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Algorithms for the Maximum Independent Set Problem

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Thesis

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(Dr.)

submitted by **MSc. Ngoc C. Lê**

born on the 11th July 1982 in Hanoi, Vietnam

Assessor: Prof. Dr. Rer. Nat. Habil. Ingo Schiermeyer
Technische Universität Bergakademie Freiberg, Freiberg, Germany
Prof. Dr. Rer. Nat. Habil. Jochen Harant
Technische Universität Ilmenau, Ilmenau, Germany

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Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such.
In the selection and use of materials and in the writing of the manuscript I received support from the following persons:

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I did not seek the help of a professional doctorate-consultant. Only those persons identified as having done so received any financial payment from me for any work done for me. This thesis has not previously been published in the same or a similar form in Germany or abroad.

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Dedication

To my parents, my sister, my beloved wife, and my two children.

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Abstract

This thesis focuses mainly on the Maximum Independent Set (MIS) problem. Some related graph theoretical combinatorial problems are also considered. As these problems are generally NP-hard, we study their complexity in hereditary graph classes, i.e. graph classes defined by a set $\mathcal{F}$ of forbidden induced subgraphs.
We revise the literature about the issue, for example complexity results, applications, and techniques tackling the problem. Through considering some general approach, we exhibit several cases where the problem admits a polynomial-time solution. More specifically, we present polynomial-time algorithms for the MIS problem in:

- some subclasses of $S_{2,j,k}$-free graphs (thus generalizing the classical result for $S_{1,2,k}$-free graphs);
- some subclasses of tree$_k$-free graphs (thus generalizing the classical results for subclasses of P_5-free graphs);
- some subclasses of P_7-free graphs and $S_{2,2,2}$-free graphs; and
- various subclasses of graphs of bounded maximum degree, for example subcubic graphs.

Our algorithms are based on various approaches. In particular, we characterize augmenting graphs in a subclass of $S_{2,k,k}$-free graphs and a subclass of $S_{2,2,5}$-free graphs. These characterizations are partly based on extensions of the concept of redundant set [125]. We also propose methods finding augmenting chains, an extension of the method in [99], and finding augmenting trees, an extension of the methods in [125]. We apply the augmenting vertex technique, originally used for P_5-free graphs or banner-free graphs, for some more general graph classes.
We consider a general graph theoretical combinatorial problem, the so-called Maximum Π-Set problem. Two special cases of this problem, the so-called Maximum $\mathcal{F}$-(Strongly) Independent Subgraph and Maximum $\mathcal{F}$-Induced Subgraph, where $\mathcal{F}$ is a connected graph set, are considered. The complexity of the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem is revised and the NP-hardness of the Maximum $\mathcal{F}$-Induced Subgraph problem is proved. We also extend the augmenting approach to apply it for the general Maximum Π-Set problem.
We revise on classical graph transformations and give two unified views based on pseudo-boolean functions and α-redundant vertex. We also make extensive uses of α-redundant vertices, originally mainly used for P_5-free graphs, to give polynomial solutions for some subclasses of $S_{2,2,2}$-free graphs and tree$_k$-free graphs.
We consider some classical sequential greedy heuristic methods. We also combine classical algorithms with α-redundant vertices to have new strategies of choosing the next vertex in greedy methods. Some aspects of the algorithms, for example forbidden induced subgraph sets and worst case results, are also considered.

Finally, we restrict our attention on graphs of bounded maximum degree and subcubic graphs. Then by using some techniques, for example α-redundant vertex, clique separator, and arguments based on distance, we general these results for some subclasses of $S_{i,j,k}$-free subcubic graphs.

1 Introduction

In a simple graph $G = (V, E)$, a set of vertices is *independent* (or *stable*) if no two vertices in this set are adjacent. An independent set, original called as *internal stable set* by Korshunov [109], is sometimes also called a *vertex packing*. In this thesis, when we say about maximalilty or minimality, we use inslusion sense and for maximum or minimum, we mention about cardinality. The cardinality of a maximum independent set in G is called the *independence number* (or the *stability number*) of G, denoted by $\alpha(G)$. The problem of determining a maximum independent set (called MIS problem for short) and/or compute the independence number of a particular graph finds important applications in a wide range of practical problems arising in many aspects of human activities, including not only computer science, but also information theory, biology, transport management, telecommunications, and finance.
In this chapter, we give an overview of literature about the issue. First, we start with some notations used throughout the thesis. Then, in Section 1.2, we formulate the problem. Section 1.3 is devoted to descriptions of some other optimization in graph theory related to the MIS problem. In Section 1.4, a systematic concept of graph classes is described. Then, some known results about the polynomial solvability of the problem in some special graph classes are revised in Section 1.5. We also discuss some polynomially computable bound of the independence number in Section 1.6. In Section 1.7, we describe briefly some selected applications of the MIS problem. Finally, in Section 1.8, there is a brief description about the main contributions of the thesis.

1.1 Notation

In this section, we want to collect most of the terminology and notations used in the thesis. For those not given here, they will be defined when needed. For those not given in the thesis, we refer to [23, 34, 168].
All graphs considered are finite, simple, and undirected. Moreover, given a graph G consisting of k connected components $G_1, G_2, \ldots, G_k$, every maximum independent set I of G can be partitioned into k parts $I_1, I_2, \ldots, I_k$ such that I_i is a maximum independent set of G_i for every $i = 1, 2, \ldots, k$. Hence, we suppose that every graph considered in this thesis is connected unless stated otherwise.
For a graph $G = (V, E)$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Unless stated otherwise, let us denote by $n(G) = |V(G)|$ the *order* of G and by $m(G) = |E(G)|$ the *size* of G. If the graph G is defined explicit, then we can write V, E, n, and m, instead of $V(G)$, $E(G)$, $n(G)$, and $m(G)$ for short, respectively.
An edge (u, v) of a graph is denoted by uv. For vertices $u, v \in V(G)$, we write $u \sim v$ if $uv \in E(G)$ and $u \nsim v$ if $uv \notin E(G)$.
For a vertex u in a graph G, we denote by $N_G(u) := \{v \in V : uv \in E\}$ the *neighborhood* of u in G, by $N_G[u] := N_G(u) \cup \{u\}$ the *closed neighborhood* of u, by $\deg_G(u) := |N_G(u)|$,

the *degree* of u, and by $A_G(u) = V(G)\backslash N_G[u]$, the *anti-neighborhood* of u. We write $N(u)$, $N[u]$, $\deg(u)$, and $A(u)$ instead of $N_G(u)$, $N_G[u]$, $\deg_G(u)$, and $A_G(u)$, respectively if no confusion can arise.

For a subset $U \subset V(G)$, we denote by $N(U) := (\bigcup_{u\in U} N(u))\backslash U$ the neighborhood of U, i.e. the set of vertices of G outside U that have at least one neighbor in U, and the closed neighborhood of U is denoted by $N[U] := N(U) \cup U$. Also, $N_U(v) := N(v) \cap U$ and if W is another subset of $V(G)$, then $N_W(U) := N(U) \cap W$. We also denote $A(U) := V(G)\backslash N[U]$ and call as the *anti-neighborhood* of U.

The maximum degree and minimum degree of vertices of a graph G are $\Delta(G) := \max_{v\in V(G)} \deg(v)$ and $\delta(G) := \min_{v\in V(G)} \deg(v)$, respectively. For an integer k, G is called k-regular if $\Delta(G) = \delta(G) = k$, i.e. every vertex of G is of degree k. For a graph class $\mathcal{G}$, we denote $\Delta(\mathcal{G}) := \sup_{G\in\mathcal{G}} \Delta(G)$ and $\delta(\mathcal{G}) := \min_{G\in\mathcal{G}} \delta(G)$. If $\Delta(\mathcal{G}) \leq \Delta_0$ for some finite intger Δ_0, then we say that $\mathcal{G}$ is a graph class of *maximum degree* at most Δ_0. *Subcubic* graphs are graphs of maximum degree at most three.

For two graphs G_1, G_2, we denote by $G_1 + G_2$ the disjoint union of G_1 and G_2. Especially, for a nonnegative integer m and a graph G, we denote by mG the graph consisting of m disjoint copies of G. We also denote $G_1 \times G_2$ as the graph including induced copies of G_1 and G_2 together with edges connecting each vertex of the copy of G_1 and each vertex of the copy of G_2.

For a graph G, we denote $\overline{G}$ as the completement of G, i.e. the graph that has the same vertex set as G and two vertices in $\overline{G}$ are adjacent if and only if two coresponding vertices in G are not adjacent.

P_n and C_n denote the induced path (also called a *chain*) and the chordless cycle on n vertices, respectively.

We say that a graph H is an *induced* subgraph of G or G *induces* H if H can be obtained from G by deletion of some (possibly none) vertices (together with incident edges). The subgraph of G induced by a vertex subset $U \subset V(G)$ is the graph obtained from G by deleting all the vertices of $V(G)\backslash U$ and denoted by $G[U]$. For a vertex subset $W \subset V(G)$, we also say that W induces H if $G[W]$ induces H. We denote $G - U := G[V(G)\backslash U]$ and $G - u := G - \{u\}$ for short. For H is an induced subgraph of G, we also denote $G - H := G - V(H)$. Given a graph G, we denote $G - e$ as the graph obtained from G by deleting an arbitrary edge if no confusion arises. We also denote $G + e$ by a similar way.

A graph $G = (L, R, E)$ is bipartite if its vertex set admits a bipartition $V(G) = L \cup R$ such that $E(G) \subset \{uv : u \in L, v \in R\}$. A *clique* is a complete graph, i.e. a graph such that every pair of vertices is adjacent. By K_n we denote the clique of n vertices and by $K_{s,t}$ the complete bipartite graph with parts of size s and t. We call $K_{1,m}$ *star*, where the vertex of degree m is called the *center vertex*.

The *distance* between two vertices u and v, denoted as $dist(u, v)$, in a connected graph G is the length (i.e., the number of edges) of a shortest path connecting them. The distance between two vertex sets U, W, denoted by $dist(U, V)$, in a connected graph G is the minimum distance between two arbitrary vertices $u \in U$ and $v \in V$.

1.2 Definitions of the Problems

1.2.1 Weighted Case

Although most of the contents of this thesis is about the unweighted case of the problem, sometimes we discuss about the weighted case. If each vertex $v \in V$ is associated with a posititve weight $w(v)$, then for a subset $S \subset V$, its weight $w(S)$ is defined as the sum of weights of all vertices in S, i.e. $w(S) := \sum_{v \in S} w(v)$. The *Maximum Weight Independent Set* (WIS for short) problem seeks for independent sets of maximum weight.

1.2.2 Formal Definition of the Problems

Maximum Independent Set (MIS)
Instance: Graph $G = (V, E)$,
Output: Largest integer k such that G has an independent set of size k.

Maximum Weighted Independent Set (WIS)
Instance: A pair (G, w), where $G = (V, E)$ is a graph and $w : V \to \mathbb{R}$ is a weighted function,
Output: Largest number r such that G has an independent set of weight r.

1.2.3 Decision Formulation

Maximum Independent Set (MIS)
Instance: A pair (G, k), where $G = (V, E)$ is a graph and k is an integer,
Question: Is there an independent set S in G such that $|S| \geq k$?

Maximum Weighted Independent Set (WIS)
Instance: A triple (G, k, w), where $G = (V, E)$ is a graph, $w : V \to \mathbb{Z}$ is a weighted function, and k is an integer,
Question: Is there an independent set S in G such that $\sum_{v \in S} w(v) \geq k$?

1.2.4 Integer Programming Formulations

In this subsection, each vertex of V is associated with an integer $i = 1, \ldots, n$ ($|V| = n$).
Maximum Independent Set (MIS)

$$\max f(x) = \sum_{i=1}^{n} x_i$$

subject to

$$x_i + x_j \leq 1, \; \forall (i, j) \in E$$

and

$$x_i \in \{0, 1\}, i = 1, \ldots, n.$$

Given a vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, one of the simplest integer programming formulations of the WIS problem is the following *edge formulation*.

Maximum Weighted Independent Set (WIS)

$$\max f(x) = \sum_{i=1}^{n} w_i x_i$$

subject to

$$x_i + x_j \leq 1, \forall (i,j) \in E$$

and

$$x_i \in \{0,1\}, i = 1, \ldots, n.$$

1.3 Related Problems

Along with the MIS problem, in literature, some other related problems about graph parameters are also considered. In this section, we give a brief overview on them.

1.3.1 Maximum k-Independent Set Problem

The MIS problem can be generalized to the Maximum k-Independent problem. A k-independent set of a graph G is a vertex subset S inducing a subgraph such that every vertex is of degree at most $k-1$. In other words, the MIS problem is the Maximum 1-Independent Set problem.

1.3.2 Maximum Clique Problem

The Maximum Clique problem asks for a maximum clique. The clique number is the cardinality of a maximum clique in G, denoted by $\omega(G)$. It is easy to see that a vertex subset $S \subset V$ is a maximum independent if and only if S induces a maximum clique in $\bar{G}$. Hence, it is also considered as a dual version of the MIS problem.

1.3.3 Minimum Vertex Cover Problem

A vertex cover V' is a subset of V such that every edge of E has at least one end-vertex in V'. The Minimum Vertex Cover problem is to find a minimum vertex cover. Denote by $\beta(G)$ the minimum size of vertex cover of G, we have the following observation.

Lemma 1.1. *[168] In a graph G, $S \subset V(G)$ is an independent set if and only if $V \backslash S$ is a vertex cover and hence, $\alpha(G) + \beta(G) = n(G)$.*

This implies that the minimum vertex cover problem is polynomially equivalent to the MIS problem.

1.3.4 Minimum Vertex k-Path Cover Problem

The Minimum Vertex Cover problem was generalized as the Minimum Vertex k-Path problem. This problem was introduced by Brešar et al. [38] and motivated by the secured communication problem in wireless sensor networks [147]. It asks for a minimum vertex subset I such that every path (not neccessarily induced) of order k contains at least one vertex in I.

1.3.5 Minimum Feedback Vertex Cover Problem and Some Related Problems

Another generalization of the Minimum Vertex Cover problem is the Minimum Feedback Vertex Cover problem. It asks for a minimum vertex subset I such that every cycle contains at least one vertex in I. This problem finds application in VLSI chip design [63]. Some other combinatorial problems in graph theory related to this problem follow.

- The Maximum Induced Bipartite Subgraph problem asks for a maximum vertex subset inducing a bipartite graph.
- The Maximum k-Acyclic Set problem asks for a maximum vertex subset inducing a graph containing no cycle of length at most k.
- The Minimum Vertex k-Cycle Cover problem asks for a minimum vertex subset I such that every cycle of length k contains at least one vertex in I.
- The Maximum k-Chordal Set problem asks for a minimum vertex subset inducing a graph containing no cyvle of length larger than k.

1.3.6 Maximum Matching Problem and Maximum Induced Matching Problem

Given a graph G, a matching $E' \subset E(G)$ is an edge subset such that there is no pair of edges sharing an end-vertex. A Maximum Matching problem asks for a maximum matching. This problem and the MIS problem are related through the concept of *line graph*.
Given a graph G, its line graph $L(G) = (V', E')$ is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G. It is easy to see that a matching in G is maximum if and only if the corresponding vertex subset is a maximum independent set in $L(G)$.
Edmonds [59] described the so-called *blossom algorithm* for this problem. Then this algorithm was developed to *augmenting technique* for the MIS problem. We discuss about this technique later in Chapters 3, 4.
A matching is called induced if there exists no edge (in G) connecting end-vertices of two edges of the matching. The Maximum Induced Matching problem asks for the maximum induced matching of some graph G. Unlike the Maximum Matching problem, this problem is shown to be NP-hard in general [170] and in bipartite graph [45].

1.3.7 Maximum k-Regular Induced (Bipartite) Subgraph

Given a graph G, this problem asks for a maximum induced (bipartite) subgraph in which every vertex has degree k. It can be considered as a generalization of the Maximum Induced Matching problem ($k = 1$) and the MIS problem ($k = 0$). The NP-hardness of the problem has been shown for general and for bipartite graphs [46].

1.3.8 Maximum Dissociative Set Problem

In the mathematical discipline of graph theory, a subset of vertices in a graph G is called *dissociative* if it induces a subgraph with maximum degree one. The number of vertices in a maximum dissociative set in G is called the dissociation number of G. The problem of computing dissociation number was firstly studied by Yannakakis [170].
The Maximum Dissociative Set problem asks for a maximum dissociative set and generalizes two other graph problems: MIS and Maximum Induced Matching. The first one asks to find in a graph a maximum induced subgraph with vertex degree equal zero. The second one is to find a maximum induced subgraph with vertex degree equal one.
On the other hand, the Maximum Dissociative Set problem can also be considered as the Maximum 2-Indepedent Set problem or a dual version of the Minimum Vertex 3-Path Cover problem.

1.3.9 Minimum Dominating Set Problem

Given a graph $G = (V, E)$, $U \subset V$, a vertex $v \in V \backslash U$ is called *dominating* U if $v \sim u$ for every $v \in U$. A vertex subset $W \subset V \backslash U$ is called dominating U if every vertex of U is adjacent to at least one member of W. A *dominating set* for G is a subset D of V such that D dominates $V \backslash D$. The Minimum Dominating Set problem asks for a minimum dominating set. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G.
Dominating sets are closely related to independent sets: an independent set is also a dominating set if and only if it is maximal, so any maximal independent set in a graph is necessarily also a minimal dominating set.

1.3.10 Vertex Coloring Problem

In graph theory, *graph coloring* is a special case of graph labeling. It is an assignment of labels, traditionally called colors, to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color. This is called a *vertex coloring*. The Vertex Coloring problem ask for a method of coloring vertices of graph using least colors, say *chromatic number* of some graph G, denoted by $\chi(G)$.
The Vertex Coloring problem is related to independent sets in the sense that it asks for a partition of the set of vertices into minimum number of independent sets. It is also worth to notice that $\chi(G) \geq \omega(G)$.

1.4 Graph Classes

For the systematic viewpoint, it is useful to study representative families of graph classes rather than individual classes. In the present articles, the *hereditary classes* are under investigation, i.e. the classes with the following nice property: whenever they contain a graph G, they contain all induced subgraphs of G. A set of graphs $\mathcal{X}$ is called a hereditary class if it is closed under vertex deletion (together with all incident edges). In other words, $\mathcal{X}$ is hereditary if every graph isomorphic to an induced subgraph of a graph in $\mathcal{X}$ belongs to $\mathcal{X}$. For a comprehensive survey on graph classes, we refer the reader to [34]. Many graph classes of theoretical or practical importance are hereditary, which includes, among others,

- *bipartite* graphs;
- *planar* graphs;
- *subcubic* graphs;
- graphs of bounded vertex degree;
- forests;
- graphs of bounded *treewidth*;
- graphs of bounded *clique-width*;
- *chordal* graphs;
- *perfect* graphs;
- *line* graphs.

An important property of hereditary classes is that these and only these classes admit a uniform description in terms of forbidden induced subgraphs, which provides a systematic way to investigate various problems associated with graph classes. For a set of graphs $\mathcal{Y}$, the class of all graphs having no induced subgraphs isomorphic to graphs in $\mathcal{Y}$ is called $\mathcal{Y}$-free. Alekseev [3] obtained the following result.

Theorem 1.2. *[3] For every hereditary class $\mathcal{X}$, there is a set $\mathcal{Y}$ such that $\mathcal{X}$ is $\mathcal{Y}$-free.*

In the above theorem, graph class $\mathcal{Y}$ is also called the forbidden induced subgraph set of $\mathcal{X}$. Let $\mathcal{X}$ be a graph class. If there exists a finite set of graphs $\mathcal{Y}$ such that $\mathcal{X}$ is $\mathcal{Y}$-free, then the class $\mathcal{X}$ is called *finitely* defined (by $\mathcal{Y}$).
Moreover, in literature, we also have a concept of *strong* hereditary graph class. A graph class $\mathcal{X}$ is called strong hereditary if it is closed under vertex deletion and edge deletion, i.e. every graph isomorphic to a subgraph (not necessarily included) of a graph in $\mathcal{X}$ belongs to $\mathcal{X}$.

1.5 Complexity Results

1.5.1 Hardness of the Problem

The Maximum Clique problem (and hence the MIS problem also) is one of the first problems shown to be NP-complete [107]. The interest, therefore, has soon shifted towards characterizing the approximation properties of this problems. Papadimitriou and Yannakakis [148, 149] introduced the complexity classes *MAX NP* and *MAX SNP*. They showed that all problem in MAX NP admit a polynomial time approxiamation algorithm and that many natural problems are complete in MAX SNP. For example the MIS-Δ (the MIS problem for graphs of maximum degree at most Δ, for a given Δ) and the MAX 3-SAT problem. A breakthrough in approximation complexity is the

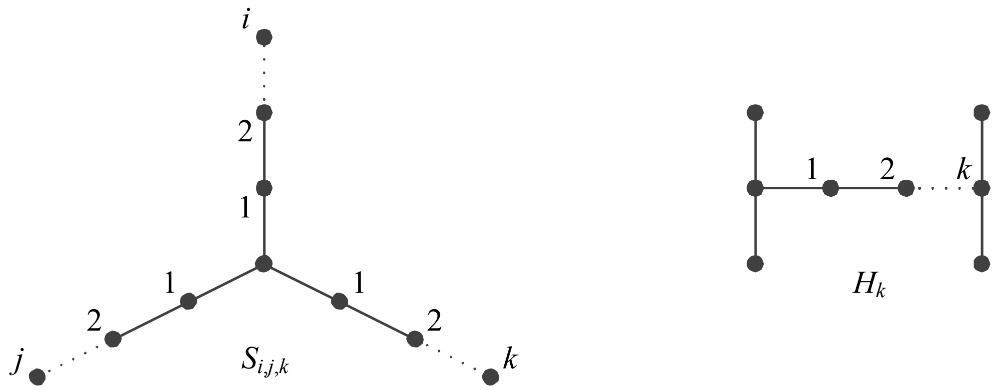

Fig. 1.1: $S_{i,j,k}$ and H_k

result by Arora et al. [10–13]. It is shown that the MAX 3-SAT problem cannot be approximated to arbitrary small constants (unless $P = NP$). This immediately shows the difficulty of finding good approximate solutions for the MIS-Δ problem, i.e. no polynomial-time algorithm can approximate the maximum independent set size within a factor of n^ϵ ($\epsilon > 0$), (unless $P = NP$).
The best polynomial time approximation algorithm for the MIS problem was developed by Boppana and Halldórsson [24]. They achieved an approximation ratio of $\mathrm{O}\left(\frac{n}{(\log n)^2}\right)$. Moreover, Håstad [86] showed that unless any problem in NP can be solved in probabilistic polynomial time, the Maximum Clique problem cannot be approximated in polynomial time within a factor $n^{1-\epsilon}$ for any $\epsilon > 0$.
On the other hand, the problem remains intractable under substantial restriction, for instance for planar graphs [72], graphs with large girth [144], triangular free graphs [150], bistellar graphs [90], or subcubic graphs [124]. We shall call such classes MIS-*hard*. Besides, in some special classes, like bipartite graphs, the problem has a polynomial time solution [85], say in time $\mathrm{O}(n^d)$. We shall refer to such classes as MIS-*easy* or MIS-*solvable* in time $\mathrm{O}(n^d)$. If $d = 1$, then we call such classes as MIS-*linear*. Although this thesis is mostly devoted to proposing some new algorithms to find some new MIS-easy hereditary graph classes, it is worth to summarize results about MIS-hard graph classes in follows.
Let $S_{i,j,k}$ be the graph consisting of three induced paths of lengths i, j and k with a common initial vertex (Fig. 1.1). Let $\mathcal{S}$ be the graph class in which every connected component is of the form $S_{i,j,k}$. We denote by H_k the graph consisting of two disjoint P_3's and a chain of length k connecting the two mid-vertices (Fig. 1.1). We associate to every graph G a parameter $\kappa(G)$, the *chordality* of G, i.e. the length of the largest chordless cycle in G and $\eta(G)$, the largest value k such that G contains an induced copy of H_k. If G is a tree, we let $\kappa(G) = 0$. If G contains no induced graph of the form H_j, we also let $\eta(G) = 0$. Let $\mathcal{G}$ be a graph class, we denote $\kappa(\mathcal{G}) = \sup_{G\in\mathcal{G}} \kappa(G)$ and $\eta(\mathcal{G}) = \sup_{G\in\mathcal{G}} \eta(G)$. Alekseev [5] observed the following result.

Theorem 1.3. *[5] Let $\mathcal{X}$ be a hereditary graph class finitely defined by $\mathcal{F}$ and $\mathcal{F}\cap\mathcal{S} = \emptyset$. Then $\mathcal{X}$ is MIS-hard.*

Moreover, Lozin and M. Milanič [124] obtained the following result.

Theorem 1.4. *[124] Let $\mathcal{X}$ be a hereditary subcubic graph class defined by $\mathcal{F}$. If*

1. *$\kappa(\mathcal{F}) < \infty$,*
2. *$\eta(\mathcal{F}) < \infty$, and*
3. *$\mathcal{F} \cap \mathcal{S} = \emptyset$,*

then $\mathcal{X}$ is MIS-hard.

Note that the question whether the family of all hereditary classes has other conditions about the forbidden subgraphs set under which, the problem is NP-hard is still open. However, the previous results suggest that the MIS problem is solvable in polynomial time for graphs in a class $\mathcal{F}$-free only if

1. $\mathcal{F}$ contains graphs with arbitrarily large induced cycles or
2. $\mathcal{F}$ contains graphs with arbitrarily large induced copies of H_i or
3. $\mathcal{F}$ contains a graph from the class $\mathcal{S}$.

1.5.2 Some MIS-Easy Graph Classes

First, we review some polynomially solvable cases of the problem.

Finite Induced Forbidden Subgraph Set

Minty [137] and Sbihi [156] independently showed that the problem is polynomial solvable in claw-free ($S_{1,1,1}$-free) graphs. This result was generalized by Alekseev in [2] for fork-free ($S_{1,1,2}$-free) graphs. Corneil et al. showed that for P_4-free graphs (i.e. cographs), $\alpha(G)$ can be determined in linear time using the co-tree structure of cographs [52]. The problem is also polynomially solvable in P_5-free graphs, a result of Lokshtanov et al. [115]. Note that the fork ($S_{1,1,2}$) and P_5 ($S_{0,1,3}$) are special cases of the general form $S_{i,j,k}$, where $i + j + k = 4$. For larger $i + j + k$ cases, there are only solutions for subclasses. Some example are followed:

- ($S_{1,2,5}$,banner)-free graphs [125];
- ($S_{2,2,2}$,banner)-free graphs [77];
- ($S_{1,2,k}$,banner,$K_{1,m}$)-free graphs and ($S_{1,2,3}$,banner$_k$, $K_{1,m}$)-free graphs [98];
- ($P_k, K_{1,m}$)-free graphs [131];
- (P_6,diamond)-free graphs [138], ($P_6, K_{2,3}$)-free graphs [142], and (P_6,co-banner)-free graphs [139]; and
- $S_{1,2,k}$-free planar graphs [123].

Alekseev [6] showed that the problem is polynomially solvable in mK_2-free graphs. The similar results for the cases (claw $+K_2$)-free graphs and ($2P_3$)-free graphs were obtained by Lozin and Mosca [127, 129].
Here, we denote apple$_k^p$ as the graph consisting of a chordless cycle of length p and an induced path of length k whose an end-vertex lies in the cycle (see Fig. 1.2). In the case $p = 4$, we call it banner$_k$ and for $p = 3$, we denote it by Z_k. If $k = 1$, then we denoted it simply by apple$_p$. Banner$_1$ is known as banner and Z_1 is known as paw.

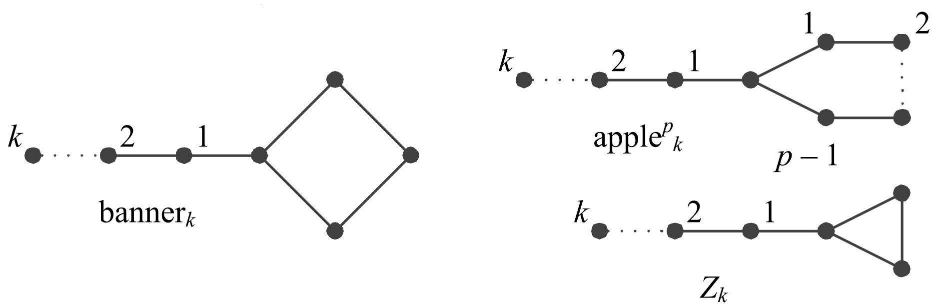

Fig. 1.2: Apples

Induced Forbidden Subgraph Set Inducing Unbounded Length Chordless Cycle

In 1976, Frank [69] showed that the chordal graph class, i.e. $(C_4, C_5, \ldots)$-free graphs, is MIS-easy. Grötschel et al. [85] showed the polynomial solvability of the problem in perfect graphs, i.e. the graphs without odd holes (odd-length induced cycles) nor odd antiholes (complements of odd holes) (strong perfect graph theorem [49]). Some other examples are a subclass of odd-apples-free (apples whose the cycles are of odd length)-free graphs [158], a subclass of $(C_5, C_6, \ldots)$-free graphs [100], (banner,$C_5, C_6, \ldots$)-free graphs [77], AH-free graphs [101], and hole- and co-chair-free graphs [26].

Induced Forbidden Subgraph Set Inducing Arbitrarily Large H_k

There are still not many results about forbidden subgraphs sets of infinite η. One example, of course, is the AH-free graph class [101]. Another example is the large H-free graph class of bounded maximum degree [124].

1.6 Bounds

In view of its computational hardness, various bounds on the independence number have been proposed.

1.6.1 Lower Bounds

The following may be the oldest non-trivial bound and implied by Turán's theorem [164].

$$\alpha(G) \geq \frac{n}{1+\bar{d}},$$

where $\bar{d}$ is the *average degree* of the graph.
Perhaps the best known lower bound based on degrees of vertices is a so-called Caro-Wei bound given independently by Caro [47] and Wei [166]:

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v)+1}.$$

Then a bunch of lower-bounds were described as improvement of the Caro-Wei bound [93, 94, 96, 145, 157].

1.6.2 Upper Bounds

Unlike lower bounds, there are not many results about upper bounds for the independence number of some graph G. Some examples can be found in [92, 96].

1.7 Applications

Practical applications of the considered optimization problems are abundant. They appear in information retrieval, signal transmission analysis, classification theory, economics, scheduling, experimental design, computer vision, and many other fields. In this section, we describe briefly selected applications related to the MIS problem.

1.7.1 Map Labelling

When designing maps, an important question is how to place the names of the regions on the map such that each name appears close to the corresponding region and no two names overlap. The basic map labelling problem can be described as follows: given a set $P = \{p_1, p_2, \ldots, p_n\}$ of n distinct points in $\mathbb{R}^n$, determine the supremum of all reals σ, for which there exist n pairwise disjoint, axis-parallel $\sigma \times \sigma$ squares $Q_1, Q_2, \ldots, Q_n \subset \mathbb{R}^2$, where p_i is a (top-left) corner of Q_i for all $i = 1, \ldots, n$. By *pairwise disjoint squares*, we mean that no overlap between any two squares is allowed. Manual label placement is a time-consuming task and it is natural to try to automate it.

The decision variant of the map labelling problem is to decide, for any given σ, whether there exists a set of squares $Q_1, \ldots, Q_n$ as described above. This problem was shown to be NP-complete by Formann and Wagner [67]. The optimization variant of the problem was described by Verweij and Aardal [165] as follows. Given the label size σ as input, the problem asks for as many pairwise disjoint squares of the desired characteristic as possible. Then clearly, the optimal solution corresponds with the solution of the MIS problem in corresponding *conflict* graph $G = (V, E)$ built as follows. The vertex set V is the set of squares and two vertices are adjacent if the two corresponding squares are overlapped.

1.7.2 Molecular Biology

Oftentimes, in Computational Biology, one must compare objects which consist of a set of elements arranged in a linearly ordered structure. In bio-informatics, a sequence alignment is a way of arranging the sequences of DNA, RNA, or protein to identify regions of similarity that may be a consequence of functional, structural, or evolutionary relationships between the sequences [143]. If two sequences in an alignment share a common ancestor, mismatches can be interpreted as point mutations and gaps as indels (i.e. insertion or deletion mutations) introduced in one or both lineages in the time since they diverged from one another. In sequence alignments of proteins, the degree of similarity between amino acids occupying a particular position in the sequence can be interpreted as a rough measure of how conserved a particular region or sequence motif is among lineages. The absence of substitutions, or the presence of only very conservative substitutions (i.e. the substitution of amino acids whose side chains have

similar biochemical properties) in a particular region of the sequence, suggest that this region has structural or functional importance [146]. Although DNA and RNA nucleotide bases are more similar to each other than amino acids, the conservation of base pairs can indicate a similar functional or structural role.
Aligning two (or more) such objects consists in determining subsets of corresponding elements in each. The correspondence must be order-preserving, i.e. if the i-th element of Object 1 corresponds to the k-th element of Object 2, then no element following i in Object 1 can correspond to an element preceding k in Object 2. Very short or very similar sequences can be aligned by hand. However, most interesting problems require the alignment of lengthy, highly variable or extremely numerous sequences that cannot be aligned solely by human effort.
The following construction was described by Lancia [111]. Given two objects, where the first has n elements, denoted by $[n] := (1, \ldots, n)$ and the second has m elements, denoted by $[m] = (1, \ldots, m)$, we consider the complete bipartite graph $W_{n,m} := ([n], [m], L)$, where $L = [n] \times [m]$. Then we called a pair (i, j) with $i \in [n]$ and $j \in [m]$ a *line*. Two lines (i, j) and (i', j') are said *cross* each other if either $i' \geq i$ and $j' \leq j$ or vice versa. A matching is a subset of set of lines L such that no two of which share an end-vertex. An alignment is identified by a *noncrossing* matching, i.e. a matching for which no two lines cross each other. Then a noncrossing matching in $W_{n,m}$ corresponds to an independent set in a so-called *line conflict graph*, constructed as follows. $G_L = (V, E)$ such that $V := L$ and two vertices l and h are adjacent if the lines l and h cross. The problem asking for a maximum alighnment obviously can be considered as a problem of finding a maximum independent set in the conflict graph. Another application of the MIS problem about pairwise structure of proteins was also described in [111].

1.7.3 Computer Vision

Brendel et al. [37] described the method of applying the MIS problem in the multiobject tracking problem. The problem was addressed is simultaneous tracking of multiple targets in a complex scene, captured by a non-static camera. Targets are occurrences of known object classes, such as cars, pedestrians, and bicycles. First, detectors of a set of object classes are applied to all video frames. Each detection is characterized by a descriptor. Then the best matching detections are transitively linked across video into distinct tracks. This is done under the hard constraint that no two tracks may share the same detection to prevent implausible video interpretations. In addition, the linking is informed by spatio-temporal relationships between the tracks which provide for soft constraints. To this end, a graph is built, where vertices represent candidate matches from every two consecutive frames, referred to as tracklets. Vertices weights encode the similarity of the corresponding matches. Edges connect vertices whose corresponding tracklets violate the hard constraints. Given this attributed graph, data association is formulated as the WIS problem.

1.7.4 Railways Dispatching

Flier et al. [65] formulated the problem of dispatching in railways as follows. During operations, railway dispatchers face the challenging problem of rerouting and rescheduling trains in the presence of delays. Once a train is delayed, it might be in conflict

with other trains that are planned to use the same track resources. The dispatcher then has to find a new feasible plan in a very short amount of time. Interestingly enough, these complicated decisions are carried out mostly by humans today, with only basic computer support such as graphical monitoring tools.
Typically, a railway station is modeled as a graph with vertices representing points on the tracks and edges representing track segments that connect such points. We study the case where the resulting graphs are planar, which is the case for many junctions and stations. Considering only the aspect of routing, two trains are in conflict if their routes share a point on the tracks. Hence, conflict free routes correspond to vertex disjoint paths. Not every route which is physically feasible is desirable in practice, though. Therefore, railway planners allow for each train only a small set of alternative paths for each train. Let say, for each pair of terminals (s_i, t_i) of some train i, there exists a set of feasible route $\mathcal{P}_i$. Then we want to find a maximum number of vertex disjoint paths $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$, where $P_{i_j} \in \mathcal{P}_{i_j}$.
We construct a conflict graph $G = (V, E)$ as follows. $V = \bigcup \mathcal{P}_i$. Two vertices u, v are adjacent if u, v belong to the same set $\mathcal{P}_i$ for some i or they are two routes sharing an inner point. Then the problem of finding a maximum number of vertex disjoint paths is the MIS problem.

1.7.5 Coding Theory

Error correcting codes lie in the heart of digital technology. Butenko et al. [42, 43] described the relations between this problem and the MIS problem as follows. Given a positive integer n, for a binary vector $u \in B^n$, where $B = \{0, 1\}$, we denote by $F_e(u)$ the set of all vectors (not necessary of dimension n) which can be obtained from u as a consequence of certain error e, such as deletion or transposition of bits. A subset $C \subset B^n$ is said to be an e-correcting code if $F_e(u) \cap F_e(v) = \emptyset$, for all $u, v \in C$, $u \neq v$. The problem here is to find the largest correcting codes.
Consider a graph G_n having a vertex for every vector $u \in B^n$ with an edge joining the vertices corresponding to $u, v \in B^n$, $u \neq v$, if and only if $F_e(u) \cap F_e(v) \neq \emptyset$. Then a correcting code corresponds to an independent set in G_n. Hence, the largest e-correcting code can be found by solving the MIS problem in the considered graph.

1.7.6 Scheduling in Wireless Networks

Scheduling is one of the most fundamental functionalities of wireless networks. It determines which links should transmit at what time and at what data rate. Joo et al. [106] formulated this problem as follows. Consider a wireless network with N nodes and L directed links. Assume that time is slotted and that a single frequency channel is shared by all the links. Multiple link transmissions at the same time slot may fail due to wireless interference. We suppose that there is no link error, i.e. a link transmission is successful if there is no simultaneous interfering transmission. We denote the (global) channel state by h. When the channel is in state h, link l can transfer r_l^h unit of data if its transmission is successful. Then consider a conflict graph $G^h = (V, E^h)$ as follows. V is the set of links and the two vertex $k, l \in V$ are adjacent if they interfere each other. For each vertex $v \in V$, define the weight of v, $w(v)$ as the product of the length of the queue of link v at time slot t and the transmission rate r_v^h. Now, for a particular time slot t and a chanel state h, we want to find a set links such that as much as possible data can be transfered in consideration the queue lengths of links. It is clear that this problem corresponds with the WIS problem of G^h.

1.8 Main Contributions of the Thesis

In this chapter, we gave the introduction about the problem. Definitions and related problems were reviewed. Some results about complexity and bounded were listed. We also described some real applications of the problem. In the next chapter, we review some main algorithmic approaches for the problem. Beside giving an overview on literature about the issue, the main contributions of this thesis are the following.

1.8.1 Augmenting Technique Applied in some Subclasses $S_{2,j,k}$-free Graphs Class

The method of *augmenting* graphs is a general approach to the MIS problem. In Chapter 3, we consider to apply this technique in some subclasses of $S_{2,j,k}$-free graph class. Some structural properties of augmenting graphs are described in Section 3.2. Then in Section 3.3, we describe methods to find augmenting graphs of some special classes. Based on that, polynomial solutions for some hereditary graph classes are obtained.

1.8.2 Augmenting Technique for Other Graph Threoretical Problems

In Chapter 4, we apply this approach for some other combinatorial problems in graph theory. The concept of augmenting graph is generalized. Then we characterize the ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free augmenting graphs and describe the method to find such augmenting graphs. Through that, a polynomial solution is obtained.

1.8.3 New Sufficient Conditions for α-redundant Vertices

In Chapter 5, we focus on graph transformations method, i.e. the techniques transforming a given graph G into a new graph G' in such a way that the difference $\alpha(G) - \alpha(G')$ is easy to compute. The revision of classical graph transformations based on pseudo-boolean function is given. We focus on α-redundant vertices in Section 5.2. The method of α-redundant vertices is a general approach to extend polynomial results of the MIS problem. Some new sufficitent conditions to recognize if a vertex is α-redundant in polynomial time are described. Based on that, some MIS-easy graph classes are obtained.

1.8.4 Heuristic Methods and Hybrid Methods

In Chapter 6, we discuss about (sequential search) heuristic methods for the MIS problem. One question arises with heuristic methods is when produced maximal independent sets become maximum. In [132], Mahadev and Reed also gave a forbidden induced subgraph set under which the VO algorithm always gives a maximum independent set. Harant et al. [95] and Zverovich [171] described forbidden induced subgraphs sets for the MIN algorithm. We combine classical heuristic methods with some reduction techniques to obtain so-called hybrid algorithms in Section 6.3. The forbidden induced subgraph sets for heuristic algorithms are described in Section 6.4. We also compare the performance of these algorithms in Section 6.5.

1.8.5 Graphs of Low Degree

In Chapter 7, we consider some results about the MIS problem for graphs of bounded maximum degree. By combining techniques mentioned in the thesis, we develop some polynomially solvable cases of the MIS-Δ problem (Section 7.2) and the MIS problem for subcubic graphs (Section 7.3).

2 Techniques for Finding Maximum Independent Sets

In this chapter, we provide an overview on techniques and algorithmic tools that have been used in order to tackle the MIS problem. We revise first on exact methods (Sections 2.1 and 2.2) and then on heuristic methods (Section 2.3). In Section 2.4, we focus on graph transformations. Augmenting technique is considered in Section 2.5. Section 2.6 is devoted to decomposition methods. In Section 2.7, we revise on methods based on bounded graph parameters. Finally, a brief list of other methods can be found in Section 2.8.

2.1 Enumerating All Maximal Independent Sets

A maximum independent set certainly is maximal. Hence, a possible approach for solving the MIS problem is enumerating all maximal independent sets. Tsukiyama et al. [163] proposed an algorithm listing all maximal independent sets in time $\mathrm{O}(n{\cdot}m{\cdot}n_\alpha)$, where n, m, and n_α are the number of vertices, edges, and maximal independent sets of a graph, respectively. A similar approach was described by Leung [113] for some special graph classes. This algorithm for interval, circular-arc, and chordal graphs runs in time $\mathrm{O}(n^2 + n_s)$, $\mathrm{O}(n^2 + n_s)$, and $\mathrm{O}((n + m) \cdot n_\alpha)$, respectively, where n_s is the sum of the numbers of vertices of all maximal independent sets.

Loukakis and Tsouros [117] proposed a depth-first enumerative algorithm that generates all maximal independent sets lexicographically. They compared their algorithm with the algorithm of Tsukiyama et al. and claimed that their algorithm is three times faster. Two years later, Loukakis [116] claimed an additional improvement of three folds of time saving over the algorithm in [117].

Note that, if one can enumerate all maximal independent set of a graph in some graph class $\mathcal{X}$ in polynomial time, then $\mathcal{X}$ is MIS-easy. The structure of $(2K_2, C_4)$-free graphs has been characterized by Blázsik et al [20]. In particular, it has been proved that any graph with n vertices in this class has at most n maximal independent sets. A more general result has been proved by Alekseev in [6], where he showed that the number of maximal independent sets in mK_2-free graphs is bounded by a polynomial for any fixed m. In combination with the algorithm of Tsukiyama et al. [163], this leads to a polynomial algorithm to find a maximum independent set in mK_2-free graphs with a fixed m. Following the same idea, Lozin and Mosca [127] proposed the algorithm based on Farber's argumentation [61] generating all maximal indepent sets of a $2K_2$-free graph. They extended this algorithm to solve the problem in subclass of $Y_{m,m}$-free graphs based on the idea of *anti-neighborhood* of edge.

2.2 Others Exact Methods

If our goal is to find just one maximum independent set or just the independence number, a lot of work can be saved in comparing with the above enumerative algorithms. Because once we find an independent set, we only need to enumerate independent sets better than the current best. Modifying the enumerative algorithms based on this argument results in various implicit enumerative methods. The most well known and commonly used implicit enumerative technique for the MIS problem is the branch and bound method. Tarjan and Trojanowski [162] proposed a recursive algorithm for the MIS problem with the time comlexity $O(2^{\frac{n}{3}})$. Later, this result was improved by Robson [153], who modified the algorithm of Tarjan and Trojanowski to obtain the time complexity of $O(2^{0.276n})$. Besides, the work of Houck and Vemuganti [105] exploited the relationship between the maximum independent set and a special class of bipartite graphs. They used this relationship to find an initial solution in their algorithm for the maximum independent set problem.
Many exact algorithms in the literature for the MIS problem were proposed in the 1980's. For example, in 1982, Loukakis and Tsouros [118] proposed a tree search algorithm that finds the size of a maximum independent set. Then in 1984, Ebenegger et al. [58] proposed another algorithm for finding the independence number of a graph. Their approach is based on the relationship between the maximization of a pseudo-Boolean function and the independence number of a graph. Computational tests on graphs with up to 100 vertices were also reported in [58]. We come back to this technique in Section 5.1. Besides, Formin et al. [66] described a *mesure and conquer* method to achieve an $O(2^{0.288n})$ complexity algorithm.

2.3 Heuristic Methods

Although exact approaches provide an optimal solution, they become impractical even on graphs with several hundreds of vertices. Therefore, when one deals with the MIS problem on very large graphs, in which the exact approaches cannot be applied, heuristics provide a possible option.

2.3.1 Greedy Heuristics

Sequential Greedy

The majority of approximation algorithms in the literature for the MIS problem are called *sequential greedy* heuristics. These heuristics generate a maximal independent set through repeated addition of a vertex into an independent set or repeated deletion of a vertex from the original graph. Borowiecki et al. [25] called the two strategies *best-in* and *worst-out* strategies, respectively. Decisions on which vertex to be added in or moved out next are based on certain indicators associated with candidate vertices. For example, a possible best-in heuristic constructs a maximal independent set by repeatedly adding in a vertex that has the smallest degree among candidate vertices. In this case, the indicator is the degree of a vertex. On the other hand, a possible worst-out heuristic can start with the whole vertex set V and then repeatedly remove a vertex out of V until V becomes independent. Three well known heuristic algorithms are Vertex Order (VO) [132], MIN [145], and MAX [83]. Algorithm MAX follows worst-out strategy using degree indicator while MIN and VO follow best-in strategy with the same indicator. Moreover, while MIN and MAX update the indicators every time when a vertex is added in or moved out, we call this approach as *new* strategy, while VO does not, but follows so-called *old* strategy. All three algorithms give a maximal independent set in polynomial time. However, under some restrictions, these maximal independent sets become maximum. Borowiecki et al. [25] suggested a more general indicator, so-called *potential function* for greedy algorithms. We describe these methods more detail in Chapter 6.

Local Search Heuristic

A common feature of the sequential heuristics is that they all find only one maximal independent set. Once a maximal independent set is found, the search stops, hoping it is (close to) the optimal solution. This suggests us a possible way to improve our approximation solutions by expanding the search. For example, once we find a solution S, we can search its 'neighbors' to improve S. This leads to the class of the *local search* heuristics. It is worth to notice that this improvement technique also leads to so-called *augmenting* methods, which are described more in Section 2.5 and Chapter 3.

Greedy Randomized Adaptive Heuristic

A class of heuristics designed to search random various neighbors of some maximal solution S is called the *randomized* heuristics. A *greedy randomized adaptive search procedure* (GRASP) is an iterative randomized sampling technique, in which, each iteration provides an heuristic solution to the problem at hand. The best solution over all GRASP iterations is kept as the final result. An elaborated implementation of the randomized heuristic for the MIS problem was described by Feo et al. [62].

Continuous-based Heuristics

Recently, continuous formulations of discrete optimization problems turn out to be particularly attractive. They not only allow us to exploit the full arsenal of continuous techniques, thereby leading to the development of new algorithms, but may also reveal unexpected theoretical properties. In 2002, Burer et al. [40] derived two continuous optimization formulations for the MIS problem. Based on these formulations, they developed and tested new heuristics for finding large independent sets. In the same year, Busygin et al. [41] proposed a heuristic for the MIS problem which utilizes classical results for the problem of optimization of a quadratic function over a sphere.

2.3.2 Advanced Search Heuristics

Local search algorithms are only capable of finding local solutions of an optimization problem. In the past few years, many powerful variations of the basic local search procedure have been developed and applied in the MIS problem to avoid this problem. Many of which are inspired from various natural phenomena, which we describe briefly in this subsection.

Simulated Annealing

In condensed-matter physics, the term "annealing" refers to a physical process to obtain a pure lattice structure, where a solid is first heated up in a heat bath until it melts, and next cooled down slowly until it solidifies into a low-energy state. During the process, the free energy of the system is minimized, which we suppose that it coresponds to the optimal solution of the problem. *Simulated annealing* was introduced in 1983 by Kirkpatrick et al. [108]. Here, the solutions of the problem correspond to the states of the physical system, and the evaluation value of a solution is equivalent to the energy of the state.
Aarts and Korst [1], without presenting any experimental result, suggested the use of simulated annealing for solving the MIS problem using a *penalty function* approach.

Neural Networks

Artificial neural networks (or simply, neural networks) represent an attempt to imitate some of the useful properties of biological nervous systems, such as adaptive biological learning. A neural network consists of a large number of parallel, highly interconnected processing elements emulating neurons, which are tied together with weighted connections analogous to synapses. In the mid-1980's, Hopfield and Tanks [104] showed that certain feedback continuous neural models are capable of finding approximate solutions to difficult optimization problems. Aarts and Korst [1] provided an excellent introduction to a particular class of neural networks (so-called the Boltzmann machine) for the MIS problem. Other examples about attempts at encoding the MIS problem of a neural network were given by Ballard et al. [15], Ramanujam and Sadayappan [152], and Takefuji et al. [160].

Genetic Algorithms

Genetic algorithms is an optimization method motivated by evolution processes in natural systems. They work on a population of solutions which are called *chromosomes* or *individuals*. Each individual has an associated *fitness* value which determines its probability of survival in the next *generation*. The higher the fitness, the higher the probability of survival. The genetic algorithm starts out with an initial population of members generally chosen at random and makes use of three basic operators *reproduction*, *crossover*, and *mutation*. Reproduction consists of choosing the chromosomes to be copied in the next generation according to a probability proportional to their fitness. The crossover operator is applied between pairs (or more) of selected individuals to produce new ofsprings having properties from their parents. The mutation operator is applied which randomly changes a chromosome. An introduction about genetic algorithms and some practical examples can be found in [135]. One of the first attempts to solve the MIS problem using genetic algorithm was done in Bäck and Khuri [14]. Hifi [102] also modified the basic genetic algorithm and applied it to the MIS problem.

Tabu Search

Tabu search, introduced independently by Glover [78, 79] and Hansen and Jaumard [91], is a modified local search algorithm, in which, a prohibition (tabu) based strategy is employed to avoid cycles in the search trajectories and to explore new regions in the search space. In 1989, Friden et al. [70] proposed a heuristic for the MIS problem based on tabu search. The tabu-searh-based branch and bound algorithm presented by the same authors in [71]. A various version of tabu search successfully applied to the MIS problem was also introduced by Mannino and Stefanutti [134].

2.4 Graph Transformations

To solve the MIS problem we can use the technique transforming a given graph G into a new graph G' in such a way that the difference $\alpha(G) - \alpha(G')$ is easy to compute. By making successive transformations, the goal is to obtain a graph that belongs to some graph class, for which a polynomial-time algorithm is already known.
A trivial example is given by the deletion of an isolated vertex which reduces the independence number by exactly one. A more sophisticated example comes from matching theory and is known as the *cycle shrinking* [119] and is a key tool to solve the Maximum Matching problem.
The literature provides many more examples of graph transformations that can be useful for the MIS problem. A very good review on such transformations was given by Lozin [122]. Here, we give a brief revision on some examples.

2.4.1 Edge Deletion and Edge Insertion

Some independence number preserving transformations reducing the number of edges have been proposed by Butz et al. [44]. Given two adjacent vertices a and b, let c be a vertex such that $c \nsim b$ and every neighbor of b except a is adjacent to a or c. Then the removal (or adding) of the edge ac does not change the independence number of the graph. Next, we consider some graph transformations deleting vertex (together with all incident edges).

2.4.2 Removal of constantly many Vertices

First, we start with techniques repeatedly removing a vertex. May be, the simplest example is above isolated vertex deletion. Following are some more complicated examples.

Simplicial Vertex Reduction

A vertex u is said to be *simplicial* if the neighborhood of u is a clique. Obviously, every independent set S contains at most one vertex v in $N_G[u]$. Moreover, if S is an independent set containing v, then $S\backslash\{v\} \cup \{u\}$ is an independen set. Hence, deletion of any neighbor of a simplicial vertex does not change the independence number or the deletion of the simplicial vertex together with its neighborhood reduces the independence number by one. It is worth noticing that the simplicial vertex reduction leads to efficient algorithms for the MIS problem in some special graph classes. A well-known example is given by the chordal (triangulated) graphs [80]. This reduction provides a linear-time solution for the MIS problem in the class of chordal graphs [154]. In some cases, this reduction allows to simplify the problem substantially. For instance, it has been proven by Brandstädt and Hammer [28] that the independence number of a (P_5, $K_{1,4}$, fork, banner)-free graph without simplicial vertices is at most three, and hence can be computed efficiently.

Neighborhood Reduction

Let a and b be two adjacent vertices in a graph G. If $N[a] \subset N[b]$, then for any independent set S with $b \in S$, the set $(S \backslash \{b\}) \cup \{a\}$ is also independent. Therefore, the removal of b from the graph does not change its independence number. Clearly, the simplicial reduction can be considered as a sequence of neighborhood reductions (of neighbors of the simplicial vertex) followed by the deletion of an isolated vertex (the simplicial vertex itself). Neighborhood reduction has been discovered independently by many researchers under various names such as neighborhood reduction or elementary compression. The neighborhood reduction has been used by Golumbic and Hammer [81] to reduce any circular arc graph to a special canonical form which allows a simple solution to the MIS problem, thus providing an efficient algorithm to solve the problem in the class of circular arc graphs.

Twin Reduction

Two adjacent vertices a, b of G are called *twin* if $N_G[a] = N_G[b]$. Clearly, twin reduction is a special case of neighborhood reduction. Twin reduction was used by Corneil [51] to determine the independence number of cographs.

Vertex Deletion

Billionet [19] gave another vertex reduction. Let (a, b, c) be a P_3. It has been observed in [19] that if $(N(a) \cup N(c)) \backslash N[b]$ is a clique, then the removal of b does not change the graph independence number.

α-redundance

All above vertex removal techniques can be considered as special cases of a so-called α-redundant technique [27]. A vertex is called α-redundant if its removal does not change the independence number. We revise this method more detail in Section 5.2.

2.4.3 Transformations based on Boolean Identities

An efficient method to build up transformations are Boolean identities. STRUCTION introduced by Ebenegger et al. [58] for example can be used to solve the MIS problem in circular-arc graphs [81], in CAN-free graphs [88], and in CN-free graphs [89]. Some restricted version of the STRUCTION method have been applied to the MIS problem by Beigel [17] and Formin et al. [66]. Other graph transformations based on Boolean identities are magnet reduction [87] and BAT reduction [97]. A special case of BAT reduction is vertex folding used to improve the worst case time complexity for the vertex cover and independent set problems by Chen et al. [48]. Moreover, the transformation inverse to vertex folding, was described by Alekseev [5] under the name *vertex splitting* in order to reduce in polynomial time the maximum independent set problem from the class of all graphs to some restricted classes. A weaker version of vertex splitting was used by Murphy [144] to prove NP-hardness of the problem in graph with large girth. We describe these methods more detail in Section 5.1.

2.4.4 Clique Reduction and Edge Projection

Clique Reduction

For a graph $G = (V, E)$ and a clique K in G, Lovász and Plummer [119] defined $G|K$ as the graph obtained from G by deleting the vertices in K and connecting two non-adjacent vertices u and v in $V - K$ by an edge if and only if $K \subset N(u) \cup N(v)$. The conditions for K and G such that $\alpha(G) = \alpha(G-K)+1$ were described by Sassano [155] and Hertz and de Werra [101] and were used to solve the MIS problem for (bull,fork)-free graphs [159] and for AH-free graphs [101]. An edge can be considered as a special clique of cardinality two and similar technique for an edge is the following.

Edge Projection

Let $G = (V, E)$ be a graph, and let $e = uv \in E$. Mannino and Sassano [133] described a reduction, so-called *edge projection* as follows. Denote $G|e = (V|e, E|e)$ as the projection of e in G obtained by deleting all vertices (together with all incident edges) of $\{u, v\} \cup (N(u) \cap N(v))$ and adding edges connecting non-adjacent pairs of vertices, in which, one is adjacent only with u and one is adjacent only with v. The authors also described the conditions, under which $\alpha(G) = \alpha(G|e) + 1$. Using this technique, the authors developed a new upper bound procedure for the MIS problem.

2.4.5 Conic Reduction

Lozin [120] developed another reduction, so-called *conic* reduction as follows. Let a be a vertex of a graph G and $I(a)$ be the family of non-trivial (of cardinality at least two) independent set in $G[N(x)]$. Let us define a similar relation on $I(a)$ as follows: two sets $X, Y \in I(a)$ are similar if and only if $N_{A(a)}(X) = N_{A(a)}(Y)$. In each similarity class, we choose a maximum independent set and denote the family of all chosen sets by $F(a)$. Note that $F(a)$ can be constructed not uniquely. Then a vertex a is said to be conic if under any construction of $F(a)$ and for any $X, Y \in F(a)$ such that $X \cup Y \in I(a)$, $X \cap Y \neq \emptyset$ implies either $X \subset Y$ or $Y \subset X$. For any set $X \in F(a)$, denote by $S(X)$ the family of all maximal sets in $F(a)$ properly included in X, and set $r(X) := |X| - 1 - \sum_{Y \in S(X)}(|Y| - 1)$. Particularly, if $S(X) = \emptyset$, then $r(X) = |X| - 1$. Then the conic reduction of a graph G centered at a conic vertex a as the three following steps.

1. Remove the conic vertex together with its neigborhood from the graph.
2. For every set $X \in F(a)$, add to the remainder $G[A(a)]$ a set X^T of $r(X)$ new vertices.
3. For every new vertex $x \in X^T$, link x to each vertex in $N_{A(a)}(X)$; link x to a new vertex $y \in Y^T$ if and only if there is no set $Z \in F(a)$ such that $X, Y \subset Z$.

Denote by G^T the graph produced by conic reduction of G. Lozin showed in [120] that $\alpha(G) = \alpha(G^T)+1$ and used this reduction to solve the MIS problem in (fork,parachute, butterfly,kite)-free graphs.

2.5 Augmenting Graph

It is well-known that finding a maximum matching in a given graph can be done in polynomial time. This is due to Berge's idea of augmenting (alternating) chains [18] and the celebrated so-called Blossom algorithm of Edmonds [59] that finds augmenting chains in order to construct maximum matchings in graphs in polynomial time. This result can be immediately translated into a polynomial solution to the MIS in the class of line graphs. Rephrasing Berge's idea in terms of independent sets, we can say that in a line graph, an independent set is maximum if and only if there are no augmenting chains with respect to this set. This idea can be extended to a general approach for finding maximum independent sets, the method of finding *augmenting graphs* as follows.

Definition 2.1. *[98] Let S be an independent set in a graph G. A bipartite graph $H = (W, B, E)$ with the vertex set $W \cup B$ and the edge set E is called augmenting for S (and we say that S admits the augmenting graph H) if*

1. $W \subset S$, $B \subset V(G) \backslash S$,
2. $N(B) \cap (S \backslash W) = \emptyset$,
3. $|B| > |W|$.

Clearly, if $H = (W, B, E)$ is an augmenting graph for S, then S is not a maximum independent set in G, because the set $S' = (S \backslash W) \cup B$ is independent and $|S'| > |S|$. We shall say that the set S' is obtained from S by *H-augmentation*. Conversely, if S is not a maximum independent set, and S' is an independent set such that $|S'| > |S|$, then the subgraph of G induced by the vertices subset $(S \backslash S') \cup (S' \backslash S)$ is augmenting for S. Therefore, we have the following key result.

Theorem 2.1. *[98] An independent set S in a graph G is maximum if and only if there are no augmenting graphs for S.*

This theorem suggests the following general approach to find a maximum independent set in a graph G. Begin with any independent set S (may be empty) in G and as long as S admits an augmenting graph H, apply H-augmentation to S. Clearly, the problem of finding augmenting graphs is generally NP-hard, as the MIS problem is NP-hard. For a polynomial time solution to some graph class, one has to solve the two following problems:

(P1) Find a complete list of augmenting graphs in the class under consideration.

(P2) Develop polynomial time algorithms for detecting all augmenting graphs in the class.

This technique was developed for claw-free graphs independently by Minty [137] and Sbihi [156]. Recently, the approach has been successfully applied to develop polynomial-time algorithms to solve the MIS problem in many other special graph classes. Some examples are (P_6,diamond)-free graphs [138], ($P_6, K_{2,3}$)-free graphs [142], ($S_{1,2,3}$,banner$_k$, $K_{1,m}$)-free graphs and ($S_{1,2,j}$,banner,$K_{1,m}$)-free graphs [98], and ($S_{1,2,5}$,banner)-free graphs [125]. In Chapter 3, we revise this method and apply in some subclasses of $S_{2,j,k}$-free graphs. We also describe how to apply this method to some other combinatorial problems in graph theory in Chapter 4.

2.6 Modular Decomposition and Decomposition by Clique Separators

2.6.1 Modular Decomposition

Another useful method to solve the MIS problem in special graph classes is the *modular decomposition* technique. Let $G = (V, E)$ be a graph, U be a subset of V and u be a vertex of G outside U. We say that u *distinguishes* U if u has both a neighbor and a non-neighbor in U. A subset $U \subset V(G)$ is called a *module* in G if it is indistinguishable for any vertex outside U. A module U is *trivial* if U is a single vertex or V itself, otherwise it is *non-trivial*. A graph whose each module is trivial is called *prime*. It has been shown (for example in [136]) that if the problem is polynomially solvable for every prime graph of a graph class $\mathcal{X}$, then it is also polynomial solvable in $\mathcal{X}$.
In the simplest case, when a graph is disconnected or the complement of a disconnected graph, this technique leads to a linear algorithm for the MIS problem in P_4-free graphs (i.e. cographs) [52]. Recently, the technique has been applied to a more general class: Fouquet et al. [68] defined the class of ($P_5, \overline{P_5}$,fork)-free graphs as semi-P_4-sparse graphs, where the fork is $S_{1,1,2}$. Using modular decomposition, the authors proposed a linear time recognition algorithm for semi-P_4-sparse graphs. They solved, among other problems, the MIS problem by adapting the linear algorithms of Chvátal et al. [50] designed for the class of perfect graphs that are $(P_5, \overline{P_5}, C_5)$-free. In addition, the authors proposed an algorithm to solve the MIS problem for ($P_5, \overline{P_5}$,fork)-free graphs. Brandstädt and Kratsch [30] also used this technique to solve the problem in (P_5,gem)-free graphs.

2.6.2 Clique Separator

A *clique separator* in a connected graph G is a subset K of vertices of G which induces a complete graph, such that the graph $G - K$ is disconnected. It is well-known that the MIS problem can be reduced in polynomial-time to graphs without clique separators. Such graphs are called *atom*. The corresponding divide-and-conquer approach providing such a reduction is known as decomposition by clique separators. It was originally developed by Whitesides [169], and adapted for the WIS and the MIS problems by Tarjan [161] and Alekseev [4], respectively. More specifically, decomposition by clique separators can be used to efficiently solve the WIS problem for a graph class $\mathcal{X}$, once we know how to solve it on certain subgraphs of the atoms. This technique was used in [4] for $(P_2 + P_3, K_{1,m})$-free graphs and of $(P_5, \overline{P_2 + P_3})$-free graphs.

2.6.3 Combined Technique

Brandstädt and Hoàng [29] combined decomposition by clique separators with modular decomposition into a more general decomposition scheme as following. Given a graph class $\mathcal{X}$, if the MIS problem is polynomially solvable for those induced subgraphs of graphs in $\mathcal{X}$ which are prime atoms, then $\mathcal{X}$ is MIS-easy. However, Brandstädt and Hoàng haven't given the full proof of the technique and Brandstädt and Giakoumakis [26] stated that latter attempts for proving it failed. The authors proposed another combined approach so called atoms of prime graphs. Using it, the MIS-easiness for hole- and co-chair-free graphs is obtained.

2.7 Graphs of Bounded Parameters

2.7.1 Treewidth

Graphs of *treewidth* at most k, also known as partial k-trees, generalize trees and are very important from an algorithmic viewpoint, since many graph problems that are NP-hard for general graphs including the MIS problem are showed by Arnborg [9] being solvable in linear time when restricted to graphs of treewidth at most k. In particular, showing that a graph class is of uniformly bounded treewidth implies that such a class is MIS-linear. For example, together with the conclusion that graphs of bounded degree and bounded chordality have bounded treewidth, this argument leads to polynomial solution for the MIS problem for graphs of bounded maximum degree and bounded chordality. This technique was also used by Broersma et al. to show the polynomial solvability of the MIS problem in asteroidal-triple-free graphs [39].

2.7.2 Diameter

Given a graph G, the *diameter* of G is the largest distance between two vertices of G and denoted as $diam(G)$. From the observation that the treewidth of a *planar* graph is bounded above by a function of its diameter [57, 60], the MIS problem is polynomially solvable in bounded diameter planar graphs. This technique was used by Lozin and Milanič [123] to reduce the problem from $S_{1,2,k}$-free planar graphs to $S_{1,2,2}$-free planar graphs. We also use this technique in Section 7.3 for subcubic graphs.

2.7.3 Clique-width

Clique-width can be considered as an extension of the concept treewidth in the sense that if a graph G has bounded treewidth, then G also has bounded clique-width [53, 55]. Moreover, Courcelle et al. [54] described a unified approach to the efficient solution of many problems on graph classes of bounded clique-width via the expressibility of the problems in terms of certain logical expression. Together with modular decomposition, this technique was used to solve efficiently the problem in some subclasses of the fork-free graph class [32, 36] the P_5-free graph class [29, 35].

2.8 Other Techniques

We conclude this chapter by mentioning several other ways of tackling the MIS problem in particular graph classes:

- In bipartite graphs, the maximum weight independent set problem can be solved by network flow techniques.

- In *perfect* graphs, the WIS problem can be solved by semi-definite programming [84].

- *Anti-neighborhood.* Brandstädt and Hoàng [29] showed that if the MIS problem is polynomially solvable in the anti-neighborhood of each vertex of any graph G of the graph class $\mathcal{X}$, then $\mathcal{X}$ is MIS-easy. Lozin and Mosca [127] extended this technique to anti-neighborhood of edge for $Y_{m,m}$-free graphs.

- *Dynamic Programming.* Special dynamic programming approaches have been designed for graphs in particular classes based on their structural properties and characterizations. Example include interval graphs [151], distance-hereditary graphs [16], and AT-free graphs [39].

- By showing that two claws in a large H-free graph are of distance finte, Lozin and Milanič showed that the class of large H-free graphs of bounded maximum degree is MIS-easy [124].

2.9 Discussion

Over the past four decades, research on the maximum independent set and related problems has yielded many interesting and profound results. However, a great deal remains to be learned about the MIS problem. In the two first chapters, we have provided an expository survey on complexity algorithms and applications of the problem. Furthermore, an extensive up-to-date bibliography is included. We have also revised on main approaches to tackle the problem. However, the present activity in work related to the MIS problem is so extensive that a survey of this nature is outdated before it is written.

3 Augmenting Methods

Our objective in this chapter is to employ the augmenting graphs approach to develop polynomial time algorithms for the MIS problem on some special subclasses of $S_{2,j,k}$-free graphs for some given integers j and k. In the next section, we revise the augmenting graph method and main approaches for the MIS problem using this technique. Augmenting graphs for some subclasses of $S_{2,k,l}$-free graphs are characterized in Section 3.2. Methods for finding such augmenting graphs are described in Section 3.3. In Section 3.4, we summarize some discussion about the issue.

3.1 Augmenting Graphs Method Revision

Given a graph G and an independent set S, we call vertices of S *white* and remaining vertices *black*. Recall that an augmenting graph H for S is an induced bipartite subgraph $H = (B, W, E)$ of G such that (i) $B \subset V(G)\backslash S$, $W \subset S$, (ii) $|B| > |W|$, and (iii) $N_S(B) \subset W$. For a polynomial time solution for the MIS problem, one has to solve the two following problems:

(P1) Find a complete list of augmenting graphs in the class under consideration.

(P2) Develop polynomial time algorithms for detecting all augmenting graphs in the class.

Now, we give a brief summary on the two problems (P1) and (P2) in the literature.

3.1.1 Characterization of Augmenting Graphs

Obviously, we may restrict our consideration on minimal augmenting graphs. The following observations describe several necessary conditions for an augmenting graph to be minimal.

Lemma 3.1. *[98] If $H = (B, W, E)$ is a minimal augmenting graph for an independent set S of a graph G, then*

1. *H is connected;*
2. *$|W| = |B| - 1$;*
3. *for every subset $U \subset W$, $|U| < |N_B(U)|$.*

The following observation is a consequence of Lemma 3.1 and was obtained in [125].

Corollary 3.2. *[125] Let $H = (B, W, E)$ be a minimal augmenting graph for an independent set S of a graph G. Then for every vertex $b \in M$, there exists a perfect matching between $B\backslash\{b\}$ and W in H, i.e. a matching consists of every vertex of $B\backslash\{b\}$ and W.*

Remark. By the above corollary, from now on, given a minimal augmenting graph $H = (B, W)$ and a black vertex $b \in B$, we denote by M is such a perfect matching and for every vertex u of H different from b and by $\mu(u)$ the matched vertex of u in M. For a subset $U \subset V(H)$, we also denote $\mu(U) := \{\mu(u) : u \in U\}$.
Minty [137] showed that a connected claw-free augmenting graph is an *alternating chain*, i.e. an induced path whose vertices are black and white alternatively and the two end-vertices are black. After that, Alekseev [2] proved that a connected fork-free augmenting graph is either an alternating chain or a *complex*, i.e. a graph obtained from a complete bipartite graph by deleting a matching. Another extension of Minty's result is the following observation of Hertz and Lozin [98].

Lemma 3.3. *[98] For any three integers l, k, and m, the class of ($S_{1,2,l}$,banner$_k$, $K_{1,m}$)-free graphs contains finitely many minimal augmenting graphs different from chains.*

Then characterizations of augmenting graphs mainly followed the two following directions.
In the first approach, researchers characterized augmenting graphs of ($S_{1,2,k}$,banner)-free graphs based on the observation that a banner-free bipartite graph is either C_4-free or complete. First, Alekseev and Lozin [7] have shown that an augmenting graph in ($S_{1,2,3}$,banner)-free graphs is either a chain, complete, or a simple tree (tree^1, see Fig. 3.2) or a plant. Then Gerber et al. [74] extended this result by showing that in ($S_{1,2,4}$,banner)-free graphs, there are only nine augmenting graphs different from those of ($S_{1,2,3}$,banner)-free graphs. Finally, Lozin and Milanič [125] described the concept of redundant set as follows. In an augmenting graph $H = (W, B, E)$, a subset vertices U is called redundant if (i) $|U \cap W| = |U \cap B|$ and (ii) H contains no edges betwwen black vertices of U and vertices of $H - U$. Then the authors showed that in ($S_{1,2,5}$,banner)-free graphs, there are only finitely many augmenting graphs which are different from chain, not complete, different from or cannot be reduced to tree^1, …, tree^6 (see Fig. 3.2) by a redundant set of size at most ten. The proof of Lozin and Milanič based on the result of Hertz and Lozin (Lemma 3.3).
In the second approach, researchers characterized augmenting graphs of subgraphs of P_5-free graphs based on the observation showed indepedently by many researchers (for example [76]) that every connected P_5-free bipartite graph is $2K_2$-free. A $2K_2$-free bipartite graph is a *bipartite-chain* graph, i.e. the vertices can be ordered under inclusion of their neighborhood [76]. Based on this property, Mosca [140] showed that every augmenting graph $H = (B, W, E(H))$ in P_5-free graphs is associated with a so-called augmenting vertex, i.e. a black vertex $b \in B$ such that $W = N_S(b)$. Also using this observation, Boliac and Lozin [22] showed that in $(P_5, K_{2,m} - e)$-free graphs, there are only finitely minimal augmenting graphs not complete for a given integer m. Similarly, Gerber et al. [75] showed that in $(P_5, K_{3,3} - e)$-free graphs, there are only finitely many minimal augmenting graphs not complete nor of the form $K^+_{m,m}$, i.e. the graph obtained from $K_{m,m}$ by adding a pendant vertex.
It is also worth to notice that Mosca [142] also characterited minimal augmenting graphs in $(P_6, K_{2,3})$-free graphs.

3.1.2 Finding Augmenting Graphs

Now, we give a brief review on methods finding augmenting graphs characterized in the above subsection.

Augmenting Chain

Of course, a trick to avoid finding augmenting chains is to restrict ourselves in P_k-free graphs for some given integer k. This trick was used to obtain polynomial solution for the MIS problem in $(P_k, K_{1,m})$-free graphs [131] and in (P_8,banner)-free graphs [74]. Alekseev [2] avoided this by a reduction on fork-free graph containing both claw and P_8.
The first algorithm for finding augmenting chains was developed for claw-free graph by Minty [137] based on technique used by Edmonds [59] for maximum matching problem. This algorithm was extended for skew-star-free graphs by Gerber et al. [73] and for ($S_{1,2,l}$,banner)-free graphs by Hertz et al. [99].

Augmenting Complete Graphs and Nearly Complete Graphs

For nearly complete graphs here, we mean complexes or augmenting graphs of the form $K^+_{m,m}$. First, Alekseev [2] introduced the methods of finding complete augmenting graphs and complexes in fork-free graphs. Similar techniques were also developed for finding complete augmenting graphs in P_5-free graphs by Boliac et al. [22] and banner-free graph by Alekseev and Lozin [7] and finding augmenting graphs of the form $K^+_{m,m}$ in $(P_5, K_{3,3} - e)$-free graphs by Lozin and Mosca [128]. Then Hertz and Lozin [98] combined the two approaches of finding complete augmenting graphs in banner-free graphs and in P_5-free graphs and developed a method for banner$_2$-free graphs.

Augmenting Trees and Redundant Set

First, Alekseev and Lozin [7] introduced the methods of finding simple augmenting tree (tree^1) and plant in ($S_{1,2,3}$,banner)-free graphs. These techniques were extended for ($S_{1,2,5}$,banner)-free graphs by Lozin and Milanič [125]. They argued that Problem P2 of augmenting graph technique can be substituted by the following problem, where $\mathcal{A}$ is the set of all augmenting graphs of S.

Problem Augmentation($\mathcal{A}$): Find an augmenting graph if S admits an augmenting graph in $\mathcal{A}$.

Note that it is not necessary that a found augmenting graph belongs to $\mathcal{A}$. Then the authors showed that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two classes of augmenting graphs such that for every graph $H = (W, B, E) \in \mathcal{A}_2$, there is a redundant subset U of size at most k such that $H - U \in \mathcal{A}_1$, for some given integer k, then Problem Augmentation($\mathcal{A}_2$) is polynomially reducible to Problem Augmentation($\mathcal{A}_1$). They showed that augmenting graphs in ($S_{1,2,5}$,banner)-free graphs is an augmenting chain or belongs to some finite set or is of the form augmenting trees of the form tree^1, $\ldots$, tree^6 (see Fig. 3.2) or can be reduced by a redundant set of size at most ten to augmenting trees. Then the methods of finding augmenting trees in ($S_{1,2,5}$,banner)-free graphs were described.
Now, for the rest of this chapter, we try to unify all above approaches.

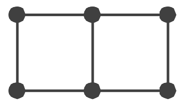

Fig. 3.1: Domino

3.2 Augmenting Graphs in Subclasses of $S_{2,k,l}$-free Graphs

In this section, we describe some structure properties of augmenting graphs of some subclasses of $S_{2,k,l}$-free graphs. But first we obtain the following obvious consequence of Lemma 3.1.

Corollary 3.4. *Let $H = (B, W)$ be a minimal augmenting graph. Then every white vertex of H is of degree at least two.*

Since all augmenting graphs are bipartite, we revise and describe some properties of bipartite graphs, which will be used later in the thesis.

Lemma 3.5. *Let $G = (X, Y, E)$ be a bipartite graph such that there exists a vertex $x \in X$ and $N_Y(x) = Y$. Assume that $|X| = m + 1$. Then at least one of the following statements is true.*

1. *$\{x_i, y_i, y_j, x, y_k, x_k\}$ induces a banner$_2$ or a domino (see Fig. 3.1) for some $x_i, x_k \in X$ and $y_i, y_j, y_k \in Y$, where x is a vertex of degree three in both cases.*
2. *We can linearly order $X = (x, x_1, x_2, \dots, x_m)$ such that there exists some integer p, $0 \le p \le m$, $N_Y(x_i) \supset N_Y(x_j)$ for every $1 \le i \le p$ and $i \le j \le m$ and $|N_Y(x_i)| = 1$ for every $i > p$. Moreover, if $p \ge m-1$, then G is a bipartite-chain.*

Proof. First, assume that Case 1 does not happen. We linearly order X by construction method.
Assume that we already have choosen $x_1, \dots, x_p$. Let $U = X \backslash \{x, x_1, \dots, x_p\}$. Let $x_{p+1} \in U$ be a vertex such that $|N_Y(x_{p+1})|$ is largest among vertices in U.
Suppose that $|N_Y(x_{p+1})| \ge 2$ and there exists a vertex $x_i \in U \backslash \{x_{p+1}\}$ such that $x_i \sim y_i$ and $x_{p+1} \nsim y_i$ for some $y_i \in Y$. By the choice of x_{p+1}, $x_i \nsim y_j$ for some $y_j \in N_Y(x_{p+1})$. Then $\{x, y_k, y_i, y_j, x_{p+1}, x_j\}$ induces a domino or a banner$_2$ for some $y_k \in N_Y(x_{p+1}) \backslash \{y_j\}$, x is a vertex of degree three in both cases, depending on $x_i \sim y_k$ or not, a contradiction.
Now, assume that $p \ge m-1$. Then $N_Y(x) \supset N_Y(x_i) \supset N_Y(x_j)$ for every $1 \le i < j \le m$. We show that for $y_i, y_j \in Y$, either $N_X(y_i) \subset N_X(y_j)$ or $N_X(y_j) \subset N_X(y_i)$. Indeed, suppose that $y_i \sim x_i$ and $y_j \sim x_j$ for some $x_i \in X \backslash N(y_j)$ and $x_j \in X \backslash N(y_i)$. Then $N_Y(x_i) \not\subset N_Y(x_j)$ and $N_Y(x_j) \not\subset N_Y(x_i)$, a contradiction. □

Lemma 3.6. *[56] For any natural numbers t and p, there is a number $\nu := \nu(t, p)$ such that every bipartite graph with a matching at least ν contains either a complete bipartite graph $K_{t,t}$ or an induced matching on p edges.*

3.2.1 Redundant Sets

In this subsection, we extend the concept of redundant sets of Lozin and Milanič [125] and describe some applications.

Definition 3.1. *In an augmenting graph $H = (W, B, E)$, a vertex subset U is called redundant if*

1. $|U \cap W| = |U \cap B|$ *and*
2. *for every vertex* $b \in B \backslash U$, $N_{W \backslash U}(U \cap B) \subset N_{W \backslash U}(b)$.

Then we have the following observation as an extension of Theorem 3 in [125].

Theorem 3.7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two classes of augmenting graphs. If there is a constant k such that for every augmenting graph $H = (W, B, E) \in \mathcal{A}_2$, there is a redundant subset U of size at most k such that $H - U \in \mathcal{A}_1$, then Problem Augmentation($\mathcal{A}_2$) is polynomially reducible to the problem Augmentation($\mathcal{A}_1$).*

Proof. The proof mimics the proof of Theorem 3 in [125]. Let $Augment_1(G, S)$ be a procedure that solves the problem Augmentation($\mathcal{A}_1$) for a graph G and an independent set S. Assume that the procedure outputs a subset $V' \subset V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathcal{A}_1$ (and perhaps even if this is not the case). The procedure also returns $\emptyset$ if no augmenting graph is found. To prove the theorem, we present Procedure $Augment_2(G, S)$ (see Algorithm 1) that solve the problem Augmentation($\mathcal{A}_2$).

Asssume that S admits an augmenting graph $H = (B, W, E) \in \mathcal{A}_2$. Then by the

---

Algorithm 1 $Augment_2(G, S)$

Input: A graph G and an independent set S of G
Output: A subset $V' \subset V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathcal{A}_2$. Return $\emptyset$ if no augmenting graph is found.

for all $U \subset V(G)$ of size at most k such that

    1. $B_0 := U \cap (V(G) \backslash S)$ is independent in H,
    2. $|B_0| = |U \cap S|$

  do
  $G' := G - N_G(B_0) \cap (V(G) \backslash S)$ {Remove the (black) neighbors of B_0 in $V(G) \backslash S$};
  $G'' := G' - \{b \in V(G') \backslash S : N_{S \backslash U}(B_0) \backslash N_{S \backslash U}(b) \neq \emptyset\}$ {Remove the (black) vertices of $V(G') \backslash S$ whose the neighborhood in $S \backslash U$ does not cover the neighborhood of B_0 in $S \backslash U$};
  $T := Augment_1(G'' - U, S \backslash U)$;
  if $T \neq \emptyset$ **then**
    return $U \cup T$ {We have an augmenting graph for S}
  end if
end for
return $\emptyset$

---

theorem's assumption, H contains a redundant set U of size at most k such that

$H - U \in \mathcal{A}_1$. It is obvious that the graph $H - U$ is augmenting for $S \backslash U$. Moreover, since U is redundant, G'' contains every vertex of $H - U$, i.e. Steps 2 and 3 have not removed any vertex of $H - U$. Therefore, Procedure *Augment*$_1$ must output a non-empty set T. Consequently, Procedure *Augment*$_2$ also output a non-empty set $U \cup T$. We show that $G[U \cup T]$ is augmenting for S. Indeed, by Step 2, $G[U \cup T]$ is a bipartite graph. Since T is augmenting for $S \backslash U$ in G'', $|T \cap S \backslash U| < |T \cap V(G'')|$. Moreover, since $|U \cap S| = |U \cap V(G) \backslash S|$, $|(T \cup U) \cap S| < |(T \cup U) \cap V(G) \backslash S|$. By Step 3, $N_S(U \backslash S) \subset T \cap S$, i.e. $N_S((T \cup U) \backslash S) \subset (T \cup U) \cap S$. Hence, the graph $G[U \cup T]$ is augmenting for S, even if $G[T]$ does not coincide with $H - U$. Therefore, whenever S admits an augmenting graph in $\mathcal{A}_2$, Procedure *Augment*$_2$ finds an augmenting graph. To this end, the procedure inspects polynomially many subsets of vertices of the input graph, which results in polynomially many calls of Procedure *Augment*$_1$. The construction of the graph G'' also is performed in polynomial time. Hence, Problem Augmentation($\mathcal{A}_2$) is polynomially reducible to Problem Augmentation($\mathcal{A}_1$). □

Remark. Recall the remark after the proof of the similar theorem, say Theorem 3, in [125]. Let T be the graph produced by Procedure *Augment*$_1$, i.e. T induces an augmenting graph for $S \backslash U$ in $G'' - U$. Let S' be the set of (white) neighbors of black vertices of U in the graph $G'' - U$. Then $T \cup U$ is augmenting for S in G if and only if $S' \subset V(T)$. This is ensured by 2. of Definition 3.1 and Step 3 of Procedure *Augment*$_2$. Moreover, we can also extend the redundant set concept more as follows. If Procedure *Augment*$_1$ start with some initialization process where a finite vertex set whose the neighbor of the black vertices cover the neighbor in $S \backslash U$ of U is computed, then we can process this initialization procedure in *Augment*$_2$ and remove the condition that every neighbor in $S \backslash U$ of black vertices in $B \backslash U$ cover the neighbor U in $S \backslash U$. More precisely, assume that we have Procedure Augment$_1$ as in Algorithm 2, i.e. it starts by generating enumeratively some candidate C, a finite induced subgraph contained in all augmenting graphs of $\mathcal{A}_1$ and then Procedure *Generate*$_1$ return an augmenting graph containing C or an empty set if such augmenting graph not exists. Then we have Augment$_2$ as in Algorithm 3. And hence, Problem Augmentation($\mathcal{A}_2$) is polynomially reducible to Problem Augmentation($\mathcal{A}_1$). More precisely, we have the following definition.

Definition 3.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the two augmenting graph classes. Then $\mathcal{A}_2$ is polynomially reducible to $\mathcal{A}_1$ by a reduction set U associated with a key set B^* if we have the following conditions.*

1. *There exists a polynomial procedure finding an augmenting graph in $\mathcal{A}_1$ (or deciding such augmenting graph does not exist) and such procedure has a form as in Algorithm 2, i.e. starts by generating some candidate graph C, where $|C| \leq k$, for some integer k.*
2. *For every augmenting graph $H = (B, W, E) \in \mathcal{A}_2$, there is a copy of U in $V(H)$ and a copy of B^* in $B \cap C$ (for convenience, also called U and B^*, respectively) in $V(H)$ such that $|U \cap B| = |U \cap W|$ and $N_{W \backslash U}(U \cap B) \subset N_{W \backslash U}((B^* \backslash U) \cap B)$.*

And by the above arguments, we have the following observation.

Theorem 3.8. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the two augmenting graph classes. Then Problem Augmentation($\mathcal{A}_2$) is polynomially reducible to Problem Augmentation($\mathcal{A}_1$) if there are two integers k_1, k_2 such that for every augmenting graph $H = (B, W, E) \in \mathcal{A}_2$, there* *is a reduction set U of size at most k_1 associated with a key set B^* of size at most k_2 such that $H - U \in \mathcal{A}_1$.*

Algorithm 2 *Augment*$_1(G, S)$

Input: A graph G and an independent set S of G

Output: A subset $V' \subset V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathcal{A}_1$. Return $\emptyset$ if no augmenting graph is found.

for all Candidates C **do**
    $T := Generate_1(C, G, S)$;
    if $T \neq \emptyset$ **then**
        return T {We have an augmenting graph for S}
    end if
end for
return $\emptyset$

Algorithm 3 *Augment*$_2(G, S)$

Input: A graph G and an independent set S of G.

Output: A subset $V' \subset V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathcal{A}_2$. Return $\emptyset$ if no augmenting graph is found.

for all $U \subset V(G)$ of size at most k such that

1. $B_0 := U \cap (V(G) \backslash S)$ is independent in H,
2. $|B_0| = |U \cap S|$

do
    $G' := G - N_G(B_0) \cap (V(G) \backslash S)$ {Remove the (black) neighbors of B_0 in $V(G) \backslash S$};
    for all Candidates C of G' such that $N_{S \backslash U}(B_0) \subset N_{S \backslash U}(C \cap (V(G' - U) \backslash S))$ **do**
        $T := Generate_1(C, G' - U, S \backslash U)$;
        if $T \neq \emptyset$ **then**
            return $U \cup T$ {We have an augmenting graph for S}
        end if
    end for
end for
return $\emptyset$

3.2.2 ($S_{2,k,l}$,Even Apples)-free Graphs

We say that G is an (k, m)-extended-chain if G is a tree and contains two vertices a, b such that there exists an induced path $P \subset G$ connecting a, b, every vertex of $G - P$ is of distance at most $k - 1$ from either a or b, and every vertex of $G - P$ has no neighbor in P except possibly a or b and every vertex of G is of degree at most $m - 1$. The following observation is an extension of Lemma 3.3.

Lemma 3.9. *For any three integers k, l, and m such that $4 \leq 2k \leq l$ and $m \geq 3$, in ($S_{2,2k,l}$,$apple^l_4$, $apple^l_6$,. . .,$apple^l_{2k+2}$,$K_{1,m}$)-free graphs, there are only finitely many* *minimal augmenting graphs different from augmenting $(2k,m)$-extended-chains and not of the form apple$_{2p}$. Moreover, if H is of the form augmenting $(2k,m)$-extended-chain, then every white vertex is of degree two.*

Note that in an augmenting graph of the form apple$_{2p}$ (or augmenting apple for short), the vertex of degree three is white. However, given an augmenting apple $H = (B, W, E(H))$, where b is the black vertex of degree one and w is the white vertex of degree three. Then $U := \{b, w\}$ is a redundant set such that $H - U$ is an augmenting chain, a special case of augmenting (k,m)-extended-chain.

Proof. Let $H = (B, W, E)$ be a minimal augmenting graph. If $\Delta(H) = 2$, then H is a cycle or a chain. Since H is bipartite and $|B| = |W| + 1$ (Lemma 3.1), H cannot be a cycle. Now, assume that H is not a chain. We show that either (i) there exists some vertex a such that there is no vertex of distance $2k + l + 1$ from a or (ii) H is an augmenting extended-chain or augmenting apple. Note that, every vertex of H is of degree at most $m - 1$, otherwise an induced $K_{1,m}$ appears, a contradiction. Since H is connected, if we have (i), then

$$|V(H)| \le \sum_{i=0}^{2k+l+1} (m-1)^i = \frac{1-(m-1)^{2k+l+2}}{2-m},$$

i.e. H belongs to some finite set of augmenting graphs.
If a white vertex $w \in W$ has two black neighbor b_1, b_2 of degree one, then $\{b_1, a, b_2\}$ is an augmenting P_3, a contradiction. Hence, we have the following observation.

Claim 3.9.1. *Every white vertex of H has at most one black neighbor of degree one. In particular, if a white vertex w is of degree at least four, then there are at least three neighbors of w of degree two.*

Claim 3.9.2. *Either H contains a vertex, say a, of degree at least three and a has at least three neighbors of degree at least two or H is an augmenting apple.*

Proof. Since H is neither a chain or a cycle, there eixsts at least one vertex of degree at least three.
By Corollary 3.4, every white vertex of H is of degree at least two, i.e. every white neighbor of a black vertex has another black neighbor. Hence, if H contains a black vertex of degree three, then this vertex is a desired vertex a.
Hence, we assume that **(1)** every black vertex of H is of degree at most two. If there exist two black vertices of degree one, then by (1), the path connecting these two black vertices is an augmenting chain, a contradiction. Hence, we assume that **(2)** there exists at most one black vertex of degree one.
By Claim 3.9.1, there exists no white vertex of degree four or we have a desired vertex a. Moreover, if there exist two white vertices of degree three, then either one of them has three neighbors of degree two, i.e. we have a desired vertex a, or we have two black vertex of degree one.
Now, if every white vertex of H is of degree two except one of degree three whose one black neighbor is of degree one, then H is an augmenting apple. □

Let a be a vertex in the conclusion of the above claim. Denote by V_i the subset of vertices of H of distance i from a. Let a_p be the vertex of maximum distance from a

and assume that $p \geq 2k + l + 1$. Let $P := (a_0, a_1, \ldots, a_p)$, where $a_i \in V_i$, be a shortest path connecting $a = a_0$ and a_p. Let $V_1 = \{a_1, b_{1,1}, b_{1,2}, \ldots\}$, and $b_{i+1,j}$ be a vertex of $N_{V_{i+1}}(b_{i,j})$, if such one exists. By the assumption about a, $b_{2,1}$, and $b_{2,2}$ exist (note that they may coincide).
We show that $a_i \nsim b_{i+1,1}$ and $a_{i+1} \nsim b_{i,1}$ for $i = 1, 2, \ldots, 2k$ by induction. Note that it also implies that $b_{i,j} \neq a_i$ for every i, j.
If $a_2 \sim b_{1,1}$, then $\{b_{1,1}, a, a_1, a_2, a_3, \ldots, a_{l+2}\}$ induces a banner$_l$, a contradiction.
If $a_1 \sim b_{2,1}$, then either $\{b_{2,1}, b_{1,1}, a, a_1, a_2, \ldots, a_{l+1}\}$ or $\{b_{2,1}, a_1, a_2, a_3, a_4, \ldots, a_{l+3}\}$ induces a banner$_l$ depending on $a_3 \sim b_{2,1}$ or not, a contradiction.
Now, by induction hypothesis, consider $2 \leq i \leq k$. If $a_i \sim b_{i+1,1}$, then either $\{b_{i+1,1}, a_i, a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{i+l+2}\}$ induces a banner$_l$ or $\{b_{i+1,1}, b_{i,1}, \ldots, b_{1,1}, a, a_1, \ldots, a_i, a_{i+1}, \ldots, a_{i+l}\}$ induces an apple$^l_{2i+2}$ depending on $a_{i+2} \sim b_{i+1}$ or not, a contradiction.
If $a_{i+1} \sim b_{i,1}$ for $2 \leq i \leq k$, then $\{b_{i,1}, b_{i-1,1}, \ldots, b_{1,1}, a, a_1, a_2, \ldots, a_{i+1}, a_{i+2}, \ldots, a_{i+l+1}\}$ induces an apple$^l_{2i+2}$, a contradiction.
Again, by induction hypothesis, consider $k + 1 \leq i \leq 2k$. If $a_i \sim b_{i+1,1}$, then either $\{b_{i+1,1}, a_i, a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{i+l+2}\}$ induces a banner$_l$ or $\{a_{i-1}, a_{i-2}, \ldots, a_1, a, b_{1,1}, b_{2,1}, \ldots, b_{i+1,1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+l}\}$ induces an $S_{2,2k,l}$ depending on $a_{i+1} \sim b_{i,1}$ or not, a contradiction. If $a_{i+1} \sim b_{i,1}$, then $\{a_i, a_{i-1}, a_{i-2}, \ldots, a_1, a, b_{1,1}, b_{2,1}, \ldots, b_{i,1}, a_{i+1}, a_{i+2}, \ldots, a_{i+l+1}\}$ induces an $S_{2,2k,l}$, a contradiction.
Hence, a_i has only one neighbor, say a_{i+1}, in V_{i+1} and only one neighbor, say a_{i-1}, for $i = 1, 2, \ldots, 2k$.
If $b_{i,1} \sim b_{i+1,2}$ for some $1 \leq i \leq 2k - 1$ (if such two vertices exists), then $\{b_{1,1}, \ldots, b_{i,1}, b_{i+1,2}, b_{i,2}, \ldots, b_{1,2}, a, a_1, \ldots, a_l\}$ induces an apple$^l_{2i+2}$, a contradiction. Hence, $b_{i,j}$ (if such vertex exists) has at most one neighbor in V_{i+1} for $1 \leq i \leq 2k - 1$. It also implies that $b_{i,j} \neq b_{i,k}$ for every $1 \leq i \leq 2k$ and $j \neq k$ if such vertices exist.
If V_{2k} contains at least two vertices, say a_{2k} and, without loss of generality, $b_{2k,1}$, then $\{b_{2,2}, b_{1,2}, a, b_{1,1}, b_{2,1}, \ldots, b_{2k,1}, a_1, a_2, \ldots, a_l\}$ induces an $S_{2,2k,l}$, a contradiction.
To summarize, $V_{2k} = \{a_{2k}\}$, every vertex of V_i has only one neighbor in V_{i-1}, for every $1 \leq i \leq p$.
Let T be the connected component of $H - a_1$ containing a. Then T is a tree by the above arguments. We show that a is black. Indeed, for contradiction, suppose that a is white. Let a_1 be the black vertex b of Corollary 3.2. Then there is a perfect matching between $B \cap T$ and $W \cap T$. Let b be a leaf of T. Then by Corollary 3.4, b is black and hence $\mu(b)$ be the (only) white neighbor of b. It also implies that $\mu(b)$ has only one neighbor being a leaf. Indeed, if $\mu(b)$ has another black neighbor being a leaf b', then there exists no $\mu(b')$, a contradiction. Then by induction on T, a has only one black neighbor in T, a contradiction with a is of degree at least three. Hence, we have the following claim.

Claim 3.9.3. *If a is a vertex of the conclusion of Claim 3.9.2, then a is black. Moreover, there exists a neighbor w of a such that the connected component of $H - w$ containing a is a tree T, every vertex of T is of distance at most $2k - 2$ to a, and every white vertex of T is of degree two.*

Let a be the black vertex b of Corollary 3.2. Then there is a perfect matching between $B \cap T \backslash \{a\}$ and $W \cap T$, i.e. $|B \cap T| = |W \cap T| + 1$. Claims 3.9.1 and 3.9.3 lead to the following observation.

Claim 3.9.4. *Every white vertex w of H is either of degree two or three. Moreover,* *in the latter case, exactly one black neighbor of w is of degree one.*

Let j be the largest number such that $|V_j| \geq 2$. Then $2 \leq j \leq 2k-2$. Moreover, j is even, since every leaf of T is black.
Note that every black vertex a_q such that $2k-j < q < p-2k$ is of degree two, otherwise a_q becomes a vertex of the conclusion of Claim 3.9.2 and there exist at least two vertices of degree $2k$ from a_q, a contradiction with Claim 3.9.3.
Let T_1 and T_2 be the two connected component of $H - a_{2k-j+1} - a_{p-2k-1}$ containing a_{2k-j} and a_{p-2k}, respectively. Then by Claim 3.9.3, T_1 and T_2 are trees such that the most distance between a vertex of T_1 (respectively, T_2) to a_{2k-j} (respectively, a_{p-2k}) is $2k-2$. Moreover $|W \cap (T_1+T_2)| + 2 = |B \cap (T_1+T_2)|$.
Now, every white vertex a_q, where $2k-j < q < p-2k$, is of degree two or three, and in the later case a black neighbor of a_q different from a_{q-1} and a_{q+1} is of degree one. Hence, every such white vertex is of degree two, otherwise we have a contradiction with $|W|+1 = |B|$.
Thus, H is an augmenting $(2k-1, m)$-extended-chain. □

We denote tree$_r$ as the graph consisting r induced P_3's sharing a common end-vertex (see Fig. 3.2, tree^1) Note that the tree$_3$ is an $S_{2,2,2}$.

Lemma 3.10. *For any three integers k, l, and m such that $4 \leq 2k \leq l$ and $m \geq 3$, in ($S_{2,2k,l}$,banner, apple$^l_6$,. . .,apple$^l_{2k+2}$,tree$_m$)-free graphs, there are only finitely many minimal augmenting graphs different from augmenting $(2k, m)$-extended-chains, not of the form apple$_{2p}$, nor complete.*

Proof. Let H be a minimal ($S_{2,2k,l}$,banner, apple$^l_6$,. . .,apple$^l_{2k+2}$,tree$_m$)-free augmenting graph. By Lemma 3.9, there exists a vertex x of degree at least $m+2$, otherwise H belongs to some finite set of graphs or is of the form $(2k, m+1)$-extended-chains or augmenting apple. Let b be an arbitrary black vertex different from x and b be the black vertex b in Lemma 3.2.
Let $X = N_H(x) \backslash \{b, \mu(x)\}$, i.e. X contains at least m vertices. Since H is banner-free, either H is C_4-free or H is complete. Suppose that H is C_4-free, i.e. every vertex in $\mu(X)$ has only one neighbor in X. It implies $H[X \cup \mu(X)]$ is an induced matching on at least m edges. This induced matching together with x induce a tree$_m$, a contradiction. □

3.2.3 Augmenting Graphs for $S_{2,2,5}$-free Graphs

In this section, we inspect on ($S_{2,2,5}$,banner$_2$,domino)-free augmenting graphs. We extend the consideration of Section 4 in [125].

Lemma 3.11. *If a minimal augmenting ($S_{2,2,5}$,banner$_2$)-free graph H contains no black vertex of degree more than k ($k \geq 3$), then the degree of each white vertex is at most $p = \max(k^2+k+1, \nu(k+1, 2k^2-2k+2)) + 1$, where ν is the function of Lemma 3.6.*

Proof. Suppose that H contains a white vertex w of degree more than p. Denote by V_j the set of vertices of H at distance j from w. Hence, $|V_1| \geq p+1$.

Claim 3.11.1. *$H[V_1 \cup V_2]$ contains an induced matching of size at least $2k^2-2k+1$ $\{b_1w_1, \ldots, b_{2k^2-2k+2}w_{2k^2-2k+2}, \ldots\}$. Moreover, every w_i has only one neighbor in V_1, i.e. having a neighbor in V_3, and $|N_{V_3}(\{w_i's\})| \geq 2k-1$.*

Proof. Let an arbitrary vertex $b \in V_1$ be the b in Corollary 3.2. Then there exists a perfect matching between $V_1 \backslash \{b, \mu(w)\}$ and V_2, i.e. a matching of size at least $\nu(k+1, 2k^2-2k+2)$. Since every black vertex of H is of degree at most k, H contains no $K_{k+1,k+1}$. By Lemma 3.6, there exists an induced matching on $2k^2-2k+2$ between $V_1 \backslash \{b, \mu(w)\}$ and V_2. Let this matching be $\{b_1 w_1, \ldots, b_{2k^2-2k+2} w_{2k^2-2k+2}, \ldots\}$, where $b_i \in V_1$ and $w_i \in V_2$.
We show that every w_i has only one neighbor, say b_i, in V_1. Indeed, suppose that $w_i \sim c$ for some $c \in V_1$ and $c \neq b_j$ for every j. Then $w_j \sim c$ for every $j \neq i$, otherwise $\{w, b_i, w_i, b_j, w_j, c\}$ induces a banner$_2$, a contradiction. But now, c is a black vertex having at least $2k^2-2k+2$ white neighbors, a contradiction.
Hence, V_2 contains at least $2k^2-2k+2$ vertices having only one neighbor in V_1, i.e. having a neighbor in V_3. Moreover, every black vertex in V_3 has at most k white neighbors in V_2, i.e. $|N_{V_3}(\{w_i's\})| \geq 2k-1$. □

Claim 3.11.2. *$V_4 = \emptyset$, i.e. $|V_3| + |V_1| = |V_2| + 2$.*

Proof. Suppose that V_4 contains a (white) vertex x and let y be its neighbor in V_3, without loss of generality, assume that $y \sim u \in V_2$ and $u \sim b \in V_1$. We show that b is the only one neighbor of u in V_1. Indeed, suppose that $u \sim b'$ for some $b' \in V_1 \backslash \{b\}$. Then $\{b, w, b', u, y, x\}$ induces a banner$_2$, a contradiction.
By Corollary 3.4, x has at least one more black neighbor, say z ($z \in V_3$ or $z \in V_5$). Note that $z \nsim u$, otherwise $\{y, x, z, u, b, w\}$ induces a banner$_2$, a contradiction.
Since y, z have at most $2k$ neighbors in V_1, there exist at least two vertices w_i, w_j non-adjacent to y, z. Then $\{w_i, b_i, w, w_j, b_j, b, u, y, z, x\}$ induces an $S_{2,2,5}$, a contradiction.
Therefore, $V_4 = \emptyset$ and $|V_3| + |V_1| = |V_2| + 2$ by Lemma 3.1. □

Let an arbitrary vertex $b \in V_3$ be the vertex b in Corollary 3.2. Consider the induced matching $\{b_1 w_1, \ldots, b_{2k^2-2k+2} w_{2k^2-2k+2}, \ldots\}$ of the conclusion of Claim 3.11.1. Let $A := \{b_i's\} \backslash \{\mu(w)\}$, without loss of generality assume that $A = \{b_1, b_2, \ldots, b_{2k^2-2k+1}, \ldots\}$. Then $\mu(A) = \{w_1, w_2, \ldots, w_{2k^2-2k+1}, \ldots\}$. Let $D = N_{V_3}(\mu(A)) \backslash \{b\}$. Then similar to Claim 3.11.1, $|D| \geq 2k-1$.

Claim 3.11.3. *There exist two vertices $d, d' \in V_3$ such that d' has a neighbor $u \in V_2$, $u, \mu(d)$ are non-adjacent to $\mu(w)$, $\mu(d), u$ share a neighbor $a \in V_1$.*

Proof. Since $\mu(w)$ has at most $k-1$ neighbors in $\mu(D)$, let $d_1, \ldots, d_k, \ldots \in D$ such that $\mu(d_i) \nsim \mu(w)$. For contradiction, let a_i be the neighbor of $\mu(d_i)$ in V_1 and $a_i \neq a_j$ for every $i \neq j$.
Consider $\mu(a_i)$ for an arbitrary i. If $\mu(a_i) \sim a$ for some $a \in V_1 \backslash \{a_1\}$, then $\mu(a_i) \sim b_j$ for every $b_j \in A \backslash \{a_i, a\}$, otherwise $\{a_i, \mu(a_i), a, w, b_j, w_j\}$ induces a banner$_2$, a contradiction. However, every $b_j \in A$ has at most $k-1$ neighbors in V_2. Hence, there exists at least one integer i, such that $\mu(a_i)$ has no neighbor in V_1. Then $\mu(a_i)$ has a neighbor, say d' in V_3. Moreover, $d' \neq d_i$, otherwise $\{\mu(d_i), d_i, \mu(a_i), a_i, w, \mu(w)\}$ induces a banner$_2$, a contradiction. Now, d_i, d' are two desired d, d' of the claim. □

Now, we have the following observations.
(1) $d \nsim u$, otherwise $\{u, d, \mu(d), a, w, \mu(w)\}$ induces a banner$_2$, a contradiction. Similarly, $d' \nsim \mu(d)$.
(2) Since d, d' has at most $2k$ neighbors in $\mu(A)$, there exists a vertex, without loss of generality, say $b_1 \in A$ such that w_1 is not adjacent to d, d'. Note that w_1 has a

neighbor, say $d_1 \in V_3$.
(3) $\mu(d) \nsim b_1$ (similarly, $u \nsim b_1$). Indeed, suppose that $\mu(d) \sim b_1$. Then $d_1 \sim \mu(d)$, otherwise $\{w, a, \mu(d), b_1, w_1, d_1\}$ induces a banner$_2$, a contradiction. But now, $\{\mu(d), d_1, w_1, b_1, w, \mu(w)\}$ induces a banner$_2$, a contradiction.
(4) $\mu(d)$ and u have no neighbor other than a in V_1. Indeed, suppose that $u \sim a' \in V_1 \backslash \{a\}$. Then $\{a', u, a, w, b_1, w_1\}$ induces a banner$_2$, a contradiction.
(5) $d \nsim \mu(d_1)$ (similarly, d_1 is not adjacent to $\mu(d), u$ and $\mu(d_1) \nsim d'$). Indeed, suppose that $d \sim \mu(d_1)$. Then $\mu(d) \nsim d_1$, otherwise $\{\mu(d_1), d, \mu(d), d_1, w_1, b_1\}$ induces a banner$_2$, a contradiction. If $\mu(d_1)$ has two neigbors $a_1, a_2 \in V_1 \backslash \{a\}$, then $\{a_1, w, a_2, \mu(d_1), d, \mu(d)\}$ induces a banner$_2$, a contradiction. Hence, $\mu(d_1)$ has at most one neigbor in V_1 different from a_1. Thus, because d and d_1 have at most $2k$ neighbors in V_2, there exist two non-neighbors $b_i, b_j \in A$ of $\mu(d_1)$ such that w_i and w_j are non-adjacent to d, d_1. Now, $\{w_i, b_i, w, b_j, w_j, a, \mu(d), d, \mu(d_1), d_1\}$ induces an $S_{2,2,5}$, a contradiction.
Now, $\{d', u, a, \mu(d), d, w, b_1, w_1, d_1, \mu(d_1)\}$ induces an $S_{2,2,5}$, a contradiction. □

Lemma 3.12. *Given a graph G and a minimal augmenting ($S_{2,2,5}$,banner$_2$,domino)-free graph $H = (B, W, E)$ for an independent set S, at least one of the following statements is true:*

1. *H belongs to some finite set of augmenting graphs;*
2. *H is an augmenting $(4, p)$-extended-chain, for some constant p, or an augmenting apple;*
3. *H is an augmenting graph of the form tree^1, tree^2, ..., tree^7 (see Fig. 3.2) or can be reduced by a redundant set containing at most 32 vertices to an augmenting graph of the form tree^1, tree^2, ..., tree^7;*
4. *there is a vertex $b \in B$ such that b is adjacent to all vertices of W.*

Such b of 4 is called the *augmenting vertex* of S, as in [140, 141]. We also call augmenting graphs of the form tree^1, tree^2, ..., tree^7 as *augmenting trees.*

Proof. We proof by contradiction. Let $p = \max(k^2 + k + 1, \nu(k + 1, 2k^2 - 2k + 2)) + 1$, where $k = 10$. Let $b \in B$ such that $|N_W(b)|$ is largest. Note that, if every black vertex is of degree one, then H is an augmenting P_3. If H contains finite number of vertices, then we have 1. If $N_W(b) = W$, then we have 4. Hence, by Lemma 3.9 and Lemma 3.11, we may assume that $10 \le |N_W(b)| \le |W| - 1$. Let b be the vertex b of Corollary 3.2. Let $A = N(b) = \{w_1, w_2, \ldots, w_k\}$ ($k \ge 10$), $C = W \backslash A$, i.e. $C \ne \emptyset$. Let $b_i = \mu(w_i)$. Let C_1 denote the set of vertices in C having at least one neighbor in $\mu(A)$ and $C_0 = C \backslash C_1$. By the connectivity of H, $C_1 \ne \emptyset$. We have the following observations.

Claim 3.12.1. *$H[A \cup \mu(A)]$ is an induced sub-matching of M.*

Proof. We show that $b_i \nsim w_j$ for every pair i, j such that $i \ne j$, $1 \le i, j \le k$. Let $z \in C_1$ and without loss of generality, assume that $z \sim b_1 \in \mu(A)$.
By the choice of b, b_1 is not adjacent to all w_i's, without loss of generality, assume that $b_1 \nsim w_2$.
Now, $b_2 \nsim w_1$, otherwise $\{b, b_1, b_2, w_1, w_2, z\}$ induces a domino or a banner$_2$ depending

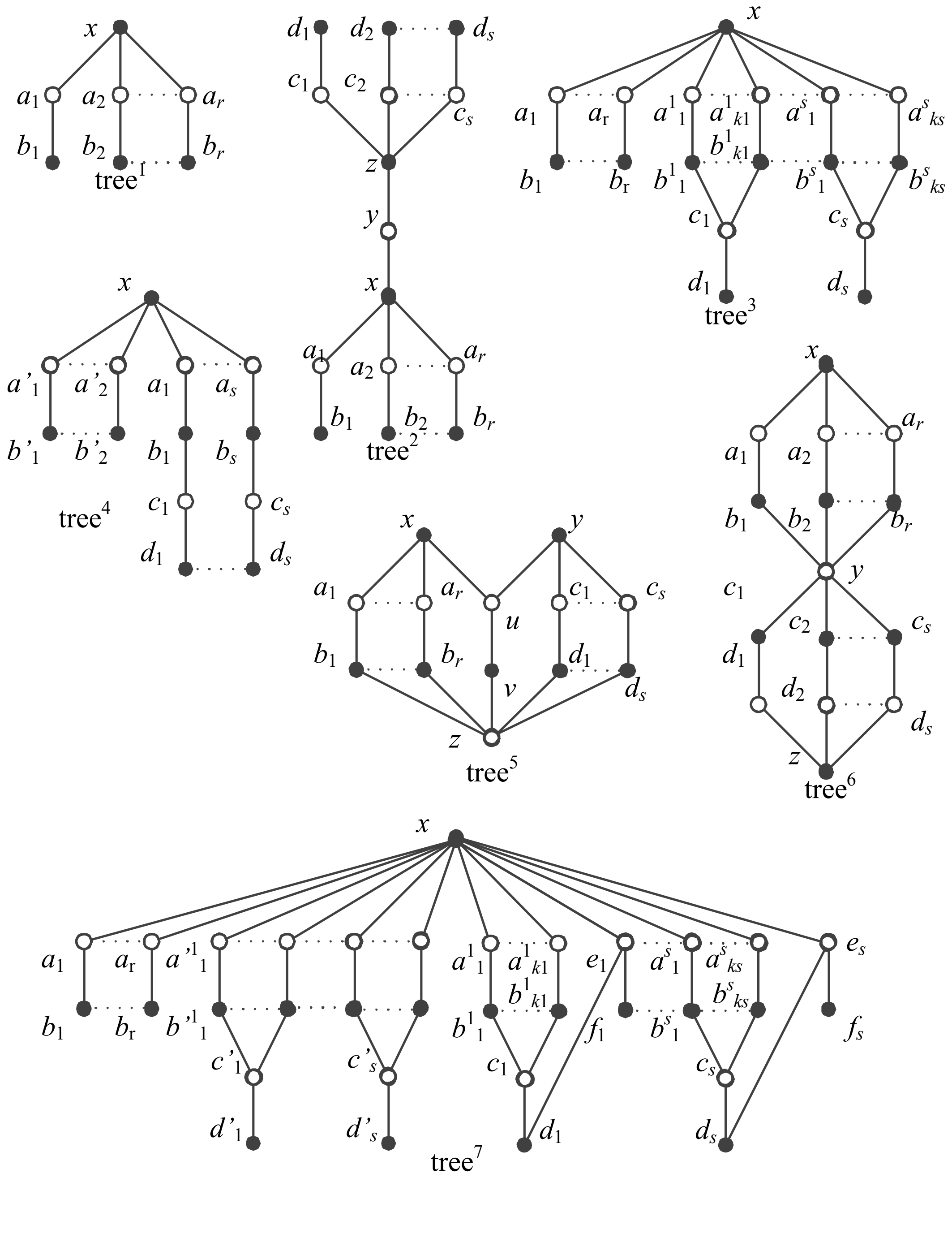

Fig. 3.2: Augmenting Simple Trees

on $b_2 \sim z$ or not, a contradiction.
Moreover, $b_2 \nsim w_i$ for every $i > 2$, otherwise $\{b, b_1, b_2, w_1, w_2, w_i\}$ induces a domino or a banner$_2$ depending on $b_1 \sim w_i$ or not, a contradiction.
Now, $b_1 \nsim w_i$, for every $i > 2$, otherwise $\{w_1, b_1, w_i, b, w_2, b_2\}$ induces a banner$_2$, a contradiction.
Hence, $b_i \nsim w_1$ for $i > 2$, otherwise $\{b, w_i, b_i, w_1, b_1, z\}$ induces a domino or a banner$_2$, depending on $z \sim b_i$ or not, a contradiction.
Thus, $b_i \nsim w_2$ for $i > 2$, otherwise $\{w_2, b_i, w_i, b, w_1, b_1\}$ induces a banner$_2$, a contradiction.
Moreover $b_i \nsim w_j$, for any $j \neq i$ and $i, j > 2$, otherwise $\{w_j, b_i, w_i, b, w_1, b_1\}$ induces a banner$_2$, a contradiction. □

Claim 3.12.2. *There exists no vertex pair $z_1, z_2 \in C_1$ sharing two neighbors in $\mu(A)$.*

Proof. Suppose that there exists a vertex pair $z_1, z_2 \in C_1$ sharing two neighbors in $\mu(A)$, without loss of generality, say b_1, b_2. Then $\{z_1, b_2, z_2, b_1, w_1, b\}$ induces a banner$_2$, a contradiction. □

Claim 3.12.3. *Given $z \in C_1$, $z \sim b_j$ for some $b_j \in \mu(A)$, a black neighbor c of z different from b_j, a black neighbor $\mu(t)$ of z for some $t \in C$, and another white neighbor $y \in C$ of $\mu(t)$ different from z, the following statements are true:*

1. *$c \nsim w_j$;*
2. *$y \nsim b_j$ and $\mu(y) \nsim z$; and*
3. *if $\mu(t) \sim w_i$ for some $i \neq j$, then y, z are not adjacent to b_i and $\mu(y) \nsim w_i$;*
4. *in particular, $\mu(y)$ and $\mu(t)$ cannot share a same neighbor in A.*

Proof. Suppose that $c \sim w_j$. Then $c \sim w_i$ for every $i \neq j$, otherwise $\{b_j, z, c, w_j, b, w_i\}$ induces a banner$_2$, a contradiction. But now, we have a contradiction with the choice of b.
Now, $y \nsim b_j$, otherwise $\{z, \mu(t), y, b_j, w_j, b\}$ induces a banner$_2$, a contradiction. Moreover, $\mu(y) \nsim z$, otherwise $\{w_j, b_j, z, \mu(t), y, \mu(y)\}$ induces a domino or a banner$_2$ depending on $\mu(y) \sim w_j$ or not, a contradiction.
Assume that $\mu(t) \sim w_i$ for some $i \neq j$. Then $z \nsim b_i$, otherwise $\{\mu(t), w_i, b_i, z, b_j, w_j\}$ induces a banner$_2$, a contradiction. Hence, $y \nsim b_i$, otherwise $\{b_i, y, \mu(t), w_i, b, w_j\}$ induces a banner$_2$, a contradiction. Now, $\mu(y) \nsim w_i$, otherwise $\{w_i, \mu(y), y, \mu(t), z, b_j\}$ induces a banner$_2$, a contradiction. □

Claim 3.12.4. *Every black vertex different from b is adjacent to at most one neighbor in A.*

Proof. Clearly, every black vertex of $\mu(A)$ has only one neighbor in A by Claim 3.12.1. Now, suppose that there exists some black vertex $y \in B \backslash (\{b\} \cup \mu(A))$ having two neighbors, without loss of generality, say $w_1, w_2 \in A$. Then y is adjacent to every vertex $w_i \in A \backslash \{w_1, w_2\}$, otherwise $\{w_1, y, w_2, b, w_i, b_i\}$ induces a banner$_2$, contradiction. Now, y is adjacent to every vertex of A and $\mu(y)$, a contradiction with the choice of b. □

Claim 3.12.5. *There exists no vertex $b_j \in \mu(A)$ having two neighbors $z_1, z_2 \in C_1$ sharing another black neighbor, say $c \neq b_j$.*

Proof. Indeed, otherwise, by Claim 3.12.3, $c \nsim w_j$, then $\{z_1, c, z_2, b_j, w_j, b\}$ induces a banner$_2$, a contradiction. □

Claim 3.12.6. *Given $b_j \in \mu(A)$, let $C(b_j)$ be the set of vertices of C_1 adjacent to b_j. Then $H[C(b_j) \cup \mu(C(b_j))]$ is an induced sub-matching of M.*

Proof. For contradiction, without loss of generality, suppose that $z_1, z_2 \in C$ are two neighbors of b_j and $z_1 \sim \mu(z_2)$. By Claim 3.12.3, $\mu(z_2) \nsim w_j$. Hence, $\{z_1, \mu(z_2), z_2, b_j, w_j, b\}$ induces a banner$_2$, a contradiction. □

Claim 3.12.7. *If H contains a vertex $y \in C$ adjacent to at least $k-3$ vertices of $\mu(A)$, then either H is of the form tree^5 or tree^6 or H contains a redundant set U of size at most 32, such that $H - U$ is of the form either tree^1, tree^4, tree^5, or tree^6.*

Proof. Let D_1 be the subset of vertices of C sharing some neighbor in $\mu(A)$ with y, A_1 be the vertex subset of A such that $\mu(A_1) = N_{\mu(A)}(y)$, $A_2 = A \backslash A_1$, E_1 be the vertices subset of C_1 adjacent to some vertex in $\mu(A_2)$. Without loss of generality, assume that $w_1, w_2, \ldots, w_{k-3} \in A_1$. We have the following observations.
(1) y has no neighbor in $\mu(D_1)$ and $\mu(y)$ has no neighbor in $A_1 \cup D_1$. Indeed, by Claim 3.12.3, $\mu(y)$ has no neighbor in A_1. If for some $z \in D_1$, without loss of generality, assume that $z \sim b_1$, $y \sim \mu(z)$, then $y \nsim b_1$, by Claim 11.3, a contradiction. Moreover, since $\mu(y) \nsim w_1$, $\mu(y) \nsim z$, otherwise $\{z, \mu(y), y, b_1, w_1, b\}$ induces a banner$_2$, a contradiction.
(2) By Claim 3.12.5, every vertex of D_1 has exactly one neighbor in $\mu(A_1)$. In particular, every vertex of $C_1 \backslash \{y\}$ has at least $k-4$ non-neighbors in $\mu(A)$. Moreover, there exists only one vertex $y \in C_1$ adjacent to at least $k-3$ vertices in $\mu(A)$.
(3) Any two vertices of D_1 have different neighbors in $\mu(A_1)$. Indeed, without loss of generality, suppose that $z_1, z_2 \in D_1$ both are adjacent to b_1. By Claim 3.12.4, and $|A_1| = k-3 \geq 7$, there exist $w_i, w_j \in A_1$ different from w_1 and not adjacent to $\mu(z_1), \mu(z_2)$. By (2) and Claim 3.12.6, $\{\mu(z_1), z_1, b_1, z_2, \mu(z_2), y, b_i, w_i, b, w_j\}$ induces an $S_{2,2,5}$, a contradiction.
(4) Similar to Claim 3.12.6, let $C(y)$ be the subset of vertices of C_0 adjacent to $\mu(y)$. Then $H[C(y) \cup \mu(C(y))]$ is an induced sub-matching of M.
(5) Similarly to (3) (using (4)), there are at most one vertex of C_0 adjacent to $\mu(y)$.
(6) $H[(C_1 \backslash \{y\}) \cup \mu(C_1 \backslash \{y\})]$ is an induced sub matching of M. Indeed, suppose that for a couple of vertices $z_1, z_2 \in C_1 \backslash \{y\}$, $z_1 \sim \mu(z_2)$. Without loss of generality, assume that z_1, z_2 are adjacent to $b_{i_1}, b_{i_2} \in \mu(A)$, respectively. Then by Claim 3.12.3, $\mu(z_2) \nsim w_{i_2}$. Hence, $z_1 \nsim b_{i_2}$, otherwise $\{z_2, \mu(z_2), z_1, b_{i_2}, w_{i_2}, b\}$ induces a banner$_2$, a contradiction. By (2) and Claim 3.12.4, there exists a vertices pair $b_i, b_j \in \mu(A)$ not adjacent to z_1, z_2 such that w_i and w_j are not adjacent to $\mu(z_1)$, $\mu(z_2)$. Now, $\{b_i, w_i, b, w_j, b_j, w_{i_2}, b_{i_2}, z_2, \mu(z_2), z_1\}$ induces an $S_{2,2,5}$, a contradiction.
(7) There exists no vertex $t \in C \backslash \{y\}$ having a neighbor in $\mu(C_1 \backslash \{y, \mu(t)\})$. Indeed, if $t \in C$ is adjacent to $\mu(z)$ for some $z \in C_1 \backslash \{y, t\}$, then for the vertex b_j adjacent to z, $t \nsim b_j$ by Claim 3.12.3. By (2) and Claim 3.12.6, there exists a pair of vertices w_i, w_l non-adjacent to $\mu(z)$ such that b_i, b_l non-adjacent z, t. Now, $\{b_i, w_i, b, w_l, b_l, w_j, b_j, z, \mu(z), t\}$ induces an $S_{2,2,5}$, a contradiction.
(8) Similarly, there exists no vertex $t \in C_1 \backslash \{y\}$ having a neighbor in $\mu(C \backslash \{y, \mu(t)\})$.
(9) If $C_0 = \{z\}$, then $z \sim \mu(y)$. If $|C_0| \geq 2$, then there exists a vertex $x \in C_0$ such that $x \sim \mu(z)$. For every such vertex x, the followings are true: $y \sim \mu(x)$, $\mu(x) \nsim z$, and

$\mu(x) \nsim w_i$ for $w_i \in A_1$. Moreover, if $|C_0| \geq 2$, then $A_2 = \emptyset$, i.e. y is adjacent to every vertex of $\mu(A)$.
Indeed, if $C_0 \neq \emptyset$, then by (7) and the minimality of H, there exists a vertex $z \in C_0$ such that $z \sim \mu(y)$, otherwise $|C_0| = |N_H(C_0)|(= |\mu(C_0)|)$, a contradiction. Moreover, no other vertex of C_0 is adjacent to $\mu(y)$ by (5). Hence, if $|C_0| \geq 2$, then, again by (7) and the minimality of H, there exists a vertex $x \in C_0$ such that $x \sim \mu(z)$.
Let $x \in C_0$ such that $x \sim \mu(z)$. Since $\mu(z) \nsim y$ by Claim 3.12.3, $x \nsim \mu(y)$, otherwise $\{z, \mu(z), x, \mu(y), y, b_1\}$ induces a banner$_2$, a contradiction. Thus, $\mu(x) \nsim z$, otherwise $\{y, \mu(y), z, \mu(z), x, \mu(x)\}$ induces a domino or a banner$_2$, depending on $\mu(x) \sim y$ or not, a contradiction. Now, if $y \nsim \mu(x)$, then by Claim 3.12.4, there exists a pair of vertices $b_i, b_j \in \mu(A_1)$ such that w_i and w_j are not adjacent to $\mu(x), \mu(z)$ and $\{w_i, b_i, y, b_j, w_j, \mu(y), z, \mu(z), x, \mu(x)\}$ induces an $S_{2,2,5}$, a contradiction. Then $\mu(x) \nsim w_i$ for any $w_i \in A_1$, otherwise $\{y, b_i, w_i, \mu(x), x, \mu(t)\}$ induces a banner$_2$, a contradiction.
Asume that $|C_0| \geq 2$, we show that $A_2 = \emptyset$. Indeed, without loss of generality, assume that $y \nsim b_k$. Let $x \in C_0$ be a vertex such that $x \sim \mu(z)$. Then $\mu(y)$ or $\mu(z)$ is not adjacent to w_k, otherwise since $z \nsim w_k$ by Claim 3.12.3, $\{z, \mu(z), w_k, \mu(y), y, b_1\}$ induces a banner$_2$, a contradiction. Similarly, $\mu(x)$ or $\mu(z)$ is not adjacent to w_k. Now, $\mu(y) \nsim w_k$, otherwise since there exists a pair of vertices $w_i, w_j \in A_1$ not adjacent to $\mu(y), \mu(z)$ by Claim 3.12.4, $\{b_i, w_i, b, w_j, b_j, w_k, \mu(y), z, \mu(z), x\}$ induces an $S_{2,2,5}$, a contradiction. By similar reasons, $\mu(x) \nsim w_k$. Now, by Claim 3.12.4, there exists a vertex $w_i \in A_1$ not adjacent to $\mu(x)$ and $\{z, \mu(y), y, \mu(x), x, b_i, w_i, b, w_k, b_k\}$ induces an $S_{2,2,5}$, a contradiction.
(10) If $|D_1| \geq 2$, then no vertex of $\mu(D_1)$ has a neighbor in A. Indeed, by (3), without loss of generality, let $z_1, z_2 \in C_1$ be adjacent to b_1, b_2, respectively. To the contrary, suppose that $\mu(z_1)$ has a neighbor $w_i \in A$. By Claim 3.12.3, $w_i \neq w_1$. If $w_i = w_2$, then by (1), (6), and Claims 3.12.3, 3.12.4, $\{z_2, b_2, w_2, b, w_j, \mu(z_1), z_1, b_1, y, \mu(y)\}$ induces an $S_{2,2,5}$ for some vertex $w_j \neq w_1, w_2$ such that $w_j \nsim \mu(z_1)$, a contradiction. If $w_i \neq w_1, w_2$, then by (1) and (6), $\{w_2, b, w_i, \mu(z_2), z_2, \mu(z_1), z_1, b_1, y, \mu(y)\}$ induces an $S_{2,2,5}$ in the case that $\mu(z_2) \sim w_i$, or $\{\mu(z_2), z_2, b_2, y, \mu(y), w_2, b, w_i, \mu(z_1), z_1\}$ induces an $S_{2,2,5}$ in the case that $\mu(z_2) \nsim w_i$, a contradiction.
(11) If there exist two vertices $z_1, z_2 \in C_1$ sharing a neighbor in $\mu(A_2)$, then either H is of the form tree^5 or there is a redundant set U containing at most four vertices such that $H - U$ is of the form tree^2 or tree^5.
First, since $A_2 \neq \emptyset$, $|C_0| \leq 1$ by (9). Without loss of generality, assume that z_1, z_2 share a neighbor $b_k \in \mu(A_2)$.
If z_2 has another neighbor, say $b_l \in \mu(A)$, then by (2), there exists a pair of vertices b_i, b_j not adjacent to z_1, z_2. Hence, $\{b_i, w_i, b, w_j, b_j, w_l, b_l, z_2, b_k, z_1\}$ induces an $S_{2,2,5}$, a contradiction. Thus, b_k is the only one neighbor in $\mu(A)$ for any vertex $z \in C_1$ adjacent to b_k.
Note that, for any such z, $\mu(z) \nsim w_k$ by Claim 3.12.3. Moreover, $\mu(z) \nsim w_j \in A$ for $w_j \neq w_k$, otherwise $\{b_i, w_i, b, b_l, w_l, w_j, \mu(z), z, b_k, z'\}$ induces an $S_{2,2,5}$ for z' be another neighbor of b_k in C_1 different from z; by Claim 3.12.4 and (2), b_i, b_l not adjacent to z, z'; and w_i, w_l not adjacent to $\mu(z)$, a contradiction.
Now, y is adjacent to at least one vertex among $\mu(z_1), \mu(z_2)$, otherwise by (6), $\{\mu(z_1), z_1, b_k, z_2, \mu(z_2), w_k, b, w_1, b_1, y\}$ induces an $S_{2,2,5}$, a contradiction. Without loss of generality, assume that $y \sim \mu(z_1)$. Then $y \sim \mu(z_2)$, otherwise by (6), $\{w_1, b_1, y, b_2,$

$w_2, \mu(z_1), z_1, b_k, z_2, \mu(z_2)\}$ induces an $S_{2,2,5}$, a contradiction. Hence, y is adjacent to every vertex $z \in C_1$ adjacent to b_k.
It also implies that y has no other non-neighbor than b_k in $\mu(A)$. Indeed, without loss of generality, suppose that $y \nsim b_{k-1}$. Then $\{z_1, \mu(z_1), y, \mu(z_2), z_2, b_1, w_1, b, w_{k-1}, b_{k-1}\}$ induces an $S_{2,2,5}$, a contradiction.
Moreover, $\mu(y) \nsim z$ for every vertex $z \in C_1$ adjacent to b_k, otherwise $\{\mu(y), z, \mu(z), y, b_1, w_1\}$ induces a banner$_2$, a contradiction.
Besides, $D_1 = \emptyset$. Indeed, without loss of generality, suppose that there exists some vertex $t \in D_1$ such that $t \sim b_1$. Then $t \nsim b_k$. Moreover, $t \nsim \mu(z)$ for any $z \in C_1$ adjacent to b_k, otherwise $\{t, \mu(z), y, b_1, w_1, b\}$ induces a banner$_2$, a contradiction. Now, by (6), $\{\mu(z_1), z_1, b_k, z_2, \mu(z_2), w_k, b, w_1, b_1, t\}$ induces an $S_{2,2,5}$, a contradiction.
We consider the two following cases.
Case 1. $C_0 = \emptyset$. Then

$$U := \{y, \mu(y)\}$$

is a redundant set of size two such that $H - U$ is of the form tree^2 in the case that $\mu(y) \nsim w_k$, or H is of the form tree^5 in the case that $\mu(y) \sim w_k$.
Case 2. $C_0 = \{x\}$ and $x \sim \mu(y)$ by (9). Then $\mu(x) \nsim w_k$, otherwise $\{x, \mu(x), w_k, \mu(y), y, b_1\}$ induces a banner$_2$ or $\{w_1, b_1, y, b_2, w_2, \mu(y), x, \mu(x), w_k, b_k\}$ induces an $S_{2,2,5}$ depending on $\mu(y) \sim w_k$ or not, a contradiction. Thus, $\mu(x) \nsim z$ for any $z \in C_1$ adjacent to b_k, otherwise, by Claim 3.12.4, there exists a pair of vertices $w_i, w_j \neq w_k$ not adjacent to $\mu(x)$ and hence, $\{b_i, w_i, b, w_j, b_j, w_k, b_k, z, \mu(x), x\}$ induces an $S_{2,2,5}$, a contradiction. Moreover, $\mu(x) \nsim w_i$ for any $w_i \in A_1$, otherwise $\{z_1, \mu(z_1), y, \mu(z_2), z_2, \mu(y), x, \mu(x), w_i, b\}$ induces an $S_{2,2,5}$, a contradiction. Now,

$$U := \{y, \mu(y), x, \mu(x)\}$$

is a redundant set of size at most four such that $H - U$ is of the form tree^2, in the case that $\mu(y) \nsim w_k$, or

$$U := \{x, \mu(x)\}$$

is a redundant set of size at most two such that $H - U$ is of the form tree^5, in the case that $\mu(y) \sim w_k$.
From now on, we assume the following statement.
(11') Two diffirent vertices in $C_1 \backslash \{y\}$ share no common neighbor in $\mu(A)$. This also implies that $|E_1| \leq 3$.
(12) If $D_1 = \emptyset$, then there exists a redundant set U of size at most 24 such that $H - U$ is of the form tree^1. Indeed, if in addition, $C_0 = \emptyset$, then by Claim 3.12.4,

$$U := \{y, \mu(y)\} \cup A_2 \cup \mu(A_2) \cup E_1 \cup \mu(E_1) \cup N_A(\mu(E_1)) \cup \mu(N_A(\mu(E_1)))$$

is a redundant set of size at most 20 such that $H - U$ is of the form tree^1. Now, we consider the two following cases.
Case 1. $C_0 = \{z\}$. Then by (9) and Claim 3.12.4,

$$\begin{aligned} U \ := \ & \{y, \mu(y), z, \mu(z)\} \cup A_2 \cup \mu(A_2) \cup E_1 \cup \mu(E_1) \cup \\ & \cup N_A(\mu(E_1) \cup \{\mu(z)\}) \cup \mu(N_A(\mu(E_1) \cup \{\mu(z)\})) \end{aligned}$$

is a redundant set of size at most 24 such that $H - U$ is of the form tree^1.
Case 2. $|C_0| \geq 2$. Then y is adjacent to every vertex of $\mu(A)$ by (8). Let z be the (only)

vertex of C_0 adjacent to $\mu(y)$. Denote by C_0' the set of vertices of $C_0\backslash\{z\}$ adjacent to $\mu(z)$ and let $C_0'' := C_0\backslash(C_0' \cup \{z\})$. Then $C_0' \neq \emptyset$, otherwise $|C_0\backslash\{z\}| = |N_H(C_0\backslash\{z\})|$, a contradiction with the minimality of H. Moreover, for every $x \in C_0'$, $\mu(x) \sim y$, $\mu(x)$ is not adjacent to any vertex of A_1, and $x \nsim \mu(y)$ by (9).
2.1. $C_0'' = \emptyset$. Then H is of the form tree^5 or tree^6 depending on $\mu(z)$ has a neighbor in A or not.
2.2. $C_0'' \neq \emptyset$. Then it must contains a vertex $t \sim \mu(x)$ for some $x \in C_0'$, otherwise $|N(C_0'')| = |C_0''|$, a contradiction with the minimality of H. Now, $\mu(t) \nsim x$, otherwise $\{z, \mu(z), x, \mu(x), t, \mu(t)\}$ induces a domino or a banner$_2$ depending on $\mu(t) \sim z$ or not, a contradiction. Thus, $\mu(t) \nsim y$, otherwise $\{y, \mu(t), t, \mu(x), x, \mu(z)\}$ induces a banner$_2$, a contradiction. Now, by Claim 3.12.4, there exists a pair of vertices w_i, w_j is not adjacent to $\mu(x), \mu(t), \mu(z)$ and hence, $\{\mu(t), t, \mu(x), x, \mu(z), y, b_i, w_i, b, w_j\}$ induces an $S_{2,2,5}$, a contradiction.
From now on, we assume the following statement.
(12') $D_1 \neq \emptyset$.
(13) If $|C_0| \geq 2$, then H contains a redundant set U of size two such that $H - U$ is of the form tree^5.
By (9), y is adjacent to every vertex of $\mu(A)$. Let z be the (only) vertex of C_0 adjacent to $\mu(y)$ and $x \in C_0$ be adjacent to $\mu(z)$. Also by (9), for every such vertex x, $\mu(x) \sim y$, $\mu(x) \nsim z$. Moreover, by Claim 3.12.3, z has no neighbor in $\mu(A)$.
Since $D_1 \neq \emptyset$, without loss of generality, assume that there exists a vertex $z_1 \in D_1$ adjacent to b_1. Now, $\mu(z) \sim w_1$, otherwise $\{\mu(z_1), z_1, b_1, w_1, b, y, \mu(y), z, \mu(z), x\}$ induces an $S_{2,2,5}$, a contradiction. Moreover, by (3) and Claim 3.12.4, $D_1 = \{z_1\}$. We consider the two following cases.
Case 1. z has a neighbor $\mu(t) \in \mu(C_0)$ for some $t \in C_0$ different from z. Then by (7), (8), and Claim 3.12.3, $\mu(t) \sim w_1$, otherwise $\{\mu(z_1), z_1, b_1, w_1, b, y, \mu(y), z, \mu(t), t\}$ induces an $S_{2,2,5}$, a contradiction. But now, $\{\mu(z), w_1, \mu(t), z, \mu(y), y\}$ induces a banner$_2$, a contradiction.
Case 2. z has no neighbor in $\mu(C_0)$ other than $\mu(z)$. Let x be a vertex in C_0 adjacent to $\mu(z)$ and C_0' be the set of vertices of C_0 different from z and not adjacent to $\mu(z)$. If $C_0' \neq \emptyset$, then by (7) and (8), there exists a vertex $t \in C_0'$ adjacent to $\mu(x)$, otherwise $|C_0'| = |N_H(C_0')|$, a contradiction with the minimality of H. Now, $t \nsim \mu(z)$, otherwise $\{\mu(y), z, \mu(z), x, \mu(x), t\}$ induces a domino or a banner$_2$ depending on $t \sim \mu(y)$ or not, a contradiction. Now, by Claim 3.12.4, there exists a pair of vertices w_i, w_j different from w_1 not adjacent to $\mu(x)$ and hence, $\{b_i, w_i, b, w_j, b_j, w_1, \mu(z), x, \mu(x), t\}$ induces an $S_{2,2,5}$, a contradiction.
From above considerations, every vertex $x \in C_0$ different from z is adjacent to $\mu(z)$ and $\mu(x)$ is adjacent to y. Now,

$$U := \{z_1, \mu(z_1)\}$$

is a redundant set of size two, such that $H - U$ is of the form tree^5.
From now on, we assume the following statements.
(13') $|C_0| \leq 1$.
(14) If $|D_1| \geq 2$, then by (10) and (13'),

$$\begin{aligned} U \; := \; & \{y, \mu(y)\} \cup C_0 \cup \mu(C_0) \cup E_1 \cup \mu(E_1) \cup \\ & \cup N_A(\mu(E_1) \cup \mu(C_0)) \cup \mu(N_A(\mu(E_1) \cup \mu(C_0))) \cup \end{aligned}$$

$$\cup N_{D_1}(\mu(N_A(\mu(E_1) \cup \mu(C_0)))) \cup$$
$$\cup \mu(N_{D_1}(\mu(N_A(\mu(E_1) \cup \mu(C_0)))))$$

is a redundant set of size at most 26 such that $H - U$ is of the form tree^4.
(15) If $|D_1| = 1$, then

$$\begin{aligned} U \ := \ & \{y, \mu(y)\} \cup C_0 \cup \mu(C_0) \cup D_1 \cup \mu(D_1) \cup E_1 \cup \mu(E_1) \cup \\ & \cup N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0)) \cup \mu(N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0))) \cup \\ & \cup N_{D_1}(\mu(N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0)))) \\ & \cup \mu(N_{D_1}(\mu(N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0))))) \end{aligned}$$

is a redundant set of size at most 32 such that $H - U$ is of the form tree^1.
All above observations ((1) - (15)) finish the proof of the claim. □

From now on, asume that every vertex of C_1 has at least four non-neighbors in $\mu(A)$.

Claim 3.12.8. *$C_0 = \emptyset$, i.e. $C = C_1$.*

Proof. Suppose that $C_0 \neq \emptyset$. Then there exists some vertex $z \in C_1$, without loss of generality, say $z \sim b_1$, and $y \in C_0$ such that $y \sim \mu(z)$, otherwise $|C_0| = |N_H(C_0)|$, a contradiction with the minimality of H. Thus, $\{b_i, w_i, b, w_j, b_j, w_1, b_1, z, \mu(z), y\}$ induces an $S_{2,2,5}$, for b_i, b_j not adjacent to z and w_i, w_j not adjacent to $\mu(z)$, a contradiction. □

Claim 3.12.9. *If $|C| \leq 4$, then H contains a redundant set U of size at most 16 such that $H - U$ is of the form tree^1.*

Proof. Assume that $|C| \leq 4$, i.e. $|\mu(C)| \leq 4$. Note that every (black) vertex of $\mu(C)$ has at most one neigbor in A by Claim 3.12.4, i.e. $|N_A(\mu(C))| \leq 4$. Then

$$U := C \cup \mu(C) \cup N_A(\mu(C)) \cup \mu(N_A(\mu(C)))$$

is a redundant set of size at most 16 such that $H - U$ is of the form tree^1. □

Claim 3.12.10. *Assume that $|C| \geq 5$. Then the following statements are true.*
Case 1. *If there exist vertices $z_1, z_2 \in C$ sharing some neighbor in $\mu(A)$, then H is of the form tree^2.*
Case 2. *If for any two vertices $y, z \in C$, y, z sharing no neighbor in $\mu(A)$, then H is of the form tree^3 or tree^7 or H contains a redundant set U of size at most six such that $H - U$ is of the form tree^3.*

Proof. We consider the two above cases.
Case 1. Without loss of generality, assume that $z_1, z_2 \in C$ sharing a neighbor $b_1 \in \mu(A)$.
1.1. z_2 has another neighbor, say $b_2 \in \mu(A)$. Assume that there exist two vertices, without loss of generality, say b_3, b_4, not adjacent to z_1, z_2. Then $\{b_3, w_3, b, b_4, w_4, w_2, b_2, z_2, b_1, z_1\}$ induces an $S_{2,2,5}$, a contradiction. Hence, $|N_{\mu(A)}(\{z_1, z_2\})| \geq k - 1$. Since both z_1 and z_2 have at most $k - 4$ neighbors in $\mu(A)$, each of them has at least four neighbors in $\mu(A)$.
Let $z_3 \in C$ adjacent to some vertex $b_i \in N_{\mu(A)}(\{z_1, z_2\})$. Then z_3 has at least four neighbors in $\mu(A)$. Hence, z_3 sharing two neighbors in $\mu(A)$ with z_1 or z_2, a contradiction with Claim 3.12.2. So, there exists no other vertex in C (than z_1, z_2) having a neighbor

in $N_{\mu(A)}(\{z_1, z_2\})$. Together with $|C| \geq 5$, it implies that $|N_{\mu(A)}(\{z_1, z_2\})| \leq k-1$, i.e. $|N_{\mu(A)}(\{z_1, z_2\})| = k-1$.
Without loss of generality, assume that z_1, z_2 are not adjacent to b_k. Since $|C| \geq 5$, there exist $z_3, z_4 \in C$ such that z_3, z_4 are adjacent to b_k. Moreover, z_3, z_4 have no other neighbor in $\mu(A)$. By Claim 3.12.4, there exists a vertex b_i such that $b_i \sim z_1$ and w_i is not adjacent to $\mu(z_3), \mu(z_4)$. Hence, by Claim 3.12.6, $\{\mu(z_3), z_3, b_k, z_4, \mu(z_4), w_k, b, b_i, w_i, z_1\}$ induces an $S_{2,2,5}$, a contradiction.
1.2. Every vertex of C is adjacent to b_1 has only one neighbor, say b_1 in $\mu(A)$. Note that, for every such vertex z, $\mu(z) \nsim w_1$ by Claim 3.12.3. Moreover, $\mu(z) \nsim w_i \in \mu(A)$ for $w_i \neq w_1$, otherwise by Claim 3.12.4, there exists a pair of vertices $w_j, w_l \neq w_1$ and non-adjacent to $\mu(z)$ and hence, $\{b_j, w_j, b, w_l, b_l, w_i, \mu(z), z, b_1, z'\}$ induces an $S_{2,2,5}$ for z' be another neighbor of b_1 in C different from z, a contradiction.
Now, let C_{11} be the set of vertices of C_1 adjacent to b_1 and $C_{12} := C_1 \backslash C_{11}$. If $C_{12} = \emptyset$, then H is of the form tree^2. Let $y \in C_{12}$ and, without loss of generality, assume that $y \sim b_2 \in \mu(A)$. If y is not adjacent to two vertices, say $\mu(z_1), \mu(z_2) \in \mu(C_{11})$, then $\{\mu(z_1), z_1, b_1, z_2, \mu(z_2), w_1, b, w_2, b_2, y\}$ induces an $S_{2,2,5}$, a contradiction. If y is adjacent to two vertices $\mu(z_1), \mu(z_2) \in \mu(C_{11})$, then y is adjacent to every vertex $b_i \in \mu(A)$ different from b_1, otherwise $\{z_1, \mu(z_1), y, \mu(z_2), z_2, b_2, w_2, b, w_i, b_i\}$ induces an $S_{2,2,5}$, a contradiction. Now, y has at least $k-1$ neighbors in $\mu(A)$, a contradiction. Hence, $C_{11} = \{z_1, z_2\}$ and every vertex $y \in C_{12}$ is adjacent to exactly one vertex of $\mu(C_{11})$.
If $\mu(z_1)$ is adjacent to two vertices $y_1, y_2 \in C_{12}$, then $\{y_1, \mu(z_1), y_2, b_i, w_i, b\}$ induces a banner$_2$ in the case that y_1, y_2 sharing the same neighbor $b_i \in \mu(A)$ by Claim 3.12.3 or $\{b_{i_1}, y_1, \mu(z_1), y_2, b_{i_2}, z_1, b_1, w_1, b, w_i\}$ induces an $S_{2,2,5}$ for b_{i_1}, b_{i_2} be (different) neighbors of y_1, y_2 in $\mu(A)$, respectively, and $w_i \in A$ different from w_1, w_{i_1}, w_{i_2}, a contradiction. Hence, each $\mu(z_1), \mu(z_2)$ has at most one neighbor in C_{12}. It implies that $|C_{12}| \leq 2$ and thus, $|C| \leq 4$, a contradiction.
Case 2. If for every vertex $\mu(z) \in \mu(C_1)$, z is the only neighbor of $\mu(z)$, then H is of the form tree^3.
We show that for every pair $z_1, z_2 \in C$, $\mu(z_1) \nsim z_2$. Indeed, for contradiction, suppose that $\mu(z_1) \sim z_2$. Without loss of generality, assume that z_1, z_2 are adjacent to b_1, b_2, respectively. Then $\mu(z_2) \nsim z_1$, otherwise by Claim 3.12.3, $\{\mu(z_2), z_1, \mu(z_1), z_2, b_2, w_2\}$ induces a banner$_2$, a contradiction. Moreover, $N_{\mu(A)}(\{z_1, z_2\}) \geq k-2$, otherwise by Claim 3.12.4, there exists a pair of vertices w_i, w_j not adjacent to $\mu(z)$ such that b_i, b_j not adjacent to z_1, z_2, and hence, $\{b_i, w_i, b, w_j, b_j, w_2, b_2, z_2, \mu(z_1), z_1\}$ induces an $S_{2,2,5}$, a contradiction. Hence, the non-neighbors of z_1, z_2 in $\mu(A)$ have at most two neighbors in C, i.e. $|C| \leq 4$, a contradiction.
Now, consider the case that there exists some vertex $z \in C$, such that $\mu(z)$ is adjacent to some vertex of A. Without loss of generality, assume that $z \sim b_1$ and $\mu(z) \sim w_2$. Then $b_2 \nsim z$, by Claim 3.12.3. We consider the two following subcases.
2.1. $b_2 \sim y$ for some $y \in C$. Then for every $x \in C \backslash \{y, z\}$, $\mu(x) \sim w_2$, otherwise $\{z, \mu(z), w_2, b_2, y, b, w_i, b_i, x, \mu(x)\}$ induces an $S_{2,2,5}$ for $b_i \sim x$, a contradiction. By Claim 3.12.4, it also implies that $\mu(y)$ is not adjacent to any vertex $w_i \in A$ such that $b_i \sim x$ for some $x \in C_1$ different from y, otherwise $|C| = 2$, a contradiction. Now,

$$U := \{w_2, b_2, y, \mu(y)\} \cup N_A(\mu(y)) \cup \mu(N_A(\mu(y)))$$

is a redundant set of size at most six such that $H - U$ is of the form tree^3.
2.2. $N_C(b_2) = \emptyset$. Assume that there exists some vertex $y \in C$, without loss of

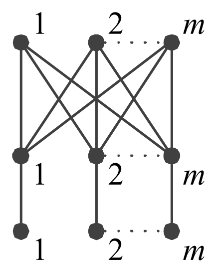

Fig. 3.3: M_m

generality, assume that $y \sim b_3$ and $\mu(y) \sim w_2$. Then for every $x \in C$ different from y, z, $\mu(x) \sim w_2$, otherwise $\{z, \mu(z), w_2, \mu(y), y, b, w_i, b_i, x, \mu(x)\}$ induces an $S_{2,2,5}$ for $b_i \sim x$, a contradiction. Now,

$$U := \{w_2, b_2\}$$

is a redundant set of size two such that $H-$ is of the form tree^3.
Now, if there exists no vertex pair $y, z \in C$, such that $\mu(y), \mu(z)$ share the same neighbor in A, then H is of the form tree^7. □

All above claims finish the proof. □

3.2.4 Augmenting Bipartite Chain

Given a graph G and an independent set S, in this subsection, we consider Case 4 of Lemma 3.12, say augmenting graphs $H = (B, W, E)$ such that there exists a vertex $b \in B$ and $N_S(b) = W$. We show that under some restrictions, these augmenting graphs have structural properties similar to P_5-free augmenting graphs, say, being a bipartite-chain. More precisely, we have the following result.

Lemma 3.13. *Given a graph G, an integer $k \geq 3$, and a (banner$_2$,domino,M_k)-free (see Fig. 3.3) minimal augmenting graphs $H = (B, W, E)$ for an independent set S such that there exists some black vertex $b \in B$ adjacent to every white vertex of W, and $|W| \geq 2k + 1$, at least one of the following statements is true.*

1. *H is of the form tree^1 or there exists a reduction set U of size at most $2k - 2$ associated with a key set of size one such that $H - U$ is of the form tree^1.*

2. *H is a bipartite-chain, or there exists a redundant set U of size at most $2k - 2$ such that $H - U$ is a bipartite-chain.*

Proof. We refer to Lemma 3.18 for the procedure finding tree^1 and note that such procedure start by finding a candidate containing b, i.e. b is adjacent to every white vertex in the augmenting tree^1 and we have the key set $B^* := \{b\}$.
Let $B = \{b, b_1, \ldots, b_q\}$, b be the vertex b in Corollary 3.2, p be the integer p in Lemma 3.5 such that $N_W(b_i) \supset N_W(b_j)$ for every $1 \leq i \leq p$, $i < j \leq q$ and $|N_W(b_i)| = 1$ for every $i \geq p + 1$.
If $p \leq k - 1$, then $U = \{b_1, \ldots, b_p, \mu(b_1), \ldots, \mu(b_p)\}$ is a reduction set of size at most $2k - 2$ associated with B^* such that $H - U$ is of the form tree^1.
If $p \geq q - k + 1$, then $U = \{b_{p+1}, \ldots, b_q, \mu(b_{p+1}), \ldots, \mu(b_q)\}$ is a redundant set of size

at most $2k-2$ such that $H-U$ is a bipartite-chain.
If $k \leq p \leq q-k$ then $\{b, b_1, \ldots, b_{k-1}, b_{q-k+1}, \ldots, b_q, \mu(b_{q-k+1}), \ldots, \mu(b_q)\}$ induces an M_k, a contradiction. □

Note that if $|W| \leq 2k$, then H contains at most $4k+1$ vertices. The following observation is a generalization of Lemma 10 in [22] and Theorem 1 in [75] about augmenting graphs in $(P_5, K_{2,m}-e)$-free graphs and $(P_5, K_{3,3,}-e)$-free graphs, respectively.

Lemma 3.14. *Given a graph G, an independent set S of G, an integer m, and a minimal augmenting $(K_{m,m}-e)$-free bipartite-chain $H=(B,W,E)$, at least one of the following statements is true.*

1. *H has at most $2m-2$ white vertices;*
2. *H is of the form $K_{l,l+1}$ or there is a redundant set of size at most $2m-4$ such that $H-U$ is of the form $K_{l,l+1}$, for some l.*

Note that if an augmenting graphs contains at most $2m-2$ white vertices, it contains at most $4m-3$ vertices.

Proof. Assume that $|W| = p \geq 2m-1$. Let $W = \{w_1, w_2, \ldots, w_p\}$ and $B = \{b_1, b_2, \ldots, b_p, b_{p+1}\}$. Assume that $N_W(b_i) \subset N_W(b_j)$ for $i<j$. Moreover, by Corollary 3.2, there exists a perfect matching between $B\backslash\{b_{p+1}\}$ and W. Without loss of generality, assume that $b_i \sim w_i$ for $1 \leq i \leq p$. Then we have $|N_W(b_i)| \geq i$ for $i = 1, 2, \ldots$.
Now, $b_i \sim w_j$ for every $b_i \in B$ and $w_j \in W$ such that $p-m+4 \geq i \geq m-1$ and $p-m+3 \geq j \geq i+1$, otherwise $\{b, b_p, \ldots, b_{p-m+3}, b_i, w_j, w_{m-1}, \ldots, w_1\}$ induces a $K_{m,m}-e$, a contradiction.
Hence, $\{b, b_p, \ldots, b_{m-1}, w_{p-m+1}, \ldots, w_1\}$ induces a $K_{p-m+3,p-m+2}$ and $U := \{b_{m-2}, \ldots, b_1, w_p, \ldots, w_{p-m+2}\}$ is a redundant of size $2m-4$ such that $H-U$ is a $K_{p-m+3,p-m+2}$. □

3.3 Finding Augmenting Graphs

In this section, we describe methods finding augmenting graphs characterized in the above section. Remind that we can enumerate all augmenting graphs of bounded size in polynomial time. Moreover, Hertz and Lozin [98] described method of finding augmenting graphs of the form $K_{m,m+1}$ in banner$_2$-free graphs. Moroever, augmenting apples can be reduced to augmenting chains by a redundant set of size two.

3.3.1 Augmenting Extended-chain

The method for finding augmenting chains in ($S_{1,2,j}$,banner)-free graphs has been described by Hertz, Lozin, and Schindl [99]. Now, we extend this method for finding augmenting (l,m)-extended-chains in ($S_{2,l,l}$,banner$_l$,R_l^1,R_l^2,R_l^3,R_l^4,R_l^5)-free graphs (see Fig. 3.4). Note that R_l^1, R_l^3, R_l^4, and R_l^5 induce $S_{1,l,l}$ and R_l^2 induces $S_{2,2,l}$. Hence, the following result is a generalization of Theorem 2 in [99].

Lemma 3.15. *Given integers l and m, where l is even, an ($S_{2,l,l}$,banner$_l$,R_l^1,R_l^2, R_l^3, R_l^4, R_l^5)-free graph G, and an independent set S in G, one can determine whether S admits an augmenting (l,m)-extended-chain in polynomial time.*

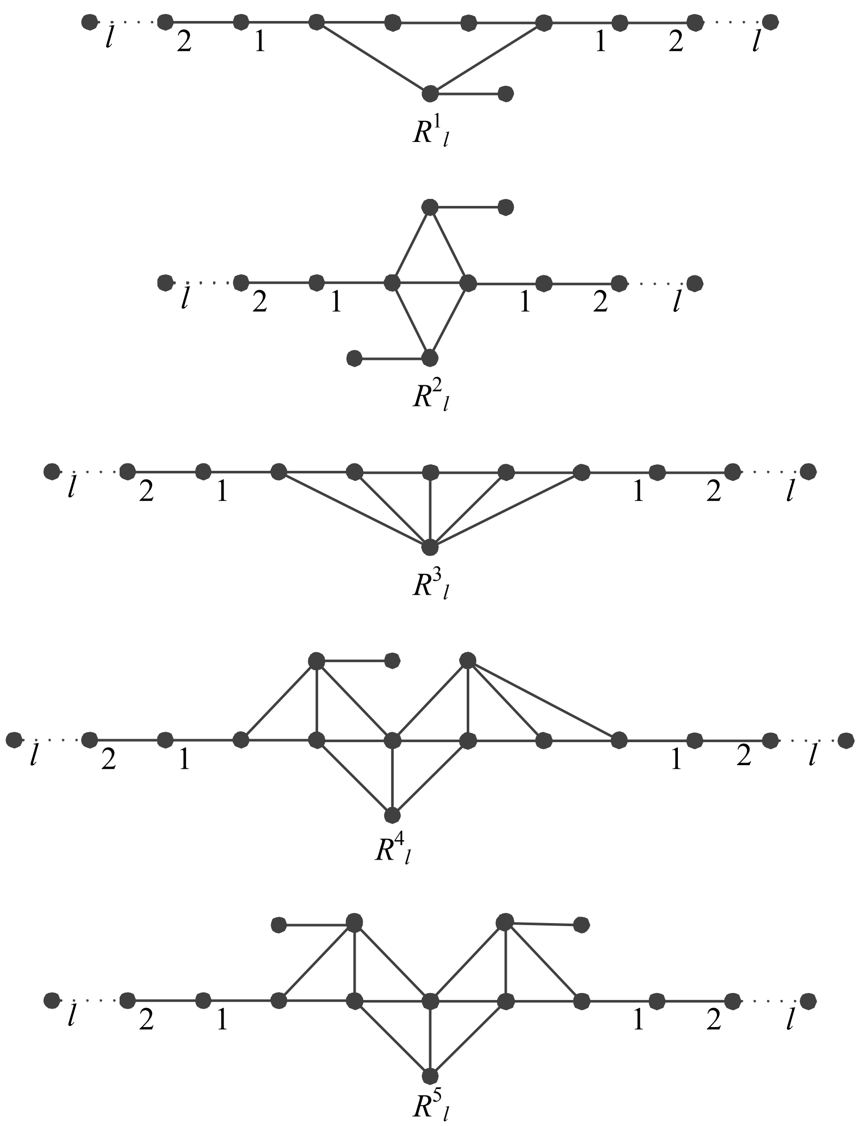

Fig. 3.4: R_l^1,R_l^2, R_l^3, R_l^4, and R_l^5

Proof. To simplify the proof, we start with a pre-processing consisting in detecting augmenting (l, m)-extended-chains whose the path part is of length at most $2l$ since such an augmenting (l, m)-extended-chain contains at most $\frac{1-(m-1)^l}{2-m} + 2l + 1$ vertices and can be enumerated in polynomial time.
In order to determine whether S admits an augmenting (l, m)-extended-chain whose the path-part is of length at least $2l + 2$, we first find a candidate, i.e. a pair (L, R), where L and R are disjoint trees consisting induced paths $x_0, x_1, \ldots, x_l$ and $x_{2p-l}, x_{2p-l+1}, \ldots, x_{2p}$, respectively $(p \geq l+1)$ and every vertex outside that path of L (R, respectively) is of distance at most $l - 1$ from x_0 (x_{2p}, respectively) and not adjacent to any vertices among $\{x_1, x_2, \ldots, x_l, x_{2p-l}, x_{2p-l+1}, \ldots, x_{2p}\}$. If such a candidate does not exist, then there is no augmenting (l, m)-extended-chain whose the path part is of length at least $2l + 2$ for S. Moreover, since such candidates contain only finite vertices, we can enumerating them in polynomial time.
Our purpose is to find an alternating chain connecting x_l and x_{2p-l}. Evidently, if there are no such chains, then there is no augmenting (l, m)-extended-chain whose the path part is of length at least $2l + 2$ for S containing L and R.
Having found a candidate (L, R), we have the following observations about vertices of G in the sense that the vertices not satisfying these assumptions can be simply removed from the graph, since they cannot occur in any valid alternating chain connecting x_l and x_{2p-l}. Let $P := (x_0, x_1, \ldots, x_{2p})$ be the path part of a desired (l, m)-extended-chain.

Claim 3.15.1. *1. Each white vertex has at least two black neighbors.*

2. Each black vertex lying outside L and R has exactly two white neighbors.

3. No black vertex outside L and R has a neighbor in L or R.

4. No white vertex outside L and R has a neighbor in L or R, except such a neighbor is x_l or x_{2p-l}.
Moreover, no white vertex outside P has a neighbor in P.

Proof. 1. and 2. are obvious since a vertex not satisfying these conditions cannot occur in any augmenting extended-chain containing L and R as sub-extended-chains.
Note that x_l and x_{2p-l} are black vertices. Hence, if a black vertex outside L and R has a neighbor in L or R, then clearly such a vertex cannot belong to the desired augmenting chain, similar for a white vertex outside L and R.
If a white vertex outside P has a neighbor in P, then clearly such a neighbor is black and hence it has at least three white neighbors, a contradiction. □

From the conditions of the above claim, we have the following observation.

Claim 3.15.2. *If S admits an augmenting (l, m)-extended-chain containing L and R, then no vertex of $P \backslash (L \cup R)$ is the center of an induced claw.*

Proof. By contradiction, suppose that G contains a claw $G[C]$, where $C = \{a, b, c, d\}$, whose center a (i.e. the vertex of degree three) is a vertex x_j on P. Without loss of generality, we choose a claw such that $|\{b, c, d\} \backslash P|$ is minimal and, among such claws, choose a claw such that j is minimum. Note that, since there exists at least one vertex of $\{b, c, d\}$ lying outside P, together with 3. of Claim 3.15.1, $l + 1 \leq j \leq 2p - l - 1$. Moreover, since every black vertex of P has all its white neighbors lying in P, every vertex of $C \backslash P$ is black.

We shall use the following convention: for a black vertex v outside P, if only one of the two white neighbors of v is defined explicitly, then the other is denoted as $\bar{v}$. Also, for a vertex v of C not belonging to P such that $N(v) \cap P \neq \emptyset$, we denote by $r(v)$ the largest index in $\{j, j+1, \dots, 2p-l-1\}$ and by $s(v)$ the smallest index in $\{l+1, l+2, \dots, j\}$ such that v is adjacent to $x_{r(v)}, x_{s(v)}$.
We now analyze three cases: exactly one (C1), two (C2), or three (C3) vertex/vertices of $\{b, c, d\}$ do(es)n't belong to P.
Case (C1). Without loss of generality, assume that $b = x_{j-1}$ and $c = x_{j+1}$. Then we have the following observations.
(1) d is not adjacent to x_{j-2}, x_{j+2}. Indeed, if $d \sim x_{j-2}$ (similar for the case $d \sim x_{j+2}$), then $\{x_{j-2}, x_{j-1}, x_j, d, x_{r(d)}, x_{r(d)+1}, \dots, x_{r(d)+l-1}\}$ induces a banner$_l$ in the case $r(d) \geq j+2$ or $\{d, x_{j-2}, x_{j-1}, x_j, x_{j+1}, \dots, x_{j+l}\}$ induces a banner$_l$ in the case $r(d) = j$, a contradiction.
(2) $r(d) = j$ or $s(d) = j$. Indeed, by (1), suppose that $r(d) \geq j+3$ and $s(d) \leq j-3$. Then $\{x_{j-1}, x_j, d, x_{s(d)}, x_{s(d)-1}, \dots, x_{s(d)-l+1}, x_{r(d)}, x_{r(d)+1}, \dots, x_{r(d)+l-1}\}$ induces an $S_{2,l,l}$, a contradiction.
(3) $s(d) \geq j-3$ and $r(d) \leq j+3$. Indeed, suppose that $s(d) \leq j-4$ (similar for the case $r(d) \geq j+4$). Then by (2), $\{x_{j-2}, x_{j-1}, x_j, x_{s(d)}, x_{s(d)-1}, \dots, x_{s(d)-l+1}, x_{j+1}, x_{j+2}, \dots, x_{j+l-1}\}$ induces an $S_{2,l,l}$, a contradiction.
(4) $r(d) = s(d) = j$. Indeed, by (2) and (3), suppose that $r(d) = j+3$ and $s(d) = j$ (similar for the case $s(d) = j-3$ and $r(d) = j$). Among $\{x_j, x_{j+3}\}$, there exists at most one white vertex. Hence, $\{x_{j+2}, x_{j+1}, \bar{d}, d, x_{j+3}, x_{j+4}, x_{j+5}, \dots, x_{j+l+3}, x_j, x_{j-1}, \dots, x_{j-l}\}$ induces an R_l^1, a contradiction.
Now, since $r(d) = s(d) = j$, $\{\bar{d}, d, x_j, x_{j-1}, x_{j-2}, \dots, x_{j-l}, x_{j+1}, x_{j+2}, \dots, x_{j+l}\}$ induces an $S_{2,l,l}$, a contradiction.
Case (C2). Without loss of generality, assume that $b = x_{j-1}$ and c and d are outside P. Then we have the following observations.
(1) x_{j+1} is adjacent both to c and d to avoid (C1).
(2) Also to avoid (C1), c is adjacent to $x_{s(c)+1}, x_{r(c)-1}$, similarly for d.
(3) It cannot happen that $s(c) = s(d) \leq j-2$ or $r(c) = r(d) \geq j+2$. Indeed, say if $s(c) = s(d) \leq j-2$, then $\{c, x_{j+1}, d, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l}\}$ induces a banner$_l$, a contradiction.
(4) Similarly, if $s(c) = s(d) = j$, then there exists no common neighbor x_i of c and d for $i \geq j+2$ and if $r(c) = r(d) = j+1$, then there exists no common neighbor x_i of c and d for $i \leq j-2$. And in both cases, c and d have no common neighbor outside P.
(5) c and d are not adjacent to x_{j-2}. Indeed, suppose that $c \sim x_{j-2}$ (similar for the case $d \sim x_{j-2}$). Then $r(c) = j+1$ (similarly, $r(d) = j+1$), otherwise $\{x_j, x_{j-1}, x_{j-2}, c, x_{r(c)}, x_{r(c)+1}, \dots, x_{r(c)+l-1}\}$ induces a banner$_l$, a contradiction, and $s(c) = j-3$, otherwise $\{x_j, x_{j-1}, x_{j-2}, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+1}\}$ induces a banner$_l$, a contradiction. Moreover, d is neither adjacent to x_{j-2} nor x_{j-3} also by (4). Hence, $s(d) = j$, otherwise $\{x_{j-1}, x_{j-2}, c, x_j, d, x_{s(d)}, x_{s(d)-1}, \dots, x_{s(d)-l+1}\}$ induces a banner$_l$, a contradiction. Now, among $\{x_j, x_{j+1}\}$, there exists exactly one white vertex. Moreover, $c \nsim \bar{d}$ by (4). Now, $\{d, \bar{d}, x_{j+1}, c, x_{j-3}, x_{j-4}, \dots, x_{j-l-2}, x_{j+2}, x_{j+3}, \dots, x_{j+l+1}\}$, induces an $S_{2,l,l}$, a contradiction.
(6) By (2) and (5), if $s(c) \leq j-3$, then $s(c) \leq j-4$.
(7) $s(c) = j$ or $r(c) = j+1$. Similarly, $s(d) = j$ or $r(d) = j+1$. Indeed, by (5) and (6), if $s(c) \leq j-4$ and $r(c) \geq j+2$, then $\{x_{j-1}, x_j, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+1}, x_{r(c)}, x_{r(c)+1},$

$\ldots, x_{r(c)+l-1}\}$ induces an $S_{2,l,l}$, a contradiction.
(8) $s(c) = j$ or $r(d) = j+1$ (similarly, $s(d) = j$ or $r(c) = j+1$). Indeed, by (5) and (6), without loss of generality, suppose that $s(c) \leq j-4$ and $r(d) \geq j+2$. Then by (7), $r(c) = j+1$ and $s(d) = j$. Hence, $\{x_{j-2}, x_{j-1}, x_j, c, x_{s(c)}, x_{s(c)-1}, \ldots, x_{s(c)-l+2}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.
(9) $s(c) = j$ or $s(d) = j$. Indeed, by (5) and (6), without loss of generality, suppose that $s(c), s(d) \leq j-4$. Then $r(c) = r(d) = j+1$, by (7). Now, by (3), without loss of generality, assume that $s(c) < s(d)$. Then by (4), $\{x_{s(d)+1}, d, x_{j+1}, c, x_{s(c)}, x_{s(c)-1}, \ldots, x_{s(c)-l+2}, x_{j+2}, x_{j+3}, \ldots, x_{j+l+1}\}$ induces an $S_{2,l,l}$, a contradiction.
(10) $r(c) = j+1$ or $r(d) = j+1$. Indeed, if $r(c), r(d) \geq j+2$, then by (7), $s(c) = s(d) = j$. Without loss of generality, by (2) and (4), assume that $r(c) > r(d)+1$. Then $\{x_{r(d)}, d, x_j, c, x_{r(c)}, x_{r(c)+1}, \ldots, x_{r(c)+l-2}, x_{j-1}, x_{j-2}, \ldots, x_{j-l}\}$ induces an $S_{2,l,l}$, a contradiction.
(11) $s(c) = s(d) = j$. Indeed, by (5) and (6), suppose that $s(c) \leq j-4$ (similar for the case that $s(d) \leq j-4$). Then by (9), (8), and (7), $s(d) = j$, $r(d) = r(c) = j+1$. Note that, among $\{x_j, x_{j+1}, x_{s(c)}, x_{s(c)+1}\}$, neighbors of c, there exist exactly two white vertices and hence, $c \nsim \bar{d}$. Now, $\{\bar{d}, d, x_{j+1}, c, x_{s(c)}, x_{s(c)-1}, \ldots, x_{s(c)-l+2}, x_{j+2}, x_{j+3}, \ldots, x_{j+l+1}\}$ induces an $S_{2,l,l}$, a contradiction.
(12) $r(c) = r(d) = j+1$. Indeed, by (10), suppose that $r(c) = j+1$ and $r(d) \geq j+2$. Among x_j, x_{j+1}, there exists only one white vertex and $d \nsim \bar{c}$ by (4). Then $\{\bar{c}, c, x_j, x_{j-1}, x_{j-2}, \ldots, x_{j-l}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.
Now, $\{\bar{c}, c, x_j, d, \bar{d}, x_{j-1}, x_{j-2}, \ldots, x_{j-l}, x_{j+1}, x_{j+2}, \ldots, x_{j+l+1}\}$ induces an R_l^2, a contradiction.
Case (C3). We have the following observations.
(1) First, note that, $r(b)$, $r(c)$, and $r(d)$ (and similarly, $s(b)$, $s(c)$, and $s(c)$) are three mutually different integers. Otherwise, suppose that $r(b) = r(c)$. Then we have the claw $\{x_{r(c)}, x_{r(c)+1}, b, c\}$ i.e. (C2).
(2) To avoid (C1), if $b \sim x_i$ for some i, then b is adjacent to at least one vertex among x_{i-1}, x_{i+1}. It implies b is adjacent to $x_{s(b)+1}, x_{r(b)-1}$. Similarly for c and d.
(3) Moreover, by the minimality of j and to avoid (C2), we know that x_{j-1} has exactly two neighbors in $\{b, c, d\}$, say b and c. To avoid (C1) and (C2), we conclude that x_{j+1} is adjacent to d and has at least one neighbor in $\{b, c\}$, say c. Moreover, $b \nsim x_{j+1}$. Indeed, if $b \sim x_{j+1}$, then $r(b), r(c), r(d) \leq j+2$, otherwise $\{x_{j-1}, b, x_{j+1}, c, x_{r(c)}, x_{r(c)+1}, \ldots, x_{r(c)+l-1}\}$ or $\{x_{j-1}, c, x_{j+1}, b, x_{r(b)}, x_{r(b)+1}, \ldots, x_{r(b)+l-1}\}$ or $\{b, x_{j-1}, c, x_{j+1}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ induces a banner$_l$ depending on which is the largest index among $r(b)$, $r(c)$, $r(d)$, a contradiction. But now, $j+1 \leq r(c), r(b), r(d) \leq j+2$, a contradition with the mutual difference of $r(b), r(c)$, and $r(d)$.
(4) It also implies that at least one of $s(b), s(c)$ is less than $j-1$ and at least one of $r(d), r(c)$ is greater than $j+1$.
(5) $b \nsim x_{j+1}$, together with $b \sim x_{r(b)-1}$, it implies that if $r(b) \geq j+2$, then $r(b) \geq j+3$. Similarly, if $s(d) \leq j-2$, then $s(d) \leq j-3$.
(6) In a pair of consecutive vertices of P, there is a black vertex and a white vertex. Hence, b, c, d are not adjacent to three pairs of consecutive vertices of P, otherwise we have a black vertex with three white neighbors, a contradiction. Together with c is adjacent to $x_{s(c)+1}$ and $x_{r(c)-1}$, it leads to either $r(c) \leq j+2$ or $s(c) \geq j-2$. Moreover, if c is adjacent to x_{j-2}, x_{j+2}, then $s(c) = j-2$ and $r(c) = j+2$. Similarly, we have the

following observations: $r(b) = j$ or $s(b) \geq j-2$, $s(d) = j$ or $r(d) \leq j+2$.
(7) c and b cannot share a neighbor x_i for some $i \leq j-2$, otherwise $\{x_i, c, x_j, b, x_{r(b)}, \ldots, x_{r(b)+l-1}\}$, $\{b, x_i, c, x_j, d, x_{r(d)}, \ldots, x_{r(d)+l-2}\}$, or $\{x_i, b, x_j, c, x_{r(c)}, \ldots, x_{r(c)+l-1}\}$ induces a banner$_l$ depending on which is the largest index among $r(b)$, $r(c)$, $r(d)$ (note that at least one of these integers is bigger than $j+1$ and they are mutually different by (1)), a contradiction. Moreover, b and c cannot share a neighbor x_i for some $i \geq j+2$, otherwise $\{x_j, c, x_i, b, x_{s(b)}, x_{s(b)-1}, \ldots, x_{s(b)-l+1}\}$ or $\{x_j, b, x_i, c, x_{s(c)}, \ldots, x_{s(c)-l+1}\}$ induces a banner$_l$ depending on which one is larger among $s(b)$ and $s(c)$. Similarly, c and b cannot share a white neighbor outside P. By similar arguments, these properties are also true for the two pairs c, d and b, d.
(8) $s(c) \geq j-2$, similarly, $r(c) \leq j+2$. Moreover, if $s(c) = j-2$, then $r(c) = j+1$. Similarly, if $r(c) = j+2$, then $s(c) = j-1$. Indeed, suppose that $s(c) \leq j-4$. Then $c \sim x_{j-2}$, otherwise $\{x_{j-1}, x_{j-2}, x_{j-3}, c, x_{r(c)}, x_{r(c)+1}, \ldots, x_{r(c)+l-1}\}$ induces a banner$_l$ or $\{x_{j-2}, x_{j-1}, c, x_{s(c)}, x_{s(c)-1}, \ldots, x_{s(c)-l+1}, x_{r(c)}, x_{r(c)+1}, x_{r(c)+l-1}\}$ induces an $S_{2,l,l}$ depending on $c \sim x_{j-3}$ or not. But now, c is adjacent to $\{x_{s(c)}, x_{s(c)+1}, x_{j+1}, x_j, x_{j-1}, x_{j-2}\}$, a contradiction with (6). Now, if $s(c) = j-3$, then $c \sim x_{j-2}$ by (2) and $r(c) = j+1$ by (6). Hence, $\{c, x_{j-l-3}, \ldots, x_{j-4}, x_{j-3}, \ldots, x_{j+1}, x_{j+2}, \ldots, x_{j+l+1}\}$ induces an R_l^3, a contradiction. Moreover, if $s(c) = j-2$ and $r(c) = j+2$, then $\{c, x_{j-l-2}, \ldots, x_{j-3}, x_{j-2}, \ldots, x_{j+1}, x_{j+2}, \ldots, x_{j+l+2}\}$ induces an R_l^3, a contradiction.
(9) $r(b) = j$ or $s(b) = j-1$, similarly, $r(d) = j+1$ or $s(d) = j$. Indeed, if $r(b) \geq j+3$ and $s(b) \leq j-2$, then $\{x_j, x_{j+1}, x_{j+2}, b, x_{s(b)}, x_{s(b)-1}, \ldots, x_{s(b)-l+1}\}$ induces a banner$_l$ or $\{x_{j+1}, x_j, b, x_{s(b)}, x_{s(b)-1}, \ldots, x_{s(b)-l+1}, x_{r(b)}, x_{r(b)+1}, \ldots, x_{r(b)+l-1}\}$ induces an $S_{2,l,l}$ depending on $b \sim x_{j+2}$ or not, a contradiction.
(10) $s(b) \geq j-3$, similarly, $r(d) \geq j+3$. Indeed, suppose that $s(b) \leq j-4$. Then $r(b) = j$, by (9). Now b is not adjacent to x_{j-2} and x_{j-3} at the same time, otherwise either $\{b, x_{j-l-4}, \ldots, x_{j-5}, x_{j-4}, \ldots, x_j, x_{j+1}, \ldots, x_{j+l}\}$ induces an R_l^3 or b is adjacent to three pairs of consecutive vertices of P, a contradiction with (6). Hence, $b \nsim x_{j-2}$, otherwise $\{x_{j-3}, x_{j-2}, b, x_{s(b)}, x_{s(b)-1}, \ldots, x_{s(b)-l+1}, x_j, x_{j+1}, \ldots, x_{j+l-1}\}$ induces an $S_{2,l,l}$, a contradiction. Suppose that $b \sim x_{j-3}$. Then $c \sim x_{j-2}$, otherwise $\{b, x_{j-3}, x_{j-2}, x_{j-1}, c, x_{r(c)}, x_{r(c)+1}, \ldots, x_{r(c)+l-2}\}$ induces a banner$_l$, a contradiction. Now, $r(c) = j+1$ by (8), $r(d) \geq j+2$ by (1), and $s(d) = j$ by (9). Hence, $\{x_{j-2}, c, x_j, b, x_{s(b), x_{s(b)-1}}, \ldots, x_{s(b)-l+2}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction. Thus, $b \nsim x_{j-3}$. Now, $\{x_{j-3}, x_{j-2}, x_{j-1}, b, x_{s(b)}, \ldots, x_{s(b)-l+2}, c, x_{r(c)}, \ldots, x_{r(c)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.
(11) $r(b) = j$, similarly, $s(d) = j$. Indeed, suppose that $r(b) \geq j+3$. Then by (9), $s(b) = j-1$. Moreover, $s(c) = j-2$, $r(c) = j+1$, $r(d) \geq j+2$, and $s(d) = j$ by (1), (8), and (9). Now, $\{x_{r(b)-1}, b, x_j, c, x_{j-2}, x_{j-3}, \ldots, x_{j-l}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ or $\{x_{r(d)}, d, x_j, c, x_{j-2}, x_{j-3}, \ldots, x_{j-l}, b, x_{r(b)}, x_{r(b)+1}, \ldots, x_{r(b)+l-2}\}$ induces an $S_{2,l,l}$ depending on $r(d) > r(b)$ or $r(b) > r(d)$ (note that by (2) and (7), if $r(b) > r(d)$, then $r(b) > r(d) + 1$).
(12) $s(c) = j-1$, similarly, $r(c) = j+1$. Indeed, suppose that $s(c) = j-2$. Then $r(c) = j+1$ by (8), $s(b) = j-1$ by (1), (2), and (7) and $r(d) \geq j+2$ by (1). Among x_j and x_{j-1}, there exists only one white vertex. Consider the other white neighbor of b, say $\bar{b}$. Then $\{\bar{b}, b, x_j, c, x_{j-2}, x_{j-3}, \ldots, x_{j-l}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.
(13) x_j is black, otherwise $\{\bar{c}, c, x_j, b, x_{s(b)}, \ldots, x_{s(b)-l+2}, d, x_{r(d)}, \ldots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction. Now, by the symmetry, we have three remaining cases, which

are considered follows.
Case 3.1. b is adjacent to x_{j-2} and x_{j-3}, d is adjacent to x_{j+2} and x_{j+3}. Then $\{x_j, x_{j-l-2}, \ldots, x_{j-3}, b, x_{j-1}, c, x_{j+1}, d, x_{j+3}, \ldots, x_{j+l+2}\}$ induces an R_l^3, a contradiction.
Case 3.2. $s(b) = j-2$ and $r(d) = j+2$. Then $\{x_j, x_{j-l-1}, \ldots, x_{j-2}, \bar{b}, b, x_{j-1}, c, x_{j+1}, d, \bar{d}, x_{j+2}, \ldots, x_{j+l+1}\}$ induces an R_l^4, a contradiction.
Case 3.3. $s(b) = j-2$ and d is adjacent to x_{j+2} and x_{j+3}. Then $\{x_j, x_{j-l-1}, \ldots, x_{j-2}, \bar{b}, b, x_{j-1}, c, x_{j+1}, d, x_{j+2}, x_{j+3}, \ldots, x_{j+l+1}\}$ induces an R_l^5, a contradiction. □

Our purpose here is to detect an augmenting extended-chain whose the path part is of length at least $2l+2$. We first find candidates (L, R) as described above. Note that such candidates can be enumerated in polynomial time. Then perform Steps (a) through (d) for each such pair:
(a) remove all black vertices that have a neighbor in L or in R,
(b) remove the vertices of L and R except for x_l and x_{2p-l}, and
(c) remove all the vertices that are the center of a claw in the remaining graph,
(d) then in the resulting claw-free graph, determine whether there exists an alternating chain between x_l and x_{2p-l} by the method described in [137, 156].
For each candidate, Steps (a) through (d) can be implemented in time $O(n^4)$. Hence, we have the conclusion of the lemma. □

Recall that augmenting complete graphs can be found in polynomial time in banner-free graphs [7]. The above lemma, together with Lemmas 3.9, 3.10, and Theorem 2.1, lead to the following observation.

Theorem 3.16. *Given three integers k, l, and m such that $4 \leq 2k \leq l$, the fowlowing graph classes are MIS-easy:*

1. *($S_{2,k,l}$,banner$_l$,apple$_6^l$,apple$_8^l$,…,apple$_{2k+2}^l$, $K_{1,m}$, R_l^1, R_l^2, R_l^3, R_l^4, R_l^5)-free graphs and*
2. *($S_{2,k,l}$,banner,apple$_6^l$,apple$_8^l$,…,apple$_{2k+2}^l$, R_l^1, R_l^2, R_l^3, R_l^4, R_l^5,tree$_m$)-free graphs.*

By considering induced subgraphs of $R_l^1, R_l^2, R_l^3, R_l^4, R_l^5$, we have the following consequence.

Corollary 3.17. *Given three integers k, l, m, the following graph classes are MIS-easy:*

1. *($S_{1,k,l}$,banner$_l$,apple$_6^l$,apple$_8^l$,…,apple$_{2k+2}^l$, R_l^2, $K_{1,m}$)-free graphs,*
2. *($S_{1,k,l}$,banner,apple$_6^l$,apple$_8^l$,…,apple$_{2k+2}^l$, R_l^2,tree$_m$)-free graphs,*
3. *($S_{2,2,l}$,banner$_l$,R_l^3, R_l^4, R_l^5, $K_{1,m}$)-free graphs, and*
4. *($S_{2,2,l}$,banner,R_l^3, R_l^4, R_l^5,tree$_m$)-free graphs.*

These results are generalizations of the result of Lozin and Rautenbach for $(P_l, K_{1,m})$-free graphs [131] and the results of Hertz and Lozin for ($S_{1,2,l}$,banner,$K_{1,m}$)-free graphs and ($S_{1,2,3}$,banner$_k$,$K_{1,m}$)-free graphs [98].

3.3.2 Augmenting Trees

In this subsection, we present methods for finding in polynomial time augmenting graphs from the seven basic families represented in Fig. 3.2. These methods was developed from the techniques presented in [125] (finding augmenting trees of the form tree^1, …, tree^6 in ($S_{1,2,5}$,banner)-free graphs). Like in [125], we first check whether G contains a certain small induced subgraph (candidate) and then try to extend it to the whole augmenting graph. In this subsection, we consider a graph G which is an ($S_{2,2,5}$,banner$_2$,domino)-free graph. Given a black vertex b, we denote by $W(b)$ be the set of white neighbors of b. For a non-negative integer i, denote by B^i the set of black vertices having exactly i white neighbors. We refer to Fig. 3.2 for the indices.

Lemma 3.18. *If G contains no augmenting P_3, then an augmenting tree^1 (if any) can be found in time* $\mathrm{O}(n^{17})$.

Proof. Refer to Fig. 3.2, tree^1 for r. If $r = 1$, then tree^1 is a P_3. Assume that G contains an augmenting graph tree^1, for some $r \geq 2$. Therefore, G contains an induced $P_5 = (b_1, a_1, x, a_2, b_2)$, where $b_1, b_2 \in B^1$. If G contains no such an initial structure, then it contains no augmenting tree^1. If such a structure exists, then we proceed as follows.
Let us denote $A = W(x) \backslash \{a_1, a_2\}$ and for $a \in A$, let $K(a)$ denote the set of black neighbors of a in B_1 not adjacent to any vertex of $\{x, b_1, b_2\}$. Notice that a desired augmenting tree exists only if $K(a) \neq \emptyset$ for every $a \in A$. Finally, let $V' = \bigcup_{a \in A} K(a)$.
Since $K(a) \subset B^1$ for every $a \in A$, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.
Consider any vertex $a \in A$, we show that $K(a)$ induces a clique for every $a \in A$. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b_1, b_2. Then $\{b_1, a, b_2\}$ induces an augmenting P_3, a contradiction. It follows that a desired augmenting tree^1 exists if and only if $\alpha(G[V']) = |A|$.
We show that $G[V']$ must be P_5-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$ and let $a \in A$ be such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. Thus, $p_2 \in K(a)$, otherwise $\{b_1, a_1, x, a_2, b_2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$, a contradiction. Hence, if $G[V']$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction with when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.
Since the P_5-free graph class is MIS-solvable in time $\mathrm{O}(n^{12})$ [115], one can find a simple augmenting tree containing the P_5 (b_1, w_1, b, w_2, b_2) in $\mathrm{O}(n^{12})$. With an exhaustive search, all candidate P_5 of augmenting trees can be found in time $\mathrm{O}(n^5)$. For such candidates P_5's, V' can be build in $\mathrm{O}(n^3)$. Hence, we have the conclusion of the lemma. □

Lemma 3.19. *If G contains neither augmenting P_3 nor P_7, then an augmenting tree^2 (if any) can be found in time* $\mathrm{O}(n^{14})$.

Proof. Refer to Fig. 3.2, tree^2 for r and s. We may restrict ourselves to find a tree^2 with $r, s \geq 2$, since any tree^2 with, say $r = 1$, either equals to P_7 or contains a redundant subset U of size two such that tree$^2 - U$ is of the form tree^1.
As a candidate, consider the subgraph of tree^2 (see Fig. 3.2) induced by $\{a_1, a_2, b_1, b_2, c_1,$

$c_2, d_1, d_2, x, y, z\}$ such that $b_1, b_2, d_1, d_2 \in B^1$ and x, z share no common white neighbor other than y.
Let us denote $A = (W(x) \cup W(z)) \backslash \{a_1, a_2, c_1, c_2, y\}$. For $a \in A$, let $K(a)$ denote the set of black neighbors of a in B^1 not adjacent to any vertex of $\{x, b_1, b_2, d_1, d_2\}$. Note that, by the assumption, every vertex of A is either adjacent to x or y. Notice that a desired augmenting tree exists only if $K(a) \neq \emptyset$ for every $a \in A$.
We show that $K(a)$ induces a clique. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b_1, b_2. Then $\{b_1, a, b_2\}$ induces an augmenting P_3, a contradiction.
Since for every $a \in A$, $K(a) \in B^1$, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.
Finally, let $V' = \bigcup_{a \in A} K(a)$. It follows that a desired augmenting tree^2 exists if and only if $\alpha(G[V']) = |A|$.
We now show that $G[V']$ is P_3-free. Suppose, to the contrary, that (p_1, p_2, p_3) is an induced P_3 in $G[V']$. Let $a \in A$ such that $p_1 \in K(a)$. Since $K(a)$ is a clique, p_3 is not adjacent to a. Assume that $p_3 \sim a'$. Then since $p_2 \in B^1$, p_2 is not adjacent to at least one vertex among a, a'. Without loss of generality, assume that $p_2 \nsim a$, and a is adjacent to x, but not to z. Then $\{d_2, c_2, z, c_1, d_1, y, x, a, p_1, p_2\}$ induces an $S_{2,2,5}$, a contradiction.
Hence, $G[V']$ is a disjoint union of cliques, i.e. a maximum independent set in $G[V']$ can be found in linear time. All candidates of the form tree^2 whose $r = s = 2$ can be found by an exhaustive search in time $O(n^{11})$. For such candidates P_5's, V' can be build in $O(n^3)$. Hence, we have the conclusion of the lemma. □

Lemma 3.20. *If G contains neither augmenting P_3 nor P_5, then an augmenting tree^3 or an augmenting tree^4 (if any) can be found in time $O(n^{31})$.*

Proof. First, note that tree^4 is a special case of tree^3. We refer to Fig. 3.2, tree^3 for indices. Moreover, we may restrict ourselves to finding a tree^3 with $s \geq 3$, since any tree^3 with, say, $s \leq 2$ is either of the form tree^1 or contains a redundant subset U of size four such that tree$^3 - U$ is of the form tree^1.
As a candidate, consider the subgraph of tree^3 (see Fig. 3.2) induced by $\{d_1, c_1, b_1^1, a_1^1, x, a_1^2, b_1^2, c_2, d_2, a_1^3, b_1^3, c_3, d_3\}$ such that $b_1^1, b_1^2, b_1^3 \in B^2$, $d_1, d_2, d_3 \in B^1$. Let us denote $A = W(x) \backslash \{a_1^1, a_1^2, a_1^3\}$. For $a \in A$, let $K(a)$ denote the set of black neighbors b of a in $B^1 \cup B^2$ and not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3\}$ such that if $b \in B^2$, then G contains a pair of adjacent vertices c_b and d_b such that $c_b \notin W(x)$, $W(b) = \{a, c_b\}$, $d_b \in B^1$, and d_b is not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3, b\}$ (note that d_b may coincide with d_1, d_2, or d_3). Let $V' = \bigcup_{a \in A} K(a)$. And again, by the existence of a desired augmenting tree^3, $K(a)$ is not empty for all $a \in A$. Note that by the asumption, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.
Consider any vertex $a \in A$, we show that $K(a)$ induces a clique. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b, b'. By the symmetry, we consider the three following cases.
Case 1. $b, b' \in B^1$. Then $\{b, a, b'\}$ induces an augmenting P_3, a contradiction.
Case 2. $b' \in B^1$ and $b \in B^2$. Then $\{b', a, b, c_b, d_b\}$ induces an augmenting P_5, a contradiction.
Case 3. $b, b' \in B^2$. Then $c_b \neq c_{b'}$, otherwise $\{b, c_b, b', a, x, a_1^1\}$ induces a banner$_2$, a contradiction. Now, $\{c_{b'}, b', a, b, c_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is among

c_1, c_2, c_3 different from $c_b, c_{b'}$, a contradiction.
It follows that a desired augmenting tree^3 exists if and only if $\alpha(G[V']) = |A|$.
Given $a, a' \in A$ and $b \in K(a) \cap B^2$, $b' \in K(a')$ such that $b \nsim b'$ and if $b' \in B^2$, assume that $d_b \neq d_{b'}$, we show that $b' \nsim d_b$. Indeed, suppose that $b' \sim d_b$. Then $b' \nsim c_b$, otherwise $c_{b'} = c_b$, and hence, $d_{b'} = d_b$, a contradiction. Thus, $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a', b', d_b, c_b, b\}$ induces an $S_{2,2,5}$, a contradiction. Now, if $b' \in B^2$, then $d_b \nsim d_{b'}$, otherwise $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a', b', c_{b'}, d_{b'}, d_b\}$ induces an $S_{2,2,5}$, a contradiction.
Hence, for every pair of non-adjacent vertices b, b' such that $b \in K(a) \cap B^2$, $b' \in K(a')$ for two distinct vertices $a, a' \in A$, $\{b, b', d(b)\}$ is independent. Moreover, if $b' \in B^2$, then $\{b, b', d_b, d_{b'}\}$ is independent.
Now, assume that B' is a maximum independent set of $G[V']$. Let $C' := \{c_b : b \in B' \cap B^2\}$, $D' := \{d_b : b \in B' \cap B^2\}$. Then by above arguments, $B' \cup D'$ is independent. And in the case that $|B'| = |A|$, $H := G[A \cup B' \cup C' \cup D']$ is an augmenting graph of the form tree^3 of G.
As in Lemma 3.18, we show that $G[V']$ is P_5-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$ and let $a \in A$ such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. But now, $p_2 \in K(a)$, otherwise $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$, a contradiction. Hence, if $G[V']$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction with when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.
All candidates can be found by an exhaustive search in time $\mathrm{O}(n^{19})$. For such candidates, V' can be build in $\mathrm{O}(n^3)$. Again, by the solution for the MIS problem in P_5-free graphs [115], we have the conclusion of the lemma. □

Lemma 3.21. *An augmenting tree^5 (if any) can be found in time* $\mathrm{O}(n^{14})$.

Proof. Refer to Fig. 3.2, tree^5 for r and s. We may restrict ourselves to find a tree^5 with $r, s \geq 1$ and $r \geq 2$, since a tree^5 with, say, $r = 0$ contains a redundant set U of size four such that tree$^5 - U$ is of the form tree^1, and a tree^5 with $r = s = 1$ can be found in time $\mathrm{O}(n^9)$.
As a candidate, consider the subgraph of tree^5 (see Fig. 3.2) induced by $\{a_1, a_2, b_1, b_2, c_1, d_1, u, v, x, y, z\}$ such that $b_1, b_2, v, d_1 \in B^2$ and x, y share no common white neighbor other than u. Let us denote $A_x = W(x) \backslash \{a_1, a_2, u\}$ and $A_y = W(y) \backslash \{c_1, u\}$ and for $a \in A := A_x \cup A_y$, let $K(a)$ denote the set of common black neighbors of a and z in B^2 not adjacent to any vertex of $\{x, y, b_1, b_2, v, d_1\}$.
Note that by the assumption, every vertex of A is either adjacent to x or y. Since $K(a) \subset B^2$ for every $a \in A$, $K(a) \cap K(a') = \emptyset$, for every pair of distinct vertices $a, a' \in A$.
Consider a pair of distinct vertices $b, b' \in K(a)$ for some $a \in A$. If $b \nsim b'$, then $\{b, a, b', z, v, u\}$ induces a banner$_2$, a contradiction. Hence, $K(a)$ is a clique for all $a \in A$.
Now, let $V'(x) := \bigcup_{a \in A_x} (K(a))$, $V'(y) := \bigcup_{a \in A_y} (K(a))$, and $V' := V'(x) \cup V'_y$. Note that, $V'(x) \cap V'(y) = \emptyset$ by the definition. Then a desired augmenting tree^5 exists if and only if $K(a) \neq \emptyset$ for every $a \in A$ and $\alpha(G[V']) = |A|$.
As in Lemma 3.19, we show that $G[V']$ is P_3-free. Suppose, to the contrary, that (p_1, p_2, p_3) is an induced P_3 in $G[V']$. Let $a \in A$ such that $p_1 \in K(a)$. Since $K(a)$ is a clique, p_3 is not adjacent to a. Assume that $p_3 \sim a'$. Since $p_2 \in B^2$, p_2 is not adjacent

to at least one vertex among a, a'. Without loss of generality, assume that $p_2 \nsim a$ and a is adjacent to y, but not to x. Then $\{b_2, a_2, x, b_1, a_1, u, y, a, p_1, p_2\}$ induces an $S_{2,2,5}$, a contradiction. Hence, a maximum independent set can be found in $G[V']$ in linear time.
All candidates can be found by an exhaustive search in time $O(n^{11})$. For such candidates, V' can be build in $O(n^3)$. Hence, we have the conclusion of the lemma. $\square$

Lemma 3.22. *An augmenting tree*^{6} *(if any) can be found in time* $O(n^{27})$.

Proof. Refer to Fig. 3.2, tree^6 for r and s. We may restrict ourselves to find a tree^6 with $r, s \geq 2$, since a tree^6 with, say, $r = 1$, contains a redundant set U of size four such that tree$^6 - U$ is of the form tree^1.
As a candidate, consider the subgraph of tree^6 (see Fig. 3.2) induced by $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, x, y, z\}$ such that $b_1, b_2, c_1, c_2 \in B^2$ and x, z share no common white neighbor. Let us denote $A_x = W(x)\backslash\{a_1, a_2\}$ and $A_z = W(z)\backslash\{d_1, d_2\}$. For $a \in A := A_x \cup A_z$, let $K(a)$ denote the set of common black neighbors of a and y in B^2 and not adjacent to any vertex of $\{x, b_1, b_2, c_1, c_2, z\}$. Note that $A_x \cap A_z = \emptyset$ by the assumption. Since for every $a \in A$, $K(a) \subset B^2$, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$. Consider a pair of distinct vertices $b, b' \in K(a)$ for some $a \in A$. If $b \nsim b'$, then $\{b, a, b', y, c_1, d_1\}$ induces a banner$_2$ in the case that $a \in A(x)$ (similar for the case $a \in A(z)$), a contradiction. Hence, $K(a)$ is a clique for all $a \in A$.
Now, let $V'(x) := \bigcup\limits_{a \in A_x} (K(a))$, $V'(z) := \bigcup\limits_{a \in A_z} (K(a))$, and $V' := V'(x) \cup V'_z$. Note that, $V'(x) \cap V'(z) = \emptyset$. Then a desired augmenting tree^6 exists if and only if $K(a) \neq \emptyset$ for every $a \in A$ and $\alpha(G[V']) = |A|$.
As in Lemma 3.18, we show that $G[V'_x]$ and $G[V'_z]$ are P_5-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V'_x]$ or $G[V'_z]$, let $a \in A$ be such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. But now, $p_2 \in K(a)$, otherwise $\{b_1, a_1, x, a_2, b_2, a, p_1, p_2, p_3, p_4\}$ or $\{c_1, d_1, z, d_2, c_2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$ depending on $a \in A(x)$ or $a \in A(z)$, a contradiction. Hence, if $G[V'_x]$ or $G[V'_z]$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction with when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.
Moreover, assume that there exists a pair of vertices b, b' such that $b \in K(a)$, $b' \in K(a')$ for some $a \in A(x)$, $a' \in A(z)$, and $b \sim b'$. Then $\{b_1, a_1, x, a_2, b_2, a, b, b', a', z\}$ induces an $S_{2,2,5}$, a contradiction. Hence, there is no edge connecting a vertex in $G[V'_x]$ and a vertex in $G[V'_z]$. So, $G[V']$ is P_5-free.
Note that all candidates can be found by an exhaustive search in time $O(n^{15})$. For such candidates, V' can be build in $O(n^3)$. Hence, by the result of Lokshtanov et al. [115] we have the conclusion of the lemma. $\square$

Lemma 3.23. *If G contains no augmenting P_3, nor P_5, nor P_7, then an augmenting tree*^{7} *(if any) can be found in time* $O(n^{19})$.

Proof. Refer to Fig. 3.2 for indices. We may restrict ourselves to find a tree^7 with $s \geq 3$, since a tree^7 with $s \leq 2$ is of the form tree^3 or contains a redundant set U of size at most eight such that tree$^7 - U$ is of the form tree^3.
As a candidate, consider the subgraph of tree^7 (see Fig. 3.2) induced by $\{x, a_1^1, b_1^1, c_1, d_1, e_1, f_1, a_1^2, b_1^2, c_2, d_2, e_2, f_2, a_1^3, b_1^3, c_3, d_3, e_3, f_3\}$ such that $b_1^1, d_1 \in B^2$ and $f_1 \in B^1$. Let us denote $A = W(x)\backslash\{a_1^1, a_1^2, a_1^3, e_1, e_2, e_3\}$. For $a \in A$, let $K(a)$ denote the set of black

neighbors b of a in $B^1 \cup B^2$ not adjacent to any vertex of $\{x, b_1^1, d_1, e_1, f_1, b_1^2, d_2, e_2, f_2, b_1^3, d_3, e_3, f_3\}$ and such that if $b \in B^2$, then G contains either

- two vertices c_b, d_b such that $c_b \notin W(x)$, $W(b) = \{a, c_b\}$, $d_b \in B^1$, and d_b is not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3, f_1, f_2, f_3, b\}$ or
- an induced alternating (black white vertices) P_4 (c_b, d_b, e_b, f_b) such that $e_b \in W(x)\backslash\{a_1^1, c_1, a_1^2, c_2, a_1^3, c_3\}$, $c_b \notin W(x)$, $W(b) = \{a, c_b\}$, $W(d_b) = \{c_b, e_b\}$, $W(f_b) = \{e_b\}$, and d_b, f_b are not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3, f_1, f_2, f_3, b\}$.

Let $V' = \bigcup_{a \in A} K(a)$.
By the existence of a desired augmenting tree[7], $K(a)$ is not empty for all $a \in A$. Note that, by assumption, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.
Given a vertex $b \in K(a) \cap B^2$ for some $a \in A$, we show that $d_b \notin K(e_b)$. Indeed, suppose that $d_b \notin K(e_b)$. Since $d_b \in B^2$, $c_b = c_{d_b}$, $d_{d_b} = b$, and $e_{d_b} = a$. Hence, there exists some vertex $b' \in B^1$, such that $f_{d_b} = b'$, i.e. $b' \sim a$ and b' is not adjacent to b, d_b. Hence, $b' \nsim f_b$, otherwise $\{c_b, b, a, b', f_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is a vertex among c_1, c_2, c_3 different from c_b, a contradiction. Now, $\{b', a, b, c_b, d_b, e_b, f_b\}$ induces an augmenting P_7, a contradiction.
Suppose that there exist two vertices b, b' such that $b \in K(a) \cap B^2$ and $b' \in K(a') \cap B^2$ for two distinct vertices $a, a' \in A$ and $d_b, d_{b'}$ are different and adjacent to some vertex $a'' \in W(x)\backslash\{a, a', a_1^1, a_1^2, a_1^3\}$ different from a, a'. Then $\{c_b, d_b, a'', d_{b'}, c_{b'}, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$ where c_i is a vertex among c_1, c_2, c_3 different from $c_b, c_{b'}$, a contradiction. Hence, for every pair of vertices b, b' such that $b \in K(a) \cap B^2$, $b' \in K(a') \cap B^2$ for two distinct vertices $a, a' \in A$, $e_b \neq e_{b'}$.
Consider any vertex $a \in A$, we show that $K(a)$ induces a clique. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b, b'. By the symmetry, we consider the three following cases.
Case 1. $b, b' \in B^1$. Then $\{b, a, b'\}$ induces an augmenting P_3, a contradiction.
Case 2. $b' \in B^1$ and $b \in B^2$. We have the three following subcases.
2.1. $d_b \in B^1$. Then $\{b', a, b, c_b, d_b\}$ induces an augmenting P_5, a contradiction.
2.2. $d_b \in B^2$ and $b' \nsim f_b$. Then $\{b', a, b, c_b, d_b, e_b, f_b\}$ induces an augmenting P_7, a contradiction.
2.3. $d_b \in B^2$ and $b' \sim f_b$. Then $\{f_b, b', a, b, c_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is a vertex among c_1, c_2, c_3 different from c_b, a contradiction.
Case 3. $b, b' \in B^2$. Then $c_b \neq c_{b'}$, otherwise $\{b, c_b, b', a, x, a_1^1\}$ induces a banner$_2$, a contradiction. Now, $\{c_{b'}, b', a, b, c_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is a vertex among c_1, c_2, c_3 different from $c_b, c_{b'}$, a contradiction.
It follows that a desired augmenting tree[7] exists if and only if $\alpha(G[V']) = |A|$.
Given $a, a' \in A$, $b \in K(a) \cap B^2$, and $b' \in K(a')$ such that $b \nsim b'$, if $b' \sim d_b$, then $b' \nsim c_b$, otherwise $c_{b'} = c_b$ and then $d_{b'} = d_b$, a contradiction. Then $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a', b', d_b, c_b, b\}$ induces an $S_{2,2,5}$, a contradiction. Now, if $b' \in B^2$, then $d_b \nsim d_{b'}$, otherwise $\{b_1^i, a_1^i, x, a_1^j, b_1^j, a', b', c_{b'}, d_{b'}, d_b\}$ induces an $S_{2,2,5}$, for $i, j \in \{1, 2, 3\}$ such that c_b is different from c_i, c_j, a contradiction. Note that for every $b \in K(a) \cap B^2$ for some $a \in A$, $f_b \in K(e_b)$. Hence, for every pair of non-adjacent vertices b, b' such that $b \in K(a) \cap B^2$, $b' \in K(a')$ for two distinc vertices $a, a' \in A$, $\{b, b', d_b, f_b\}$ is independence. Moreover, if $b' \in B^2$, then $\{b, b', d_b, d_{b'}, f_b, f_{b'}\}$ is independent.
Now, assume that B' is a maximum independent set of $G[V']$. Let $C' := \{c_b : b \in$

$B' \cap B^2\}$, $D' := \{d_b : b \in B' \cap B^2\}$. Then by above arguments, $B' \cup D'$ is independent. And in the case that $|B'| = |A|$, $H := G[A \cup B' \cup C' \cup D']$ is an augmenting graph of the form tree^7 of G. Hence, a maximum independent set of $G[V']$ in the case that $\alpha(G[V']) = |A|$ gives us an augmenting of the form tree^7.
As in Lemma 3.18, we show that $G[V']$ is P_5-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$, and let $a \in A$ be such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. But now, $p_2 \in K(a)$, otherwise $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$, a contradiction. Hence, if $G[V']$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction with when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.
All candidates can be found by an exhaustive search in time $O(n^{19})$. For such candidates, V' can be build in $O(n^3)$. By the result of Lokshtanov et al. [115], we have the conclusion of the lemma. □

Lozin and Hertz [98] described the method finding augmenting graph of the form $K_{p,p+1}$ in banner$_2$-free graphs. Hence, Theorem 2.1, Lemmas 3.12, 3.14, 3.15, 3.18, ..., 3.23 lead to the following result.

Theorem 3.24. *Given integers m, l, the ($S_{2,2,5}$,banner$_2$,domino,M_m, $K_{m,m} - e$,R_l^3, R_l^4, R_l^5)-free graph class is MIS-easy.*

Corollary 3.25. *Given integers m, the ($S_{1,2,5}$,banner$_2$,domino,M_m, $K_{m,m} - e$)-free graph class is MIS-easy.*

This corollary is a generalization of the ressult of Lozin and Milanič for ($S_{1,2,5}$,banner)-free graphs [125], of the results of Lozin and Mosca for (P_5, $K_{3,3} - e$)-free graphs [128], of Boliac and Lozin [22] for (P_5, $K_{2,m} - e$)-free graphs, and of Lê et al. about for some subclasses of $S_{1,2,2}$-free graphs [112]. Note that we used redundant set and reduction set to reduce "near" augmenting complete bipartite graphs to augmenting complete bipartite graphs. This technique generalizes the method for augmenting $K_{m,m}^+$ in [128].

3.3.3 Augmenting Vertex

In this subsection, we describe the technique was used in [75, 76, 140, 141] for P_5-free graphs to apply in subclasses of ($S_{2,2,5}$,banner$_2$,domino,M_m)-free graphs. Let S be an independent set of a graph $G = (V, E)$ and $v \in V \backslash S$. We denote as in [140], $H(v, S) := \{w \in V \backslash (S \cup \{v\} \cup N(v)) : N_S(w) \subset N_S(v)\}$. Given a graph $G = (V, E)$, an independent set S, and a vertex $v \in V \backslash S$, Mosca [140] defined that v is *augmenting* for S (and that S admits an augmenting vertex), if $G[H(v, S)]$ contains an independent set S_v such that $|S_v| \geq |N_S(v)|$. This implies that $H' := (S_v \cup \{v\}, N_S(v), E(H'))$ is an augmenting graph. Then by Theorem 2.1 and Lemma 3.12, we restrict ourselves in the following problem.
Consider the problem of finding a maximum independent set, say S', of $G[H(v, S)]$, and $|N_S(v)| \geq 3$. By the definition of $H(v, S)$, one has that $N_S(v)$ is a maximal independent set of $G[N_S(v) \cup H(v, S)]$; in particular, $N_S(v)$ and S' induce a connected bipartite subgraph of G. Note that, by the results in the two previous subsections, for ($S_{2,2,5}$,banner$_2$,domino,R_l^3, R_l^4, R_l^5, M_m)-free graphs, we can find every (minimal) augmenting graph in polynomial time except augmenting bipartite-chains. And clearly,

every augmenting bipartite-chain is associated with some augmenting vertex v. Moreover, for an augmenting bipartite-chain $H = (B, W, E)$ associated with some augmenting vertex v, i.e. $v \in B$ and $W \subset N_S(v)$, there also exists some vertex $s \in W$ such that $B \subset N(s)$. Hence, instead of solving the MIS problem in $G[H(v, S)]$, it is enough to solve for $G[N(s) \cap H(v, S)]$ for every $s \in N_S(v)$. So, we modify the concept of augmenting vertex as follows.

Definition 3.3. *Let S be an independent set of a graph $G = (V, F)$ and $v \in V \backslash S$, $s \in N_S(v)$. We say that v is augmenting for S associated with s if $G[N(s) \cap H(v, S)]$ contains an independent set $S_{v,s}$ such that $|S_{v,s}| \geq |N_S(v)|$.*

Moreover, with an addition asumption that a maximum independent set of $G[N(s) \cap H(v, S)]$ can be found in polynomial time for every $s \in N_S(v)$, we can also choose s such that $\alpha(G[N(s) \cap H(v, S)])$ is maximum.

Refer to Algorithm 4, where p is a constant defined as in Lemma 3.12, an extended

Algorithm 4 MISAugVer(G)

Input: a ($S_{2,2,5}$,banner$_2$,domino,M_m)-free graph G
Output: S, A maximum independent set of G.

```
Find an arbitrary maximal independent set S in G;
while There exists an H-augmentations to S where H contains at most 2m − 1
    vertices, or H is an augmenting (4, p)-extended-chain, an augmenting apple, or H
    is of the form tree^1, ..., tree^7 or can be reduced to such forms by some redundant
    set or some reduction set of size at most 32, or S admits an augmenting vertex v
    associated with some vertex s do
  if S admits an H-augmentation then
    Apply an augmenting H for S;
  end if
  if S admits an augmenting vertex v associated with s then
    S := (S\N_S(v)) ∪ {v} ∪ S_{v,s};
  end if
end while
return S
```

version of Algorithm Alpha in [140], a maximal independent set of G can be found (say by some greedy method) in time $O(n^2)$. One can compute the set $H(v, S)$ in time $O(n^2)$. Note that an augmenting of at most $2m - 1$ vertices can be found in time $O(n^{2m+1})$. Moreover, by Lemmas 3.15, 3.18, ..., 3.23, an augmenting graph of the forms mentioned in the **while** condition can be found in polynomial time. The **while** loop is repeated at most n time. Hence, we observe the following result, an extension of Theorem 7 in [140].

Lemma 3.26. *Given two integers l and m, an ($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^3, R_l^4, R_l^5)-free graph $G = (V, E)$, a maximal independent set of G S, and $v \in V \backslash S$, if one can find a maximum independent set of $G[N(s) \cap H(v, S)]$ for every $s \in N_S(v)$ in polynomial time, then one can find a maximum independent set of G in polynomial time.*

Now, for some notations from [141], let K be a graph. Let us denote as $K^{(1)}$ the graph obtain from K by adding two new vertices v, s, such that s dominates K, while

v is adjacent only to s. In general, let $K^{(h)}$ be the graph obtained from K by adding $h+1$ new vertices $v, s_1, \ldots, s_h$ such that $\{s_1, s_2, \ldots, s_h\}$ induce an independent set, s_i's dominate K, while v is adjacent only to s_i's. We obtain the following result as an extension of similar result in [141] for P_5-free graphs.

Theorem 3.27. *Given two integers l, m and a graph K, if the ($S_{2,2,5}$,banner$_2$,domino, $M_m, R_l^3, R_l^4, R_l^5, K$)-free graph class is MIS-easy, then so is the ($S_{2,2,5}$,banner$_2$,domino, $M_m, R_l^3, R_l^4, R_l^5, K^{(1)}$)-free graph class.*

Proof. Let G be an ($S_{2,2,5}$,banner$_2$,domino,M_m, $R_l^3, R_l^4, R_l^5, K^{(1)}$)-free graph with n vertices. Moreover, let S be a maximal independent set and let v be an augmenting vertex of S. Then for every $s \in N_S(v)$, $G[N(s) \cap H(v, S)]$ is K-free, otherwise vertices v, s, and the induced subgraph K would induce a $K^{(1)}$ in G, a contradiction. Hence, by Lemma 3.26, we have the statement of the theorem. □

Now, given two integers l and m, like in [141], let us show that a result similar to Theorem 3.27 can be stated for ($S_{2,2,5}$,banner$_2$,domino,M_m, $R_l^3, R_l^4, R_l^5, K^{(h)}$)-free graphs with $h \geq 1$ as well.
Let $G = (V, E)$ be an ($S_{2,2,5}$,banner$_2$,domino,M_m, $R_l^3, R_l^4, R_l^5, K^{(h)}$)-free graph with n vertices and S be a maximal independent set of G. Assume that one can solve the MIS problem for ($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^3, R_l^4, R_l^5, K)-free graphs in polynomial time. The goal is to show that one can carry out Step 6 of Algorithm 4 in polynomial time. We use the technique described in [141]. Let us say that a vertex $v \in V$ is a *trivial augmenting vertex* for S if v is augmenting for S and $|N_S(v)| \leq h$. Then one can check if a vertex $v \in V$ is a trivial augmenting vertex for S in time $\mathrm{O}(n^{h+1})$, by verifying if $G[H(v, S)]$ contains an independent set S^* of $|N_S(v)|$ vertices. Such S^* is called the independent set associated with the augmenting vertex v.
Assume that G admit no trivial augmenting vertex for S and that there exists $v \in V \backslash S$ augmenting for S (in particular, $h < |N_S(v)|$). Thus, $G[H(v, S)]$ contains an independent set T with $|N_S(v)| \leq |T|$. Since G is ($S_{2,2,5}$,banner$_2$,domino,M_m)-free together with an addition asumption that G contains no augmenting graph contains at most $2m-1$ vertices, no augmenting graph of the forms tree^1, ..., tree^7, no augmenting $(4, p)$-extended-chain, no augmenting apple, no augmenting graph that can be reduced to such forms by some redundant set or reduction set, by Lemmas 3.12 and 3.13, $H' := (T \cup \{v\}, N_S(v), E(H'))$ is an augmenting bipartite-chain.
Let us write $T = \{t_1, \ldots, t_r\}$ ($r \geq |N_S(v)| \geq h$), with $N_S(t_i) \subset N_S(t_{i+1})$ for any index i. Since G admit no trivial augmenting vertex for S, one has $|N_S(t_k)| \geq k$ for $k = 1, \ldots, h$. For any $t \in H(v; S)$, let us write $M(t) = \{w \in H(v, S) : N_S(w) \supset N_S(t), |N_S(w)| \geq h\}$. Then $T \subset \{t_1, \ldots, t_h\} \cup (M(t_h) \backslash N(\{t_1, \ldots, t_h\}))$. Note that $M(t_h)$ is K-free, otherwise $M(t_h) \cup \{s_1, s_2, \ldots, s_h\} \cup \{v\}$ induces a $K^{(h)}$ for $s_1, \ldots, s_h \in N_S(t_h)$, a contradiction.
Now, since Step 6 of Algorithm 4 considers all the vertices in $V \backslash S$, to check if S admits an augmenting vertex one has not to solve the MIS problem in $H(v, S)$ for every $v \in V \backslash S$. In fact, for every $v \in V \backslash S$, it is sufficient to verify: (i) if v is a trivial augmenting vertex for S, and then (ii) if v is augmenting, by assuming that S admit no trivial augmenting vertex. That can be formalized by the procedure Algorithm 5 [141], whose the input is any vertex v of $V \backslash S$ which can be executed in time $\mathrm{O}(n^{h+d+1})$.

Note that, given an augmenting vertex v (for S), Procedure Green(v) could not recognize it as an augmenting vertex: that can happen whenever $H(v, S)$ contains a

Algorithm 5 Procedure Green (v)

Input: a vertex $v \in V \backslash S$

Output: a possible proof that v is augmenting associated with $T = \{t_1, \ldots, t_h\}$ and an independent set S^* associated with v.

$S^* := \emptyset$; $T := \emptyset$;
if $|N_S(v)| \leq h$ **then**
  if $H(v;S)$ contains a independent set Q of $|N_S(v)|$ vertices **then**
    set $S^* := Q$; $\{v$ is (trivially) augmenting for $S\}$;
  end if
else
  for all independent set U of h vertices of $G[H(v,S)]$, i.e. $U = \{t_1, \ldots, t_h\}$, with $N_S(t_i) \subset N_S(t_{i+1})$, and $|N_S(t_i)| \geq i$ **do**
    $S' :=$ MISAugVer$(G[M(t_h) \backslash N(\{t_1, \ldots, t_h\})])$;
    if $|S' \cup \{t_1, \ldots, t_h\}| > |S^*|$ **then**
      $S^* := S' \cup \{t_1, \ldots, t_h\}$; $T := \{t_1, \ldots, t_h\}$;
    end if
  end for
end if
if $|S^*| \geq |N_S(v)|$ **then**
  return v is augmenting for S associated with T and S^*
end if

trivial augmenting vertex. Now, we give the new definition for augmenting vertex v as following.

Definition 3.4. *Let S be an independent set of a graph $G = (V, E)$, h be an integer, and $v \in V \backslash S$, $t_1, t_2, \ldots, t_h \in H[v, S]$. We say that v is h-augmenting for S associated with $\{t_1, \ldots, t_h\}$, where $N_S(t_i) \subset N_S(t_{i+1})$ for every index i, if $G[M(t_h) \backslash N(\{t_1, \ldots, t_h\})]$ contains an independent set $S_{v,t_1,\ldots,t_h}$ such that $|S^*| \geq |N_S(v)|$ where $S^* := S_{v,t_1,\ldots,t_h} \cup \{t_1, t_2, \ldots, t_h\}$. S^* is called the independent set associated with the augmenting v.*

To summarize, in order to define an efficient method to solve the MIS problem in ($S_{2,2,5}$,banner$_2$,domino,M_m, $K^{(h)}$)-free graphs, one can rewrite Step 6 of Algorithm 4 as in Algorithm 6.

By the structure of the new Step 6, one can formally state the following result as an

Algorithm 6 New Step 6

for all $v \in V \backslash S$ **do**
  Procedure Green(v);
  if v is augmenting for S associated with S^* **then**
    $S := (S \backslash N_S(v)) \cup S^*$; **stop**;
  end if
end for

extension of Theorem 4.3 in [141].

Theorem 3.28. *Given three integers h, l, m and a graph K, if the ($S_{2,2,5}$,banner$_2$, domino,M_m, R_l^3, R_l^4, R_l^5, K)-free graph class is MIS-easy, then so is the ($S_{2,2,5}$,banner$_2$, domino,M_m, $R_l^3, R_l^4, R_l^5, K^{(h)}$)-free graph class.*

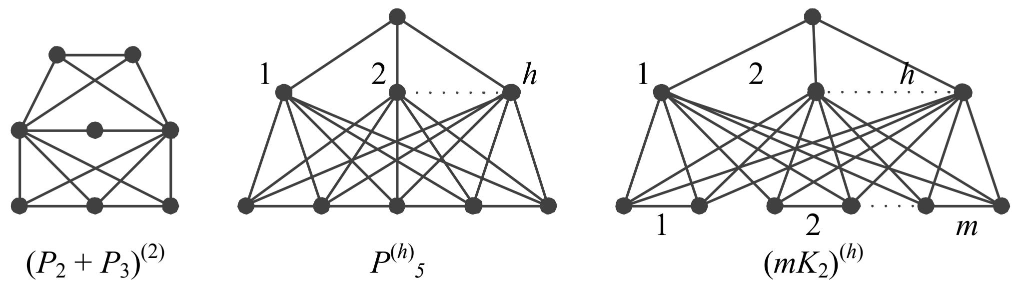

Fig. 3.5: Special Graphs in Corollary 3.30

Corollary 3.29. *Given two integers h, m and a graph K, if the ($S_{1,2,5}$,banner$_2$,domino, M_m, K)-free graph class is MIS-easy, then so is the ($S_{1,2,5}$,banner$_2$,domino,$M_m, K^{(h)}$)-free graph class.*

Especially, Theorem 3.28 leads to some interesting polynomially solvable graph classes of the MIS problem. Remark that the MIS problem was proved to be polynomially solvable in P_5-free graphs [115], (P_2+P_3)-free graphs [127], pK_2-free graphs [6], we have the following consequence.

Corollary 3.30. *Given four integers h, l, m, p, the following graph classes (see Fig. 3.5) are MIS-easy:*

1. *($S_{2,2,5}$,banner$_2$,domino,$M_m, R_l^3, R_l^4, R_l^5, P_5^{(h)}$)-free graphs,*
2. *($S_{2,2,5}$,banner$_2$,domino,$M_m, R_l^3, R_l^4, R_l^5, (P_2+P_3)^{(2)}$)-free graphs, and*
3. *($S_{2,2,5}$,banner$_2$,domino,$M_m, R_l^3, R_l^4, R_l^5, (pK_2)^{(h)}$).*

Now, we use the technique described in [33] for P_5-free graphs to extend 3., the case $h = 2$ of the above corollary.

Corollary 3.31. *Given four integers l, m, p, and r, the ($S_{2,2,5}$,banner$_2$,domino,M_m, tree$_r$,R_l^3, R_l^4, R_l^5, Q_p)-free graph class (see Fig. 3.6) is MIS-easy.*

Proof. Recall the modular decomposition technique. We show that a prime (Q_p,tree$_r$)-free graph is $((2p+r-2)K_2)^{(2)}$-free. Indeed, let G be a prime (Q_p,tree$_r$)-free graph, and suppose that G contains an induced subgraph Q' isomorphic to $((2p+r-2)K_2)^{(2)}$. Let $T \subset V(G)$ be the subset of vertices of G adjacent to every vertex of the $(2p+r-2)K_2$ of Q'. Since T contains at least two non-adjacent vertices, $\bar{G}[T]$, the complement subgraph of G induced by T, contains a non-trivial component C. Because G is prime, C is not a module. Hence, there exists a vertex $v \in V(G)\backslash C$ distinguishing C, i.e. $v \sim c_1$ and $v \nsim c_2$ for some vertices c_1, c_2 in C. Moreover, since $\bar{G}[C]$ is connected, we can substitute c_1, c_2 by two vertices of the path connecting them and can assume that $c_1 \nsim c_2$ in G.
If v is adjacent to every vertex of the $(2p+r-2)K_2$ of Q', then $v \in T$ and since $v \nsim c_2$, $v \in C$, a contradiction. Hence, there exists a vertex c' of the $(2p+r-2)K_2$ of Q' such that $c' \nsim v$.
Since G is tree$_r$-free, v is distinguish at most $r-1$ edges of the $(2p+r-2)K_2$ of Q'. Then we have the two following cases.

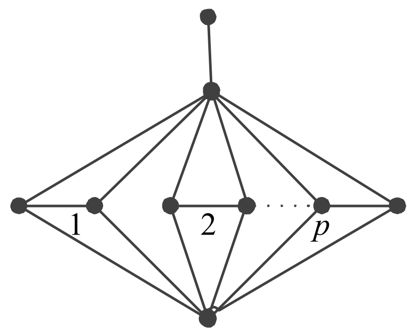

Fig. 3.6: Q_p

Case 1. v is adjacent to both end-vertices of at least p edges of the $(2p+r-2)K_2$ of Q'. Then $\{v, c', c_2\}$ together with these p edges induce a Q_p, a contradiction.
Case 2. v is non-adjacent to both end-vertices of at least p edges of the $(2p+r-2)K_2$ of Q'. Then $\{v, c_1, c_2\}$ together with these p edges induce a Q_p, a contradiction. □

3.4 Discussion

In this chapter, we have reviewed the augmenting graph method. Our motivation here is to combine the methods applied for P_5-free graphs and ($S_{1,2,k}$,banner)-free graphs to generalize known results.
First, we extended the result of Hertz and Lozin about augmenting chains of ($S_{1,2,l}$, banner$_l$,$K_{1,m}$)-free graphs [98] to augmenting (l,m)-extended-chains and augmenting apples in ($S_{2,2k,l}$,banner$_l$,apple$^l_6$,...,apple$^l_{2k+2}$,$K_{1,m}$)-free graphs. Then the method of finding such augmenting graphs have been extended from the method of Hertz et al. [99] finding augmenting chain in ($S_{1,2,l}$,banner)-free graphs to ($S_{2,l,l}$,banner$_l$,R^1_l, R^2_l, R^3_l, R^4_l, R^5_l)-free graphs.
Second, by extending the method of Lozin and Milanič [125] for ($S_{1,2,5}$,banner)-free graphs, we showed that the problem can be restricted to finding augmenting extended-chains, augmenting apples, and augmenting bipartite-chain in ($S_{2,2,5}$,banner$_2$,domino, M_m)-free graphs by using concepts of redundant sets (in extended sense) and reduction sets. It leads us to generalizations of results about $(P_5, K_{2,m} - e)$-free graphs [22], $(P_5, K_{3,3} - e)$-free graphs [75], and augmenting vertex in P_5-free graphs [75, 76, 140, 141]. It also leads to some interesting results in ($S_{2,2,5}$,banner$_2$,domino,M_m)-free graphs, e.g. Corollaries 3.30 and 3.31.
Note that $S_{1,1,2}$ (fork) and $S_{0,1,3}$ (P_5) are the largest single known forbidden subgraphs, for which the MIS problem is polynomially solvable. For larger $S_{i,j,k}$, even for subclasses, to our knowledge, there are still not many known results except in some subclasses of P_6-free graphs, graphs of bounded maximum degree, planar graphs, and ($S_{1,2,5}$,banner)-free graphs (see [112, 123–125, 142]).
Moreover, by applying a technique, which has been used for P_5-free graphs, for a larger graph class, say $S_{2,2,5}$-free graphs, we believe that it is possible to apply other techniques which were used in P_5-free graphs in $S_{2,2,l}$-free graphs. Let us remark that the P_5-free graph class has been shown to be MIS-easy [115].

4 Augmenting Technique for Some Related Problems

In this chapter, we describe the method of augmenting graphs for some other graph combinatorial problems. In the Section 4.1, some particular problems and a general version of these problems are described. Then we consider two general cases, so-called the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem and the Maximum $\mathcal{F}$-Induced Subgraph problem in Sections 4.2 and 4.3. We summarize some discussion about the issue in Section 4.4.

4.1 Maximum Set Problems

Krishnamoorthy and Deo [110] and then Lewis and Yannakakis [114] have considered the Node-Deletion problem as follows. For a fixed graph property Π, find a minimum subset of vertices which must be deleted (together with incident edges) from a given graph G so that the resulting graph satisfies Π. In [114], the authors showed that if Π is *non-trivial*, i.e. true for infinitely many graphs and false for infinitely many graphs, and *hereditary*, i.e. true for any induced subgraph of a graph satisfying Π, then the problem is NP-hard in general.
For a vertex subset $S \subset V(G)$, S is called a Π-*set* if $G[S]$ satisfies Π, where $G[S]$ is the subgraph of G induced by S. Now, we consider the dual-problem of the node-deletion problem, i.e. the problem asking for a maximum Π-set of G. This problem is called Maximum Π-Set or Π-MS for short.
In this chapter, we consider two special cases of this problem, so-called the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem and the Maximum $\mathcal{F}$-Induced Subgraph problem as follows. Given a connected graph set $\mathcal{F}$ and a graph G, the first problem asks for a maximum induced subgraph H of G such that H contains no graph of $\mathcal{F}$ as a subgraph (for the Strongly Indpendence) or an induced subgraph (for the Independence). This problem was described by Göring et al. [82]. The author also described some bounds of the cardinality of a maximum (strongly) independent subgraph. The second problem asks for a maximum induced subgraph H of G such that every connected component of H is some graph of $\mathcal{F}$.
In this chapter, we consider the following non-trivial Maximum Π-Set problems.
a1. Maximum Independent Set [109] Π: The graph contains no edge.
a2. Maximum k-Independent Set. [64] Π: Every vertex is of degree at most $k-1$.
a3. Maximum k-Path Free Set. Π: The graph contains no path (not neccessarily induced) of k vertices ($k \geq 2$), also called k-path free. This problem is a dual version of the Minimum Vertex k-Path Cover problem [38].
a4. Maximum Forest. Π: The graph contains no cycle. This problem is a dual version of the Minimum Feedback Vertex Cover problem [63].
a5. Maximum Induced Bipartite Subgraph. Π: The graph contains no cycle of

odd length.
a6. Maximum k-Acyclic Set. Π: The graph contains no cycle of length at most k.
a7. Maximum k-Chordal Set. Π: The graph contains no cycle of length larger than k.
a8. Maximum k-Cycle Free Set. Π: The graph contains no cycle of length k ($k \geq 3$), also called k-cycle free. This problem is a dual version of the Minimum Vertex k-Cycle Cover problem
Note that cycles considered in Problems a4., ..., a8. are not neccessarily induced. The eight above problems can be considered as special cases of the Maximum $\mathcal{F}$-(Strongly) Indpendent Subgraph problem. For example, Problem a3. is the Maximum $\mathcal{F}$-Strongly Indpendent problem, where $\mathcal{F} = \{P_k\}$ and Problem a4. is the Maximum $\mathcal{F}$-Independent problem, where $\mathcal{F} = \{C_3, C_4, \ldots\}$. Note that Problem a4. can also be considered as a special case of the Maximum $\mathcal{F}$-Induced Subgraph problem where $\mathcal{F}$ is the class of all trees.
The four following problems are special cases of the Maximum $\mathcal{F}$-Induced Subgraph problem.
b1. Maximum Induced Matching. [45] Π: Every vertex is of degree one.
b2. Maximum k-Regular Induced Subgraph. [46] Π: Every vertex is of degree k.
b3. Maximum k-Regular Induced Bipartite Subgraph. [46] Π: The graph is bipartite and every vertex is of degree k.
b4. Maximum Induced k-Cliques. Π: Every connected component is a k-clique. This problem is a generalization of Problem b1 ($k = 2$).
The $\mathcal{F}$-(strongly) independence property obviously is hereditary. Hence, by the result of Lewis and Yannakakis [114], the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem is NP-hard if $\mathcal{F}$ is non-trivial, i.e. there exist infinitely many graphs not containing any graph of $\mathcal{F}$ as a(n) (induced) subgraph and there exist infinitely many graphs containing some graph of $\mathcal{F}$ as a(n) (induced) subgraph. In particular, Problems a1. - a8. are NP-hard in general (see also [110]).
The Maximum $\mathcal{F}$-Induced Subgraph and in particular, Problems b1. - b4., are not hereditary. For example, given a vertex subset $S \subset V(G)$ for some graph G, and $G[S]$ is a k-regular induced subgraph. Then for $S' \subset S$, it is not neccessary that $G[S']$ is a k-regular induced subgraph. Actually, it is neccessary that S' is $(k+1)$-independent set, i.e. every vertex of $G[S']$ is of degree at most k. Unlike the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem, so far, the NP-hardness of the Maximum $\mathcal{F}$-induced Subgraph problem hasn't been shown in general yet. To our knowledge, the NP-hardness of Problem b1. was shown for bipartite graphs [45] and of Problems b2., b3. [46] in general.
Let $\mathcal{B}$ be the set of all bipartite graphs. Clearly, the Problem a5. is trivial in $\mathcal{B}$. We also know that Problem a1. is polynomially solvable in $\mathcal{B}$ [80]. We say that a property Π is *connected* if for every graph G, G satisfies Π if and only if every connected component of G satisfies Π. It has been shown by Yannakakis [170] that except the MIS problem, the Π-MS problem is still NP-hard for $\mathcal{B}$ if Π is non-trivial, hereditary, and connected in $\mathcal{B}$. Clearly the $\mathcal{F}$-(strongly) independence property is connected in $\mathcal{B}$, i.e. the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem is NP-hard for $\mathcal{B}$ if it is non-trivial in $\mathcal{B}$.
In particular, recall that $\mathcal{S}$ is the class of graphs whose every connected component is

of the form $S_{i,j,k}$. Beside the result of Alekseev [5] about the NP-hardness of the MIS problem (Theorem 1.3), Lozin [121] and Boliac and Lozin [21] also have shown that the Maximum Induced Matching problem and the Maximum Dissociative Set problem are NP-hard in $\mathcal{F}$-free bipartite graphs, where $\mathcal{F}$ is a finite graph set, if $\mathcal{F} \cap \mathcal{S} = \emptyset$. It leads us to the motivation of developing methods for solving the Π-MS problems in a subclass of the $S_{1,2,k}$-free graph class.

4.2 Maximum $\mathcal{F}$-(Strongly) Independent Subgraphs

4.2.1 Augmenting Graph Techniques

We start with the following obvious observation which is used implicitly throughout the chapter.

Lemma 4.1. *Given a graph $G = (V, E)$, a hereditary property Π, and a Π-set $S \subset V$, every subset S' of S satisfies Π.*

Now, we extend the concept of bipartite graphs as follows.

Definition 4.1. *Given a property Π, a graph $G = (V, E)$ is called Π-bipartite if the vertex set $V(G)$ can be partitioned into two subsets B and W such that both $G[B]$ and $G[W]$ satisfy Π.*

Given a graph G and a Π-set S of G, we call vertices in S *white* and the others *black*. Let S' be a subset of S. For a (white) vertex $w \in S'$, we denote $c_{S'}(w)$ as the connected component in $G[S']$ containing w. For a (black) vertex $b \notin S$, we denote $N^e_{S'}(b) := \bigcup_{w \in N_{S'}(b)} c_{S'}(w)$ as the *extended neighborhood* of b in S'. We also denote $N^e_{S'}(B) := \bigcup_{b \in B} N^e_{S'}(b)$ for $B \subset V(G)\backslash S$. For a white vertex subset W of S, we denote $N^e_{S'}(W) := \bigcup_{u \in N_{S'}(W)} c_{S'\backslash W}(u)$ for $W \subset S$ as the extended neighborhood of W in S. If $W = \{w\}$, then we write $N^e_{S'}(w)$ for short. Next, we present an extension of the concept of augmenting graphs originally used for the MIS problem.

Definition 4.2. *Given a graph $G = (V, E)$, a hereditary, connected property Π, and a Π-set S, a Π-bipartitie graph $H = (B, W \cup U, E(H))$, where $U := N^e_{S\backslash W}(B)$, is called augmenting for S (or S has an H-augmentation) if*

1. *$W \subset S$, $B \subset V\backslash S$;*
2. *$|B| > |W|$; and*
3. *$B \cup U$ is a Π-set.*

In the case that the graph G is already defined, for convenience, we also denote H as $H = (B, W, U)$. Now, similarly as observed for the MIS problem, we have the following key theorem.

Theorem 4.2. *Given a graph $G = (V, E)$ and a hereditary connected property Π, a Π-set S is maximum if and only if there exists no augmenting graph for S.*

Proof. Suppose that $H = (B, W, U)$ is an augmenting graph for S. Consider the set $S' = (S\backslash W) \cup B$. Then clearly, $|S'| > |S|$ by 1. and 2. of Definition 4.2. Moreover, 3. of Definition 4.2, the definition of U, and the connectedness of Π ensure that S' satisfies Π.
For the converse direction, suppose that there exists a Π-set S' such that $|S'| > |S|$, we show that $H = (B, W, U)$, where $B = S'\backslash S$, $W = S\backslash S'$, and $U = N^e_{S\backslash W}(B)$, is an augmenting graph for S. Indeed, H is Π-bipartite, $|B| > |W|$, and $W \subset S$, $B \subset V\backslash S$. Since $U \subset S\backslash W = S \cap S' \subset S'$ and $B \subset S'$, $B \cup U \subset S'$ is a Π-set. □

Like for the MIS problem, Theorem 4.2 suggests the following general approach to find a maximum Π-set for a hereditary connected property Π in a graph G. Start with some Π-set S (may be the empty set) in G and, as long as S admits an augmenting graph H, apply H-augmentation to S. Clearly, the problem of finding augmenting graphs is polynomially equivalent to the Π-MS problem and hence, is NP-hard in general. Moreover, it's also enough for us to restrict our consideration on minimal augmenting graphs only. We have the following observation about minimal augmenting graphs.

Lemma 4.3. *Given a graph G, a hereditary, connected graph property Π, a Π-set S, and an augmenting graph for S $H = (B, W, U)$, if H is minimal, then*

1. *H is connected and*
2. *$|B| = |W| + 1$.*

Proof. For contradiction, suppose that H is not connected. Then there exists a connected component H' of H such that $|B \cap H'| > |W \cap H'|$. Let $B' = B \cap H'$, $U' = U \cap H'$, and $W' = W \cap H'$. We show that $H' = (B', U' \cup W', E(H'))$ is an augmenting graph for S which leads to a contradiction.
Indeed, since H is a Π-bipartite graph, H' is a Π-bipartite graph. Moreover, $W' \subset W \subset S$ and $B' \subset B \subset V(G)\backslash S$. By the connectivity of H' and the definitions of U and U', $N^e_{S\backslash W'}(B') = N^e_{S\backslash W}(B') = U'$. Obviously, $|B'| > |W'|$. Finally, $B' \cup U' \subset B \cup U$ leads to $B' \cup U'$ is a Π-set.
For contradiction, suppose that $|B| > |W| + 1$. Let b be an arbitrary vertex of B, $B' = B\backslash\{b\}$, $W' = W$, and $U' = N^e_{S\backslash W'}(B') \subset N^e_{S\backslash W}(B) = U$. Then clearly, $H' = (B', U' \cup W', E(H'))$ is an augmenting graph for S, a contradiction. □

(P1) Find a complete list of (minimal) augmenting graphs in the class under consideration.

(P2) Develop polynomial time algorithms for detecting all (minimal) augmenting graphs in the class.

4.2.2 Minimal Augmenting Graph in ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free Graphs

Let G be a connected graph. In this chapter, we say that G is an extended-chain if G consists of a chain P and the neighborhoods of the two end-vertices u, v such that $N_2(u)\backslash P = N_2(v)\backslash P = \emptyset$ and $N[u]$, $N[v]$ induce two stars. The two vertices u, v are also called end-vertices of G.

Lemma 4.4. *Given a connected ($S_{1,2,l}$,banner$_l$, Z_l)-free graph G, $\Delta(G) \leq p$ for some positive integer p if and only if at least one of the following statements is true.*

1. *G is a cycle.*
2. *G is an extended-chain whose the two end-vertices are of degree at most p.*
3. *There is a positive integer q such that $|V(G)| \leq q$.*

Proof. First, $\Delta(G) \leq |V(G)| - 1$. Moreover, $\Delta(G) = 2$ in the case that G is a cycle and $\Delta(G) \leq p$ in the case that G is an extended-chain whose two end-vertices are of degree at most p.
Now, assume that $3 \leq \Delta(G) \leq p$. Let a be a vertex of degree at least three in G. It is enough to show that every other vertex of G is of distance at most $l+2$ from a or $|N_2(a)| = 1$ and $N(a)$ is independent.
Let V_i be the set of vertices of distance i from a. Assume that there exists an induced path $(a, a_1, a_2, \ldots a_{l+3})$, $a_i \in V_i$. Let $b_1, b_2, \ldots \in V_1 \backslash \{a_1\}$. For b_i, in the case that $N_{V_2}(b_i) \backslash \{a_2\} \neq \emptyset$, let c_i be a vertex of that set. Clearly, a_i has no neighbor in V_1 nor V_2 for $i \geq 4$ and a_3 has no neighbor in V_1.
If $a_3 \sim c_1$, then $\{a_1, a_2, c_1, a_3, a_4, \ldots, a_{l+3}\}$ induces an $S_{1,2,l}$, a banner$_l$, or a Z_l depending on the adjacency between c_1 and $\{a_1, a_2\}$, a contradiction. Hence, a_3 has only one neighbor, say a_2, in V_2.
If $a_2 \sim c_1$, then $\{b_1, c_1, a_1, a_2, a_3, \ldots, a_{l+2}\}$ induces an $S_{1,2,l}$, a banner$_l$, or a Z_l depending on the adjacency between $\{a_1, a_2\}$ and $\{b_1, c_1\}$, a contradiction. Hence, a_2 has no neighbor in V_2.
If $a_2 \sim b_1$, then $\{a, a_1, b_1, a_2, a_3, \ldots, a_{l+2}\}$ induces a Z_l or a banner$_l$ depending on $a_1 \sim b_1$ or not, a contradiction. Hence, a_2 has only one neighbor, say a_1, in V_1.
If $a_1 \sim b_1$, then $\{a, b_1, a_1, a_2, \ldots, a_{l+1}\}$ induces a Z_l, a contradiction. Hence, a_1 has no neighbor in V_1.
If $a_1 \sim c_1$, then $\{c_1, b_1, a, a_1, a_2, \ldots, a_{l+1}\}$ induces a banner$_l$, a contradiction. Hence, a_1 has only one neighbor, say a_2, in V_2.
If $b_1 \sim b_2$, then $\{b_1, b_2, a, a_1, \ldots, a_l\}$ induces a Z_l, a contradiction. Hence, $N(a)$ is independent.
Now, $\{b_2, c_1, b_1, a, a_1, \ldots, a_l\}$ induces an $S_{1,2,l}$ or a banner$_l$ depending on $c_1 \sim b_2$ or not, a contradiction. Hence, $V_2 = \{a_2\}$. □

Lemma 4.4 implies that a connected ($S_{1,2,l}$,banner$_l$, Z_l)-free graph G is of bounded maximum degree if and only if G is a cycle, or is an extended-chain, or belongs to some finite graph set.
Remark. If we restrict ourselves to bipartite graphs, then obviously, we need not to forbid Z_l in the above lemma. However, if G is a $K_{1,m}$-free bipartite graph for some m, then $\Delta(G) \leq m$. So, by the above lemma, there are only finitely many connected ($S_{1,2,l}$,banner$_l$, $K_{1,m}$)-free bipartite graphs different from cycle and extended-chain. Moreover, Problems a1. - a8. and b1. - b4. are obviously polynomially solvable for cycles and extended-chains, and of course, trivial for a finite graph set. That means the problems are polynomially solvable for the ($S_{1,2,l}$,banner$_l$, $K_{1,m}$)-free bipartite graphs.
Now, we describe graph properties, under which we can use the previous result to characterize augmenting graphs in ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs. We say that

a property Π is *connected-degree-bounded* by a positive number p if $\Delta(G) \leq p$ for every connected graph G satisfying Π. The following observation describes structural properties of augmenting graphs of a Π-maximum set problem, where Π is connected-degree-bounded.

Lemma 4.5. *Given a property Π connected-degree-bounded by p, there are only finitely many connected augmenting ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs which are neither a cycle nor an extended-chain whose two end-vertices are of degree at most $r := \min(m-1, p)$.*

Proof. Let $H = (B, W, U)$ be a connected augmenting graph. By Lemma 4.4, it is enough to show that for an arbitrary black vertex $b \in B$ (and similarly for the white vertices), $\deg(b) \leq \Delta$ for some positive integer $\Delta = \Delta(m, l, p)$.
Since H is Π-bipartite, the connected component in B containing b is either an extended-chain or a cycle or contains at most q vertices for some positive integer $q = q(p, l)$, i.e. $\deg_B(b) \leq \max(\min(p, m-1), q-1)$.
Now, consider a connected component W' of the white part of H. If W' is a cycle or an extended-chain, then b has at most $2m-1$ neighbors in W', otherwise we have an induced $K_{1,m}$, a contradiction. If W' is neither a cycle nor an extended-chain, then $|V(W')| \leq q$ by Lemma 4.4, i.e. b has at most q neighbors in W'.
Note that b has neighbors from at most $m-1$ connected components of the white part of H, otherwise we have an induced $K_{1,m}$, a contradiction. Hence, $\deg_W(b) \leq (m-1) \cdot \max(2m-1, q)$.
All above considerations give us the statement of the lemma. □

Clearly, the graph properties of Problems a1. and a2. are connected-degree-bounded by the definitions. The following observations are for Problems a3. - a8.

Lemma 4.6. *There is a function $f : \mathbb{N}^2 \to \mathbb{N}^*$ such that for an arbitrary connected k-path free and $K_{1,m}$-free graph G, $|V(G)| \leq f(k, m)$.*

Proof. For $k = 2$ or 3, let $f(k, m) := k-1$. For larger k, we define $f(k, m)$ by induction. Consider an arbitrary vertex $v \in V(G)$, since G is k-path free, every connected component C of $N[v]$ is $(k-1)$-path free, i.e. $|V(C)| \leq f(k-1, m)$. Because G is $K_{1,m}$-free, $N(v)$ has at most $m-1$ connected components, i.e. $\deg(v) \leq (m-1) \cdot f(k-1, m)$.
Hence, $\Delta(G) \leq (m-1) \cdot f(k-1, m)$. Again by the k-path freeness, in particular, the P_k-freeness of G,

$$|V(G)| \leq \frac{1 - ((m-1) \cdot f(k-1, m))^{k-1}}{1 - (m-1) \cdot f(k-1, m)} =: f(k, m).$$

□

This result and Lemma 4.4 ensure the connected-degree-boundedness of k-path freeness property.

Lemma 4.7. *There is a function $h : \mathbb{N}^2 \to \mathbb{N}^*$, such that for an arbitrary connected k-cycle free and $K_{1,m}$-free graph G, $\Delta(G) \leq h(k, m)$ $(k \geq 3)$.*

Proof. Consider an arbitrary vertex $v \in V(G)$. Since G is k-cycle free, every connected component C of $N(v)$ is $(k-1)$-path free, i.e. $|V(C)| \leq f$ for some integer $f := f(k-1, m)$ by Lemma 4.6. Since G is $K_{1,m}$-free, $N(v)$ has at most $m-1$ connected components, i.e. $d(v) \leq (m-1) \cdot f(k-1, m) =: h(k, m)$. □

Now, we describe some properties of augmenting extended-chains and augmenting cycles. First, we extend the concept of alternating chain of Minty [137] as follows. An *alternating chain* is an induced path connecting single black vertices separated by segments of white vertex/vertices. The following observation describe minimal augmenting graphs which are extended-chain or cycle in more detail.

Lemma 4.8. *Given a graph $G = (V, E)$, a Π set S, and a minimal augmenting graph $H = (B, W, U)$, the following statements are true.*

1. *If H is an extended-chain (or a cycle, respectively), then the path part of H (or H, respectively) contains no segment of white vertex/vertices all belonging to U and lying between two white vertices belonging to W.*

2. *If H is an extended-chain, then H contains no edge whose both end-vertices are black or both end-vertices belong to W. The path part also contains no segment of white vertices all belonging to U and lying between two black vertex.*

3. *If H is a cycle, then H contains either exactly one edge whose both end-vertices are black or exactly one segment of white vertices all belonging to U and lying between two black vertices. More precisely, these two black vertices are connected by an alternating chain.*

Proof. 1. is obvious by the definition of U.
Suppose that H is an extended-chain containing an edge whose both end-vertices are black, or both end-vertices belong to W. Then we can divide H at that edge into two parts such that there exists (at least) one part being an augmenting graph, a contradiction. Similarly, suppose that the path part of H contains a segment of white vertex/vertices all belonging to U and lying between two black vertices. Divide H into three parts L, R, and M, where M is that segment, L and R are the two connected components of $H - M$. Then at least one of $L \cup M$ or $R \cup M$ is an augmenting graph, a contradiction.
Suppose that H is a cycle and contains two edges whose both end-vertices are black or both belong to W or segment(s) of white vertex/vertices all belonging to U and lying between two black vertices. Then we also can divide H, at those edge(s)/segment(s), into two parts, such that there exists (at least) one part being an augmenting graph, a contradiction. Besides, H must contain either one edge whose both end-vertices are black or one segment of white vertices all belonging to U and lying between two black vertices to ensure the condition $|B| > |W|$. □

We can find augmenting graphs belonging to some finite set in polynomial time, say by an exhaustive search. In the next subsection, we describe methods of finding augmenting extended-chains and augmenting cycles.

4.2.3 Finding Augmenting Extended-Chains and Augmenting Cycles

Given two integers l and m, in this section, we describe method finding augmenting extended-chains and augmenting cycles in ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs. A $K_{1,m}$-free augmenting extended-chain whose the path part is of length at most $l + 1$ contains at most $2m + l - 2$ vertices and hence, can be found in polynomial time. So, from now on, we restrict ourselves on the problem of finding augmenting extended-chains whose the path part is of length at least $l+2$.

Step 1 looks for candidates which are a pair (L, R), where L is an induced subgraph containing an induced path of length l and an induced star K_{1,m_1} $(2 \le m_1 < m)$ such that the center vertex of the star is an end-vertex of the path and R is an induced star K_{1,m_2} $(1 \le m_2 < m-1)$ such that $V(L)$ and $V(R)$ are disjoint and no vertex in L is adjacent to a vertex in R. Such candidates can be enumerated in polynomial time. Moreover, such candidate must exist, otherwise we have no augmenting extended-chain of length at least $l+2$. Assume that the path part of L is $(a_1, a_2, \ldots, a_l)$, where a_1 is the center vertex of the star part. Let a be the center vertex of R. Now, we try to find an alternating chain $a_1, a_2, \ldots, a_l, a_{l+1}, \ldots, a_p = a$, connecting a_1 and a.

In **Step 2**, we remove from G all neighbors (together with incident edges) of L or R except those adjacent to only a_l or a because these vertices cannot appear in the desired alternating chain.
Now, we try to extend L, from a_l to a_{l+1} and so on until we meet a or conclude that the process cannot succeed. Assume that we have extended to $a_{l'} \neq a$ for some $l' \ge l$ and every vertex a_i for $2 \le i \le l'-1$ is of degree two. We show that $a_{l'}$ is of degree at most two or we can conclude that the process fails, i.e. we cannot find an extended-chain containing L and R. Indeed, let b, c be two neighbors of $a_{l'}$ different from $a_{l'-1}$. If $b \sim c$, then $\{b, c, a_{l'}, a_{l'-1}, \ldots, a_{l'-l}\}$ induces a Z_l, a contradiction. Hence, $N[a_{l'}]$ induces a $K_{1,m'}$ for some $m' \le m-1$. Now, if b has another neighbor, say b' different from $a_{l'}$, then $\{c, b', b, a_{l'}, a_{l'-1}, \ldots, a_{l'-l}\}$ induces a banner$_l$ or an $S_{1,2,l}$ depending on $b' \sim c$ or not, a contradiction. This argument also implies that $a \notin N(a_{l'})$ and hence, the process fails. It also implies that for a candidate (L, R), there exists at most one alternating chain connecting them.

Hence, **Step 3** finding a desired alternating chain connecting a_l and a or deciding that such a path does not exist can be performed in linear time. Moreover, enumerating all candidate pairs (Step 1) and checking and removal vertices (Step 2) can be performed in polynomial time.
Similarly, we also can find potential augmenting cycles as follows. First, we restrict ourselves to look for only augmenting cycles of length at least $l+5$. So, we start with candidates which are chains of length $l+4$ and contain at least one black end-vertex. Then similarly to arguments for augmenting extended-chain, we can find an alternating chain (or show that such a chain does not exist) connecting the two end-vertices of a given candidate in a polynomial time.
Note that, so far, we only find a potential augmenting extended-chain or a potential augmenting cycle. We also have to assign white vertices of that extended-chain or cycle to the sets U and W to have a valid augmenting extended-chain or a valid augmenting cycle. This problem depends on Property Π.
Let H be a potential minimal augmenting extended-chain or a potential augmenting cycle. Then a valid asignment for white vertices of H must satisfy the following conditions.

1. Every white vertex of H whose a white neighbor does not belong to H must belong to W (by the definition of U).

2. Assume that H is a cycle. Recall that H is generated from a candidate which is a chain of length $l + 4$ whose an end-vertex is black. Then every white vertex from this black vertex along the candidate to the next black vertex of the cycle is assigned to U. This condition ensures Condition 3. of Lemma 4.8.

3. Assume that H is an extended-chain. Let $P = (a_1, a_2, \ldots, a_p)$ be the path part of H. If P contains no black vertex, then every white vertex of P is assigned to U. Otherwise, we can break H at a vertex of P belonging to W and at least one part is an augmenting graph, a contradiction. Moreover, the number of white vertices of H assigned to W is less than the number of black vertices of H exactly one. Now, let i and j be the minimum and maximum integers, respectively, such that a_i and a_j are black, respectively. Let $H' := H - \{a_i, \ldots, a_j\}$. Then white vertices of H' are assigned to W and U such that the number of white vertices assigned to W equals to the number of black vertices of H' by Conditions 2. and 3. of Lemma 4.8.

4. Except the path in Condition 2., on the path segment of white vertex/vertices between two black vertices, there is exactly one white vertex belonging to W.

5. In general, for the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem, we also have to check if $H[B \cup U]$ contains an (induced) forbidden subgraph. We list out some examples as follows.

   a) For Problem a2., if H is an extended-chain whose an end-vertex a is black or has been assigned to U, then a has at most $k - 1$ neighbors which are black or assigned to U.

   b) For Problem a3., H contains no path of length k whose every vertex is black or assigned to U.

   c) For Problems a4. - a8., if H is a cycle containing only one black vertex, i.e. every white vertex is assigned to U, then the length of H shouldn't violate Property Π.

Clearly, we can do an assignment satisfying the above conditions or conclude that it is impossible in polynomial time. In summary, we have the following observation.

Theorem 4.9. *Given two postive integers l, m, Problems a2. - a8. are polynomially solvable in ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs.*

4.3 Maximum $\mathcal{F}$-Induced Subgraph Problem

For convenience, unless some confusions may arise, we use the same notations for an $\mathcal{F}$-induced subgraph and its vertex set. The following obvious observation is used implicitly through this section.

Lemma 4.10. *Given a connected graph set $\mathcal{F}$, the following statements are true.*

1. *If G is an $\mathcal{F}$-induced graph and H be a collection of some connected components of G, then H is an $\mathcal{F}$-induced subgraph.*

2. *If G_1 and G_2 are $\mathcal{F}$-induced graphs, then the disjoint union of G_1 and G_2 is $\mathcal{F}$-induced.*

4.3.1 An NP-hard Result

Theorem 4.11. *Given a connected graph set $\mathcal{F}$, if $\Delta(\mathcal{F})$ is finite, then the Maximum $\mathcal{F}$-induced Subgraph problem is NP-hard. In particular, Problem b4. is NP-hard.*

Proof. The proof follows the idea of the proof of Theorem 2.1 in [46].
If $\delta(\mathcal{F}) = 0$, i.e. $\mathcal{F}$ contains a single vertex graph, then clearly, the problem is NP-hard because the Maximum Independent Set problem is NP-hard in general. For the case $\delta(\mathcal{F}) \geq 1$, we use a reduction from the MIS problem.
Denote $\Delta := \Delta(\mathcal{F})$. Let G be any graph and assume that $V(G) = \{v_1, \dots, v_n\}$. Let $F \in \mathcal{F}$ be an arbitrary graph and H be the union of p disjoint copies of F for some large enough p such that $t := |V(H)| \geq n\Delta$. We construct an auxiliary graph $G(H)$ by replacing each vertex of G with a copy of H. More formally, $G(H)$ is obtained from the union of n disjoint copies of H, denoted $H_1, \dots, H_n$, by connecting every vertex of H_i to every vertex of H_j whenever $v_i \sim v_j$ in G.
With some abuse of terminology, we say that H_i is adjacent to H_j, denoted $H_i \sim H_j$, in $G(H)$ if $v_i \sim v_j$ in G. We prove that G has an independent set of size at least α if and only if $G(H)$ has an $\mathcal{F}$-induced subgraph of at least $t\alpha$ vertices.
Let I be an independent set of G such that $|I| \geq \alpha$. Replacing each vertex in I by its corresponding copy of H results in an $\mathcal{F}$-induced subgraph $G(H)$ with at least $t\alpha$ vertices.
Conversely, suppose that Q is an maximum $\mathcal{F}$-induced subgraph of $G(H)$ with at least $t\alpha$ vertices. Let C be a connected component of Q. We claim that the vertices of C cannot belong to more than one copy of the graph H in $G(H)$. Indeed, suppose that C intersects more than one copy of H. Then, due to the connectivity of C, for each such copy H_i, there must exists another such copy which is adjacent to H_i. This implies that H_i contains at most Δ vertices of C, otherwise any vertex of C in an adjacent copy of H would have degree more than Δ in C, a contradiction. Therefore, C has at most $n\Delta$ vertices. Now, consider a copy H_i containing some vertex of C and a vertex $v \in V(Q - C - H_i)$. Then v is not adjacent to any vertex of H_i, otherwise v is adjacent to every vertex of H_i which leads to v is adjacent to some vertex of C, i.e. $v \in V(C)$ by the connectivity of C, a contradiction. That means if we replace in Q the connected component C by any copy H_i containing some vertex of C, we obtain an $\mathcal{F}$-induced subgraph of $G(H)$, which is strictly larger than Q. This contradiction shows that every connected component of Q intersects exactly one copy of the graph H_i in $G(H)$.
Moreover, the maximality of Q implies that each of its components coincides with a connected component of the copy of H that it intersects. Besides, if Q contains a connected component C of some copy H_i of H, then by the same argument, for every vertex $v \in V(Q - H_i)$, v is not adjacent to any vertex of H_i. That means, again, by the maximality of Q, if Q contains a connected component C of some coppy H_i of H, then Q contains H_i. Clearly, the vertices of G corresponding to H_i's of Q form an independent set and this set contains at least α vertices as $|Q| \geq t\alpha$. This completes the reduction from the MIS problem to the problem of finding a maximum $\mathcal{F}$-induced subgraph. This reduction is polynomial in the size n of the input graph whenever the size of the graph H is bounded by a polynomial in n. $\square$

4.3.2 Augmenting Graphs

In this subsection, we develop the augmenting technique to solve the Maximum $\mathcal{F}$-induced Subgraph problem. We use the notation about black and white vertices as well as extended-neighborhood as in Section 4.2. First, we start with the description of augmenting graphs.

Definition 4.3. *Given a connect graph set $\mathcal{F}$, a graph $G = (V, E)$, and an $\mathcal{F}$-induced subgraph S, an induced subgraph H of G will be called an augmenting graph for S if $V(H)$ can be partitioned as $V(H) = B \cup U \cup W$ such that*

1. *$B \subset V(G)\backslash S$, $W \subset S$, and $U = N^e_{S\backslash W}(B) \cup N^e_{S\backslash W}(W)$;*
2. *$B \cup U$ is an $\mathcal{F}$-induced subgraph; and*
3. *$|B| > |W|$.*

In the case that graph G is already defined, for convenience, we also denote H as $H = (B, U, W)$. Like in the previous section, we also have the following key theorem.

Theorem 4.12. *Given a connected graph set $\mathcal{F}$, a graph G, and an $\mathcal{F}$-induced subgraph S, S is maximum if and only if there exists no augmenting graph for S.*

Proof. Suppose that $H = (B, U, W)$ is an augmenting graph for S. We show that $S' = (S\backslash W) \cup B$ is an $\mathcal{F}$-induced subgraph. Then $|B| > |W|$ leads to $|S'| > |S|$. Indeed, by the definition of U, $W \cup U$ is a collection of some connected components of S, i.e. $W \cup U$ and $S\backslash(W \cup U)$ is an $\mathcal{F}$-induced subgraph. Since $N^e_{S\backslash W}(B) \subset U$, $N_{S\backslash(W\cup U)}(B \cup U) \subset N_S(B \cup U) = \emptyset$. Hence, $B \cup U$ and $S\backslash(W \cup U)$ are two $\mathcal{F}$-induced subgraphs such that there exists no edge connecting them, i.e. S' is an $\mathcal{F}$-induced subgraph.
Now, for the converse direction. Let S' be an $\mathcal{F}$-induced subgraph such that $|S'| > |S|$. Let $B := S'\backslash S$, $W := S\backslash S'$, and $U := N^e_{S\cap S'}(B) \cup N^e_{S\cap S'}(W)$. We have $|B| = |S'\backslash S| > |S\backslash S'| = |W|$. Moreover, $B \subset V(G)\backslash S$ and $W \subset S$ by the definition. By the definition of U, $B \cup U$ is a collection of some connected components of S', i.e. $B \cup U$ is an $\mathcal{F}$-induced subgraph. It leads to that $H = (B \cup U \cup W, E(H))$ is an augmenting graph. □

Clearly, we can restrict ourselves to minimal (inculsion sense) augmenting graph only. We have the following observation about the connectivity of minimal augmenting graphs.

Lemma 4.13. *Given a connected graph set $\mathcal{F}$, a graph G, an $\mathcal{F}$-induced subgraph S, and an augmenting graph for S, $H = (B, U, W)$, if H is minimal, then H is connected.*

Proof. Suppose that H is not connected. Then there exists a connected component H' of H such that $|B \cap H'| > |W \cap H'|$. Let $B' := H' \cap B$, $W' := H' \cap W$, and $U' := U \cap H'$. We show that $H' := (B', U', W')$ is an augmenting graph for S, which leads to a contradiction.
Indeed, by the connectivity of H', $U' = N^e_{S\backslash W}(B') \cup N^e_{S\backslash W}(W') = N^e_{S\backslash W'}(B') \cup N^e_{S\backslash W'}(W')$. Moreover, $B' \subset B \subset V(G)\backslash S$ and $W' \subset W \subset S$. Again, by the connectivity of H, $B' \cup U'$ is a collection of some connected components of $B \cup U$, i.e. an $\mathcal{F}$-induced subgraph. □

Note that, the $\mathcal{F}$-induced property is connected by the definition. Besides, if $\Delta(\mathcal{F})$ is finite, i.e. the $\mathcal{F}$-induced property is connected-bounded-degree by some integer Δ, then similarly to Lemma 4.5, there exist only finitely many connected ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free augmenting graphs which are neither an augmenting extended-chain nor a cycle. Moreover, we have the following observation.

Lemma 4.14. *Given a connected graph set $\mathcal{F}$, if $\delta(\mathcal{F}) \geq 2$, then there exists no augmenting extended-chain. Moreover, if H is an augmenting cycle, then every vertex of H is black.*

Proof. Let G be a graph whose S is an $\mathcal{F}$-induced subgraph and $H = (B, U, W)$ is an augmenting graph. Since $U = N^e_{S\setminus W}(B) \cup N^e_{S\setminus W}(W)$, $U \cup W$ is a collection of some connected components of S, i.e. an $\mathcal{F}$-induced subgraph. Moreover, $B \cup U$ is also an $\mathcal{F}$-induced subgraph. That means, $\delta(H) \geq 2$, i.e. H is not an extended-chain.
Now, assume that H is a cycle. Since $d_{U\cup B}(b), d_{U\cup W}(w), d_{U\cup B}(u), d_{U\cup W}(u) \geq 2$ for every $b \in B$, $w \in W$, and $u \in U$ and $|B| > |W|$, every vertex of H is black. □

Hence, Theorem 4.12, Lemma 4.13, and the above lemma lead us to the following observation.

Theorem 4.15. *Given two integers l, m and a connected graph set $\mathcal{F}$, such that $\delta(\mathcal{F}) \geq 2$ and $\Delta(\mathcal{F})$ is finite, for ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs, the Maximum $\mathcal{F}$-induced Subgraph problem is polynomially reducible to the problem of detecting cycles belonging to $\mathcal{F}$. In particular, Problems b2. and b3., the case $k \geq 3$, and Problem b4., the case $k \geq 4$ are polynomially solvable for ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs.*

Clearly, we can detect a cycle belonging to a finite set in polynomial time. Now, assume that there exist only finitely many cycles not belonging to $\mathcal{F}$, i.e. there exists some integer k such that $C_p \in \mathcal{F}$ for every $p \geq k$. Then the following procedure detects these cycles in some graph G in polynomial time. First, we start by finding an induced copy of P_k and let u, v be the two end-vertices. If such copy does not exist, then there exists no induced cycle of length at least k. We delete from G all vertices of $V(P_k)\setminus\{u, v\}$ and all their neighbors, except u and v and find in the resulting graph the shortest path connecting u, v. It is not difficult to see that this procedure and enumerating of all candidates P_k can be implemented in polynomial time. It leads us to the following observation.

Corollary 4.16. *Given two integers l, m and a connected graph set $\mathcal{F}$, such that $\delta(\mathcal{F}) \geq 2$ and $\Delta(\mathcal{F})$ is finite, and $\mathcal{F}$ contains only finitely many cycles or does not contain only finitely many cycles, for ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs, the Maximum $\mathcal{F}$-induced Subgraph problem is polynomially solvable. In particular, Problems b2., the case $k \geq 2$, and b4., the case $k \geq 3$, are polynomially solvable for ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs.*

4.3.3 Maximum Induced Matching Problem

In this subsection, we focus on the Maximum Induced Matching problem, the special case of Problems b2., b3. ($k = 1$), and b4. ($k = 2$), i.e. $\mathcal{F} = \{P_2\}$ and $\delta(\mathcal{F}) = \Delta(\mathcal{F}) = 1$. Note that if S is an induced matching, then every vertex subset of S is a dissociative set, i.e. consists of an induced matching M and an independent set

I such that there exists no edge connecting them. First, recall Definition 4.3, let $H = (B, U, W)$ be an augmenting graph. Then $H = (B, U \cup W, E(H))$ is a bipartite dissociative graph. Moreover, $W \cup U$ is an induced matching. Let B_I, B_M and W_I, W_M are independent sets, induced matchings of B and W, respectively. Since $U \cup B$ is an induced matching, $B_I \cup U$ is an induced matching. Since $U \cup W$ is an induced matching, $U \cup W_I$ is an induced matching. Hence, $|B_I| = |U| = |W_I|$, i.e. Condition 3. of Definition 4.3 can be substituted by $|B_M| > |W_M|$. For convenience, from now on, we also write $H = (B_I, B_M, U, W_I, W_M)$. The following observation is obvious based on the definition of augmenting graph and is used implicitly through this subsection.

Lemma 4.17. *Let $H = (B_I, B_M, U, W_I, W_M)$ be an augmenting graph. Then the following statements are true.*

- *Each white vertex in H has exactly one white neighbor.*
- *Each (white) vertex in U has exactly one black neighbor.*
- *Each black vertex in H has at most one black neighbor.*
- *If a black vertex b in H has no black neighbor, then it has exactly one white neighbor in U.*
- *If a black vertex b in H has an black neighbor, then it has no neighbor in U.*

Again, we can restrict ourselves in considering only minimal augmenting graphs. Beside the connectedness, we have the following observation about minimal augmenting graphs.

Lemma 4.18. *Given a graph G, an induced matching S, and an augmenting graph for S, $H = (B_I, B_M, U, W_I, W_M)$, if H is minimal, then $|B_M| := |W_M| + 2$.*

Proof. Suppose that $|B_M| > |W_M| + 2$. Let b_1b_2 be an arbitrary edge of B_M and $B'_M = B \backslash \{b_1, b_2\}$. Then obviously, $H' = (B_I \cup B'_M, U \cup W_I \cup W_M, E(H'))$ is an augmenting graph for S, a contradiction. □

Lemma 4.19. *Let $H = (B_I, B_M, U, W_I, W_M)$ be an augmenting extended-chain. Then H is an augmenting chain.*

Proof. Let a be an end-vertex of H. We show that a is of degree two. For contradiction, suppose that a_1 is a neighbor of a in the parth part of H and b_1, b_2 are two others neighbors of a. Note that H does not contain a P_3 whose vertices are of the same color. If a is black, then at least one vertex among b_1, b_2, say b_1, is white. But now, b_1 is a white vertex having no white neighbor, a contradiction. If a is white, then among a_1, b_1, b_2, there are at least two black vertices. Without loss of generality, assume that b_1 is black. Since b_1 has no black neighbor, $a \in U$. But now, a has at least two white neighbors or at least two black neighbors, a contradiction. Hence, every augmenting extended-chain is an augmenting chain. □

From the definition of augmenting graphs, if H is an augmenting chain or an augmenting cycle, then H contains edges whose both vertices are white and these edges are separated by single black vertices or single edges whose both vertices are black. In this subsection, we call chains satisfying this property as *alternating chains*.

Lemma 4.20. *Let $H = (B_I, B_M, U, W_I, W_M)$ be an augmenting chain or an augmenting cycle. Then the following statements are true.*

1. *From an edge, whose both (black) vertices belong to B_M, go along H following its neighbor(s), we meet at least one edge whose both vertices belong to W_M before an edge whose both vertices belonging to B_M.*
2. *From an edge, whose both (white) vertices belong to W_M, go along H following its neighbor(s), we meet at least one edge whose both vertices belong to B_M before an edge whose both vertices belonging to W_M.*
3. *From a (black) vertex belonging to B_I, go along H following the neighbor belonging to U, we meet at least one edge whose both vertices belonging to B_M before an edge whose both vertices belonging to W_M.*
4. *From a (white) vertex belonging to W_I, go along H following its (white) neighbor belonging to U, we meet at least one edge whose both vertices belongs to W_M before an edge whose both vertices belonging to B_M.*

Proof. Consider a vertex $b \in B_M$, its white neighbor, say a, belongs to W_I or W_M. Assume that $a \in W_I$. The white neighbor of a, say a', belongs to U. The black neighbor of a', say b', belongs to B_I. If the other white neighbor of b', say a'', belongs to W_I, then obviously, the white neighbor of a'' belongs to U and so on. Hence, we have 1.
Consider a white vertex $a \in W_M$, asume that its black neighbor, say b, belongs to B_I. Then the other white neighbor of b, say a', belongs to U. The white neighbor of a', say a'' belongs to W_I. If the black neighbor of a'', say b', belongs to B_I, then the other white neighbor of b' belongs to U and so on. Hence, we have 2.
Consider a black vertex $b \in B_I$, both neighbors of b are white and exactly one neighbor, say a, belongs to U. The white neighbor of a, say a' belongs to W_I. If the black neighbor of a', say b', belongs to B_I, then the other white neighbor of b' (different from a') belongs to U and so on. Hence, we have 3.
Consider a white vertex $a \in W_I$, its white neighbor, say a', belongs to U and the black neighbor of a', say b, belongs to B_I. Assume that the other white neighbor of b, say a'' belongs to W_I. Then the white neighbor of a'', say a''', belongs to U, and the black neighbor of a''' belongs to B_I and so on. Hence, we have 4. □

Lemma 4.21. *There exists no augmenting cycle.*

Proof. Suppose that $H = (B_I, B_M, U, W_I, W_M)$ is an augmenting cycle. Then by 1. and 2. of the above lemma, H consists of alternating edges whose both end-vertices belong to B_M and W_M (separated by vertices belonging to B_I, W_I, or U). Then we have a contradiction with $|B_M| > |W_M|$. □

Moreover, if H is an augmenting chain, then to ensure $|B_M| > |W_M|$, we have the following observation.

Lemma 4.22. *Let H be an augmenting chain. Then the two end-vertices of H are black. Moreover, H contains at least one edge whose both vertices are black.*

From the above subsection, the problem of finding maximum induced matching in ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs is polynomially equivalent to the problem of finding augmenting chains. Like in Subsection 4.2.3, we start with generating candidates (L, R) where R is a single black vertex b and L is a chain $(x_0, x_1, \ldots, x_l)$, where x_0 is black. Then we delete all vertices adjacent with $x_0, x_1, \ldots, x_{l-1}$. Next, we try to extend from x_l to b, which can be done (like in Subsection 4.2.3) in linear time. Note that in the extending process, we require that the found potential augmenting chain is an alternating chain and contains at least one edge whose both vertices are black. Assume that we found a potential augmenting chain H. Note that a black vertex belongs to B_I if it has no black neighbor and to B_M otherwise. Algorithm 7 assigns white vertices to U, W_I, or W_M. It is easy to see that this algorithm is of polynomial complexity. Then we have the following observation.

Algorithm 7 *Assign*(G, S, P)

Input: A graph G, an induced matching S of G, and a potential augmenting chain $P = (x_0, x_1, \ldots, x_p)$ (whose x_0, x_p are black).

Output: Assign white vertices of P to U, W_I, or W_M.

$i := 1$; $j := 1$;
while $i < p$ **do**
    while $i \leq p$ AND ((x_i is white) OR (x_{i+1} is white)) **do**
        $i := i + 1$;
    end while
    if ($i < p$) AND (x_{j+1} is black) **then**
        $x_{j+2} \to W_M$; $x_{j+3} \to W_M$; $j := j + 4$;
    end if
    while $j < i$ **do**
        $x_{j+1} \to U$; $x_{j+2} \to W_I$; $j := j + 3$;
    end while
    $i := i + 1$;
end while

Theorem 4.23. *Given integers l and m, the Maximum Induced Matching problem is polynomially solvable in ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs.*

4.4 Discussion

In this chapter, we consider a general combinatorial graph theoretical problem, the so-called Maximum Π-Set problem. Two special cases of this problem, so-called the Maximum $\mathcal{F}$-(Strongly) Independent Subgraph problem and the Maximum $\mathcal{F}$-Induced Subgraph problem were considered. We proved the NP-hardness of the second problem for the case $\Delta(\mathcal{F})$ is finite.

We have introduced the augmenting graphs approach for solving the Maximum Π-set problem. This technique was used successfully to show polynomial solutions in many graph classes for the MIS problem. By this technique, we found a graph class, say ($S_{1,2,l}$,banner$_l$, Z_l, $K_{1,m}$)-free graphs, for given integers l, m, for which some maximum Π-set problems have polynomial solutions. The problems include hereditary problems and non-hereditary problems. Like for the MIS problem, this method potentially offers a general approach to solve these problems in other graph classes. We also expect that it is possible to apply this technique for other graph combinatorial problems.

5 Graph Transformations

In this chapter, we describe graph transformations mentioned in Section 2.4 under general points of view. In the first section, we revisit the pseudo-boolean function method and show the relationship to other reductions method. Then in the second section, we focus on α-redundant technique. We give an overview on some reductions in the sense of α-redundant vertices and give polynomial solutions for some hereditary graph classes. In the third section, we summarize some discussion about the issue.

5.1 Pseudo-Boolean Functions

In this section, we review the method used by Ebenegger et al. [58] and Hammer et al. [88, 89] for STRUCTION, by Hammer and Hertz for magnet reduction [87], and by Hertz for BAT reduction [97] to give a unified look on some graph reductions.

5.1.1 Posiform and Conflict Graph

It is known that a pseudo-Boolean function f (i.e. a function of the form $f : \{0,1\}^n \to \mathbb{R}$) can always be written in a polynomial form:

$$f(x_1, x_2, \ldots, x_n) = K + \sum_{i=1}^{p} w_i T_i,$$

where $T_i = \prod_{j \in A_i} x_j \prod_{k \in B_i} \overline{x}_k$ with $A_i, B_i \subset \{1, 2, \ldots, n\}$ and $A_i \cap B_i = \emptyset$.
If all $w_i (1 \leq i \leq p)$ are strictly positive and $K = 0$, we say that f is a *posiform*. In this chapter, we mainly consider only unweighted graphs, i.e. $w_i = 1 \, \forall i$.
To a posiform f, we associate a *conflict graph* $G = (V, E)$ defined as follows:

$$V = \{1, 2, \ldots, p\} \qquad E = \{ij : \exists k \in ((A_i \cap B_j) \cup (A_j \cap B_i))\}$$

In other words, two vertices i, j of G are linked by an edge if x_k appears in T_i (or T_j) while $\overline{x}_k$ appears in T_j (or T_i). It is clear from the definition of G that $\max f = \alpha(G)$, i.e. the maximum of f is equal to the independence number of G.
Conversely, for each simple graph $G = (V, E)$, there exists a posiform f such that G is the conflict graph of f. Indeed, consider an arbitrary covering of the edge set E by complete bipartite partial subgraphs $G_i = (V_{i_1}, V_{i_2}, E_i)$ of $G, i = 1, 2, \ldots, q$. Then we set

$$f = \sum_{u \in V} T_u,$$

where $T_u = \prod_{j \in A_u} x_j \prod_{k \in B_u} \overline{x}_k$ with $A_u = \{i : u \in V_{i_1}\}, B_u = \{i : u \in V_{i_2}\}$.
Let T_u and T_v be two terms of the posiform f such that x_i appears in T_u and $\overline{x}_i$ appears

in T_v. Then $u \in V_{i_1}$ and $v \in V_{i_2}$. Hence, the edge uv belongs to $E_i \subset E$ showing that G is the conflict graph associated with f.
Note that given a graph $G = (V, E)$, there might exist different coverings of E by complete bipartite partial subgraphs, hence, we may have different posiforms for one conflict graph.

5.1.2 Reductions Based on Pseudo-Boolean Functions

In this subsection, we review some graph transformations based on pseudo-boolean functions.

STRUCTION

The *STRUCTION* (for STability number RedUCTION) method, introduced by Ebenegger et al. [58] and named by Hammer et al. [88], is a procedure which, given a graph $G = (V, E)$, constructs a new graph $G' = (V', E')$ with $\alpha(G') = \alpha(G) - 1$ as follows.
Let a_0 be an arbitrary vertex and $a_1, a_2, \ldots, a_p$ be its neighbors. The remaining vertices are $a_{p+1}, a_{p+2}, \ldots, a_n$, where $n = |V|$. We associate the term

$$T_0 = \overline{x}_1 \overline{x}_2 \ldots \overline{x}_p$$

with vertex a_0. Furthermore, for each neighbor a_i of a_0 ($i \leq p$), we define a term

$$T_i = x_i \prod_{\substack{j \,:\, a_j \in N(a_i) \\ j < i}} \overline{x}_j .$$

For every remaining vertex a_i of G ($i > p$), we introduce a term

$$T_i = x_i \prod_{j \in N(i)} \overline{x}_j .$$

Finally, we put

$$f = \sum_{i : a_i \in X} T_i .$$

It is proved in [58] that

$$\sum_{i=0}^{p} T_{a_i} = 1 + \sum_{\substack{q,r \\ q<r<\leq p \\ a_q \notin N(a_r)}} T_{qr},$$

where $T_{qr} = x_q x_r \prod_{s<q} \overline{x}_s \prod_{\substack{q<t<r \\ a_t \in N(a_r)}} \overline{x}_t$.

Let $f' = \sum_{q,r} T_{qr} + \sum_{i=p+1}^{n} T_{a_i}$. Then f' is also a posiform. Let G' be a conflict graph of f'. Then $\alpha(G') = \alpha(G) - 1$.
The construction of the posiform f is based on the following cover of the edge set (by complete bipartite partial subgraphs). For each vertex $a_i \in N_G(a_0)$, let $G_i = (V_{i_1}, V_{i_2}, E_i)$ be a bipartite graph, where $V_{i_1} = \{a_i\}$ and $V_{i_2} = \{a_j \in N_G(a_i) : j < i\}$.

For each vertex $a_i \in V\backslash N_G[a_0]$, let $G_i = (V_{i_1}, V_{i_2}, E_i)$ be a bipartite graph, where $V_{i_1} = \{a_i\}$ and $V_{i_2} = \{a_j \in N_G(a_i)\}$.
From the above construction and the conflict graph of f', we have the direct transformation of STRUCTION as follows. The vertex set V' of G' consists of $a_{p+1}, a_{p+2}, \ldots, a_n$, as well as a set of "new" vertices a_{ij} associated to all the pair i, j of non-adjacent vertices a_i, a_j in the neighborhood of a_0. The edge set of G' consists of all the edges of the subgraphs of G induced by $\{a_{p+1}, a_{p+2}, \ldots, a_n\}$; all the edges of the form $a_{i_1j_1}a_{i_2j_2}$ where $i_1 \neq i_2$; all the edges of the form $a_{ij_1}a_{ij_2}$ if $a_{j_1}a_{j_2} \in E(G)$; and all the edges of the form $a_{ij}a_r(r > p)$ if a_ia_r or $a_ja_r \in E(G)$.
Let W be the set of new vertices in the above construction. Then from [58], we have the following observations. Let S' be a maximum independent set of G'. If $S' \cap W = \emptyset$, then $S' \cup \{a_0\}$ is a maximum independent set of G. Otherwise, the new vertices in S' must be of the form $a_{ij_1}, a_{ij_2}, \ldots, a_{ij_r}$ and

$$S = (S'\backslash\{a_{ij_1}, a_{ij_2}, \ldots, a_{ij_r}\}) \cup a_i, a_{j_1}, a_{j_2}, \ldots, a_{j_r}$$

is a maximum independent set of G.
The problem with STRUCTION is the appearances of new vertices and the number of vertices can exponentially grow. It has been demonstrated by Hammer et al. [88, 89] that for a certain families of graphs one can avoid the potentially exponential growth, thus giving a polynomial time algorithm for those families. Some restricted version of the STRUCTION method have been applied to the MIS problem by Beigel [17] and Formin et al. in [66]. More on the STRUCTION method can be also found in [103, 167]. A generalization of the STRUCTION method can be found in [8].

Magnet

Hammer and Hertz [87] introduced a transformation based on pseudo-Boolean method, which, when applicable, builds from a graph $G = (V, E)$, a new graph $G' = (V', E')$ with $|V'| = |V| - 1$ and $\alpha(G') = \alpha(G)$. They described that a *magnet* in a graph $G = (V, E)$ is a pair (a, b) of adjacent vertices such that each vertex in $N_G(a)\backslash N_G(b)$ is adjacent to each vertex in $N_G(b)\backslash N_G(a)$. The edges incident to a or b can be covered by the two following complete bipartite partial subgraphs:

$$G_1 = (V_{1_1}, V_{1_2}), \text{ where } V_{1_1} = N_G(b)\backslash N_G(a) \text{ and } V_{1_2} = N_G(a)\backslash N_G(b) \text{ and}$$
$$G_2 = (V_{2_1}, V_{2_2}), \text{where } V_{2_1} = \{a, b\} \text{ and } V_{2_2} = N_G(a) \cap N_G(b).$$

The remaining edges are covered by arbitrary complete bipartite partial subgraphs. Then in the associated posiform f, we have $T_a = x_1x_2$ and $T_b = \overline{x}_1x_2$. Hence,

$$T_a + T_b = x_1x_2 + \overline{x}_1x_2 = (x_1 + \overline{x}_1)x_2 = x_2.$$

It follows that f can be reduced to a posiform g which $f = g$ and g has one summand less than f, so the correspondent conflict graph G' has one vertex less than G.
The graph G' can be obtained directly from G by replacing the vertex a and b by a new vertex $\widetilde{ab}$ linking to every common neighbor of a and b in G. Hertz and de Werra [100] characterized a graph class such that by repeated use of magnets the graph is reduced to an independen set, and hence, such the graph class is MIS-easy.
A special case of magnet is that $N_G(a) \cap N_G(b) = \emptyset$ and each vertex of $N_G(a)$ is adjacent to each vertex of $N_G(b)$, i.e. we can use the equality $x_1 + \overline{x}_1 = 1$ directly, i.e. the graph G' is obtained from G by removal both a and b and $\alpha(G') = \alpha(G) - 1$.

BAT

Magnet is based on the Boolean equality $\overline{x} + x = 1$ and the consequence $\overline{x}y + xy = y$. Hertz [97] introduced another graph transformation based on the Boolean equality $\overline{xy} + x + y = 1 + xy$ and the consequence

$$\overline{x}_1\overline{x}_2x_3 + x_1x_3 + x_2x_3 = x_3 + x_1x_2x_3.$$

A *BAT* in a graph $G = (V, E)$ is a triple (a, b, c) such that (b, a, c) induces a P_3. Denote

$$C_{abc} := N_G(a) \cap N_G(b) \cap N_G(c),$$

$$\begin{aligned} C_{ab} &:= (N_G(a) \cap N_G(b) \backslash C_{abc}), \\ C_{ac} &:= (N_G(a) \cap N_G(c) \backslash C_{abc}), \\ C_{bc} &:= (N_G(b) \cap N_G(c)) \backslash (C_{abc} \cup \{a\}), \\ C_a &:= N_G(a) \backslash (C_{abc} \cup C_{ab} \cup C_{ac} \cup \{b, c\}), \\ C_b &:= N_G(b) \backslash (C_{abc} \cup C_{ab} \cup C_{bc} \cup \{a\}), \text{ and} \\ C_c &:= N_G(c) \backslash (C_{abc} \cup C_{bc} \cup C_{ac} \cup \{a\}). \end{aligned}$$

Assume that we have the following conditions:

- C_a can be partitioned into two subsets C_{a_b} and C_{a_c} such that each vertex in C_{a_b} is adjacent to each vertex of $C_b \cup C_{bc} \cup C_{ab}$ and each vertex of C_{a_c} is adjacent to each vertex of $C_c \cup C_{bc} \cup C_{ac}$ and
- each vertex of C_{ac} is adjacent to each vertex of $C_b \cup C_{bc} \cup C_{ab}$ and each vertex of C_{ab} is adjacent to each vertex of $C_c \cup C_{bc} \cup C_{ac}$.

Let correspond vertices with $\overline{x}_1\overline{x}_2x_3, x_1x_3, x_2x_3$ be a, b, c respectively. Then the edges incident to a, b or c can be covered by the three following complete bipartite partial subgraphs.

$$G_1 = (V_{1_1}, V_{1_2}), \text{ where } V_{1_1} = \{b\} \cup C_{a_b} \cup C_{ac} \text{ and } V_{1_2} = \{a\} \cup C_b \cup C_{bc} \cup C_{ab};$$

$$G_2 = (V_{2_1}, V_{2_2}), \text{ where } V_{2_1} = \{c\} \cup C_{a_c} \cup C_{ab} \text{ and } V_{2_2} = \{a\} \cup C_c \cup C_{bc} \cup C_{ac}; \text{ and}$$

$$G_3 = (V_{3_1}, V_{3_2}), \text{ where } V_{3_1} = \{a, b, c\} \text{ and } V_{3_2} = C_{abc}.$$

The remaining edges are covered by arbitrary complete bipartite partial subgraphs. Then in the associated posiform f, we have $T_a = \overline{x}_1\overline{x}_2$ and $T_b = x_1$, $T_c = x_2$. Now,

$$\begin{aligned} T_a + T_b + T_c &= \overline{x}_1\overline{x}_2x_3 + x_1x_3 + x_2x_3 \\ &= (\overline{x}_1\overline{x}_2 + x_1 + x_2)x_3 \\ &= (1 + x_1x_2)x_3 = x_3 + x_1x_2x_3. \end{aligned}$$

It follows that f can be reduced to a posiform g such that $f = g$ and g has one summand less than f, so the correspondent conflict graph G' has one vertex less than G.

The graph G' can be obtained directly from G by replacing the vertex a and b, c by two new vertices $\widetilde{a}$ and $\widetilde{bc}$ such that $\widetilde{a}$ is adjacent to every vertex in C_{abc} and $\widetilde{bc}$ is

adjacent to every vertex in $(N_G(a) \cup N_G(b) \cup N_G(c)) \backslash \{a, b, c\}$. Hertz [97] characterized some graph classes such that by repeated use of BAT, the MIS problem is polynomially solvable in which.
Similar to magnet, we have a special case of BAT, that is when $N_{abc} = \emptyset$, i.e. we can use the equality $\overline{x}_1\overline{x}_2 + x_1 + x_2 = 1 + x_1x_2$ directly. So, we can substitute a, b and c by a vertex $\widetilde{bc}$ such that $\widetilde{bc}$ is adjacent to every vertex of $(N_G(a) \cup N_G(b) \cup N_G(c)) \backslash \{a, b, c\}$, and $\alpha(G') = \alpha(G) - 1$, i.e. the vertex folding reduction.

Weighted Version

These technique can be modified to use for the WIS problem. For example, STRUCTION can be applied on vertex a_0 such that every vertex of $N_G[a_0]$ has the same weight. Similarly, the concept of magnet can be extended by adding the requirement that the two vertices a, b have the same weight and for BAT is that the three vertices b, a, c have the same weight.

5.1.3 Other Related Reductions

In this subsection, we show the relations between some other graph transformations and pseudo-boolean functions method.

Simplicial Vertex Reduction

Consider a simplicial vertex x as the vertex a_0 in the STRUCTION method, since every two neighbors of a_0 are adjacent, we have the set of new vertices is empty. It can be inferred that a_0 belongs to some maximum independent set. Hence, simplicial vertex reduction can be considered as a special case of STRUCTION.

Neighborhood Reduction and Twin Reduction

It is obviously that neighborhood reduction, and hence, twin reduction also are special cases of magnet reduction. Moreover, in the special case when $N_G(a) \cap N_G(b)$ is a clique, the neighborhood reduction coincides with the simplicial vertex reduction.

Vertex Folding and Vertex Splitting

Consider the special case of the BAT-reduction when $N_{abc} = \emptyset$. Then the new vertex $\widetilde{a}$ is isolated in the new graph G', i.e. the removal of a from G' decreases its independence number by exactly one. The composition of the two reductions (BAT and removal of a) is known as *vertex folding*. The transformation inverse to vertex folding, i.e. *vertex splitting*, is applicable to any graph.

Edge Deletion and Edge Insertion

It is mentioned by Lozin [122] that the magnet simplification can be obtained as a combination of the edge deletion and the neighborhood reduction. In the same manner, the neighborhood reduction can be considered as a combination of the edge deletion and the twin reduction. Now, we take a deeper consider on this transformation using pseudo-Boolean methods. Consider the following equality:

$$x_1x_2 + \overline{x}_1 = x_2 + \overline{x}_1\overline{x}_2.$$

Let a and b are two adjacent vertices. Assume that $N_G(a)$ can be partitioned into two subsets N_{a_1} and N_{a_2} and $N_G(b)$ can be partitioned into two subsets N_{b_1} and N_{b_2} such that $N_{a_2} \cap N_{b_2} = \emptyset$ and each vertex in N_{a_2} is adjacent to each vertex in N_{b_2}, $N_{b_1} \subset N_{a_1}$. The edges incident with a or b can be covered by the following complete bipartite subgraphs:

$$G_1 = (V_{1_1}, V_{1_2}), \text{ where } V_{1_1} = N_{b_2} \text{ and } V_{1_2} = N_{a_2} \text{ and}$$

$$G_2 = (V_{2_1}, V_{2_2}), \text{ where } V_{2_1} = \{a\} \text{ and } V_{2_2} = N_{a_1}.$$

The remaining edges are covered by arbitrary complete bipartite partial subgraphs. In the associated posiform f, we have $T_a = x_1x_2$ and $T_b = \overline{x}_1$. Now,

$$T_a + T_b = x_1x_2 + \overline{x}_1 = x_2 + \overline{x}_1\overline{x}_2.$$

It follows that f can be reduced to a posiform g, which $f = g$. Moreover, g is associated with the graph G' coming from G by substitution a and b by two adjacent vertices $\widetilde{a}$ and $\widetilde{b}$ such that $\widetilde{b}$ is adjacent to every vertex of $N_G(b)\backslash\{a\}$ and $\widetilde{a}$ is adjacent to every vertex of N_{a_1}. In other words, we removed all edges of the form ac such that c is adjacent to every vertex of $N_G(b)$. In the converse direction, if there exists no vertex of $N_G(a)\backslash\{b\}$ adjacent to every vertex of $N_G(b)$, then we can insert a bunch of edges of the form ac such that c is adjacent to every vertex of $N_G(b)$. Combine the two steps, we have the edge deletion and edge insertion, i.e. the inverse transformation.
In the same manner as with magnet and BAT, we can extend this result. Instead of above equality, we use the following equality:

$$x_1x_2x_3 + \overline{x}_1x_3 = x_2x_3 + \overline{x}_1\overline{x}_2x_3.$$

It leads us to the following edge deletion (insertion). Given two adjacent vertices a and b, let c be a vertex such that c is adjacent to every vertex of $N_G(b)\backslash \cup N_G[a]$, the removal (or insertion) of the edge ac does not change the independence number of the graph.

5.2 Alpha-redundant Vertex

In this section, we describe some new conditions to recognize α-redundant vertices and use this technique to solve the MIS problem in some hereditary graph classes. Some results of this section have been published in [112].

5.2.1 Some Related Results

Recall that an α-redundant vertex v of a graph G is a vertex which can be deleted from G without changing the independence number of G. Formally, Brandstädt and Hammer gave the following definition.

Definition 5.1. *[27] Given a graph $G = (V, E)$, a vertex $v \in V(G)$ is called α-redundant if $\alpha(G - v) = \alpha(G)$.*

The problem of recognizing α-redundant vertices is obviously polynomially equivalent to the problem of finding an independent set of maximum size and hence is NP-complete in general. However, in some cases, α-redundant vertices can be recognized efficiently.
The concept of α-redundant vertex was introduced in [27]. The authors used this technique to extend polynomial solutions of the MIS problem from P_4-free graphs to (P_5,banner)-free graphs, and then to ($P_5, K_{3,3} - e$,twin-house)-free graphs. Then this technique was used to generalize many polynomial results in subclasses of P_5-free graphs. Some examples are extending from $(P_5, K_1 \times mK_2)$-free graphs [76] to $(P_5, (m + 1)K_1 \times mK_2)$ and extend to (P_5, F_{19},twin-house)-free graphs [31] (see Fig. 6.1 for F_{19}). Zverovich [172] extended the result of (P_5, F_{19},twin-house)-free graphs. Gerber and Lozin extended from $(P_5, K_{1,m})$-free graphs [140] to $(P_5, K_{m,m})$-free graphs [76] and from (banner,fork)-free graphs to($S_{2,2,2}$,banner)-free graphs [77]. In [77], the authors also used this technique to show a polinomial solvability of the problem in (banner,$C_5, C_6, \ldots$)-free graphs.
Among classical reduction techniques, there are some vertices deletions which are special cases of α-redundant vertex. More precisely, we have the following summary.

Proposition 5.1. *[19, 52, 81, 154] Given a graph $G = (V, E)$, a vertex $b \in V(G)$ is α-redundant if it satisfies one of the following conditions.*

1. *b is a neighbor of a simplicial vertex .*
2. *There exists a neighbor a of b such that $N[a] \subset N[b]$ (neighborhood reduction).*
3. *There exist a and c being two non-adjacent neighbors of b such that $(N(a) \cup N(c)) \backslash N[b]$ is a clique (vertex deletion).*

In the next subsection, we describe an application of α-redundant technique for $K_{1,m}$-free graphs. Another unified look about above vertex removal reductions based on α-redundant vertices is given. First, the following obvious proposition will be used implicitly through the thesis.

Proposition 5.2. *Given a graph $G = (V, E)$ and a vertex $u \in V(G)$, if there exists some maximum independent set S not containing u, then u is α-redundant.*

5.2.2 An α-redundant Vertex in an Induced $K_{1,m}$

Using the result of Mosca [140] that the $(P_5, K_{1,m})$-free graph class is MIS-solvable in time $O(n^{m+1})$ and α-redundant vertex technique, Gerber and Lozin [77] showed that the $(P_5, K_{m,m})$-free graph class is MIS-solvable in time $O(n^{2m})$. This result is based on the following observation.

Lemma 5.3. *[77] Given a graph G containing an induced $K_{1,m}$, $\{u, v_1, v_2, \ldots, v_m\}$, where u is the center vertex (i.e. the vertex of degree m), there exist some vertices $u_1, u_2, \ldots, u_m$ such that $\{u, u_1, u_2, \ldots, u_m\}$ is independent and there is a perfect matching between $\{u_i\}$ and $\{v_i\}$ or u is α-redundant vertex.*

Note that vertex deletion and neighborhood reduction (and hence simplicial reduction and twin reduction also) are consequences of Lemma 5.3 for the cases $m = 2$ and $m = 1$, respectively. The following result is a consequence of the above lemma and Lemma 3.6.

Lemma 5.4. *Given two integers m_1, m_2 and a (tree$_{m_2}$, K_{m_1,m_1})-free graph, there exists a number $\nu = \nu(m_1, m_2)$ such that for every star $K_{1,\nu}$, the center vertex u is α-redundant.*

Recall that tree$_r$ is the graph consisting of r P_3 sharing an end-vertex (see Fig. 3.2, tree^1).

Proof. Let $\nu = \nu(m_1, m_2)$ be the number ν in Lemma 3.6. For contradiction, suppose that $\{u, v_1, \ldots, v_\nu\}$ is an induced $K_{1,\nu}$ whose u is the center vertex and u is not α-redundant.
By Lemma 5.3, there exist some vertices $u_1, u_2, \ldots, u_\nu$ such that $\{u, u_1, u_2, \ldots, u_\nu\}$ is independent and there is a perfect matching between $\{u_i\}$ and $\{v_i\}$. In other words, there exists a matching of size ν between $\{u_i\}$ and $\{v_i\}$. Then, by Lemma 3.6, $\{u, u_1, \ldots, u_\nu, v_1, \ldots, v_\nu\}$ induces a K_{m_1,m_1} or a tree$_{m_2}$, a contradiction. □

The following obsevation is a consequence of Lemmas 5.3 and 3.5.

Corollary 5.5. *Let G be a graph and $\{u, v_1, v_2, \ldots, v_m\}$ be an induced $K_{1,m}$, where u is the center vertex. Then u is α-redundant or there exist some vertices $u_1, u_2, \ldots, u_m$ such that $\{u, u_1, \ldots, u_m\}$ is independent, $u_i \sim v_i$ for $1 \le i \le m$, and at least one of the following statements is true.*

1. *$\{u_i, v_i, v_j, u, v_k, u_k\}$ induces a banner$_2$ or a domino for some i, j, k, where u is a vertex of degree three in both cases.*

2. *There exists a permutation $\sigma = (i_1, i_2, \ldots, i_m)$ of $(1, 2, \ldots, m)$ such that for some p, $0 \le p \le m$, $u_{i_j} \sim v_{i_k}$ for every $1 \le j \le p$ and $j \le k \le m$ and u_{i_j} has only one neighbor in $\{v_1, v_2, \ldots, v_m\}$ for $j > p$.*

This observation is weaker than Lemma 5.4 in the sense of forbidden induced subgraph but is useful in the next subsection. Given a graph G, an integer m, we can find an induced $K_{1,\nu}$ of G in $O(n^{\nu+1})$. Together with Theorem 3.16, it leads us to the following result.

Theorem 5.6. *Given three integers k, l, and m such that $4 \le 2k \le l$, the ($S_{2,k,l}$, banner$_l$, apple$^l_6$, apple$^l_8$, …, apple$^l_{2k+2}$, R^1_l, R^2_l, R^3_l, R^4_l, R^5_l, $K_{m,m}$, tree$_m$)-free graph class is MIS-easy.*

Corollary 5.7. *Given three integers k, l, m, the following graph classes are MIS-easy:*

1. *($S_{1,k,l}$, banner$_l$, apple$^l_6$, apple$^l_8$, …, apple$^l_{2k+2}$, R^2_l, $K_{m,m}$, tree$_m$)-free graphs,*

2. *($S_{2,2,l}$, banner$_l$, R^3_l, R^4_l, R^5_l, $K_{m,m}$, tree$_m$)-free graphs.*

These results are generalizations of the result of Gerber and Lozin about $(P_5, K_{m,m})$-free graphs [76].

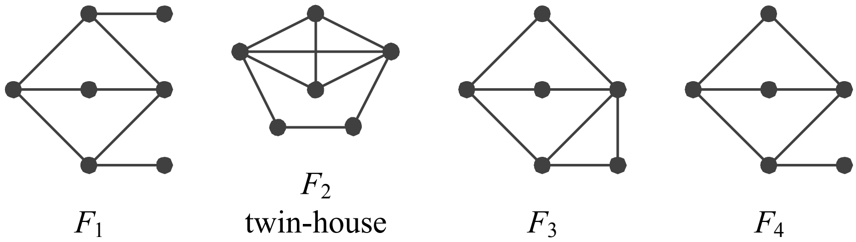

Fig. 5.1: Graphs considered in Subsection 5.2.3

5.2.3 Applications in $S_{2,2,2}$-free Graphs

In this subsection, we apply Corollary 5.5 in $S_{2,2,2}$-free graphs with the remark that $S_{2,2,2}$ is tree$_3$. We refer to Fig. 5.1 for graphs $F_1, \dots, F_4$. Note that F_2 is also known as *twin-house.*

Lemma 5.8. *Given an integer $m \geq 3$ and an ($S_{2,2,2}$,banner$_2$,domino)-free graph, let $\{u, v_1, v_2, \dots, v_m\}$ induces a $K_{1,m}$, where u is the center vertex. Then u is α-redundant or the following statements are true.*

1. *There exists some vertex u' such that u' is adjacent to $v_1, \dots, v_m$ and not to u.*
2. *If G is F_1-free, then there exists some vertex v_i, $1 \leq i \leq m$, such that v_i is the center vertex of some $K_{1,m+1}$ and G induces a $K_{m,m}$.*

Proof. Since G is (banner$_2$,domino)-free, by Corollary 5.5, let p be an integer number, $0 \leq p \leq m$, and $u_1, \dots, u_m$ such that $\{u, u_1, \dots, u_m\}$ is independent and, without loss of generality, $u_i \sim v_j$, for every $1 \leq i \leq p$, $i \leq j \leq m$, u_i has only one neighbor $v_i \in \{v_1, \dots, v_m\}$ for every $i > p$.
Since G is $S_{2,2,2}$-free, $p \geq m - 2$, i.e. u_1 is adjacent to $\{v_1, \dots, v_m\}$ and not to u.
Now, assume that G is F_1-free. Then $p \geq m-1$, otherwise $\{u, u_1, u_{m-1}, u_m, v_1, v_{m-1}, v_m\}$ induces an F_1, a contradiction. Hence, $\{v_m, u, u_1, \dots, u_m\}$ induces a $K_{1,m+1}$, where v_m is the center vertex. Moreover, $\{u, u_1, \dots, u_{m-1}, v_1, \dots, v_m\}$ induces a $K_{m,m}$. □

Given a graph G, to find an induced $K_{1,m}$ or to show that such an induced graph does not exist can be performed in time $\mathrm{O}(n^{m+1})$. The reduction of all $K_{1,m}$ can be performed in time at most $\mathrm{O}(n)$. Hence, we obtain the following observation.

Lemma 5.9. *If the ($S_{2,2,2}$,banner$_2$,domino,$F_1, K_{1,m}$)-free graph class is MIS-solvable in time at most $\mathrm{O}(n^{m+1})$, then the ($S_{2,2,2}$,banner$_2$,domino,$F_1, K_{1,m+1}$)-free graph class is MIS-solvable in time at most $\mathrm{O}(n^{m+2})$.*

Minty [137] and Sbihi [156] independently showed that the MIS problem is solvable for claw free graphs, and hence for ($S_{2,2,2}$,banner$_2$,domino,$F_1, K_{1,3}$)-free graphs in time $\mathrm{O}(n^3)$. Using the above lemma, induction method, and Lemma 5.8, we obtain the following observation.

Theorem 5.10. *The ($S_{2,2,2}$,banner$_2$,domino,$F_1, K_{1,m}$)-free graph class and the ($S_{2,2,2}$, banner$_2$,domino,$F_1, K_{m,m}$)-free graph class are MIS-solvable in time $\mathrm{O}(n^{m+1})$ and in time $\mathrm{O}(n^{m+2})$, respectively.*

The above theorem is a generalization of Mosca's result for $(P_5, K_{1,m})$-free graphs [140] and the result of Gerber and Lozin for $(P_5, K_{m,m})$-free graphs [76].

Lemma 5.11. *In an ($S_{2,2,2}$,banner$_2$,domino,$K_{3,3}-e$,twin-house)-free graph G, every vertex of degree three of some induced banner is α-redundant.*

Proof. Let $\{u, v_1, v_2, v_3, u_1\}$ induces a banner where u and u_1 are vertices of degree three and two, respectively. If u is not α-redundant, then, by Lemma 5.8, there exists a vertex u' adjacent to v_1, v_2, v_3 and non-adjacent to u. Now, $\{u, u_1, u', v_1, v_2, v_3\}$ induces a twin-house or a $K_{3,3}-e$ depending on $u' \sim u_1$ or not, a contradiction. □

Gerber and Lozin [77] showed that the ($S_{2,2,2}$, banner)-free graph class is MIS-solvable in time $\mathrm{O}(n^5)$. Finding and reduction of all banners can be done in time $\mathrm{O}(n^6)$. Hence, we obtain the following result.

Theorem 5.12. *The ($S_{2,2,2}$,banner$_2$,domino,$K_{3,3}-e$,twin-house)-free graph class is MIS-solvable in time $\mathrm{O}(n^6)$.*

Lemma 5.13. *In an ($S_{2,2,2}$,banner$_2$,domino,F_3, F_4)-free graph G, every vertex of degree three of some induced fork is α-redundant.*

Proof. Let $\{u, v_1, v_2, v_3, u_1\}$ be an induced fork in G, where u is of degree three and v_1 is adjacent to u and u_1. If u is not α-redundant, then, by Lemma 5.8, there exists a vertex u' such that u' is adjacent to v_1, v_2, v_3 and non-adjacent to u. Now, $\{u, u_1, u', v_1, v_2, v_3\}$ induces an F_3 or F_4 depending on $u' \sim u_1$ or not, a contradiction. □

Together with Alekseev's result saying that the fork-free graph [2] is MIS-easy, it leads to the following observation.

Theorem 5.14. *The ($S_{2,2,2}$,banner$_2$,domino,F_3, F_4)-free graph class is MIS-easy.*

5.2.4 Applications in ($S_{i,j,k}$,apples)-free Graphs

In this subsection, we apply α-redundant technique to subclasses of $S_{i,j,k}$-free graphs. We start with the following observation.

Lemma 5.15. *Given two integers k, l, $2 \le k \le l$ and an ($S_{2,k,l}$,banner,apple$_5$,…, apple$_{l+3}$)-free graph G, let $\{a, b_1, c_1, \dots, c_k, d_1, \dots, d_l\}$ induces an $S_{1,k,l}$, where a is of degree three and (a, b_1), $(a, c_1, \dots, c_k)$, and $(a, d_1, \dots, d_l)$ are three induced paths of length $2, k$, and l, respectively. Then a is α-redundant.*

Proof. Indeed, if a is not α-redundant, then b_1 has a neighbor, say b_2 such that $b_2 \nsim a$. We show that b_2 is non-adjacent to c_i, d_i by induction.
If $b_2 \sim c_1$, then $b_2 \sim d_1$, otherwise $\{b_1, b_2, c_1, d_1\}$ induces a banner, a contradiction. Hence, $b_2 \sim d_2$, otherwise $\{a, c_1, b_2, d_1, d_2\}$ induces banner, a contradiction. Now, $\{b_1, a, c_1, b_2, d_2\}$ induces a banner, a contradiction. Hence, $b_2 \nsim c_1$ and similarly, $b_2 \nsim d_1$.
Now, assume that b_2 is non-adjacent to $c_1 \dots, c_{i-1}, d_1, \dots, d_{i-1}$. Then $b_2 \nsim c_i$, otherwise $\{b_1, b_2, c_1, \dots, c_i, a, d_1\}$ induces an apple$_{i+3}$, a contradiction. Similarly, $b_2 \nsim d_i$.
But now, $\{a, b_1, b_2, c_1, \dots, c_k, d_1, \dots, d_l\}$ induces an $S_{2,k,l}$, a contradiction. □

The above lemma and Theorems 3.16, 3.24 give us the following observation.

Theorem 5.16. *Given two integers k, m, the following graph classes are MIS-easy:*

1. *($S_{2,2,5}$,banner,apple$_5$,...,apple$_8$)-free graphs,*

2. *($S_{2,2,k}$,banner,apple$_5$,...,apple$_{k+3}$,tree$_m$)-free graphs,*

3. *($S_{2,k,k}$, R_k^2,banner,apple$_5$,...,apple$_{k+3}$,tree$_m$)-free graphs.*

Lemma 5.17. *Given two integers k, l, $2 \leq k \leq l$ and an ($S_{3,k,l}$,banner,apple$_5$,..., apple$_{l+4}$)-free graph G, let $\{a, b_1, b_2, c_1, \ldots, c_k, d_1, \ldots, d_l\}$ induces an $S_{2,k,l}$, where a is of degree three and (a, b_1, b_2), $(a, c_1, \ldots, c_k)$, and $(a, d_1, \ldots, d_l)$ are three induced paths of length $3, k$, and l, respectively. Then b_1 is α-redundant.*

Proof. We show that there exists a maximum independent set not containing b_1. Let S be a maximum indpendent set of G and $b_1 \in S$. Then S contains a vertex b_3 adjacent to b_2, otherwise $(S\backslash\{b_1\}) \cup \{b_2\}$ is a desired set. We consider the two following cases.
Case 1. $b_3 \nsim a$. We show that b_3 is non-adjacent to c_i, d_i by induction.
If $b_3 \sim c_1$, then $b_3 \sim d_1$, otherwise $\{b_1, b_2, b_3, a, c_1, d_1\}$ induces an apple$_5$, a contradiction. Now, $\{c_1, b_3, d_1, a, b_1\}$ induces a banner, a contradiction. Hence, $b_3 \nsim c_1$ and similarly, $b_3 \nsim d_1$.
Now, assume that b_3 is non-adjacent to $c_1 \ldots, c_{i-1}, d_1, \ldots, d_{i-1}$. Then $b_3 \nsim c_i$, otherwise $\{b_1, b_2, b_3, c_1, \ldots, c_i, a, d_1\}$ induces an apple$_{i+4}$, a contradiction. Similarly, $b_3 \nsim d_i$.
But now, $\{a, b_1, b_2, b_3, c_1, \ldots, c_k, d_1, \ldots, d_k\}$ induces an $S_{3,k,l}$, a contradiction.
Case 2. $b_3 \sim a$. Then b_3 is adjacent to c_1, d_1, otherwise $\{b_1, b_2, b_3, a, c_1\}$ or $\{b_1, b_2, b_3, a, d_1\}$ induces a banner, a contradiction. This implies that $a, c_1, d_1 \notin S$. We claim now that
(1) b_3 is the only neighbor of b_2 in S different from b_1 and
(2) b_3 is the only neighbor of a different from b_1 in S.
To prove (1), suppose that b is such another neighbor. Then similarly, b is adjacent to a, c_1, d_1. But, now $\{c_1, b, b_3, b_2, b_1\}$ induces a banner, a contradiction.
To show (2), suppose that a has another neighbor, say b' in S. Then $b' \sim b_2$, otherwise $\{b', a, b_1, b_3, b_2\}$ induces a banner, a contradiction. But now we have a contradiction with (1).
By (1) and (2), we have $S' = (S\backslash\{b_1, b_3\}) \cup \{a, b_2\}$ is a desired maximum independent set. □

The above lemma and Theorem 5.16 give us the following observation.

Theorem 5.18. *Given two integers k, m, the following graph classes are MIS-easy:*

1. *($S_{3,3,5}$,banner,apple$_5$,...,apple$_9$)-free graphs,*

2. *($S_{3,3,k}$,banner,apple$_5$,...,apple$_{k+4}$,tree$_m$)-free graphs,*

3. *($S_{3,k,k}$, R_k^2,banner,apple$_5$,...,apple$_{k+4}$,tree$_m$)-free graphs.*

Now, we extend the above result to $S_{j,k,k}$.

Lemma 5.19. *Given three integers j, k, l, $3 \leq j$, $2 \leq k \leq l$ and an ($S_{j,k,l}$,apple$_3$,..., apple$_{l+j+1}$)-free graph G, let $\{a, b_1, \ldots, b_{j-1}, c_1, \ldots, c_k, d_1, \ldots, d_l\}$ induces an $S_{j-1,k,l}$, where a is of degree three and $(a, b_1, \ldots, b_{j-1})$, $(a, c_1, \ldots, c_k)$, and $(a, d_1, \ldots, d_l)$ are three induced paths of length $j-1, k$, and l, respectively. Then b_{j-2} is α-redundant.*

Proof. We show that there exists a maximum independent set not containing b_{j-2}. Let S be a maximum indpendent set of G and $b_{j-2} \in S$. Then S contains a vertex b_j adjacent to b_{j-1}, otherwise $(S \backslash \{b_{j-2}\}) \cup \{b_{j-1}\}$ is a desired set. It also implies that $b_j \nsim b_{j-2}$. We show that b_j is non-adjacent to b_{j-i} for $3 \le i \le j-1$ by induction.
For convenience, let $b_0 = a$. If $b_j \sim b_{j-i}$, then $b_j \sim b_{j-i-1}$, otherwise $\{b_{j-i}, b_{j-i+1}, \ldots, b_j, b_{j-i-1}\}$ induces an apple$_{i+1}$, a contradiction. But now, $\{b_j, b_{j-i}, b_{j-i-1}, b_{j-i+1}\}$ induces an apple$_3$, a contradiction. Now, we consider the two following cases.
Case 1. $b_j \nsim a$. We show that b_j is non-adjacent to c_i, d_i by induction.
If $b_j \sim c_1$, then $b_j \sim d_1$, otherwise $\{a, b_1, \ldots, b_j, c_1, d_1\}$ induces an apple$_{j+2}$, a contradiction. Now, $\{c_1, b_j, d_1, a, b_1\}$ induces a banner, a contradiction. Hence, $b_j \nsim c_1$ and similarly, $b_j \nsim d_1$.
Now, assume that b_j is non-adjacent to $c_1 \ldots, c_{i-1}, d_1, \ldots, d_{i-1}$. Then $b_j \nsim c_i$, otherwise $\{b_1, \ldots, b_j, c_1, \ldots, c_i, a, d_1\}$ induces an apple$_{i+j+1}$, a contradiction. Similarly, $b_j \nsim d_i$.
But now, $\{a, b_1, \ldots, b_j, c_1, \ldots, c_k, d_1, \ldots, d_l\}$ induces an $S_{j,k,l}$, a contradiction.
Case 2. $b_j \sim a$. Then b_j is adjacent to c_1, d_1, otherwise $\{b_1, \ldots, b_j, a, c_1\}$ or $\{b_1, \ldots, b_j, a, d_1\}$ induces an apple$_{j+1}$, a contradiction. This implies that $a, c_1, d_1 \notin S$. We claim now that
(1) b_j is the only neighbor of b_{j-1} in S different from b_1 and
(2) b_j is the only neighbor of a in S.
To prove (1), suppose that b is such another neighbor. Then similarly, b is adjacent to a, c_1, d_1. But, now $\{c_1, b, b_j, b_{j-1}, b_{j-2}\}$ induces a banner, a contradiction.
To show (2), suppose that a has another neighbor, say b' in S. Then $b' \sim c_1$, otherwise $\{b', a, c_1, b_j\}$ induces an apple$_3$, a contradiction. Similarly $b' \sim d_1$. Hence, $b' \sim b_{j-1}$, otherwise $\{c_1, b', d_1, b_j, b_{j-1}\}$ induces a banner, a contradiction. But now we have a contradiction with (1).
By (1) and (2), we have $S' = (S \backslash \{b_{j-2}, b_j\}) \cup \{a, b_{j-1}\}$ is a desired maximum independent set. □

The above lemma, Theorem 5.18, and induction method give us the following observation.

Theorem 5.20. *Given two integers k, l ($k \le l$), the ($S_{k,l,l}$,apple$_3$,...,apple$_{k+l+1}$)-free graph class is MIS-easy.*

5.3 Discussion

In this chapter, we have revisited graph transformation techniques. Two unified views about such reductions, pseudo-boolean function and α-redundant vertex, and some basic properties are considered. Both of the two methods are very potential to give new reduction techniques.
We also used the α-redundant method to obtain polynomial solution for the MIS problem some special subclasses. These results generalize some previous known results in literature for some subclasses of P_5-free graphs and some subclasses of $S_{j,k,l}$-free graphs of the previous chapter. Moreover, this method also can support classical heuristic algorithms for the MIS problem like Vertex Order (VO) [132], MIN [145], and MAX [83] which are considered in the next chapter.
Note that in the literature, the α-redundant technique was used mainly in subclasses of P_5-free graphs. In this chapter, we extended the method with the same motivation

as in Subsection 3.3.3 that it is possible to apply techniques which were used in P_5-free graphs in more general classes, e.g. $S_{2,2,2}$-free graphs and tree$_m$-free graphs, and $S_{k,l,l}$-free graphs.

6 Greedy Heuristic Methods

Heuristic methods can give maximal independent sets in polynomial time. In this chapter, we focus on sequential greedy methods. Some classical techniques are reviewed in the first section. We also consider some properties of these algorithms, for example: lower bounds of the computed maximal independent sets (Section 6.2), forbidden induced subgraph sets, under which a maximum indepent set is given (Section 6.4). We also describe new algorithms and some combined methods with α-redundant technique in the previous chapter (Section 6.3). Performances of new algorithms are also considered (Section 6.5). In the last section, we summarize some discussion about the issue.

6.1 Classical Methods

In this section, we review on three well-known heuristic algorithms, so-called MIN, MAX, and VO (Vertex Ordering).

6.1.1 Algorithm MIN

The MIN algorithm was described many times in literature, an example is [145]. It starts with an empty independent set I. Then the algorithm repeatedly chooses a vertex of minimum degree from a graph G, adds this vertex to I, and removes the vertex from G until G contains no remaining vertex.

Algorithm 8 MIN(G)

Input: A graph G

Output: A maximal independent set of G.

1: $I := \emptyset$; $i := 1$; $H_i := G$;
2: **while** $V(H_i) \neq \emptyset$ **do**
3: Choose $u \in V(H_i)$ such that $\deg_{H_i}(u) = \delta(H_i)$;
4: $I := I \cup \{u\}$; $i := i + 1$; $H_i := H_{i-1} - N_{H_{i-1}}[u]$;
5: **end while**
6: **return** I

6.1.2 Algorithm MAX

The MAX algorithm [83] repeatedly chooses a vertex of maximum degree from a graph G, removes the vertex from G until G contains no remaining edge. Then the remaining vertices compose a desired maximal independent set.

Algorithm 9 MAX(G)

Input: A graph G
Output: A maximal independent set of G.
$i := n$; $H_i := G$;
while $E(H_i) \neq \emptyset$ **do**
  Choose $u \in V(H_i)$ such that $\deg_{H_i}(u) = \Delta(H_i)$;
  $i := i - 1$; $H_i := H_{i+1} - u$;
end while
return $V(H_i)$

6.1.3 Algorithm VO (Vertex Order)

The VO algorithm [132] first orders the vertex set of a graph G in increasing degree order. Then it proceeds through the list and adds vertices to the being constructed independent set if they are non-adjacent to any vertices in the current set.

Remark. Based on the three above algorithms, one can think about a greedy heuristic method based on old worst-out strategy (see Subsection 2.3.1) working like first order the vertex set of a graph G in decreasing degree order. Then the algorithm proceeds through the list, adds a vertex to the being constructed independent set if it has no neighbor in the remaining graph and removes it from G. The process was repeated until the list is empty. However, a deeper analysis leads to that actually this algorithm and Algorithm VO produce the same maximal independent set for every graph.

Algorithm 10 VO(G)

Input: A graph G
Output: A maximal independent set of G.
$I := \emptyset$;
Order $V(G)$ as a list of increasing degree order (u_i);
for $i := 1$ **to** $n(G)$ **do**
  if $N_I(u_i) = \emptyset$ **then**
    $I := I \cup \{u_i\}$;
  end if
end for
return I

6.2 Caro-Wei Bound

Given a graph G, we denote $k_{MIN}(G)$, $k_{MAX}(G)$, and $k_{VO}(G)$ as the smallest cardinalities of the maximal independent sets obtained by the MIN, MAX, and VO algorithms, respectively. Wei [166] used MIN algorithm to discover a lower bound on $\alpha(G)$ in terms of the degree sequence of G, i.e.:

$$\alpha(G) \geq k_{MIN}(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}.$$

Caro [47] also indepently proved this result. As Wei observed, the above bound is sharp, i.e. we have the equality if G is a union of disjoint cliques. Griggs [83] also showed that

Algorithm MAX can be used to prove the Caro-Wei bound. Surprisingly, Algorithm VO also can be employed to obtain this bound as in the following observation.

Proposition 6.1. $k_{VO}(G) \geq \sum\limits_{v \in V(G)} \frac{1}{\deg(v)+1}$.

Proof. The proof mimics the similar proofs for Algorithms MIN [166] and MAX [83]. We consider the VO algorithm. Let (u_i), $i = 1, 2, \ldots, k_{VO}$, be the (ordered) vertices added in the result maximal independent set. Let $H_i := G$ and $H_{i+1} := H_i - N_{H_i}[u_i]$, for $i = 1, 2, \ldots, k_{VO}$. It is obvious that each vertex v belongs to $N_{H_i}[u_i]$ for some u_i. Moreover, if $v \in N_{H_i}[u_i]$, then v appears after u_i in the list generated by the algorithm, i.e. $\deg_{H_i}(u_i) \leq \deg_G(u_i) \leq \deg_G(v)$. Hence,

$$k_{VO}(G) = \sum_{i=1}^{k_{VO}} \frac{\deg_{H_i}(u_i) + 1}{\deg_{H_i}(u_i) + 1} \geq \sum_{i=1}^{k_{VO}} \sum_{v \in N_{H_i}[u_i]} \frac{1}{\deg(v) + 1} \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}.$$

□

We refer the readers the result of Borowiecki et al. [25] about a Caro-Wei-like bound using potential function of vertices instead of degree.

6.3 Hybrid Methods

In this section, we describe some modified versions of classical greedy algorithms. They are combinations of the MIN algorithm, the MAX algorithm and some reductions in Chapter 5.

6.3.1 MIN and α-redundance

We recall a reduced version of Lemma 5.3 (the case $m = 2$) as follows.

Corollary 6.2. *Given a graph $G = (V, E)$, a vertex $u \in V(G)$ is α-redundant if there exist two vertices $v_1, v_2 \in N(u)$ such that $v_1 \nsim v_2$ and there exist no two vertices u_1, u_2 such that $\{u, u_1, u_2\}$ is independent and $\{u, u_1, u_2, v_1, v_2\}$ induces a $K_{2,3}$ or a banner or a P_5.*

The MMIN algorithm (see Algorithm 11) is a combination of simplicial reduction, α-redundant technique, and Algorithm MIN.

Consider an arbitrary graph G, let $n = |V(G)|$. Then Algorithm MMIN gives a maximal independent set. The algorithm repeatedly chooses a minimum degree vertex u, then it checks and removes u if it is α-redundant by applying Corollary 6.2. We can find a minimum degree vertex of G in time $O(n^2)$. Given that (v_1, u, v_2) induces a P_3, we can check if there exist vertices u_1, u_2 such that $\{u, u_1, u_2, v_1, v_2\}$ induces a $K_{2,3}$, a banner, or a P_5 in time $O(n^2)$. For each u, such a test can be performed in time at most $O(n^2)$. Hence, we have the following result.

Theorem 6.3. *For a graph $G = (V, E)$, Algorithm MMIN gives a maximal independent set in time $O(n^5)$, where $n = |V(G)|$.*

Algorithm 11 MMIN(G)

Input: A graph G
Output: A maximal independent set of G.

```
I := ∅; i := 1; H_i := G;
while V(H_i) ≠ ∅ do
    Choose u ∈ V(H_i) such that deg_{H_i}(u) = δ(H_i);
    for all v_1, v_2 ∈ N_{H_i}(u) such that v_1 ≁ v_2 do
        if There exist no u_1, u_2 ∈ V(H_i) such that {u, u_1, u_2} is independent and
           {u, u_1, u_2, v_1, v_2} induces a P_5 or a banner or a K_{2,3} then
            H_{i+1} := H_i − u; i := i + 1; Break;
        end if
    end for
    I := I ∪ {u}; i := i + 1; H_i := H_{i−1} − N_{H_{i−1}}[u];
end while
return I
```

6.3.2 MAX and α-redundance

We describe the method of combining Algorithm MAX and α-redundant technique in Algorithm 12. Like with Algorithm MMIN, the idea is picking a maximum degree vertex $u \in V(G)$, before removing it, we check if some neighbor of u is α-redundant and remove such neighbor instead.

Consider an arbitrary simple graph G, let $n = |V(G)|$. Then Algorithm MMAX

Algorithm 12 MMAX(G)

Input: A graph G
Output: A maximal independent set of G.

```
I := ∅; i := n; H_i := G;
while E(H_i) ≠ ∅ do
    Choose u ∈ V(H_i) such that deg_{H_i}(u) = Δ(H_i);
    for all v ∈ N_{H_i}(u) do
        if There exists u_1 ∈ N_{H_i}(v)\N_{H_i}[u] such that there exists no v_1 ∈
           N_{H_i}(u_1)\N_{H_i}[v] then
            H_{i−1} := H_i − v; i := i − 1; Break;
        else if There exist u_1, u_2 ∈ N_{H_i}(v)\N_{H_i}[u] such that u_1 ≁ u_2 and there exist no
           v_1, v_2, v_3 such that {v, v_1, v_2, v_3} is independent, and v_1 ∼ u, v_2 ∼ u_1, v_3 ∼ u_2
           then
            H_{i−1} := H_i − v; i := i − 1; Break;
        end if
    end for
    H_{i−1} := H_i − u; i := i − 1;
end while
return V(H_I)
```

gives a maximal independent set. The algorithm repeatedly checks if the remaining graph still contains edges and chooses a maximum degree vertex u. Then it checks and removes a vertex $v \in N(u)$ if v is α-redundant by applying Lemma 5.3 for the case

$m = 1$ and $m = 3$. If no vertex in $N(u)$ is α-redundant in this sense, then u is removed with the assumption that u is α-redundant. In the case that there is no remaining edge, the remaining vertices form a maximal independent set.
We can find a maximum degree vertex of G in time $\mathrm{O}(n^2)$. $|N(u)|$ is at most $n-1$. Let $v \in N(u)$, we can check if v is α-redundant by using Lemma 5.3 in time $\mathrm{O}(n^2)$ for the case $m = 1$, and in time $\mathrm{O}(n^5)$ for the case $m = 3$. The removal of vertices will be performed at most n times. Therefore, we obtain the following result.

Theorem 6.4. *For an arbitrary graph G, Algorithm MMAX finds a maximal independent set in time* $\mathrm{O}(n^7)$.

6.3.3 MAX and $K_{1,l}$-reduction

In literature (and also in Chapters 3 and 5), there are some results about polynomial time solution for the MIS problem in subclasses of $K_{1,l}$-free graphs, for example $(P_k, K_{1,l})$-free graphs [131], ($S_{1,2,j}$,banner,$K_{1,l}$)-free graphs, ($S_{1,2,3}$,banner$_k$, $K_{1,l}$)-free graphs [98], and Theorems 3.16, 5.10. Thus, one possible heuristic approach for the MIS problem is to remove all maximum degree vertices which are the center vertex of some $K_{1,l}$ and then apply one polynomial solution for some subclass of $K_{1,l}$-free graphs. This idea leads us to Algorithm MAX-l (see Algorithm 13).

Algorithm 13 Algorithm MAX-l

Input: $G = (V, E)$
Output: S, an independent set of G.

```
H_n := G; i := n; S := ∅
while H_i contains an induced K_{1,l} do
    Choose a vertex u ∈ V(H_i) such that u is the center vertex of some K_{1,l} and u
    is of maximum degree among center vertices of all induced copies of K_{1,l} in H_i
    i := i − 1; H_i := H_{i+1} − u
end while
Let S be the maximum independent set of H_i obtained by some technique for
(K_{1,l})-free graphs;
return S
```

6.4 Forbidden Induced Subgraphs

In this section, we describe sufficient conditions for heuristic algorithms mentioned in the above sections. First, we revise some previous known results.

6.4.1 Previous Known Results

Mahadev and Reed [132] characterized a graph class, for which a maximum independent set can be obtained by Algorithm VO as in the following theorem.

Theorem 6.5. *[132] Algorithm VO always gives us a maximum independent set for $\mathcal{F}_1$-free graphs, where (see Fig. 6.1)*

$$\mathcal{F}_1 = \{F_1, F_2, F_3, F_4, F_5, F_6\}.$$

A set of forbidden induced subgraphs $\mathcal{F}_2$, under which Algorithm MIN always results in finding a maximum independent is given by Harant et al. [95]. Zverovich [172] also obtained another set of forbidden subgraphs, $\mathcal{F}_3$, for the MIN algorithm. The two results are summarized in the following theorem.

Theorem 6.6. *[95, 172] Given the two following finite graph sets (see Fig. 6.1):*

$$\mathcal{F}_2 = \{F_1, F_3, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}\} \text{ and}$$

$$\mathcal{F}_3 = \{F_1, F_4, F_5, F_6, F_7, F_8, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}, F_{20}, F_{21}, F_{22}, F_{23}, F_{24}\},$$

Algorithm MIN always gives us a maximum independent set for $\mathcal{F}_2$-free graphs and for $\mathcal{F}_3$-free graphs.

6.4.2 Algorithm MAX

The following result describes a set of forbidden induced subgraphs, under which Algorithm MAX gives a maximum independent set.

Theorem 6.7. *Let G be an $\mathcal{F}_4$-free graph of order $n \geq 7$, where (see Fig. 6.1)*

$$\mathcal{F}_5 = \{F_4, F_{15}, F_{19}, F_{20}, F_{21}, F_{24}, F_{25}, F_{26}, F_{27}\}.$$

Then

$$k_{MAX}(G) = \alpha(G).$$

Proof. For contradiction, suppose that there exists some connected $\mathcal{F}_5$-free graph G, $n(G) \geq 7$ and $E(G) \neq \emptyset$, such that there exists a vertex $u \in V(G)$ and u is of maximum degree but u is not α-redundant, i.e. Algorithm MAX will fail if it chooses and removes u.
It is obvious that u is α-redundant if there exists a maximum independent set S not containing u. We start by considering a maximum independent set S containing u and let $T = V(G)\backslash S$.

Claim 6.7.1. *There exist some vertices $v_1, v_2 \in N(u)$ such that $v_1 \nsim v_2$.*

Proof. Suppose that $N(u)$ is a clique. Then for every $v \in N(u)$, $\deg(v) \geq \deg(u) \geq \deg(v)$. Hence, $N[u]$ is a clique and a connected component of G, i.e. $G[N[u]] = G$ and there exists a maximum independent set S of G containing a neighbor of u and not containing u, i.e. u is α-redundant, a contradiction. □

Claim 6.7.2. *If there exist some vertices $v_1, v_2 \in N(u)$ such that $v_1 \nsim v_2$, then there exists a vertex $u' \in S$ such that $\{u, u', v_1, v_2\}$ induces a $K_{2,2}$ (i.e. a C_4).*

Proof. By Lemma 5.3 ($m = 2$), there exists vertices $u_1, u_2 \in S$ such that $u_1 \sim v_1$ and $u_2 \sim v_2$. Since $\{u, u_1, u_2, v_1, v_2\}$ does not induce a P_5 (i.e. an F_{25}), $u_2 \sim v_1$ or $u_1 \sim v_2$, i.e. we have the claim. □

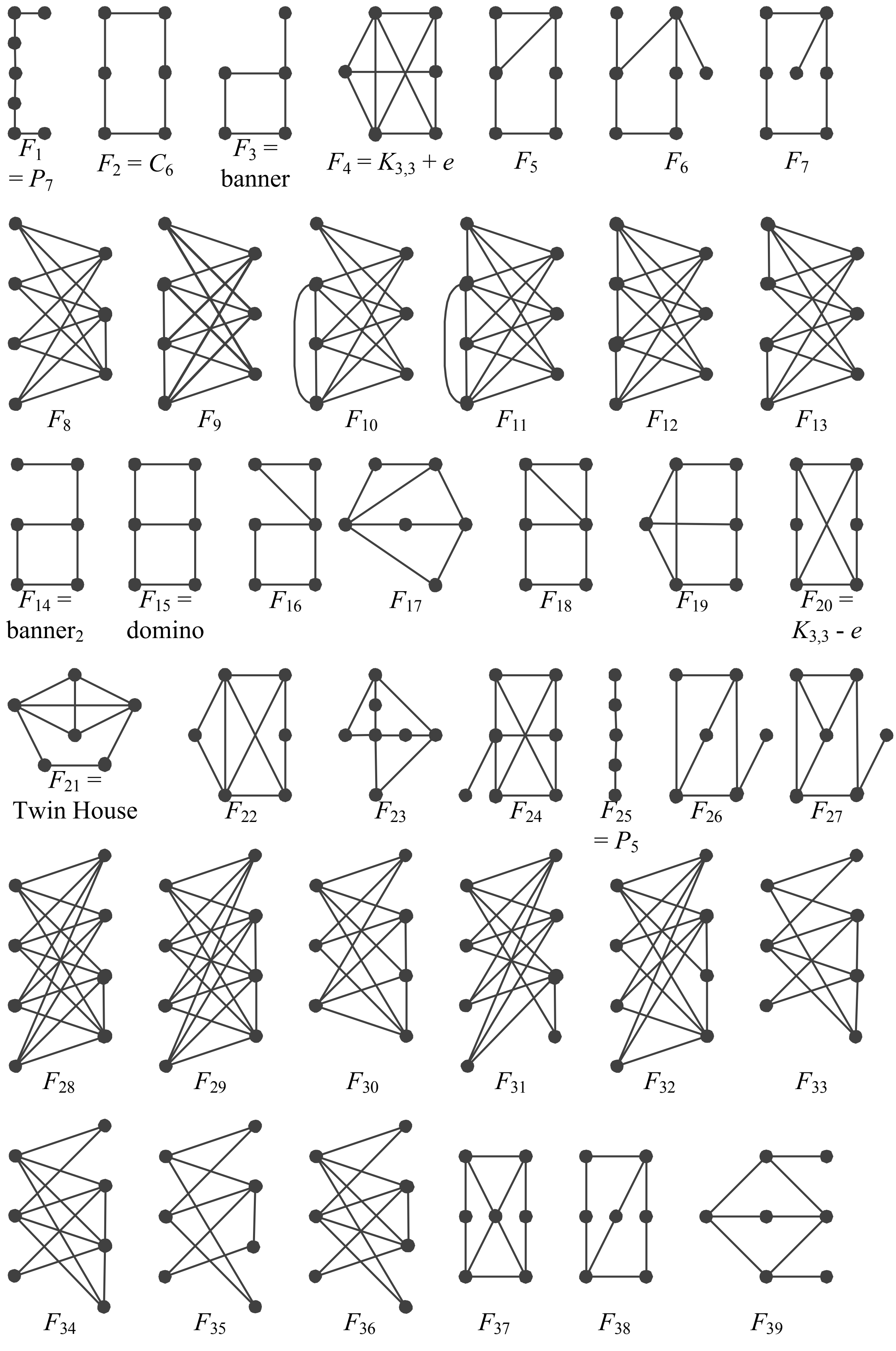

Fig. 6.1: Forbidden Induced Subgraphs for some Heuristic Greedy Algorithms

Claim 6.7.3. *There exist no vertices* $v_1, v_2 \in T$ *and* $u_1, u_2 \in S$ *such that* $\{u, u_1, u_2, v_1, v_2\}$ *induces a* $K_{2,3}$.

Proof. Suppose that there exist such vertices. Let H be a maximal induced complete bipartite subgraph of G with parts A and B such that $\{u, u_1, u_2\} \subset A \subset S$ and $\{v_1, v_2\} \subset B \subset T$.
Case 1. $|B| < |A|$.
Since $\deg(u) \geq \deg(v_2)$, there exists $v_3 \in T \backslash V(H)$ such that $v_3 \sim u$ and $v_3 \nsim v_2$. If $N_A(v_3) = A$, then $v_3 \sim v$ for some $v \in B$ (otherwise, H is not maximal). Without loss of generality, let $v = v_1$. Then $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces a $K_{3,3} + e$ (i.e. F_4), a contradiction.
Now, without loss of generality, we assume that there exists some $v_3 \in T \backslash V(H)$ such that $v_3 \sim u$, $v_3 \nsim v_1$, and $v_3 \nsim u'$ for some $u' \in A$, say $v_3 \nsim u_1$.
We show that $v_3 \sim u'$ for some $u' \in A \backslash \{u\}$. Indeed, if $v_3 \sim u'$ for every $u' \in A \backslash \{u\}$, then, by Claim 6.7.2, there exists some $u_3 \in S \backslash A$ such that u_3 is adjacent to v_1, v_3. Moreover, $v \nsim u_3$ for some $v \in B$ (otherwise we have a contradiction with the maximality of H). Assume that $v_2 \nsim u_3$. Then $v_3 \sim v_2$, otherwise $\{u_2, v_2, u, v_3, u_3\}$ induces a P_5, a contradiction. Now, $\{u, u_1, u_3, v_1, v_2, v_3\}$ induces an F_{19}, a contradiction. Now, $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20} or an F_{21}, depending on $v_3 \sim v_2$ or not, a contradiction.
Case 2. $|B| \geq |A|$, i.e $\exists v_3 \in B \backslash \{v_1, v_2\}$.
The set $S' = (S \backslash A) \cup B$ cannot be an independent set of G, otherwise, since $|S'| \geq |S|$ and $u \notin S'$, u is α-redundant. Hence, there exists some $u_3 \in S \backslash A$ such that $u_3 \sim v$ for some $v \in B$, say $v = v_1$. Moreover, the maximality of H implies that u_3 cannot be adjacent to every vertex of B. Without loss of generality, assume that $u_3 \nsim v_2$. Then $\{u, u_1, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{24} or an F_{20}, depending on $v_3 \sim u_3$ or not, a contradiction. □

By Claim 6.7.1, there exist vertices $v_1, v_2 \in N(u)$ such that $v_1 \nsim v_2$. Then, by Lemma 5.3 ($m = 2$), there exist vertices $u_1, u_2 \in S$ such that $u_1 \sim v_1$ and $u_2 \sim v_2$. Moreover, $\{u, u_1, u_2, v_1, v_2\}$ does not induce a P_5 nor a $K_{2,3}$ (by the Claim 6.7.3), hence it induces a banner. Without loss of generality, assume that u_1 is the vertex of degree one of the banner.
Let $H = G[\{u, u_1, u_2, v_1, v_2\}]$. Since $\deg_H(v_1) > \deg_H(u)$, there exists some $v_3 \notin V(H)$ such that $u \sim v_3$ and $v_3 \nsim v_1$. By Claim 6.7.3, $v_3 \nsim u_1$ or $v_3 \nsim u_2$, otherwise $\{u, u_1, u_2, v_1, v_3\}$ induces a $K_{2,3}$, a contradiction.
If $v_3 \sim u_1$ and $v_3 \nsim u_2$, then $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{15} or an F_{19}, depending on $v_2 \sim v_3$ or not, a contradiction.
If $v_3 \sim u_2$ and $v_3 \nsim u_1$, then $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{26} or an F_{27}, depending on $v_2 \sim v_3$ or not, a contradiction.
If v_3 is not adjacent to u_1, u_2, then, by Claim 6.7.2, there exists some $u_3 \in S$ such that $u_3 \sim v_3, u_3 \sim v_1$. Moreover, by Claim 6.7.3, $v_2 \nsim u_3$, otherwise $\{u, u_1, u_3, v_1, v_2\}$ induces a $K_{2,3}$, a contradiction. Now, $\{u, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{15} or an F_{20}, depending on $v_2 \sim v_3$ or not, a contradiction. □

6.4.3 Algorithm MMIN

In this subsection, we consider a forbidden induced subgraphs for Algorithm MMIN. Denote $k_{MMIN}(G)$ as the smallest cardinality of the maximal independent set obtained by applying the MMIN algorithm on the graph G. Then we obtain the following theorem.

Theorem 6.8. *Let G be an $\mathcal{F}_5$-free graph of order $n \geq 9$, where (see Fig. 6.1)*

$$\mathcal{F}_6 = \{F_1, F_7, F_{14}, F_{15}, F_{24}, F_{28}, F_{29}, F_{30}, F_{31}, F_{32}, F_{33}, F_{34}, F_{35}, F_{36}\}.$$

Then

$$k_{MMIN}(G) = \alpha(G)$$

Proof. We basically follow the idea used in [172] with replacing the MIN algorithm by MMIN algorithm. Note that in this proof, when we say that some vertex u is α-redundant, we refer to Corollary 6.2, i.e. there exist vertices v_1, v_2, where (v_1, u, v_2) induces a P_3, but there exist no vertices u_1, u_2 such that $\{u, u_1, u_2, v_1, v_2\}$ induces a P_5 or a banner or a $K_{2,3}$.
Suppose that G is an $\mathcal{F}_6$-free connected graph and the algorithm fails for G. That means there exists some $u_0 \in V(G)$ such that

1. u_0 is of minimum degree in G,
2. u_0 is not α-redundant, and
3. u_0 not belongs to any maximum independent set of G.

Without loss of generality, we may assume that G is a minimal graph (inclusive sense) containing such a vertex u_0.

Claim 6.8.1. *Every maximum independent set of G contains $N(u_0)$.*

Proof. If the statement does not hold, then there is a maximum independent set I of G and a vertex $v \in N(u_0)\backslash I$. Let $G' = G - v$. Then clearly, I is independent in G'. Hence, $\alpha(G') \geq |I| = \alpha(G)$. So, $\alpha(G') = \alpha(G)$, i.e. every maximum independent set of G' is a maximum independent set of G.
Note that $u_0 \in V(G')$ and $v \in N_G(u_0)$, hence, u_0 is of minimum degree in G'. We show that u_0 is not α-redundant in G'. Then by the minimality of G, u_0 belongs to some maximum independent set J of G' which is also a maximum independent set of G, a contradiction.
To show that u_0 is not α-redundant in G', we have to show that for arbitrary vertices $v_1, v_2 \in N_{G'}(u_0) \subset N_G(u_0)$ such that (v_1, u_0, v_2) induces a P_3, there exist vertices u_1, u_2 in G' such that $\{u_0, u_1, u_2, v_1, v_2\}$ induces a P_5 or a banner or a $K_{2,3}$. Since u_0 is not α-redundant in G, for such v_1, v_2 there exist vertices $u_1, u_2 \in V(G)$ such that $\{u_0, u_1, u_2, v_1, v_2\}$ induces a $K_{2,3}$ or a banner or a P_5 in G. Note that, such $u_1, u_2 \notin N_G(u_0)$, hence, $u_1, u_2 \in V(G')$. Thus, u_0 is not α-redundant in G'. □

Let S be a maximum independent set of G and $T = V(G)\backslash S$. Then $u_0 \in T$ and $N(u_0) \subset S$.

Claim 6.8.2. *Let $v \in T$ be at distance two from u_0. Then $|N_S(v)| \geq 2$*

Proof. Since the distance between u_0 and v is two, there exists some $w \in N_S(u_0) \cap N_S(v)$. If the statement is not true, then $S' = (S\backslash\{w\})\cup\{v\}$ is a maximum independent set of G and $N(u_0) \nsubseteq S'$, a contradiction. $\square$

Claim 6.8.3. *There exist vertices $u_1, u_2 \in T$ and $v_1, v_2 \in S$ such that $\{v_1, v_2, u_0, u_1, u_2\}$ induces a $K_{2,3}$.*

Proof. Since u_0 not belongs to any maximum independent set, u_0 is not simplicial (see Simplicial Vertex Reduction, Subsection 2.4.2), there exist vertices $v_1, v_2 \in N(u_0)$ such that (v_1, u_0, v_2) induces a P_3. Because u_0 is not α-redundant, there exists some u_1, u_2 such that $\{u_0, u_1, u_2, v_1, v_2\}$ induces a $K_{2,3}$ or a banner or a P_5 (Corollary 6.2). By the symmetry, we have to consider only the two following cases.
Case 1. $\{u_0, u_1, u_2, v_1, v_2\}$ induces a P_5 and $u_1 \sim v_1$, $u_2 \sim v_2$. Since both u_1, u_2 are of distance two from u_0, by Claim 6.8.2, $|N_S(u_1)|, |N_S(u_2)| \geq 2$.
1.1. There exists some $v_3 \in N_S(u_1)\cap N_S(u_2)$. We have $v_3 \nsim u_0$, otherwise $\{u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{15}, a contradiction. Since $(S\backslash\{v_1, v_2, v_3\}) \cup \{u_0, u_1, u_2\}$ is not independent, there exists some $v_4 \in S\backslash\{v_1, v_2, v_3\}$ such that v_4 is adjacent to at least one of vertices u_0, u_1, u_2. Now, $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_7 or an F_{14} or an F_{15} depending on whether v_4 is adjacent to exactly one vertex or two or three vertices of $\{u_0, u_1, u_2\}$, a contradiction.
1.2. $N_S(u_1) \cap N_S(u_2) = \emptyset$. Then there exists some $v_3 \in N_S(u_1)\backslash N(u_2)$ and $v_4 \in N_S(u_2)\backslash N(u_1)$. We have $v_3 \nsim u_0$ (similarly, $v_4 \nsim u_0$), otherwise $\{u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{14}, a contradiction. Now, $\{v_3, u_1, v_1, u, v_2, v_4\}$ induces an F_1, a contradiction.
Case 2. $\{u_0, u_1, u_2, v_1, v_2\}$ induces a banner and $u_2 \nsim v_1$.
Since u_2 is of distance two from u_0, there exists some $v_3 \in N_S(u_2)\backslash\{v_2\}$. Then $\{v_1, v_3, u_0, u_1, u_2\}$ induces an F_{14} or an F_{15}, a contradiction, or a $K_{2,3}$, depending of v_3 is adjacent to none, one, or two vertices among $\{u_0, u_1\}$. $\square$

Claim 6.8.4. *There exist no vertices $u_1, u_2 \in T$ and $v_1, v_2, v_3, v_4 \in S$ such that $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$ induces a $K_{3,4}$.*

Proof. For contradiction, suppose that there exist vertices $u_1, u_2 \in T$ and $v_1, v_2, v_3, v_4 \in S$ such that $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$ induces a $K_{3,4}$. Let H be a maximal induced complete bipartite subgraph of G with parts A and B such that $\{v_1, v_2, v_3, v_4\} \subset A \subset S$ and $\{u_0, u_1, u_2\} \subset B \subset T$. Consider the two following cases.
Case 1. $|B| < |A|$. Since $\deg(u_0) \leq \deg(v_1)$, there exists some $t \in N(v_1)\backslash(N(u_0)\cup B)$. This also implies $t \in T$.
1.1. t is adjacent to every vertex of A. Then t is adjacent to some $u_i \in B$, otherwise we have a contradiction with the maximality of H. Without loss of generality, suppose that $t \sim u_1$. Then $\{u_0, u_1, u_2, t, v_1, v_2, v_3, v_4\}$ induces an F_{29} or an F_{28} depending on $t \sim u_2$ or not, a contradiction.
1.2. t is non-adjacent to some vertex of A. Without loss of generality, assume that $t \nsim v_2$. We show that $t \nsim v_j$ for every $v_j \in A\backslash\{v_1\}$. Indeed, suppose that $t \sim v_j$ for some $v_j \in A\backslash\{v_1, v_2\}$. Then $t \sim u_k$ for every $u_k \in B\backslash\{u_0\}$, otherwise $\{u_0, u_k, t, v_1, v_2, v_j\}$ induces an F_{20}, a contradiction. Now, $\{u_0, u_1, u_2, t, v_1, v_2, v_j\}$ induces an F_{30}, a contradiction.
Hence, $\{u_0, u_1, u_2, t, v_1, v_2, v_3, v_4\}$ induces one of the following induced subgraphs F_{24}, F_{31}, or F_{32} depending on the adjacency between t and $\{u_1, u_2\}$, a contradiction.

Case 2. $|B| \geq |A| \geq 4$, i.e. there exists some $u_3 \in B \backslash \{u_0, u_1, u_2\}$. Since $S' = (S \backslash A) \cup B$ is not independent (otherwise S' is a maximum independent set containing u_0, a contradiction), there exists some $w \in S \backslash A$ such that w is adjacent to at least one vertex of B, assume that $w \sim u_j$. Note that w cannot be adjacent to all u_i belonging to B because of the maximality of H. Assume that $w \nsim u_k$ for some $u_k \in B$. If w is adjacent to some vertex $u_l \in B \backslash \{u_j, u_k\}$, then $\{u_j, u_k, u_l, w, v_1, v_2\}$ induces an F_{20}, a contradiction. If w is non-adjacent to every vertex of B but u_j, then $V(H) \cup \{w\}$ induces an F_{24}, a contradiction. □

Now, by Claim 6.8.3, let $u_1, u_2 \in T$ and $v_1, v_2 \in S$ such that $\{v_1, v_2, u_0, u_1, u_2\}$ induces a $K_{2,3}$. Let $A = N_S(\{u_0, u_1, u_2\})$. Since $|(S \backslash A) \cup \{u_0, u_1, u_2\}| < |S|$ (otherwise we have a maximum independent set containing u_0, a contradiction), $|A| \geq 4$. Moreover, since $(S \backslash \{v_1, v_2\}) \cup \{u_i, u_j\}$ is not independent for every two vertices u_i, u_j of u_0, u_1, u_2 (otherwise we have a maximum independent set not containing all neighbors of u_0, a contradiction with Claim 6.8.1), there exist vertices $v_3, v_4 \in N_S(\{u_0, u_1, u_2\}) \backslash \{v_1, v_2\}$ such that $|N_{\{u_0,u_1,u_2\}}(\{v_3, v_4\})| \geq 2$. By Claim 6.8.4, $|N_{\{u_0,u_1,u_2\}}(v_3)|$ or $|N_{\{u_0,u_1,u_2\}}(v_4)|$ is smaller than three.

If $N_{\{u_0,u_1,u_2\}}(v_3) = 2$ (similarly for $|N_{\{u_0,u_1,u_2\}}(v_4)| = 2$), then $\{u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20}, a contradiction.

If $|N_{\{u_0,u_1,u_2\}}(v_3)| = 1$ and $|N_{\{u_0,u_1,u_2\}}(v_4)| = 3$ (or vice versa), then $\{u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{24}, a contradiction.

The remaining case is $|N_{\{u_0,u_1,u_2\}}(v_3)| = |N_{\{u_0,u_1,u_2\}}(v_4)| = 1$. Without loss of generality, we assume that v_3 is adjacent to u_1 but neither to u_0 nor u_2. Since $\deg(u_0) \leq \deg(v_3)$, there exists some $u_3 \in N(v_3) \backslash N(u_0)$.

If $u_3 \nsim u_1$, then $\{u_0, u_1, u_3, v_1, v_2, v_3\}$ induces an F_{14} or an F_{15} or an F_{20} depending on the adjacency between u_3 and $\{v_1, v_2\}$, a contradiction.

If u_3 is adjacent to u_1, u_2 and not adjacent to v_1, v_2, then $\{u_0, u_1, u_2, v_1, v_2, u_3\}$ induces an F_{20}, a contradiction.

If u_3 is adjacent to u_1, u_2 and adjacent to exactly one of v_1, v_2, then $\{u_0, u_1, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{33}, a contradiction.

If u_3 is adjacent to u_1, u_2, v_1, v_2, then $\{u_0, u_1, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{34}, a contradiction.

If $u_3 \nsim u_2$ and u_3 is adjacent to exactly one vertex of v_1, v_2, then $\{u_0, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{14}, a contradiction.

If $u_3 \sim u_1$ and u_3 is not adjacent to v_1, v_2, u_2, then $\{u_0, u_1, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{35}, a contradiction.

If $u_3 \nsim u_2$ and u_3 is adjacent to v_1, v_2, u_1, then $\{u_0, u_1, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{36}, a contradiction. □

6.4.4 Algorithm MMAX

Denote $k_{MMAX}(G)$ as the smallest cardinality of the maximal independent set obtained by applying the MMAX algorithm on the graph G, the following theorem provides a set of forbidden induced subgraphs for Algorithm MMAX.

Theorem 6.9. *Let G be an $\mathcal{F}_7$-free graph of order $n \geq 9$, where (see Fig. 6.1)*

$$\mathcal{F}_7 = \{F_1, F_5, F_7, F_8, F_{14}, F_{15}, F_{18}, F_{20}, F_{21}, F_{24}, F_{37}, F_{38}, F_{39}\}.$$

Then

$$k_{MMAX}(G) = \alpha(G).$$

Proof. We basically follow the idea of the proof of Theorem 6.7. Suppose that there exists some connected $\mathcal{F}_7$-free graph G, $n(G) \geq 7$, and some vertex $u_0 \in V(G)$ such that u_0 is of maximum degree in G and u_0 is not α-redundant.
In this proof, for every vertex $v \in N(u_0)$, we call v not α-redundant in the following sense (Lemma 5.4, the case $m = 1$ and $m = 3$):

1. There exists some vertex $u_1 \in N(v)\backslash N[u]$, then $N(u_1)\backslash N[v] \neq \emptyset$ or
2. there exist some vertices $u_1, u_2 \in N(v)\backslash N[u]$ and $u_1 \nsim u_2$, then there exist vertices $v_1, v_2, v_3 \in N(u)\backslash N[v]$ such that $\{v_1, v_2, v_3\}$ is independent and $v_1 \sim u_0$, $v_2 \sim u_1$, $v_3 \sim u_2$.

Like in the proof of Theorem 6.7, we have the following observation.

Claim 6.9.1. *There exist some vertices $v_1, v_2 \in N(u_0)$ such that $v_1 \nsim v_2$.*

Let S be a maximum independent set of G. Then $u_0 \in S$. Denote $T = V(G)\backslash S$. Clearly, $N(u_0) \subset T$.

Claim 6.9.2. *There exist vertices $v_0 \in N(u_0)$ and $u_1, u_2 \in S$ such that $\{v_0, u_0, u_1, u_2\}$ induces a $K_{1,3}$.*

Proof. By Claim 6.9.1, there exist vertices $v_1, v_2 \in N(u_0)$ such that $v_1 \nsim v_2$. Moreover, u_0 is not α-redundant. By Lemma 5.3 ($m = 2$), there exist vertices $u_1, u_2 \in S$ such that $u_1 \sim v_1$, $u_2 \sim v_2$. Then $(u_1, v_1, u_0, v_2, u_2)$ induces a P_5, otherwise we have a desired $K_{1,3}$. Since v_1, v_2 are not α-redundant, $N(u_1)\backslash N[v_1]$ and $N(u_2)\backslash N[v_2]$ are not empty. We consider the two following cases.
Case 1. There exists some $v_3 \in (N(u_1)\backslash N[v_1]) \cap (N(u_2)\backslash N[v_2])$. If $v_3 \sim u_0$, then v_3 is such a vertex v_0 of the conclusion of the claim. Hence, we assume that $v_3 \nsim u_0$. Since $(S\backslash\{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_3\}$ is not independent, there exists some $u_3 \in S\backslash\{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one of the vertices v_1, v_2, v_3. If u_3 is adjacent to only one vertex among $\{v_1, v_2, v_3\}$, then $\{u_0, u_1, u_2, u_3, v_1, v_2\}$ induces an F_7, a contradiction. If u_3 is adjacent to two or more vertices among $\{v_1, v_2, v_3\}$, then v_1 or v_2 is such a vertex v_0 of the conclusion of the claim.
Case 2. There exist some vertices $v_3 \in N(u_1)\backslash(N[v_1] \cup N(u_2))$ and $v_4 \in N(u_2)\backslash(N[v_2] \cup N(u_1))$. This case is processed through considered all following subcases.
2.1. $v_3 \sim v_4$.
2.1.1. $v_3 \sim u_0$ (similar for the case $v_4 \sim u_0$). Then $v_3 \sim v_2$, otherwise $\{u_2, v_2, u_0, v_1, u_1, v_3\}$ induces an F_{14}, a contradiction. Moreover, $v_4 \nsim v_1$, otherwise $\{u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{15}, a contradiction. Hence, $v_4 \sim u_0$, otherwise $\{u_2, v_4, v_3, u_1, v_1, u_0\}$ induces an F_{14}, a contradiction. Now, $\{u_1, v_1, u_0, v_2, u_2, v_4\}$ induces an F_{14}, a contradiction.
2.1.2. u_0 is neither adjacent to v_3 nor v_4. If $v_3 \sim v_2$ (similar for the case $v_4 \sim v_1$), then $v_1 \sim v_4$, otherwise $\{v_1, u_0, v_2, u_2, v_4, v_3\}$ induces an F_{14}, a contradiction. Now, $\{u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{15}, a contradiction.
Hence, we assume that $v_1 \nsim v_4$ and $v_2 \nsim v_3$. Since $(S\backslash\{u_1, u_0, u_2\}) \cup \{v_1, v_2, v_3\}$ (and $(S\backslash\{u_1, u_0, u_2\}) \cup \{v_1, v_2, v_4\}$ either) is not independent, there exists some $u_3 \in S\backslash\{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one vertex among $\{v_1, v_2, v_3, v_4\}$. Hence, u_3 is adjacent to v_1 or v_2, otherwise $\{u_0, u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ induces an F_1,

a contradiction. Now, v_1 or v_2 is such a vertex v_0 of the conclusion of the claim.
2.2. $v_3 \nsim v_4$.
2.2.1. $v_3 \sim u_0$ (similar for the case $v_4 \sim u_0$). Then $v_3 \sim v_2$, otherwise $\{u_2, v_2, u_0, v_1, u_1, v_3\}$ induces an F_{14}, a contradiction. Now, $v_1 \sim v_4$ if and only if $v_4 \sim u_0$, otherwise $\{u_2, v_4, u_0, v_1, u_1, v_3\}$ induces an F_{14}, a contradiction. Hence, we have the two following subcases.
(i) v_4 is adjacent to v_1 and u_0. Then $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{37}, a contradiction.
(ii) v_4 is non-adjacent to u_0 and v_1. Since $(S \backslash \{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_4\}$ is not independent, there exists some $u_3 \in S \backslash \{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one vertex among v_1, v_2, v_4. Hence, u_3 is adjacent to v_1 or v_2, otherwise $u_3 \sim v_4$ and $\{u_3, v_4, u_2, v_2, u_0, v_1, u_1\}$ induces an F_1, a contradiction. Now, v_1 or v_2 is such a vertex v_0 of the conclusion of the claim.
2.2.2. u_0 is not adjacent to v_3, v_4.
(i) $v_3 \sim v_2$ (similar for the case $v_4 \sim v_1$). Then $v_4 \nsim v_1$, otherwise $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{38}, a contradiction. Since $(S \backslash \{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_4\}$ is not independent, there exists some $u_3 \in S \backslash \{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one vertex among v_1, v_2, v_4. Hence u_3 is adjacent to v_1 or v_2, otherwise $u_3 \sim v_4$ and $\{u_3, v_4, u_2, v_2, u_0, v_1, u_1\}$ induces an F_1, a contradiction. Now, v_1 or v_2 is such a vertex v_0 of the conclusion of the claim.
(ii) $v_3 \nsim v_2$, i.e. v_3 is not adjacent to u_0, v_2, and v_4 and similarly, v_4 is not adjacent to v_1 and u_0. Now, $\{v_3, u_1, v_1, u_0, v_2, u_2, v_4\}$ induces an F_1, a contradiction. □

Claim 6.9.3. *Let $v_0 \in N(u_0)$ and $u_1, u_2 \in N_S(v_0) \backslash \{u_0\}$. Then there exist some vertices v_1, v_2 such that $\{u_0, u_1, u_2, v_0, v_1, v_2\}$ induces a $K_{3,3}$.*

Proof. Since $\{v_0, u_0, u_1, u_2\}$ induces a $K_{1,3}$ and v_0 is not α-redundant, by Lemma 5.3 ($m = 3$) there exist some vertices $v_1, v_2, v_3 \in V(G)$ such that $\{v_0, v_1, v_2, v_3\}$ is independent and $u_i \sim v_{i+1}$ for $i = 0, 1, 2$. Let $X = \{u_0, u_1, u_2\}$. By the symmetry, we consider the following cases.
Case 1. $|N_X(v_i)| = 3$ for at least two integers i among $0, 1, 2$. Then $\{u_0, u_1, u_2, v_0, v_1, v_2, v_3\}$ induces a $K_{3,3}$.
Case 2. $|N_X(v_2)| = 2$ and $N_X(v_2) = \{u_1, u_2\}$. Then $\{u_0, u_1, u_2, v_0, v_1, v_2\}$ induces an F_{14}, an F_{15}, or an F_{20} depending on $|N_X(v_1)|$ is one, two, or three, respectively.
Case 3. $|N_X(v_1)| = |N_X(v_2)| = 1$ and $|N_X(v_3)| = 3$. Then $\{u_0, u_1, u_2, v_0, v_1, v_2, v_3\}$ induces an F_{39}, a contradiction.
Case 4. $|N_X(v_i)| = 1$ for $i = 1, 2, 3$. Let H be the maximal graph consisting of k induced paths of lengths 2 of the form (v_0, u_i, v_{i+1}), where v_0 is the common initial vertex. Since $(S \backslash \{u_0, u_i\}) \cup \{v_0, v_{i+1}\}$ is not independent for every i, for each i $(2 \le i \le k)$, there exists some vertex $w_i \in S \backslash \{u_0, u_{i-1}\}$ such that w_i is adjacent to v_0 or v_i. The rest of the proof is processed by considering the following subcases.
4.1. There exists some index i, such that $w_i \sim v_i$, without loss of generality, assume that $w_2 \sim v_2$. By Lemma 5.3 ($m = 1$), there exists some $u \in N_S(v_1) \backslash \{u_0\}$. If $u = w_2$, then $\{u_0, u_1, u_2, w_2, v_0, v_1, v_2\}$ induces an F_{15} or an F_7 depending on $w_2 \sim v_0$ or not, a contradiction. If $u \neq w_2$, then $\{u, v_1, u_0, v_0, u_1, v_2, w_2\}$ induces an F_1, an F_7, an F_{14}, or an F_{15} depending on the adjacency between $\{u, w_2\}$ and $\{v_0, v_1, v_2\}$, a contradiction.
4.2. There exists some vertex $w \in S \backslash \{u_0, u_1, \ldots, u_m\}$ such that $w \sim v_0$ and $w \nsim v_i$ for $i \ge 2$. Then $w \nsim v_1$, otherwise $\{v_2, u_1, v_0, u_0, v_1, w\}$ induces an F_{15}, a contradic-

tion. Since v_0 is not α-redundant, by Lemma 5.3 ($m = 1$), there exists some vertex $t \in N(w) \backslash N[v_0]$ and by the maximality of H, t is adjacent to some vertex u_i or v_i. Moreover, since $(S \backslash \{u_0\}) \cup \{v_1\}$ is not independent, there exists some vertex $u \in S \backslash V(H)$ such that $v_1 \sim u$.

4.2.1. t is not adjacent to any v_i. Then $t \sim u_i$ for some u_i and t is adjacent to all others u_j, otherwise $\{v_{j+1}, u_j, v_0, u_i, t, w\}$ induces an F_{14}, a contradiction. Thus, $\{u, u_0, w, v_0, v_1, t\}$ induces an F_{15} or an F_{14} depending on $t \sim u$ or not, a contradiction.

4.2.2. $t \sim v_i$ for some $i \geq 1$.

(i) $t \nsim v_1$. Then, without loss of generality, assume that $t \sim v_2$. Hence, t is not adjacent to u_0, u_1, and u, otherwise $\{u_0, u_1, u, w, v_0, v_1, t\}$ induces an F_7, an F_{14}, or an F_{15}, a contradiction. Now, $\{t, v_2, u_1, v_0, u_0, v_1, u\}$ induces an F_1, a contradiction.

(ii) $t \sim v_1$.

(a) t is adjacent to u_i for some $i \geq 1$. Then $t \sim u$, otherwise $\{u, v_1, t, w, v_0, u_i\}$ induces an F_{14}, a contradiction. Now, $\{t, u, v_0, v_1, u_0, u_i\}$ induces an F_{18} or an F_5 depending on $t \sim u_0$ or not, a contradiction.

(b) $t \nsim u_i$ for any $i \geq 1$ and $t \sim u_0$. Then $t \sim v_{i+1}$ for every $i \geq 1$, otherwise $\{v_{i+1}, u_i, v_0, u_0, t, w\}$ induces an F_{14} or $\{v_1, t, v_0, v_2, v_3, u_1, u_2\}$ induces an F_7, a contradiction.

(c) $t \nsim u_i$ for every i. Then $t \nsim u$, otherwise $\{t, w, v_0, u_0, v_1, u\}$ induces an F_5, a contradiction. Now, $\{u, v_1, t, w, v_0, u_1, v_2\}$ induces an F_1, a contradiction. □

Claim 6.9.4. *There exist no vertices $u_1, u_2 \in S$ and v_1, v_2, v_3 such that $\{u_0, u_1, u_2, v_1, v_2, v_3\}$ induces a $K_{3,3}$.*

Proof. Suppose there exist such vertices. Let H be a maximal induced complete bipartite subgraph of G with parts A and B such that $A = \{u_0, u_1, \ldots u_p\} \subset S$ and $B = \{v_1, v_2, \ldots, v_q\} \subset T$ ($p \geq 2$ and $q \geq 3$).

Case 1. $p < q$. Since $S' = (S \backslash A) \cup B$ is not an independent set of G, there exists some vertex $u \in S \backslash A$ such that $u \sim v_i$ for some $v_i \in B$, say $u \sim v_1$. Moreover, the maximality of H implies that u is not adjacent to every vertex of B. Without loss of generality, assume that $u \nsim v_2$. Then u is not adjacent to any vertex $v_i \in B \backslash \{v_1\}$, otherwise $\{u, u_0, u_1, v_1, v_2, v_i\}$ induces an F_{20}, a contradiction. Now, $\{u, u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{24}, a contradiction.

Case 2. $3 \leq q \leq p$. Since $\deg(u_0) \geq \deg(v_2)$, there exists some vertex $v \in T \backslash V(H)$ such that $v \sim u_0$ and $v \nsim v_2$.

2.1. $N_A(v) = A$. By the maximality of H, $v \sim v_i$ for some $v_i \in B$. Without loss of generality, assume that $v \sim v_1$. Now, $\{u_0, u_1, u_2, u_3, v_1, v_2, v\}$ induces an F_8, a contradiction.

2.2. $v \nsim u_i$ for some $u_i \in A$. Without loss of generality, assume that $v \nsim u_1$. Then v is not adjacent to any vertex u_i, $i \geq 2$, otherwise $\{u_0, u_1, u_i, v, v_1, v_2\}$ induces an F_{21} or an F_{20} depending on $v \sim v_2$ or not, a contradiction. Now, since $(S \backslash \{u_0\}) \cup \{v\}$, there exists some vertex $u \in N_S(v) \backslash A$.

2.2.1. u is adjacent to some vertex v_i of B. Without loss of generality, assume that $u \sim v_1$. By the maximality of H, u is not adjacent to some vertex v_i of B different from v_1. Without loss of generality, assume that $u \nsim v_2$. Thus, $\{u, u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20} or an F_{24} depending on $u \sim v_3$ or not, a contradiction.

2.2.2. u is not adjacent to any vertex $v_i \in B$. So, v is adjacent to every vertex v_i of B different from v_1, otherwise $\{u, v, u_0, v_1, u_1, v_i\}$ induces an F_{14}. Now,

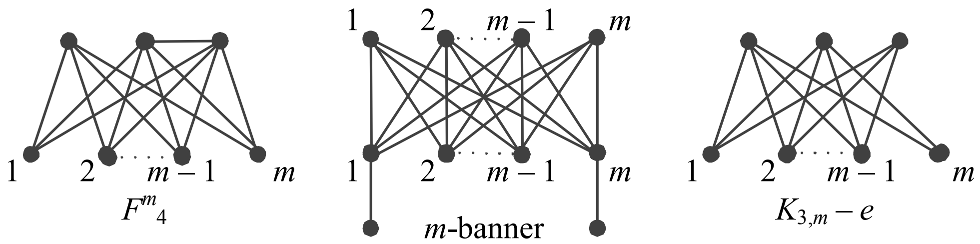

Fig. 6.2: Some Forbidden Induced Subgraphs for Algorithm MAX-l

$\{v, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20}, a contradiction. □

Together, all above claims prove the statement of the theorem. □

6.4.5 Algorithm MAX-l

Assume that we have implemented a polynomial algorithm which can find a maximum independent set in $\mathcal{F} \cup \{K_{1,m}\}$-free graph (Step 6 in Algorithm 13) for some graph class $\mathcal{F}$. Then the following theorem describes a set of forbidden induced subgraphs for Algorithm MAX-l.

Lemma 6.10. *Given an integer $l \geq 6$, let G be an $\mathcal{F}_8$-free graph of order $n \geq 2l + 1$, where*

$$\mathcal{F}_8 = \mathcal{F} \cup \{F_{14}, F_{15}, S_{2,2,2}, K_{3,m} - e, F_4^m, m\text{-banner}\}.$$

Then $k_{MMAX-l}(G) = \alpha(G)$ for $l \geq 2m - 2$ and $m \geq 4$ (see Fig. 6.1 and Fig. 6.2).

Proof. We follow the proof of Theorem 6.7. Suppose that there exists some connected $\mathcal{F}_8$-free graph G such that $n(G) \geq 7$. Moreover, there exists some vertex $u \in V(G)$ such that

1. u is the center vertex of some $K_{1,l}$,
2. u is of maximum degree among such center vertices of $K_{1,l}$'s in G, and
3. and u is not α-redundant.

Let S be a maximum independent set of G (hence, $u \in S$), and $T = V(G) \backslash S$. Let $W = \{v_1, v_2, \ldots, v_p\}$ be the maximal vertex subset such that $\{u, v_1, \ldots, v_p\}$ is a star whose the center vertex is u and $p \geq l$.

Claim 6.10.1. *There exist some vertices $u_1, u_2, \ldots, u_p \in S$ such that $\{u, u_2, \ldots, u_p, v_1, \ldots, v_p\}$ induces a $K_{p,p}$.*

Proof. By Lemma 5.3 ($m = p$) and Lemma 3.5, together with the $S_{2,2,2}$-freeness of G, there exist some vertices $u_1, u_2, \ldots, u_p \in S$, and without loss of generality, we may assume that for every $1 \leq i \leq p - 2$, $i \leq j \leq p$, $u_i \sim v_j$.
Moreover, $u_i \sim v_j$ for every $1 \leq i \leq p - m + 2$ and $j < i$, otherwise $\{u, u_1, u_i, v_j, v_{p-m+2}, \ldots, v_p\}$ induces a $K_{3,m} - e$, a contradiction. Then $u_i \sim v_j$ for every $m - 1 \leq i \leq p - 2$ and $j < i$, otherwise $\{u, u_1, \ldots, u_{m-2}, u_i, v_j, v_{p-1}, v_p\}$ induces a $K_{3,m} - e$, a contradiction.
Now, to avoid m-banner, u_{p-1} or u_p has at least two neighbors among $\{v_1, \ldots, v_p\}$.

Without loss of generality, assume that $u_{p-1} \sim v_i$ for some $i \neq p-1$. Then $u_{p-1} \sim v_j$ for every $j \neq i, p-1$, otherwise $\{u, u_1, \dots, u_{m-2}, u_{p-1}, v_{p-1}, v_i, v_j\}$ induces a $K_{3,m} - e$, a contradiction. Now $\{u, u_1, \dots, u_{p-1}, v_1, \dots, v_p\}$ induces a $K_{p,p}$. □

Without loss of generality, let H be a maximal induced complete bipartite subgraph of G with parts A and W such that $\{u, u_2, \dots, u_p\} \subset A \subset S$, it implies $q := |A| \geq p$. Let $A = \{u, u_2, \dots, u_p, \dots, u_q\}$. Then we have the following observation.

Claim 6.10.2. *There exists some vertex $v \in T \backslash W$ such that $v \sim u$ and $v \nsim v_i$ for some $v_i \in W$.*

Proof. We consider the two following cases.
Case 1. $p = |W| < |A|$. Since $\deg(u) \geq \deg(v_1)$ (note that v_1 also is a center vertex of a $K_{1,l}$, say $\{v_1, u, u_2, \dots, u_l\}$), there exists some vertex $v \in T \backslash V(H)$ such that $v \sim u$ and $v \nsim v_1$.
Case 2. $p = |W| = |A|$. Since the set $S' = (S \backslash A) \cup W$ cannot be an independent set, there exists some vertex $u' \in S \backslash A$ such that $u' \sim v$ for some $v \in W$. Assume that $u' \sim v_1$. Since $\deg(u) \geq \deg(v_1)$, there exists some vertex $v \in T \backslash V(H)$ such that $v \sim u$ and $v \nsim v_1$. □

Without loss of generality, assume that $v \nsim v_2$. Moreover, by the maximality of W, $v \sim v_i$ for some $v_i \in W$. Wihout loss of generality, assume that $v \sim v_1$. Note that v has at most $m-1$ neighbors in A, otherwise m neighbors of v in A, together with v, v_1, v_2 induce an F^4_m, a contradiction. Moreover, v has at most $m-1$ neighbors in W, otherwise $m-1$ neighbors of v in W, together with v, u_i, u_j, v_2 induces a $K_{3,m} - e$ for u_i, u_j are two non-neighbors of v in A, a contradiction. We consider the two following cases.
Case 1. v is not adjacent to any vertex of $A \backslash \{u\}$. Then there exists some vertex $u' \in S \backslash A$ such that $u' \sim v$, otherwise $(S \backslash u) \cup \{v\}$ is a maximum independent set not containing u, a contradiction. By the maximality of H, there exists some vertex $v_i \in W$ such that $u' \nsim v_i$. If u' is adjacent to two vertices $v_j, v_k \in W$, then $\{u, u', u_2, \dots, u_{m-1}, v_i, v_j, v_k\}$ induces a $K_{3,m} - e$, a contradiction. Hence, u' is adjacent to at most one vertex v' of W. Now, $\{v_j, u_2, v_k, u, v, u'\}$ induces an F_{14}, for $v_j, v_k \in W$ are two non-neighbors of both v and u', a contradiction.
Case 2. v is adjacent to some vertex of $A \backslash \{u\}$, without loss of generality, assume that $v \sim u_2$. Then $m-1$ non-neighbors of v in W, together with v, u, u_2, u_i induces a $K_{3,m} - e$ for some non-neighbor u_i of v in A, a contradiction. □

Together with Theorem 5.14, the above lemma leads to the following result.

Theorem 6.11. *The MIS problem is polynomially solvable (by Algorithm MAX-l) in ($S_{2,2,2}$,banner$_2$,domino,F_{39}, $K_{3,m} - e$, F^4_m)-free graphs.*

6.4.6 Comparision

The following results are obvious.

Proposition 6.12.

- *F_1 induces F_{25}.*

- *F_7 induces F_2 and $F_8, \dots, F_{13}$ induce F_4.*
- *$F_{14}, \dots, F_{24}$ induce F_3.*
- *F_{28}, F_{29} induce F_8 and $F_{30}, \dots, F_{36}$ induce F_{21}.*
- *F_{26}, F_{27} induce F_3.*
- *$F_1, F_5, F_7, F_{14}, F_{15}, F_{37}, F_{38}, F_{39}$ induce F_{25}.*

Proposition 6.13.

- *Every $\mathcal{F}_4$-free graph is $\mathcal{F}_1$-free and $\mathcal{F}_5$-free.*
- *Every $\mathcal{F}_1$-free graph is $\mathcal{F}_2$-free and $\mathcal{F}_3$-free.*
- *Every $\mathcal{F}_2$-free graph and every $\mathcal{F}_3$-free graph are $\mathcal{F}_6$-free.*
- *Every $\mathcal{F}_5$-free graph is $\mathcal{F}_7$-free.*

6.5 Performance of Algorithms

In Subsection 6.4.6, we compared greedy heuristics algorithms in the sense of forbidden induced subgraph sets. Some observations in this approach are: MIN is better than VO; MMIN is better than MIN; and MMAX is better than MAX; all in forbidden induced subgraph set sense. In this section, we compare the greedy heuristic algorithms mentioned in this chapter by considering their performances on some special graphs.

Proposition 6.14. *For every integer p, there exist graphs G such that:*

$$k_{MMIN}(G) - k_{MIN}(G) > p \qquad \text{and} \qquad k_{MMAX}(G) - k_{MAX}(G) > p.$$

Proof. Let H_1 and H_3 be two K_p's and H_2 be a $\overline{K_p}$. Let

$$G := H_1 \times H_2 \times H_3.$$

Then

$$k_{MAX}(G) = k_{MIN}(G) = 2 \qquad \text{while} \qquad k_{MMAX}(G) = k_{MMIN}(G) = p = \alpha(G).$$

□

6.6 Discussion

In the sense of forbidden induced subgraph sets, heuristic methods mentioned in this chapter perform not so well in comparing with, for example, the result of Lokshtanov et al. [115] and the results of the two previous chapters. However, there are not many results about polynomial time solution for the MIS problem in some subclasses of P_7-free graphs (Theorem 6.8) except for (P_7,banner)-free graphs [7], and $(P_7, K_{1,m})$-free graphs [131], and not many results about polynomial time solution for the MIS problem in subclasses of $S_{2,2,2}$-free graphs (Theorem 6.11) except ($S_{2,2,2}$,banner)-free graphs [76],

and some results of Chapters 3 and 5. Our results in this chapter can be considered as a contribution in subclasses of P_7-free graph and of $S_{2,2,2}$-free graphs. Remind that the complexity of the problem for the class of P_7-free graphs or the class of $S_{2,2,2}$-free graphs is still an open question. Our results in this chapter also follow the approach of Mahadev and Reed [132], Harant et al. [95], and Zverovich [171].
Moreover, greedy heuristic methods can be easily implemented and they also have low complexity in comparing with method of Lokshtanov et al. [115] or methods mentioned in Chapter 3. Our combined methods also suggest that we can combine other (conditionally) exact methods with greedy methods to obtain interesting algorithms, especially in chosing the next vertex in general by best-in or worst-out strategies.

7 Graphs of Bounded Maximum Degree

In this chapter, we consider the MIS-Δ problem, i.e. the MIS problem restricted on graphs of maximum degree at most Δ for a given integer Δ. In the first section, we start by reviewing some known results. In Section 7.2, we describe some results about NP-easy classes for the MIS-Δ problem. Section 7.3 is devoted to sucubic graphs, i.e. the MIS-3 problem. In the last section, we summarize some results of the chapter. Given an integer Δ and a graph class $\mathcal{X}$, we also call $\mathcal{X}$ MIS-Δ-easy or MIS-Δ-hard as the concepts MIS-easy, MIS-hard, respectively, restricted on graphs of $\mathcal{X}$ of maximum degree at most Δ .

7.1 Known Results

Lozin and Milanič [124] used distance argument to show that the $(H_k, H_{k+1}, \ldots)$-free graph (see Fig. 1.1) class is MIS-Δ-easy for given integers k and Δ. By using combination technique of modular decomposition and clique separators of Brandstädt and Hoàng [29], Lozin and Milanič also showed the MIS-Δ-easiness for (apple$_k$,apple$_{k+1}$,...)-free graphs. Based on this result, Lozin et al. [126] obtained the MIS-3-easiness for $S_{2,2,2}$-free subcubic graphs. However, as mentioned in Subsection 2.6.3, we haven't got a full proof for the combined method of Brandstädt and Hoàng [29] yet. Hence, in this chapter, we always use these results with some remarks.
Lozin and Rautenbach [130] showed that for a given integer k and Δ, $(S_{k,k,k}, T_{k,k,k})$-free graphs of maximum degree at most Δ (see Fig. 7.1) are of bounded tree-width. It leads to the following consequence.

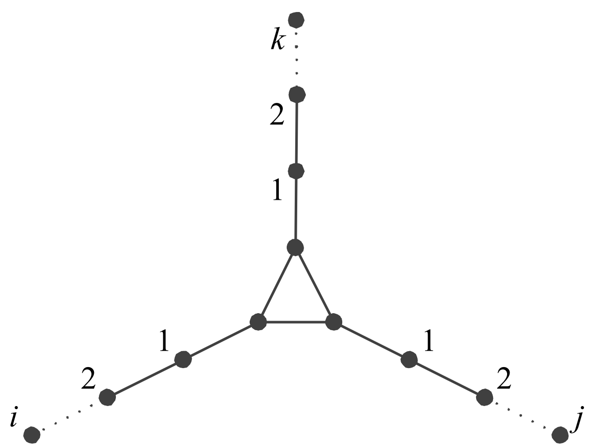

Fig. 7.1: $T_{i,j,k}$

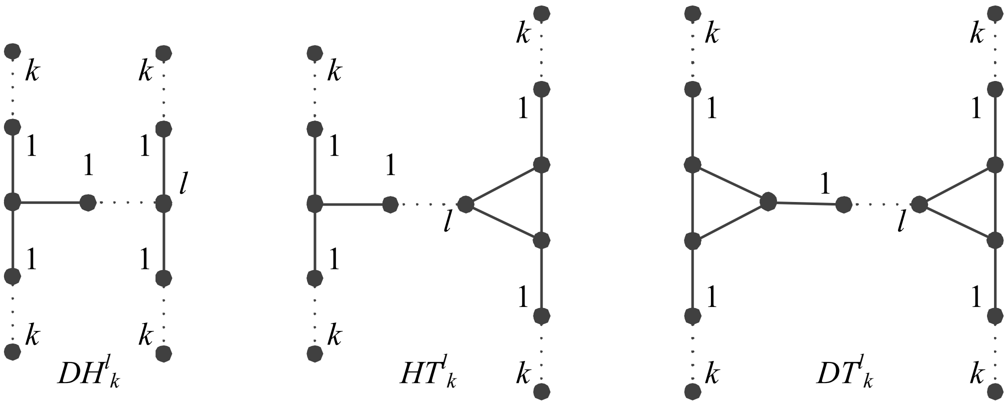

Fig. 7.2: DH^l_k, HT^l_k, and DT^l_k

Corollary 7.1. *Given two integers k, Δ, the MIS-Δ problem is polynomially solvable in ($S_{k,k,k}, T_{k,k,k}$)-free graphs.*

By Theorem 3.16, we also obtain the following consequence.

Corollary 7.2. *Given an integer k, the ($S_{2,2,k}$,banner$_k$)-free graph class is MIS-4-easy.*

7.2 Graphs of Maximum Degree at Most Δ

In this section, we use the technique used by Lozin and Milanič [124] for the MIS-Δ problem in $(H_k, H_{k+1}, \ldots)$-free graphs to extend Corollary 7.1. Denote by DH^l_k, HT^l_k, and DT^l_k the graphs in Fig. 7.2. We have the following observations.

Lemma 7.3. *Given three integers k, l, Δ and a $(DH^l_k, DH^{l+1}_k, \ldots, HT^l_k, HT^{l+1}_k, \ldots, DT^l_k, DT^{l+1}_k, \ldots)$-free graph G of maximum degree at most Δ, let (S_1, S_2) be a pair of induced $S_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$'s in G and P be the shortest path connecting S_1, S_2. Then P is of length at most $l+1$.*

Proof. For contradiction, suppose that $P = (c_0, \ldots, c_m)$, $m \geq l+1$, and $c_0 \in S_1$, $c_m \in S_2$. Let $P_1 = (a_0, a_1, \ldots, a_{4k\cdot\Delta}) \subset S_1$ and $P_2 = (b_0, b_1, \ldots, b_{4k\cdot\Delta}) \subset S_2$ be the two induced paths such that $P_1 \cap (P \backslash \{a_{2k\cdot\Delta}\}) = P_2 \cap (P \backslash \{b_{2k\cdot\Delta}\}) = \emptyset$. Note that $a_{2k\cdot\Delta}$ and $b_{2k\cdot\Delta}$ are the vertices of degree three of S_1 and S_2, respectively. Let $P_3 = (d_0, \ldots, d_p)$ be an induced path such that $d_0 = a_{2k\cdot\Delta}$, $d_p = b_{2k\cdot\Delta}$, and for some $0 \leq i \leq p$, $d_{i+j} = c_j$ for every $0 \leq j \leq m$, $d_0, \ldots, d_i \in S_1 \backslash (P_1 \backslash \{a_{2k\cdot\Delta}\})$, and $d_{i+m}, \ldots, d_p \in S_2 \backslash (P_2 \backslash \{b_{2k\cdot\Delta}\})$. It also implies $p \geq m \geq l+1$.

Claim 7.3.1. *d_1, d_2 together with P_1 induce (a.1) an $S_{1,k,k}$ or (a.2) a $T_{1,k,k}$.*

Proof. Let $a_{i_1}, a_{i_2}, \ldots, a_{i_l}$ be neighbors of d_1 in P_1 and $i_j \leq i_{j+1}$ for every j $1 \leq j \leq l-1$. Then $l \leq \Delta - 1$. Note that $d_1 \sim a_{2k\cdot\Delta}$. If $l = 1$, i.e. $d_1 \sim a_{2k\cdot\Delta}$, then $\{a_{2k\cdot\Delta}, d_1, a_{2k\cdot\Delta-1}, \ldots, a_{2k\cdot\Delta-k}, a_{2k\cdot\Delta+1}, \ldots, a_{2k\cdot\Delta+k}\}$ induces an $S_{1,k,k}$.
Hence, we assume that $l \geq 2$. If $i_{j+1} - i_j \geq 2k$ for some $1 \leq j \leq l-1$, then $\{d_1, d_2, a_{i_j}, \ldots, a_{i_j+k-1}, a_{i_{j+1}}, \ldots, a_{i_{j+1}-k+1}\}$ induces an $S_{1,k,k}$.
Thus, we assume that $i_{j+1} - i_j < 2k$ for each $1 \leq j \leq l-1$. Since $i_l \geq 2k\cdot\Delta$, $i_1 > 2k\cdot\Delta - 2k\cdot(\Delta-2) = 4k$. Similarly, $i_l < 4k\cdot\Delta - 4k$. Now, $\{d_1, d_2, a_{i_1}, \ldots, a_{i_1-k+1}, a_{i_l}, \ldots, a_{i_l+k-1}\}$ induces a $T_{1,k,k}$ or an $S_{1,k,k}$ depending on $i_l - i_1 = 1$ or not. □

Similarly, we obtain the following claim.

Claim 7.3.2. *d_{p-1}, d_{p-2} together with P_2 induce (b.1) an $S_{1,k,k}$ or (b.2) a $T_{1,k,k}$.*

By the two above claims, $P_1 \cup P_2 \cup P_3$ induces a DH_k^q or a TH_k^q or a DT_k^q for some $q \geq l$ depending on the combination of Cases (a.1) and (a.2) with Cases (b.1) and (b.2), a contradiction. □

Hence, given three integers k, l, Δ and a $(DH_k^l, DH_k^{l+1}, \ldots, HT_k^l, HT_k^{l+1}, \ldots, DT_k^l, DT_k^{l+1}, \ldots)$-free graph G of maximum degree at most Δ, if (S_1, S_2) be a pair of induced $S_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$'s in G, then the distance between the two vertices of degree three of S_1 and S_2 is at most $l + 1 + 4k \cdot \Delta$. We also have similar results for a pair of induced $T_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$'s and for a pair of an induced $S_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$ and an induced $T_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$. Then, we obtain the following consequence.

Corollary 7.4. *For every fixed positive integers k, l, Δ, there exists a constant $\rho = \rho(k, l, \Delta)$ such that any connected $(DH_k^l, DH_k^{l+1}, \ldots, HT_k^l, HT_k^{l+1}, \ldots, DT_k^l, DT_k^{l+1}, \ldots)$-free graph G of maximum degree at most Δ contains an induced subgraph containing neither induced $S_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$ nor induced $T_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$ with at least $|V(G)| - \rho$ vertices.*

Proof. Assume that G contains an induced $S_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$ S (or an induced $T_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$ T), the distance from a vertex of degree three of S (or T) to a vetex of degree three of any other $S_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$ or $T_{2k\cdot\Delta,2k\cdot\Delta,2k\cdot\Delta}$ is at most $l+1+4k\cdot\Delta$. Since G is a connected graph of maximum degree at most Δ, there is a constant $\rho = \rho(k, l, \Delta)$ bounding the number of vertices of G of distance at most $l + 1 + 4k \cdot \Delta$ from the vertex of degree three of S (or T). Deletion of all these vertices leaves a desired subgraph of G. □

Lozin and Milanič [124] also showed the following technical result.

Lemma 7.5. *[124] Let $\mathcal{X}$ be a graph class such that there exits an integer ρ and a hereditary graph class $\mathcal{Y}$ such that:*

- *$\mathcal{Y}$ is MIS-easy and*
- *for each $G \in \mathcal{X}$, we can find in polynomial time a subset U of its vertex set of cardinality at most ρ such that $G - U \in \mathcal{Y}$.*

Then $\mathcal{X}$ is MIS-easy.

Together with Corollary 7.1 and Corollary 7.4, we obtain the following result.

Theorem 7.6. *For every fixed positive integers k, l, Δ, the $(DH_k^l, DH_k^{l+1}, \ldots, HT_k^l, HT_k^{l+1}, \ldots, DT_k^l, DT_k^{l+1}, \ldots)$-free graph class is MIS-Δ-easy.*

7.3 Subcubic Graphs

We start with the result of Lozin et al. [126].

Theorem 7.7. *[126] The MIS-3 problem is polynomially solvable for $S_{2,2,2}$-free graphs.*

Lemma 7.8. *Given an integer k and a $(DH_2^k, DH_2^{k+1}, \ldots)$-free subcubic graph G, the distance between two induced $S_{2,2,2}$'s is at most $k+1$. In particular, for every fixed positive integer k, there exists a constant $\rho = \rho(k)$ such that any connected $(DH_2^k, DH_2^{k+1}, \ldots)$-free subcubic graph G contains an induced $(S_{2,2,2})$-free subgraph with at least $|V(G)| - \rho$ vertices.*

Proof. For contradiction, suppose that S_1 and S_2 are two induced $S_{2,2,2}$'s and $P = (v_0, v_1, \ldots, v_{k+1})$ are the shortest path connecting S_1 and S_2 such that $v_0 \in S_1$ and $v_{k+1} \in S_2$. Similar to Lemma 7.3 and Corollary 7.4, we only have to show that v_1, v_2 together with S_1 induces an $S_{1,2,2}$. Assume that $V(S_1) = \{a, b_1, b_2, c_1, c_2, d_1, d_2\}$ such that (a, b_1, b_2), (a, c_1, c_2), and (a, d_1, d_2) are induced P_3's. Note that v_1 has at most two neighbors in S_1 and non-adjacent to a. Without loss of generality, we consider the following cases.
Case 1. v_1 has neighbors only in (b_1, b_2). Then $S_1 \cup \{v_1, v_2\}$ induces an $S_{1,2,2}$.
Case 2. v_1 has a neighbor in $\{c_1, c_2\}$ and a neighbor in $\{d_1, d_2\}$. Then $\{v_1, v_2, c_1, c_2, d_1, d_2\}$ induces an $S_{1,2,2}$. □

Now, together with Lemma 7.5, we obtain the following result.

Lemma 7.9. *Given a positive integer k, if the $S_{2,2,2}$-free subcubic graph class is MIS-easy, then so is the $(DH_2^k, DH_2^{k+1}, \ldots)$-free subcubic graph class.*

7.3.1 Bounded Diameter

Lozin and Milanič [123] used bounded diameter technique to extend their result from $S_{1,2,2}$-free planar graphs to $S_{1,2,k}$-free planar graphs ($k \geq 3$). Given a positive integer ρ, if a connected subcubic graph G is of diameter at most ρ, then G contains at most φ vertices for some integer $\varphi = \varphi(\rho)$. Hence, given a hereditary subcubic graph class $\mathcal{X}$, if every connected graph of $\mathcal{X}$ is of diameter at most ρ, then $\mathcal{X}$ is MIS-easy.

Lemma 7.10. *Given an integer $k \geq 2$ and a connected $S_{2,2,k}$-free graphs G containing an induced copy $S_{2,2,2}$, $diam(G) \leq 2k + 4$.*

Proof. Consider an induced copy F of $S_{2,2,2}$, $\{a, b_1, b_2, c_1, c_2, d_1, d_2\}$ in G, where (a, b_1, b_2), (a, c_1, c_2), (a, d_1, d_2) are induced paths, in a connected $S_{2,2,k}$-free subcubic graph H. We show that no vertex in H has distance greater than k from $V(F)$. In turn, this implies that no vertex in H has distance greater than $k+2$ from a. By the triangle inequality, this implies that the diameter of H is at most $2k + 4$.
For a positive integer i, let V_i denote the set of all vertices in H at distance i from $V(F)$. For contradiction, suppose that there exists some vertex $v \in V_{k+1}$. Let $P = \{v_0, v_1, \ldots, v_{k+1}\}$ be a shortest path connecting $V(F)$ and v in H with $v_0 \in V(F), v = v_{k+1}$, and $v_i \in V_i$ for all $1 \leq i \leq k+1$. Since $\deg(a) = 3$, v_1 is not adjacent to a. Moreover, $\deg(v_1) \leq 3$, hence, v_1 has at most two neighbors in F. We consider the two following cases.
Case 1. $dist(v_1, a) = 3$, i.e. v_1 is adjacent to at least one vertex among $\{b_2, c_2, d_2\}$ and not adjacent to any vertex among $\{b_1, c_1, d_1\}$. If v_1 is adjacent to only one vertex among $\{b_2, c_2, d_2\}$, say b_2 then $\{a, b_1, b_2, v_1, \ldots, v_{k-2}, c_1, c_2, d_1, d_2\}$ induces an $S_{2,2,k}$, a contradiction. If v_1 is adjacent to two vertices among $\{b_2, c_2, d_2\}$, say b_2, c_2, then $\{v_1, v_2, v_3, \ldots, v_{k+1}, b_2, b_1, c_2, c_1\}$ induces an $S_{2,2,k}$, a contradiction.

Case 2. $dist(v_1, a) = 2$, i.e. v_1 is adjacent to at least one vertex among $\{b_1, c_1, d_1\}$. Without loss of generality, we consider the following subcases.
(2.1) v_1 is adjacent to b_1 and not adjacent to any vertex among $\{c_1, c_2, d_1, d_2\}$, then $\{a, b_1, v_1, \dots, v_{k-1}, c_1, c_2, d_1, d_2\}$ induces an $S_{2,2,k}$, a contradiction.
(2.2) v_1 is adjacent to b_1 and c_1, then $\{v_1, v_2, v_3, \dots, v_{k+1}, b_1, b_2, c_1, c_2\}$ induces an $S_{2,2,k}$, a contradiction.
(2.3) v_1 is adjacent to b_1 and c_2 then $v_1(v_2v_3 \dots v_{k+1}, b_1b_2, c_2c_1)$ induces an $S_{2,2,k}$, a contradiction. □

Then we have the following observation. Note that, this is also a consequence of Lemma 7.9.

Corollary 7.11. *Given an integer k, if the $S_{2,2,2}$-free subcubic graph class is MIS-easy, then so is the $S_{2,2,k}$-free subcubic graph class.*

Now, we extend Lemma 7.10 as in the following result.

Lemma 7.12. *Given two integers $k \geq 3$, $l \geq k+1$ and a connected ($S_{k,k,l}$,apple$^l_5$,..., apple$^l_{2k+1}$)-free subcubic graph G containing an induced copy of $S_{k,k,k}$, $diam(G) \leq 2l+4$.*

Proof. Consider an induced copy F of $S_{k,k,k}$, $\{a, b_1, \dots, b_k, c_1, \dots, c_k, d_1, \dots, d_k\}$ in G, where $(a, b_1, \dots, b_k)$, $(a, c_1, \dots, c_k)$, and $(a, d_1, \dots, d_k)$ are induced paths, in a connected ($S_{k,k,l}$,apple$^l_5$,apple$^l_6$, ..., apple$^l_{2k+1}$)-free subcubic graph G. Similar to Lemma 7.10, it is enough to show that no vertex in G has distance greater than l from $V(F)$. For a positive integer i, let V_i denote the set of all vertices in H at distance i from $V(F)$.
For contradiction, suppose that there exists a vertex $v \in V_{l+1}$. Let $P = (v_0, v_1, \dots, v_{l+1})$ be a shortest path connecting $V(F)$ and v in H with $v_0 \in V(F), v = v_{l+1}$ and $v_i \in V_i$ for all $1 \leq i \leq l+1$. Since $\deg(a) = 3$, v_1 is not adjacent to a. Moreover since $\deg(v_1) \leq 3$, v_1 is adjacent to at most two vertices among $\{b_i's, c_i's, d_i's\}$. Because of the symmetry, we consider two following cases.
Case 1. v_1 is adjacent to only vertices among $\{b_i's\}$. Let p be the smallest positive integer such that v_1 is adjacent to b_l, then $\{a, b_1, \dots, b_p, v_1, \dots, v_{l-p}, c_1, \dots, c_k, d_1, \dots, d_k\}$ induces an $S_{k,k,l}$, a contradiction.
Case 2. v_1 is adjacent to b_p and c_q, $1 \leq p \leq q \leq k$.
If $p = q = k$, then $\{v_1, v_2, \dots v_{l+1}, b_k, \dots, b_1, c_k, \dots, c_1)\}$ induces an $S_{k,k,l}$, a contradiction.
If $p = q = 1$, then $\{v_1, v_2, \dots, v_{l+1}, b_1, \dots, b_k, c_1, \dots, c_k\}$ induces an $S_{k,k,l}$.
For the remaining subcase, $\{v_{l+1}, \dots, v_1, b_p, \dots, b_1, a, c_1, \dots, c_q\}$ induces an apple$^l_{p+q+2}$. Note that, $p+q+2 = 4$ if and only if $p = q = 1$ and $p+q+2 = 2k+2$ if and only if $p = q = k$. Hence, in this subcase, $5 \leq p+q+2 \leq 2k+1$. □

7.3.2 α-redundant Vertex

In this section, we use the technique used in Subsection 5.2.4 to extend the results of Lemmas 7.10 and 7.12.

Lemma 7.13. *Given two integers k, l, $2 \leq k \leq l$ and an ($S_{3,k,l}$,apple$_5$,...,apple$_{l+4}$)-free graph G, let $\{a, b_1, b_2, c_1, \dots, c_k, d_1, \dots, d_l\}$ induces an $S_{2,k,l}$, where (a, b_1, b_2), $(a, c_1, \dots, c_k)$, and $(a, d_1, \dots, d_l)$ are three induced paths of length $3, k$, and l, respectively. Then b_1 is α-redundant.*

Proof. We show that there exists a maximum independent set not containing b_1. Let S be a maximum indpendent set of G and $b_1 \in S$. Then S contains a vertex b_3 adjacent to b_2, otherwise $(S \backslash \{b_1\}) \cup \{b_2\}$ is a desired set. Since $\deg(a) = 3$, $b_3 \nsim a$. We consider the two following cases.
Case 1. b_3 is not adjacent to c_1, d_1. We show that b_3 is non-adjacent to c_i, d_i by induction. Assume that b_3 is non-adjacent to $c_1 \ldots, c_{i-1}, d_1, \ldots, d_{i-1}$. Then $b_3 \nsim c_i$, otherwise $\{b_1, b_2, b_3, c_1, \ldots, c_i, a, d_1\}$ induces an apple$_{i+4}$, a contradiction. Similarly, $b_3 \nsim d_i$. But now, $\{a, b_1, b_2, b_3, c_1, \ldots, c_k, d_1, \ldots, d_l\}$ induces an $S_{3,k,l}$, a contradiction.
Case 2. b_3 is adjacent to c_1 or d_1. Note that either $c_1 \in S$ or $d_1 \in S$, otherwise $(S \backslash \{b_1\}) \cup \{a\}$ is a desired set. Hence, b_3 is adjacent to only one vertex among c_1, d_1. Now, $\{a, b_1, b_2, b_3, c_1, d_1\}$ induces an apple$_5$, a contradiction. □

Together with Corollary 7.11, it leads us to the following consequence.

Corollary 7.14. *Given an integer $k \geq 2$, if $S_{2,2,2}$-free subcubic graph class is MIS-easy, then so is the ($S_{3,3,k}$,apple$_5$,...,apple$_{k+4}$)-free subcubic graph class.*

Theorem 5.18 leads to a polynomial solution for ($S_{3,k,k}$,banner,apple$_5$, ...,apple$_{k+4}$)-free subcubic graphs. Now, we extend this result using α-redundant technique as in the following observations.

Lemma 7.15. *Given an integer $k \geq 2$ and an ($S_{4,k,k}$,banner,apple$_5$,...,apple$_{k+5}$)-free graph G, let $\{a, b_1, b_2, b_3, c_1, \ldots, c_k, d_1, \ldots, d_k\}$ induces an $S_{3,k,k}$, where a is of degree three and (a, b_1, b_2, b_3), $(a, c_1, \ldots, c_k)$, and $(a, d_1, \ldots, d_k)$ are three induced paths of length $3, k$, and k, respectively. Then b_2 is α-redundant.*

Proof. We show that there exists a maximum independent set not containing b_2. Let S be a maximum indpendent set of G and $b_2 \in S$. Then S contains a vertex b_4 adjacent to b_3, otherwise $(S \backslash \{b_2\}) \cup \{b_3\}$ is a desired set. Since $\deg(a) = 3$, $b_4 \nsim a$. Suppose that $b_4 \nsim b_1$, we consider the two following cases.
Case 1. b_4 is not adjacent to c_1, d_1. We show that b_4 is non-adjacent to c_i, d_i by induction. Assume that b_4 is non-adjacent to $c_1 \ldots, c_{i-1}, d_1, \ldots, d_{i-1}$. Then $b_4 \nsim c_i$, otherwise $\{b_1, \ldots, b_4, c_1, \ldots, c_i, a, d_1\}$ induces an apple$_{i+5}$, a contradiction. Similarly, $b_4 \nsim d_i$. But now, $\{a, b_1, \ldots, b_4, c_1, \ldots, c_k, d_1, \ldots, d_k\}$ induces an $S_{4,k,k}$, a contradiction.
Case 2. b_4 is adjacent to c_1 or d_1. Then b_4 is adjacent to both c_1, d_1, otherwise $\{a, b_1, \ldots, b_4, c_1, d_1\}$ induces an apple$_6$, a contradiction. Moreover, $b_4 \sim c_2$, otherwise $\{a, b_1, \ldots, b_4, c_1, c_2\}$ induces an apple$_6$, a contradiction. But now, we have a contradiction with $\deg(b_4) \leq 3$. Hence, we have the following observation.

Claim 7.15.1. $b_4 \sim b_1$.

But now, $\{a, b_1, \ldots, b_4\}$ induces a banner, a contradiction. □

Lemma 7.16. *Given an integer $k \geq 2$ and an ($S_{5,k,k}$,banner,apple$_5$,...,apple$_{k+6}$)-free graph G, let $\{a, b_1, \ldots, b_4, c_1, \ldots, c_k, d_1, \ldots, d_k\}$ induces an $S_{4,k,k}$, where a is of degree three and $(a, b_1, \ldots, b_4)$, $(a, c_1, \ldots, c_k)$, and $(a, d_1, \ldots, d_k)$ are three induced paths of length $4, k$, and k, respectively. Then b_3 is α-redundant.*

Proof. We show that there exists a maximum independent set not containing b_3. Let S be a maximum indpendent set of G and $b_4 \in S$. Then S contains a vertex b_5 adjacent to b_4, otherwise $(S \backslash \{b_3\}) \cup \{b_4\}$ is a desired set. Similarly to Lemma 7.15, we have the following observation.

Claim 7.16.1. *b_5 is adjacent to b_1 or b_2.*

Then b_5 is adjcent to both b_1, b_2, otherwise either $\{b_1, \ldots, b_5\}$ induces a banner or $\{a, b_1, \ldots, b_5\}$ induces an apple$_5$, a contradiction. It implies that b_5 is the only neighbor of b_4 in S, since $\deg(b_1) = \deg(b_2) = 3$. Now, $(S \backslash \{b_3, b_5\}) \cup \{b_4, b_2\}$ is a desired set. □

Lemma 7.17. *Given an integer $k \geq 2$ and an ($S_{6,k,k}$,banner,apple$_5$,…,apple$_{k+7}$)-free graph G, let $\{a, b_1, \ldots, b_5, c_1, \ldots, c_k, d_1, \ldots, d_k\}$ induces an $S_{5,k,k}$, where a is of degree three and $(a, b_1, \ldots, b_5)$, $(a, c_1, \ldots, c_k)$, and $(a, d_1, \ldots, d_k)$ are three induced paths of length $5, k$, and k, respectively. Then b_4 is α-redundant.*

Proof. We show that there exists a maximum independent set not containing b_4. Let S be a maximum indpendent set of G and $b_4 \in S$. Then S contains a vertex b_6 adjacent to b_5, otherwise $(S \backslash \{b_4\}) \cup \{b_5\}$ is a desired set. Similarly to Lemma 7.15, we have the following observation.

Claim 7.17.1. *b_6 is adjacent to a vertex among $\{b_1, b_2, b_3\}$.*

Note that b_6 has at most two neighbors in $\{b_1, b_2, b_3\}$. To avoid an induced banner, apple$_5$, and apple$_6$, b_6 is adjcent to both b_1 and b_2 or to both b_2 and b_3. It implies that b_6 is the only neighbor of b_5 in S, since $\deg(b_1), \deg(b_2), \deg(b_3) \leq 3$. Now, $(S \backslash \{b_4, b_6\}) \cup \{b_3, b_5\}$ or $(S \backslash \{b_4, b_6\}) \cup \{b_2, b_5\}$ is a desired set. □

Lemma 7.18. *Given an integer $k \geq 2$ and an ($S_{7,k,k}$,banner,apple$_5$,…,apple$_{k+8}$)-free graph G, let $\{a, b_1, \ldots, b_6, c_1, \ldots, c_k, d_1, \ldots, d_k\}$ induces an $S_{6,k,k}$, where a is of degree three and $(a, b_1, \ldots, b_6)$, $(a, c_1, \ldots, c_k)$, and $(a, d_1, \ldots, d_k)$ are three induced paths of length $6, k$, and k, respectively. Then b_5 is α-redundant.*

Proof. We show that there exists a maximum independent set not containing b_5. Let S be a maximum indpendent set of G and $b_5 \in S$. Then $A = N_S(b_6) \neq \emptyset$, otherwise $(S \backslash \{b_5\}) \cup \{b_6\}$ is a desired set. Similar to Lemma 7.15, we have the following observation.

Claim 7.18.1. *For every $b \in A$, b is adjacent to a vertex among $\{b_1, \ldots, b_4\}$.*

To avoid induced banner, apple$_5$, apple$_6$, and apple$_7$, every vertex $b \in A$ is adjcent to b_1, b_2 or to b_2, b_3 or to b_3, b_4. Hence, if $b \in A$ is adjacent to b_2 and b_3, then b is the only one neighbor of b_6 in S. Note that each vertex of $\{b_1, \ldots, b_4\}$ has at most one neighbor in A. Thus, $(S \backslash \{b_5, b\}) \cup \{b_6, b_3\}$ is a desired set. We consider the following cases.

Case 1. $\{b_1, b_2, b_3, b_4\} \subset N(A)$. Then $|A| = 2$ and $(S \backslash A \backslash \{b_5\}) \cup \{b_6, b_4, b_2\}$ is a desired set.

Case 2. $N(A) = \{b_3, b_4\}$. Then $|A| = 1$ and $(S \backslash A \backslash \{b_5\}) \cup \{b_6, b_4\}$ is a desired set.

Case 3. $N(A) = \{b_1, b_2\}$. Then $|A| = 1$.

3.1. $b_3 \notin S$. Then $(S \backslash A \backslash \{b_5\}) \cup \{b_6, b_2\}$ is a desired set.

3.2. $b_3 \in S$. Then b_3 and b_5 are the only neighbors of b_4 in S, otherwise for the other neighbor b', $\{b', b_2, \ldots, b_7\}$ induces an apple$_6$, a contradiction. Thus, $(S \backslash A \backslash \{b_5, b_3\}) \cup \{b_6, b_4, b_2\}$ is a desired set. □

It leads us to the following observation.

Theorem 7.19. *Given an integer k, the ($S_{7,k,k}$,banner,apple$_5$, …,apple$_{k+8}$)-free subcubic graph class is MIS-easy.*

7.3.3 Clique Separators

This subsection is based on the following result of Alekseev [4].

Lemma 7.20. *[4] Given a hereditary graph class $\mathcal{X}$, if the MIS problem is polynomially solvable for every graph containing no clique separator of $\mathcal{X}$, then it is polynomially solvable in $\mathcal{X}$.*

Lemma 7.21. *Given two integers p, l, $l \geq 2$ and an apple^l_p-free subcubic graph G containing an induced copy of apple$_p$ and containing no clique separator, $diam(G) \leq 2l + \lfloor \frac{p}{2} \rfloor + 1$.*

Proof. Consider an induced copy F of apple$_p$, $\{b, a, c_1, \ldots, c_{p-1}\}$ in G, where a and b are vertices of degree three and one, respectively. We show that no vertex in G has distance greater than l from $V(F)$. And by the triangle inequality, this implies that no vertices pair in H has distance greater than $2l + \lfloor p/2 \rfloor + 1$, hence the diameter of H is at most $2l + \lfloor p/2 \rfloor + 1$. For a positive integer i, let V_i denote the set of all vertices in H at distance i from $V(F)$.
For contradiction, suppose that there exists a vertex $v \in V_{l+1}$. Let $P = \{v_0, v_1, \ldots, v_{l+1}\}$ be a shortest path connecting $V(F)$ and v in G with $v_0 \in V(F), v = v_{l+1}$ and $v_i \in V_i$ for all $1 \leq i \leq l+1$. Since $\deg(a) = 3$, v_1 is not adjacent to a and since $\deg(v_1) \leq 3$, v_1 is adjacent to at most two vertices among $\{b, c_1, c_2, \ldots, c_{p-1}\}$. Because of the symmetry, we consider the following cases.
Case 1. v_1 is adjacent to only one vertex among $\{c_1, c_2, \ldots, c_{p-1}\}$, say c_1. Then $\{v_l, v_{l-1}, \ldots, v_1, c_1, \ldots, c_{p-1}, a\}$ induces an apple^l_p, a contradiction.
Case 2. v_1 is adjacent to only b among $\{b, c_1, \ldots, c_{p-1}\}$. Then $\{v_{l-1}, \ldots, v_1, b, a, c_1, \ldots, c_{p-1}\}$ induces an apple^l_p, a contradiction.
Case 3. v_1 is adjacent to two vertices among $\{c_1, c_2, \ldots, c_{p-1}\}$. Since v_1 is not a clique separator of G, there exists an induced path P' not containing v_1, $(u_0, u_1, \ldots, u_q)$ such that $q \geq p$, $u_q = v_p$, $u_0 \in V(F)$ and u_i is not adjacent to any vertex of F, for $i \geq 2$. Moreover, since a is not a clique separator of G, among such those paths, we can choose a path P' such that $u_1 \sim b$ and hence, u_1 has at most one neighbor among $\{c_1, c_2, \ldots, c_{p-1}\}$. If u_1 is not adjacent to any vertex among $\{c_i\}$ then $\{u_{l-1}, \ldots, u_1, b, a, c_1, c_2, \ldots, c_{p-1}\}$ induces an apple^l_p, a contradiction. If u_1 is adjacent to some c_i, then $\{u_{l+1}, \ldots, u_1, c_1, \ldots, c_{p-1}, a\}$ induces an apple^l_p, a contradiction. □

Together with Corollary 7.14, it leads us to the following consequence.

Corollary 7.22. *Given an integer l, if the $S_{2,2,2}$-free subcubic graph class is MIS-easy, then so is the ($S_{3,3,l}$,apple$^l_5$,apple$^l_6$, apple$^l_7$)-free subcubic graph class.*

Moreover, together with Theorems 5.20, 7.19, and Lemma 7.12, it leads us to the following result.

Theorem 7.23. *Given two integers k, l, the following graph classes are MIS-easy*

1. *($S_{7,k,l}$,banner$_l$,apple$^l_5$,…,apple$^l_{k+8}$)-free subcubic graphs and*
2. *($S_{k,k,l}$,Z_l,banner$_l$,apple$^l_5$,…,apple$^l_{2k+1}$)-free subcubic graphs.*

7.4 Summary

In this chapter, we have revised some techniques used to solve the MIS-Δ problem. By using distance arguments, we have obtained an NP-easy class for the MIS-Δ problem. This technique is also used for subcubic graphs to obtain the result for ("larger" H's)-free graphs. We also have combined α-redundant vertex technique, bounded diameter technique, and clique separator to apply on subcubic graphs. We have shown that the MIS problem is solvable in polynomial time in some subclasses $S_{i,j,k}$-free subcubic graphs. Note that so far, there are not many results in this area for the case $i, j, k \geq 3$. Our results about $S_{i,j,k}$-free subcubic graphs generalize the results about subcubic graphs in [124, 126].

8 Conclusion

In this thesis, we have presented several complexity results for the interrelated problems of finding maximum independent sets and some related graph combinatorial problem. The common natural assumption was that the input graphs belong to a hereditary class of graphs. Now, we summarize some results we have obtained in the thesis. Several open-ended questions and challenges are left for future research. In Section 8.1, we summarize some results of the thesis about NP-easy graph classes for the MIS problem and some graph classes, in which, the complexity status is still unknown. Then we discuss about main results of the thesis and possible future research about augmenting graph technique (Section 8.2), graph transformations (Section 8.3), heuristic method (Section 8.4), the MIS-Δ problem (Section 8.5), and other possible algorithmic improvements for the problem (Section 8.6).

8.1 Complexity Question

8.1.1 Open Complexity Question

First, let us informally observe how widely open remains the gap between the polynomial and the NP-hard side of the MIS problem in hereditary graph classes. Recall that $\mathcal{S}$ is the graph class consisting of graphs whose every connected component is of the form $S_{i,j,k}$ for some integers i, j, k. Recall the result of Alekseev [5] that if the MIS problem is polynomially solvable for F-free graphs, where F is a finite graph, then every connected component of $F \in \mathcal{S}$. After the MIS problem is showed to be polynomially solvable for claw-free graphs independently in 1980 by Minty [137] and Sbihi [156], for P_4-free graphs in 1985 by Corneil [52], fork-free graphs in 1999 by Alekseev [2], and for $2P_3$-free graphs by Lozin and Mosca [129], P_5-free graphs became the only one graph class defined by a single induced forbidden subgraph containing at most five vertices, for which the polynomial solvability was unknown. This question was solved in 2013 by Lokshtanov et al. [115].
By the result of Lokshtanov et al., the complexity status of the MIS problem is solved for the F-free graph class, where F is a graph consisting of at most five vertices, i.e. the F-free graph class is MIS-easy if and only if $F \in \mathcal{S}$. The naturally next step is to consider larger forbidden induced subgraphs. The areas of unknown complexity status of the problem in F-free graphs occur already when F consists of a single graph on six or seven vertices. For single forbidden induced subgraphs containing six vertices, we already have NP-easiness for $3P_2$-free graphs [6], for (claw+P_2)-free graphs [127] graphs, and for $2P_3$-free graphs [129]. More specifically, there are five minimal classes defined by a single forbidden induced subgraph for which the complexity status of the maximum independent set problem is unknown. These are P_6-free graphs, $S_{1,2,2}$-free graphs, $(P_4 + P_2)$-free graphs, (claw+P_3)-free graphs, and $(P_3 + 2P_2)$-free graphs.
For hereditary classes defined by infinitely many forbidden induced subgraphs, recall

the results of Lozin and Milanič [124] for H-free graphs of bounded maximum degree, of Hertz and de Werra [101] for AH-free graphs, and of Gerber et al. [77] for (banner,$C_5, C_6, \ldots$)-free graphs. Let us emphasize that the complexity of the maximum independent set problem is still unknown for $(C_k, C_{k+1}, \ldots)$-free graphs, for every $k \geq 5$, as well as for $(H_k, H_{k+1}, \ldots)$-free graphs, for every $k \geq 1$ without any condition about the maximum degree of the graphs.
Although the results of Alekseev [5] and of Lozin and Milanič [124] provide sufficient conditions for the set $\mathcal{F}$ which guarantee that the MIS problem remains NP-hard for $\mathcal{F}$-free graphs, it is not known whether the converse versions of these results hold true. For example, to the best of our knowledge, no graph $S \in \mathcal{S}$ is known such that the MIS problem is NP-hard in the class of S-free graphs, where $\mathcal{S}$ is the graph class consisting all finite graphs whose every connected component is of the form $S_{i,j,k}$, for some integers i, j, k. Moreover, the question whether other forbidden induced subgraph conditions for the NP-hardness of the problem exist or not remains open.

8.1.2 Some New MIS-easy Graph Classes

In the thesis, we have obtained polynomial solutions for some subclasses of $S_{2,2,2}$-free graphs (Chapters 5 and 6), of $S_{2,2,5}$-free graphs, $S_{2,k,k}$-free graphs, and $S_{k,k,k}$-free graphs (Chapters 3 and 5). So far, there are still not many results in these areas (see Subsection 1.5.2) in the literature.
For graphs of maximum degree at most Δ (Chapter 7), a result for $(DH_k^l, DH_k^{l+1}, \ldots, HT_k^l, HT_k^{l+1}, \ldots, DT_k^l, DT_k^{l+1})$-free graphs has been obtained. Moreover, also in this chapter, using some techniques, we have generalized a result for $(H_k, H_{k+1}, \ldots)$-free subcubic graphs and some results for $S_{2,2,k}$-free subcubic graphs, $S_{3,3,k}$-free subcubic graphs, $S_{7,k,k}$-free subcubic graphs, and $S_{k,k,k}$-free subcubic graphs. Again, there are still not many results in this area in the literature.

8.2 Augmenting Graph

The method of augmenting graphs can generally be applied only to unweighted graphs. However, this approach has proved useful in designing polynomial time algorithms to some weighted cases as well (for instance for claw-free graphs [137, 156]). More generally, the future research question here we want to propose: To what extent can the method of augmenting graphs be applied to weighted graphs?
In Chapter 4, we also describe the method to apply augmenting graph technique to some other graph combinatorial problems. Some questions which arise here are the following. Can we apply this technique for other problems? If yes, then how can we construct augmenting graph concepts? What are the structural properties for augmenting graphs in other hereditary graph classes? Can we apply this technique for weighted versions? Another motivation of research on augmenting graph is to combine results in literature in more general graph classes. In Chapter 3, we have combined methods for P_5-free graphs, for banner$_2$-free graphs, and for $S_{2,k,k}$-free graphs. By the way, there are still some results in literature that we could not integrate, for example Alekseev's method for finding augmenting complex [2], and Mosca's method for P_6-free graphs [142].

8.3 Graph Transformations

8.3.1 Pseudo-Boolean Function Method

About 20-30 years ago, pseudo-boolean functions were used to obtain graph reductions, through which, we can simplify the problem substantially. In Section 5.1, we give a unified overview on other reduction methods. The question here is: Can we still compose other graph transformations using posiforms and how can we apply these transformations to reduce the complexity of the problem in some special graph classes?

8.3.2 α-redundant Technique

In literature, α-redundant vertex technique are recently used mainly for subclasses of P_5-free graphs. In some sense, these approaches are useless after the result of Lokshtanov et al. [115]. Our motivation is trying to apply this useful technique in more general graph classes, say in tree$_m$-free graphs. This approach gives us a method to generalize some results for $K_{1,m}$-free graphs, $S_{2,2,2}$-free graphs, and $S_{j,k,l}$-free graphs. In the future, we may put our effort in applying this technique in other graph classes, for example P_6-free graphs.
Moreover, through successful applications of a technique used for P_5-free graphs, possible approaches that we can think about is trying to apply other methods, for example modular decomposition, clique separator, ..., to other graph superclasses of P_5-free graphs. We have shown some examples following this direction in Corollary 3.31 and in Subsection 7.3.3.

8.4 Heuristic Methods

Greedy heuristic methods not always give us maximum independent sets, but they can give us maximal independent sets in acceptable comlexity, especially when graphs are of large order and/or size. In Chapter 6, we investigate on some properties of some greedy heuristic methods, for example lower bounds of cardinality of the obtained maximal independent sets, performance on some special graphs, and especially, forbidden induced subgraph sets, underwhich the obtained maximal independet sets become maximum. Another consideration is try to combine graph transformations to develop new strategies in choosing the next vertex in the greedy sequence.
Note that so far, forbidden induced subgraph sets for heuristic methods are sufficient conditions. Hence, one question arising is: Can we sharpen out these sets more? Can we reduce the number of forbidden induced subgraphs or make them simpler? In another direction, we can investigate on combining other graph transformations to improve lower bounds and performances of algorithms.

8.5 Graph of Bounded Maximum Degree

In Chapter 7, we have used some technique mentioned in other chapters under the restriction of vertex degrees. It leads us to some interesting results. Some questions arise here: How can we apply other methods under this restriction? How can we apply them for other kinds of restriction, for example planar graphs?

8.6 Other Improvements

There are still many aspects about algorithmic approach for the MIS problem which we have not considered in our work yet. We recall Chapter 2 for a review on main methods tackling the problem. We believe that it is possible to apply our results to reduce the complexity of exact methods (Sections 2.1 and 2.2) or to improve the performance of heuristic methods (Section 2.3).

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