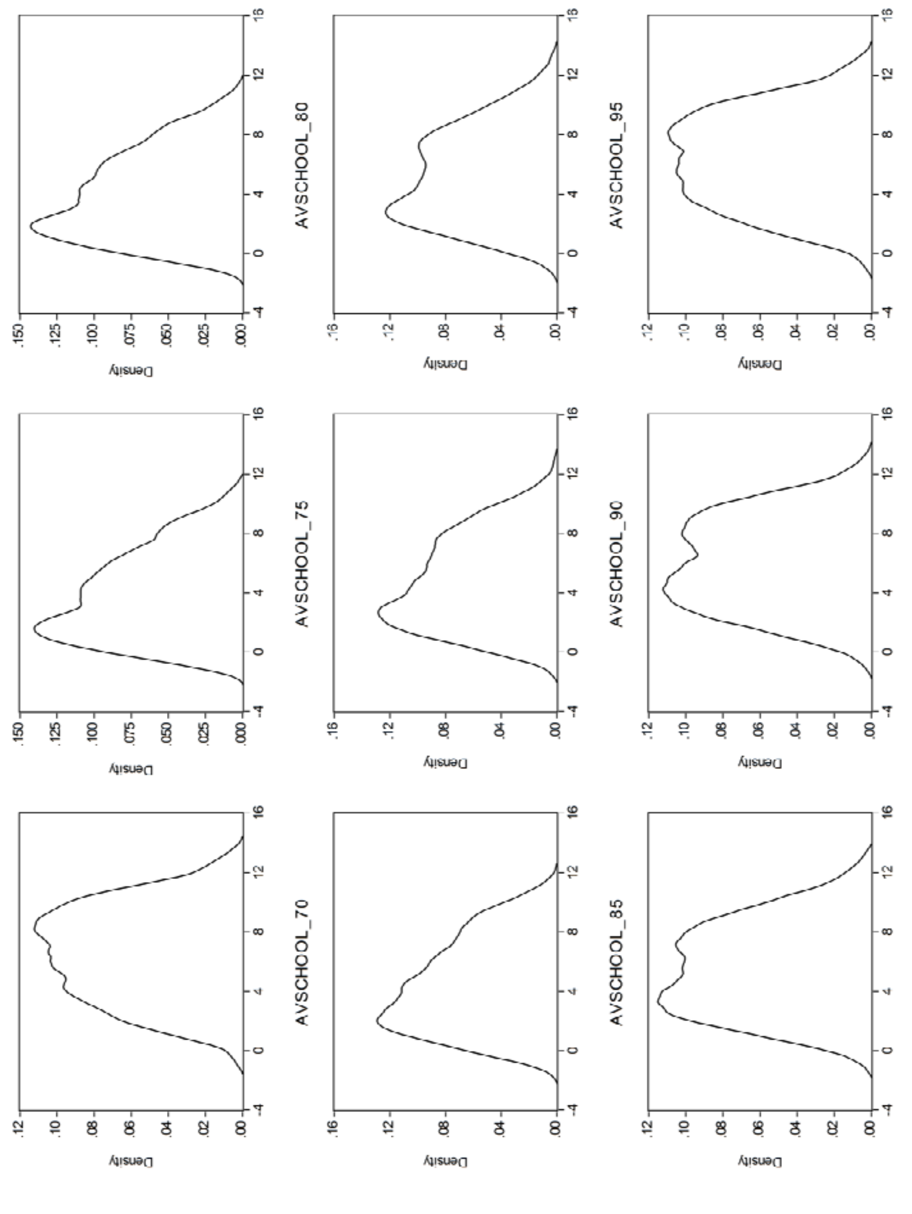
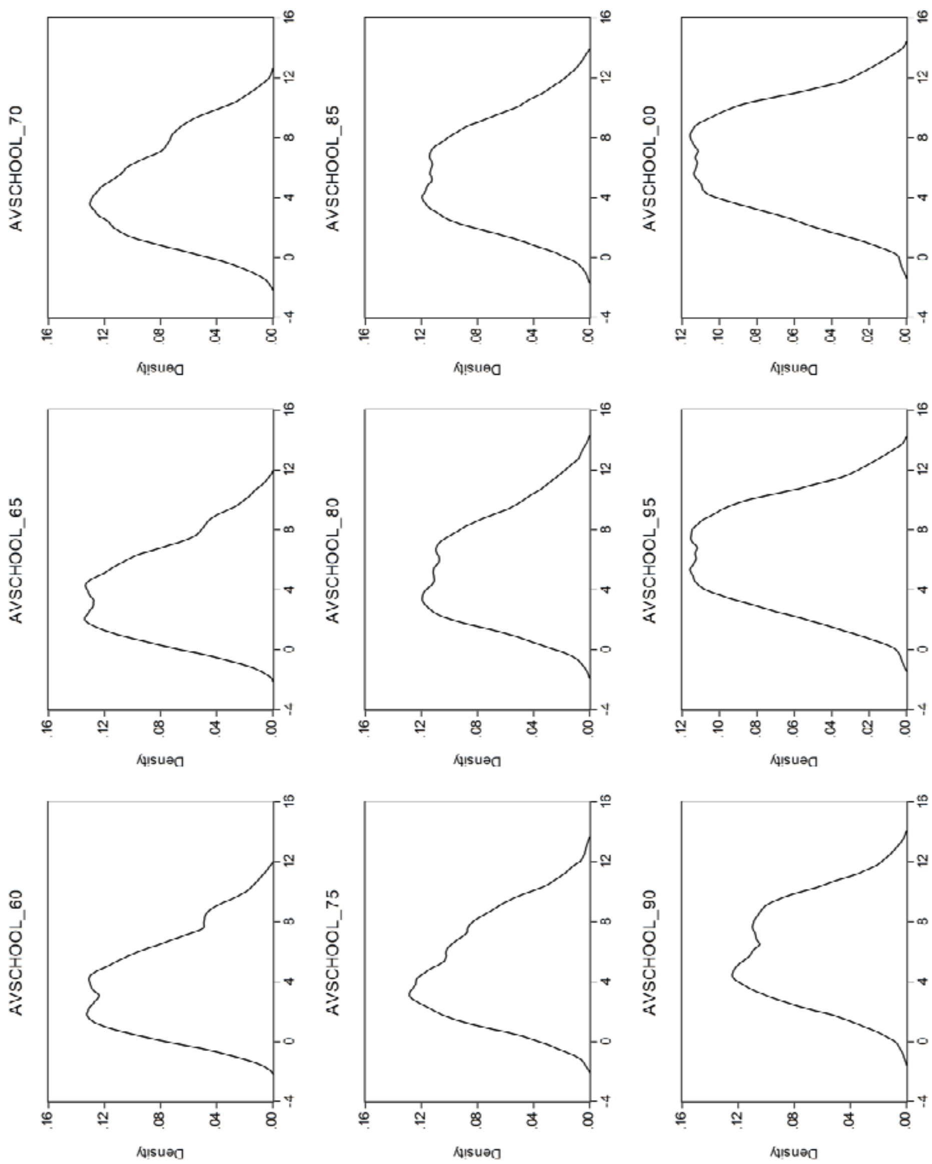


Figure 5.13 The Distribution of the Average Years of Schooling (65 Countries)



5.6 The Loess Fit Method

After having examined the data in detail by presenting the descriptive statistics and the kernel densities in the previous section, this section will deal with loess fit curves of the respective variables compared to real per capita GDP. By use of EViews, scatter plots including a loess fit curve will be presented. In this way, it is possible to get an idea of what the function[141] looks like and in how far a connection between the investment rate, the population growth rate, or human capital on the one hand and real per capita GDP on the other hand is existent. Furthermore, it gives a hint on whether the respective variable might indeed be an explanation for the emergence of twin peaks in real per capita GDP. In the following section, first the investment rate will be considered, while thereafter population growth and human capital will be under consideration.

5.6.1 The Investment Rate

The loess fit curves to be presented here are, as mentioned above, based on the nearest neighbor method.[142] On the y-axis, the investment rate can be found as it shall be examined in how far real per capita income influences the investment rate. On the x-axis, real per capita GDP is measured. Figure 5.14[143] (see p. 129) shows the results of this method. As for the kernel densities, only the figures for every fifth year will be shown here while the complete set of graphs will be presented in the Appendix (A.8).

Figure 5.14 shows clearly that there is no linear relationship between the investment rate and real per capita GDP. It looks more like a function which is at least quadratic. Hence, when further analyzing the data by use of regression analysis, which is done in Chapter 7 in more detail, it is necessary to include polynomials of GDP on the right hand side of the equation. Though, how many of these polynomials need to be included depends on the results of the regression analysis. It needs to be checked how many of them are statistically significant and which model is optimal. But this will be the subject in the following chapter. In the next subsections, the loess fit analysis shall be repeated for the population growth rate and for human capital.

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[141] For example, the investment function (or rather say the savings function in terms of the Solow growth model) or the population growth function.

[142] The loess fit curves are estimated by the statistical computer package EViews. For using this method, the analyst has to decide on some elements. The bandwidth is kept at 0.3. The polynomial degree is equal to 1, and the weight function is local weighting (tricube). The evaluation method used is the Cleveland subsampling by using 100 evaluation points.

[143] RGDPL_60 stands for real per capita GDP in the year 1960. For the other graphs, this holds respectively.

Figure 5.14 The Loess Fit Curves – Investment Rate and Real per Capita GDP

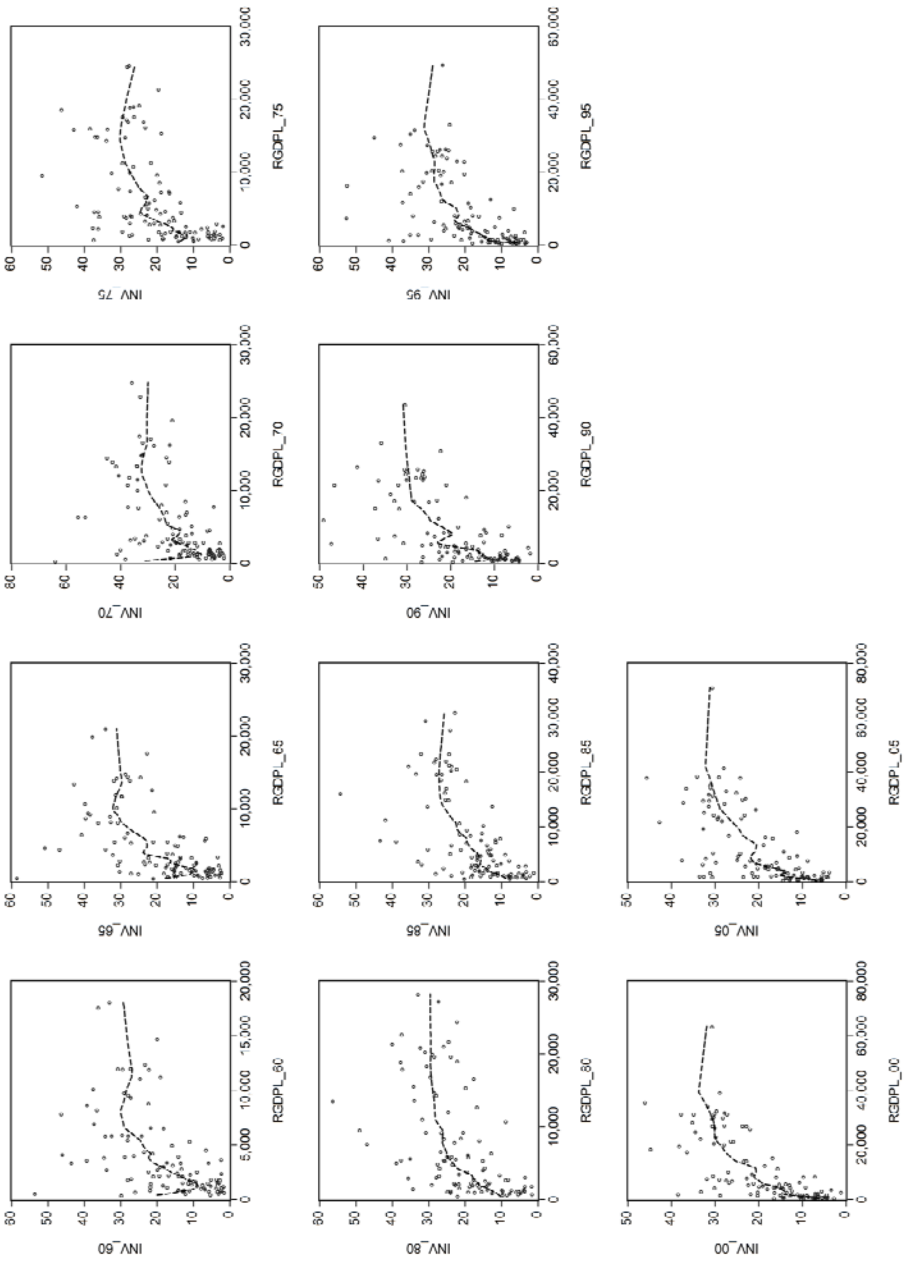


Figure 5.15 The Loess Fit Curves – Population Growth Rate and Real per Capita GDP

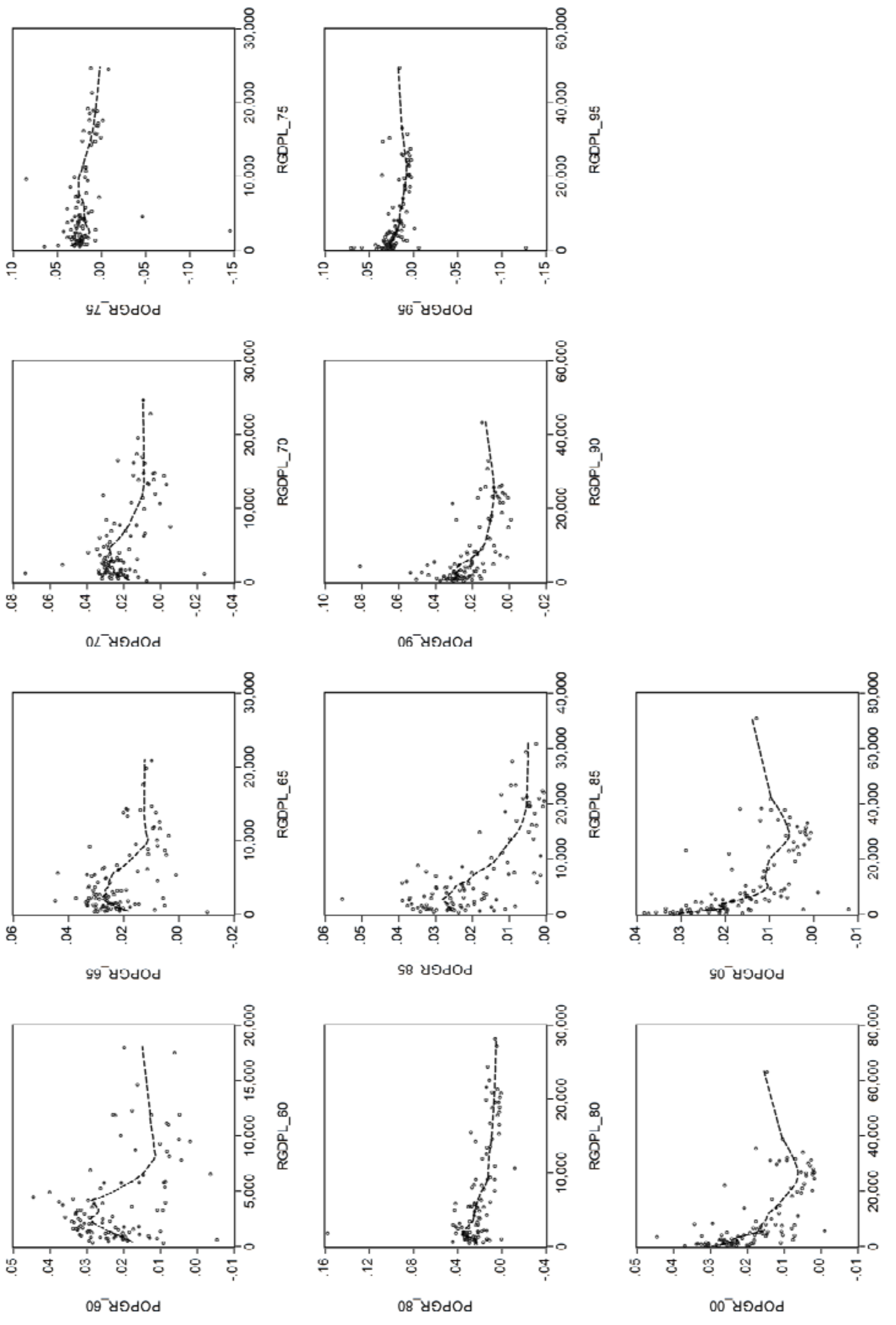
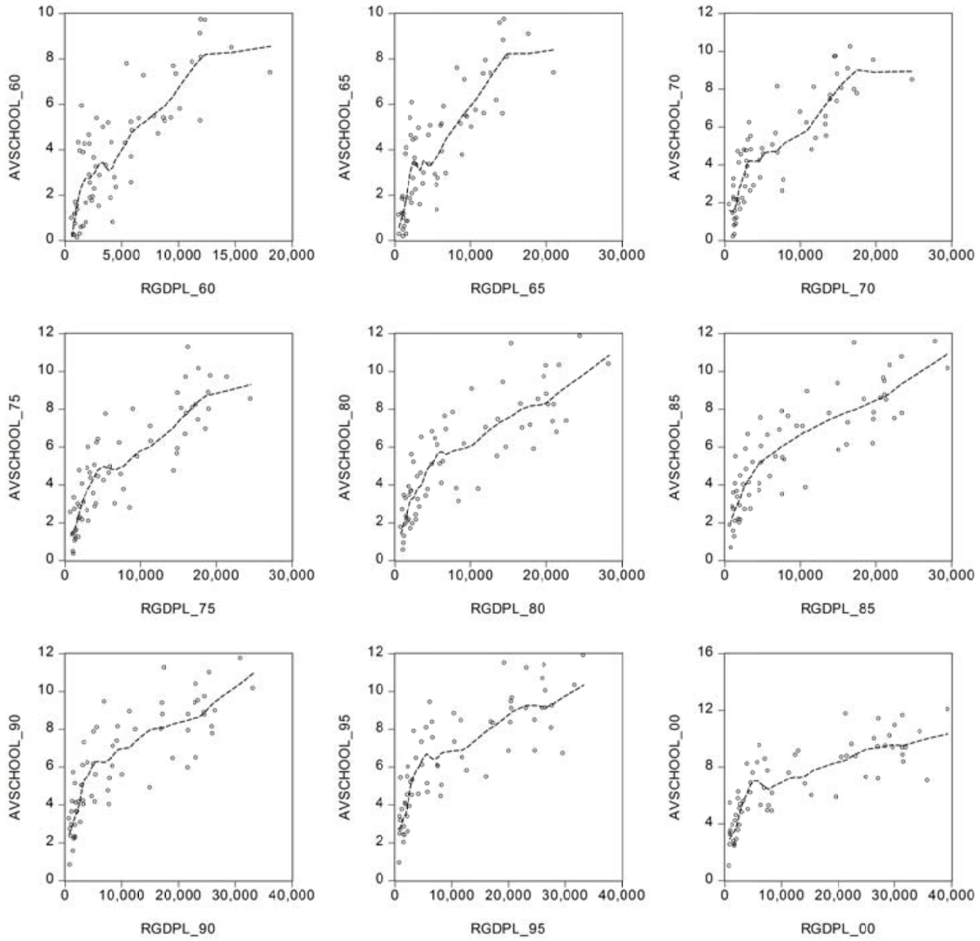


Figure 5.16 The Loess Fit Curves – Human Capital and Real per Capita GDP



5.6.2 The Population Growth Rate

In this subsection, the loess fit curves for the population growth rate will be presented. Figure 5.15 (see p. 130) shows the results. The figure shows very obvious outliers for some years.[144] The distribution changed over time. In the first years under consideration, there is no clear relationship (a rather horizontal line appears). In the following years, the distribution first rises and then starts to decrease at a diminishing rate. From 1995 on, the curve is rather U-shaped. It becomes obvious that the population growth rate alone is not able to explain the emergence of twin peaks. Yet, when combined with the investment rate, it is rather likely that twin peaks emerge. This idea will be taken up in Chapter 7. However, before this is discussed further, human capital will be analyzed by the loess fit method. This will be done in the following subsection.

5.6.3 Human Capital

As described above, human capital is given by the average years of schooling. As these data out of the Barro-Lee-dataset are only available for every fifth year, the complete set of possible graphs will be presented here. Figure 5.16 (see p. 131) shows the loess fit curves for human capital.[145] What becomes apparent is that real per capita GDP seems to positively correlate with the average years of schooling. Furthermore, the loess fit curves are nonlinear and seem to obey diminishing returns in the first four years considered. After 1980, the curve remains upward sloping. Sometimes there are little peaks in it so that also here the conclusion would be that the average years of schooling might be influenced by real per capita GDP in a polynomial way.

5.7 Conclusion

In this chapter, the focus was on the empirical analysis of the twin peaks phenomenon. After an overview of existing literature on the empirical side of the polarization debate, the data sources being used in this doctoral thesis were described in more detail. The GDP data as well as the investment rate as an approximation of the savings rate were taken out of the Penn World Table 6.3. As human capital data are not included in this dataset, another source needed to be found. It was decided to follow the habit to use the Barro-Lee dataset. It was shown that the best option is to use the average years of schooling as an indicator of human capital. After looking at the data in more detail by use of descriptive

[144] Just to remind the reader, they were left inside for reasons outlined above.

[145] For these graphs, the dataset covering only 65 countries needs to be used, as the other 17 countries being in the first dataset do not offer data for GDP in all years. Hence, they need to be excluded. This means that in the end only 65 countries offer data.

statistics, the data were further analyzed by the kernel method. It was shown that, when following the method of Danny Quah, namely when using only those countries which offer data in all years under consideration, twin peaks indeed arise and remain a common feature of the world income distribution. The future distribution of real per capita GDP was examined by use of the Markov chain analysis. Additionally, it was shown that neither the investment rate nor the population growth rate nor human capital are as clearly twin peaked as GDP so that it cannot directly be concluded that this yielded the twin peaks in income.

For this reason, it was decided to present a new form of analysis in the research on the bimodality in the world income distribution: the loess fit method. This more descriptive statistical method was applied in order to get further insight in the role of the investment rate, the population growth rate, and human capital. The loess fit curves were presented as scatter plots together with a regression line based on the nearest neighbor method. From these figures, it could be concluded that the investment rate, the population growth rate, and human capital indeed are nonlinearly influenced by GDP. Based on these findings, the following chapter will focus on the main question of this doctorate from a theoretical point of view: is the Solow growth model indeed able to capture bimodality?

6 A Neoclassical Growth Model Capturing Bimodality

In the previous chapter, an empirical analysis was presented which showed that bimodality is indeed a common feature in the real per capita GDP distribution. In Chapter 3, it was shown that from a graphical and verbal point of view, the Solow growth model is indeed able to capture multiple steady states. Therefore, Solow's hypothesis was confirmed. In this chapter, his claim shall be examined further from a different perspective. Here, the focus is on the theoretical model.

A number of authors criticize physical capital as a source of economic growth (for example, Galor and Moav, 2004, who argue that using human capital accumulation instead of physical capital accumulation as a prime engine of growth yields much better results). Nevertheless, in this doctoral thesis the idea of physical capital as a source of economic growth will be pursued. By use of an endogenous savings rate, the model shall be solved analytically in order to find out whether indeed two stable equilibria result. This is based on the achievement of Azariadis (2006) that the only robust variable for explaining the existence of poverty traps, and hence of bimodality in the real per capita income distribution across the countries of the world, is investment. As outlined above, investment is assumed to be equal to savings. Consequently, within the framework of the Solow growth model this implies that an endogenous savings rate is likely to yield twin peaks.

To begin with, Section 6.1 will deal with the determination of the endogenous savings rate.[146] Thereafter, the assumptions underlying the modified Solow growth model of this chapter will be discussed. These will be used in Section 6.3 for the formulation of the model. Thereafter, in Section 6.4, the steady states will be determined. For this analytical determination of the steady states, iteration methods like the Newton method need to be applied. Section 6.5 will conclude this chapter.

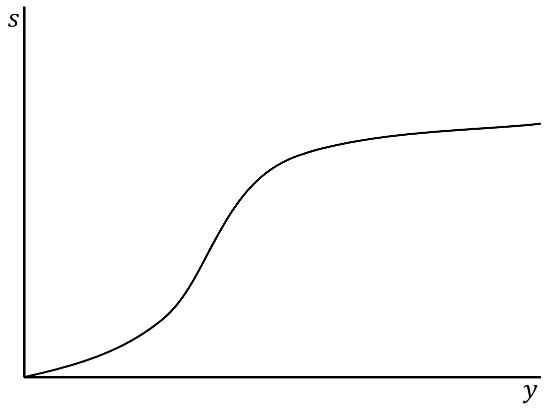
6.1 The Endogenous Savings Rate

In Chapter 3, the idea of an endogenous savings rate was already discussed. It was shown that a number of authors point to the positive relationship between savings and income (for example Steger 2001). Additionally, there are analyses indicating that the savings rate might rather be zero or at very low levels (see Harms and Lutz, 2004). These findings in combination with a look at the data provided by Ogaki, Ostry, and Reinhart (1996) show that a logistic savings function is quite realistic. The authors argue that subsistence consumption plays a crucial role at least at the lowest income levels. For more details, the reader is referred

[146] It should be kept in mind that even though the Solow growth model includes the savings rate, in this doctoral thesis the data on the investment rate are used instead for the above-mentioned reasons. Hence, for reasons of correctness the term "investment rate" will be used in this chapter instead.

back to Chapter 3. Figure 6.1 shows the savings function underlying the analyses of this chapter. This function has a segment of exponential growth and in addition a degree of saturation. These are characteristics of the differential equation which will be the subject of the following subsection, where the theoretical background to the differential equation will be given.

Figure 6.1 The Savings Rate



6.1.1 The Differential Equation

A differential equation is an equation including a function y, which has to be determined, as well as one or more of its derivatives with respect to one of the other variables. If the differential equation is dependent on only one independent variable, it is called ordinary differential equation; otherwise it is called partial differential equation. Often, the function y is not identifiable, if it exists at all. Thus, further conditions for this function need to be formulated (Bronstein and Semendjajew, 1979).

Usually, differential equations allow for growth of the variable to be considered. Though, if for any reason there is something like an upper limit to this growth, a degree of saturation needs to be included in the equation. In this case, logistic growth is described by the differential equation. This equation may have the following form:

$$\frac{dP}{dt} = \lambda P(K - P), \tag{6.1}$$

where P stands for the variable, here population, $K - P$ is the remaining distance to the upper limit, hence to the degree of saturation K, and λ is a parameter with $\lambda > 0$. Getting closer to K, the population will start to stagnate and thus remain stable. Using Equation (6.1), it can be verified that the function P is given as follows (Heuser, 2009):[147]

$$P(t) = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-\lambda K t}}. \tag{6.2}$$

This means that if a population has the initial value P_0 and changes according to the logistic law of Equation (6.1) for small values, then its size $P(t)$ will necessarily be given by Equation (6.2) at time $t \geq 0$ (Heuser, 2009). If t goes to infinity, then $P(t)$ approaches the degree of saturation, here denoted by K.

[147] The proof is shown in the Appendix A.17.

Basing the savings rate on a degree of saturation is very important, because it cannot reach more than 100 percent of income in a closed economy. Whether the degree of saturation should be 100 percent or rather less than that will be discussed later. At this point, it is sufficient to note that it is much more plausible to assume logistic growth of the savings rate as opposed to exponential growth.

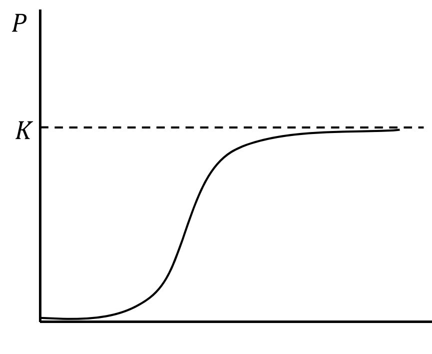
The function given in Equation (6.2) is static. Furthermore, it is strictly growing with a small initial population, that is if $\frac{K}{P_0} > 1$, hence if $P_0 < K$. On the contrary, if the initial population is large, that is if $\frac{K}{P_0} < 1$, hence if $P_0 > K$, the function is strictly falling. Of course, if $P_0 = K$, the equation is constant. In order to see how the derivative of P evolves over time, thus how Equation (6.1) evolves over time, the second derivative with respect to time will be calculated. The result is given as follows:

$$\ddot{P} = \lambda(K - 2P)\dot{P}, \tag{6.3}$$

where a dot on a variable indicates its first derivative with respect to time and two dots indicate the second derivative with respect to time. This is the Newton-denotation (Heuser, 2009). If there is a very small initial population of $P_0 = \frac{K}{2}$, then also $P < \frac{K}{2}$ so that $K - 2P > 0$ holds. According to Equation (6.3), the second derivative will be positive. Starting with a value above $\frac{K}{2}$, the second derivative will be negative. This can be summarized as follows: the growth rate $\dot{P}$ increases until population reaches the size $\frac{K}{2}$, half of the possible maximum size K – this is the period of accelerated growth. Then $\dot{P}$ decreases continuously in the period of delayed growth (Heuser, 2009). The most important feature is the initial situation. The result is an S-curve as given in Figure 6.2.[148]

The differential equation can be applied to many phenomena in different sciences, for example bacteria growth in biology, vibrations of components in physics, or courses of celestial bodies in astronomy. In this doctorate it shall be applied to the development of the savings rate. Yet, contrary to the other fields of application, not the evolvement of the variable over time but rather on its evolvement with respect to changes in income is of concern. This will be discussed in more detail in the next subsection.

Figure 6.2 The Logistic Function



Source: Own representation according to Heuser, 2009

[148] For a more detailed description of the differential equation as well as a description of a number of examples for its applicability to reality, the interested reader is referred to Heuser (1991) or any other theoretical literature on differential equations.

6.1.2 The Savings Rate

In Chapter 3, theoretical considerations concerning the savings function were presented. It was shown that by looking at the existing literature, it is quite plausible to assume a logistic function determining savings dependent on income. Additionally, in Chapter 5, the investment rate as an approximation of the savings rate was examined empirically. It was demonstrated that the investment rate exhibited twin peaks in 1965 and in 2003. Though the development was interrupted in 1975, twin peaks reappeared so that it could be concluded that the savings rate is likely to yield twin peaks in the neoclassical growth model.[149] Additionally, the correlation coefficient was quite high, which, despite of its shortages due to the measurement of the savings rate, implies that income indeed has an influence on the savings rate. As the correlation coefficient is positive, it is feasible to conclude that the higher income, the higher the savings rate.[150]

Table 6.1 Maximum Values of the Savings Rate

Year	1955	1965	1975	1985	1995	2003
Maximum value	38.0	46.3	47.8	47.7	43.6	43.8

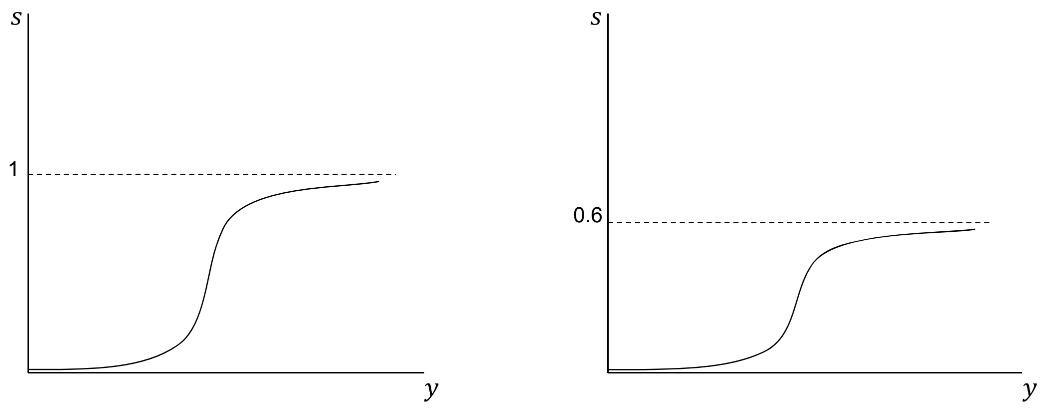
Following the argumentation of Section 3.2.1 (see p. 32), at low levels of income the savings rate can be assumed to be very low. If income starts to increase, the evolvement of the savings rate can be described by exponential growth up to a certain level of income. However, the savings rate cannot be higher than 100 percent; consequently, there clearly is a degree of saturation. In addition, looking at Table 6.1 shows that the maximum values of the savings rate in the respective years (rounded to the first decimal) do not exceed 47.8 percent. Taking into consideration that in one of the years not considered in this doctoral thesis the savings rate might even be slightly higher than what was measured here, the degree of saturation is assumed to be at 60 percent instead. Figure 6.3 shows the two different possibilities of the form of the savings rate, Panel (a) with a degree of saturation of 1 (hence 100 percent) and Panel (b) with a degree of saturation of 0.6 (hence 60 percent).

In the previous section, a function of this form was identified as a logistic curve being the solution to a differential equation with logistic growth. Yet, here the x-axis does not capture time but income instead.

[149] Even though in the empirical analyses the investment rate was used to approximate the savings rate, the savings rate will be used in this doctorate when writing about the theoretical model. The reason is that in the Solow growth model the variable s is included, and not the investment rate. In this way, the argumentation concerning finding twin peaks in the Solow growth model can be followed easier.

[150] Or – vice versa – correlation values just indicate that two variables correlate with each other, but not in which direction.

Figure 6.3 The Degree of Saturation



Panel (a): Degree of saturation = 1 Panel (b): Degree of saturation = 0.6

On the basis of the previous section together with the findings of this section, Equations (6.1) and (6.2) will be adapted to the savings rate. Replacing P by s, t by y in Equation (6.1), and additionally K by s_{max} in Equation (6.2) yields the corresponding equations for the endogenous savings rate:

$$\frac{ds}{dy} = \lambda s(1-s) \tag{6.4}$$

$$s(y) = \frac{s_{max}}{1+\left(\frac{s_{max}}{s_0}-1\right)e^{-\lambda s_{max} y}}, \tag{6.5}$$

with $s_0 > 0$. As in Section 6.1.1, it can be shown that Equation (6.5) is indeed the logistic savings function fitting the differential equation given by Equation (6.4).[151] To improve the handling, the following replacements will be made:

$$\beta_1 = \lambda \tag{6.6}$$

$$\beta_0 = \frac{1}{\beta_1} ln \frac{1-s_0}{s_0}, \tag{6.7}$$

where $\beta_1 > 0$. Apart from this, the degree of saturation will be set equal to one for convenience, so that

$$s_{max} = 1. \tag{6.8}$$

Consequently, the Equations (6.4) and (6.5) can be rewritten as follows:

$$\frac{ds}{dy} = \beta_1 s(1-s), \tag{6.9}$$

$$s(y) = \frac{1}{1+e^{-\beta_1(y-\beta_0)}}. \tag{6.10}$$

Equation (6.10) can be plotted, yet for this to be possible, the parameters have to be assigned specific values. In Figure 6.4, the savings function is plotted for $\beta_1 = 1$, $s_0 = 0.1$, and hence $\beta_0 = ln9$. For the simplified version of the savings

[151] The proof will be presented later on for a simplified version of the equations. At this stage, the proof is equal to the one above using K and P instead.

function it can be shown that Equation (6.10) is indeed the logistic function belonging to the differential Equation (6.9). First of all, the derivative of the logistic function (6.10) needs to be calculated:

$$\frac{ds}{dy} = \frac{-1(-\beta_1)e^{-\beta_1(y-\beta_0)}}{\left(1+e^{-\beta_1(y-\beta_0)}\right)^2}. \tag{6.11}$$

On the other hand, inserting Equation (6.10) into Equation (6.9) yields:

$$\frac{ds}{dy} = \beta_1 \frac{1}{1+e^{-\beta_1(y-\beta_0)}}\left(1 - \frac{1}{1+e^{-\beta_1(y-\beta_0)}}\right). \tag{6.12}$$

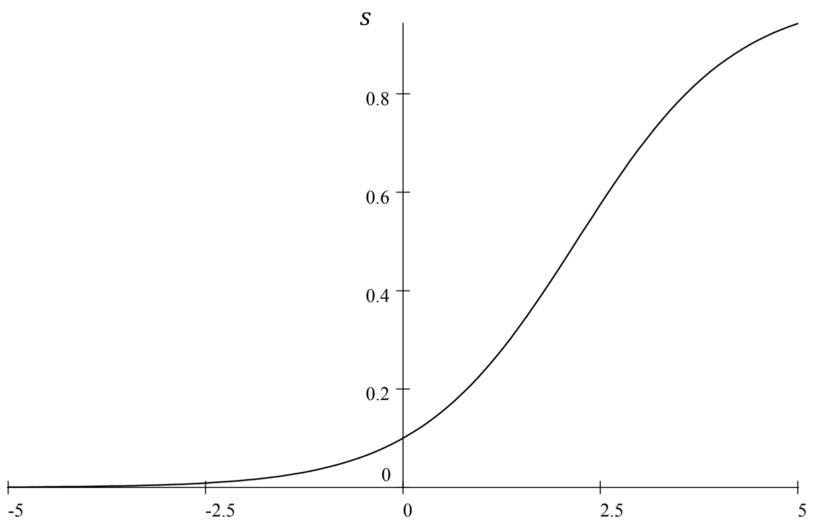
Equations (6.11) and (6.12) can then be equated:

$$\Rightarrow \frac{-1(-\beta_1)e^{-\beta_1(y-\beta_0)}}{\left(1+e^{-\beta_1(y-\beta_0)}\right)^2} = \frac{\beta_1}{1+e^{-\beta_1(y-\beta_0)}} \frac{1+e^{-\beta_1(y-\beta_0)}-1}{1+e^{-\beta_1(y-\beta_0)}}. \tag{6.13}$$

$$\Leftrightarrow \frac{\beta_1 e^{-\beta_1(y-\beta_0)}}{\left(1+e^{-\beta_1(y-\beta_0)}\right)^2} = \frac{\beta_1 e^{-\beta_1(y-\beta_0)}}{\left(1+e^{-\beta_1(y-\beta_0)}\right)^2}. \tag{6.14}$$

In consequence, it can be concluded that Equation (6.10) is indeed the logistic function fitting the differential Equation (6.9). Knowing what the savings rate should look like, the next section will present the assumptions underlying the model developed in this doctoral thesis.

Figure 6.4 The Savings Function with $\beta_1 = 1$, $s_0 = 0.1$, and $\beta_0 = \ln 9$



6.2 The Assumptions

6.2.1 The Basics

Most of the assumptions of the basic Solow growth model remain the same. Nevertheless, for completeness of the description of the model, they will be repeated in detail and partly be extended to give a complete picture of the model's assumptions.

The model is assumed to be based on a closed economy without state activity. Output is denoted by Y. There are two factors of production, both being possessed by the households, namely capital K and labor L. The term A denotes Harrod-neutral, thus labor-augmenting technological progress. This means that technological progress appears to be coupled with labor, namely as effective labor AL. The households are paid wages and rent for their input factors. This income can then be used for consumption C or investment I in the capital stock K. The production function is only indirectly influenced by time t as the input factors change over time and this in turn influences output.

6.2.2 The Production Function

The production function is neoclassical of the form:

$$Y = F(K, AL). \tag{6.15}$$

It has the following features:

$$F(0,0) = 0 \tag{6.16}$$

$$\frac{\partial F}{\partial K} > 0 \; and \; \frac{\partial F}{\partial AL} > 0 \tag{6.17}$$

$$\frac{\partial^2 F}{\partial K^2} < 0 \; and \; \frac{\partial^2 F}{\partial (AL)^2} < 0 \tag{6.18}$$

$$F(\lambda K, \lambda AL) = \lambda F(K, AL) \tag{6.19}$$

$$\lim_{K\to\infty}\left(\frac{\partial F}{\partial K}\right) = \lim_{AL\to\infty}\left(\frac{\partial F}{\partial AL}\right) = 0 \tag{6.20}$$

$$\lim_{K\to 0}\left(\frac{\partial F}{\partial K}\right) = \lim_{AL\to 0}\left(\frac{\partial F}{\partial AL}\right) = \infty. \tag{6.21}$$

Equation (6.16) indicates that without capital input and effective labor input no output will be produced. The inequalities given by the Conditions (6.17) and (6.18) ensure that the production function exhibits diminishing marginal products of capital and of effective labor respectively. Equation (6.19) guarantees that the production function exhibits constant returns to scale. Furthermore, the Conditions (6.20) and (6.21) are also called the Inada conditions. They indicate that the marginal product of capital (or effective labor) goes to zero as K or AL approach infinity. On the contrary, if K or AL approach zero, the marginal product of capital (or effective labor) goes to infinity (Barro and Sala-i-Martin, 2004).

The most prominent production function having the above mentioned characteristics is the Cobb-Douglas production function (Barro and Sala-i-Martin, 2004). Then, the production function is given as follows:

$$\mathrm{Y} = \mathrm{K}^{\alpha}(\mathrm{AL})^{1-\alpha}. \quad (6.22)$$

It can also be defined in terms of effective labor, hence by dividing both sides of Equation (6.22) by AL. Then the production function looks as follows:[152]

$$y = k^{\alpha}. \quad (6.23)$$

6.2.3 The Dynamics of A, L, and K

As in the basic Solow growth model it is assumed that labor evolves according to Equation (6.24):

$$\dot{L}(t) = nL(t), \quad (6.24)$$

where n stands for the population growth rate. In this model, n is assumed to be exogenously given.[153] Technological progress evolves symmetrical to labor:

$$\dot{A}(t) = gA(t). \quad (6.25)$$

From the basic Solow growth model it is known that the capital stock evolves as follows:

$$\dot{K}(t) = s(Y)Y(t) - \delta K(t). \quad (6.26)$$

The parameter δ depicts depreciation. It is assumed to be constant and destroys the capital stock over time. Even though the savings rate, s, is assumed to be exogenous in the basic Solow growth model as presented in Chapter 5, it will be endogenized in this chapter. The assumptions concerning the savings rate will be dealt with in the following subsection. Overall, there are no specific features of n, g, and δ. However, in sum they have to be greater than zero:[154]

$$n + g + \delta > 0. \quad (6.27)$$

6.2.4 The Savings Rate

As stated before, the savings rate is assumed to be endogenous. In Section 6.1, the savings rate was elaborated on. It was argued that the savings rate first experiences exponential growth and then approaches a degree of saturation. This behavior is well described by a differential equation and the corresponding logistic

152 $y = \frac{Y}{AL}$, $k = \frac{K}{AL}$

153 There seem to be good reasons to assume it to be endogenously determined by income, too, as was argued in Chapter 5. Yet, it was decided to endogenize only one variable keeping the other exogenous.

154 This assumption is necessary to ensure the positive slope of the $(n + g + \delta)k$-line in Figure 6.6. Otherwise the steady state would not exist as indicated in Figure 6.6.

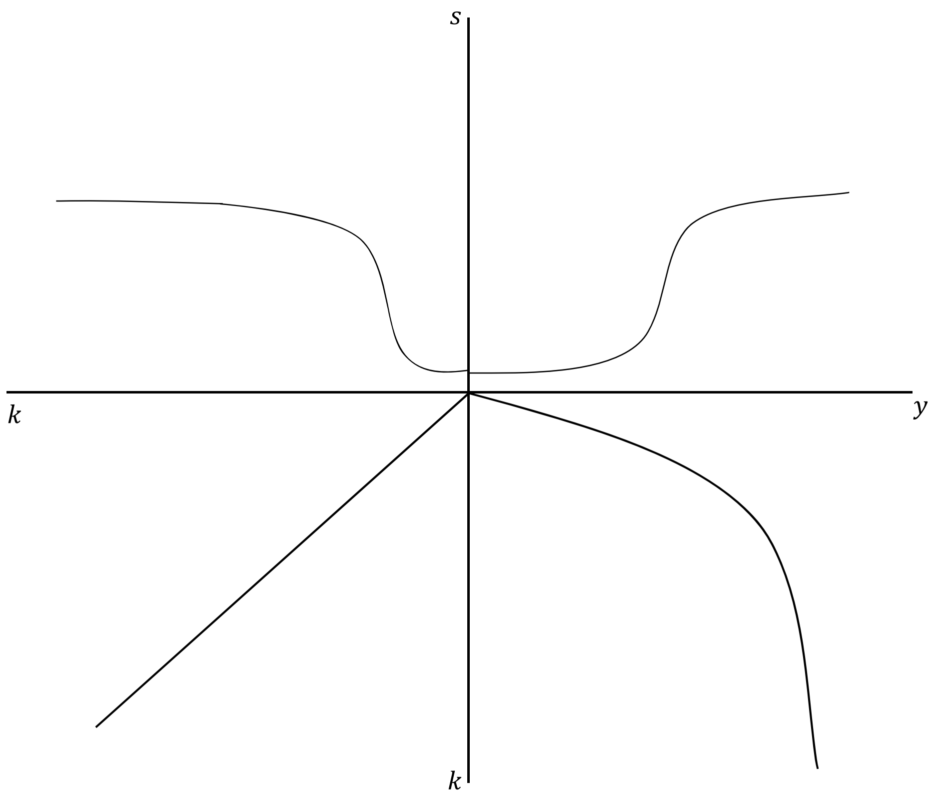
function. The former was given by Equation (6.4) and the latter by Equation (6.5). For simplicity, further assumptions were made so that the simplified versions of these equations were given by the Equations (6.9) and (6.10) respectively, which are repeated here:

$$\frac{ds}{dy} = \beta_1 s(1-s), \tag{6.28}$$

$$s(y) = \frac{1}{1+e^{-\beta_1(y-\beta_0)}}. \tag{6.29}$$

As explained before, the degree of saturation is assumed to be equal to one.[155] Knowing the basic assumptions, the model can be formulated. This will be done in the following section.

Figure 6.5 The Determination of the s(y)f(k)-Curve[156]



[155] This implies that all income would be saved, which is not realistic from an economic point of view. A more sensible assumption would be a value of, for example, 0.6, as was argued above. However, in order to facilitate the calculations, it will be kept at one.

[156] Such a 4-quadrants graph helps to find a function $s(k)$ if one only knows what $s(y)$ and $y(k)$ look like. $s(y)$ is the logistic curve. $y(k)$ is known from the Solow growth model. Starting at a point on $s(y)$, one can walk through the quadrants and then mark the respective points in the s, k-quadrant. By connecting the points, the $s(k)$ curve can be found as shown in Figure 6.5.

6.3 The Model

In Section 6.2, the endogenous savings rate was determined. It will now be inserted in the neoclassical growth model to form the basis of the twin peaks model. The purpose of this section is to formulate the basic equations which determine the growth model. Several of these equations are already known from Chapter 5. However, for a complete description of the model, they will be repeated here.

The production function per efficient unit of labor is given as follows:

$$y = f(k) = k^{\alpha}, \tag{6.30}$$

where $y = \frac{Y}{AL}$, hence it is equal to income Y per efficient unit of labor AL, $k = \frac{K}{AL}$, indicating the capital stock K per efficient unit of labor AL, and α indicates the capital share in income.[157]

Labor is assumed to grow according to Equation (6.31). $L(0)$ represents the initial value of the labor force. Labor increases exponentially over time, so that the growth rate of the labor force is n.[158]

$$L(t) = e^{nt}L(0) \tag{6.31}$$

$$\Leftrightarrow g_L = n. \tag{6.32}$$

Accordingly, technological progress A grows at rate g as given by Equation (6.33). Again, $A(0)$ stands for the initial value of technology; it also grows exponentially.

$$A(t) = e^{gt}A(0) \tag{6.33}$$

$$\Leftrightarrow g_A = g \tag{6.34}$$

Capital accumulation is given by Equation (6.35).[159] It shows the development of the capital intensity over time.

$$\dot{k} = s(y)f(k) - (n + g + \delta)k \tag{6.35}$$

Here, $s(y)$ stands for the endogenous savings rate (in contrast to the exogenous one in Chapter 3), n is the population growth rate as described above, g is the rate of technological progress, and δ is the rate of depreciation. A dot on the variables indicates its first derivative with respect to time. It should be kept in mind that as in Chapter 3, s is still assumed to be between zero and one for the reasons outlined above.

$$s \in [0,1] \tag{6.36}$$

However, a savings rate of one is rather unrealistic because this implies that no income is spent on consumption, even on subsistence consumption. From a purely

[157] Usually, the capital share is assumed to be equal to about one third, though this is not of interest at this point of discussion.

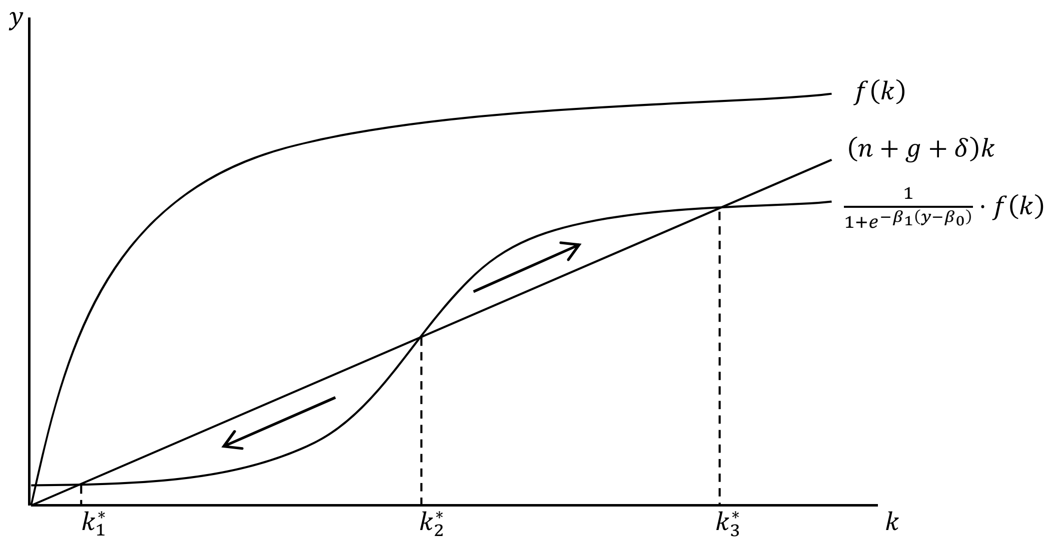
[158] For simplification, the labor force is assumed to be equal to the total population so that n denotes the population growth rate.

[159] From now on, the terms “capital stock“ and “income“ refer to the capital stock per efficient unit of labor and income per efficient unit of labor respectively.

mathematical point of view this is possible, but due to the construction of $s(y)$ implying $\lim_{y \to \infty} s(y) = 1$, a value of one will never be reached exactly even in the long run – with rising income it will, though, approach its upper limit.

Before the determination of the steady state is considered, the model will be graphed. The form of the savings function was already given in Figure 6.4 for specific values of the parameters. Applying the logistic function of the savings rate to the neoclassical production function with constant returns to scale, the $s(y)y$-curve can be determined as shown in Figure 6.5. On this basis, the growth model can then be plotted. It is given in Figure 6.6.[160]

Figure 6.6 The Growth Model



6.4 The Steady State Determination

6.4.1 Basic Considerations

In the previous section the growth model was shown. The purpose of this section is to solve the model and to prove that the neoclassical growth model including an endogenous savings rate is able to yield three steady states, two of which are stable, just as shown in Figure 6.6. The condition for a steady state is given as follows:

[160] It should be noted that the $s(y)f(k)$-curve does not have to go through the origin but might also cut the $(n+g+\delta)k$-line above, depending on the exact savings function, hence on the parameters. In this example, the origin is no intersection point of the $(n+g+\delta)k$-line and the $s(y)f(k)$-curve.

$$\dot{k} = 0. \tag{6.37}$$

Using Equation (6.35), inserting Equation (6.29), and replacing $f(k)$ by y yields:

$$s(y)f(k) - (n + g + \delta)k = 0 \tag{6.38}$$

$$\Rightarrow \frac{1}{1+e^{-\beta_1(y-\beta_0)}}y - (n + g + \delta)k = 0. \tag{6.39}$$

This equation has several parameters, namely β_0, β_1, n, g, and δ. In addition, there are two variables, y and k. For this reason, the equation is not unambiguously solvable. However, Equation (6.30) gives the interrelation between y and k. Thus, it has to be decided, which variable shall be replaced for further calculations, y or k. In order to facilitate the calculations, k will be replaced by an expression of y. As y appears in the exponent, it is easier to calculate with y rather than with k^{α}. Rearranging Equation (6.30) yields Equation (6.40) which is then inserted into Equation (6.39).

$$k = y^{\frac{1}{\alpha}} \tag{6.40}$$

$$\Rightarrow \frac{1}{1+e^{-\beta_1(y-\beta_0)}}y - (n + g + \delta)y^{\frac{1}{\alpha}} = 0 \tag{6.41}$$

An additional simplifying assumption will be formulated, which is given as follows:

$$n + g + \delta = \frac{1}{2\beta_0}. \tag{6.42}$$

As the parameter β_0 was already described by Equation (6.7), this restriction implies:

$$\beta_0 = \frac{1}{2(n+g+\delta)} = \frac{1}{\beta_1} ln\left(\frac{1-s_0}{s_0}\right). \tag{6.43}$$

It demands several parameter restrictions, some of which are already known from before:

$$n + g + \delta > 0, \tag{6.44}$$

$\beta_1 > 0$, and (6.45)

$$s_0 \in]0,1]. \tag{6.46}$$

In the course of the calculations undertaken in this section, further assumptions need to be made. They will be discussed when they apply. Inserting Equation (6.40) into Equation (6.39) and rearranging it with respect to y then yields:

$$\frac{1}{1+e^{-\beta_1(y-\beta_0)}} = \frac{1}{2\beta_0}y^{\frac{1-\alpha}{\alpha}}. \tag{6.47}$$

Such a nonlinear equation is not unambiguously solvable. It is impossible to explicitly determine the general three intersection points shown by Figure 6.5 – neither by hand, nor by any computer package. In order to be able to find the intersection points (or, if Equation (6.47) is reformulated, the roots), an approximation method needs to be applied. For this to be possible, the right-hand side should be a straight line for simplicity reasons. In order to reach this, an assumption concerning the exponent of y on the right hand side of Equation (6.42) needs to be made, namely that $\alpha = 0.5$.[161]

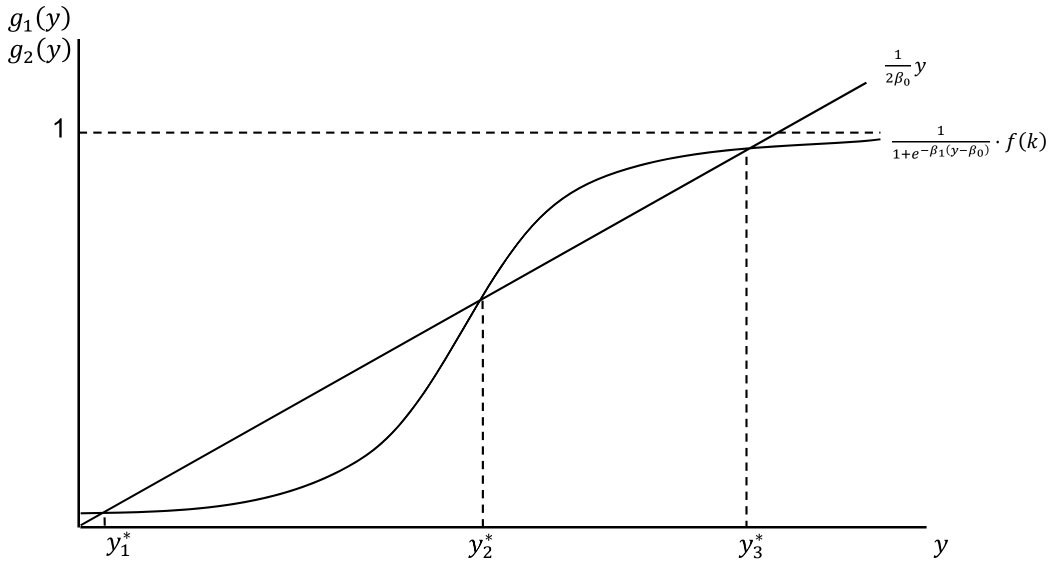
$$\frac{1}{1+e^{-\beta_1(y-\beta_0)}} = \frac{1}{2\beta_0} y \tag{6.48}$$

The intersection points are, however, still not determinable because of the structure of the equations. In contrast, it can be proved that there are three intersection points. This will be done now. For this purpose, Equation (6.48) will be split up into two separate equations in the following, namely $g_1(y)$ and $g_2(y)$.[162]

$$g_1(y) = \frac{1}{1+e^{-\beta_1(y-\beta_0)}} \tag{6.49}$$

$$g_2(y) = \frac{1}{2\beta_0} y \tag{6.50}$$

Figure 6.7 Determination of the Steady States



Equation (6.49) is already well-known, because it is the function determining the savings rate which was given in Equation (6.28) above. The behavior of this equation can be described as follows:

$$\lim_{y\to\infty} g_1(y) = 1 \text{ and} \tag{6.51}$$

[161] Even though it is widely agreed that α should be assumed to be equal to 0.3, it is assumed to be 0.5 here, which is not too implausible, though. This assumption facilitates the calculations.

[162] The functions $g_1(y)$ and $g_2(y)$ represent, from an economic perspective, the savings curve and the investment requirement line. The intersections, hence the steady states, represent points in which savings are equal to the required investment.

$$\lim_{y \to -\infty} g_1(y) = 0.^{163} \quad (6.52)$$

The first function, $g_1(y)$, is hence an S-curve, where β_1 determines the slope of $g_1(y)$ at the point $y = \beta_0$, and β_0 determines the position of this function with respect to the horizontal axis in Figure 6.7.

From mathematical perspective, the intersection point y_2 is unstable, it is called a "rejecting point of intersection", while y_1 and y_3 are "attracting points of intersection". The reason is that at point y_2, the slope of the $g_1(y)$-curve is greater than that of the $g_2(y)$-line. This corresponds to the economic basis of argumentation. To the left of y_2, savings are lower than the investment needed, so that the economy is pushed back to y_1. However, starting slightly above y_2, savings are higher, so that the country is pushed towards y_3. This was already shown for the basic Solow growth model in Chapter 3, however, there in a $k - f(k)$-diagram.

Based on the previous findings, the next step is to determine y_1, y_2, and y_3. The problem is that, as already stated above, it is not possible to explicitly solve the system of the Equations (6.49) and (6.50). Yet, the solutions can be approximated. Using the assumptions made before it is, however, possible to prove that these three points of intersection exist. This is the task of the following subsection.

6.4.2 Proof of the Intersection Points

Looking at the two equations $g_1(y)$, hence the savings function, and $g_2(y)$, one intersection point can be determined easily, namely the one in the middle.[164] Inserting $y = \beta_0$ into $g_2(y)$, the following point can be found:

$$g_2(\beta_0) = \frac{1}{2\beta_0}\beta_0 = \frac{1}{2}. \quad (6.53)$$

The same is done for $g_1(y)$:

$$g_1(\beta_0) = \frac{1}{1+e^{-\beta_1(\beta_0-\beta_0)}} = \frac{1}{1+e^{-\beta_1 \cdot 0}} = \frac{1}{1+1} = \frac{1}{2}. \quad (6.54)$$

Thus, it is indeed proved that under the assumptions mentioned above, the point $\binom{\beta_0}{1/2} \in \mathbb{R}^2$ is a point of intersection. It is also clear why this is the middle intersection point. The vertical axis covers values between zero and one. The point y_3 can be expected to be at the upper end of the scale and y_1 rather at the lower end; y_2 is exactly in the middle.

Knowing one intersection point, it still has to be examined, under which assumptions there will be two further points of intersection. Looking at Figure 6.7

[163] From an economic point of view it is implausible to let y go to $-\infty$; however, this expression is mathematically needed to be able to prove that there are indeed three intersection points.

[164] The reason is that according to Figure 6.7, the slope of $g_1(y)$ is greater than that of $g_2(y)$. It will be shown later on that this is indeed the case for this intersection point. For this reason, this is definitely the middle intersection point, as for the other two, the slope of $g_2(y)$ is greater than that of $g_1(y)$.

shows that the slope of $g_1(y)$ is steeper at point $y = \beta_0$ than the slope of $g_2(y)$. If it was the other way around, then there would not be more intersection points than the one already found. Consequently, the first condition for more than one steady state is as follows:

$$g_1'(\beta_0) > g_2'(\beta_0). \tag{6.55}$$

The second condition is given by:

$$\lim_{y\to-\infty} g_1(y) = 0. \tag{6.56}$$

Even though only positive values of y are of economic interest, this assumption is necessary to prove that $g_1(y)$ does not go through the origin as $g_2(y)$ does. Finally, the third condition is as follows:

$$\lim_{y\to\infty} g_1(y) = 1. \tag{6.57}$$

This assumption is again necessary to ensure that $g_1(y)$ does not reach an output value of one. The second and the third condition are indeed fulfilled, as they were already fundamental assumptions of the savings function mentioned above. Thus, only the first condition remains to be proved. For this purpose, the derivatives of the two functions, namely $g_1'(y)$ and $g_2'(y)$, have to be calculated. Then, $y = \beta_0$ will be inserted into these derivatives to check whether Equation (6.55) indeed holds. To start with, the derivative of $g_1(y)$ will be calculated first.

$$g_1'(y) = \beta_1 \frac{e^{-\beta_1(y-\beta_0)}}{\left(1+e^{-\beta_1(y-\beta_0)}\right)^2} \tag{6.58}$$

Now, $y = \beta_0$ will be inserted into Equation (6.58):

$$\Rightarrow g_1'(\beta_0) = \beta_1 \frac{e^{-\beta_1(\beta_0-\beta_0)}}{\left(1+e^{-\beta_1(\beta_0-\beta_0)}\right)^2} \tag{6.59}$$

$$\Leftrightarrow g_1'(\beta_0) = \frac{\beta_1}{4}. \tag{6.60}$$

The same procedure has to be repeated for $g_2(y)$:

$$g_2'(y) = \frac{1}{2\beta_0}, \tag{6.61}$$

which is independent of y. These findings can now be used to find out under which assumptions the first condition for three steady states is fulfilled:

$$\frac{\beta_1}{4} > \frac{1}{2\beta_0} \tag{6.62}$$

$$\beta_0\beta_1 > 2. \tag{6.63}$$

[165]

[165] In order to examine whether the results are economically plausible, this result may be used to determine a condition for s_0 according to Equation (6.7): $\beta_1 \frac{1}{\beta_1} ln \frac{1-s_0}{s_0} > 2$. It can easily be shown that $s_0 < 0.12$. An initial savings rate of 0.12 is within the interval mentioned above and it is a realistic value from an economic point of view. Furthermore, β_0 was also described by Equation (6.43), so that a condition for the term $(n + g + \delta)$ can be derived: $\beta_1 \frac{1}{2(n+g+\delta)} > 2$. From this the following condition can be found: $n + g + \delta < \frac{\beta_1}{4}$. Together with Equation (6.27) this implies the following restriction: $(n + g + \delta) \in \left]0; \frac{\beta_1}{4}\right[$.

Now it will be examined whether Equation (6.55) indeed holds. For this purpose, the limit of each of the two derivatives when y goes to infinity will be determined:

$$\lim_{\beta_0 \to \infty} g_1'(\beta_0) = \lim_{y \to \infty} \frac{\beta_1}{4} = \frac{\beta_1}{4} \tag{6.64}$$

$$\lim_{\beta_0 \to \infty} g_2'(\beta_0) = \lim_{\beta_0 \to \infty} \frac{1}{2\beta_0} = 0 \tag{6.65}$$

Hence, it can be concluded that Equation (6.55) indeed holds and there are two further points of intersection of the two functions $g_1(y)$ and $g_2(y)$.

Knowing that there will indeed be three points of intersection, the two remaining intersection points, y_1 and y_3 will be determined in the following. As the points cannot explicitly be determined, the points will be approximated in the next section. Here, however, the existence of these points shall be proved first.

It is well known that $g_2(y)$ is a straight line going through the origin:

$$g_2(0) = \frac{1}{2\beta_0} \cdot 0 = 0. \tag{6.66}$$

Additionally, $g_1(y)$ does not go through the origin as shown before. As the slope of $g_1(y)$ is larger than that of $g_2(y)$ at the point $y = \beta_0$, and as $\lim_{y \to -\infty} g_1(y) = 0$, it is clear that $g_1(y)$ and $g_2(y)$ have to intersect once more between $y = 0$ and $y = \beta_0$. However, it is not known where exactly this will be. Before this problem is dealt with, it will first be shown that this point of intersection has to exist. This means that at $y = 0$ the following inequality holds:

$$g_1(0) > g_2(0). \tag{6.67}$$

Inserting $y = 0$ into $g_1(y)$ and $g_2(y)$ yields:

$$g_1(0) = \frac{1}{1+e^{\beta_1 \beta_0}} > 0, \tag{6.68}$$

as $e^{\beta_1 \beta_0} > 0$ for all β_0 and β_1. Hence, Equation (6.67) holds so that both functions definitely have an intersection point somewhere between 0 and β_0; consequently $y_1 \in]0; \beta_0[$.[166]

Concerning the area to the right of β_0, there are further assumptions. From Equation (6.57), it is known that $\lim_{y \to \infty} g_1(y) = 1$. Additionally, it can be derived for which y the function $g_2(y)$ reaches the output value one, so that indeed there has to be a further point of intersection to the right of β_0.

$$g_2(y) = \frac{1}{2\beta_0} y = 1 \tag{6.69}$$

$$\Leftrightarrow y = 2\beta_0 \tag{6.70}$$

Now it has to be determined whether $g_1(y)$ is really smaller than one at $y = 2\beta_0$.

$$g_1(2\beta_0) = \frac{1}{1+e^{-\beta_1(2\beta_0 - \beta_0)}} = \frac{1}{1+e^{-\beta_1 \beta_0}} < 1, \tag{6.71}$$

[166] The exact points must not be hit. Therefore, the interval is an open interval.

as $e^{-\beta_1\beta_0} > 0$, so that the denominator is greater than one for all β_0 and β_1. For this reason the following holds:

$$g_1(2\beta_0) < g_2(2\beta_0). \tag{6.72}$$

This implies that there has to be a point of intersection between β_0 and $2\beta_0$; in consequence, $y_3 \in]\beta_0; 2\beta_0[$.[167] The question which remains is once again, where exactly this point can be found.

This will be determined by use of an approximation method. In this way, approximated values of y can be determined which might then be used to calculate the respective values of k so that the points can be marked in the growth diagram in Figure 6.6. The theoretical background of the approximation method will be given in the following subsection.

6.4.3 Approximation Methods for the Intersection Points

The problem of calculating the intersection point of two functions can be reformulated in such a way that root determination remains. For this purpose, the two functions are subtracted from each other. If the difference is equal to zero, then this point is a root of that function. In trivial cases, the corresponding value of this root can easily be calculated. However, often it is not possible to determine the roots explicitly. In such cases, they can only be approximated. Such approximation methods are part of numerical mathematics. This section deals with the determination of roots with one variable only. This task can be theoretically formulated as follows:

$$g(x) = 0. \tag{6.73}$$

The function $g(x)$ is not explicitly solvable for x, so that an approximation method needs to be applied (Preuß and Wenisch, 2001).[168] The approximation methods can be distinguished into the bisection method and the iteration method. They will be discussed in more detail in the following subsections.

6.4.3.1 The Bisection Method

The bisection method is one of the simplest methods to approximate roots. Its advantage is that it only uses the values of $g(x)$, though it is not a very effective method. The bisection method is based on the following theorem:

[167] The exact points must not be hit. Therefore, the interval is an open interval.

[168] Whatever method is chosen, it has to be kept in mind that there might also be several solutions so that it needs to be defined which of these solutions will be calculated. This means that an initial approximation is needed; this is also called the initial value problem (Preuß and Wenisch, 2001).

"Let $[g(x)]$ be a continuous function in the interval $[x_u, x_o]$. If *[*$sign(g(x_u)) \neq sign(g(x_o))$*]*[169], that means if the product of $[g(x_u)]$ and $[g(x_o)]$ is negative, then $[g]$ has at least one zero in the interval $[x_u, x_o]$."
(Preuß and Wenisch, 2001, p. 35).

The method then consists of three steps. First, a so-called test point needs to be found as follows:

$$x = \tfrac{1}{2}(x_u^k + x_o^k) \tag{6.74}$$

For this x, $g(x)$ has to be calculated. In the next step, the following substitutions need to be made:

$$x_u^{k+1} = x, x_o^{k+1} = x_o^k, \tag{6.75}$$

if $sign(g(x)) \neq sign\left(g(x_o^k)\right)$, or alternatively:

$$x_u^{k+1} = x_u^k, x_o^{k+1} = x, \tag{6.76}$$

if $sign(g(x)) \neq sign\left(g(x_u^k)\right)$.[170] Then, the third step is that the interval $I_{k+1} = [x_u^{k+1}, x_o^{k+1}]$ fulfils the role of $I_k = [x_u^k; x_o^k]$ with $sign\left(g(x_u^k)\right) \neq sign\left(g(x_o^k)\right)$, and the whole process starts all over again (Preuß and Wenisch, 2001).

This numerical method of approximation is robust if it is appropriately formulated, but it is a slow method. It is best applied if the solution to be found does not have to be very precise. Otherwise, there are more effective methods which might be used instead (Preuß and Wenisch, 2001). They will be discussed in the next subsection.[171]

6.4.3.2 Iteration Methods

An alternative to the rather ineffective bisection method is given by the class of iteration methods, which describes a stepwise approximation. The iteration methods consist of the root problem as given by Equation (6.74) and an equivalent fixed-point problem of the following form:

$$x = g(x) \; or \; x = g(x, \dots, x), \tag{6.77}$$

where in the latter case x is included $(k+1)$ times. „The term equivalent indicates that the solutions to the root problem and the fixed-point problem are the same" (Preuß and Wenisch, 2001, p. 40). The iteration rule is then given by the Equations (6.78) and (6.79):

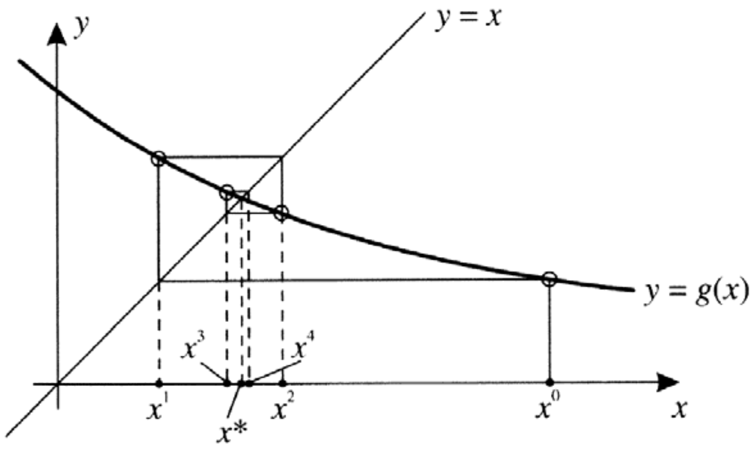
[169] The function sign indicates a function determining the sign of the function at a specific point; if it is negative, the output of this function is -1; if it is zero, the output is equal to 0; and finally, if it is positive, then the output is 1.
[170] If, by hazard, the root is hit, hence if $g(x) = 0$, the process stops immediately, of course.
[171] For more details concerning the bisection method, the interested reader is referred to Preuß and Wenisch, 2001.

$$x_0 \approx x^* \tag{6.78}$$

$$x_{i+1} = g(x_i), (i = 0, 1, \dots). \tag{6.79}$$

According to this process, a sequence of new approximations of x_i will be calculated beginning with the first approximation x_0. "If this sequence with x_i converges with $\lim_{i\to\infty} x_i = x^*$, then $x^* = g(x^*)$ holds, that means x^* is a solution to the equivalent fixed-point problem and so a solution of $g(x^*)$. [… A]n iteration rule is suitable for a root problem only if the sequence of approximations converges" (Preuß and Wenisch, 2001, p. 41). The iteration process is shown graphically in Figure 6.8.

Figure 6.8 The Iteration Process



Source: Preuß and Wenisch, 2001, p. 41

What remains open in this iteration method is how to find the fixed-point problem, under which circumstances there will be convergence in the iteration rule and when this will happen in a fast way, when should an iteration process be stopped, and what the approximation error will be (Preuß and Wenisch, 2001).

The iteration methods can be subdivided into ordinary iteration methods, the Newton method, the secant method, the Steffensen method, and the Pegasus method, just to name the most important ones. In this doctoral thesis, only the Newton method will be discussed in detail as it will be applied in the next section. For more details concerning the remaining methods the interested reader is referred to Preuß and Wenisch, 2001.

The Newton Method

The Newton method is also known as the tangent method. It is usually an efficient method to approximate the roots of a function (Sydsæter and Hammond, 2008) and is applied if "the algebraic calculation of the roots by explicitly solving an equation is not possible or too time-consuming" (Senger, 2007, p. 186). The method works as follows: there is already an initial value close to the real root, $\boldsymbol{x_0}$. This approximation can be improved by constructing a tangent to the function at $\boldsymbol{x_0}$. The new intersection point with the horizontal axis, $\boldsymbol{x_1}$, is again used to construct a new tangent to the function. This process will be repeated several times while $\boldsymbol{x}$ generally converges quickly towards the root (Sydsæter and Hammond, 2008). From basic mathematics it is known that the slope of a tangent to a function is equal to the derivative of this function at the respective point:

$$g'(x_0) = \frac{y - g(x_0)}{x - x_0}. \tag{6.80}$$

For this reason, the tangent is given by:

$$y = g(x_0) + g'(x_0)(x - x_0). \tag{6.81}$$

For the intersection point of the tangent with the horizontal axis, y is set to zero:

$$y = 0 = g(x_0) + g'(x_0)(x_1 - x_0). \tag{6.82}$$

This equation can be reformulated to yield an expression for x_1:

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}.\text{[172]} \tag{6.83}$$

The same is repeated for x_2:

$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}, \tag{6.84}$$

so that the Newton method for the approximation of roots is given by the following iteration rule:

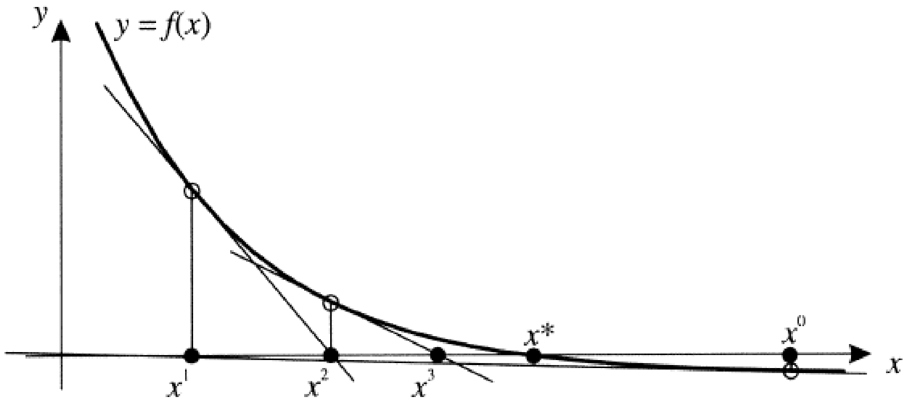
$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \tag{6.85}$$

where $n = 0, 1, 2, \dots$ and $g'(x_n) \neq 0$ (Senger, 2007). The graphical representation of the Newton method can be found in Figure 6.9.

The Newton method might also fail, for example if $g'(x_n) = 0$ because then the iteration rule cannot be used. In addition, "usually, Newton's method fails only if the absolute value of [$g'(x_n)$] becomes too small, for some n" (Sydsæter and Hammond, 2008, p. 247). The remaining question is when to abort the process. The number of approximations depends on the precision needed. Generally speaking, the process is aborted when the result does not change anymore or, if a precision up to the second decimal is needed for example, when the result does not change anymore in the second decimal (Senger, 2007).

[172] A convergence criterion is that the derivative of g at the point to be determined must not be zero, hence $g'(x^*) \neq 0$.

Figure 6.9 The Newton Method



Source: Preuß and Wenisch, 2001, p. 45

In this section, an overview of the approximation methods for the intersection points was given. The Newton method is a quite effective method for this purpose, especially when dealing with differential equations. Consequently, it will be applied in the next section.

6.4.4 Application of the Newton Method

The Newton method may be applied to the problem of this doctoral thesis. Yet, in order to facilitate the calculations, a number of assumptions for the parameters have to be made. Furthermore, based on these assumptions, the parameters have to be assigned values to allow for calculations. Though, when using values for the parameters, round-off errors might result. Nevertheless, this will be done here as there is no better alternative. In order to decrease possible errors and to keep economic plausibility, the restrictions of previous sections will be used. Table 6.2 gives an overview of the restrictions determined before and used for the parameter estimations. Examples for parameter combinations are given in Table 6.3.[173] Values in italics are underlying assumptions, the others are determined values. Example 6 is used for the following calculations.

To begin with, the function $g(y) = g_1(y) - g_2(y)$ to be used for the Newton method is given as follows:

$$g(y) = \frac{1}{1+e^{-0.65917(y-3.33333)}} - \frac{1}{2 \cdot 3.33333} y \tag{6.86}$$

$$\Leftrightarrow g(y) = \frac{1}{1+e^{-0.65917(y-3.33333)}} - 0.15y. \tag{6.87}$$

[173] They are given rounded up to the fifth decimal.

Table 6.2 Parameter Restrictions

Parameter	Restrictions	Assumed value
α		0.5
β_0	$\beta_0 = \frac{1}{\beta_1} ln \frac{1-s_0}{s_0}$	
β_1	$\beta_1 > \frac{2}{\beta_0}$	
$(n+g+\delta)$	$(n+g+\delta) = \frac{1}{2\beta_0}$ $0 < (n+g+\delta) < \frac{\beta_1}{4}$	
s_0	$0 \le s_0 < \frac{1}{e^2+1}$	

Table 6.3 Parameter Estimations

Example	$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_0\beta_1}$	$(\boldsymbol{n+g+\delta})$[174]	upper limit $(\boldsymbol{n+g+\delta})$	$\boldsymbol{s_0}$
1	3.33333	0.88333	2.94444	0.15	0.22083	0.050
2	3.33333	0.82546	2.75154	0.15	0.20637	0.060
3	3.33333	0.77601	2.58669	0.15	0.19400	0.070
4	3.33333	0.73270	2.44235	0.15	0.18318	0.080
5	3.33333	0.69409	2.31363	0.15	0.17352	0.090
6	3.33333	0.65917	2.19722	0.15	0.16479	0.100
7	3.33333	0.62722	2.09074	0.15	0.15681	0.110
8	3.33333	0.61220	2.04066	0.15	0.15305	0.115
9	3.33333	0.60490	2.01632	0.15	0.15122	0.118
10	3.33333	0.60058	2.00193	0.15	0.15015	0.119
11	3.12500	0.70311	2.19722	0.16	0.17578	0.110
12	2.94118	0.74706	2.19722	0.17	0.18676	0.110
13	2.77778	0.79100	2.19722	0.18	0.19775	0.110
14	2.63158	0.83495	2.19722	0.19	0.20874	0.110
15	2.50000	0.87889	2.19722	0.20	0.21972	0.110

[174] The rate of depreciation is widely agreed to be equal to 0.10 on average. The rate of population growth is assumed to be equal to 0.02 and the one of technological progress is set at 0.03. Then, the term $(n+g+\delta)$ will be equal to 0.15. In the last five examples of Table 6.3, this value is slightly increased to see what changes in response to changes in this term.

This function can also be graphed. Before, however, the two parts will be graphed individually. Figure 6.10 shows $g_1(y)$ for the respective parameter values and Figure 6.11 shows $g_2(y)$ accordingly. Figure 6.12 is then the graphical representation of their difference, hence of Equation (6.88). It shows that there are indeed three roots. For the underlying parameter values, the roots can approximately be read off from the graph. Hence, good approximations for the initial values of y can be found by graphical analysis. They will then be used to determine the roots by use of the Newton method.

The graphical analysis proved that there are indeed three roots which will be determined by applying the Newton method in the following. The function $g(y)$ is known, while $g'(y)$ is still to be determined:

$$g'(y) = 0.65917\frac{e^{-0.65917(y-3.33333)}}{\left(1+e^{-0.65917(y-3.33333)}\right)} - 0.15 \tag{6.88}$$

Figure 6.10 Function $g_1(y)$ for $\beta_0 = 3.33333$, $\beta_1 = 0.65917$, $s_0 = 0.10$, and $(n + g + \delta) = 0.15$

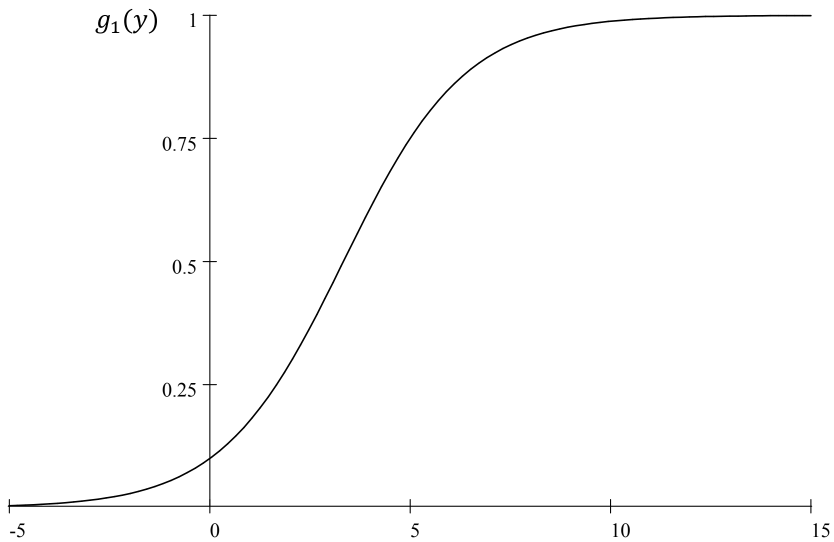


Figure 6.11 Function $g_2(y)$ for $\beta_0 = 3.33333$, $\beta_1 = 0.65917$, $s_0 = 0.10$, and $(n + g + \delta) = 0.15$

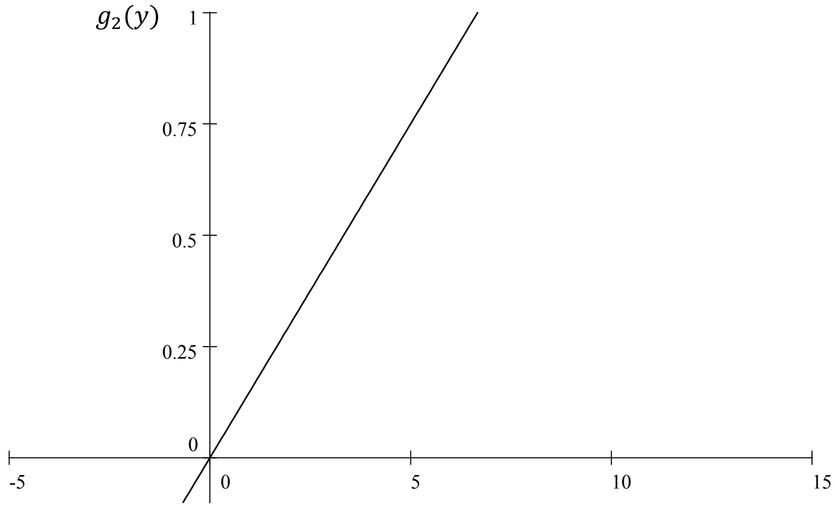
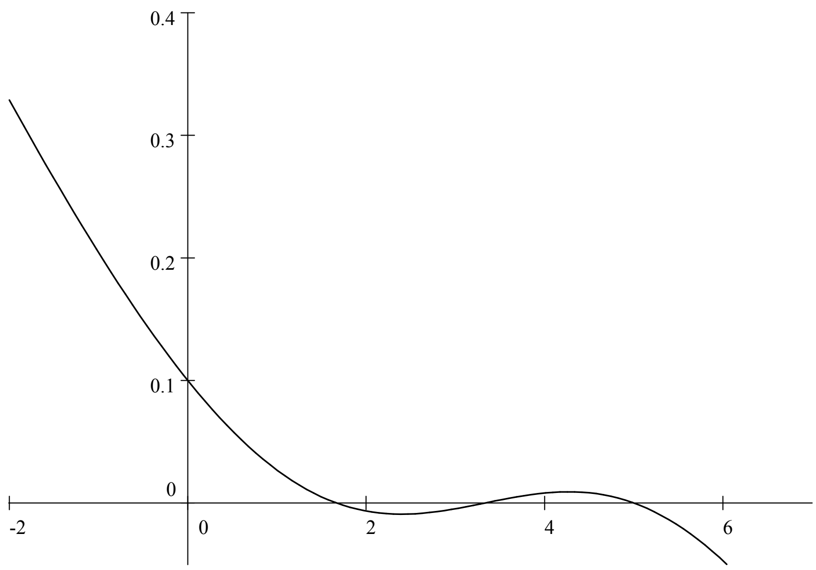


Figure 6.12 Root Determination for $\beta_0 = 3.33333$, $\beta_1 = 0.65917$, $s_0 = 0.10$, and $(n + g + \delta) = 0.15$



Finally, an initial value of y, namely y_0, needs to be fixed for the application of the Newton method. For this purpose, one of the three intersection points needs to be chosen first. The middle point was already determined before. It is known that it can be found at $y_2^* = \beta_0 = 3.33333$. Knowing this, the corresponding capital stock can be determined according to Equation (6.40):

$$k_2^* = (3.33333)^2 = 11.11111. \quad (6.89)$$

As this intersection point is known, two points remain to be determined. The Newton method will first be applied to the left point of intersection. From Figure 6.12, $y_0 = 2$ can be read off as an initial value. Then y_1 can be approximated using the Newton method:

$$y_1 = 2 - \frac{g(2)}{g'(2)} = 2 - \frac{-0.00659}{-0.01334} \approx 1.505898. \quad (6.90)$$

From Table 6.4, it becomes apparent that the Newton method converges quickly towards the root. This is due to the fact that the initial value of y was chosen sufficiently well. Already y_{IV} gives the root at a value of $y_1^* = 1.66667$.

Table 6.4 The Newton Process for $\mathbf{y_1^*}$

i	Value of $\mathbf{y_i}$	$\mathbf{g(y_t)}$	$\mathbf{g'(y_t)}$
0	2	-0.006592	-0.013342
I	1.505898	0.004776	-0.033027
II	1.650501	0.000432	-0.027065
III	1.666470	0.000005	-0.026414
IV	1.666667	0.000000	-0.026406
V	1.666667	0.000000	-0.026406
VI	1.666667	0.000000	-0.026406

The third root can be determined in the same way. Now, y_0 is set at 4.5 after inspection of Figure 6.12. The iteration rule then yields:

$$y_I = 4.5 - \frac{g(4.5)}{g'(4.5)} = 4.5 - \frac{0.008311}{-0.007358} \approx 5.629475. \quad (6.91)$$

Table 6.5 again gives the whole Newton process. It shows that the third root is to be found at y_V with $y_3^* = 5$. These two additional zeros can be used to calculate the respective capital stocks, again according to Equation (6.40):

$$k_1^* = (1.66667)^2 = 2.77778, \quad (6.92)$$

$$k_3^* = (5)^2 = 25. \quad (6.93)$$

Table 6.5 The Newton Process for $\mathbf{y_3^*}$

i	Value of $\mathbf{y_i}$	$\mathbf{g(y_t)}$	$\mathbf{g'(y_t)}$
0	4.5	0.008311	-0.007358
I	5.629475	-0.024836	-0.052532
II	5.156702	-0.004642	-0.032858
III	5.015433	-0.000412	-0.027036
IV	5.000180	-0.000005	-0.026413
V	5.000000	0.000000	-0.026406
VI	5.000000	0.000000	-0.026406

Consequently, the three intersection points of $g_1(y)$ and $g_2(y)$ are given by $y_1^* = 1.66667$, $y_2^* = 3.33333$, and $y_3^* = 5$. The corresponding values of k are $k_1^* = 2.77778$, $k_2^* = 11.11111$, and $k_3^* = 25$ respectively.

6.4.5 Inserting the Steady States in the Original Growth Model

The values of y^* and k^* will now be inserted into the equation determining the steady states in the basic model determined in this doctorate, namely in Equation (6.35), while replacing $s(y)$ by Equation (6.29):

$$\dot{k} = \frac{1}{1+e^{-\beta_1(y-\beta_0)}} y - (n + g + \delta)k. \tag{6.94}$$

The following parameter values are used in this equation: for $\beta_0 = 3.33333$ and for $\beta_1 = 0.65917$. Subsequently, $\dot{k}$ will be determined for each individual intersection point which was calculated in the previous section. If inserting the corresponding values for y and k indeed yields $\dot{k} = 0$, then this is a steady state.

6.4.5.1 First candidate: $y_1^* = 1.66667$ and $k_1^* = 2.77778$

$$\dot{k} = \frac{1}{1+e^{-0.65917(y_1^*-3.33333)}} y_1^* - 0.15k_1^* \tag{6.95}$$

$$\dot{k} = \frac{1}{1+e^{-0.65917(y_1^*-3.33333)}} 1.66667 - 0.15 \cdot 2.77778 \tag{6.96}$$

$$\dot{k} = -0.0000012995 \approx 0 \tag{6.97}$$

[175]

Considering that rounded parameters were used and hence also the variables y and k are rounded, Equation (6.101) indeed proofs that $y_1^* = 1.66667$ and

[175] The difference between the determined $\dot{k}$ and zero is almost certainly due to round-off errors.

$k_1^* = 2.77778$ represent the first steady state. As formerly argued, this steady state is stable.

6.4.5.2 *Second candidate:* $y_2^* = 3.33333$ *and* $k_2^* = 11.11111$

The same is now repeated for the second steady state candidate, namely $y_2^* = 3.33333$ and $k_2^* = 11.11111$:

$$\dot{k} = \frac{1}{1+e^{-0.65917(y_1^*-3.33333)}} y_1^* - 0.15k_1^* \tag{6.98}$$

$$\dot{k} = \frac{1}{1+e^{-0.65917(3.33333-3.33333)}} 3.33333 - 0.15 \cdot 11.11111 \tag{6.99}$$

$$\dot{k} = 0. \tag{6.100}$$

Consequently, this point is indeed the second steady state of the growth model developed in this chapter for the given parameter values indicated above. However, as seen in Figure 6.6, this second steady state is not stable. Instead, it is a rejecting equilibrium. Starting with a capital stock of less than 11.11111, a country is pushed back towards the first steady state (self-destroying growth); in contrast, an initial capital stock of more than 11.11111, drives the country towards the third steady state (self-enforcing growth).

6.4.5.3 *Third candidate:* $y_3^* = 5$ *and* $k_3^* = 25$

Finally, the values of the third steady state candidate are inserted into the equation to calculate $\dot{k}$:

$$\dot{k} = \frac{1}{1+e^{-0.65917(y_1^*-3.33333)}} y_1^* - 0.15k_1^* \tag{6.101}$$

$$\dot{k} = \frac{1}{1+e^{-0.65917(5-3.33333)}} 5 - 0.15 \cdot 25 \tag{6.102}$$

$$\dot{k} \approx 0.0000043104 \approx 0.\text{[176]} \tag{6.103}$$

As for the first steady state, rounded values for the parameters and also for the variables y and k are used. For this reason, the deviation from zero of $\dot{k}$ can again be interpreted as round-off error. Consequently, also this third steady state is a steady state of the original growth model determined in this chapter for given parameter values. As the first steady state, it is also stable.

Having shown that the model developed in Section 6.3 indeed has two stable and one unstable steady state under certain assumptions concerning the parameters used, the next section will conclude this chapter.

[176] The difference between the determined $\dot{k}$ and zero is almost certainly due to round-off errors.

6.5 Conclusion

In this chapter, first of all the endogenous savings rate was constructed. It was formulated by use of a differential equation on the basis of a logistic function describing the savings rate behavior when income changes. Thereafter, the assumptions underlying the neoclassical growth model with endogenous savings were described. Basically, they remained the same as in the Solow growth model except for the fact that savings are not growing at an exogenously given rate but rather endogenously depending on income. Thereafter, the model was formulated and also a graphical representation showed that three steady states are possible within such a model construction. In Section 6.4, the steady states should be determined. Due to the specific structure of the savings rate, the steady states could not be determined directly. It could, however, in a first step be proved that the intersections indeed exist. Yet, for this to be possible, the parameters had to fulfil a number of restrictions.

Even though the steady states could not be determined exactly, they could be approximated. Before this was done, an overview of possible methods to approximate the roots of nonlinear functions was given. Thereafter, based on the restrictions found before, parameter values could be estimated. Using these values, the Newton method was applied to determine the steady states. These steady states were used to calculate the respective steady state capital stock values. The corresponding income and capital stock data were then inserted, together with the parameter values used for applying the Newton method, into the original growth model determined in this doctoral thesis to check whether $\dot{k}$ is really equal to zero. This was the case for all three values.

The proof of this model as well as the determination of the steady state values could occur only by making quite restrictive assumptions. The steady state values of y and k seem to be much too low from an economic perspective. Yet, this might be a scaling problem which might be overcome by changing the restrictions. In spite of this, it was possible to prove that the Solow growth model is able to yield twin peaks with a savings rate as constructed in this chapter. Hence, Solow was indeed right that his model is able to capture bimodality (Solow, 1956).

In the next chapter, a third way to examine Solow's claim that his model is able to capture multiple steady states will be followed. There, the Solow growth model will be modified based on empirical data, hence an empirically determined model will be formulated and will then be checked for bimodality.

7 The Empirically Determined Model

In the previous chapter, it was shown that even though it is possible to construct a theoretical Solow growth model which captures twin peaks, this model underlies quite restrictive assumptions. In this chapter, another way to formulate his growth model capturing twin peaks shall be examined: based on the empirics it shall be examined whether there is any possibility to capture bimodality. In the previous chapters, the empirical data were analyzed in detail. It was shown that three different variables might lead to twin peaks within the framework of the Solow growth model: the investment rate, the population growth rate, and human capital. As already argued and shown in Chapter 3, the investment rate in combination with the population growth rate is very likely to be responsible for multiple steady states in the framework of the Solow growth model.[177] These findings are supported by the literature on the one hand and by the empirics on the other hand. In Chapter 5, it was shown that none of the three variables is really polarized itself. However, from the loess fit curves it could be concluded that especially the investment rate is very likely to have a functional form which would be needed in order to end up with multiple steady states. Additionally, also the population growth rate seems to be a good indicator. Before the model capturing twin peaks can be developed, an empirical basis needs to be established. For this reason, in Section 7.1 regression analyses will be presented which aim at finding out what the investment function depending on income might look like. The same will be done for the population growth function in Section 7.2. Section 7.3 will discuss the functional forms chosen for the investment rate and the population growth rate. Based on this, in Section 7.4 the theoretical growth model yielding twin peaks in real per capita GDP based on the investment function and the population growth function will be presented. Section 7.5 will conclude this chapter.

7.1 Empirical Evidence on the Investment Function

In Section 3.2.1 (see p. 32), it was already argued that it is rather unrealistic to assume the savings rate to be equal to a fixed proportion of income as used in the Solow growth model. It was claimed that it is more realistic to assume savings to be dependent on income. It should be kept in mind that for the empirical analyses

[177] Additionally, using an endogenous growth model might also yield twin peaks, here due to human capital. However, the concern of this doctoral thesis is to check the Solow growth model for its capability to capture bimodality. For this reason it was decided to stick to the basic model without human capital and examine in how far the savings rate or the savings rate together with the population growth rate yield twin peaks within the framework of the Solow growth model.

the investment rate was used as a proxy for the savings rate on which data are not easily available.[178]

The task of this section is to define the functional form of the investment rate. For this purpose, regression analysis which estimates the investment rate on the income data will be made. As indicated by the loess fit curves of Section 5.6, it is quite likely that the investment rate depends on GDP in a polynomial way. Hence, regressions will be run by including several polynomials of GDP as explanatory variables. The highest one will be the fourth polynomial, and from then on, those coefficients being statistically insignificant will be excluded until all coefficients are significant.

The dataset being used for this analysis is a data pool. Hence, pooled regressions will be performed here. The data for all countries offering data as described before will be combined for all years from 1960 to 2007. This dataset will then be examined by regression analysis with fixed effects in the cross sections, hence in the countries.[179]

When analyzing panel datasets, there are two different classes of methods. The first one is a pooled regression model which assumes homogenous cross sections. The second class covers fixed and random effects, which are to be used if the individuals of the panel are assumed to be heterogeneous instead. In this doctoral thesis, heterogeneity is to be assumed so that the second class of methods is more likely. To be more precise, fixed effects should be used as these differences between the cross sections, here the countries, are not random. A panel method with fixed effects assumes that the heterogeneity of the cross sections is covered by a movement of the constant in the regression. The coefficients in a pooled method are relatively more efficient due to smaller standard errors because of a larger number of observations. Yet, this efficiency advantage gets lost in case the parameters of the regression for the individual cross sections are significantly different. This aspect can be tested by including fixed effects and comparing this model with the one without fixed effects. The F-statistic then gives an indication of whether fixed effects should be used (Eckey, Kosfeld and Dreger, 2004).[180]

In Table 7.1, the results of four different regression models using the pooled dataset are presented. The results along with R-squared, adjusted R-squared, the F-test as well as the corresponding p-value, the Durbin-Watson statistic, and the Akaike information criterion are presented in the table. All of the four models are estimated with fixed effects in the cross sections. Models 1, 2, and 4 have the

[178] It has to be kept in mind that in this doctorate the investment rate is used as an approximation of the savings rate. Hence, when talking about the data, the term investment rate is used – on the contrary, when only talking about the theoretical model, the term savings rate is used instead sticking to the original formulation of the Solow growth model.

[179] Additionally, regressions without fixed effects were run, a time trend was included, a lagged dependent was tried, and log transformations of GDP were considered as well. The models presented in Table 7.1 turned out to yield the best results.

[180] In the regressions it turned out that the coefficients are important and hence they were kept inside.

investment rate as the dependent variable whereas in Model 3, the logarithm of the investment rate is the dependent variable. It is divided by 100 and then added by 1 in order to eliminate negative values which are not defined when using the logarithm. Model 4 includes more observations because here, as opposed to the other three models, no autoregressive terms are included.[181] Comparing the adjusted R-squared values of the four models shows that obviously Model 3 is the best model.[182] Also the F-statistic is the highest in this case, even though in all of the models all variables used are significant together.

The Durban Watson test is a test for serial correlation. The statistic which is calculated for this test has values around 0 and 4. The test examines the null hypothesis that there is no autocorrelation in the data. A value of 2 means that the null hypothesis cannot be rejected, hence there is no autocorrelation. A value of 0 indicates that the null hypothesis has to be rejected and there is positive autocorrelation. A value of 4 also leads to the rejection of the null hypothesis. In this case, there is negative autocorrelation (see for example Greene, 2011). Regarding the Durbin Watson statistic, Model 3 is again the best, while also here the other models despite for Model 4 are very close regarding their DW-values. Finally, also the Akaike criterion can be used to find out which model is good. The Akaike criterion is a criterion to be used for choosing the “optimal” model, based on how well the estimated model can adjust to the used empirical data and the complexity of the model, measured on the basis of the number of parameters (Akaike, 1973). Even though the disadvantage is that it tends to overestimate the quality of models with many parameters using large samples, it is still “the one that is commonly used (at least in nonlinear models)” (Maddala, 1992, p. 500). The smaller the value is, the better is the model. Furthermore, when dealing with negative numbers, it needs to be stated that an Akaike value of -1.0 indicates a “better model” than a value of +0.3. For this reason, Model 3 is again identified as the best model for estimating the investment rate. Summarizing the results from above and keeping in mind that the values are very close for some of the models so that there might be more than one optimal model, it can be concluded that Model 3 might well to be chosen as the optimal one, even though Models 1 and 2 are very good as well.

[181] Autoregressive models are used when it is assumed that an observation y_t is dependent on a specific number of observations y_{t-p} which are preceding the observation y_t. Hence, a specific number of periods in the past determine the present observation. This is called autoregressiveness (Wooldridge, 2008).

[182] Yet, despite for model 3, the values are very close together.

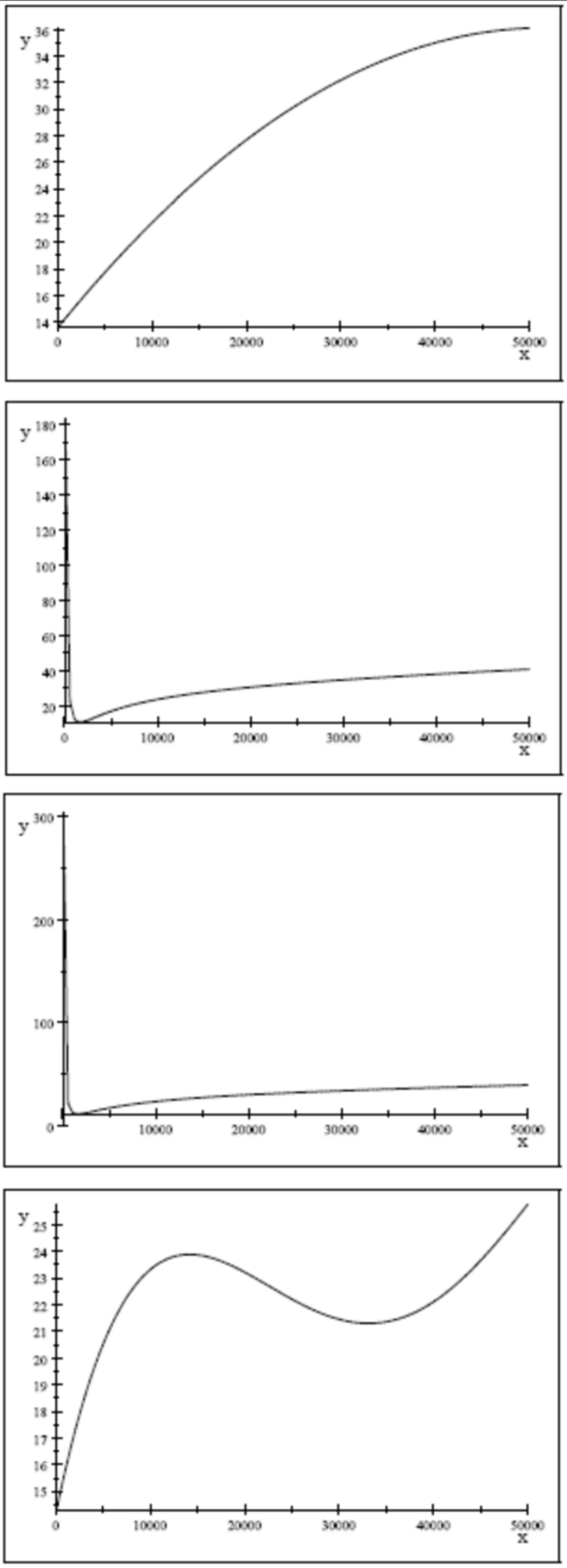
Table 7.1 Regression Results (Investment Rate)

Dependent variable: Investment share
Sample: 105 countries, 48 years (1960 to 2007)

	Model 1* **FE(CS)**	**Model 2*** **FE(CS)**	**Model 3*** **FE(CS), dependent: log((inv/100)+1)**	**Model 4*** **FE (CS)**
Constant	13.56190 (16.36904)	2496.570 (5.782045)	18.57734 (5.367528)	14.17623 (33.62826)
gdp	0.000876 (7.651932)			0.001714 (13.33981)
gdp^2	$-8.5*10^{-9}$ (-4.447980)			$-9.99*10^{-8}$ (-11.67291)
gdp^3				$2.13*10^{-12}$ (9.887106)
gdp^4				$1.45*10^{-17}$ (-8.636291)
log(gdp)		-1081.415 (-5.049995)	-8.005837 (-4.667544)	
log(gdp)^2		174.0533 (4.418313)	1.281382 (4.064309)	
log(gdp)^3		-12.32989 (-3.870119)	-0.090057 (-3.534973)	
Log(gdp)^4		0.327363 (3.424158)	0.002367 (3.098434)	
AR(1)	0.816352 (56.95431)	0.833661 (57.85254)	0.821192 (56.99756)	
AR(2)	-0.059106 (-3.207537)	-0.066342 (-3.573325)	-0.056309 (-3.049227)	
AR(3)	0.082615 (5-862919)	0.097106 (6.880102)	0.090413 (6.403486)	
# of obs.	4713	4713	4713	5040
R^2	0.918669	0.922213	0.924641	0.755862
R^2-adjusted	0.916751	0.920345	0.922831	0.750521
F-statistic	478.9076	493.7108	510.9550	141.5060
p-value	0.000000	0.000000	0.000000	0.000000
Durbin Watson	2.013725	2.016636	2.022144	0.378626
Akaike criterion	5.185417	5.141678	-4.487781	6.294282

*=(t-statistics in brackets)

Figure 7.1 The Models (Investment Rate)



Panel (a): Model 1

Panel (b): Model 2

Panel (c): Model 3

Panel (d): Model 4

Finally, knowing which model to use on the basis of the statistics reported in Table 7.1, a graphical analysis shall follow. Figure 7.1 shows the graphical representation of the four models. From the graphical analysis it becomes obvious that out of the four models, Model 4 is the one most suitable to the ideas about the investment rate mentioned earlier. Yet, it was shown that this model is rather bad in comparison to the others. For this reason, Models 1 to 3 are considered in more detail. Also Model 1 seems plausible from an economic point of view. However, the statistical analysis showed that Model 3 should slightly be preferred. The second best model is Model 2 and then, at the third position, Model 1 follows. As economic plausibility is also a decisive factor and the statistical values for R-squared adjusted and the Akaike information criterion do not differ a lot, Model 1 is chosen as the model to be used here. Before this information can be used for determining the theoretical investment function as a proxy for the savings function in the Solow growth model, the population growth rate shall be examined in the same way by use of regression analysis in order to find out what this function might look like. This is the purpose of the following section.

7.2 Empirical Evidence on the Population Growth Rate

In the previous section, an empirical analysis examined the form of the investment function as an approximation for the savings function. The idea behind this was that savings are not constant as assumed by the neoclassical Solow growth model but rather dependent on income. In order to understand how real per capita GDP might influence investments (and hence savings) a regression analysis was performed which yielded an insight into the possible functional form. This will be discussed in more detail in Section 7.3. However, before this can be done the same procedure will be followed for the population growth rate. This rate is also assumed to be constant in the Solow growth model. Yet, as found in Chapter 5, it is more realistic to assume population growth to depend on income as well.

In order to understand why population growth might as well rather be assumed to be endogenous, the reader is referred back to Section 3.2.2.2. Based on the theoretical ideas about the population growth rate as outlined above, the empirical data shall be examined in order to find the functional form of the connection between population growth and real per capita GDP in a country. This will again be done by regression analysis. Five different estimations are reported in Table 7.2 (see p. 169).[183] Models 1, 2, 3, and 5 have population growth as the dependent variable while Model 4 uses the logarithmic form of population growth (added by one due to negative values which would prevent a logarithmic form). On the right hand side, the first four models again include autoregressive terms (as far as they were significant at least at a 5 percent level). While in the first model, GDP and

[183] Of course, a lot more were run when preparing this section as well. Yet, only an overview of the most interesting functions is given here.

GDP-squared remain as significant variables, using logarithms on the right hand side yields only the first polynomial as a significant variable. Also in Model 4, only one polynomial of GDP remains. Finally, Model 5 is estimated without autoregressive terms. Instead, a lagged dependent (lagged by one year) is included and turns out to be significant alongside with the first two polynomials of GDP. Model 2 is similar to Model 1, just that it was decided to leave out y^2 as it has a very small influence and it is not significant at a 1 percent level. Thus, Model 2 will be used as an alternative to Model 1.

Looking at Table 7.2 also shows the important values in order to be able to judge on the statistical goodness of the models. The adjusted R-squared, for example, indicates Model 5 as the best one directly followed by Model 2. However, it needs to be noted that the other models differ in that value by at most 2 percent from the optimal value. This is not a large difference. The F-statistics are also very close. Only Model 5 has a significantly higher value, again followed by Model 2. Nevertheless, in all models, the variables taken together have a significant influence on the dependent variable. Concerning the Durbin Watson statistic, Model 5 is optimal as well as here the value is closest to the value 2. Though, also for this statistic the values of all five models are good and very close to 2. The models reported in Table 7.2 all indicate negative values for the Akaike information criterion. Generally speaking, the model with the lowest Akaike value is the model to be preferred.[184] In this case, Model 4 is the optimal one. However, it needs to be noted that the values of Models 1, 2, and 3 are very close and also Model 5 does not fall apart by a large amount. Summing up, even though Model 5 seems to have the best statistical indicators, the models are that close that there is no clear choice in favor of one of the models. Again, the decision should also be based on plausibility. For this purpose, these five models are graphed. The results are shown in Figure 7.2 (see p. 170).

Looking at Figure 7.2 shows that the models do not differ that much despite for Model 2.[185] Basically, they are all plausible. For poor countries, population growth is high. The richer a country the lower is the population growth rate. This fits the theoretical considerations mentioned earlier in this dissertation. As the models are all close together from a statistical point of view, other aspects need to be considered. It was decided to use Model 2, which has the second highest value for R-squared adjusted of the five models presented in Table 7.2.

[184] Hence, having negative values means that the model with the highest absolute value is to be preferred (in other words that with the most negative value).

[185] Model 5 cannot be drawn due to the lagged dependent.

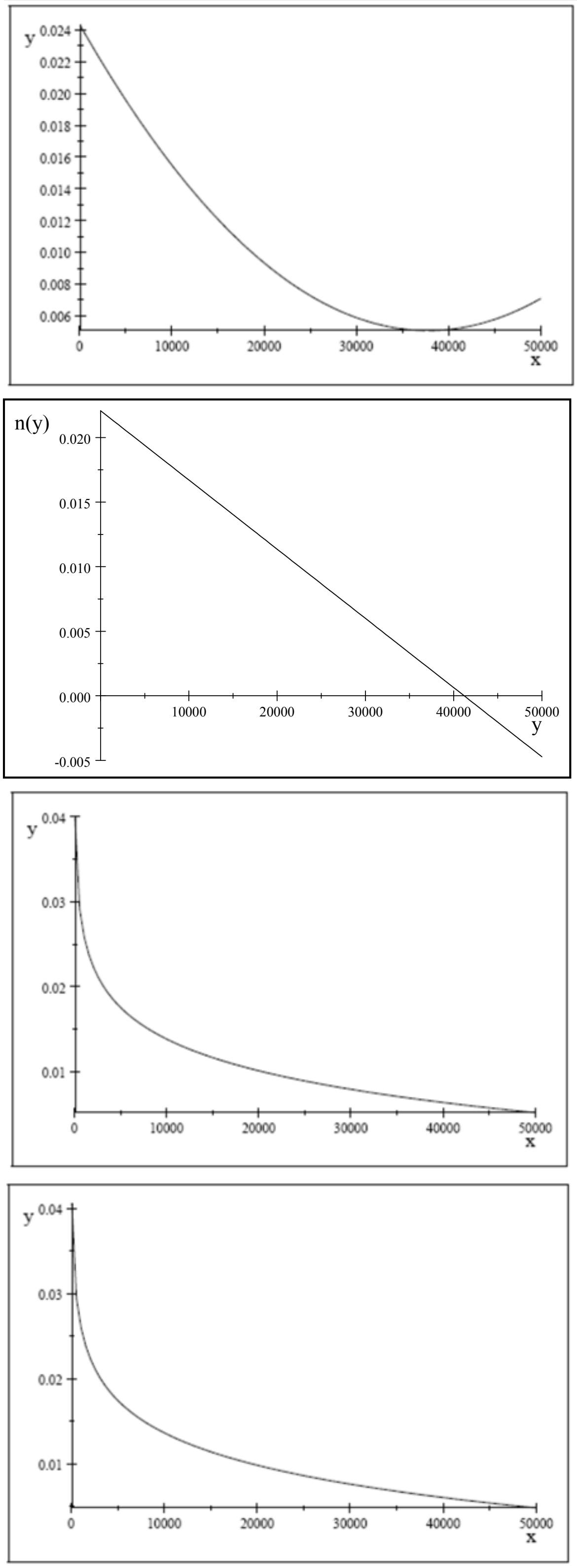
Table 7.2 Regression Results (Population Growth)

Dependent variable: population growth rate
Sample: 105 countries, 47 years (1961 to 2007)

	Model 1* FE(CS)	**Model 2*** FE(CS)	**Model 3*** FE(CS)	**Model 4*** FE (CS), dependent: log(popgr+1)	**Model 5*** FE(CS)
Constant	0.024344 (19.22060)	0.022076 (0.000775)	0.063450 (10.81687)	0.063575 (12.04628)	0.007474 (15.01958)
gdp	-0.00000102 (-4.564347)	$-5.36*10^{-7}$ ($7.22*10^{-8}$)			$-3.29*10^{-7}$ (-4.024865)
gdp^2	$1.35*10^{-11}$ (2.362456)				$4.80*10^{-12}$ (2.131589)
Popgr(-1)					0.680480 (60.04497)
log(gdp)			-0.005384 (-7.920420)	-0.005428 (-8.859332)	
AR(1)	0.638023 (36.44750)	0.671055 (0.016253)	0.635387 (36.30157)	0.620687 (36.26682)	
AR(2)	-0.066205 (-3.237331)	-0.081017 (0.019613)	-0.066533 (-3.259848)	-0.067908 (-3.459202)	
AR(3)	0.080701 (4.046209)	0.057197 (0.015632)	0.081454 (4.092900)	0.073318 (3.767379)	
AR(4)	-0.070413 (-3.549369)		-0.071541 (-3.613647)	-0.031697 (-2.018771)	
AR(5)	0.032933 (2.032662)		0.035451 (2.197015)		
# of obs.	3057	3387	3057	3221	3785
R^2	0.699700	0.705124	0.700203	0.692671	0.704951
R^2-adjusted	0.688663	0.695584	0.689283	0.682131	0.696469
F-statistic	63.39632	73.91540	64.12617	65.72038	83.10791
p-value	0.000000	0.000000	0.000000	0.000000	0.000000
Durbin Watson	2.089046	2.093792	2.090095	2.043596	1.983050
Akaike criterion	-6.703647	-6.705862	-6.705938	-6.720975	-6.668861

*=(t-statistics in brackets)

Figure 7.2 The Models (Population Growth)



Panel (a): Model 1

Panel (b): Model 2

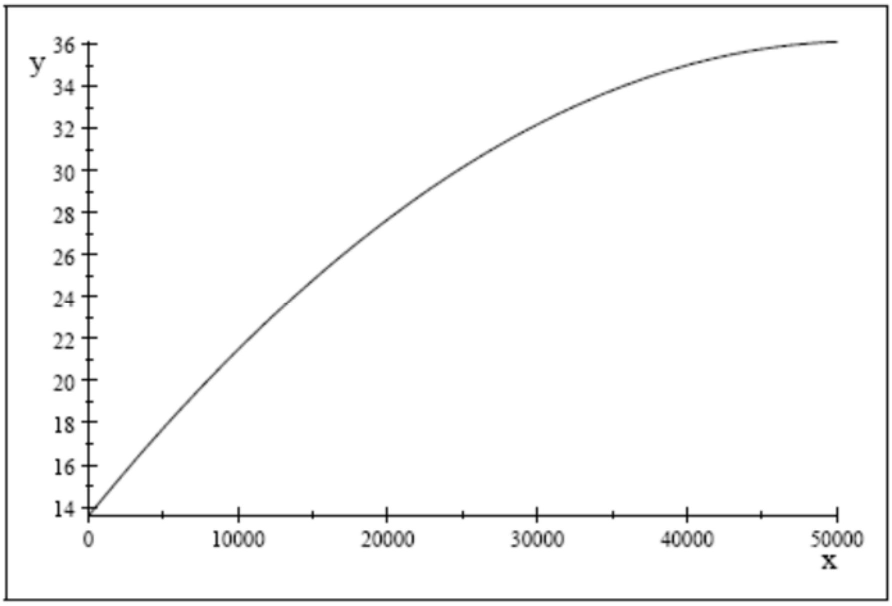
Panel (c): Model 3

Panel (d): Model 4

7.3 The Functional Forms

In the previous two sections, regressions were run as a preparation for finding the savings function (in the empirical work, investment rates were used as an approximation for savings; however, as here the theoretical model will be determined so that the term savings will be used again) and the function for population growth. The respective models were given by Tables 7.1 and 7.2. Additionally, except for Model 5 in the case of the population growth rate, all models were drawn to examine the economic plausibility of the models. In this section the probable functional forms of the savings function and the population growth function will be discussed. Section 7.3.1 will provide the solution for the savings function while Section 7.3.2 will show the same for the population growth rate.

Figure 7.3 The Savings Function



7.3.1 The Savings Function

In Section 7.2, it was argued that Model 1 is a plausible representation of the savings function.[186] This means that the empirically determined savings function is given by

$$s(y) = 13.5619 + 0.000876 \cdot y - 0.0000000085 \cdot y^2. \tag{7.1}$$

Figure 7.3 gives the graphical representation of this function. As the equation is determined by the empirics, no parameters may be chosen as it was done in the basic Solow growth model. Equation (7.1) will replace the parameter s in the Solow

[186] Yet, looking at Table 7.1 shows that in most cases, the goodness of the models does not differ a lot so that another model for the savings function might be chosen as well.

growth model. This will be done in Section 7.4 in order to find out whether multiple steady states arise.

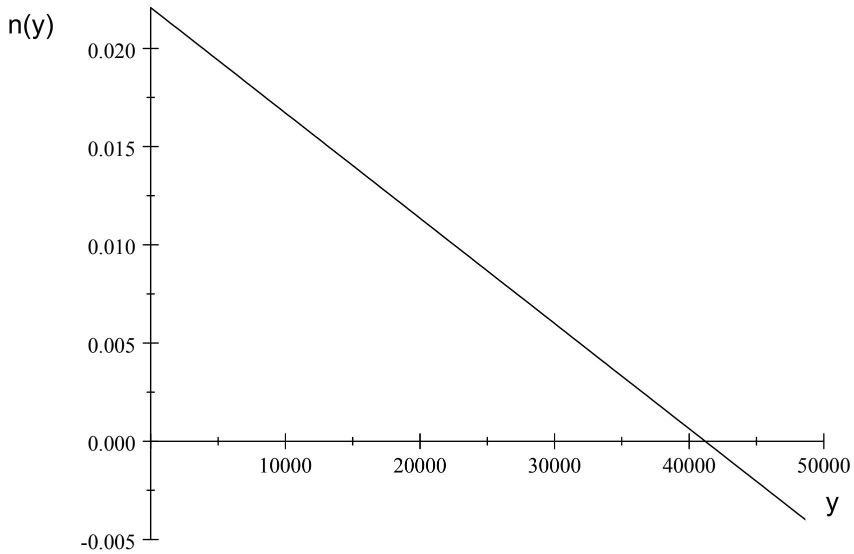
7.3.2 The Population Growth Function

In Chapter 5, it was argued that an endogenous savings function alone will not yield twin peaks in the basic Solow growth model as indicated by the empirical analyses. Yet, when combined with an endogenous population growth rate rather than assuming it to be constant, multiple steady states may arise. The endogenous savings rate was already described in the previous subsection. Now, the respective function for the population growth rate needs to be determined. Again, also for this case the goodness of the functions was very similar. Nevertheless, a decision needed to be taken, and so it was decided to choose Model 2 for the population growth rate. Hence, the population growth rate is given by

$$n(y) = 0.022076 - 5.36 \cdot 10^{-7} y. \quad (7.2)$$

Figure 7.4 shows this function. As outlined above, this is a linear function with a negative slope indicating that the population growth rate decreases as income rises. This is plausible according to the arguments mentioned in Section 3.2.2 (see p. 38)

Figure 7.4 The Population Growth Function



7.4 The Adjusted Solow Growth Model

In the previous section, it was shown what the savings function and the population growth function might look like as opposed to assuming these two variables to be constant. Consequently, this section deals with the inclusion of these two formulas into the Solow growth model. After doing this, the intersection points, hence the steady states, will be determined.[187]

7.4.1 The Model

In this subsection, the model explaining twin peaks in the real per capita income distribution shall be determined. Just as the basic Solow growth model, also this model assumes a closed economy without international trade and without state activity. The goods market as well as the factor markets are characterized by perfect competition and perfect information. Only one homogenous good is produced according to a Cobb-Douglas type of production function with constant returns to scale. As in the Solow growth model, there is technological progress which is assumed to be exogenous and labor-saving (Harrod-neutral). The two production factors are capital and labor. Consumers are maximizing utility and companies maximize their profits – both act as price takers. So far, the assumptions are equal to those of the basic Solow growth model.

The new assumptions refer to the savings rate on the one hand and the population growth rate on the other hand. While the basic Solow growth model assumes both rates to be exogenous and constant, the findings of Chapter 5 support the view that this is rather unrealistic. Instead, both rates are assumed to be endogenous, namely dependent on real per capita GDP, hence on y in terms of the Solow growth model. How this can be modeled will be shown in the following.

Production is given by Equation (7.3):

$$Y = F(K, L) = K^{\alpha}(AL)^{1-\alpha}, \tag{7.3}$$

where Y indicates income, K stands for capital, L for labor, A stands for technology, and α represents the capital share, whereby $0 < \alpha < 1$. The production function per efficient unit of labor is given by

$$y = f(k) = k^{\alpha}, \tag{7.4}$$

whereby $f'(k) > 0, f''(k) < 0, \lim_{k \to \infty} f(k)k^{-1} = 0$, $y = \frac{Y}{AL}, k = \frac{K}{AL}$.

There are decreasing marginal products of K and AL. Technology is assumed to grow exogenously while population growth is assumed to grow endogenously in contrast to the basic Solow growth model.

[187] The income values calculated here need not fit the values of the graphical analysis of Chapter 5, as the countries need not be in their steady states yet.

$$\frac{dA(t)}{dt} = gA(t) \tag{7.5}$$

$$\Leftrightarrow g_A = g \tag{7.6}$$

$$\Leftrightarrow A(t) = e^{gt}A(0) \tag{7.7}$$

$$\frac{dL(t)}{dt} = n(y)L(t) \tag{7.8}$$

$$\Leftrightarrow g_L = n(y) \tag{7.9}$$

$$\Leftrightarrow L(t) = e^{n(y)t}L(0), \tag{7.10}$$

where g_A and g_L describe the growth rates of technology and labor respectively, g is the rate of technological progress, δ is the rate of depreciation, which is assumed to be exogenous and constant, and n indicates the population growth rate, which is assumed to be endogenously determined by y.

7.4.2 Steady State Determination

For determining the steady states, the capital accumulation needs to be determined. Here, it has to be considered that the savings rate and the population growth rate are dependent on per capita income while the production function is given per efficient unit of labor. In order to clarify what the difference is like, the following equations will be written in terms of levels, not in per capita terms or per efficient units of labor.

To begin with, the capital stock evolves according to

$$\dot{K} = s\left(\frac{Y}{L}\right)Y - \delta K. \tag{7.11}$$

Capital accumulation per efficient unit of labor is then given by Equation (7.12):

$$\frac{\dot{K}}{AL} = s\left(\frac{Y}{L}\right)f\left(\frac{K}{AL}\right) - \left(n\left(\frac{Y}{L}\right) + g + \delta\right)\frac{K}{AL}. \tag{7.12}$$

Using Equation (7.4) and hence replacing $\frac{K}{AL}$ by $\left(\frac{Y}{AL}\right)^{\frac{1}{\alpha}}$, this equation can be rewritten as:

$$\frac{\dot{K}}{AL} = s\left(\frac{Y}{L}\right)\frac{Y}{AL} - (n\left(\frac{Y}{L}\right) + g + \delta)\left(\frac{Y}{AL}\right)^{\frac{1}{\alpha}}. \tag{7.13}$$

In the steady state, capital accumulation is equal to zero, so that Equation (7.14) holds:

$$\frac{\dot{K}}{AL} = 0 \tag{7.14}$$

Using stars as an indicator of the steady state values, the following equation determines the steady states:

$$s\left(\frac{Y^*}{L}\right)\frac{Y}{AL}^* = (n\left(\frac{Y^*}{L}\right) + g + \delta)\frac{Y}{AL}^{*^{1/\alpha}}. \tag{7.15}$$

In the steady state, $\frac{Y^*}{AL} = f\left(\frac{K^*}{AL}\right)$ is constant.

The savings function and the population growth function were estimated in Section 7.1. As explored in Section 7.3, it was decided to use the following equations:[188]

$$s\left(\frac{Y^*}{L}\right) = 13.562 + 0.001\frac{Y^*}{L} - 8.5 \cdot 10^{-9}\frac{{Y^*}^2}{L} \tag{7.16}$$

$$n\left(\frac{Y^*}{L}\right) = 0.022 - 5.36 \cdot 10^{-7}\frac{Y^*}{L} \tag{7.17}$$

In order to use these equations further, Equation (7.15) will be reformulated such that A is factored out and only $\frac{Y}{L}$ and $\frac{K}{L}$ are used in the equation.

$$s\left(\frac{Y^*}{L}\right)\frac{Y}{L}A^{-1} = \left(n\left(\frac{Y}{L}\right) + g + \delta\right)\left(\frac{Y}{L}\right)^{1/\alpha}A^{-1/\alpha} \tag{7.18}$$

$$\Leftrightarrow A^{\frac{1-\alpha}{\alpha}}s\left(\frac{Y}{L}\right)\frac{Y}{L} = \left(n\left(\frac{Y}{L}\right) + g + \delta\right)\left(\frac{Y}{L}\right)^{1/\alpha} \tag{7.19}$$

Now, the Equations (7.16) and (7.17) are inserted into Equation (7.19).

$$A^{\frac{1-\alpha}{\alpha}}\left(13.562 + 0.001\frac{Y^*}{L} - 8.5 \cdot 10^{-9}\frac{{Y^*}^2}{L}\right)\frac{Y^*}{L}$$
$$= \left[\left(0.022 - 5.36 \cdot 10^{-7}\frac{Y^*}{L}\right) + g + \delta\right]\frac{{Y^*}^{\frac{1}{\alpha}}}{L} \tag{7.20}$$

For simplicity, in the following, $\frac{Y^*}{L} = y^*,\ \delta = 0.03,\ g = 0.02$ and $\alpha = \frac{1}{3}$, which are values for the parameters being commonly assumed in growth analyses. Using this information, Equation (7.20) can be summarized to yield:

$$A^2\left(13.562y^* + 0.001{y^*}^2 - 8.5 \cdot 10^{-9}{y^*}^3\right) = 0.072{y^*}^3 - 5.36 \cdot 10^{-7}{y^*}^4 \tag{7.21}$$

$$13.562A^2y^* + 0.001A^2{y^*}^2 - (8.5 \cdot 10^{-9}A^2 + 0.072){y^*}^3$$
$$+5.36 \cdot 10^{-7}{y^*}^4 = 0 \tag{7.22}$$

As in every term of this equation y^* appears, one root of the equation is

$$y_1^* = 0. \tag{7.23}$$

Factoring out y^* from Equation (7.22) and rearranging the equation, the other roots can be determined:

$$5.36 \cdot 10^{-7}{y^*}^3 - (8.5 \cdot 10^{-9}A^2 + 0.072){y^*}^2 + 0.001A^2y^*$$
$$+13.562A^2 = 0. \tag{7.24}$$

This is a cubic equation for which the root shall be determined.[189] This is not an easy task. Before the right method for the determination of roots for such an

[188] From now on, for reasons of clearness, the numbers are rounded up to 3 decimals where possible. However, in the calculations, the unrounded numbers are used in order to minimize rounding errors when determining the steady states.

[189] For a proof of how to determine the zeros of a cubic equation please refer to the Appendix (A.19).

equation can be found, some transformations are necessary first. To begin with, Equation (7.24) will be transformed into the normal form

$$y^{*^3} + ry^{*^2} + sy^* + t = 0 \tag{7.25}$$

by dividing Equation (7.24) by $5.36 \cdot 10^{-7}$:

$$y^{*^3} - \frac{8.5 \cdot 10^{-9} A^2 + 0.072}{5.36 \cdot 10^{-7}} y^{*^2} + \frac{0.001 A^2}{5.36 \cdot 10^{-7}} y^* + \frac{13.562 A^2}{5.36 \cdot 10^{-7}} = 0. \tag{7.26}$$

As there are no data on technology available, A has to be estimated. From a look at the Penn World Table 6.3, for example, it becomes obvious that the poorest country experiencing zero growth was Tanzania in 1991 with an income of almost \$500. Hence, a stable steady state should be around such a value, more or less. Of course, there are more countries experiencing zero growth in some years. Nevertheless, this might also be interpreted as being temporary.[190] For that reason, a value for A was chosen which yields a stable steady state at around \$500.[191]

By trial and error it turns out that a level of $A = 40$ yields a plausible result for the second steady state. Hence, the numerical calculation shown here is based on this level of technology.[192]

Inserting $A = 40$ into Equation (7.26) yields:

$$y^{*^3} - 134{,}495.538 y^{*^2} + 2{,}614{,}925.373 y^* + 40{,}483{,}283{,}582 = 0. \tag{7.27}$$

The solutions of this equation are given by

$$y_2^* = 134473.854 \tag{7.28}$$

$$y_3^* = -537.945 \tag{7.29}$$

$$y_4^* = 559.629. \tag{7.30}$$

The determination of the roots is shown in detail in the Appendix (A.19). Even though Equation (7.22) obviously has four roots, only three are plausible from an economic point of view. Income has to be positive, so that the third root given by Equation (7.29) will not be considered further. Consequently, three roots remain, given by Equations (7.22), (7.28), and (7.30).

A look at Table 7.3 shows that the levels of the second steady state are quite close, independent of the level of A. However, for the other two steady states, the values differ more. With $A = 40$, the result for y_3^* is quite plausible according to what was stated above.

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190 Looking for a steady state around about \$500 makes sense also from looking at the definition of being poor as described in Chapter 2. Here, the absolute poverty line was set at \$1 per day; hence, in sum this would mean \$365 a year. Averaging this with the above mentioned second poverty line of \$2 a day, a result of \$1.5 per day, hence about \$540 would be a sensible value, which is very close to the above mentioned \$500.

191 If choosing a different value, of course, the result would change. Yet, this will not be reproduced here as a further possibility.

192 Table 7.3 summarizes the calculated steady states using other levels of technology.

Table 7.3 Steady States for Different Levels of A (in $)

A	y_2^*	y_3^*	y_4^*
10	134,470.396	-136.496	137.851
20	134,471.087	-271.645	277.067
30	134,472.240	-405.459	417.656
40	134,473.854	-537.945	559.629
50	134,475.929	-669.113	702.994
100	134,493.242	-1,305.520	1,441.026

The roots calculated above stem from solving the steady state condition for y. Three steady states remained, for which the respective values for k can be determined as well. However, y and k are not measured per efficient unit of labor but instead per capita, as A was factored out of the equations. By use of Equation (7.4), resubstitution yields:

$$k_1^* = 0 \tag{7.31}$$

$$k_2^* = 2.43172 \cdot 10^{15} \tag{7.32}$$

$$k_3^* = 175{,}267{,}227.3, \tag{7.33}$$

where Equation (7.31) gives the steady state value belonging to $y_1^* = 0$, Equation (7.32) that which corresponds to $y_2^* = 134{,}473.854$, and Equation (7.33) indicates the value that belongs to $y_4^* = 559.629$. Table 7.4 summarizes the three steady state values for y and k.

Table 7.4 The Steady States (in $)

Steady State	Value of $\mathbf{y}^*$	Value of $\mathbf{k}^*$
1	0	0
2	559.629	175,267,227.3
3	134,473.854	$2.43172 \cdot 10^{15}$

The values calculated above need to be interpreted, of course. Keeping in mind that y was measured as real per capita GDP in the estimations of Section 7.1, here, real per capita GDP levels were determined alike. That is, one steady state value for y can be found at a real per capita income level of \$0, the second one at a value of \$559.63, and the third one at \$134,473.85.

Before further interpretations can be given, it needs to be determined whether the steady state values are stable. This will be done in the following subsection.

7.4.3 Steady State Stability

Steady states may be stable or unstable. A steady state is an intersection of the savings line and the depreciation line.[193] The steady states were calculated by finding the roots of the function

$$f(y) = s\left(\frac{Y}{L}\right)\frac{Y}{AL} - \left(n\left(\frac{Y}{L}\right) + g + \delta\right)\frac{Y}{AL}^{\frac{1}{\alpha}}.\text{[194]} \tag{7.34}$$

Drawing this function then allows to see in more detail when the function is above zero and when it is below it in order to find out whether a steady state is stable. Alternatively, this can also be done by calculus. Generally speaking, a country is growing as long as savings are higher than depreciation. Otherwise, a country faces a shrinking economy. Thus, if the function given in Equation (7.33) lies above the horizontal axis, a country grows and hence will be pushed towards the steady state to the right of the starting value. If the function is below the horizontal axis, on the contrary, then depreciation is higher than savings and hence, a country shrinks. This means that the country will be pushed downwards to the steady state to the left of the starting value. If the dynamics are always in direction of a steady state irrespective of starting to the left or to the right of it, the steady state is said to be stable. Instead, if the dynamics are always away from this steady state, it is said to be unstable.

Inserting the known functions for $s(y)$ and $n(y)$ into Equation (7.34), Equation (7.35) results:

$$\begin{aligned} f(y) = {} & 13.562A^2y + 0.001A^2y^2 - (8.5 \cdot 10^{-9}A^2 + 0.072)y^3 \\ & +5.36 \cdot 10^{-7}y^4. \end{aligned} \tag{7.35}$$

Assuming $A = 40$ as outlined above yields Equation (7.36).

$$f(y) = 5.36 \cdot 10^{-7}y^4 - 0.072y^3 + 1.402y^2 + 21{,}699.04y. \tag{7.36}$$

Before the graph is drawn, it will be examined whether this function has extreme values. In general, it should be kept in mind that the function is only defined here for $y > 0$. Negative real per capita GDP values are not sensible from an economic point of view and in addition, they are not possible within the framework of a model

[193] The savings line is given by $s(y)y$ in this model. Depreciation is assumed to come from capital depreciation, indicated by δ, from technological progress, given by g, and from population growth depending on income, hence $n(y)$.

[194] The variable y is used instead of y^* as the purpose of this section is not to determine the steady states but to prove the steady state stability instead.

in a closed economy as there are no transfers from abroad to compensate a negative income within the country.

When looking for the extreme values of a function, the first derivative needs to be set equal to zero. In case of a maximum, the second derivative is smaller than zero at this point and in case of a minimum, it is positive.

$$f'(y) = 0 \tag{7.37}$$

$$f'(y) = 0.000002144y^3 - 0.2162688255y^2 + 2.8032y + 21{,}699.04 \tag{7.38}$$

Equation (7.38) gives the derivative of Equation (7.36). Setting it equal to zero yields the following three solutions:

$$y_1 = -309.874 \tag{7.39}$$

$$y_2 = 323.833 \tag{7.40}$$

$$y_3 = 100{,}857.695. \tag{7.41}$$

The first solution will not be pursued further as it is not defined. Hence, only the latter two will be considered. In order to find out whether the respective points are minima or maxima, the second derivative will be calculated:

$$f''(y) = 0.000006432y^2 - 0.432537651y + 2.8032. \tag{7.42}$$

Inserting the two extreme points (7.40) and (7.41) yields the following results:

$$f''(y_2) = -136.5921916 < 0 \tag{7.43}$$

$$f''(y_3) = 21{,}806.12347 > 0. \tag{7.44}$$

Obviously, there is a maximum point of Equation (7.36) at y_2 and a minimum point at y_3. The respective values of the function are:

$$f(y_2) = 4{,}731{,}604 \tag{7.45}$$

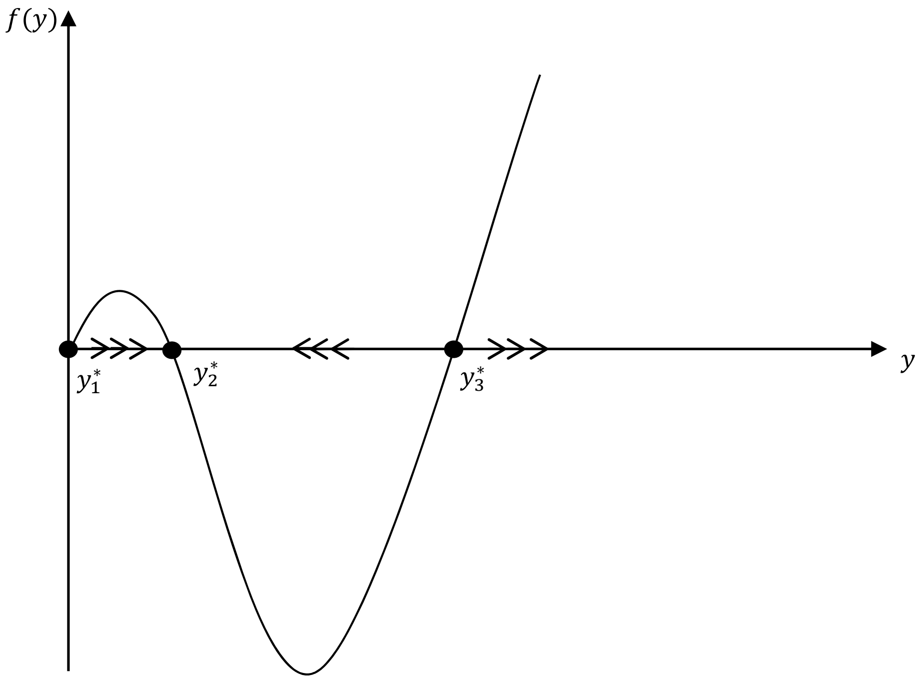
$$f(y_3) = -1.84814 \cdot 10^{13}. \tag{7.46}$$

Thus, it can be concluded that the function indicated in Equation (7.42) is not possible to be drawn along one single scale. For simplicity, it was decided to sketch the function in general terms rather than drawing it for the exact values. The result is shown in Figure 7.5.[195] Even though being aware of the fact that the scales in Figure 7.5 are not correct, it can be seen that the maximum calculated above lies between the first two steady states and the minimum between the latter two. Hence, the function needs to look similar to the one above. From this it can be concluded that y_1^* is in fact an unstable steady state. A country starting off to the right of y_1^*, will be pushed towards y_2^*. Alike, a country starting off to the right of y_2^* will be automatically pushed back towards y_2^* again. Only when reaching an income

[195] It has to be noted that the values of the roots are too far apart to have it in one graph. For this reason, it was decided to only sketch the graph. The graph is a very simplified version of the "real" graph. Nevertheless, it is easily possible to draw conclusions on the nature of the steady states.

above y_3^* a period of self-sustaining everlasting growth will arise. For this reason, y_1^* and y_3^* are unstable equilibria, while y_2^* is a stable steady state. This result is the opposite of what was needed to prove Solow's claim of being able to capture bimodality in his model. Hence, the next subsection will deal to examine whether the twin peaks phenomenon might nevertheless be explained by this empirically determined version of the Solow growth model.

Figure 7.5 The Steady States



7.4.4 Sensitivity Analysis on Different Rates of Technological Progress

In this subsection, a sensitivity analysis shall be undertaken. Table 7.5 shows the steady state values for $g = 0.01$ and $g = 0.02$ using $A = 40$.[196]

What becomes obvious is that the changes for the second steady state are minor compared to those of the third equilibrium. The lower the rate of technological progress, the closer the value of the third steady state comes to the values which can be found in the data for countries being rich today. Thus, a state of self-sustaining growth becomes more realistic for a number of already rich countries.

[196] In addition, also other values for A could be tried. However, the purpose of Table 7.5 was to show how sensitive the results are to changes in g.

Table 7.5 Influence of Different Values of g

$\mathbf{g}$	$\mathbf{y_1^*}$	$\mathbf{y_2^*}$	$\mathbf{y_3^*}$
0	0	661.318	97,150.900
0.01	0	604.170	115,813.225
0.02	0	559.629	134,473.854

Overall it should be noted that the aim of the model in this chapter is to explain why so many countries are deemed to be poor, hence why the poor peak exists. As the second steady state, which is also the stable one to which countries are always pushed back over time, is at a level which suits the idea of being poor, this model is well able to explain this dilemma even though only one stable steady state was found.

7.4.5 Further Features of the Steady State Analysis

After having shown that the results are indeed sensitive to the chosen value of technological progress, hence g[197], in this section the values of the savings part and the investment part of the model used for calculating the steady states will be determined.[198] In this way it can be examined again whether the values are indeed identical in the steady states. To begin with, the steady state values will be inserted into the following two functions:

$$f_1\left(\tfrac{Y}{L}\right) = A^{-1}s\left(\tfrac{Y}{L}\right)\tfrac{Y}{L} \tag{7.47}$$

$$f_2\left(\tfrac{Y}{L}\right) = A^{-\frac{1}{\alpha}}\left(n\left(\tfrac{Y}{L}\right) + g + \delta\right)\left(\tfrac{Y}{L}\right)^{\frac{1}{\alpha}}. \tag{7.48}$$

Now, the following values shall be inserted into the functions: $A = 40$, $\alpha = \frac{1}{3}$, $\left(\frac{Y}{L}\right)_1^* = 0$, $\left(\frac{Y}{L}\right)_2^* = 559.629$, $\left(\frac{Y}{L}\right)_3^* = 134{,}473.854$, $g = 0.02$, and $\delta = 0.03$. For $s\left(\frac{Y}{L}\right)$ and $n\left(\frac{Y}{L}\right)$, the functions estimated above will be inserted (given by the Equations (7.12) and (7.13)). Then, Equations (7.47) and (7.48) can be reformulated into:

$$f_1\left(\tfrac{Y}{L}\right) = \tfrac{1}{40}\left(13.562 + 0.001\tfrac{Y}{L} - 8.5 \cdot 10^{-9}\left(\tfrac{Y}{L}\right)^2\right)\tfrac{Y}{L} \tag{7.49}$$

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[197] The results will also be sensitive to the value for δ, which was assumed to be equal to 0.03.

[198] In the basic Solow growth model the savings part of the model is sy and the investment part, hence the investment line in this case is given by $(n + g + \delta)k$. In the modified model here, the savings part is $s(y)y$ and the investment part is $(n(y) + g + \delta)k$.

Table 7.6 Savings and Investment at the Steady States

	$\mathbf{y_1^*}$	$\mathbf{y_2^*}$	$\mathbf{y_3^*}$
$\mathbf{f_1}(\mathbf{y}^*)$	0	196.562312	-75,125.0015
$\mathbf{f_2}(\mathbf{y}^*)$	0	196.562265	-75,449.5824

$$f_2\left(\frac{Y}{L}\right) = \frac{1}{40^3}\left(0.022 - 5.36 \cdot 10^{-7}\frac{Y}{L} + 0.02 + 0.03\right)\left(\frac{Y}{L}\right)^3 \quad (7.50)$$

In Table 7.6, the results of these functions for the three steady states are summarized. What becomes apparent is that indeed the functions are equal at the steady states, as it should be. Yet, slight differences in the values stem from rounding errors when reporting the steady state values. Hence, they can be ignored.

Apart from this, some further interesting questions arise. First of all, it will be determined when the investment function becomes zero:

$$n(y) + g + \delta = 0. \quad (7.51)$$

In order to do this, the population growth function including the values for g and δ will be inserted into the equation:

$$0.022 - 5.36 \cdot 10^{-7}y + 0.02 + 0.03 = 0 \quad (7.52)$$

$$\Leftrightarrow 5.36 \cdot 10^{-7}y = 0.072 \quad (7.53)$$

$$\Leftrightarrow y = 134{,}470.149 \quad (7.54)$$

This value is very close to the steady state. Additionally, this means that the population growth rate would be -0.05. This number seems to be unrealistic. However, it ought to be kept in mind that nowadays especially the very rich countries face a new problem, namely the one of a shrinking population. This problem is often discussed, especially in connection to the social security systems and the pension schemes (for example Chand and Jäger, 1996). Hence, considering these aspects as well, it is not implausible to have a negative population growth rate once a real per capita GDP of about $134,470 is reached. As outlined above, this level of income is not reached yet by any country, even though it is not impossible that this income level might be reached once by one country or the other.

A second question which might come up is when exactly the savings function is equal to zero. This will be calculated in the same way as for the population growth function:

$$13.562 + 0.001y - 8.5 \cdot 10^{-9}y^2 = 0 \quad (7.55)$$

$$\Leftrightarrow y^2 - 103{,}058.8235y + 1{,}595{,}517{,}647 = 0. \quad (7.56)$$

The solutions to this equation are given by Equations (7.57) and (7.58):

$$y_1 = 18{,}975.416 \tag{7.57}$$

$$y_2 = 84{,}083.407. \tag{7.58}$$

The values are quite high as well, though not implausible.

Of course, many more values could be calculated by using the functions above. Yet, this will not be done here. It was shown in Section 7.4.3 where exactly the minimum and the maximum of the capital accumulation function can be found. This already implied that the function reaches very high values for y in some parts and very low values in other parts. Nevertheless, whatever the calculated values indicate, it ought to be noted that the functions for savings and population growth and hence also for capital accumulation are based on empirical findings.

7.5 Conclusion

Looking at the results of this chapter, it can be concluded that by extending the Solow growth model by empirically determined functions for the savings rate and the population growth rate, it is indeed able to explain bimodality in the real per capita income distribution. On the basis of the empirical data, savings functions and population growth functions could be estimated. This was shown in Sections 7.1 and 7.2. The regressions presented here yielded several plausible functions, two of them being chosen in order to determine whether they yield three steady states. The choice of the respective functions was the subject of Section 7.3. Finally, in Section 7.4 the complete model was presented. It was shown that three steady states can be found, one of them being stable while the other two are unstable. Hence, bimodality does not appear in the form that was expected. Instead of two stable equilibria, here the single stable equilibrium is at a low level of income, namely at a real per capita GDP of $559.63. This is a very low value which can be seen to represent the poverty trap in which many countries[199] find themselves. The high steady state is unstable. In addition, it is at a very high value of $134,473.85 which is not very realistic when looking at the real per capita GDP data. Nevertheless, it is not impossible to be reached someday. Once an income beyond this steady state is reached, self-sustaining growth results.[200]

“The purpose of a model is not to be realistic. […]. The problem with [reality] is that it is too complicated to understand. A model’s purpose is to provide insights about particular features of the world. If a simplifying assumption causes a model to give incorrect answers to the questions it is being used to address, then that lack of realism may be a defect” (Romer, 1996, pp. 11f). Summing up, even though the

[199] Mainly African countries and some Islamic countries.

[200] Self-sustaining growth here means that a country is ever growing. If there is a steady state then growth is expected to occur until this point is reached. Yet, if a country manages to get beyond the third steady state determined in this chapter, it is pushed towards infinity. Hence, there are no restrictions to growth. The country faces boundless growth then.

results were not as expected[201], also the model of this chapter might be able to explain the phenomenon of polarization of the real per capita GDP data on the basis of one stable equilibrium on the one hand and self-sustaining growth beyond the third steady state on the other hand. Consequently, also from this perspective it can be concluded that the Solow growth model is indeed able to explain the polarization of the world income distribution.

[201] It was expected to find two stable equilibria as shown in Chapter 3 and Chapter 6.

8 Conclusion

World income disparities are a prevailing issue in economic research. Over the past decades, especially the differentials of real per capita GDP across the countries of the world became a major research topic in macroeconomics. A closer look at this field of research shows that the question of convergence turns out to be significant. Especially Quah (for example 1996c) contributes extensively to the research on stratification. According to this theory, the world income distribution does appear to be twin peaked rather than being Gibrat distributed with a single peak skewed to the right. Instead, basically two clubs seem to have formed: a club consisting of the poor countries and one containing the rich ones, while the middle income group decreased sharply.

In the past, there were a lot of influential contributions to this topic, for example by Ben-David (1997), Jones (1997), Cantner et al (2001), and Chakrabarty (2012), among others. As Quah (1996c) outlines, the convergence debate including the discussion on bimodality “can be viewed either as checks on different growth models or as empirical regularities to be explained by theory” (Quah, 1996c, p. 95). This doctoral thesis is an attempt to be both.

While a large number of articles focus on the empirical analysis of the polarization phenomenon (for example Beaudry, Collard and Green (2002), Paap and van Dijk (1998), Bianchi (1997), or Semmler and Ofori (2007), just to name a few), the more theoretical checks of growth models were already pursued very early. In 1956, when Solow formulated his influential growth model, he mentioned the possibility to capture bimodality within his model framework. While several authors working on polarization or, more generally, on economic growth, concentrate on endogenous growth models (for example Chakraborty, 2004), a look at economic growth literature shows that the Solow model is still relevant. As many authors find that twin peaks are a common feature of the world income distribution, this dissertation sought to answer these two questions:

1. Is there really club convergence in the real per capita income distribution across the countries of the world?
2. Is the Solow growth model indeed able to explain the polarization phenomenon?

8.1 Summary of Contributions

The main findings were summarized within the respective chapters. Thus, this section synthesizes the answers to the above mentioned research questions. To begin with, the theoretical foundations for this doctoral thesis will be briefly reviewed. Thereafter, the results from a graphical and verbal analysis of the Solow growth model with respect to bimodality will be discussed. This will be followed by a synthesis of the empirical findings concerning the existence of convergence clubs. Thereafter, the results from an analytical examination with respect to bimodality within the Solow growth model will be presented. Finally, the findings from checking an empirically determined Solow growth model for the existence of multiple steady states will be presented.

This doctoral thesis is based on world income differentials. As outlined by Korzeniewicz and Moran (1997), income differentials between countries account for a large fraction of the overall income disparities. While the convergence debate first assumed the world income distribution to be Gibrat distributed, the polarization hypothesis gained in relevance in the past decades. Quah (1996c), Jones (1997), and Cantner et al (2001), among others, showed that there are rather twin peaks in the distribution of real per capita GDP across the countries of the world.

Considering the approach used by Solow (1956), namely graphical and verbal analysis, it could be shown that the Solow model is indeed able to yield two stable steady states. There are reasons implying multiple peaks. It is not realistic to assume a homogenous savings rate within the Solow growth model. Moreover, the savings rate might rather be a function of income, represented by an S-shaped curve which yields two stable steady states within the framework of the Solow growth model. As another possibility, population growth might also be dependent on income. Again, this changed assumption yields two stable steady states within the Solow growth model. Finally, including human capital in the neoclassical growth model and assuming that savings in physical as well as in human capital both depend on income is also able to yield bimodality. Thus, from this point of view, the hypothesis that the Solow growth model is able to explain polarization is confirmed.

There are a several alternative methods of distribution analysis. One of the most robust methods is the kernel density, because it is rather independent of the choice of origin and of the bin width. It is also the method which is mainly used in studies of the polarization of the world income distribution (see for example Bianchi (1997), Cantner et al (2001), Semmler and Ofori (2007), and Villaverde (2001), among others). In addition, Markov chains allow for a look into the future of the distribution of real per capita GDP across the countries of the world. Finally, loess fit curves help to decide on how the savings rate or the population growth rate on the one hand and income on the other hand might be correlated.

The empirical findings are based on analyses of data out of the Penn World Table 6.3 and the Barro-Lee dataset. They can be synthesized as follows. First, since the 1970s the world income distribution seems to be polarized. Based on a Markov chain analysis, this result is rather robust. Yet, whether there are two or rather three peaks depends on the choice of the starting year of the Markov chain. In most cases, bimodality turned out to be a future phenomenon of the world income distribution, yet with a very large group of poor countries and a comparatively small group of rich countries.

The empirical analysis showed that for the investment rate, which is used as an approximation of the savings rate as outlined above, bimodality became apparent after 1990. The population growth rate can be seen as mainly unipeaked while human capital, approximated by the average years of schooling, is obviously rather twin peaked. Looking at the loess fit curves helped to understand that the investment rate and real per capita GDP do not seem to be linearly related. Instead, polynomials seem to define the relationship. The conclusions to be drawn from the loess fit curves of the population growth rate and real per capita GDP are not very clear. The relationship changed over time so that the population growth rate alone is not likely to explain the existence of multiple peaks. Finally, the loess fit curves of human capital and real per capita GDP showed a nonlinear relationship with diminishing returns.

In Chapter 6, an endogenous savings rate was used to capture multiple steady states in the framework of the Solow growth model. According to Azariadis (2006), the only robust variable to explain the existence of poverty traps, and hence of bimodality in distribution of real per capita GDP across the countries of the world, is investment. Instead of using a constant savings rate, a logistic savings function depending on income was included in the Solow growth model. By use of the Newton method the steady states could be determined. For this to be possible, a number of restrictions needed to be formulated. Nevertheless, it could be shown that the Solow growth model indeed allows for two stable steady states along with an instable one. The positions of these steady states, however, are not very plausible from an economic point of view. However, this might be a scaling problem. Thus, also from this perspective it can be concluded that the Solow growth model is indeed able to explain polarization of the world income distribution.

The final way to examine the hypothesis of the Solow growth model being able to explain bimodality was based on the use of an endogenously determined neoclassical growth model. Based on the loess fit curves presented in the empirical analyses, an endogenous investment rate (as an approximation of the savings rate) together with an endogenous population growth rate were estimated. Fixed effects regressions were run using a panel dataset consisting of the real per capita GDP data for 105 countries over the period 1960 to 2007. Inserting the resulting savings function and population growth function into the Solow growth model and then solving it for the steady states yielded two instable equilibria and a stable one.

This was the opposite of what was expected. Yet, it could be shown that the stable steady state is indeed at a position which might be economically plausible. With an income of $559.63 it might well be interpreted as a poverty trap and hence as the lower peak in the kernel densities presented in Chapter 5. The higher, though instable steady state was determined at an income of $134,473.85. Looking at the data on real per capita GDP, this income level seems to be too high. Though, it is not impossible that one country or the other might reach this level once. Furthermore, also in this case there might be a scaling problem as mentioned in Chapter 6. Beyond this high instable steady state, a situation of self-sustaining growth will be reached making countries grow ever richer. Though it was expected to find two stable steady states just as in the analytical examination of the neoclassical growth model capturing twin peaks, also here it can be concluded that the Solow growth model is able to give explanations for the polarization of the distribution of real per capita GDP across the countries of the world.

This doctoral thesis offered several ways to elaborate on the capability of the Solow growth model to capture bimodality. It could be shown that the world income distribution is indeed polarized. Apart from graphical ways to prove the possibility of multimodality in the Solow growth model, inserting an endogenous savings rate and solving the model analytically also allowed for two stable steady states. Furthermore, formulating an empirically determined Solow growth model and determining the equilibria underlined the ability of the model to explain twin peaks the distribution of real per capita GDP across the countries of the world.

8.2 Implications for Future Research

There were a number of limitations in this doctoral thesis so that further research is necessary on the differentials the world income distribution between nations and especially the capability of the Solow growth model to explain them. In the analytical examination it was decided to concentrate on the inclusion of an endogenous savings rate based on the stylized facts formulated by Azariadis (2006).

However, the consideration of an endogenous population growth rate instead or even together with an endogenous savings rate should be pursued further. Another aspect which should be considered in the future would be to perform distribution analyses based on data weighted by the population of the countries rather than considering each country as one point of observation. In this way it could be examined whether the conclusions to be drawn differ compared to the results of this doctoral thesis.

In addition, in this doctorate also a version of the Solow growth model including human capital was presented. Further research should focus on solving this extended model by including an endogenous human capital savings rate. For this to be possible, a dataset with less missing values should be found.

Beyond, also the empirically determined growth model leaves open a number of questions. To begin with, future research could use one of the other estimated functions mentioned in Chapter 7. Furthermore, the Solow growth model extended by human capital should be extended by empirically determined savings rates in human capital and physical capital. However, this is again dependent on whether a dataset can be found which covers more years than the Barro-Lee dataset.

Another aspect for future research concerns the possibility to overcome the poverty trap, hence to escape the peak at a low level of income. The purpose of this doctoral thesis was to prove the existence of the twin peaks and to examine the ability of the Solow growth model to explain this phenomenon. Yet, further research should focus on policy implications for managing to overcome the poverty trap – this could be based on the findings of this thesis, namely on the savings rate (or the investment rate) as well as the population growth rate – or even human capital.

In spite of the limitations of this doctoral thesis as well as the remaining open questions, it could be proved that the world income distribution is indeed polarized. The different ways of examination showed that the Solow growth model is able to yield multiple steady states in a neoclassical framework based on realistic changes of the underlying assumptions.

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Appendices

A.1 Personal Saving Rates for Selected Countries

Table A.1 Personal Saving Rates for Selected Countries[202]

Country	GNP per equivalent adult in 1985 $ 1980-87 averages	Personal savings as a % of GDP
Low-income countries		
Tanzania	639.5	-1.0
Burkina Faso	644.6	1.0
Bangladesh	889.2	13.5
Madagascar	916.8	4.3
Togo	937.9	14.0
Somalia	1,146.4	6.2
Ghana	1,164.1	6.1
Haiti	1,210.4	4.5
Kenya	1,197.9	18.2
Sierra Leone	1,341.0	8.1
Nigeria	1,603.5	9.5
Pakistan	1,672.0	23.0
Honduras	1,679.9	7.5
Guyana	1,833.0	14.3
Sri Lanka	2,156.1	19.8
Egypt	2,158.3	29.8
Average for group	*1,324.4*	*11.2*
Lower middle-income countries		
Bolivia	2,047.9	12.2
Côte d'Ivoire	2,057.6	12.7
Cameroon	2,170.4	11.0
El Salvador	2,203.0	15.0
Philippines	2,432.0	16.2
Morocco	2,472.4	20.9
Dominican Republic	2,811.4	14.8
Thailand	2,901.4	22.8
Paraguay	3,082.4	14.6
Tunisia	3,773.4	14.5
Peru	3,786.5	24.4
Turkey	3,931.6	21.4
Iran	3,962.5	20.0
Colombia	4,164.0	12.7
Poland	4,360.6	26.8
Chile	4,587.8	12.8
Average for group	*2,805.8*	*17.1*

[202] Source: Ogaki, Ostry and Reinhart (1996), pp. 44-45

Upper middle-income countries		
Mauritius	4,406.6	24.2
Korea	4,409.5	25.4
Argentina	4,994.5	18.5
Brazil	5,099.8	17.4
Portugal	5,280.9	21.3
South Africa	5,770.9	22.8
Malaysia	5,824.4	18.6
Greece	6,232.5	25.8
Mexico	6,968.8	13-8
Venezuela	7,672.1	11.4
Trinidad and Tobago	11,161.0	15.1
Average for group	*6,165.5*	*19.5*
High-income countries		
Ireland	7,170.9	22.0
Spain	7,477.8	20.7
Israel	10,572.9	16.9
Austria	11,147.3	23.3
United Kingdom	11,462.6	15.2
Italy	11,613.1	25.7
Belgium	11.675.1	23.2
Japan	11,819.9	25.5
Netherlands	12,013.8	24.9
Finland	12,019.5	19.4
France	12,775.6	19.1
Australia	13,841.5	18.8
Switzerland	16,079.1	23.5
Canada	16,529.3	21.5
United States	18,194.5	16.4
Average for group	*12,292.9*	*21.1*

A.2 The Old Faithful Dataset

Table A.2 Eruption Lengths (in Minutes) of 107 Eruptions of Old Faithful Geyser[203]

4.37	3.87	4.00	4.03	3.50	4.08	2.25
4.70	1.73	4.93	1.73	4.62	3.43	4.25
1.68	3.92	3.68	3.10	4.03	1.77	4.08
1.75	3.20	1.85	4.62	1.97	4.50	3.92
4.35	2.33	3.83	1.88	4.60	1.80	4.73
1.77	4.57	1.85	3.52	4.00	3.70	3.72
4.25	3.58	3.80	3.77	3.75	2.50	4.50
4.10	3.70	3.80	3.43	4.00	2.27	4.40
4.25	3.58	3.80	3.77	3.75	2.50	4.50
4.10	3.70	3.80	3.43	4.00	2.27	4.40
4.05	4.25	3.33	2.00	4.33	2.93	4.58
1.90	3.58	3.73	3.73	1.82	4.63	3.50
4.00	3.67	1.67	4.60	1.67	4.00	1.80
4.42	1.90	4.63	2.93	3.50	1.97	4.28
1.83	4.13	1.83	4.65	4.20	3.93	4.33
1.83	4.53	2.03	4.18	4.43	4.07	4.13
3.95	4.10	2.72	4.58	1.90	4.50	1.95
4.83	4.12					

[203] Source: Silverman, 1986

A.3 Stationary Distributions of the Markov Chain Analysis

A.3.1 General Aspects of Stationary Distributions

The Markov chain may indeed have an implicit stationary distribution. This means that the Markov process shows already in its structure that a specific distribution will be achieved in the long run and does not come as a surprise instead. Basically, it can be said that "for any irreducible and aperiodic Markov chain, there exists at least one stationary distribution" (Häggström, 2002, p. 29). A vector $\vec{x}$ is called a stationary distribution of a Markov chain with the transition matrix M if the following holds:

$$x_j = \sum_i x_i p_{ij}, \quad \text{(A.1)}$$

and in form of a matrix:

$$M \cdot \vec{x} = \vec{x}. \quad \text{(A.2)}$$

This can be interpreted as follows: if the Markov chain has the distribution as indicated by $\vec{x}$ at a specific point of time n, then it will have the same distribution also in the future (Ching and Ng, 2006).

In this doctoral thesis – in the more theoretical Chapter 4 as well as in the application of the Markov chains in Chapter 5 – the Markov chain is assumed to be irreducible and non-empty recurrent, also called positive recurrent. "State i is said to be positive recurrent if it is recurrent and starting in state i the expected time until the process returns to state i is finite" (Ching and Ng, 2006, p. 14). A set is irreducible if the probability of reaching the status i sometime after status j is positive for all $i, j \in S$ (Langrock and Jahn, 1979). From the transition matrices in Chapter 5, it can be seen that they are indeed nonreducible and closed so that the conditions for having a stationary distribution are fulfilled. Additionally, the Markov chains are assumed to be aperiodic. "A state i is said to have period d if $[M_{ii}^{(n)} = 0]$ whenever n is not divisible by d, and d is the largest integer with this property. A state with period 1 is said to be aperiodic" (Ching and Ng, 2006, p. 14). In general, it can be stated that "for any irreducible and aperiodic Markov chain having k states, there exists at least one stationary distribution" (Ching and Ng, 2006, p. 15). This holds in this doctorate, so it is not surprising that the Markov chain analysis indeed comes to the stable distributions in the long run as shown in Table 5.11.

A.3.2 Stationary Distributions of the Markov Chains of Chapter 5

As stated before, the Markov chain of the form used in this doctoral thesis has an implicit stationary distribution. By looking at the long run distributions found in Table 5.11 (see page 111), it becomes obvious that there are indeed stationary distributions. The values of this table are calculated by ever repeating the multiplication of the transition matrix with the distribution vector. By keeping in mind that there are rounding errors (the transition probabilities are rounded to the third decimal, the distribution vector is rounded to full numbers), stabilization can be read off. In this Appendix, the stabilization shall be examined for the first example, hence for having the year 1988 as the target year in the transition matrix (see Table A.17, p. 279).

$$M\vec{x} = \overrightarrow{x_n} = \vec{x}^{204} \tag{A.3}$$

$$M = \begin{bmatrix} 0.857 & 0.520 & 0.074 & 0 & 0 & 0 \\ 0.143 & 0.320 & 0.370 & 0.042 & 0 & 0 \\ 0 & 0.160 & 0.407 & 0.250 & 0 & 0 \\ 0 & 0 & 0.148 & 0.417 & 0.143 & 0 \\ 0 & 0 & 0 & 0.250 & 0.714 & 0.500 \\ 0 & 0 & 0 & 0.042 & 0.143 & 0.500 \end{bmatrix} \tag{A.4}$$

$$\vec{\mathrm{x}} = \begin{bmatrix} 68 \\ 18 \\ 6 \\ 3 \\ 7 \\ 2 \end{bmatrix} \tag{A.5}$$

$$\begin{bmatrix} 0.857 & 0.500 & 0.074 & 0 & 0 & 0 \\ 0.143 & 0.320 & 0.370 & 0.042 & 0 & 0 \\ 0 & 0.160 & 0.407 & 0.250 & 0 & 0 \\ 0 & 0 & 0.148 & 0.417 & 0.143 & 0 \\ 0 & 0 & 0 & 0.250 & 0.714 & 0.500 \\ 0 & 0 & 0 & 0.042 & 0.143 & 0.500 \end{bmatrix} \begin{bmatrix} 68 \\ 18 \\ 6 \\ 3 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \tag{A.6}$$

$$x_1 = 0.857 \cdot 68 + 0.500 \cdot 18 + 0.074 \cdot 6 = 67.720 \approx 68 \tag{A.7}$$

$$x_2 = 0.143 \cdot 68 + 0.320 \cdot 18 + 0.370 \cdot 6 + 0.042 \cdot 4 = 17.872 \approx 18 \tag{A.8}$$

$$x_3 = 0.160 \cdot 18 + 0.407 \cdot 6 + 0.250 \cdot 3 = 6.072 \approx 6 \tag{A.9}$$

$$x_4 = 0.148 \cdot 6 + 0.417 \cdot 3 + 0.143 \cdot 7 = 3.140 \approx 3 \tag{A.10}$$

$$x_5 = 0.250 \cdot 3 + 0.714 \cdot 7 + 0.500 \cdot 2 = 6.748 \approx 7 \tag{A.11}$$

$$x_6 = 0.042 \cdot 3 + 0.143 \cdot 7 + 0.500 \cdot 2 = 2.127 \approx 2 \tag{A.12}$$

From these six equations, it can be read off what the “new” distribution vector looks like:

[204] $\vec{x}$ is the vector of the distribution in the year of stabilization taking the Markov chain with the target year 1988 as shown in Table 5.11.

$$\overrightarrow{x_n} = \begin{bmatrix} 68 \\ 18 \\ 6 \\ 3 \\ 7 \\ 2 \end{bmatrix} = \vec{x} \qquad \text{q. e. d.} \qquad \text{(A.13)}$$

Hence, it is indeed shown that the Markov chain implicitly yields a stationary distribution just as indicated above.

A.4 Sensitivity Analysis on Human Capital

As already mentioned in Chapter 5, human capital can be measured in several ways. It was decided to use the average years of schooling as an approximation for human capital in this doctorate. However, also other variables are possible. One variable mentioned before is public spending on education. Before the descriptive statistics as well as the kernel densities for this variable will be presented, the shortages of that variable will be discussed briefly. First of all, it should be kept in mind that there are countries in which education is not exclusively provided by the public sector or, in other words, in which private spending on education plays a decisive role. In such a case, having only data on public spending on education means that a large part of the spending and hence a large part of human capital is not in the data. For this reason, this variable is not really a good alternative. Furthermore, looking at the data does not really improve the data quality. Data are provided by the World Development Indicators, provided by the World Bank (for example 2009). These data are not available before 1970, and thereafter they are only attainable for every fifth year, hence 1970, 1975, and so on. From 1998 on until 2006, there are data available on a yearly basis. Table A.4 shows the descriptive statistics for the variable in all years covered by the dataset.

What becomes obvious from the descriptive statistics is that first of all, data are not available before 1970, at least by using the World Development Indicators only from one edition. The reasoning behind doing so is that the values might not be adapted to changes in the measurement and hence might not be sufficiently reliable if there were changes. Furthermore, the number of observations varies a lot between zero in 1997, for example, or very low numbers in other years (for example only four countries offering data in 1993) and very high numbers such as 122 countries in the year 1999. Hence, if one wanted to include only those countries offering data in all years under consideration, then no countries at all would remain in the dataset. Yet, this was the rule applied in the chapter so far.

Even if taking into account only every fifth year, as in the Barro-Lee dataset, then there would be far too few countries in the dataset, namely only 13.[205] In addition, the sample of the 13 countries is not really a good cross section looking at the real per capita income distribution.[206] Nevertheless, for completeness, here the kernel densities will be presented for all years covered by the dataset for nonoil countries in Figure A.4. A look at the kernel densities shows again that in some years there are no data at all and if so, sometimes there are just too few countries in the dataset to get a good result. Looking at the kernels of every fifth year should then be

[205] See Appendix (A.6) for an overview of the countries in the complete dataset (excluding the oil-producing countries as outlined in Chapter 5) as well as of those 13 countries which would remain in the dataset if only looking at those countries offering data in every fifth year starting in 1970.

[206] Starting in 1975 and again only considering countries which offer data in every fifth year would increase the number of countries to 24. The sample improves as a cross section of the income distribution. Yet, the sample is still much too small to yield reliable conclusions.

comparable to the human capital measurement used in Chapter 5. This is shown in Figure A.2. The reasoning behind it is that here, many countries are in the dataset. Yet, all countries providing data in each year separately are used for the kernels. Just as for the average years of schooling the distribution of the data on public spending on education is unipeaked. No twin peaks can be found at all, so that the conclusions to be drawn from using this indicator instead are not different from the ones drawn in Chapter 5. For all those reasons it was decided that the average years of schooling provided by the Barro-Lee dataset are the better choice as human capital variable.

Table A.4 Descriptive Statistics for Public Spending on Education

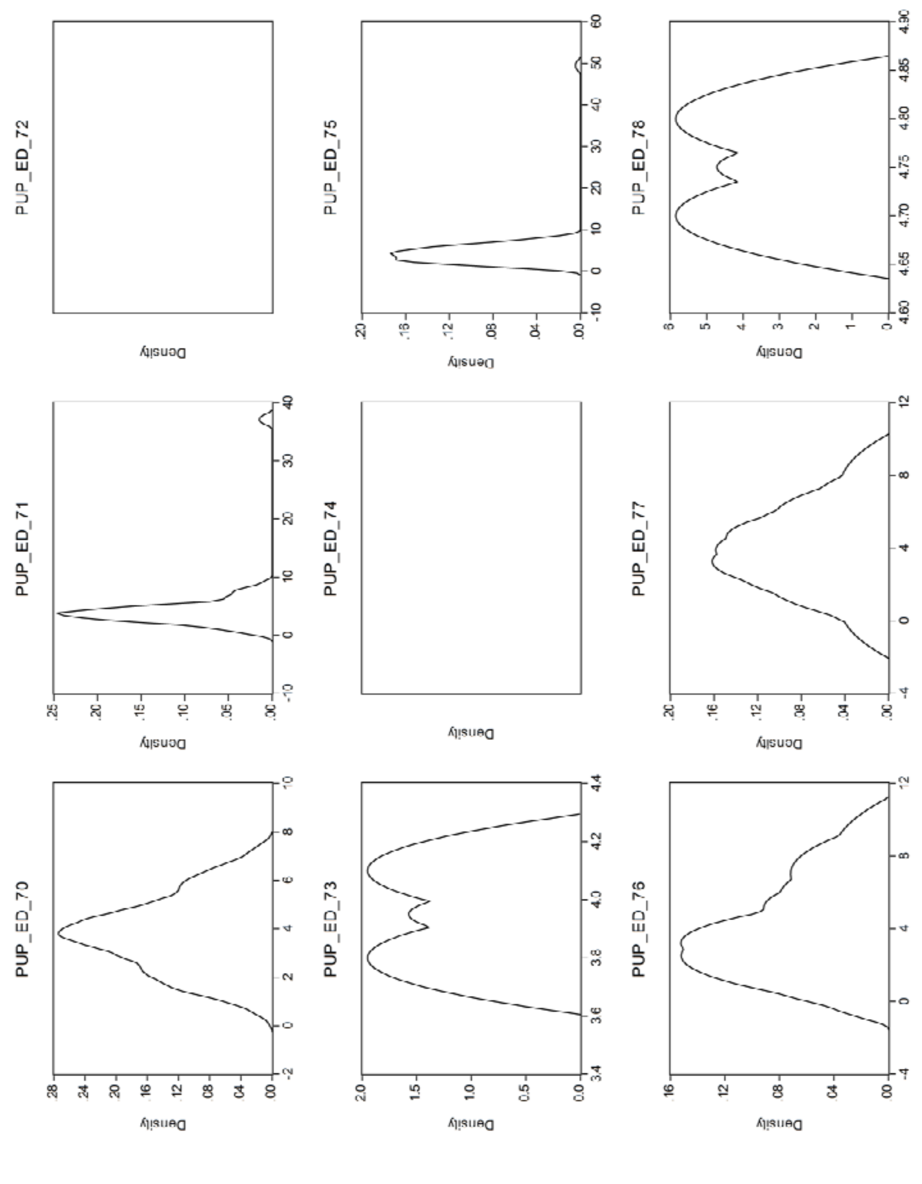
	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980
Number of observations	57	38	1	2	1	91	6	6	2	4	92
Mean ($)	3.782	4.774	3.300	3.950	3.000	4.582	4.033	3.950	4.750	5.400	4.232
Median ($)	3.800	3.800	3.300	3.950	3.000	3.900	3.250	3.800	4.750	5.400	4.250
Maximum ($)	6.800	37.100	3.300	4.100	3.000	49.500	8.100	7.400	4.800	7.000	11.900
Minimum ($)	1.000	0.600	3.300	3.800	3.000	1.100	1.700	0.800	4.700	3.800	0.900
Standard Deviation	1.453	5.659	NA	0.212	NA	5.084	2.525	2.247	0.071	1.585	1.966

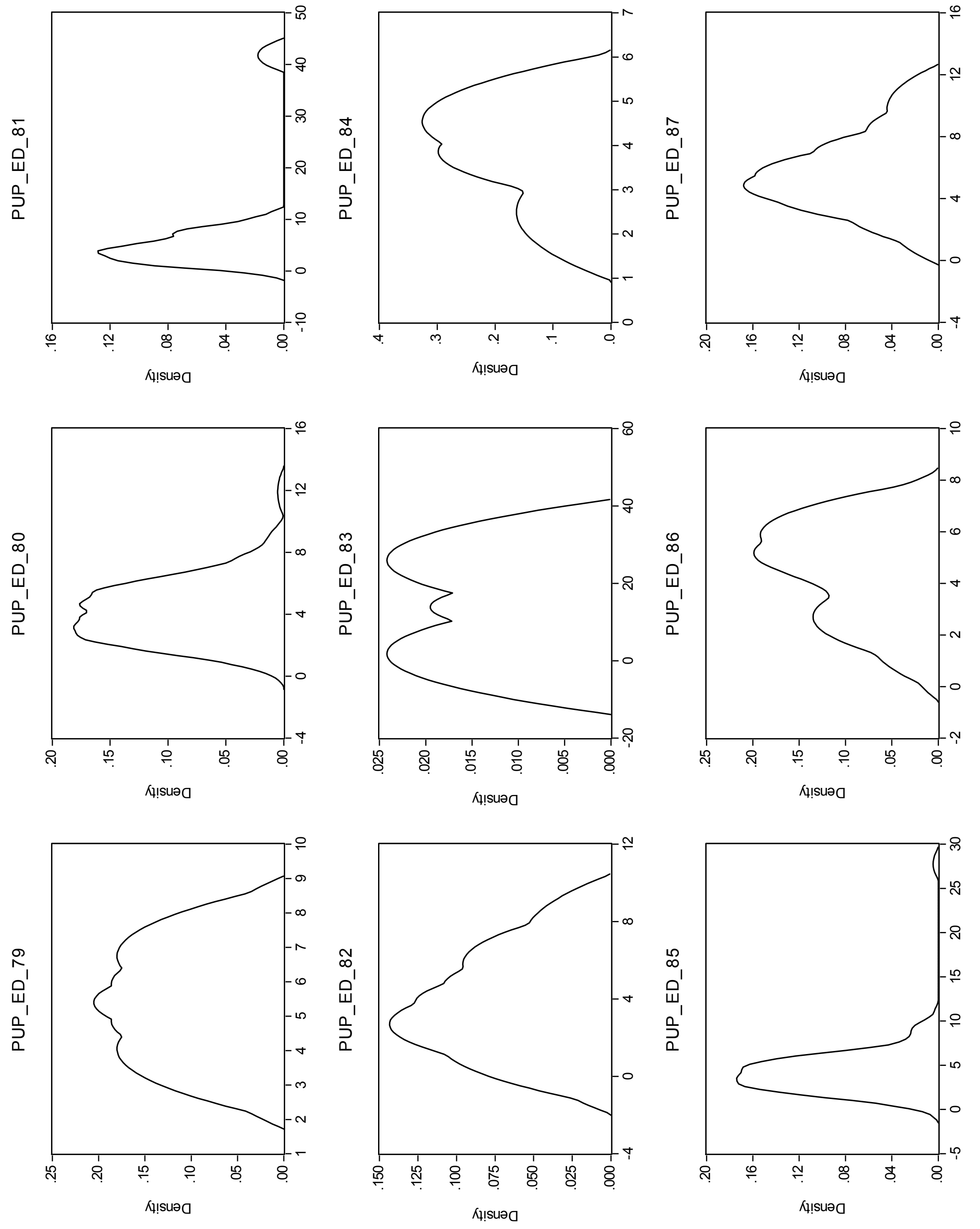
	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
Number of observations	13	4	2	3	100	9	6	1	7	113	0
Mean ($)	7.338	3.800	13.900	3.867	4.402	4.467	5.683	1.600	4.229	4.165	NA
Median ($)	3.400	3.350	13.900	4.500	3.900	5.600	5.450	1.600	4.400	3.900	NA
Maximum ($)	41.800	7.100	26.000	4.600	27.800	6.300	9.800	1.600	5.300	9.500	NA
Minimum ($)	1.500	1.400	1.800	2.500	0.400	1.600	2.600	1.600	3.200	0.900	NA
Standard Deviation	10.623	2.563	17.112	1.185	3.109	1.780	2.454	NA	0.844	1.823	NA

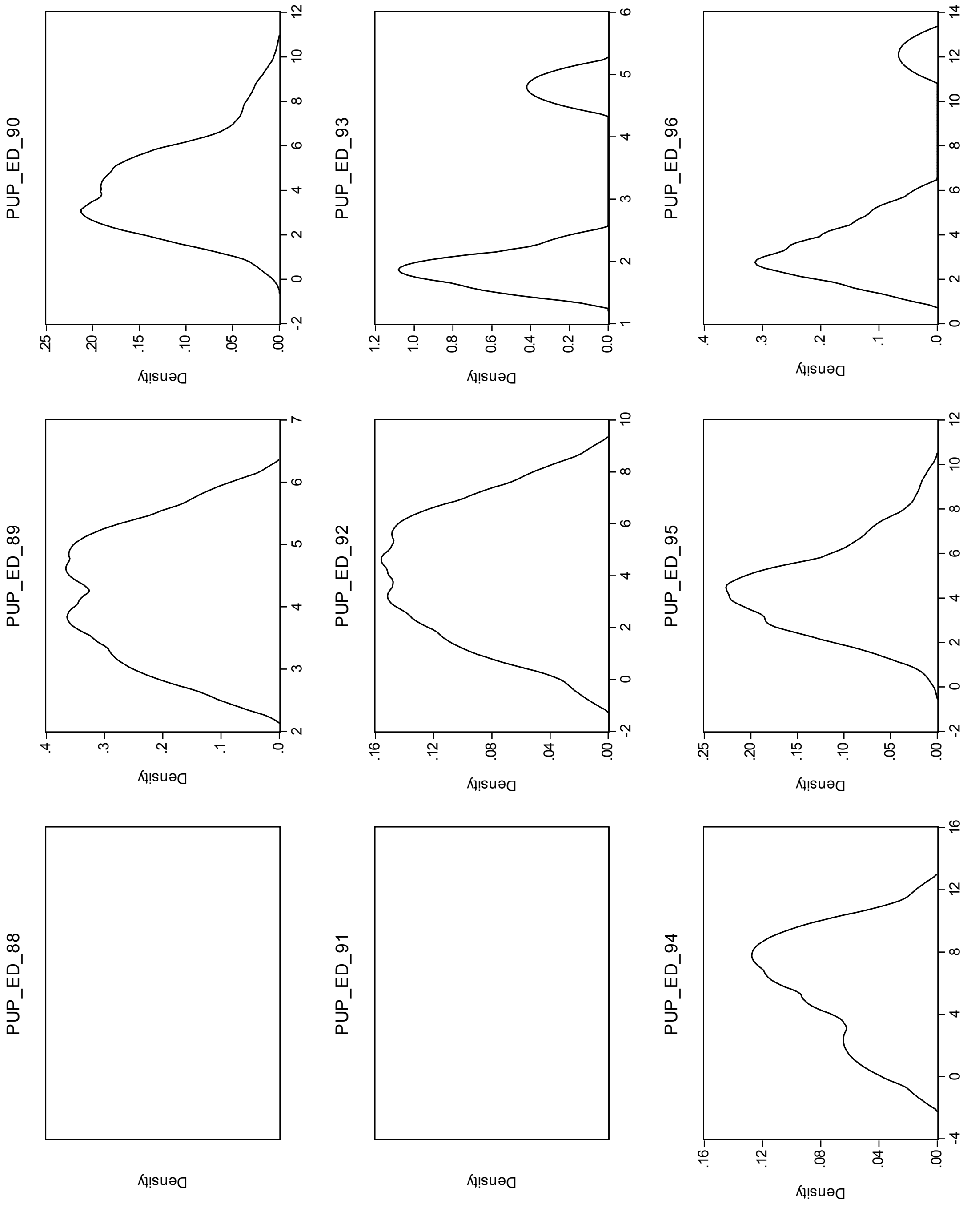
	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Number of observations	7	4	6	117	9	0	98	122	114	114	115
Mean ($)	4.143	2.600	5.983	4.406	4.256	NA	4.550	4.676	4.776	4.944	5.015
Median ($)	4.400	1.950	7.150	4.400	3.200	NA	4.400	4.500	4.450	4.700	4.700
Maximum ($)	6.800	4.800	9.200	9.200	12.100	NA	16.500	16.500	15.300	16.800	17.800
Minimum ($)	1.300	1.700	1.600	0.800	2.000	NA	1.300	0.600	0.600	0.600	0.600
Standard Deviation	2.019	1.476	2.955	1.750	3.119	NA	2.263	2.295	2.399	2.322	2.231

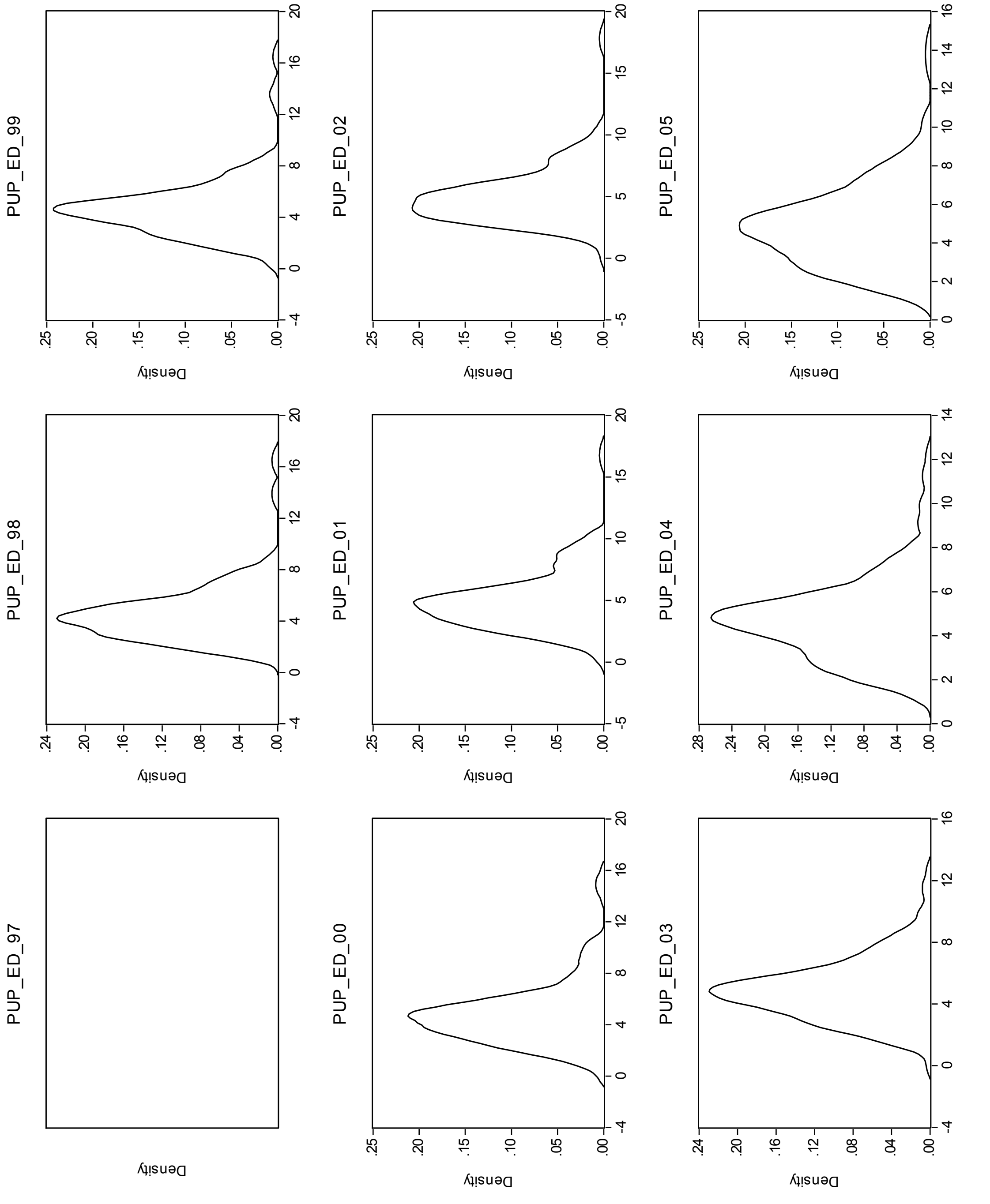
	2003	2004	2005	2006	2007
Number of observations	111	112	103	88	47
Mean ($)	4.926	4.737	4.862	4.794	4.500
Median ($)	4.700	4.600	4.800	4.700	3.900
Maximum ($)	12.100	11.800	13.800	13.300	13.300
Minimum ($)	0.600	1.600	1.700	1.200	1.500
Standard Deviation	1.977	1.841	2.038	1.924	2.096

Figure A.1 The Kernel Densities for Public Spending on Education

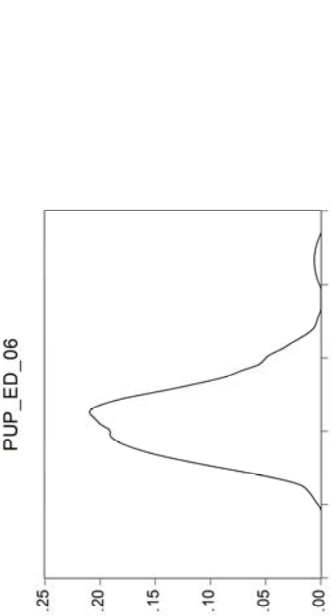


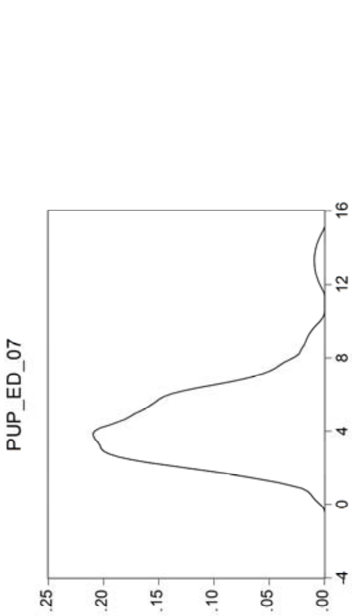

PUP_ED_79
PUP_ED_80
PUP_ED_81
PUP_ED_82
PUP_ED_83
PUP_ED_84
PUP_ED_85
PUP_ED_86
PUP_ED_87
Density


PUP_ED_88
PUP_ED_89
PUP_ED_90
PUP_ED_91
PUP_ED_92
PUP_ED_93
PUP_ED_94
PUP_ED_95
PUP_ED_96
Density



PUP_ED_97
PUP_ED_98
PUP_ED_99
PUP_ED_00
PUP_ED_01
PUP_ED_02
PUP_ED_03
PUP_ED_04
PUP_ED_05
Density


PUP_ED_06
Density
.25
.20
.15
.10
.05
.00
-4
0
4
8
12
16


PUP_ED_07
Density
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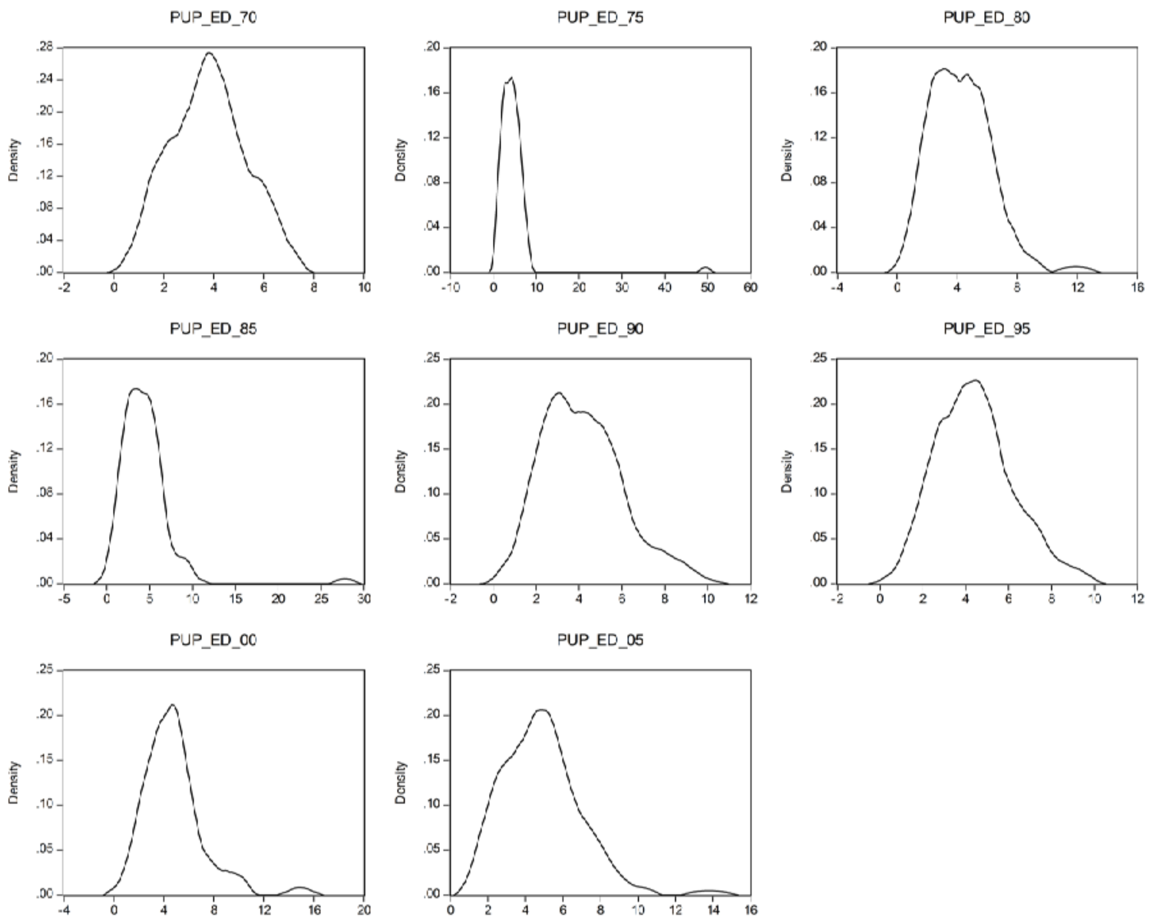
.20
.15
.10
.05
.00
-4
0
4
8
12
16

Figure A.2 The Kernel Densities for Public Spending on Education (Every Fifth Year)



A.5 The Countries Covered by the Datasets[207]

Table A.5 The Nonoil Countries Covered by the Penn World Table 6.3 with Different Starting Years[208]

Total non-oil countries	Starting 1950	Starting 1960	Starting 1970
Afghanistan	Argentina	Argentina	Afghanistan
Albania	Australia	Australia	Albania
Angola	Austria	Austria	Angola
Antigua and Barbuda	Belgium	Bangladesh	Antigua and Barbuda
Argentina	Bolivia	Barbados	Argentina
Armenia	Brazil	Belgium	Australia
Australia	Canada	Benin	Austria
Austria	Colombia	Bolivia	Bahamas
Azerbaijan	Congo, Dem. Rep.	Brazil	Bangladesh
Bahamas	Costa Rica	Burkina Faso	Barbados
Bahrain	Cyprus	Burundi	Belgium
Bangladesh	Denmark	Cameroon	Belize
Barbados	Egypt	Canada	Benin
Belarus	El Salvador	Cape Verde	Bermuda
Belgium	Ethiopia	Central African Republic	Bhutan
Belize	Finland	Chad	Bolivia
Benin	France	Chile	
Bermuda	Guatemala	China	Brazil
Bhutan	Honduras	Colombia	Bulgaria
Bolivia	Iceland	Comoros	Burkina Faso
Bosnia and Herzegovina	India	Congo, Dem. Rep.	Burundi
Brazil	Ireland	Congo, Republic of	Cambodia
Bulgaria	Israel	Costa Rica	Cameroon
Burkina Faso	Italy	Cote d`Ivoire	Canada
Burundi	Japan	Cyprus	Cape Verde
Cambodia	Kenya	Denmark	Central African Republic
Cameroon	Luxembourg	Dominican Republic	Chad
Canada	Mauritius	Ecuador	Chile

207 The names of the countries covered by the different datasets in this section might differ slightly. It was decided to report the names as they appear in the respective dataset.

208 These countries provide data on real per capita GDP as well as the investment rate and the population growth rate.

Cape Verde	Morocco	Egypt	China
Central African Republic	Netherlands	El Salvador	Colombia
Chad	New Zealand	Equatorial Guinea	Comoros
Chile	Nicaragua	Ethiopia	Congo, Dem. Rep.
China	Nigeria	Fiji	Congo, Republic of
Colombia	Pakistan	Finland	Costa Rica
Comoros	Panama	France	Cote d`Ivoire
Congo, Dem. Rep.	Peru	Gabon	Cuba
Congo, Republic of	Philippines	Gambia, The	Cyprus
Costa Rica	Portugal	Ghana	Denmark
Cote d`Ivoire	Puerto Rico	Greece	Djibouti
Croatia	South Africa	Guatemala	Dominica
Cuba	Spain	Guinea	Dominican Republic
Cyprus	Sri Lanka	Guinea-Bissau	Ecuador
Czech Republic	Sweden	Haiti	Egypt
Denmark	Switzerland	Honduras	El Salvador
Djibouti	Thailand	Hong Kong	Equatorial Guinea
Dominica	Turkey	Iceland	Ethiopia
Dominican Republic	Uganda	India	Fiji
Ecuador	United Kingdom	Indonesia	Finland
Egypt	United States	Iran	France
El Salvador	Uruguay	Ireland	Gabon
Equatorial Guinea	Venezuela	Israel	Gambia, The
Eritrea		Italy	Germany
Estonia	**52 countries**	Jamaica	Ghana
Ethiopia		Japan	Greece
Fiji		Jordan	Grenada
Finland		Kenya	Guatemala
France		Korea, Republic of	Guinea
Gabon		Lesotho	Guinea-Bissau
Gambia, The		Luxembourg	Guyana
Georgia		Madagascar	Haiti
Germany		Malawi	Honduras
Ghana		Malaysia	Hong Kong
Greece		Mali	Hungary
Grenada		Mauritania	Iceland
Guatemala		Mauritius	India
Guinea		Mexico	Indonesia
Guinea-Bissau		Morocco	Iran
Guyana		Mozambique	Ireland

Haiti		Namibia	Israel
Honduras		Nepal	Italy
Hong Kong		Netherlands	Jamaica
Hungary		New Zealand	Japan
Iceland		Nicaragua	Jordan
India		Niger	Kenya
Indonesia		Nigeria	Kiribati
Iran		Pakistan	Korea, Republic of
Ireland		Panama	Laos
Israel		Papua New Guinea	Lebanon
Italy		Paraguay	Lesotho
Jamaica		Peru	Liberia
Japan		Philippines	Luxembourg
Jordan		Portugal	Madagascar
Kazakhstan		Puerto Rico	Malawi
Kenya		Romania	Malaysia
Kiribati		Rwanda	Maldives
Korea, Republic of		Senegal	Mali
Kyrgyzstan		Seychelles	Malta
Laos		Singapore	Marshall Islands
Latvia		South Africa	Mauritania
Lebanon		Spain	Mauritius
Lesotho		Sri Lanka	Mexico
Liberia		Sweden	Micronesia, Fed. Sts.
Lithuania		Switzerland	Mongolia
Luxembourg		Syria	Morocco
Macedonia		Taiwan	Mozambique
Madagascar		Tanzania	Namibia
Malawi		Thailand	Nepal
Malaysia		Togo	Netherlands
Maldives		Turkey	New Zealand
Mali		Uganda	Nicaragua
Malta		United Kingdom	Niger
Marshall Islands		United States	Nigeria
Mauritania		Uruguay	Pakistan
Mauritius		Zambia	Palau
Mexico		Zimbabwe	Panama
Micronesia, Fed. Sts.			Papua New Guinea
Moldova		**105 countries**	Paraguay
Mongolia			Peru
Montenegro			Philippines
Morocco			Poland

Mozambique			Portugal
Namibia			Puerto Rico
Nepal			Romania
Netherlands			Rwanda
New Zealand			Samoa
Nicaragua			Sao Tome and Principe
Niger			Senegal
Nigeria			Seychelles
Pakistan			Sierra Leone
Palau			Singapore
Panama			Solomon Islands
Papua New Guinea			Somalia
Paraguay			South Africa
Peru			Spain
Philippines			Sri Lanka
Poland			St. Kitts & Nevis
Portugal			St. Lucia
Puerto Rico			St.Vincent & Grenadines
Romania			Sudan
Russia			Suriname
Rwanda			Swaziland
Samoa			Sweden
Sao Tome and Principe			Switzerland
Senegal			Syria
Serbia			Taiwan
Seychelles			Tanzania
Sierra Leone			Thailand
Singapore			Togo
Slovak Republic			Tonga
Slovenia			Tunisia
Solomon Islands			Turkey
Somalia			Uganda
South Africa			United Kingdom
Spain			United States
Sri Lanka			Uruguay
St. Kitts & Nevis			Vanuatu
St. Lucia			Vietnam
St.Vincent & Grenadines			Zambia
Sudan			Zimbabwe
Suriname			**149 countries**

Swaziland			
Sweden			
Switzerland			
Syria			
Taiwan			
Tajikistan			
Tanzania			
Thailand			
Timor-Leste			
Togo			
Tonga			
Tunisia			
Turkey			
Turkmenistan			
Uganda			
Ukraine			
United Kingdom			
United States			
Uruguay			
Uzbekistan			
Vanuatu			
Vietnam			
Yemen			
Zambia			
Zimbabwe			
175 countries			

Table A.6 The Nonoil Countries Covered by the Barro-Lee Dataset

Complete Dataset	Countries offering data in all years	Dataset in all years (human capital and income)
Afghanistan	Afghanistan	Argentina
Antigua & Barb.	Argentina	Australia
Argentina	Australia	Austria
Australia	Austria	Bangladesh
Austria	Bahrain	Barbados
Bahrain	Bangladesh	Belgium
Bangladesh	Barbados	Bolivia
Barbados	Belgium	Brazil
Belgium	Bolivia	Cameroon
Belize	Brazil	Canada
Benin	Bulgaria	Central Afr. R.
Bolivia	Cameroon	Cuba
Brazil	Canada	Dominican Rep.
Bulgaria	Central Afr. R.	Fiji
Burma	Cuba	Finland
Burundi	Cyprus	Germany, West
Cameroon	Czech Republic	Ghana
Canada	Dominican Rep.	Greece
Central Afr. R.	Fiji	Guyana
Chile	Finland	Haiti
China	Germany, West	Honduras
Colombia	Ghana	Hungary
Congo	Greece	Iceland
Costa Rica	Guyana	India
Croatia	Haiti	Indonesia
Cuba	Honduras	Ireland
Cyprus	Hong Kong	Israel
Czech Republic	Hungary	Italy
Denmark	Iceland	Jamaica
Dominica	India	Japan
Dominican Rep.	Indonesia	Kenya
Ecuador	Iran, I.R. of	Malawi
Egypt	Iraq	Malaysia
El Salvador	Ireland	Mauritius
Estonia	Israel	Nepal
Fiji	Italy	Netherlands
Finland	Jamaica	New Zealand

France	Japan	Nicaragua
Gambia	Kenya	Niger
Germany, West	Lesotho	Pakistan
Ghana	Malawi	Panama
Greece	Malaysia	Papua New Guin.
Guatemala	Mauritius	Paraguay
Guinea-Bissau	Mozambique	Peru
Guyana	Nepal	Philippines
Haiti	Netherlands	Portugal
Honduras	New Zealand	Romania
Hong Kong	Nicaragua	Senegal
Hungary	Niger	Singapore
Iceland	Pakistan	South Africa
India	Panama	Spain
Indonesia	Papua New Guin.	Sri Lanka
Iran, I.R. of	Paraguay	Sweden
Iraq	Peru	Switzerland
Ireland	Philippines	Syria
Israel	Poland	Taiwan
Italy	Portugal	Thailand
Jamaica	Romania	Togo
Japan	Senegal	Turkey
Jordan	Sierra Leone	Uganda
Kazakhstan	Singapore	United Kingdom
Kenya	South Africa	United States
Korea	Spain	Uruguay
Latvia	Sri Lanka	Zambia
Lesotho	Sudan	Zimbabwe
Liberia	Swaziland	
Lithuania	Sweden	**65 countries**
Malawi	Switzerland	
Malaysia	Syria	
Mali	Taiwan	
Mauritania	Tanzania	
Mauritius	Thailand	
Mexico	Togo	
Moldova	Tunisia	
Mozambique	Turkey	
Namibia	U.S.S.R.	
Nepal	Uganda	
Netherlands	United Kingdom	
New Zealand	United States	

Nicaragua	Uruguay	
Niger	Zaire	
Pakistan	Zambia	
Panama	Zimbabwe	
Papua New Guin.		
Paraguay	**82 countries**	
Peru		
Philippines		
Poland		
Portugal		
Reunion		
Romania		
Rwanda		
Senegal		
Seychelles		
Sierra Leone		
Singapore		
Slovakia		
Slovenia		
Solomon Islands		
South Africa		
Spain		
Sri Lanka		
St.Kitts& Nevis		
St.Lucia		
St.Vincent & G.		
Sudan		
Swaziland		
Sweden		
Switzerland		
Syria		
Taiwan		
Tajikistan		
Tanzania		
Thailand		
Togo		
Tunisia		
Turkey		
U.S.S.R.		
Uganda		
United Kingdom		
United States		

Uruguay		
Vanuatu		
Viet Nam		
Western Samoa		
Yemen, N.Arab		
Yugoslavia		
Zaire		
Zambia		
Zimbabwe		
129 countries		

Table A.7 The Nonoil Countries Covered by the World Development Indicators

Complete Dataset	**Countries offering data in every fifth year (starting in 1970)**
Afghanistan	Chile
Andorra	Colombia
Angola	Hungary
Antigua & Barb.	Iceland
Argentina	Malaysia
Armenia	Mexico
Aruba	Morocco
Australia	Portugal
Austria	Spain
Azerbaijan	St. Kitts and Nevis
Bahamas, The	Switzerland
Bahrain	Ukraine
Bangladesh	Zambia
Barbados	
Belarus	**13 countries**
Belgium	
Belize	
Benin	
Bermuda	
Bhutan	
Bolivia	
Brazil	
Bulgaria	
Burkina Faso	
Burundi	
Cambodia	
Cameroon	
Canada	
Cape Verde	
Cayman Islands	
Central African Republic	
Chad	
Channel Islands	
Chile	
China	
Colombia	
Comoros	

Congo	
Costa Rica	
Cote d'Ivoire	
Croatia	
Cuba	
Cyprus	
Czech Republic	
Denmark	
Djibouti	
Dominica	
Dominican Rep.	
Ecuador	
Egypt	
El Salvador	
Equatorial Guinea	
Eritrea	
Estonia	
Ethiopia	
Faeroer Islands	
Fiji	
Finland	
France	
Gabon	
Gambia	
Georgia	
Germany	
Ghana	
Greece	
Grenada	
Guatemala	
Guinea	
Guinea-Bissau	
Guyana	
Haiti	
Honduras	
Hong Kong	
Hungary	
Iceland	
India	
Indonesia	
Iran, I.R. of	
Ireland	

Isle of Man	
Israel	
Italy	
Jamaica	
Japan	
Jordan	
Kazakhstan	
Kenya	
Kiribati	
Korea	
Kyrgyz Republic	
Lao PDR	
Latvia	
Lebanon	
Lesotho	
Liberia	
Lithuania	
Luxembourg	
Macedonia FYR	
Madagascar	
Malawi	
Malaysia	
Maldives	
Mali	
Malta	
Marshall Islands	
Mauritania	
Mauritius	
Mexico	
Micronesia Fed. Sts.	
Moldova	
Morocco	
Mozambique	
Myanmar	
Namibia	
Nepal	
Netherlands	
New Zealand	
Nicaragua	
Niger	
Nigeria	
Pakistan	

Palau	
Panama	
Papua New Guinea	
Paraguay	
Peru	
Philippines	
Poland	
Portugal	
Romania	
Russian Federation	
Rwanda	
Samoa	
San Marino	
Sao Tome and Principe	
Senegal	
Serbia	
Seychelles	
Sierra Leone	
Singapore	
Slovakia	
Slovenia	
Solomon Islands	
Somalia	
South Africa	
Spain	
Sri Lanka	
St.Kitts& Nevis	
St.Lucia	
St.Vincent & the Grenadines	
Sudan	
Suriname	
Swaziland	
Sweden	
Switzerland	
Syria	
Tajikistan	
Tanzania	
Thailand	
Togo	
Tonga	
Tunisia	
Turkey	

Turkmenistan	
Uganda	
Ukraine	
United Kingdom	
United States	
Uruguay	
Uzbekistan	
Vanuatu	
Vietnam	
Yemen, Rep.	
Yugoslavia	
Zambia	
Zimbabwe	
177 countries	

A.6 Descriptive Statistics for the Complete Dataset

Table A.7 Descriptive Statistics for Real per Capita GDP

	1950	1951	1952	1953	1954	1955	1956	1957	1958	1959	1960	1961
Number of observations	51	57	59	61	64	68	68	68	68	72	106	108
Mean ($)	4716.31	4632.41	4502.59	4572.48	4578.94	4589.25	4723.65	4827.08	4835.04	4780.76	3980.34	4071.11
Median ($)	3343.62	3451.24	3308.89	3271.71	3339.82	3203.40	3412.25	3533.96	3491.06	3303.28	2380.07	2315.87
Maximum ($)	15058.47	13810.54	14678.17	15162.00	14861.16	15396.52	16300.78	17057.24	16996.57	16939.99	18102.63	19133.77
Minimum ($)	648.33	720.90	375.76	393.09	395.68	406.84	448.15	458.06	500.32	505.99	418.64	432.28
Standard Deviation	3634.56	3521.88	3517.22	3614.59	3686.02	3826.62	3926.56	3990.56	3942.57	4056.69	3905.58	4021.35

	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973
Number of observations	108	108	108	108	108	108	108	108	150	150	150	150
Mean ($)	4220.05	4370.03	4582.92	4727.20	4855.84	4970.50	5146.85	5406.50	5619.91	5789.68	5996.02	6313.80
Median ($)	2457.93	2512.57	2610.72	2580.30	2653.05	2764.66	2897.05	2962.06	2955.06	3132.06	3118.71	3575.97
Maximum ($)	19512.62	19942.55	20823.82	21012.32	21320.04	21796.38	22294.79	23178.56	28309.82	27232.80	29949.97	35692.40
Minimum ($)	417.14	447.44	469.69	480.77	399.17	439.91	443.03	439.25	336.48	326.14	365.40	388.56
Standard Deviation	4138.15	4266.25	4540.32	4700.09	4827.65	4897.39	5084.22	5410.25	5958.99	6074.20	6330.12	6789.04

	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985
Number of observations	150	150	150	150	150	150	150	150	150	150	150	150
Mean ($)	6535.74	6455.82	6542.88	6711.44	6878.76	7061.38	7021.13	6997.33	6978.65	7037.76	7224.37	7327.19
Median ($)	3765.28	3841.26	4011.90	4022.91	4108.13	4238.99	4385.21	4479.72	4346.94	4337.90	4557.07	4600.71
Maximum ($)	51302.88	53713.12	39042.92	36963.47	37071.21	33358.89	33093.42	31314.62	32250.03	32392.11	32418.93	33552.81
Minimum ($)	482.06	617.27	652.50	549.06	635.58	609.25	495.44	639.25	593.15	564.16	594.05	595.13
Standard Deviation	7384.50	7292.57	6930.28	7012.55	7112.14	7266.59	7172.97	7039.32	7021.02	7115.24	7388.18	7530.71

	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
Number of observations	150	151	151	152	161	161	163	173	174	174	174	174
Mean ($)	7505.92	7761.33	8009.34	8151.14	8310.74	8279.08	8160.17	8055.86	8207.65	8396.58	8634.52	8925.80
Median ($)	4680.46	4736.30	4909.80	4763.15	4807.79	4920.88	4931.97	4977.48	4895.16	5071.59	5052.88	5192.60
Maximum ($)	34333.32	35774.02	38276.22	41965.96	43562.70	47177.97	47083.22	48521.08	49896.16	49723.61	49563.98	52066.05
Minimum ($)	618.31	514.84	440.64	498.74	510.60	489.82	314.53	203.36	160.35	153.44	163.97	309.48
Standard Deviation	7757.89	8027.48	8343.54	8646.37	8620.55	8677.65	9651.28	8530.89	8784.45	8979.95	9173.16	9522.43

	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
Number of observations	174	174	175	174	174	174	174	176	174	173
Mean ($)	9133.05	9388.99	9713.29	9484.04	10008.27	10210.70	10640.35	10969.34	11513.32	11894.93
Median ($)	5311.97	5356.79	5577.72	5338.99	5496.268	5729.28	5993.55	6281.94	6868.51	7030.65
Maximum ($)	55109.43	59336.08	63419.39	64202.51	65441.51	66065.33	68381.84	71209.28	74366.13	77766.19
Minimum ($)	359.68	341.61	312.41	314.08	361.72	344.81	353.56	359.85	370.21	385.62
Standard Deviation	9793.93	10206.75	10607.09	10700.75	10840.23	10954.59	11366.54	11734.17	12187.02	12594.58

Table A.8 Descriptive Statistics for the Investment Rate

	1950	1951	1952	1953	1954	1955	1956	1957	1958	1959	1960	1961
Number of observations	51	57	59	61	64	68	68	68	68	72	106	108
Mean ($)	19.15	20.11	19.07	19.74	19.87	20.96	21.02	21.36	20.57	19.70	18.09	18.13
Median ($)	19.07	20.67	19.64	18.95	18.83	19.72	19.89	19.80	20.24	20.23	17.05	16.05
Maximum ($)	54.05	51.50	49.79	48.85	59.27	62.94	58.71	55.10	44.05	45.04	53.36	47.48
Minimum ($)	0.86	1.22	1.39	2.49	3.25	3.60	3.11	2.62	2.79	2.12	1.32	1.74
Standard Deviation	10.29	10.53	9.75	9.69	10.33	11.04	10.93	10.75	9.82	10.51	12.03	12.02

	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973
Number of observations	108	108	108	108	108	108	108	108	150	150	150	150
Mean ($)	17.93	18.07	18.39	19.14	18.94	18.71	18.93	19.31	21.82	21.84	21.06	21.69
Median ($)	15.87	15.16	15.01	16.24	17.71	16.70	17.23	17.80	18.51	19.15	18.58	19.96
Maximum ($)	56.56	50.89	45.42	58.26	48.96	48.56	47.71	51.61	63.71	57.13	54.60	66.78
Minimum ($)	1.31	1.78	1.81	2.18	2.08	2.09	2.44	1.93	1.84	2.27	1.03	1.50
Standard Deviation	12.27	11.91	12.08	12.55	11.58	11.52	11.63	11.73	13.97	13.42	12.87	13.20

	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985
Number of observations	150	150	150	150	150	150	150	150	150	150	150	150
Mean ($)	22.37	21.72	21.87	22.32	22.18	22.53	23.27	23.36	22.63	21.51	20.62	20.62
Median ($)	21.19	21.75	22.00	21.71	21.58	21.87	22.80	22.50	21.62	18.92	18.35	17.86
Maximum ($)	62.30	69.94	80.91	69.56	54.39	70.14	69.80	79.69	82.80	78.73	58.46	72.87
Minimum ($)	1.81	1.62	1.34	1.10	0.89	-14.33	1.47	1.68	2.00	1.47	1.61	0.98
Standard Deviation	13.01	12.35	12.32	12.14	11.95	12.97	12.92	13.67	14.28	13.68	12.52	13.00

	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
Number of observations	150	151	151	152	161	161	163	173	174	174	174	174
Mean ($)	20.52	20.82	21.40	21.62	22.46	21.68	21.48	20.66	21.25	21.84	22.29	22.28
Median ($)	17.24	17.79	20.28	18.81	20.65	18.90	19.72	19.08	20.81	20.65	21.16	21.04
Maximum ($)	88.96	88.18	77.62	75.93	105.68	81.01	72.44	72.44	72.44	72.44	72.44	72.91
Minimum ($)	1.86	2.36	2.03	1.04	1.59	2.15	2.01	0.50	1.47	1.97	0.93	1.51
Standard Deviation	14.02	13.89	13.84	14.19	15.67	14.33	13.71	13.09	12.79	13.06	13.62	13.34

	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
Number of observations	174	174	174	174	174	174	174	176	174	173
Mean ($)	22.36	21.84	21.91	21.67	21.45	21.95	22.79	23.37	24.16	24.56
Median ($)	21.96	19.75	20.56	20.38	20.57	21.62	22.04	22.82	22.40	23.33
Maximum ($)	72.30	73.55	79.14	71.97	78.48	76.93	90.28	93.60	91.76	89.43
Minimum ($)	2.14	2.01	1.00	-18.87	-7.36	1.52	2.61	3.31	3.55	3.03
Standard Deviation	12.86	13.72	13.79	13.50	13.68	13.67	14.07	14.22	14.62	14.61

Table A.9 Descriptive Statistics for the Population Growth Rate

	1951	1952	1953	1954	1955	1956	1957	1958	1959	1960	1961	1962
Number of observations	175	175	175	175	175	175	175	175	175	175	175	175
Mean ($)	0.0199	0.0206	0.0202	0.0209	0.0216	0.0217	0.0219	0.0223	0.0224	0.0225	0.0225	0.0228
Median ($)	0.0192	0.0194	0.0197	0.0201	0.0206	0.0210	0.0214	0.0215	0.0218	0.0223	0.0229	0.0227
Maximum ($)	0.1585	0.0879	0.0704	0.0663	0.0686	0.0716	0.0826	0.1034	0.1156	0.1154	0.1133	0.1012
Minimum ($)	-0.0991	-0.0142	-0.0127	-0.0104	-0.0169	-0.0115	-0.0073	-0.0093	-0.0068	-0.0219	-0.0404	-0.0109
Standard Deviation	0.0174	0.0128	0.0115	0.0110	0.0117	0.0119	0.0120	0.0127	0.0134	0.0139	0.0154	0.0145

	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973	1974
Number of observations	175	175	175	175	175	175	175	175	175	175	175	175
Mean ($)	0.0235	0.0230	0.0230	0.0227	0.0229	0.0227	0.0225	0.0219	0.0220	0.0215	0.0213	0.0199
Median ($)	0.0231	0.0234	0.0231	0.0239	0.0228	0.0219	0.0222	0.0222	0.0221	0.0216	0.0213	0.0207
Maximum ($)	0.1001	0.0999	0.1009	0.1011	0.1334	0.1120	0.1408	0.1446	0.1555	0.1672	0.1636	0.1581
Minimum ($)	-0.0027	-0.0373	-0.0137	-0.0094	-0.0397	0.0013	-0.0045	-0.0244	-0.0059	-0.0186	-0.0227	-0.1016
Standard Deviation	0.0136	0.0147	0.0141	0.0144	0.0169	0.0161	0.0169	0.0177	0.0161	0.0167	0.0166	0.0201

	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986
Number of observations	175	175	175	175	175	175	175	175	175	175	175	175
Mean ($)	0.0197	0.0194	0.0203	0.0200	0.0210	0.0202	0.0197	0.0197	0.0207	0.0197	0.0195	0.0193
Median ($)	0.0208	0.0209	0.0205	0.0192	0.0195	0.0196	0.0202	0.0204	0.0210	0.0207	0.0204	0.0204
Maximum ($)	0.1530	0.1444	0.1432	0.1390	0.1372	0.1568	0.0694	0.0634	0.1264	0.1096	0.0973	0.0877
Minimum ($)	-0.1467	-0.1054	-0.0366	-0.0380	-0.0395	-0.0411	-0.0860	-0.0674	-0.0351	-0.0844	-0.0124	-0.0100
Standard Deviation	0.0221	0.0203	0.0182	0.0185	0.0209	0.0215	0.0166	0.0151	0.0167	0.0174	0.0147	0.0141

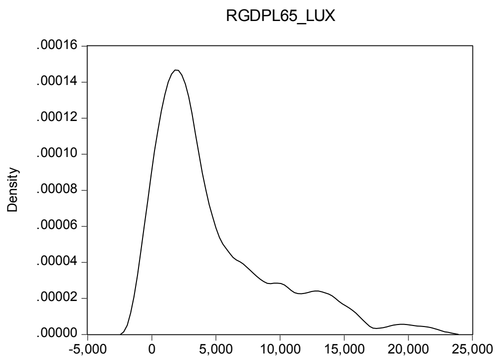
	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998
Number of observations	175	175	175	175	175	175	175	175	175	175	175	175
Mean ($)	0.0188	0.0190	0.0187	0.0185	0.0145	0.0200	0.0168	0.0150	0.0159	0.0167	0.0164	0.0161
Median ($)	0.0202	0.0205	0.0201	0.0199	0.0186	0.0188	0.0180	0.0175	0.0169	0.0162	0.0160	0.0157
Maximum ($)	0.0580	0.0659	0.0715	0.0805	0.1130	0.4864	0.1259	0.0832	0.0698	0.1923	0.1643	0.1157
Minimum ($)	-0.0189	-0.0288	-0.0539	-0.1225	-0.5546	-0.0539	-0.0707	-0.1713	-0.1283	-0.0271	-0.0135	-0.0153
Standard Deviation	0.0135	0.0142	0.0152	0.0175	0.0460	0.0382	0.0190	0.0229	0.0185	0.0185	0.0171	0.0145

	1999	2000	2001	2002	2003	2004	2005	2006	2007
Number of observations	175	175	175	175	175	175	175	175	175
Mean ($)	0.0157	0.0144	0.0150	0.0149	0.0146	0.0142	0.0141	0.0139	0.0137
Median ($)	0.0157	0.0147	0.0145	0.0142	0.0141	0.0138	0.0137	0.0133	0.0132
Maximum ($)	0.0581	0.0534	0.0620	0.0595	0.0709	0.0537	0.0496	0.0536	0.0523
Minimum ($)	-0.0107	-0.1244	-0.0102	-0.0099	-0.0096	-0.0099	-0.0104	-0.0106	-0.0103
Standard Deviation	0.0125	0.0157	0.0127	0.0128	0.0129	0.0123	0.0120	0.0120	0.0120

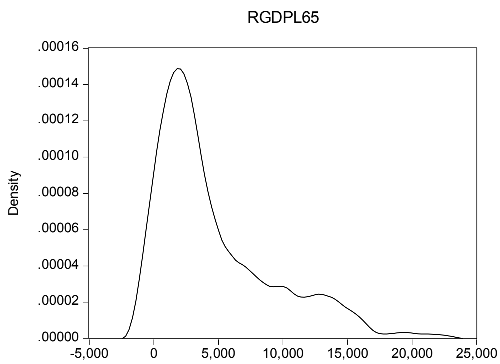
A.7 Sensitivity Analysis on the Elimination of Luxembourg

Due to the large dependence on foreign workers and foreign capital, Luxembourg should probably better be eliminated from the data set. Yet, Figure A.2 shows that it does not make a difference for the kernel densities and the conclusions to be drawn from them. Panel (a) shows the kernel density for the year 1965 taking the dataset including Luxembourg. In Panel (b), Luxembourg is eliminated. The only differences that appear are that the scale reaches lower values and perhaps a very slight peak that might appear at the upper end of the scale does not appear anymore. Yet, this peak was never interpreted as individual peak. As there are no significant differences, also for the other years not presented here, it was decided to keep Luxembourg within the dataset.

Figure A.3 Sensitivity in Response to the Elimination of Luxembourg



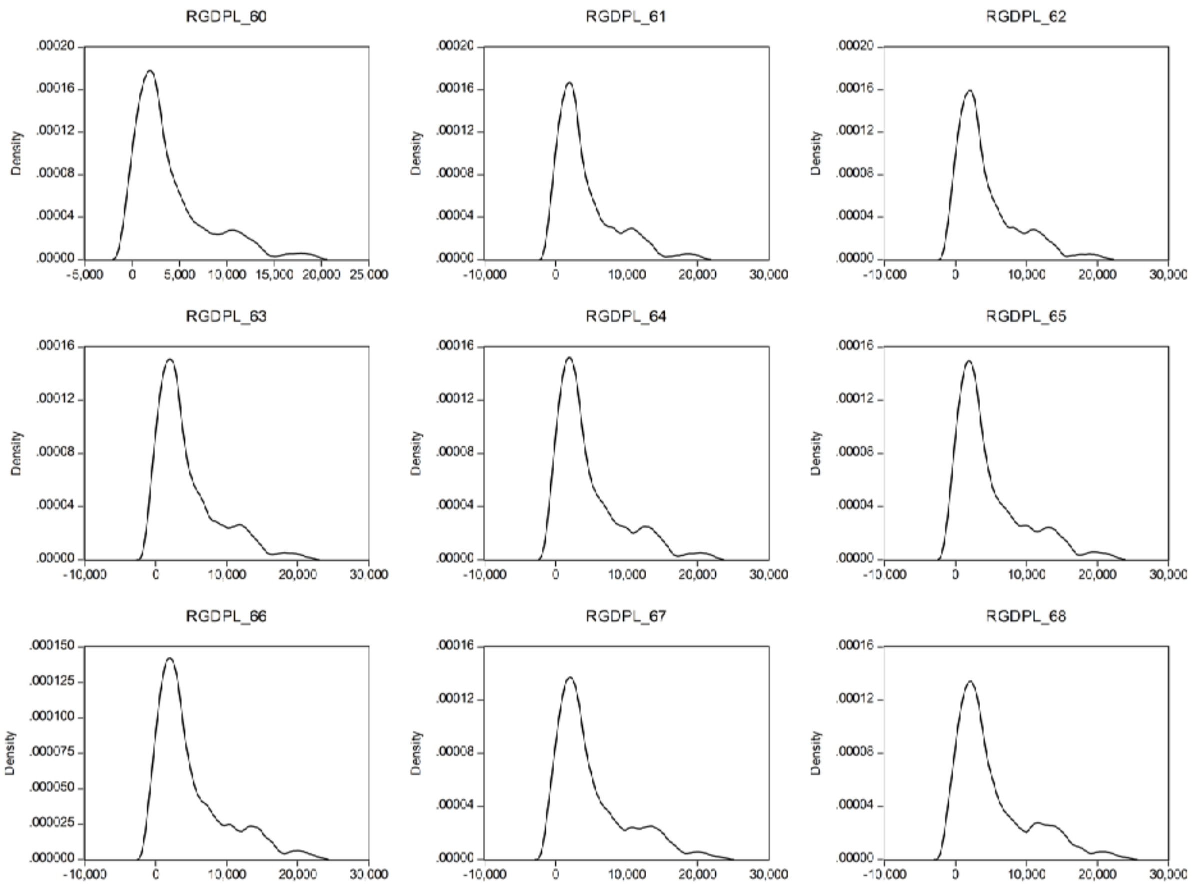
Panel (a): Dataset with Luxembourg

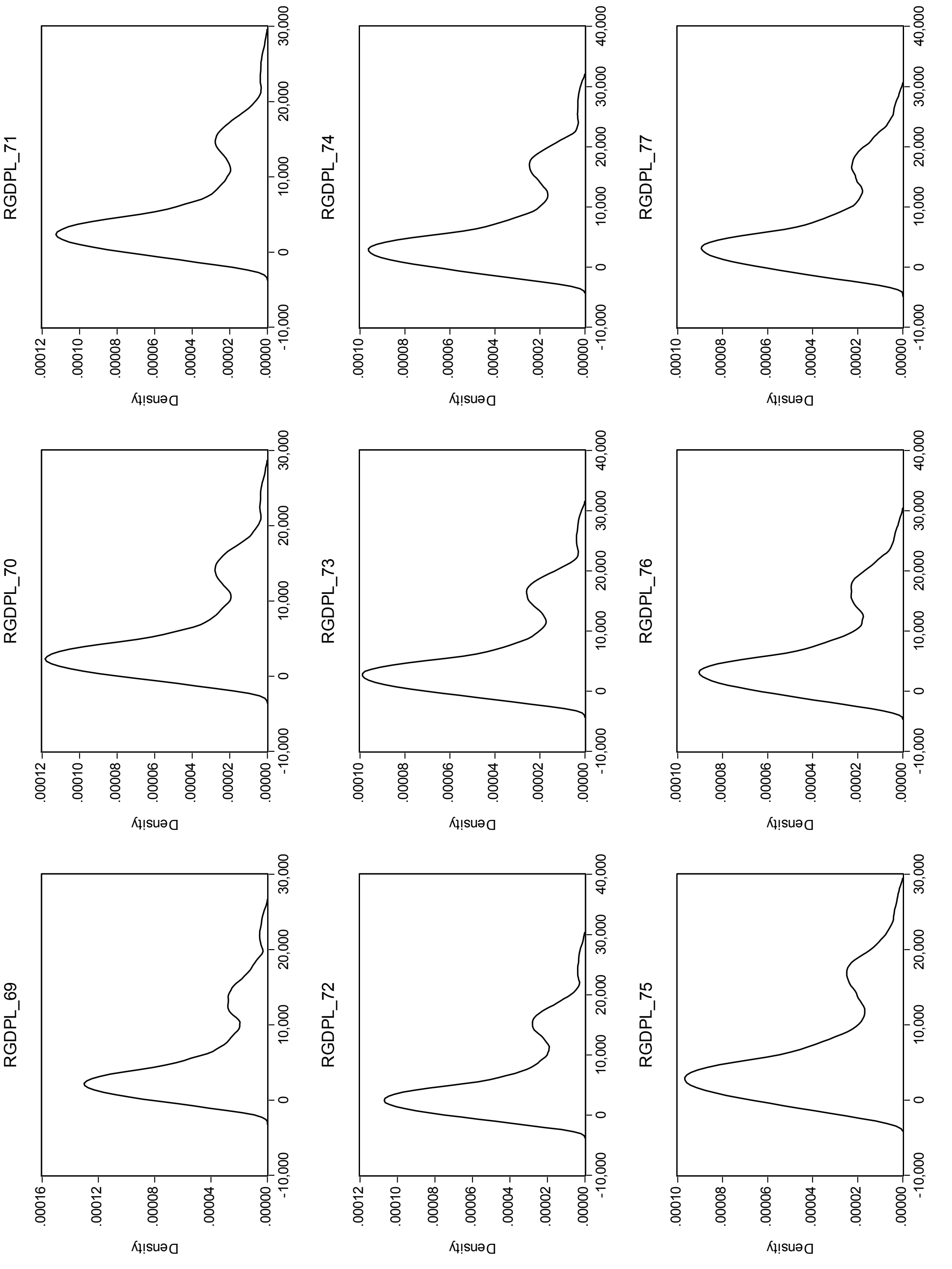


Panel (b): Dataset without Luxembourg

A.8 Kernel Densities for the Complete Dataset Offering Data in each Year from 1960 on

Figure A.4 The Kernel Densities for Real Per Capita GDP (105 Countries)

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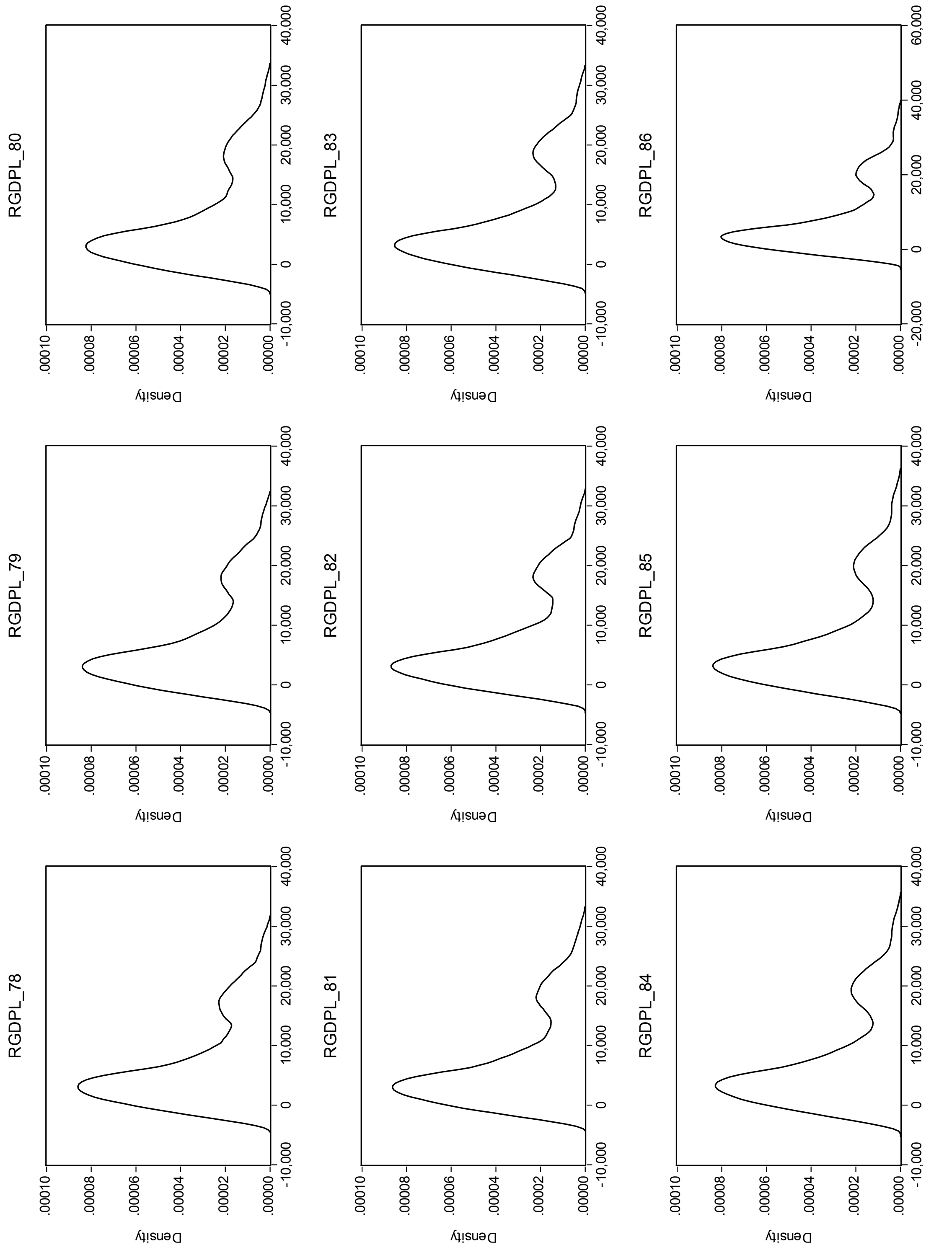
RGDPL_70
RGDPL_71
RGDPL_72
RGDPL_73
RGDPL_74
RGDPL_75
RGDPL_76
RGDPL_77
Density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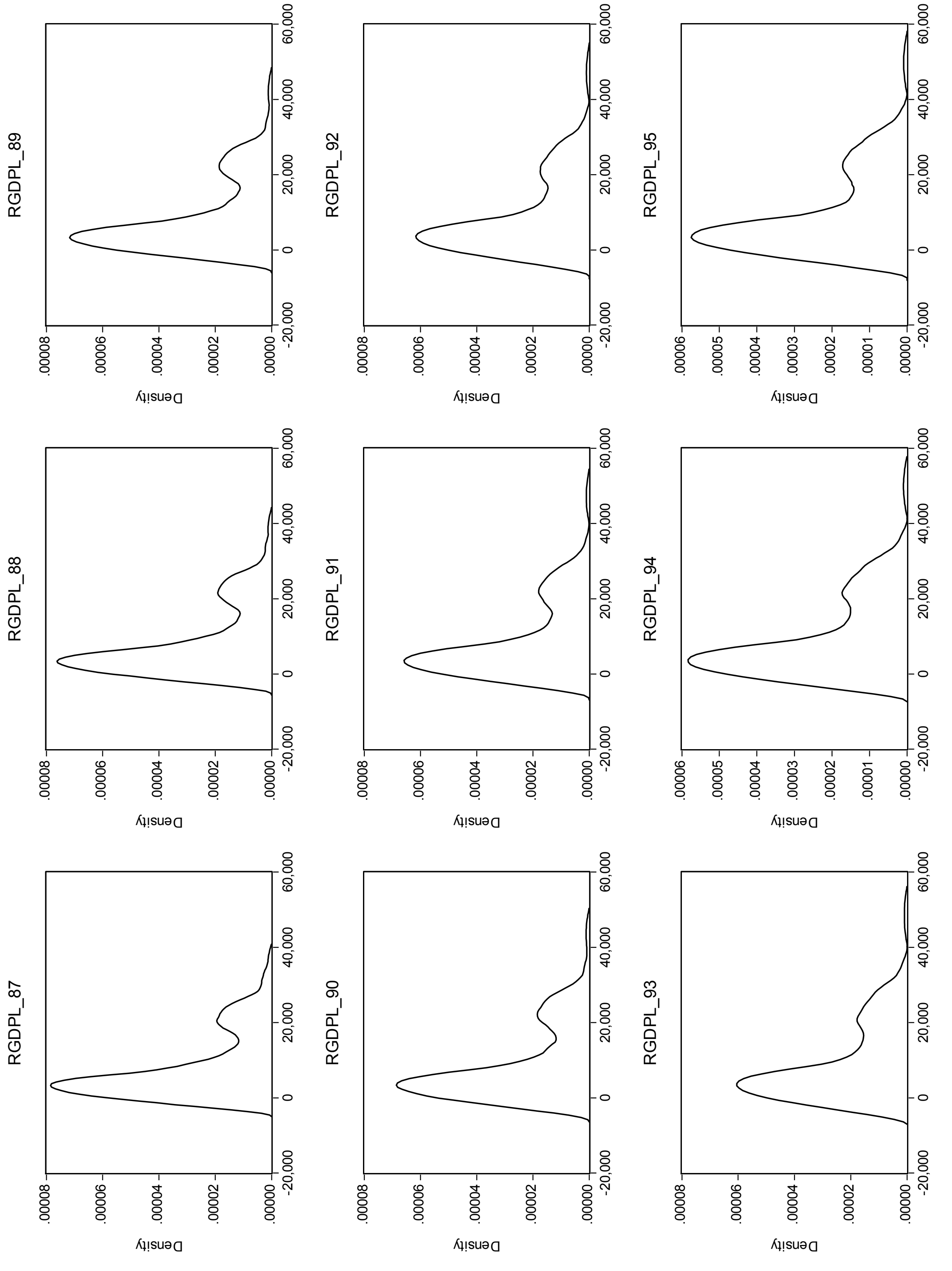
RGDPL_84
RGDPL_85
RGDPL_86
Density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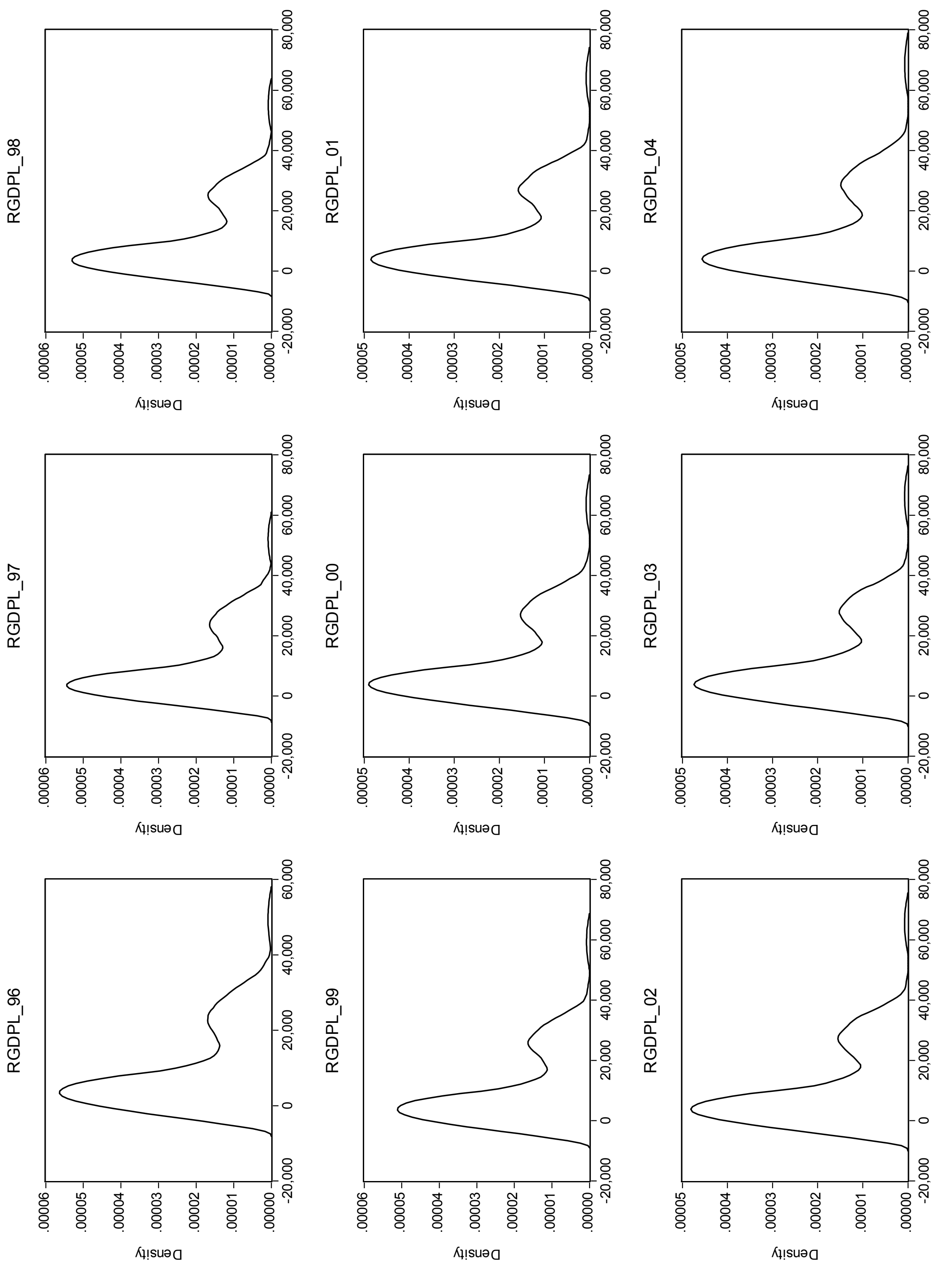
RGDPL_89
RGDPL_90
RGDPL_91
RGDPL_92
RGDPL_93
RGDPL_94
RGDPL_95
Density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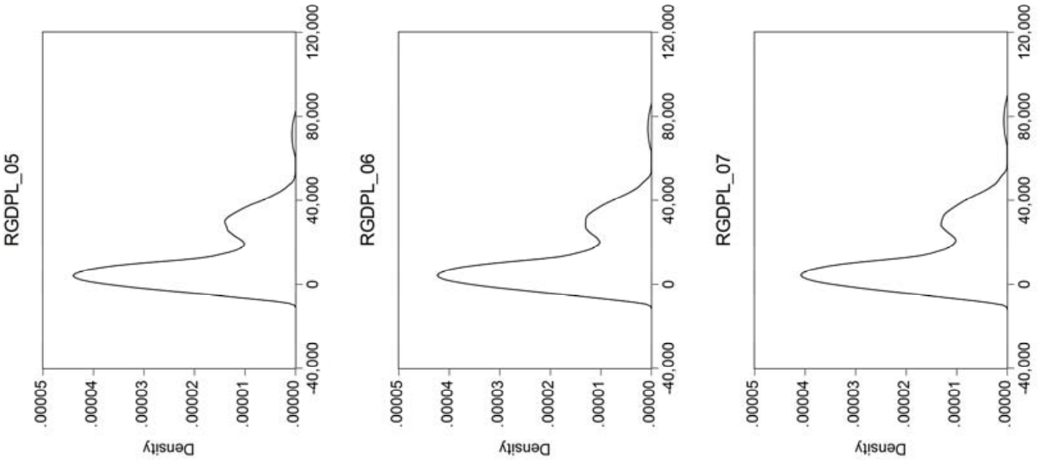
RGDPL_04
Density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RGDPL_07
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.00002
.00003
.00004
.00005
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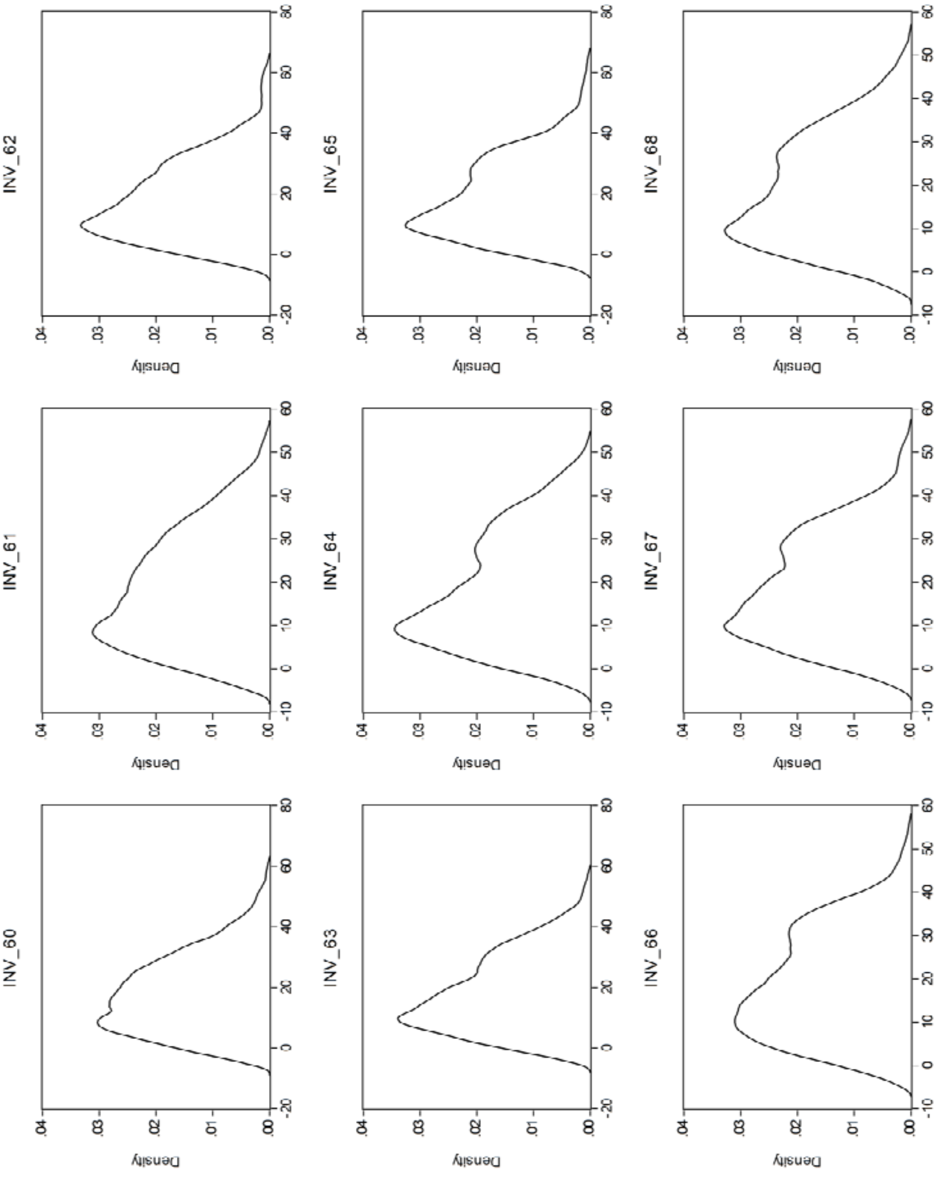
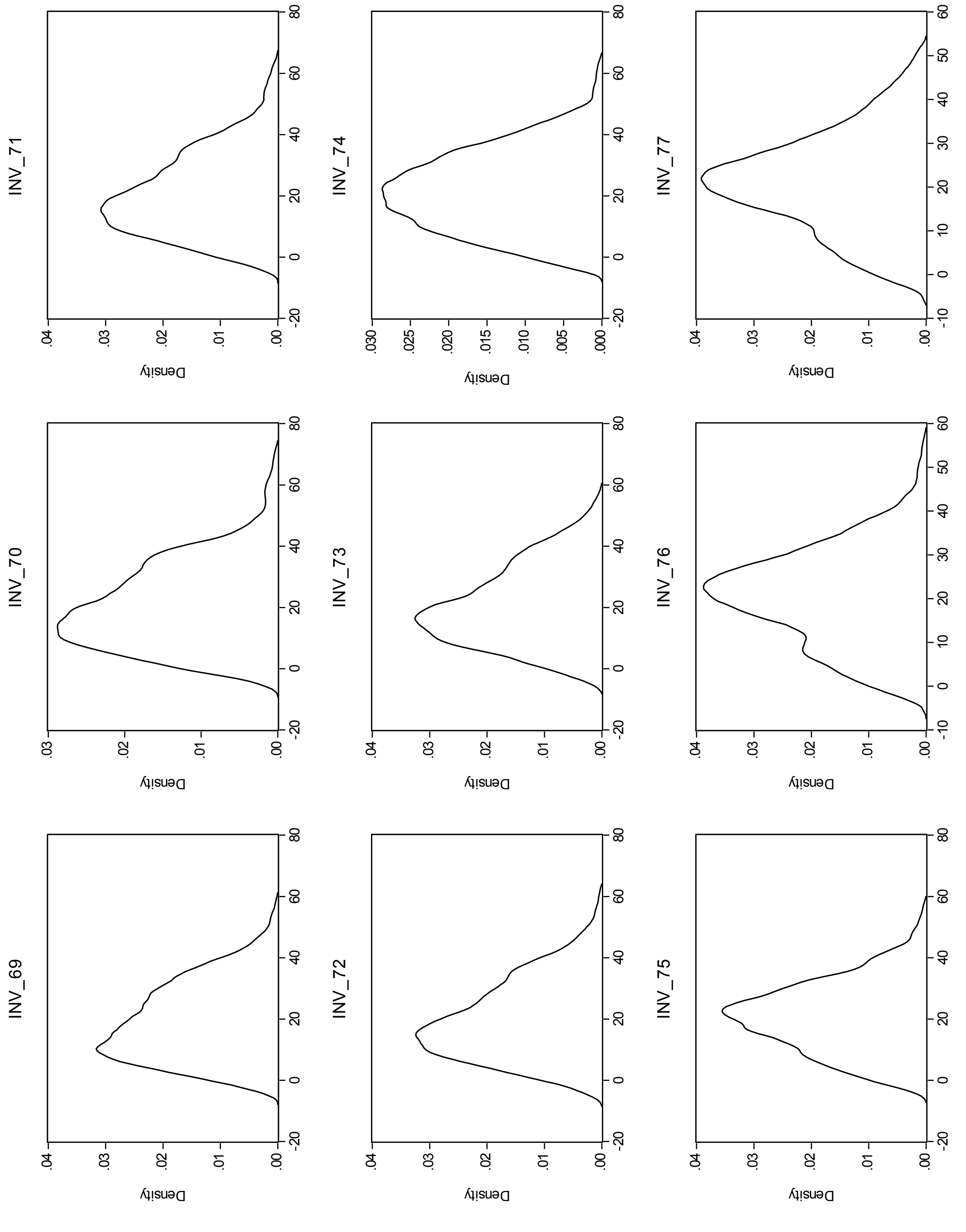
0
40,000
80,000
120,000

Figure A.5 The Kernel Densities for the Investment Rate (105 Countries)

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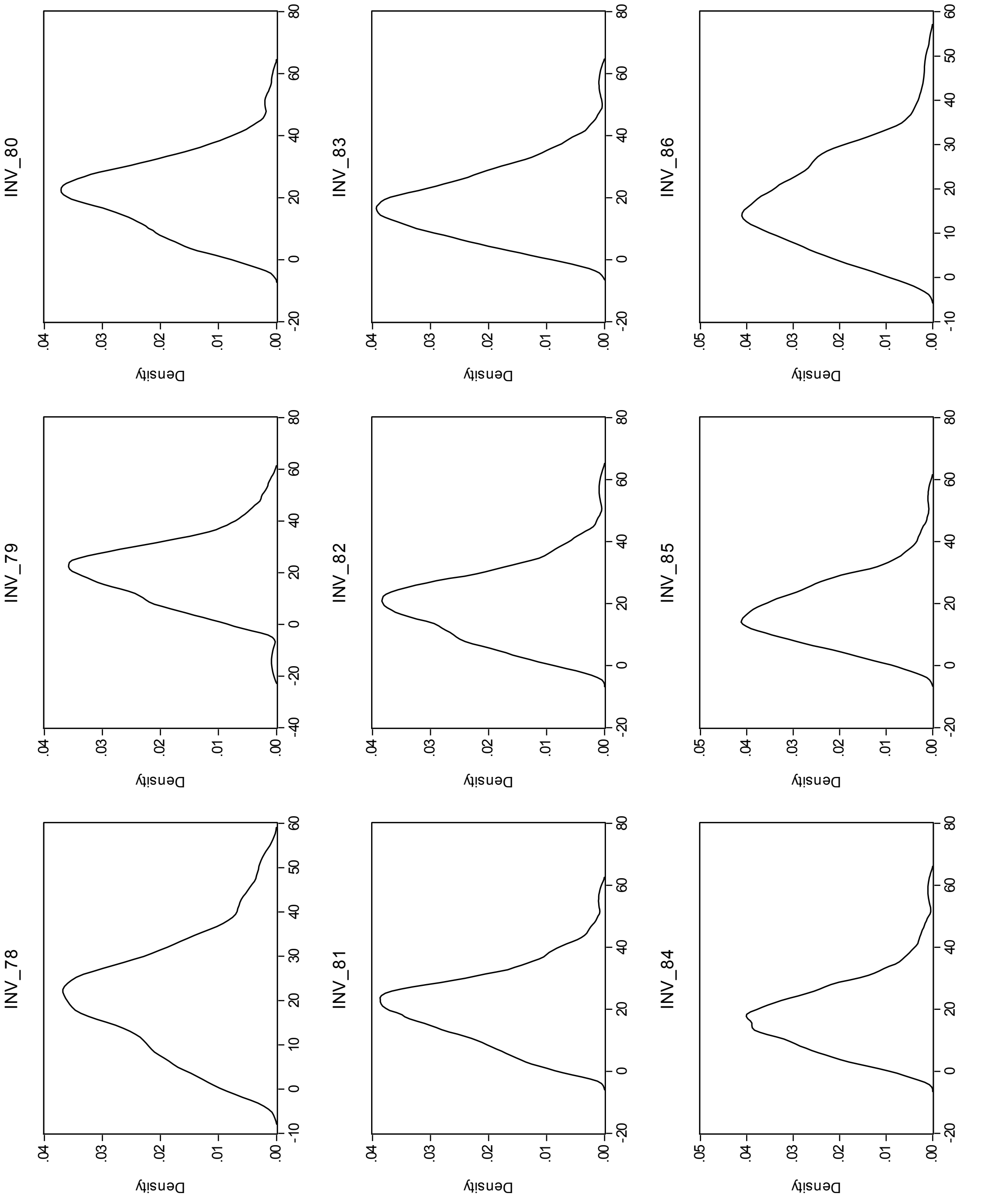
INV_74
INV_75
INV_76
INV_77
Density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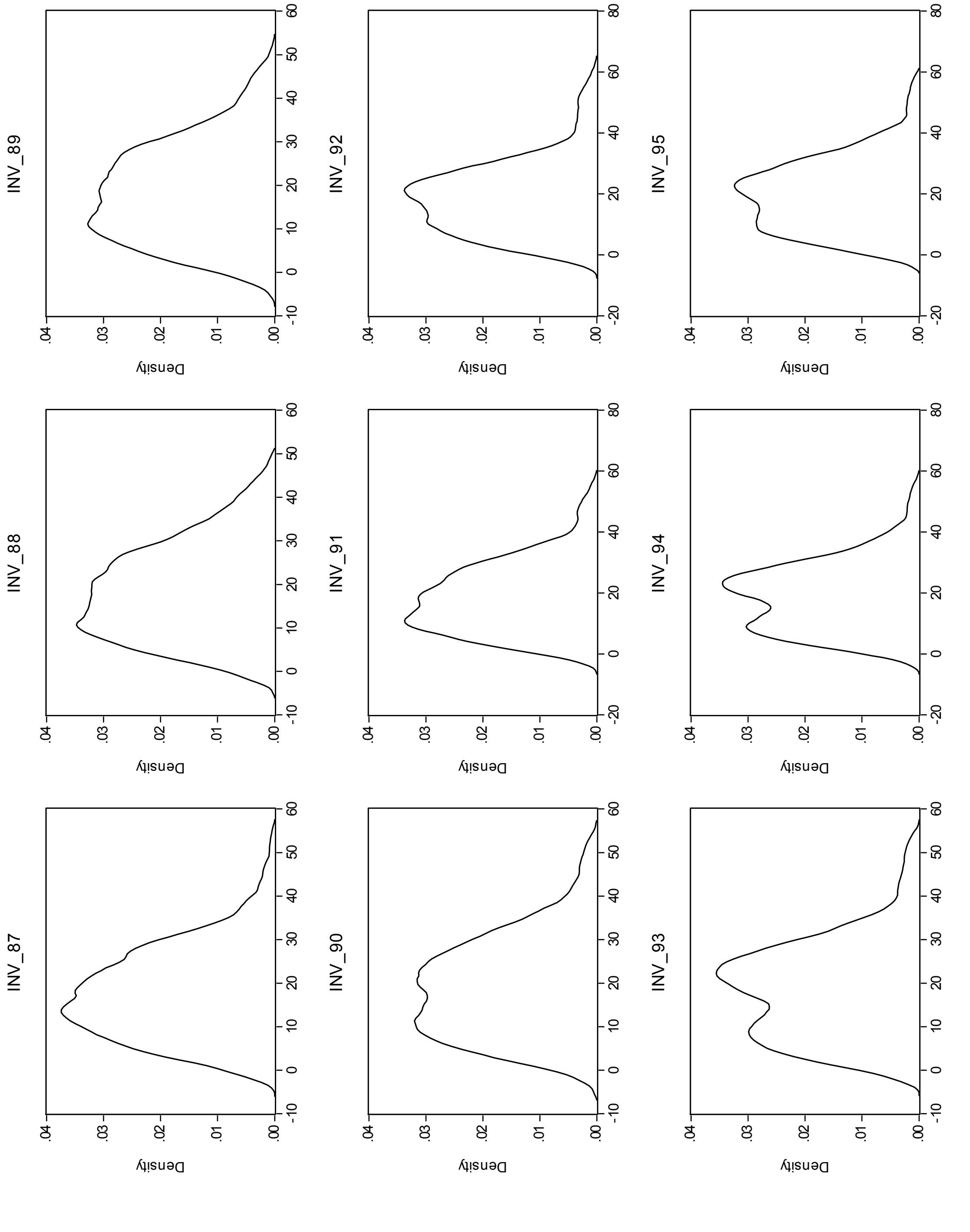
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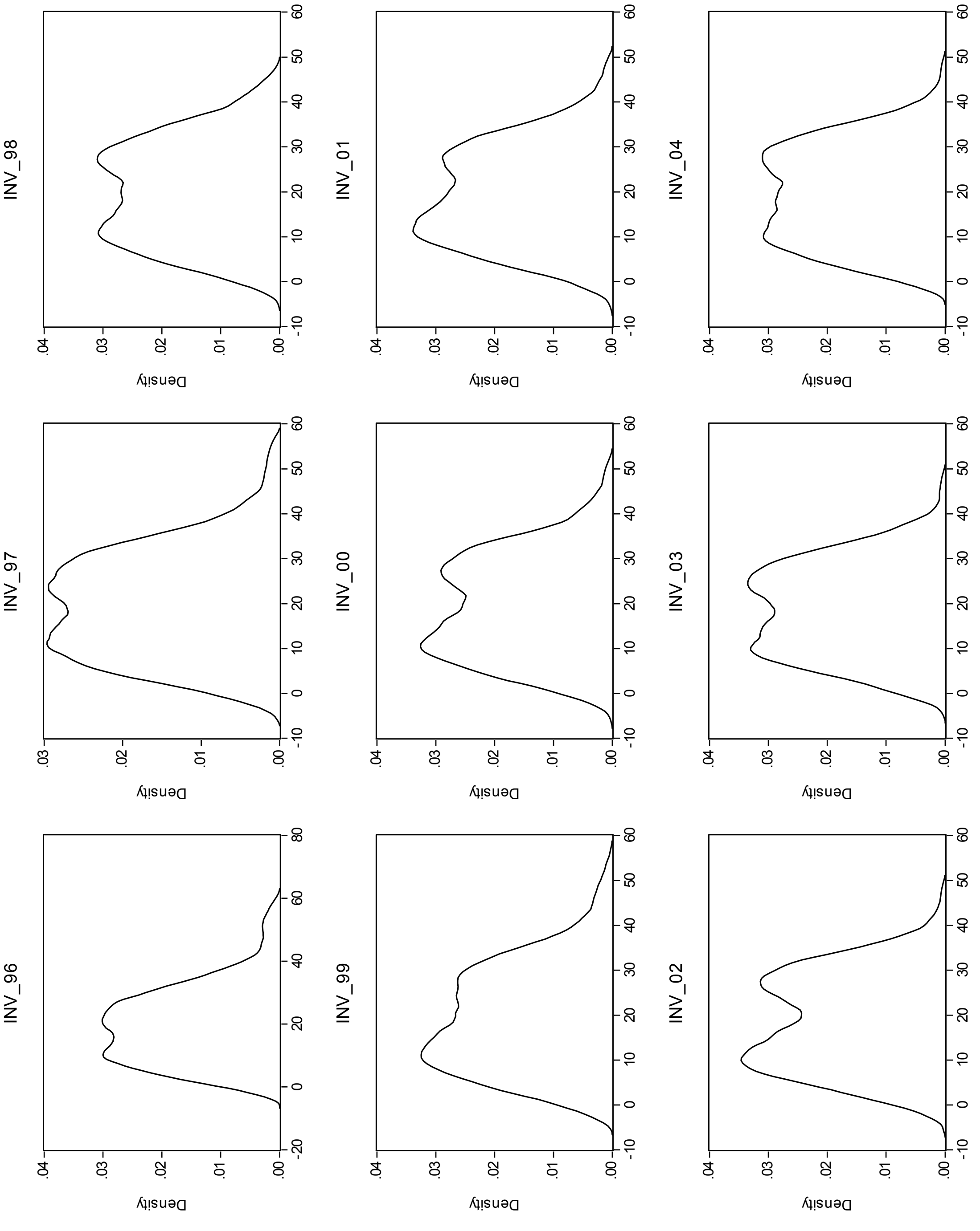
INV_95
Density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INV_01
INV_02
INV_03
INV_04
Density

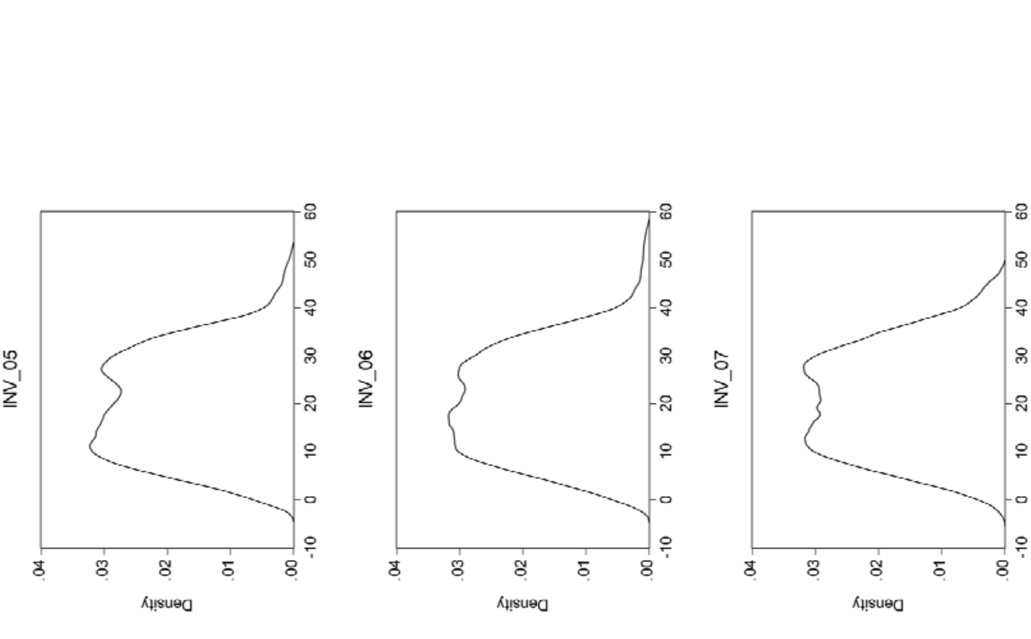
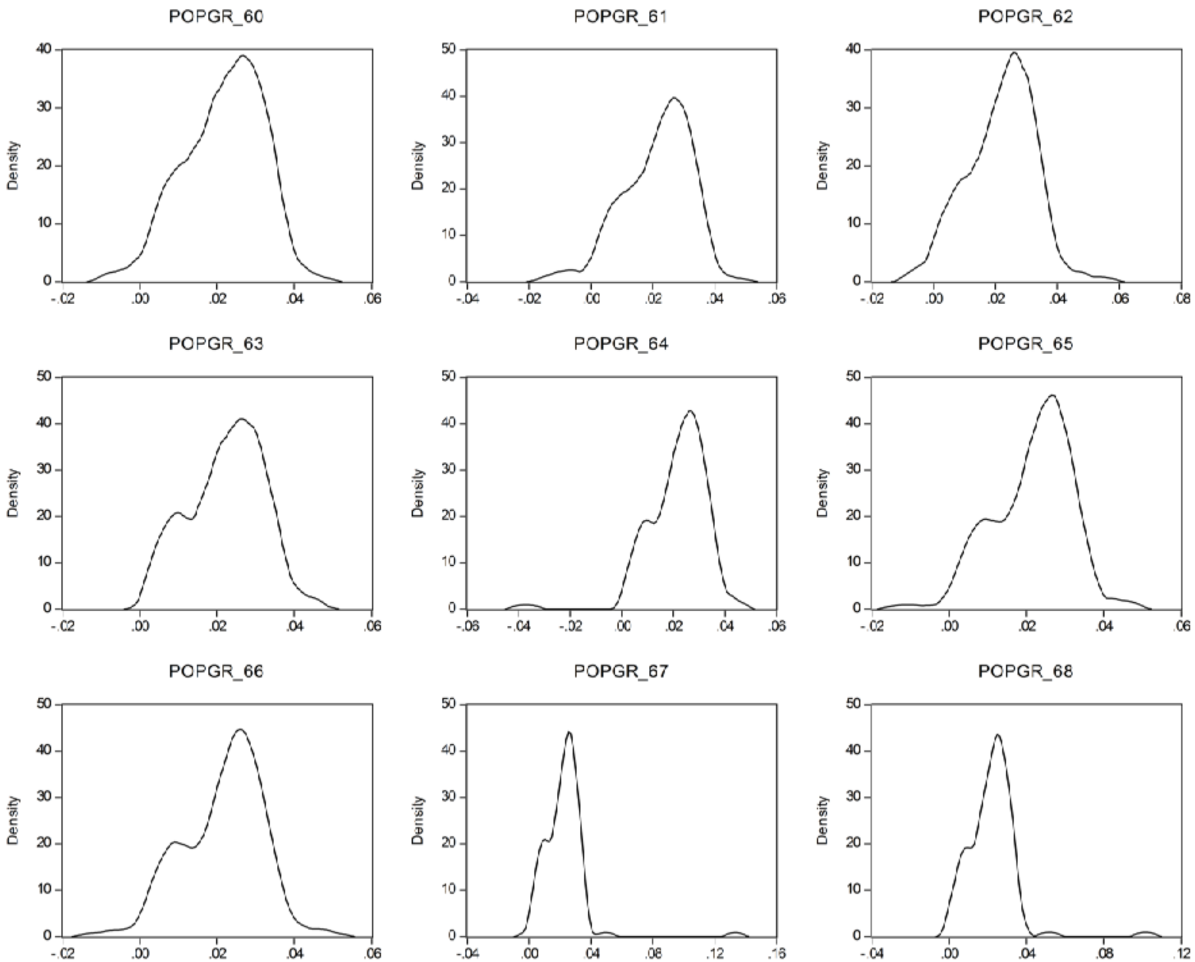
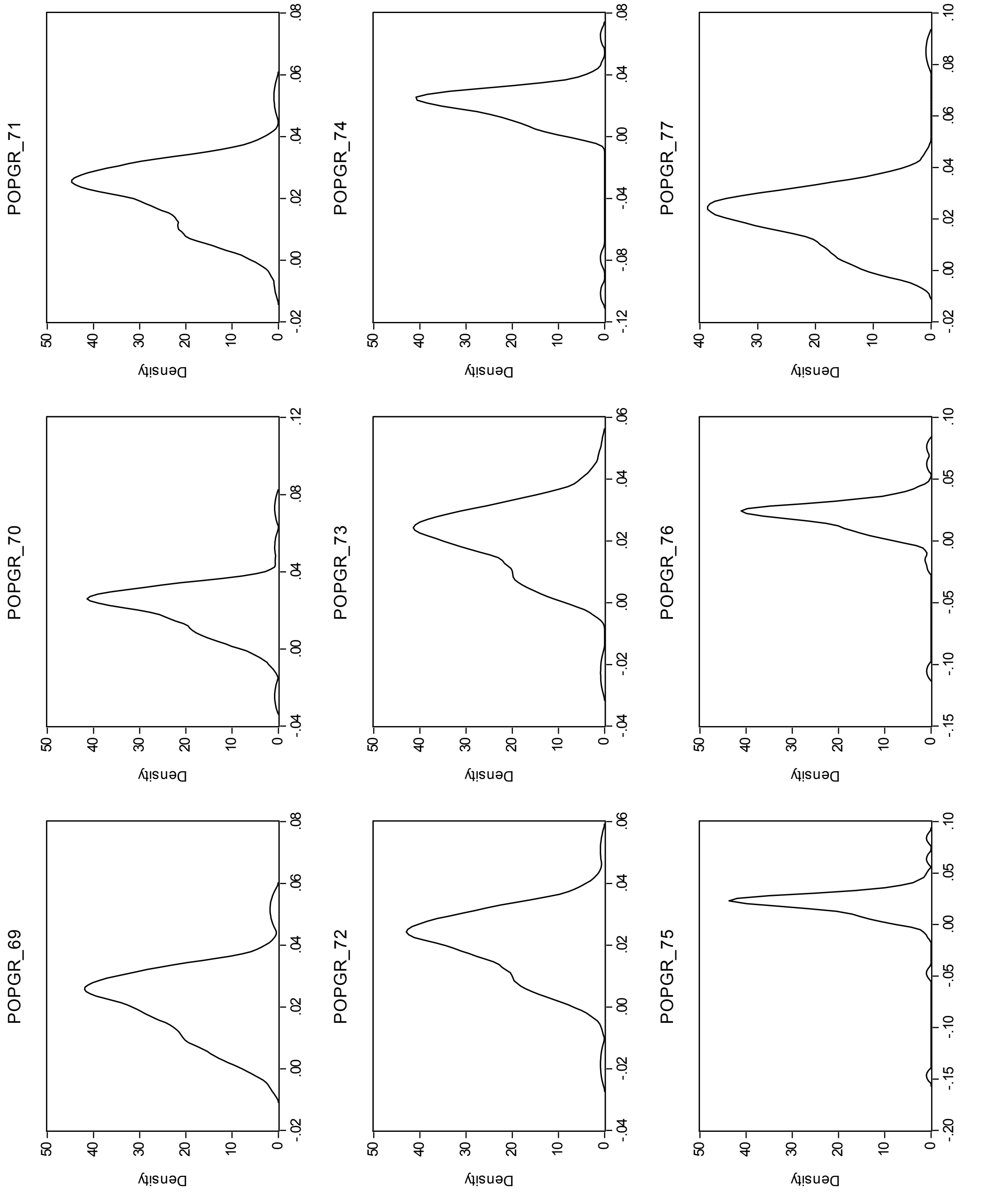

INV_05
INV_06
INV_07
Density
.00
.01
.02
.03
.04
-10
0
10
20
30
40
50
60

Figure A.6 The Kernel Densities for the Population Growth Rate (105 Countries)

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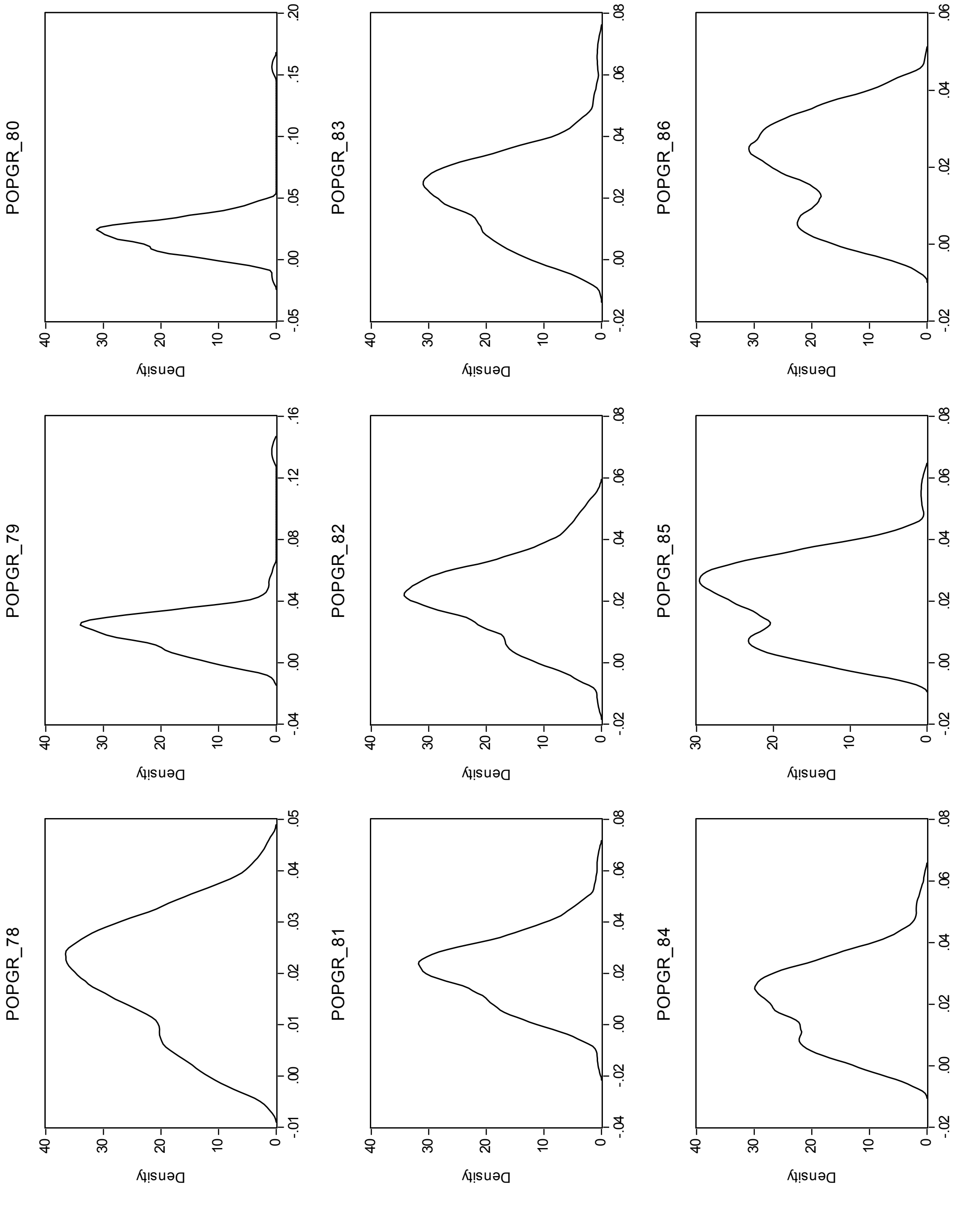
POPGR_76
POPGR_77
Density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POPGR_85
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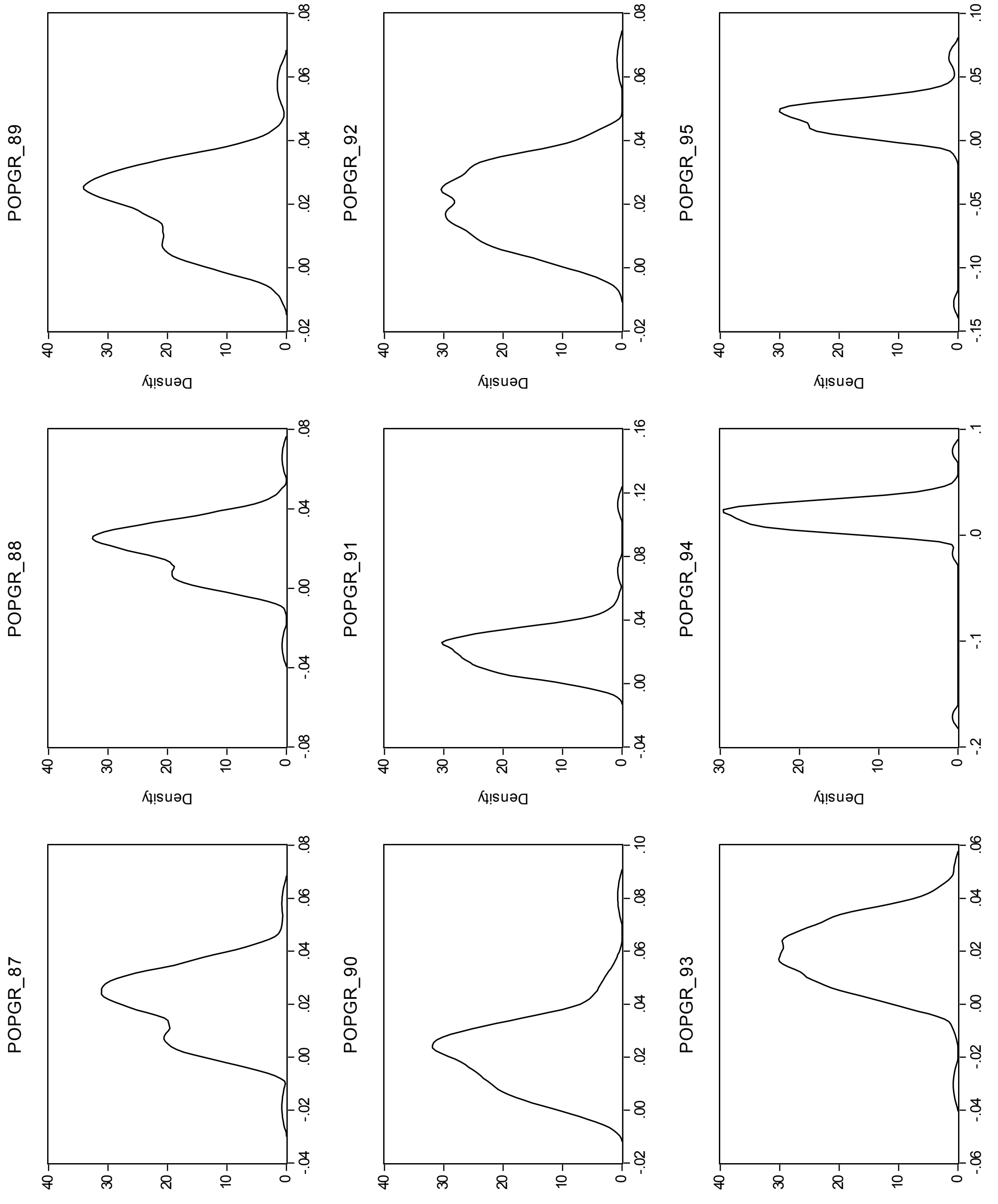
Density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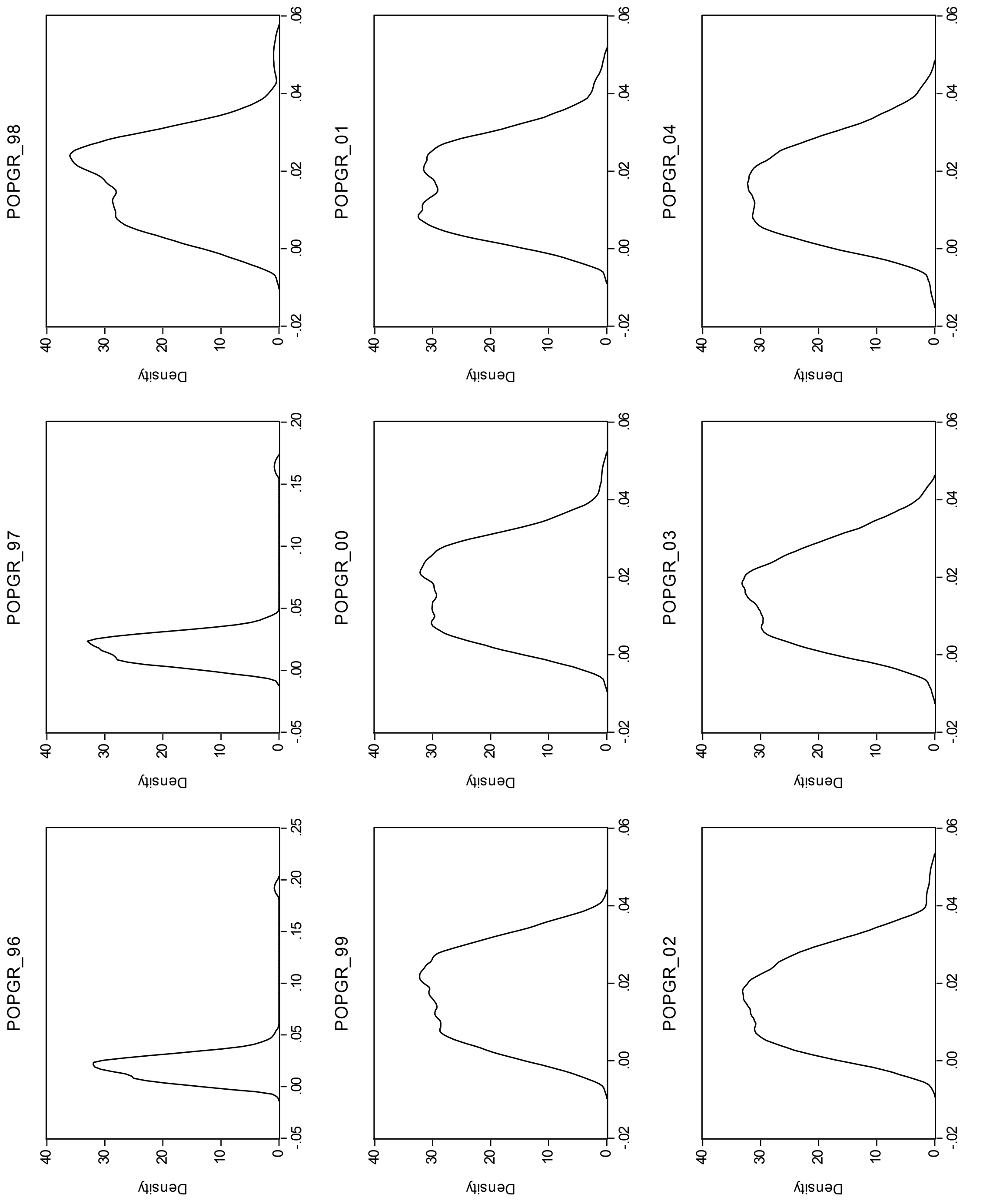
POPGR_88
POPGR_89
POPGR_90
POPGR_91
POPGR_92
POPGR_93
POPGR_94
POPGR_95
Density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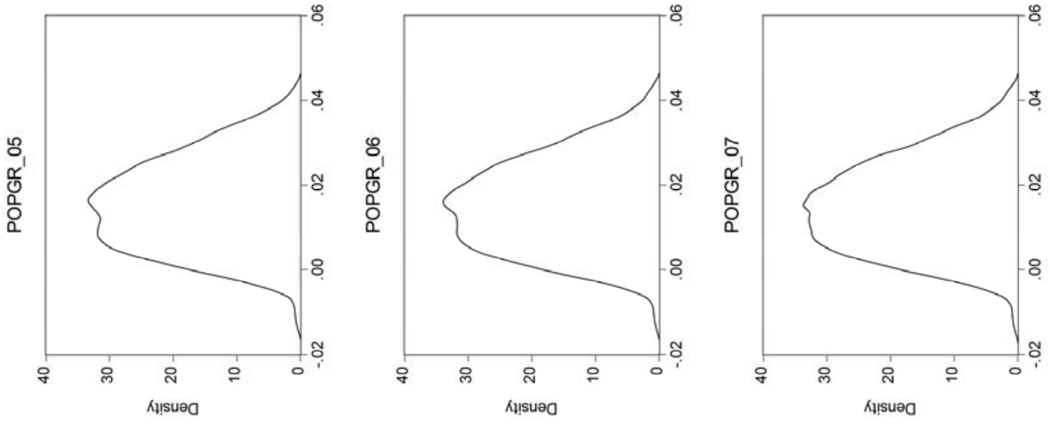
POPGR_99
POPGR_00
POPGR_01
POPGR_02
POPGR_03
POPGR_04
Density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POPGR_07
Density
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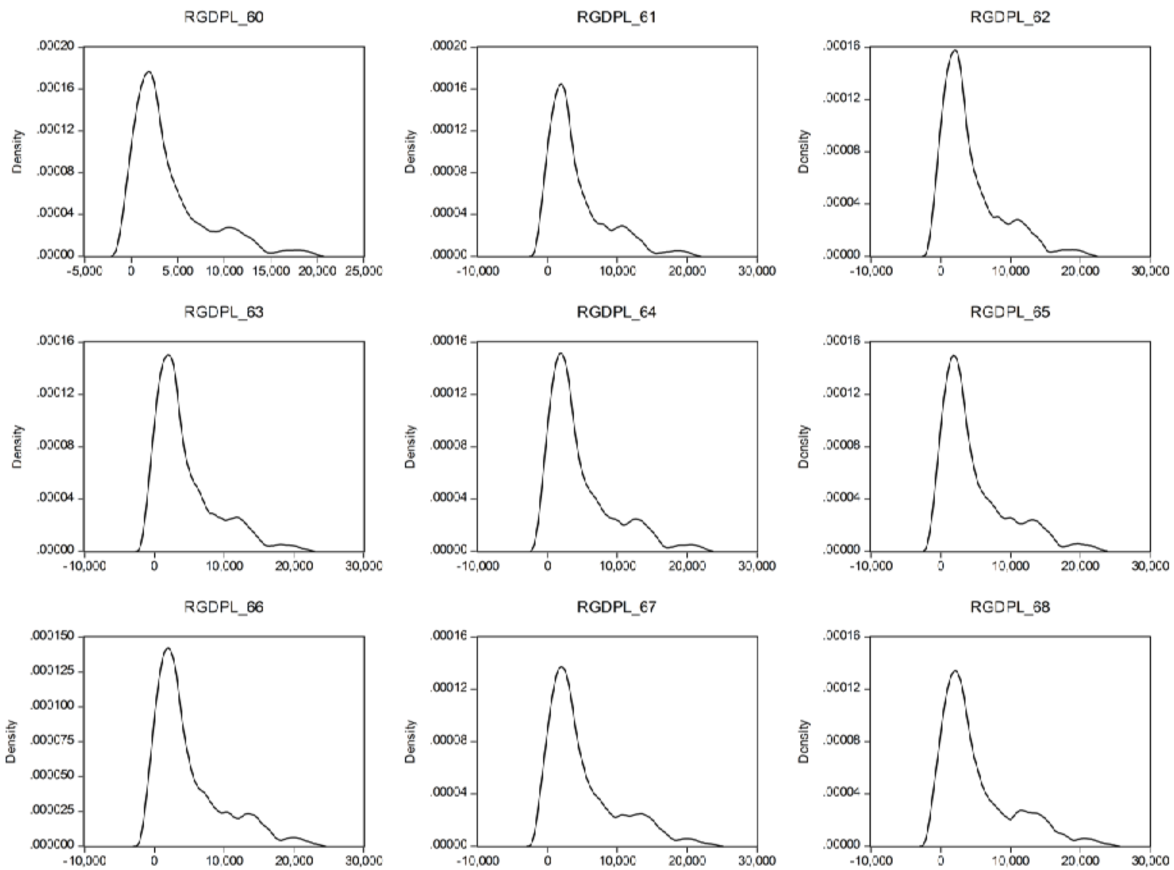
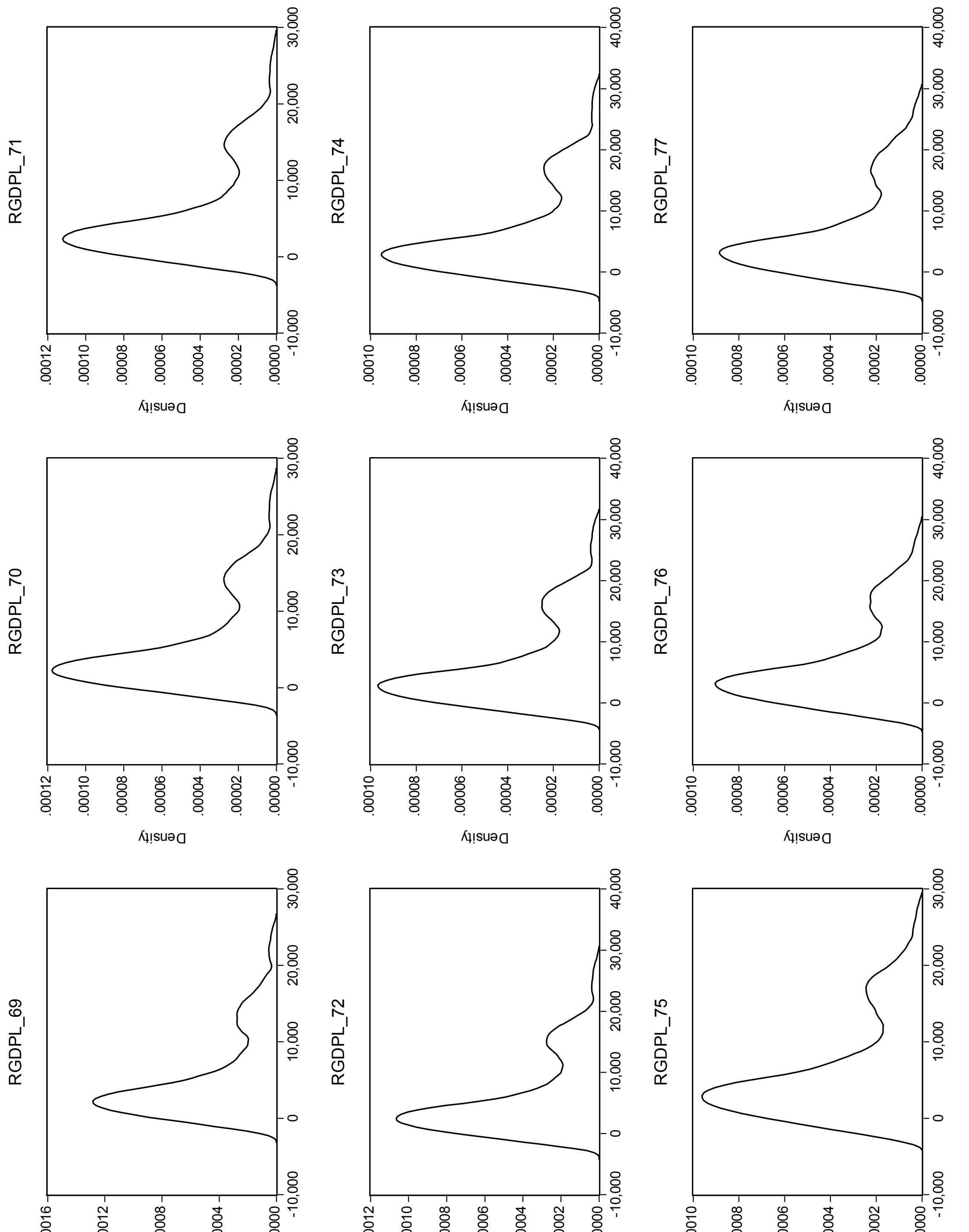
10
20
30
40
-.02
.00
.02
.04
.06

A.9 Sensitivity Analysis: Influence of Considering all Countries Offering Data in each Year Individually

Figure A.7 Kernel Densities for Real per Capita GDP in each Year Individually

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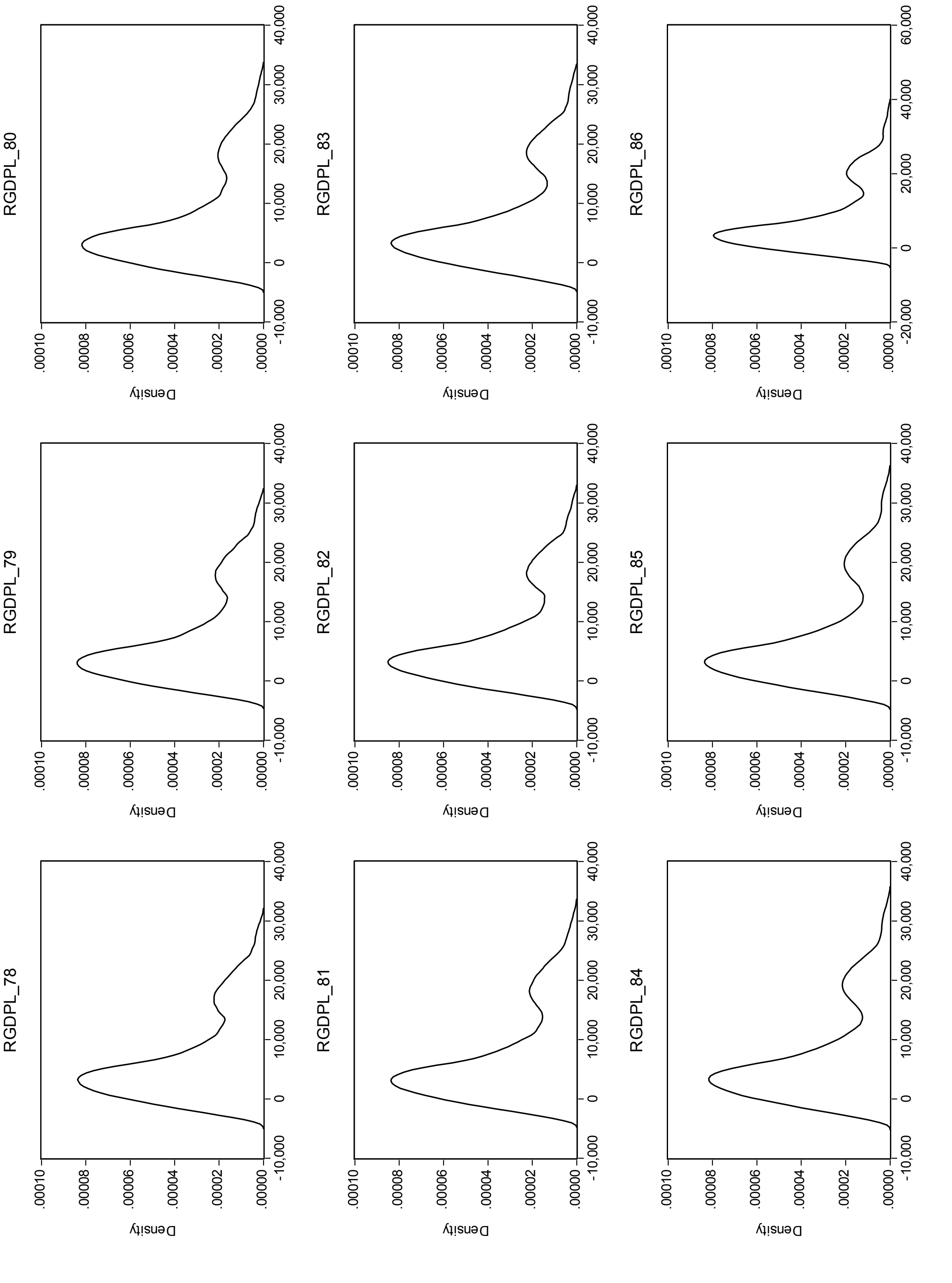
RGDPL_72
RGDPL_73
RGDPL_74
RGDPL_75
RGDPL_76
RGDPL_77
Density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RGDPL_79
RGDPL_80
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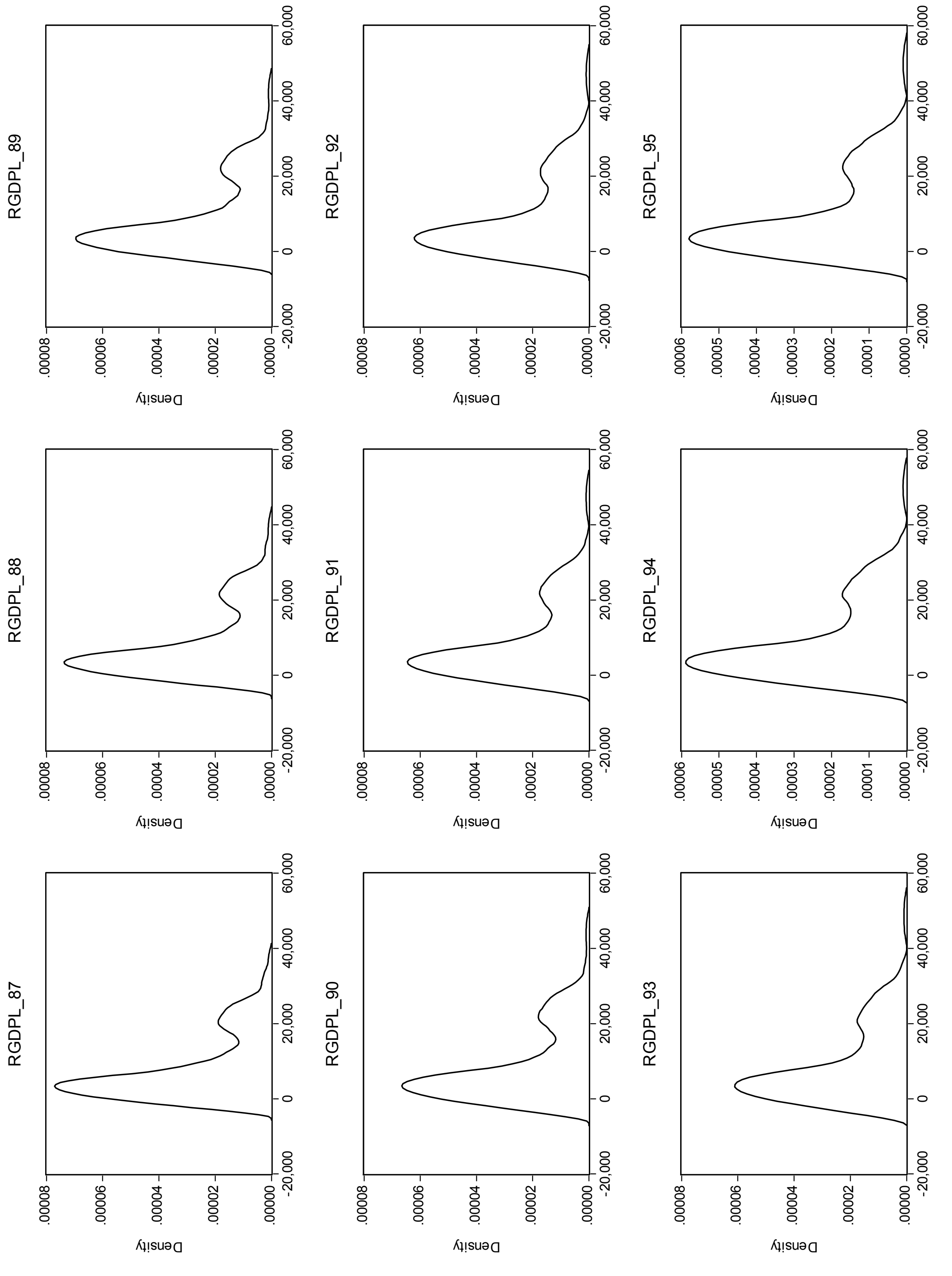
RGDPL_82
RGDPL_83
RGDPL_84
RGDPL_85
RGDPL_86
Density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RGDPL_88
RGDPL_89
RGDPL_90
RGDPL_91
RGDPL_92
RGDPL_93
RGDPL_94
RGDPL_95
Density
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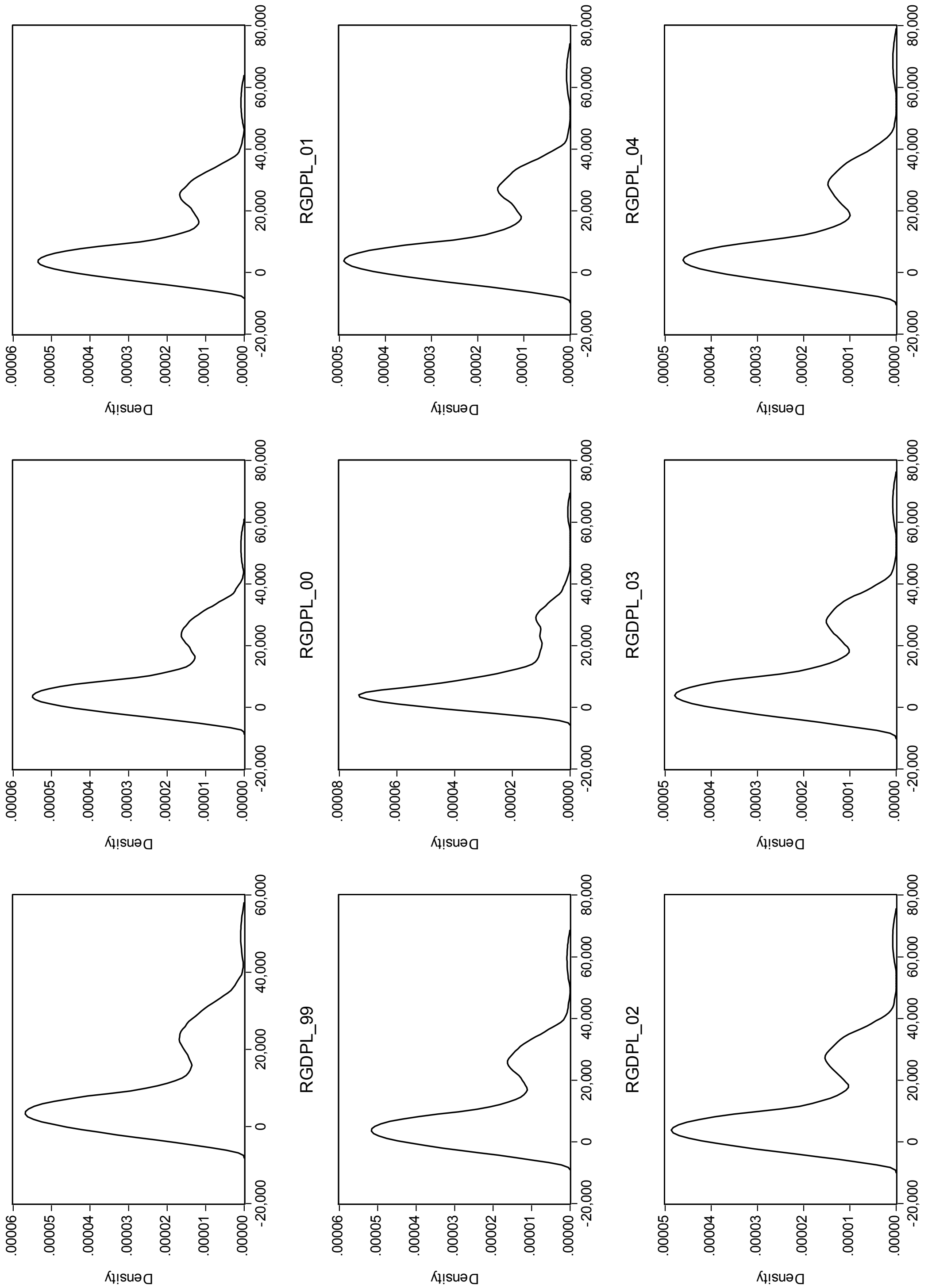
0
20,000
40,000
60,000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RGDPL_98
RGDPL_99
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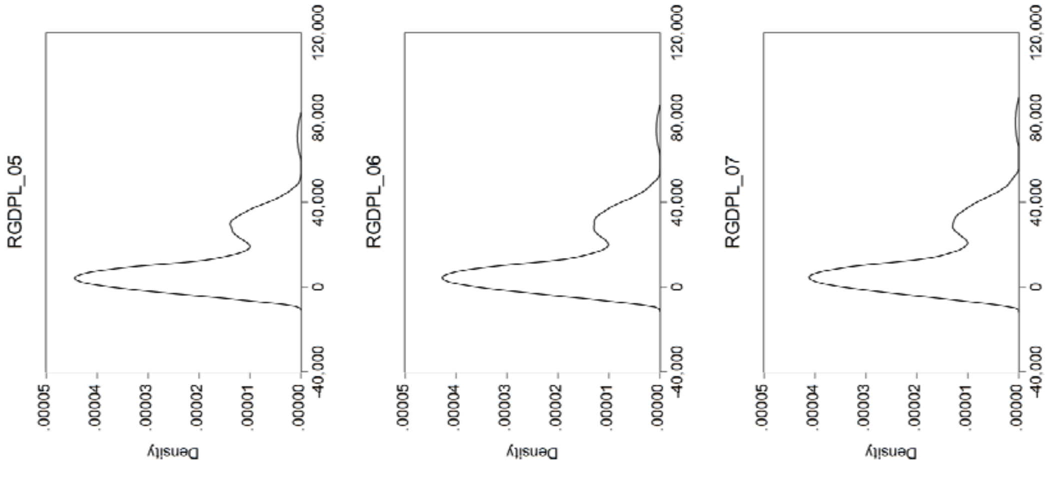
RGDPL_01
RGDPL_02
RGDPL_03
RGDPL_04
Density


RGDPL_05
RGDPL_06
RGDPL_07
Density
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.00001
.00002
.00003
.00004
.00005
-40,000
0
40,000
80,000
120,000

A.10 Sensitivity Analysis on Real per Capita GDP – Influence of Other Starting Years

Which countries are included when having the largest possible dataset? The analyst has to decide on whether to start already in 1960 having only 105 countries in the dataset, or instead whether to choose a later starting point – possible are 1965 or 1970. Table A.10 gives an overview of the new countries entering the analysis in each of the two years together with their levels of real per capita GDP. Obviously, the sample of new countries is not random but it is rather biased towards the poor countries so that this choice should have an influence on the conclusions to be drawn. In 1965, only two countries enter the dataset. In 1970, the dataset is increased by additional 42 countries.

Table A.10 The New Countries in 1965 and 1970[209]

New Countries in 1965	GDP level	New Countries in 1970	GDP level
Sierra Leone	2361.12	Afghanistan	862.79
Tunisia	2456.48	Albania	2547.41
		Angola	3007.93
		Antigua and Barbuda	4716.65
		Bahamas	19017.75
		Bahrain	21260.17
		Belize	4415.73
		Bermuda	25870.78
		Bhutan	801.23
		Bulgaria	2642.71
		Cambodia	1884.37
		Cuba	5033.08
		Djibouti	9053.14
		Dominica	1625.64
		Germany	15490.93
		Grenada	2962.64
		Guyana	2205.70
		Hungary	6999.81
		Kiribati	2783.91
		Laos	705.42
		Lebanon	13794.21
		Liberia	1873.32
		Maldives	781.12
		Malta	4299.91
		Marshall Islands	5240.94
		Micronesia, Fed. Sts.	2132.54

209 Measured in international dollars with constant prices of 2005.

		Mongolia	1305.90
		Palau	28309.82
		Poland	5685.21
		Samoa	4123.86
		Sao Tome and Principe	5332.45
		Solomon Islands	1338.98
		Somalia	921.94
		St. Kitts & Nevis	2073.12
		St. Lucia	4489.09
		St.Vincent & Grenadines	1612.92
		Sudan	1228.88
		Suriname	7177.52
		Swaziland	2384.30
		Tonga	2745.21
		Vanuatu	3038.19
		Vietnam	905.91

Additionally, Table A.11 summarizes the values of the mean and the standard deviation of the old countries and the new ones in the respective years. The means (the first number in a cell) as well as the standard deviations differ quite a lot in 1965 and 1970. In Figure A.7, the kernel densities are presented to show the differences for old and new countries in 1965 and 1970. For 1960, no kernel density is shown here as it can be looked up in Figure A.3 in the Appendix (A.5), for example.

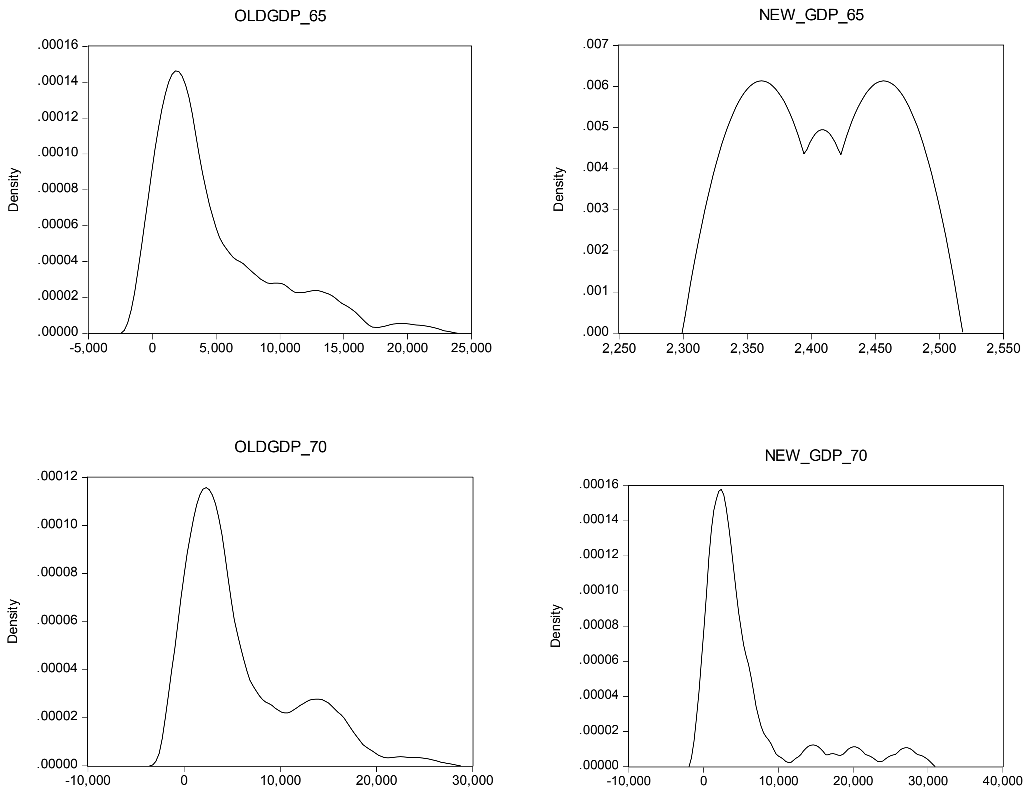
Table A.11 Comparison of the Mean and the Standard Deviation of the Old and the New Countries

		Old Countries	New Countries
1960	Mean St. Dev.	4026.10 3883.63	
1965	Mean St. Dev.	4807.39 4705.58	2408.80 67.43
1970	Mean St. Dev.	5729.51 5633.29	5587.69 6795.68

When comparing the two kernel densities in 1965, it should to be kept in mind that the structure of the new countries entering the dataset stems from the fact that only two countries enter the dataset. Hence, the result is a symmetric kernel density as shown in Figure A.7.

The kernel densities for 1970 show that the countries entering the dataset are obviously biased towards the poor countries. There is a very high peak at an income of less than $5,000 and there are several very small peaks at higher levels of income.

Figure A.8 Kernel Densities for the Old and the New Countries in 1965 and 1970



Additionally, a scatter plot for the old and the new countries in both years is shown in Figure A.8. It enables another look at what type of countries enters the dataset – poor ones or rich ones.

Finally, Figure A.9 shows histograms which again offer a different view on the elaboration on the new countries. It becomes apparent that the distributions already include poor countries as the largest group, while the new countries (especially in 1970) are almost entirely poor countries, despite for some exceptions.

Hence, including these countries into the dataset starting in 1960 would bias this dataset as well. The new countries are no random draw. Yet, comparing this to the kernel density in 1960 in Figure A.3 shows that adding these countries would only increase the already large peak at the low income level. Yet, it does not bring about any gain for the analysis. Thus, it was decided to stick to 1960 as the starting year of the analyses and use only those countries offering data in all years under consideration in this doctoral thesis.

Figure A.9 Scatter Plots for the Old and the New Countries in 1965 and 1970

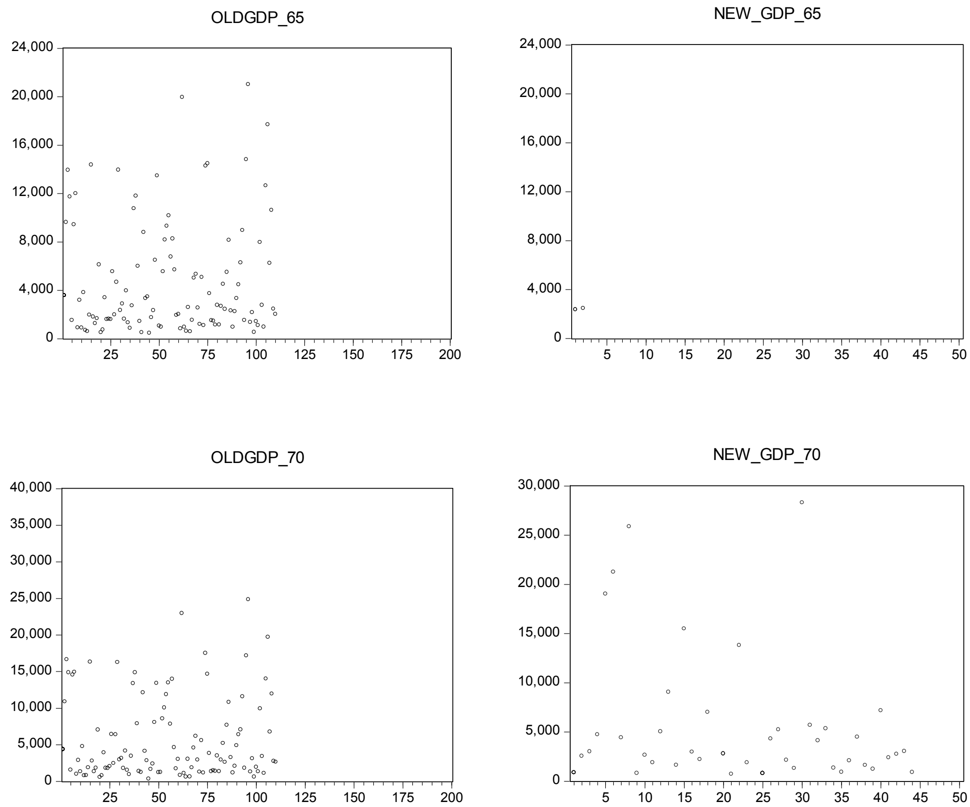
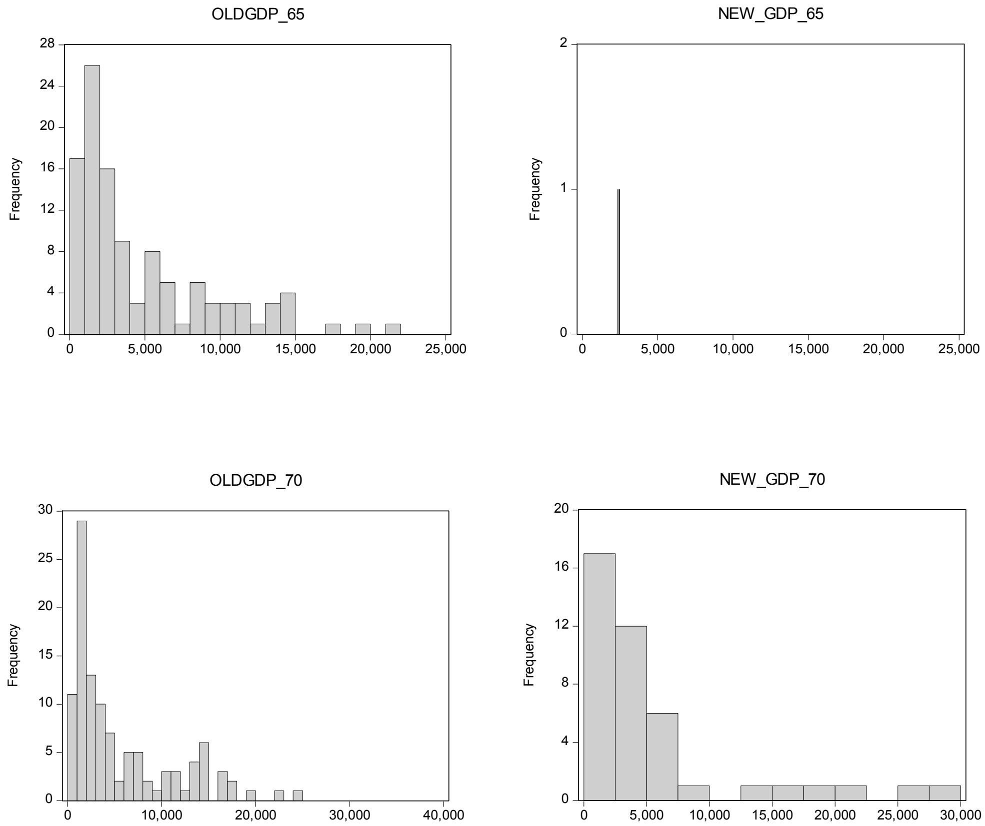


Figure A.10 Histograms for the Old and the New Countries in 1965 and 1970



A.11 The Outliers

Table A.12 The Outliers

Country	Number of years out	% out (total = 58 years)
Bermuda	15	26%
Luxembourg	33	57%
Palau	10	17%
Switzerland	11	19%

Underlying the rule mean ± 3 · standard deviation, especially Luxembourg should be excluded in all years. However, as outlined in Chapter 5, it was decided to keep Luxembourg in the dataset. Also Switzerland was decided to stay in the dataset even though it turns out to be an outlier in almost every fifth year. Bermuda was out in 26 percent of the years. It was decided to exclude it as it is also very small and should not have a large influence. The same accounts for Palau. In the end, these two countries are excluded for the kernel analyses of the real per capita GDP data.

A.12 Club Membership

Table A.13 Club Membership

Country	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Argentina*	1	0	2	2	1	1	1	1	1	1
Australia	0	0	2	2	2	2	2	2	2	2
Austria	0	0	2	2	2	2	2	2	2	2
Bangladesh	1	1	1	1	1	1	1	1	1	1
Barbados	1	0	2	2	2	2	2	2	2	2
Belgium	0	0	2	2	2	2	2	2	2	2
Benin	1	1	1	1	1	1	1	1	1	1
Bolivia	1	1	1	1	1	1	1	1	1	1
Brazil	1	1	1	1	1	1	1	1	1	1
Burkina Faso	1	1	1	1	1	1	1	1	1	1
Burundi	1	1	1	1	1	1	1	1	1	1
Cameroon	1	1	1	1	1	1	1	1	1	1
Canada	0	0	2	2	2	2	2	2	2	2
Cape Verde	1	1	1	1	1	1	1	1	1	1
Central African Rep.	1	1	1	1	1	1	1	1	1	1
Chad	1	1	1	1	1	1	1	1	1	1
Chile	1	1	1	1	1	1	1	1	1	1
China	1	1	1	1	1	1	1	1	1	1
Colombia	1	1	1	1	1	1	1	1	1	1
Comoros	1	1	1	1	1	1	1	1	1	1
Congo, Dem. Rep.	1	1	1	1	1	1	1	1	1	1
Congo, Republic of	1	1	1	1	1	1	1	1	1	1
Costa Rica	1	1	1	1	1	1	1	1	1	1
Cote d`Ivoire	1	1	1	1	1	1	1	1	1	1
Cyprus**	1	1	1	1	1	1	1	2	2	2
Denmark	0	0	2	2	2	2	2	2	2	2
Dominican Republic	1	1	1	1	1	1	1	1	1	1
Ecuador	1	1	1	1	1	1	1	1	1	1
Egypt	1	1	1	1	1	1	1	1	1	1
El Salvador	1	1	1	1	1	1	1	1	1	1
Equatorial Guinea**[210]	1	1	1	1	1	1	1	1	1	2

[210] It may seem strange to define Equatorial Guinea as a growth miracle. Yet, it depends on the definition of the clubs. According to the definitions in this doctoral thesis, it is a growth miracle, even though this only occurred in the last year considered. Of course it is also possible that the country moves back to Club 1 already in the next five years, so that the membership in Club 1 was only a temporary one.

Ethiopia	1	1	1	1	1	1	1	1	1	1
Fiji	1	1	1	1	1	1	1	1	1	1
Finland	1	0	2	2	2	2	2	2	2	2
France	0	0	2	2	2	2	2	2	2	2
Gabon	1	1	1	1	1	1	1	1	1	1
Gambia, The	1	1	1	1	1	1	1	1	1	1
Ghana	1	1	1	1	1	1	1	1	1	1
Greece	1	1	2	2	2	2	2	2	2	2
Guatemala	1	1	1	1	1	1	1	1	1	1
Guinea	1	1	1	1	1	1	1	1	1	1
Guinea-Bissau	1	1	1	1	1	1	1	1	1	1
Haiti	1	1	1	1	1	1	1	1	1	1
Honduras	1	1	1	1	1	1	1	1	1	1
Hong Kong**	1	1	1	1	2	2	2	2	2	2
Iceland	0	0	2	2	2	2	2	2	2	2
India	1	1	1	1	1	1	1	1	1	1
Indonesia	1	1	1	1	1	1	1	1	1	1
Iran	1	1	1	1	1	1	1	1	1	1
Ireland*/**	1	1	2	2	1	1	2	2	2	2
Israel	1	0	2	2	2	2	2	2	2	2
Italy	1	0	2	2	2	2	2	2	2	2
Jamaica	1	1	1	1	1	1	1	1	1	1
Japan	1	1	2	2	2	2	2	2	2	2
Jordan	1	1	1	1	1	1	1	1	1	1
Kenya	1	1	1	1	1	1	1	1	1	1
Korea, Republic of**	1	1	1	1	1	1	1	2	2	2
Lesotho	1	1	1	1	1	1	1	1	1	1
Luxembourg	0	0	2	2	2	2	2	2	2	2
Madagascar	1	1	1	1	1	1	1	1	1	1
Malawi	1	1	1	1	1	1	1	1	1	1
Malaysia	1	1	1	1	1	1	1	1	1	1
Mali	1	1	1	1	1	1	1	1	1	1
Mauritania	1	1	1	1	1	1	1	1	1	1
Mauritius	1	1	1	1	1	1	1	1	1	1
Mexico	1	1	1	1	1	1	1	1	1	1
Morocco	1	1	1	1	1	1	1	1	1	1
Mozambique	1	1	1	1	1	1	1	1	1	1
Namibia	1	1	1	1	1	1	1	1	1	1
Nepal	1	1	1	1	1	1	1	1	1	1
Netherlands	0	0	2	2	2	2	2	2	2	2
New Zealand	0	0	2	2	2	2	2	2	2	2
Nicaragua	1	1	1	1	1	1	1	1	1	1
Niger	1	1	1	1	1	1	1	1	1	1
Nigeria	1	1	1	1	1	1	1	1	1	1

Pakistan	1	1	1	1	1	1	1	1	1	1
Panama	1	1	1	1	1	1	1	1	1	1
Papua New Guinea	1	1	1	1	1	1	1	1	1	1
Paraguay	1	1	1	1	1	1	1	1	1	1
Peru	1	1	1	1	1	1	1	1	1	1
Philippines	1	1	1	1	1	1	1	1	1	1
Portugal**	1	1	1	1	1	1	1	1	2	2
Puerto Rico*/**	1	1	2	1	1	1	2	2	2	2
Romania	1	1	1	1	1	1	1	1	1	1
Rwanda	1	1	1	1	1	1	1	1	1	1
Senegal	1	1	1	1	1	1	1	1	1	1
Seychelles**/*	1	1	1	1	1	1	1	1	2	1
Singapore**	1	1	1	1	1	2	2	2	2	2
South Africa	1	1	1	1	1	1	1	1	1	1
Spain	1	0	2	2	2	2	2	2	2	2
Sri Lanka	1	1	1	1	1	1	1	1	1	1
Sweden	0	0	2	2	2	2	2	2	2	2
Switzerland	0	0	2	2	2	2	2	2	2	2
Syria	1	1	1	1	1	1	1	1	1	1
Taiwan**	1	1	1	1	1	1	1	2	2	2
Tanzania	1	1	1	1	1	1	1	1	1	1
Thailand	1	1	1	1	1	1	1	1	1	1
Togo	1	1	1	1	1	1	1	1	1	1
Turkey	1	1	1	1	1	1	1	1	1	1
Uganda	1	1	1	1	1	1	1	1	1	1
United Kingdom	0	0	2	2	2	2	2	2	2	2
United States	0	0	2	2	2	2	2	2	2	2
Uruguay	1	1	1	1	1	1	1	1	1	1
Zambia	1	1	1	1	1	1	1	1	1	1
Zimbabwe	1	1	1	1	1	1	1	1	1	1

1 = member in the club of the poor

2 = member in the club of the rich

0 = member in no club (in the years 1960 and 1965 there is only a club of poor, none of the rich)

* Growth disaster (defined as a movement of Group 2 to Group 1)[211]

** Growth miracle (defined as a movement of Group 1 to Group 2)

[211] Decisive is the overall movement from 1970 on, when there were two income groups, hence when the twin peaks became a common feature of the real per capita income distribution across the countries of the world.

A.13 The General Form of Table 5.9

Table A.14 The Movement Among Income Classes (General Form)

Target year	Source year						Total
	1	2	3	4	5	6	
1	p_{11}	p_{21}	p_{31}	p_{41}	p_{51}	p_{61}	∑
2	p_{12}	p_{22}	p_{32}	p_{42}	p_{52}	p_{62}	∑
3	p_{13}	p_{23}	p_{33}	p_{43}	p_{53}	p_{63}	∑
4	p_{14}	p_{24}	p_{34}	p_{44}	p_{54}	p_{64}	∑
5	p_{15}	p_{25}	p_{35}	p_{45}	p_{55}	p_{65}	∑
6	p_{16}	p_{26}	p_{36}	p_{46}	p_{56}	p_{66}	∑
Total	∑	∑	∑	∑	∑	∑	∑

A.14 Membership in the Income Groups (Markov Chain, Jones)

Table A.15 Membership in the Income Groups (Jones Distribution)

Country	1960	1970	1980	1988	1990	2000	2007
Argentina	5	5	5	4	4	4	4
Australia	6	6	6	5	5	5	6
Austria	5	5	6	6	6	6	6
Bangladesh	3	2	2	2	2	1	2
Barbados	5	5	6	5	5	5	5
Belgium	5	5	6	5	5	5	5
Benin	2	2	1	1	1	1	1
Bolivia	3	3	3	2	2	2	2
Brazil	4	4	4	4	4	4	4
Burkina Faso	2	1	1	1	1	1	1
Burundi	1	1	1	1	1	1	1
Cameroon	3	2	3	3	2	2	2
Canada	6	6	6	6	6	6	6
Cape Verde	3	3	2	3	3	3	3
Central African Republic	2	2	2	1	1	1	1
Chad	3	2	1	2	1	1	2
Chile	4	4	4	4	4	4	5
China	1	1	1	2	2	3	3
Colombia	4	3	4	3	3	3	3
Comoros	2	2	2	2	2	1	1
Congo, Dem. Rep.	3	2	2	1	1	1	1
Congo, Republic of	2	3	3	3	3	2	2
Costa Rica	4	4	4	4	4	4	4
Cote d`Ivoire	3	3	3	2	2	2	2
Cyprus	4	4	4	5	5	5	5
Denmark	5	6	5	5	5	5	5
Dominican Republic	3	3	3	3	3	3	4
Ecuador	3	3	4	3	3	3	3
Egypt	2	2	2	3	3	3	3
El Salvador	4	4	3	3	3	3	3
Equatorial Guinea	2	2	2	2	1	4	5
Ethiopia	2	1	1	1	1	1	1
Fiji	3	3	4	3	3	3	3
Finland	5	5	5	5	5	5	5
France	5	5	6	5	5	5	5
Gabon	4	4	5	4	4	4	3
Gambia, The	2	2	2	1	1	1	1
Ghana	1	2	2	1	1	1	1

Greece	5	5	5	5	5	5	5
Guatemala	4	4	4	3	3	3	3
Guinea*	4	3	3	2	2	2	2
Guinea-Bissau	1	1	1	1	1	1	1
Haiti	3	2	2	2	2	1	1
Honduras	3	3	3	3	3	2	2
Hong Kong**	4	5	5	6	6	6	6
Iceland	5	5	6	6	6	6	6
India	2	2	2	2	2	2	2
Indonesia	2	2	2	2	3	3	3
Iran	4	5	4	3	3	4	4
Ireland	5	5	5	5	5	5	6
Israel	5	5	5	5	5	5	5
Italy	5	5	5	5	5	5	5
Jamaica	4	4	4	4	4	3	3
Japan	4	5	5	5	6	5	5
Jordan	4	4	4	4	3	3	3
Kenya	3	2	2	2	2	1	1
Korea, Republic of	3	3	4	4	4	5	5
Lesotho	1	1	2	1	1	1	2
Luxembourg	6	6	6	6	6	6	6
Madagascar	2	2	1	1	1	1	1
Malawi	1	1	1	1	1	1	1
Malaysia	3	3	4	4	4	4	5
Mali	2	1	1	1	1	1	1
Mauritania	2	2	2	2	2	2	2
Mauritius	4	4	4	4	4	4	5
Mexico	4	4	4	4	4	4	4
Morocco	3	3	3	3	3	3	3
Mozambique	2	2	2	1	1	1	2
Namibia	4	4	4	3	3	3	3
Nepal	2	2	1	1	1	1	1
Netherlands	6	6	6	5	5	6	6
New Zealand	6	5	5	5	5	5	5
Nicaragua	3	3	3	2	2	2	2
Niger	2	2	1	1	1	1	1
Nigeria*	3	2	2	1	1	1	2
Pakistan	2	2	2	2	2	2	2
Panama	3	3	4	3	3	3	4
Papua New Guinea	2	2	2	2	2	2	2
Paraguay	3	3	3	3	3	3	3
Peru	4	4	4	3	3	3	3
Philippines	3	3	3	3	3	3	3
Portugal	4	4	5	5	5	5	5

Puerto Rico	4	5	5	5	5	5	5
Romania	3	3	4	4	4	3	4
Rwanda	2	2	1	1	1	1	1
Senegal	3	3	2	2	2	1	1
Seychelles	4	4	4	4	5	5	5
Singapore	4	4	5	5	5	6	6
South Africa	4	4	4	4	4	4	4
Spain	4	5	5	5	5	5	5
Sri Lanka	2	2	2	3	3	3	3
Sweden	6	6	6	5	5	5	5
Switzerland	6	6	6	6	6	6	6
Syria	2	2	2	2	2	2	2
Taiwan	3	3	4	4	5	5	5
Tanzania	1	1	1	1	1	1	1
Thailand	2	2	3	3	3	3	4
Togo	2	2	2	1	1	1	1
Turkey	3	3	3	3	3	3	3
Uganda	2	2	1	1	1	1	1
United Kingdom	5	5	5	5	5	5	5
United States	6	6	6	6	6	6	6
Uruguay	5	4	4	4	4	4	4
Zambia	3	3	2	2	1	1	1
Zimbabwe	3	3	3	2	2	2	1

* Growth disaster (defined as a downward movement of two or more income groups from 1960 to 1988)

** Growth miracle (defined as an upward movement of two or more income groups from 1960 to 1988)

A.15 Markov Chains for the Long Run Income Distribution

Table A.16 The Movement among Income Classes (1960 to 1988)

Target (1988)	Source (1960)						
	1	2	3	4	5	6	Total
1	6	13	2	0	0	0	21
2	1	8	10	11	0	0	20
3	0	4	11	6	0	0	21
4	0	0	4	10	2	0	16
5	0	0	0	6	10	4	20
6	0	0	0	1	2	4	7
Total	7	25	27	24	14	8	105

Table A.17 Transition Matrix[212] (1960 to 1988)

Target (1988)	Source (1960)					
	1	2	3	4	5	6
1	0.857	0.520	0.074	0	0	0
2	0.143	0.320	0.370	0.042	0	0
3	0	0.160	0.407	0.250	0	0
4	0	0	0.148	0.417	0.143	0
5	0	0	0	0.250	0.714	0.500
6	0	0	0	0.042	0.143	0.500

[212] Transition probability indicates, for example, the probability that a country currently in Group 1 (in the source year) will be in Group 1 also in the target year. Hence, the position Group 1 – Group 1 will be divided by the total number of countries in this group in the source year.

Table A.18 The Movement among Income Classes (1960 to 1990)

Target (1990)	Source (1960)						
	1	2	3	4	5	6	Total
1	6	14	4	0	0	0	24
2	1	6	9	1	0	0	17
3	0	5	10	7	0	0	22
4	0	0	3	8	2	0	13
5	0	0	1	6	10	4	21
6	0	0	0	2	2	4	8
Total	7	25	27	24	14	8	105

Table A.19 Transition Matrix[213] (1960 to 1990)

Target (1990)	Source (1960)					
	1	2	3	4	5	6
1	0.857	0.560	0.148	0	0	0
2	0.143	0.240	0.333	0.042	0	0
3	0	0.200	0.370	0.292	0	0
4	0	0	0.111	0.333	0.143	0
5	0	0	0.037	0.250	0.714	0.500
6	0	0	0	0.083	0.143	0.500

[213] Transition probability indicates, for example, the probability that a country currently in Group 1 (in the source year) will be in Group 1 also in the target year. Hence, the position Group 1 – Group 1 will be divided by the total number of countries in this group in the source year.

Table A.20 The Movement among Income Classes (1960 to 2000)

Target (2000)	Source (1960)						
	1	2	3	4	5	6	Total
1	6	14	8	0	0	0	28
2	0	6	6	1	0	0	13
3	1	4	10	7	0	0	22
4	0	1	1	8	2	0	12
5	0	0	2	6	10	3	21
6	0	0	0	2	2	5	9
Total	7	25	27	24	14	8	105

Table A.21 Transition Matrix[214] (1960 to 2000)

Target (2000)	Source (1960)					
	1	2	3	4	5	6
1	0.857	0.560	0.296	0	0	0
2	0	0.240	0.222	0.042	0	0
3	0.143	0.160	0.370	0.292	0	0
4	0	0.040	0.037	0.333	0.143	0
5	0	0	0.074	0.250	0.714	0.375
6	0	0	0	0.083	0.146	0.625

[214] Transition probability indicates, for example, the probability that a country currently in Group 1 (in the source year) will be in Group 1 also in the target year. Hence, the position Group 1 – Group 1 will be divided by the total number of countries in this group in the source year.

Table A.22 The Movement among Income Classes (1960 to 2007)

Target (2007)	Source (1960)						
	1	2	3	4	5	6	Total
1	5	13	6	0	0	0	24
2	1	7	8	1	0	0	17
3	1	3	7	8	0	0	19
4	0	1	3	5	2	0	11
5	0	1	3	8	9	2	23
6	0	0	0	2	3	6	11
Total	7	25	27	24	14	8	105

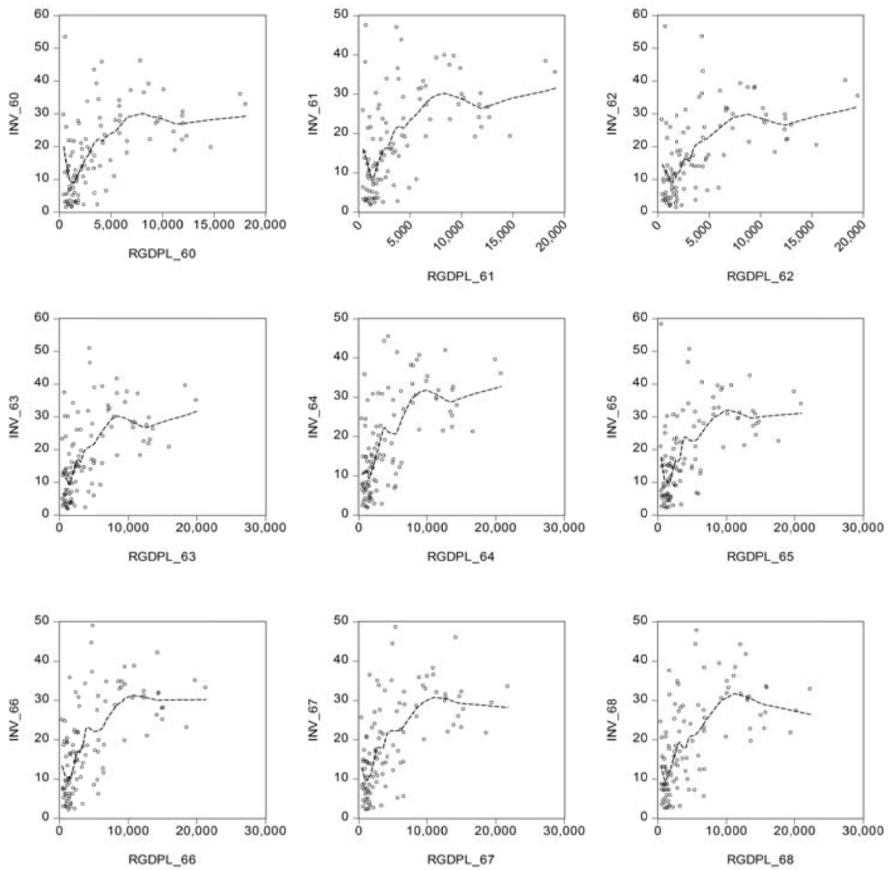
Table A.23 Transition Matrix[215] (1960 to 2007)

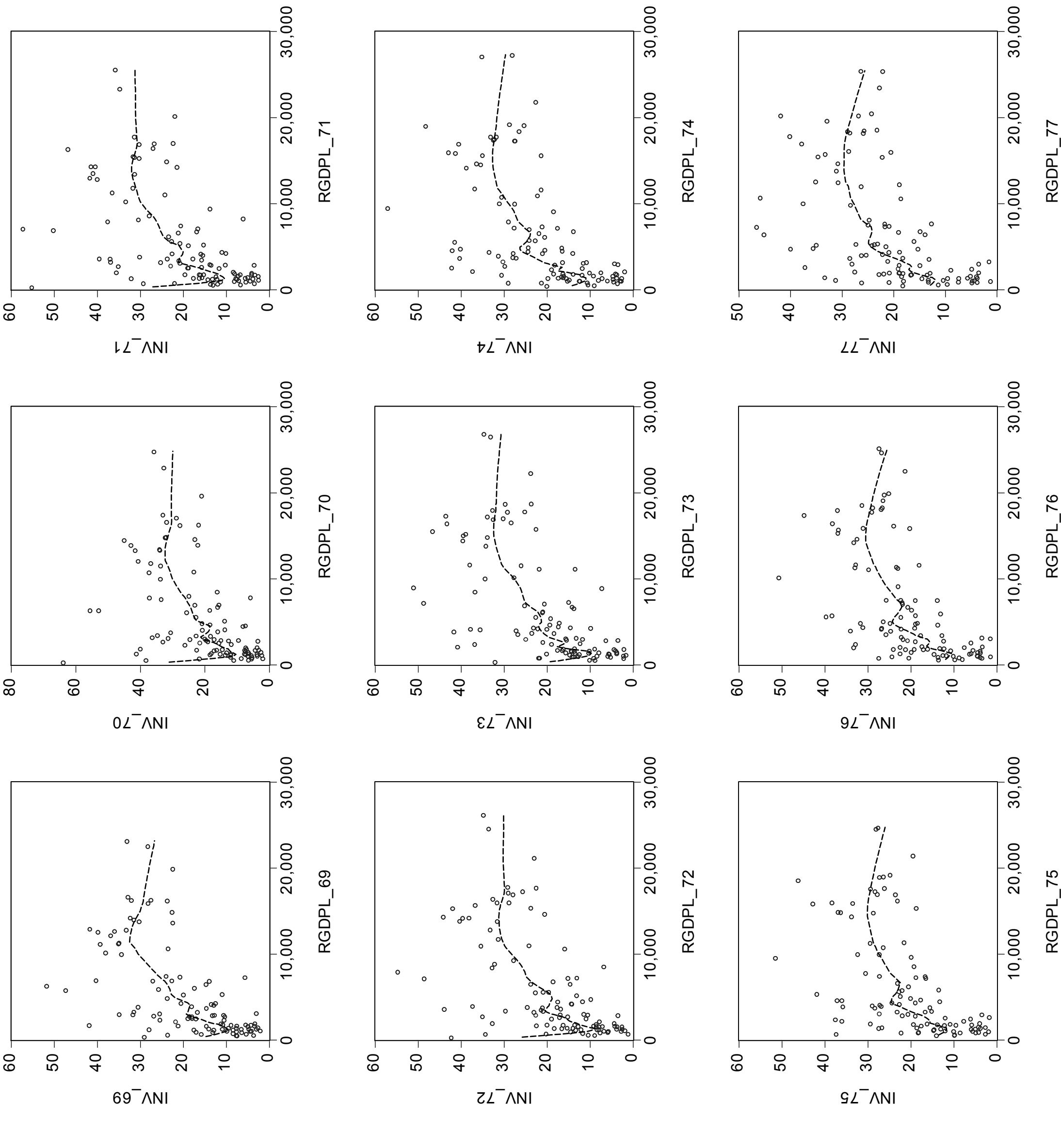
Source (1960)	Target (2007)					
	1	2	3	4	5	6
1	0.714	0.520	0.222	0	0	0
2	0.143	0.280	0.296	0.042	0	0
3	0.143	0.120	0.259	0.333	0	0
4	0	0.040	0.111	0.208	0.143	0
5	0	0.040	0.111	0.333	0.643	0.250
6	0	0	0	0.083	0.214	0.750

[215] Transition probability indicates, for example, the probability that a country currently in Group 1 (in the source year) will be in Group 1 also in the target year. Hence, the position Group 1 – Group 1 will be divided by the total number of countries in this group in the source year.

A.16 The Loess Fit Curves for all Years

Figure A.11 Savings versus Real per Capita GDP





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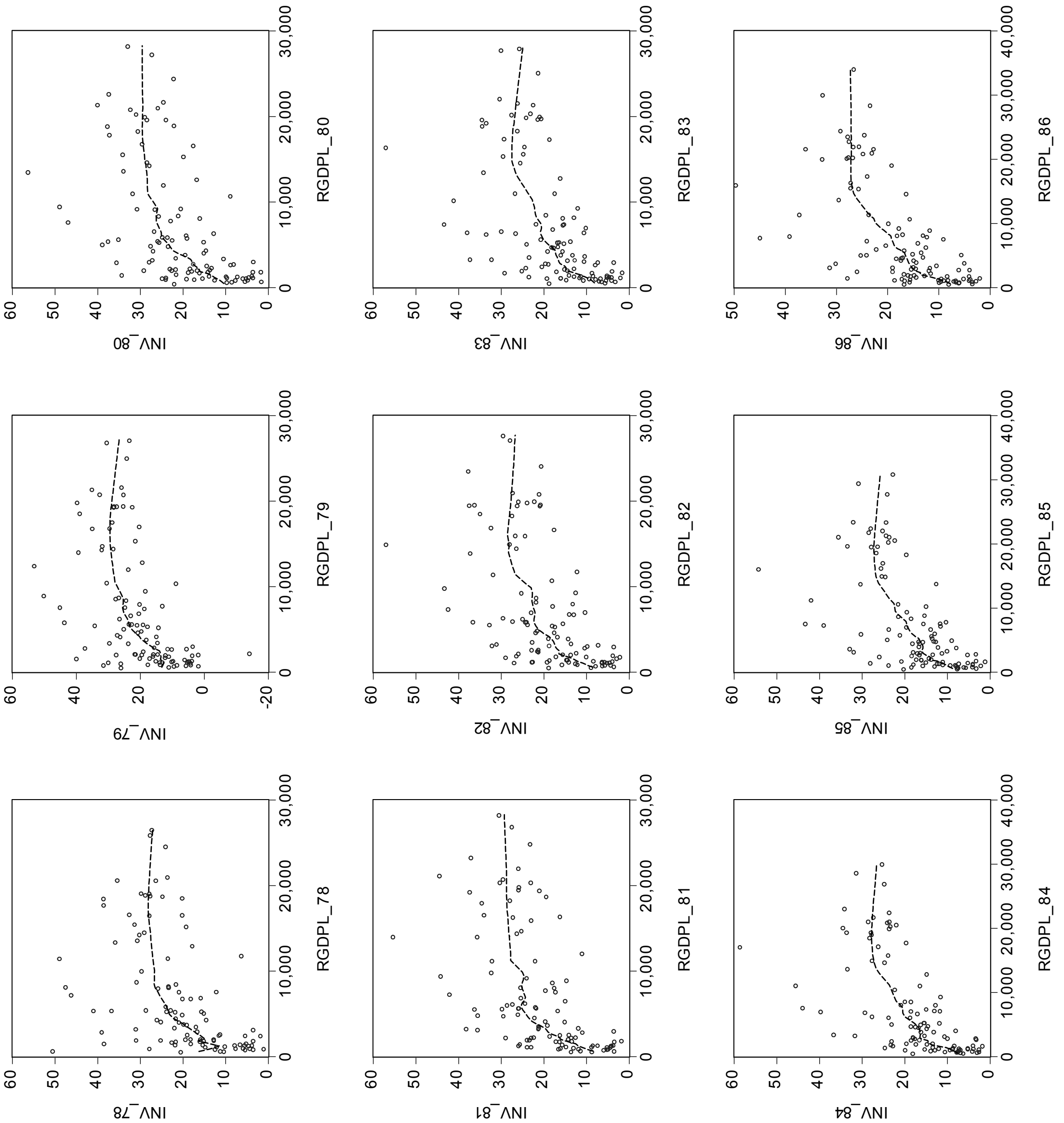
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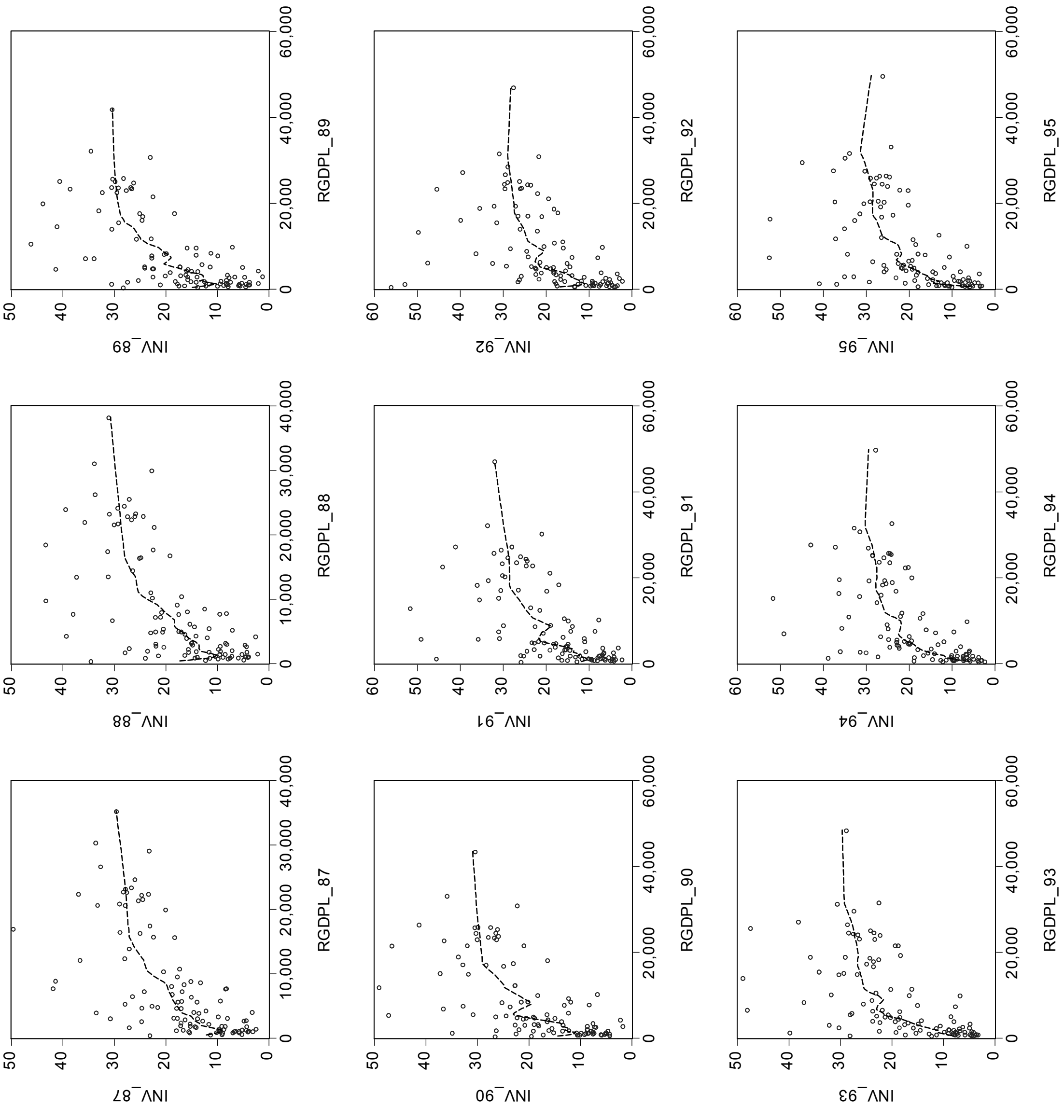
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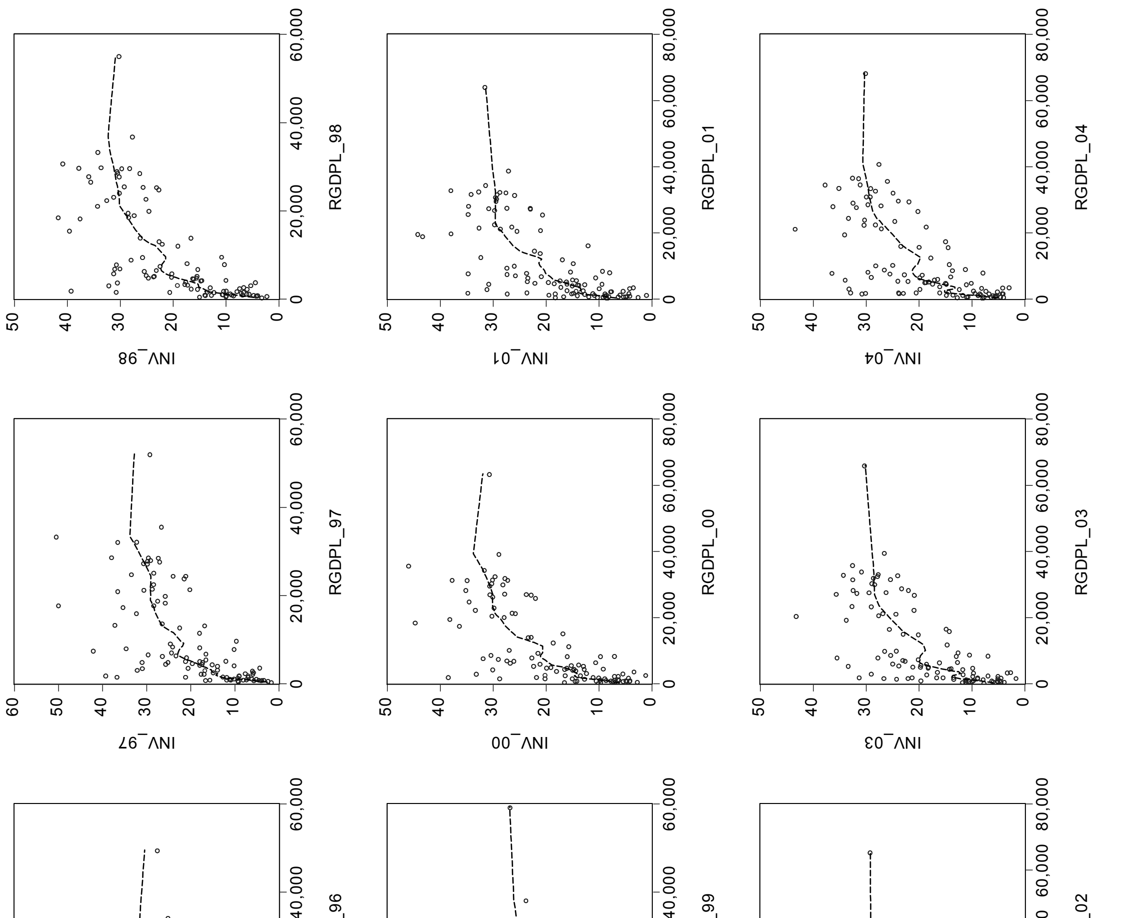
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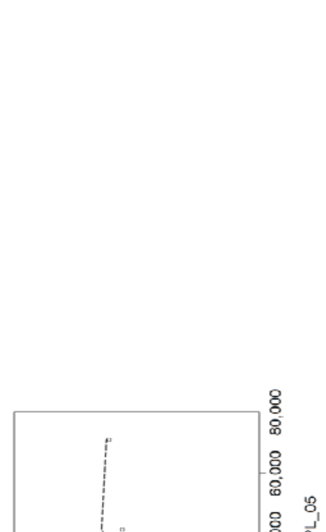
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INV_06
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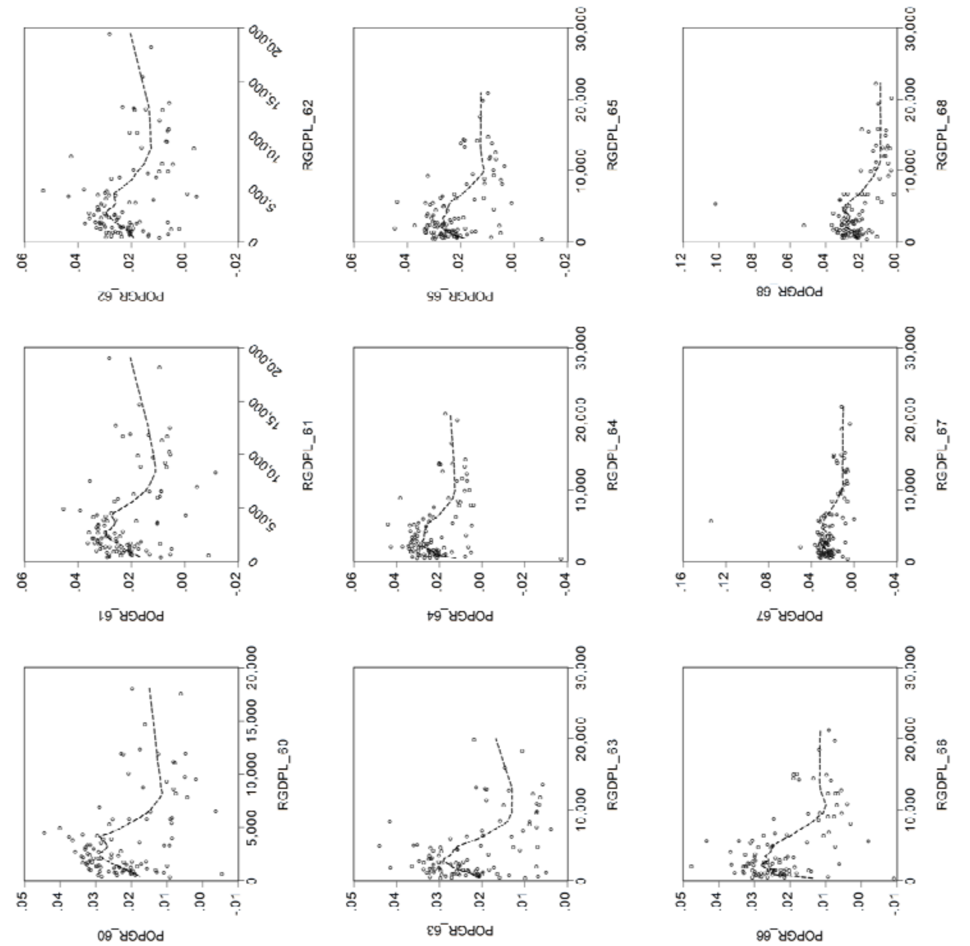
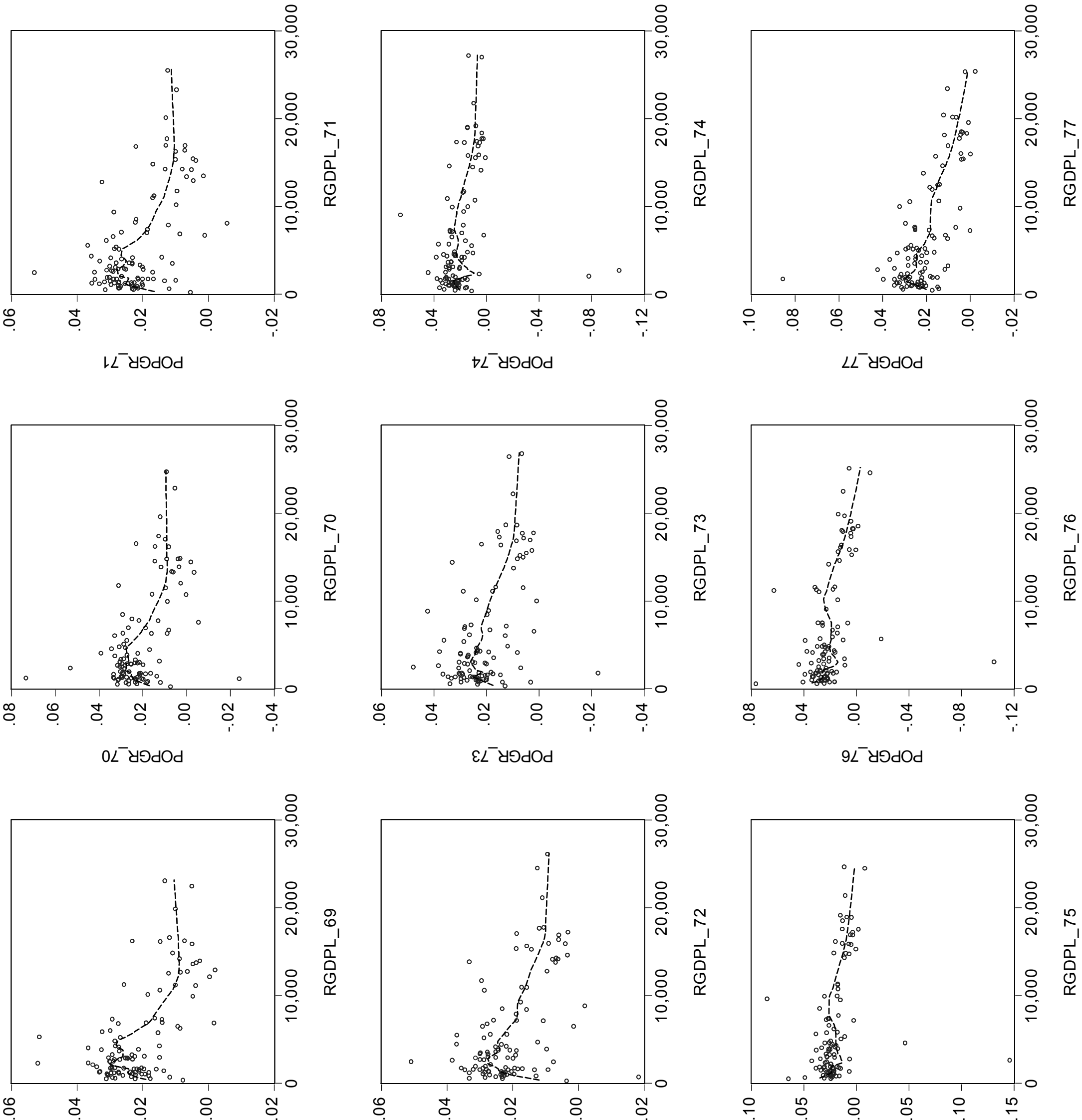
INV_07
RGDPL_07

Figure A.12 Population Growth versus Real per Capita GDP

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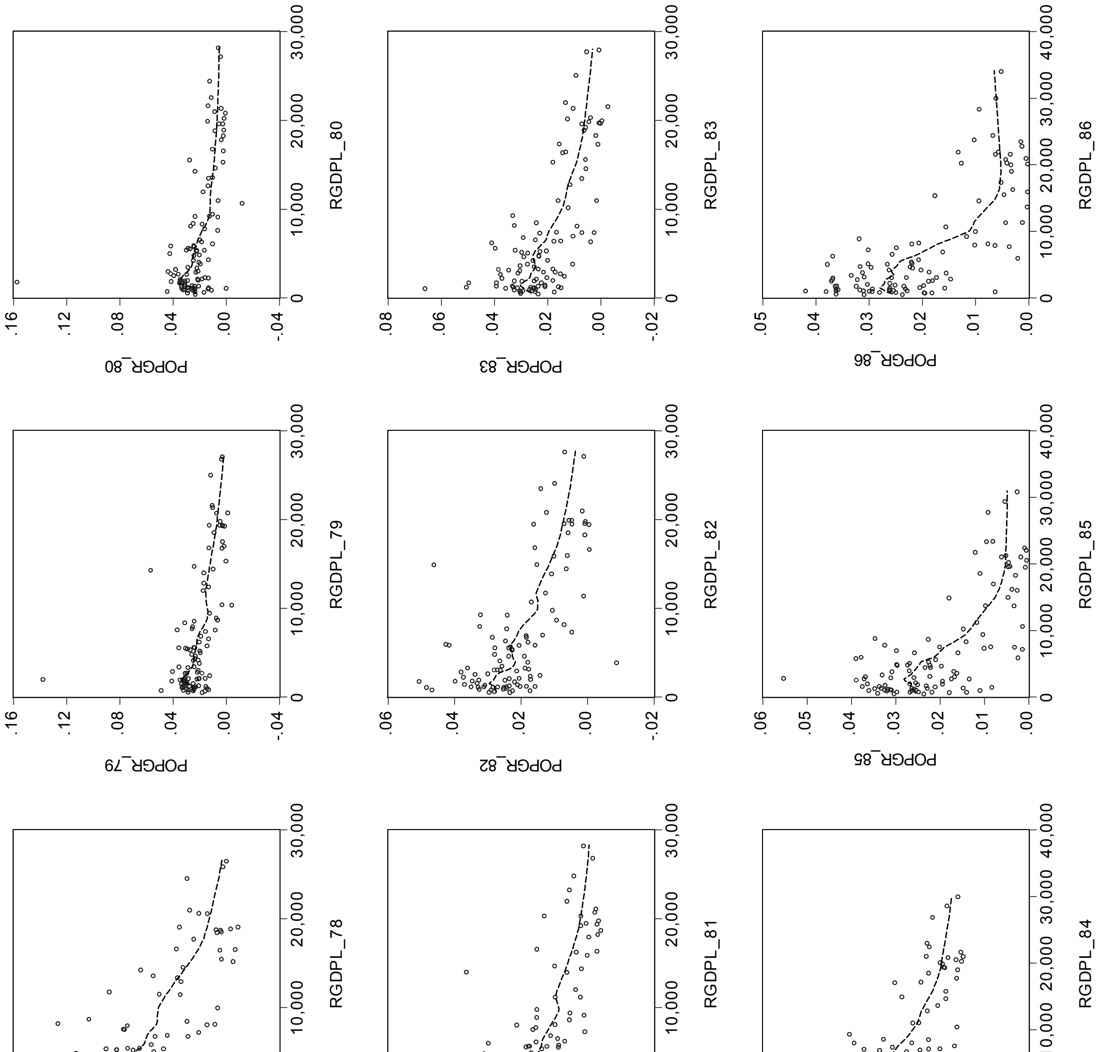
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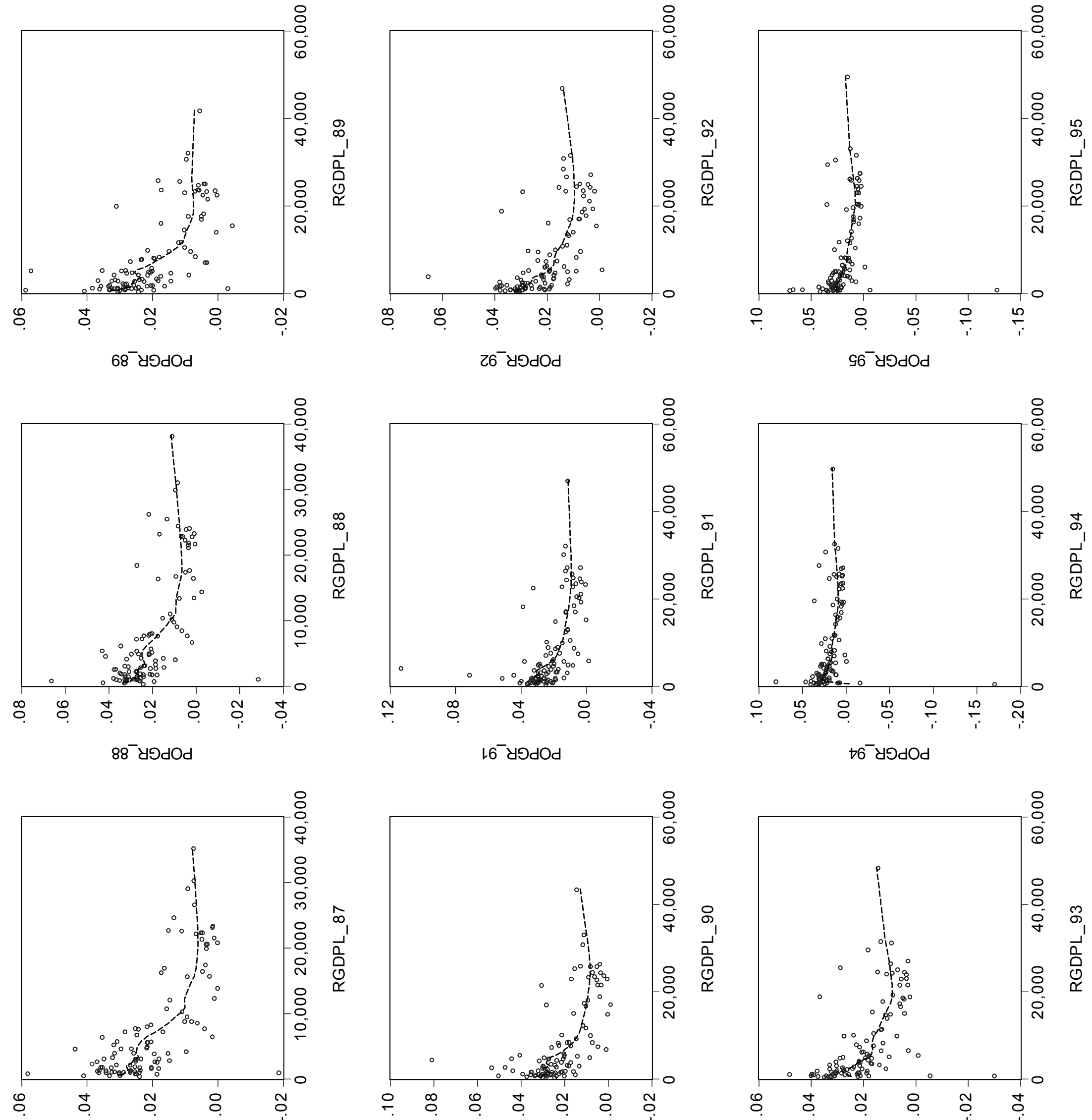

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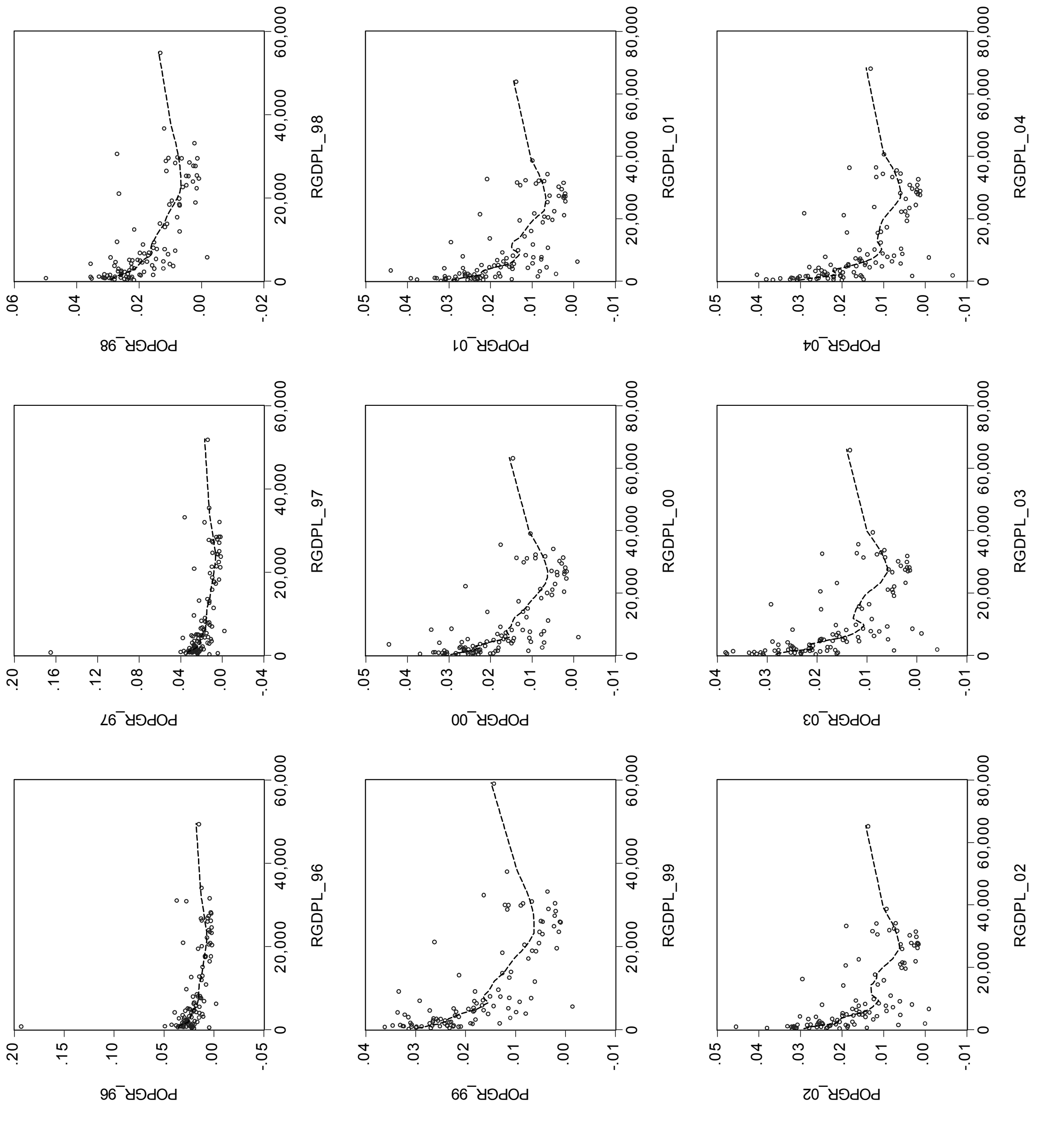
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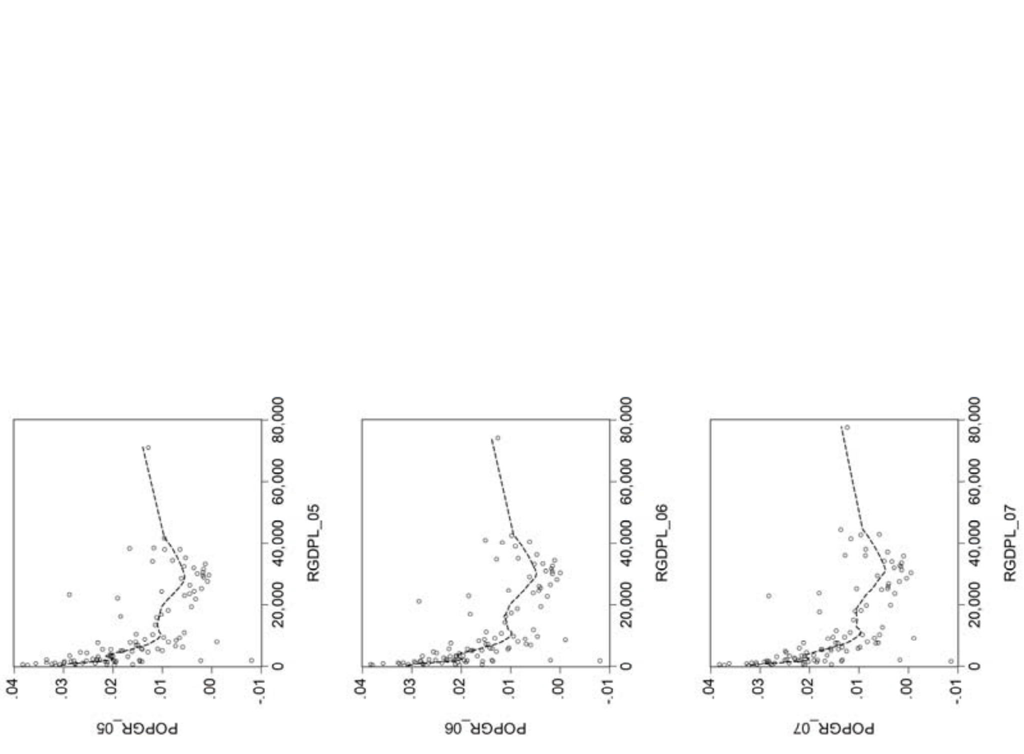
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RGDPL_05
POPGR_06
RGDPL_06
POPGR_07
RGDPL_07

A.17 Determination of the Logistic Function

It is a widely used assumption that population growth underlies the idea of logistic growth. This logistic growth has the form

$$\frac{dP}{dt} = \lambda P(K - P). \tag{A.14}$$

The solutions of this differential equation are called logistic functions. One form of these functions is

$$P(t) = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-\lambda K t}} . \tag{A.15}$$

The determination of this logistic function shall be presented here (based on Heidhorn, 2014).

The basic idea for finding the solution of the differential equation given in Equation (A.14) is based on the following relationship: a primitive of $\frac{P'(t)}{P(t)}$ is given by

$$\int_0^t \frac{P'(z)}{P(z)} dz = |ln(P(z))|_0^t = ln\left(\frac{P(t)}{P(0)}\right). \tag{A.16}$$

In order to be able to use this relationship, the differential equation needs to be reformulated:

$$\frac{P\prime(t)}{P(t)(K-P(t))} = \lambda. \tag{A.17}$$

This fraction, ignoring for the time being the numerator, can then be subdivided into:

$$\frac{1}{P(t)(K-P(t))} = \frac{A}{P(t)} + \frac{B}{K-P(t)}. \tag{A.18}$$

Reformulation yields:

$$\frac{1}{P(t)\big(K-P(0)\big)} = \frac{AK+(B-A)P(t)}{P(t)(K-P(t))}. \tag{A.19}$$

In order to fulfill Equation (A.18), the following conditions need to hold:

$$AK = 1 \tag{A.20}$$

$$B = A. \tag{A.21}$$

Otherwise, the numerator would not become 1. Using this information, the following condition results:

$$A = \frac{1}{K} = B. \tag{A.22}$$

This equation can then be substituted into Equation (A.17):

$$\frac{1}{K}\frac{P\prime(t)}{P(t)} + \frac{1}{K}\frac{P\prime(t)}{K-P(t)} = \lambda. \tag{A.23}$$

Now, the relationship given in Equation (A.16) will be used. First, Equation (A.23) will be integrated:

$$\frac{1}{K} ln\left(\frac{P(t)}{P(0)}\right) - \frac{1}{K} ln\left(\frac{K-P(t)}{K-P(0)}\right) = \lambda t. \quad \text{(A.24)}$$

Using another mathematical rule, Equation (A.24) can be reformulated:

$$ln(x_1) - ln(x_2) = ln\left(\frac{x_1}{x_2}\right). \quad \text{(A.25)}$$

$$\Rightarrow \frac{1}{K} ln\left(\frac{P(t)}{P(0)}\frac{K-P(0)}{K-P(t)}\right) = \lambda t. \quad \text{(A.26)}$$

Antilogging now yields:

$$\frac{P(t)}{P(0)}\frac{K-P(0)}{K-P(t)} = e^{K\lambda t}. \quad \text{(A.27)}$$

This equation needs to be solved for $P(t)$ in order to derive the underlying logistic function as given in Equation (A.15).

$$\Leftrightarrow P(t)\frac{K-P(0)}{P(0)} = e^{K\lambda t}(K - P(t)) \quad \text{(A.28)}$$

$$\Leftrightarrow P(t)\left(\frac{K-P(0)}{P(0)} + e^{K\lambda t}\right) = Ke^{K\lambda t} \quad \text{(A.29)}$$

$$\Leftrightarrow P(t) = \frac{P(0)Ke^{K\lambda t}}{K-P(0)+P(0)e^{K\lambda t}} \quad \text{(A.30)}$$

This fraction on the right-hand side will now be augmented by $e^{-K\lambda t}$ in order to get rid of this expression in the numerator:

$$P(t) = \frac{P(0)K}{P(0)+(K-P(0))e^{-K\lambda t}}. \quad \text{(A.31)}$$

Finally, $P(0)$ is eliminated from the numerator:

$$P(t) = \frac{K}{1+\left(\frac{K}{P(0)}-1\right)e^{-K\lambda t}} \qquad \text{q.e.d.} \quad \text{(A.32)}$$

A.18 How to Determine Roots of Cubic Equations – a Proof[216]

A cubic equation contains polynomials up to the third order and has the following form:

$$f(y) = ay^3 + by^2 + cy + d, \tag{A.33}$$

where $a \neq 0$ and $a, b, c,$ and d are real numbers. ay^3 is called the cubic term, by^2 is the quadratic term, cy is the linear term, and d is the absolute term. Now, the roots of Equation (A.33) ought to be determined.

$$ay^3 + by^2 + cy + d = 0 \tag{A.34}$$

To begin with, the equation needs to be divided by a in order to reach the normal form given in Equation (A.36).

$$y^3 + \frac{b}{a}y^2 + \frac{c}{a}y + \frac{d}{a} = 0 \tag{A.35}$$

$$\Leftrightarrow y^3 + ry^2 + sy + t = 0, \tag{A.36}$$

where $r = \frac{b}{y}, s = \frac{c}{a}, t = \frac{d}{a}$. By substituting Equation (A.37) into Equation (A.36), the reduced cubic function (A.45) can be reached. This will be shown stepwise in the following.

$$z \overset{\text{def}}{=} y + \frac{r}{3} \Leftrightarrow y = z - \frac{r}{3} \tag{A.37}$$

$$\left(z - \frac{r}{3}\right)^3 + r\left(z - \frac{r}{3}\right)^2 + s\left(z - \frac{r}{3}\right) + t = 0 \tag{A.38}$$

$$\Leftrightarrow \left(z^2 - \frac{2r}{3}z + \frac{r^2}{9}\right)\left(z - \frac{r}{3}\right) + r\left(z^2 - \frac{2r}{3}z + \frac{r^2}{9}\right) + sz - \frac{sr}{3} + t = 0 \tag{A.39}$$

$$\Leftrightarrow z^3 - \frac{2r}{3}z^2 + \frac{r^2}{9}z - \frac{r}{3}z^2 + \frac{2r^2}{9}z - \frac{r^3}{27} + rz^2 - \frac{2r^2}{3}z + \frac{r^3}{9} + sz - \frac{sr}{3} + t = 0 \tag{A.40}$$

$$\Leftrightarrow z^3 + \left(-\frac{2r}{3} - \frac{r}{3} + r\right)z^2 + \left(\frac{r^2}{9} + \frac{2r^2}{9} - \frac{2r^2}{3} + s\right)z - \frac{r^3}{27} + \frac{r^3}{9} - \frac{sr}{3} + t = 0 \tag{A.41}$$

$$\Leftrightarrow z^3 + \left(-\frac{3r}{3} + r\right)z^2 + \left(\frac{3r^2}{9} - \frac{6r^2}{9} + s\right)z + \left(-\frac{r^3}{27} + \frac{3r^3}{27} - \frac{sr}{3}\right) + t = 0 \tag{A.42}$$

$$\Leftrightarrow z^3 + 0z^2 + \left(-\frac{3r^2}{9} + s\right)z + \left(\frac{2r^3}{27} - \frac{sr}{3} + t\right) = 0 \tag{A.43}$$

$$\Leftrightarrow z^3 + \left(\frac{3s - r^2}{3}\right)z + \left(\frac{2r^3}{27} - \frac{rs}{3} + t\right) = 0 \tag{A.44}$$

$$z^3 + pz + q = 0, \tag{A.45}$$

where $p = \frac{3s - r^2}{3}$ and $q = \frac{2r^3}{27} - \frac{rs}{3} + t$. If this reduced equation is to be solved in order to find the roots, the solution is dependent on the sign of the discriminant.

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \tag{A.46}$$

Table A.24 shows the possible solutions dependent on the sign of the discriminant and the type of the variable y.

[216] These rules for determining roots of cubic equations are taken out of Bronstein and Semendjajew (1979), pp. 183f. However, any other mathematics book could be used as well. These rules are proved here in order to show how the individual formulas were determined.

Table A.24 The Possible Solutions

	y is a real number	y is a complex number
$D > 0$	one real solution	one real, two conjugated complex solutions
$D < 0$	three real solutions	three real solutions
$D = 0$	one real solution and one real “double solution” or a real “threefold” solution if $p = q = 0$	one real solution and one real “double solution” or a real “threefold” solution if $p = q = 0$

Source: Bronstein and Semendjajew, 1979, p.183

There are four possible ways to find the roots of a cubic equation. The first one is the easiest way, namely by reformulating the function into linear factors which then yield the roots. A second method would be to use approximation methods such as the Newton method, for example. The third method is to use auxiliary quantities which may be calculated by use of a table. These three methods are possible but are not applied in this doctoral thesis. For this reason, they are not discussed in detail here. The interested reader is referred to Bronstein and Semendjajew (1979), for example.

The final method for finding the roots of a cubic equation is the one which was applied in Chapter 7 of this doctorate. Hence, it will be discussed in detail here. Roots of a cubic equation can be found by use of the Cardanic formulas.[217] In order to be able to apply these formulas, the cubic function needs to have a normal form.

$$y^3 + ry^2 + sy + t = 0 \tag{A.47}$$

This equation then has to be reduced as described above to yield

$$z^3 + pz + q = 0. \tag{A.48}$$

Then, the three roots are given by

$$z_1 = u + v \tag{A.49}$$

$$z_2 = -\frac{u+v}{2} + \frac{u+v}{2} i\sqrt{3} = \varepsilon_1 u + \varepsilon_2 v \tag{A.50}$$

$$z_3 = -\frac{u+v}{2} - \frac{u+v}{2} i\sqrt{3} = \varepsilon_2 u + \varepsilon_1 v, \tag{A.51}$$

[217] They may be compared to the $p - q$-formula for finding the roots of a quadratic function.

where $u = \sqrt[3]{-\frac{q}{2} + \sqrt{D}}$, $v = \sqrt[3]{-\frac{q}{2} - \sqrt{D}}$, $D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$, and $\varepsilon_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. By resubstituting $y_n = z_n - \frac{r}{3}$ $(n = 1, 2, 3)$, the original roots y_n of the cubic equation can be determined.

From the above equations for u and v, it becomes obvious that the Cardanic formulas are not defined for $D < 0$. D is used under a square root which, by definition, demands a number bigger than or equal to zero beneath it. If $D < 0$, as in Chapter 7, other formulas have to be used (Bronstein and Semendjajew, 1979, p. 184).

$$z_1 = 2\sqrt[3]{\varrho}\, cos\left(\frac{\varphi}{3}\right) \quad \text{(A.52)}$$

$$z_2 = 2\sqrt[3]{\varrho}\, cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) \quad \text{(A.53)}$$ [218]

$$z_3 = 2\sqrt[3]{\varrho}\, cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) \quad \text{(A.54)}$$ [219]

with $\varrho = \sqrt{-\frac{p^3}{27}}$ and $cos\,\varphi = -\frac{q}{2\varrho}$. Again, by resubstituting $y_n = z_n - \frac{r}{3}$, the roots of the original cubic equation can be determined.

218 $\frac{2\pi}{3} = 120°$

219 $\frac{4\pi}{3} = 240°$

A.19 Determination of the Steady States in Chapter 7

After having seen the general proof of how to determine the roots of a cubic equation, the detailed steps for determining the roots of Equation (7.26) shall be shown based on the theoretical way described in Appendix (A.18).

To begin with the reduced equation needs to be determined:

$$z^3 + pz + q = 0, \tag{A.55}$$

where $p = \frac{3s-r^2}{3}$ and $q = \frac{2r^3}{27} - \frac{rs}{3} + t.$

The specific cubic equation for which the roots ought to be determined is given by:

$$\mathrm{y}^{*^3} - \frac{8.5\cdot 10^{-9}\cdot 0.072\mathrm{A}^2}{5.36\cdot 10^{-7}}\mathrm{y}^{*^2} + \frac{0.001\mathrm{A}^2}{5.36\cdot 10^{-7}}\mathrm{y}^* + \frac{13.562\mathrm{A}^2}{5.36\cdot 10^{-7}} = 0 \tag{A.56}$$

$$\Leftrightarrow y^{*^3} - 134{,}495.538y^{*^2} + 2{,}614{,}925.373y^* + 40{,}483{,}283{,}582 = 0. \tag{A.57}$$

Now, the parameters r, s, and t can be determined. Looking at the above equation shows that $r = -134{,}495.538$, $s = 2{,}614{,}925.373$,[220] and $t = 40{,}483{,}283{,}582$. Now, the reduced form of the function shall be determined by substituting $y^* = z - \frac{r}{3}$.

$$\left(z - \frac{r}{3}\right)^3 - 134{,}495.538\left(z - \frac{r}{3}\right)^2 + 2{,}614{,}925.373\left(z - \frac{r}{3}\right)$$
$$+40{,}483{,}283{,}582 = 0 \tag{A.58}$$

$$\left(z^2 - 2\frac{r}{3}z + \frac{r^2}{9}\right)\left(z - \frac{r}{3}\right) - 134{,}495.538\left(z^2 - 2\frac{r}{3}z + \frac{r^2}{9}\right)$$
$$+2{,}614{,}925.373\left(z - \frac{r}{3}\right) + 40{,}483{,}283{,}582 = 0 \tag{A.59}$$

$$z^3 - \frac{r}{3}z^2 - 2\frac{r}{3}z^2 + 2\frac{r^2}{9}z + \frac{r^2}{9}z - \frac{r^3}{27} - 134{,}495.538z^2$$
$$+134{,}495.538 \cdot 2\frac{r}{3}z - 134{,}495.538\frac{r^2}{9} + 2{,}614{,}925.373z$$
$$-2{,}614{,}925.373\frac{r}{3} + 40{,}483{,}283{,}582 = 0 \tag{A.60}$$

$$z^3 + (-r - 134{,}495.538)z^2 + \left(\frac{r^2}{3} + 134{,}495.538 \cdot 2\frac{r}{3} + 2{,}614{,}925.373\right)z$$
$$+\left(-\frac{r^3}{27} - 134{,}495.538\frac{r^2}{9} - 2{,}614{,}925.373\frac{r}{3} + 40{,}483{,}283{,}582\right) = 0 \tag{A.61}$$

As the term z^2 needs to be dropped out in order to reach the reduced form of the cubic equation, the bracket in front of this term has to be set equal to zero.

$$-r - 134{,}495.538 = 0 \tag{A.62}$$

$$r = -134{,}495.538 \tag{A.63}$$

[220] According to Equation (7.26): $r = -\frac{8.5\cdot 10^{-9}A^2+0.072}{5.36\cdot 10^{-7}}$ and $s = \frac{0.001A^2}{5.36\cdot 10^{-7}}$.

Substituting this into Equation (A.61) yields:

$$z^3 + 0 \cdot z^2 + \left(\frac{(-134{,}495.538)^2}{3} + 134{,}495.538 \cdot 2 \cdot \frac{-134{,}495.538}{3} + 2{,}614{,}925.373\right) z + \left(-\frac{(-134{,}495.538)^3}{27} - 134495.538\frac{(-134{,}495.538)^2}{9} - 2{,}614{,}925.373\frac{-134{,}495.538}{3} + 40{,}483{,}283{,}582\right) = 0 \quad \text{(A.64)}$$

$$z^3 - 6{,}027{,}068{,}339.863z - 180{,}056{,}839{,}436{,}494.06 = 0. \quad \text{(A.65)}$$

Knowing this reduced function, the values for p and q can be read off which then will be used to calculate the discriminant D.

Now, the term $D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$ needs to be calculated. If $D > 0$, the cubic equation has one real and two complex solutions. If $D = 0$, there are three real solutions, two of which are identical. Finally, if $D < 0$, there are three different real solutions. In the first two cases, the Cardano formulas[221] have to be applied in order to find the roots of the equation. In the latter case, also called "casus irreducibilis", the solutions have to be found by use of trigonometric functions.

$$p = -6{,}027{,}068{,}339.863 \quad \text{(A.66)}$$

$$q = -180{,}056{,}839{,}436{,}494.06 \quad \text{(A.67)}$$

$$D = \left(\frac{-6{,}027{,}068{,}339.863}{3}\right)^3 + \left(\frac{-180{,}056{,}839{,}436{,}494.06}{2}\right)^2 \quad \text{(A.68)}$$

$$\Leftrightarrow D = -3.646 \cdot 10^{24} \quad \text{(A.69)}$$

Looking at the cubic function to be analyzed here, D turns out to be negative:

$$D = -3.646 \cdot 10^{24}. \quad \text{(A.70)}$$

As $D < 0$, case three (casus irreducibilis) applies. The solutions can then be found by

$$z_1 = 2\sqrt[3]{\varrho} \cdot cos\left(\frac{\varphi}{3}\right) \quad \text{(A.71)}$$

$$z_2 = 2\sqrt[3]{\varrho} \cdot cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right)^{222} \quad \text{(A.72)}$$

$$z_3 = 2\sqrt[3]{\varrho} \cdot cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right), \quad \text{(A.73)}$$

where $\varrho = \sqrt{-\frac{p^3}{27}}$ and $cos\,\varphi = -\frac{q}{2\varrho}$ (Bronstein and Semendjajew, 1979). Using the results from above, the respective values for ϱ and $cos(\varphi)$ are:

$$\varrho = \sqrt{-\frac{(-6{,}027{,}068{,}339.863)^3}{27}} = 90{,}048{,}667{,}715{,}573.22 \quad \text{(A.74)}$$

[221] The formulas will not be discussed here as they will not be used. The interested reader is referred to Gabriel (1996).

[222] $\frac{2\pi}{3} = 120°$

$$cos\,\varphi = -\frac{-180{,}056{,}839{,}436{,}494.06}{2*90{,}048{,}667{,}715{,}573.22} = 0.9997751438 \quad \text{(A.75)}$$

$$\Rightarrow \varphi = \arccos(0.99999986) = 0.021206821. \quad \text{(A.76)}$$

Hence, using the Equations (A.74) and (A.75) allows determining the roots of the Equations (A.71) to (A.73).

$$z_1 = 2\sqrt[3]{90{,}048{,}667{,}715{,}573.22}\,cos\left(\frac{0.021206821}{3}\right) = 89{,}642.008 \quad \text{(A.77)}$$

$$z_2 = 2\sqrt[3]{90{,}048{,}667{,}715{,}573.22}\,cos\left(\frac{0.021206821}{3} + \frac{2\pi}{3}\right)$$
$$= -45{,}369.791 \quad \text{(A.78)}$$

$$z_3 = 2\sqrt[3]{90{,}048{,}667{,}715{,}573.22}\,cos\left(\frac{0.021206821}{3} + \frac{4\pi}{3}\right)$$
$$= -44{,}272.217 \quad \text{(A.79)}$$

By resubstituting $y_n = z_n - \frac{r}{3}$ and knowing that $r = -134{,}473.854$, the respective roots of the original equation can be determined.

$$y_1 = 89{,}642.008 - \frac{-134{,}495.538}{3} = 134{,}473.854 \quad \text{(A.80)}$$

$$y_2 = -45{,}369.791 - \frac{-134{,}495.538}{3} = -537.945 \quad \text{(A.81)}$$

$$y_3 = -44{,}272.217 - \frac{-134{,}495.538}{3} = 559.629 \quad \text{(A.82)}$$

These are the solutions given in Chapter 7.