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Semi-linear waves with time-dependent speed and dissipation

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Thesis

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submitted by **M.Sc. Tang Bao Ngoc, Bui**

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- Prof. Dr. rer. nat. habil. Michael Reissig
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Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such.
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Persons other than those above did not contribute to the writing of this thesis.
I did not seek the help of a professional doctorate-consultant. Only those persons identified as having done so received any financial payment from me for any work done for me. This thesis has not previously been published in the same or a similar form in Germany or abroad.

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1. Introduction

1.1. Background

From the physical point of view hyperbolic equations describe processes in which disturbances propagate with finite velocity and it also plays a central role in physical modeling in several areas of science, for instance, in fluid dynamics, electrodynamics, astrophysics, optics, acoustic, theory of electromagnetic waves. We can list up here physical phenomena which are related with wave equations, see Graham-Schiesser [G-S09]:

- Acoustic waves in fluids: shock waves in a gases, transmission of waves in liquids, applications of ultrasound, audible sound, underwater sonar applications, etc., see Elmore [Elm69].
- Chemical waves: concentration variations of chemical species propagating in a system, for instance, Ross-Muller-Vidal, [Ros88].
- Electromagnetic waves: electricity in various forms, radio waves, light waves in optic fibers, etc., see A. Shadowitz [Sha75].
- Gravitational waves: The transmission of variations in a gravitational field in the form of waves, as predicted by Einstein's theory of general relativity. Undisputed verification of their existence is still awaited, see in Chapter 5, Ohanian-Ruffini [Oha94] .
- Seismic waves: Arising from movements in the earth's crust, passing through the interior of the earth, studying of various of components of seismic waves from distant earthquakes, [Elm69].
- Traffic flow waves: Small local changes in velocity occurring in high density situations can result in the propagation of waves and even shocks, see LeVeque [Lev07].

One of a simplified model for a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$) is the free wave equation

$$u_{tt} - c^2 \Delta u = 0, \tag{1.1.1}$$

where c denotes the speed of propagation and the Laplacian Δ is taken with respect to the spatial variables. The d'Alembert's representation formula is a well-known formula in one space dimension. Whereas, in two space dimensions we have the Poisson's formula. For three space dimensions the explicit representation of solutions was investigated by G. R. Kirchhoff.
One of the methods of studying the Cauchy problem for hyperbolic equations is the energy method. The wave energy is defined by

$$E_c(u)(t) := \frac{1}{2} \int_{\mathbb{R}^n} \left(c^2 |\nabla u(t,x)|^2 + |u_t(t,x)|^2\right) dx \tag{1.1.2}$$

for a solution $u = u(t,x)$ of the free wave equation (1.1.1). Physically (in the case $n = 3$) we have the following conservation of energy

$$E_c(u)(t) = E_c(u)(0).$$

Actually, the mathematical model for the free wave equation (1.1.1) is only suitable under ideal conditions, that is, we idealize the analysis by neglecting the effect of friction, the effect of stiffness, the effect of gravity, etc. Evidently, the model of elastic waves in many media are not described by the simple wave equation, but at least they give us some predicable and reasonable mathematical properties.

1.1.1. $L^p - L^q$ decay estimates for free wave equations

In this thesis we are not only interested in energy estimates but also in $L^p - L^q$ decay estimates on the conjugate line. Let us therefore introduce here various related papers which state results about $L^p - L^q$ decay estimates for the solutions to the following Cauchy problem:

$$u_{tt} - \Delta u = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x), \tag{1.1.3}$$

with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, such that the energy solution satisfies the following a priori estimate

$$\|(\partial_t, \nabla)u(t,.)\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_1\|_{W_p^{N_p+1}} + \|u_2\|_{W_p^{N_p}}\right) \tag{1.1.4}$$

for $n \geq 2$, $1 < p \leq 2$, (p,q) lying on conjugate line, i.e $pq = p+q$, and N_p is an integer number satisfies $N_p > n\left(\frac{1}{p} - \frac{1}{q}\right)$. The first paper we want to mention here is the paper of von Wahl [vW71], in that paper he used the explicit representation of solutions in the three-dimensional case. Using another methods which applied Fourier integral operators and stationary phase, we can see these estimates in papers of W.Littman [Lit73], R.S. Strichartz [Str70], P. Brenner [Bre75], and H. Pecher, [Pec76], to cite only a few.

1.1.2. $L^p - L^q$ decay estimates for damped wave equations

We next devote to the Cauchy problem for the damped wave equation

$$u_{tt} - \Delta u + u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x), \tag{1.1.5}$$

with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. In [Mat76, Mat77], A. Matsumura established better decay estimates by the aid of the dissipation term on $L^p - L^q$-estimates level as follows:

$$\|(\partial_t, \nabla)u(t,.)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \left(\|u_1\|_{W_p^{N_p+1}} + \|u_2\|_{W_p^{N_p}}\right) \tag{1.1.6}$$

for $1 < p \leq 2$, (p,q) lying on conjugate line and integer $N_p > n\left(\frac{1}{p} - \frac{1}{q}\right)$.
We will complete this part by pointing readers out that these decay estimates coincide with the corresponding estimates for the heat equations (see G. Ponce [Pon85]).

1.2. Motivation and some problems of this thesis

The recent papers of J. Wirth, [W06] and [W07a], are devoted to the study of the Cauchy problem for the wave equation with time-dependent dissipation

$$u_{tt} - \Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{1.2.1}$$

A description of the influence of the coefficient $b = b(t)$ on the qualitative behavior of solutions is given due to the following classification:

- *Scattering:* If $b(t)$ has a very weak influence, then there is a relation to the free wave equation. Such relations are described by so-called scattering results.
- *Non-effective:* If $b(t)$ has a weak influence, then the classical energy decays to 0 and corresponding L^p-L^q decay estimates are the classical Strichartz decay estimates with an additional term as a time-dependent coefficient coming from the decay of the energy itself. Such weak dissipations will be called non-effective.
- *Effective:* If $b(t)$ has a stronger influence, then $L^p - L^q$ decay estimates are similar to those ones for the classical damped wave equation but with an additional decay function related to the dissipation itself. Such dissipations will be called effective.
- *Over-damping:* If $b(t)$ has a "very strong influence", then in general we can not expect any decay estimate of the classical wave type energy.

In both cases, *scattering* or *over-damping*, we have in general no energy decay. Roughly speaking, energy decay only appears for dissipations $b(t)u_t$ with coefficient "between" the conditions $b \notin L_1(\mathbb{R}^+)$ and $1/b \notin L_1(\mathbb{R}^+)$ in (1.2.1). But we have to be more precise. This leads to distinguish between *non-effective* and *effective* dissipation. Correspondingly, we only cite here two results from J. Wirth [W05]: Assuming the coefficient function $b = b(t)$ is a positive, smooth and monotone function of t, which satisfies

$$|b^{(k)}(t)| \le C_k b(t) \left(\frac{1}{1+t}\right)^k$$

for all $k \in \mathbb{N}_0$.

Result 1.2.1. *Assume* $\limsup_{t\to\infty} tb(t) < 1$. *Then the solution* $u = u(t,x)$ *of (1.2.1) satisfies the* $L^p - L^q$ *decay estimate*

$$\|(\partial_t, \nabla)u(t,\cdot)\|_{L^q} \le C\frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_1\|_{W_p^{N_p+1}} + \|u_2\|_{W_p^{N_p}}\right) \tag{1.2.2}$$

for $p \in (1,2]$, q *is the corresponding dual index,* $N_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$ *and* $\lambda(t)$ *is an auxiliary function which is defined by*

$$\lambda(t) := \exp\left(\frac{1}{2}\int_0^t b(\tau)d\tau\right).$$

Result 1.2.2. *Assume* $tb(t) \to \infty$ *as* $t \to \infty$. *Then the solution* $u = u(t,x)$ *of (1.2.1) satisfies the* $L^p - L^q$ *decay estimate*

$$\|(\partial_t, \nabla)u(t,\cdot)\|_{L^q} \le C\left(1 + \int_0^t \frac{d\tau}{b(\tau)}\right)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\left(\|u_1\|_{W_p^{N_p+1}} + \|u_2\|_{W_p^{N_p}}\right) \tag{1.2.3}$$

for $p \in (1,2]$, q *is the corresponding dual index and* $N_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$.

What about wave models in (1.2.1) without any dissipation? In a series of papers of M. Reissig-K. Yagdjian-F. Hirosawa (see [R-Y99], [R-Y00a], [R-Y00b] or [H-W09]) the authors have obtained results about decay estimates for solutions to the Cauchy problem

$$u_{tt} - a^2(t)\Delta u = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{1.2.4}$$

Therein $a(t)$ is chosen as $a^2(t) = \lambda^2(t)b^2(t)$, where $\lambda(t)$ is a monotonously increasing function and $b(t)$ is an oscillating function. In this thesis we shall limit to treat only the case of an increasing propagation speed and we are not interested in special oscillating parts in the coefficient $a(t)$. We recall that some results from M. Reissig [Rei11] are obtained under the following assumptions to the coefficient $a = a(t)$:

- (A1) $a(t) > 0$, $a'(t) > 0$, for $t \in [0, \infty)$,
- (A2) $a_0 \frac{a(t)}{A(t)} \leq \frac{a'(t)}{a(t)} \leq a_1 \frac{a(t)}{A(t)}, \ a_0, a_1 > 0$,
- (A3) $|a''(t)| \leq a_2 a(t) \Big(\frac{a(t)}{A(t)} \Big)^2$, $a_2 \geq 0$,
- (A4) $t + \frac{C}{\sqrt{a(t)}}$ is strictly increasing with a positive constant C and for large t.

Here $A(t) = 1 + \int_0^t a(s)ds$ is a primitive of $a(t)$. As an example for this kind of model we can choose the Anti-de Sitter model of the universe that appears in the Mathematical Cosmology:

$$u_{tt} - e^{2t}\Delta u = 0.$$

If we inverse the time variable, $t \to -t$, it becomes

$$u_{tt} - e^{-2t}\Delta u = 0.$$

This equation describes particle in the so-called de Sitter model of the universe. Both of these examples were introduced in A. Galstian [Gal03].
Our first main goal of the thesis is to combine our knowledge about wave models with time-dependent speed and without any dissipation with those for wave models with time-dependent dissipation term. For this reason it seems to be reasonable to devote to the wave model

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0, \ u(0,x) = u_1(x), \ u_t(0,x) = u_2(x) \tag{1.2.5}$$

with time-dependent increasing speed of propagation and dissipation. An interesting issue is to introduce *precise descriptions* for *scattering*, *non-effective*, *effective dissipations* and *over-damping* in model (1.2.5). Such a classification we shall propose in Sections 3.1, 3.2, 3.3 and 3.4, respectively. In particularly, in the case *non-effective* and *effective dissipations* we will derive energy estimates not only on $L^2 - L^2$ scale but also on $L^p - L^q$ scale, in both cases.
Recently, X. Gang and Y. Huicheng, [G-H13], they investigated the global existence and stability of a smooth supersonic flow with vacuum state at infinity in a $3-D$ infinitely long divergent nozzle of the form

$$u_{tt} - \frac{1}{(1+t)^{2(\gamma-1)}}\Delta u + \frac{2(\gamma-1)}{1+t}u_t = 0,$$

for $1 < \gamma < 2$. A further topic of interest is the theory for non-linear wave equations which are demonstrated in the form

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = f(u, u_t, \nabla_x u, \nabla_x u_t, \nabla_x^2 u), \ u(0,x) = u_1(x), \ u_t(0,x) = u_2(x), \tag{1.2.6}$$

with $u = u(t,x)$, time variable $t \in \mathbb{R}_+$ and space variable $x \in \mathbb{R}^n$. Recently, there are several papers which are devoted to the Cauchy problem for the following non-linear wave equations

$$u_{tt} - a(t)^2\Delta u = u_t^2 - a(t)^2|\nabla u|^2, \ u(0,x) = u_1(x), \ u_t(0,x) = u_2(x). \tag{1.2.7}$$

In particular, in two papers of K. Yagdjian, [Yag01] and [Yag05], it is explained how the above class of special semi-linear Cauchy problems can be reduced by Nirenberg's transformation to a linear model with constrain condition. The above papers and the paper Ebert and Reissig [E-R11] concern with the problem of global existence (in time) for small data solutions to the semi-linear Cauchy problem

$$u_{tt} - a(t)^2\Delta u = u_t^2 - a(t)^2|\nabla u|^2,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{1.2.8}$$

It would be a challenge to apply this approach to the case of *non-effective* dissipations to the following semi-linear Cauchy problem

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = u_t^2 - a(t)^2|\nabla u|^2,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{1.2.9}$$

This is done in Section 5.1.
Another interesting application to the case of *effective* dissipations is the question for global small data solutions to the following semi-linear model

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = f(u),\ u(0,x) = u_1(x),\ \ u_t(0,x) = u_2(x), \tag{1.2.10}$$

where $f(u) \approx |u|^p$. This is done in Section 5.2. In a recent paper of D'Abbicco and Lucente, [D-L12], the authors have constructed counter-examples which provide a nonexistence result for weak solutions to (1.2.10).

Outline of this thesis. In Chapter 2 we introduce primarily the WKB-analysis, the method of zones. In particularly, in the second part of this chapter we study very important examples, the scale-invariant models, by using a lot of techniques from the theory of *special functions.* Among other things properties of solutions to the *Bessel equation* and *confluent hypergeometric equation* are used. These examples give us a lot of ideals and some predictions for more general results which are proved later. The emphasis in Chapter 3 and Chapter 4 is on concentrating a precise description of classification under the influence of $a(t)$ and $b(t)$ and their applications to derive $L^p - L^q$ decay estimates. In these chapters some techniques are applied, for example, WKB-analysis, the method of zones. Besides, we shall also use different micro-energies in different parts of the extended phase space, the diagonalization procedure, symbol classes and their hierarchies. Theory of Fourier multipliers or stationary phase method imply the desired a-priori estimates. Afterwards, in Chapter 5 we investigate the global existence of small data solutions of two semi-linear models by applying directly *non-effective* and *effective* results from Chapters 3 and 4. Finally, we introduce in Chapter 6 some further and open problems which are related to the results of the thesis.

1.3. Selected results of this thesis

In order to make our results more understandable here and hereafter we use the notation $b(t) = \mu(t)\dfrac{a(t)}{A(t)}$.

1.3.1. $L^2 - L^2$ estimates for linear models

Non-effective dissipation [Theorem 3.2.1]

Let us assume:

(B1) $b(t) > 0, \quad b \notin L^1(\mathbb{R}_+)$,

(B2) $|\mu'(t)| \le C_\mu \mu(t)\dfrac{a(t)}{A(t)}$, (B3) $\limsup_{t\to\infty} \mu(t) < 1$.

(B3)' $\liminf_{t\to\infty} \mu(t) > 1$.

(C) $\limsup_{t\to\infty} \big(\mu(t)+\alpha(t)\big) < 2$, where $\alpha(t)$ is defined by

$$\frac{a'(t)}{a(t)} =: \alpha(t)\frac{a(t)}{A(t)}.$$

Result 1.3.1. *Let us consider the Cauchy problem (1.2.5) under the assumptions (A1) to (A3), (B1), (B2), (B3) or (B3)' and (C). Then we have the following estimates for the energy solution:*

$$\|(\partial_t, a(t)\nabla)u(t,\cdot)\|_{L^2} \le C\frac{\sqrt{a(t)}}{\lambda(t)}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big).$$

Here $\lambda = \lambda(t)$ is defined by

$$\lambda(t) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big). \tag{1.3.1}$$

Effective dissipation [Theorem 3.3.14]

We assume:

(B'1) $b(t) > 0$,

(B'2) $\left|d_t^k \mu(t)\right| \le C_k \mu(t)\left(\dfrac{a(t)}{A(t)}\right)^k$ for $k = 1, 2$,

(B'3) $\mu(t)/A(t)$ is monotonic and $\mu(t) \to \infty$ as $t \to \infty$,

(B'4) $a^2(t)/b(t) = a(t)A(t)/\mu(t) \notin L^1(\mathbb{R}_+)$.

Result 1.3.2. *Let us assume the conditions (A1) to (A3) and (B'1) to (B'4). Then we have the following $L^2 - L^2$ estimates:*

$$\|(\partial_t, a(t)\nabla)u(t,\cdot)\|_{L^2} \le C \lesssim a(t)\left(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big).$$

1.3.2. $L^p - L^q$ estimates for linear models

The case of non-effective dissipation [Theorem 4.1.10]

We assume more regularity for $a(t)$ and $b(t)$:

(A3)$^\infty$ $|a^{(k)}(t)| \lesssim a(t)\left(\dfrac{a(t)}{A(t)}\right)^k$, $k = 1, 2, \cdots$,

(B2)$^\infty$ $|\mu^{(k)}(t)| \lesssim \mu(t)\left(\dfrac{a(t)}{A(t)}\right)^k$, $k = 1, 2, \cdots$.

Result 1.3.3. *If the conditions (A1), (A2), (A3)$^\infty$, (B1), (B2)$^\infty$, (B3) or (B3)' and (C) hold, then we have the following $L^p - L^q$ estimates for the kinetic and the "elastic" energy:*

$$\|(\partial_t, a(t)\nabla)u(t,\cdot)\|_{L_q} \lesssim \frac{1}{\lambda(t)}\sqrt{a(t)}A(t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_1\|_{L^{p,r+1}} + \|u_2\|_{L^{p,r}}\right)$$

with regularity $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u_0(\cdot), u_1(\cdot) \in S(\mathbb{R}^n)$. Here $A(t) = 1 + \int_0^t a(\tau)d\tau$.

The case of effective dissipation [Theorem 4.2.2]

Result 1.3.4. *Assume the conditions (B'1) to (B'4). Then for all times t we have the $L^p - L^q$ decay estimates*

$$\big\|(u_t(t,\cdot), a(t)\nabla_x u(t,\cdot))\big\|_{L^q} \lesssim a(t)\Big(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_1\|_{L^{p,r+1}} + \|u_2\|_{L^{p,r}}\big),$$

where $r > n\Big(\dfrac{1}{p} - \dfrac{1}{q}\Big)$ with $1 < p \le 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

1.3.3. Results for non-linear models

Semi-linear models with non-effective dissipation

We introduce a new assumption which is a modification of assumption (B3) as follows:

(B3) $\limsup_{t\to\infty}\mu(t) < \max\big\{\limsup_{t\to\infty}\alpha(t), 1\big\}$ in the case of space dimension $n > 1$ and

(B3)' $1 - \delta\alpha(t) - \varepsilon_0 \le \liminf_{t\to\infty}\mu(t) \le \limsup_{t\to\infty}\mu(t) < 1$ in the case of space dimension $n = 1$.

Result 1.3.5. *Assume $a(t)$ satisfies (A1) to (A3) and $b(t)$ satisfies (B1) to (B3) ($n > 1$) or (B1) to (B3)' ($n = 1$). Then there exists a unique global (in time) classical solution $u = u(t,x)$ to*

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = u_t^2 - a(t)^2|\nabla_x u|^2, \;\; u(0,x) = \epsilon u_1(x), \;\; u_t(0,x) = \epsilon u_2(x)$$

for given $u_0, u_1 \in C_0^\infty(\mathbb{R}^n), n > 1$, and all $\epsilon \in [0,\epsilon^*)$ with an in general suitable positive and small ϵ^*.

Semi-linear models with effective dissipation [Theorem 5.2.7]

Result 1.3.6. *We assume the Hypotheses (A1) to (A3), (B'1) to (B'5) and (R1). Let us assume $n \le 4$ and*

$$\begin{cases} p > \bar{p} \text{ and } p \ge 2 & \text{if } n=1,2, \\ 2 \le p \le 3 = p_{GN}(3) & \text{if } n=3, \\ p = 2 = p_{GN}(4) & \text{if } n=4. \end{cases} \tag{1.3.2}$$

Moreover, if we assume

$$\lambda > \frac{\nu(\lambda)}{a_0}\frac{(2+M)n}{8}. \tag{1.3.3}$$

Then there exists a constant $\epsilon_0 > 0$ such that data with

$$\|(u_1, u_2)\|_{\mathcal{A}_{1,1}} \le \epsilon_0,$$

imply the existence of a unique solution to (5.2.1) in $\mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$. Furthermore, there exists a constant $C > 0$ such that this solution satisfies the estimates

$$\|u(t,\cdot)\|_{L^2} \le C \;\; \|(u_1,u_2)\|_{\mathcal{A}_{1,1}}(1 + B_a(0,t))^{-\frac{n}{4}}, \tag{1.3.4}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \le C \;\; \|(u_1,u_2)\|_{\mathcal{A}_{1,1}}(1 + B_a(0,t))^{-\frac{n}{4}-\frac{1}{2}}, \tag{1.3.5}$$

$$\|u_t(t,\cdot)\|_{L^2} \le C \;\; \|(u_1,u_2)\|_{\mathcal{A}_{1,1}}(1 + B_a(0,t))^{-\frac{n}{4}-1}a^2(t)(b(t))^{-1}. \tag{1.3.6}$$

Here λ, $\nu(\lambda)$, M, p_{GN} and $\bar{p}$ are introduced in Section 5.2 and in Theorem 5.2.7.

2. Wave models without any dissipation

2.1. Wave models with strictly increasing speed of propagation

Let us devote to the Cauchy problem

$$u_{tt} - a^2(t)\Delta u = 0, \;\; u(0,x) = u_1(x), \;\; u_t(0,x) = u_2(x). \tag{2.1.1}$$

In special cases for a this was done in M. Reissig [Rei97] or A. Galstian [Gal03].

Theorem 2.1.1. *Let us consider the Cauchy problem (2.1.1) under the following assumptions to the coefficient $a = a(t)$:*

(A1) $a(t) > 0$, $a'(t) > 0$, *for* $t \in [0, \infty)$,

(A2) $a_0 \dfrac{a(t)}{A(t)} \le \dfrac{a'(t)}{a(t)} \le a_1 \dfrac{a(t)}{A(t)}, \;\; a_0, a_1 > 0,$

(A3) $|a''(t)| \le a_2 a(t)\Big(\dfrac{a(t)}{A(t)}\Big)^2$, $a_2 \ge 0$,

(A4) $t + \dfrac{C}{\sqrt{a(t)}}$ *is strictly increasing with a positive constant C and for large t.*

Here $A(t) = 1 + \int_0^t a(s)ds$ is a primitive of $a(t)$.
For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2} \le C\sqrt{a(t)}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

For the "elastic" energy we have

$$\|a(t)\nabla u(t,\cdot)\|_{L^2} \le C\sqrt{a(t)}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

Proof. Applying partial Fourier transformation we have $\hat{u}_{tt} + a^2(t)|\xi|^2\hat{u} = 0$. Introducing the function $A(t) = 1 + \int_0^t a(\tau)d\tau$ we denote by t_ξ a function of $|\xi|$ such that $A(t_\xi)|\xi| = N$ with a suitable constant N. The function $A(t)$ is increasing, so t_ξ is a decreasing function in $|\xi|$. By the aid of t_ξ we divide the extended phase space $\{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n_\xi\}$ into two zones, the pseudo-differential zone $Z_{pd}(N) := \{(t,\xi) : A(t)|\xi| \le N\}$ and the hyperbolic zone $Z_{hyp}(N) := \{(t,\xi) : A(t)|\xi| \ge N\}$.

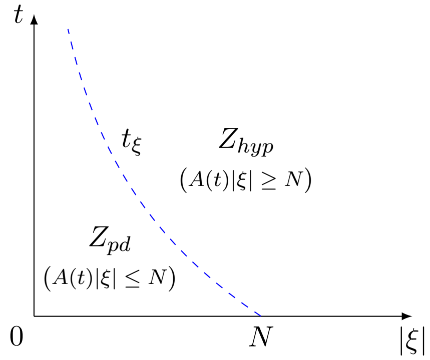


Fig. 2.1.: Description for the definition of zones.

Considerations in the pseudo-differential zone

Let us define the micro-energy $U = \Big(N\frac{a(t)}{A(t)}\hat{u}, D_t\hat{u}\Big)^T$. Then the transformed equation can be written in the form of a system of first order (in D_t)

$$D_tU = A(t,\xi)U,\ A(t,\xi) = \begin{pmatrix} -i\frac{\partial_t \frac{a}{A}}{\frac{a}{A}} & N\frac{a(t)}{A(t)} \\ \frac{A(t)a(t)|\xi|^2}{N} & 0 \end{pmatrix}.$$

Thus the solution $U = U(t,\xi)$ can be represented as $U(t,\xi) = E(t,s,\xi)U(s,\xi)$, where $E(t,s,\xi)$ is the fundamental solution, that is, the solution to the system

$$D_tE(t,s,\xi) = A(t,\xi)E(t,s,\xi),\ E(s,s,\xi) = I,\ 0 \leq s \leq t \leq t_\xi.$$

Denoting by $E^{(jk)}$ the entries of E we get for $k = 1,2$ the system

$$D_tE^{(1k)} = -i\frac{\partial_t \frac{a}{A}}{\frac{a}{A}}E^{(1k)} + N\frac{a(t)}{A(t)}E^{(2k)},\ D_tE^{(2k)} = \frac{A(t)a(t)|\xi|^2}{N}E^{(1k)},\ E^{(jk)}(s,s,\xi) = \delta_{jk}.$$

Integration yields

$$\begin{aligned} E^{(1k)}(t,s,\xi) &= \frac{a(t)}{A(t)}\frac{A(s)}{a(s)}E^{(1k)}(s,s,\xi) + iN\frac{a(t)}{A(t)}\int_s^t E^{(2k)}(\tau,s,\xi)d\tau, \\ E^{(2k)}(t,s,\xi) &= E^{(2k)}(s,s,\xi) + \frac{i|\xi|^2}{N}\int_s^t A(\tau)a(\tau)E^{(1k)}(\tau,s,\xi)d\tau. \end{aligned}$$

We are going to prove the following lemma:

Lemma 2.1.2. *We have the following estimates for the entries $E^{(kl)}(t,0,\xi)$ of the fundamental matrix $E(t,0,\xi)$:*

$$|E^{(11)}(t,0,\xi)| + |E^{(21)}(t,0,\xi)| \leq C_N\frac{a(t)}{A(t)} \text{ for all } t \in [0,t_\xi],$$
$$|E^{(12)}(t,0,\xi)| + |E^{(22)}(t,0,\xi)| \leq C_N\frac{a(t)}{A(t)}t \text{ for all } t \in [0,t_\xi].$$

Proof. If we introduce $y(t,s,\xi) := \dfrac{a(s)}{A(s)}\dfrac{A(t)}{a(t)}E^{(11)}(t,s,\xi)$, then we conclude from the above system

$$y(t,s,\xi) = 1 - |\xi|^2 \int_s^t \int_s^\tau a^2(\theta) y(\theta,s,\xi)\, d\theta d\tau, \quad |y(t,s,\xi)| \le 1 + \int_s^t \int_s^\tau a^2(\theta)|\xi|^2 |y(\theta,s,\xi)|\, d\theta d\tau,$$

respectively. The desired estimates are basing on the following lemma.

Lemma 2.1.3. *Let us assume that a function $y = y(t,s,\xi)$ satisfies the inequality*

$$|y(t,s,\xi)| \le 1 + \int_s^t \int_s^\tau a^2(\theta)|\xi|^2 |y(\theta,s,\xi)|\, d\theta d\tau.$$

Then the function satisfies the estimate

$$|y(t,s,\xi)| \le \exp\Big(\int_s^t \int_s^\tau a^2(\theta)|\xi|^2 d\theta d\tau \Big).$$

Proof. By the method of successive approximation it holds

$$|y_{k+1}(t,s,\xi)| \le 1 + |\xi|^2 \int_s^t \int_s^\tau a^2(\theta)|y_k(\theta,s,\xi)|\, d\theta d\tau.$$

Hence,

$$|y(t,s,\xi)| \le 1 + \sum_{k=1}^\infty \int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) \cdots \int_s^{t_{2k-2}} \int_s^{t_{2k-1}} |\xi|^2 a^2(t_{2k}) dt_{2k} dt_{2k-1} \cdots dt_2 dt_1.$$

We will show by induction principle that

$$\int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) \cdots \int_s^{t_{2k-2}} \int_s^{t_{2k-1}} |\xi|^2 a^2(t_{2k}) dt_{2k} dt_{2k-1} \cdots dt_2 dt_1 \le \frac{1}{k!}\Big(\int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) dt_2 dt_1 \Big)^k.$$

Then the statement of the lemma follows immediately. For $k = 1$ $(t = t_0)$ the statement is clear. Assume that the statement is true for $k = p$:

$$\int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) \cdots \int_s^{t_{2p-2}} \int_s^{t_{2p-1}} |\xi|^2 a^2(t_{2p}) dt_{2p} dt_{2p-1} \cdots dt_2 dt_1 \le \frac{1}{p!}\Big(\int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) dt_2 dt_1 \Big)^p.$$

To prove that the statement is valid for $k = p+1$ we conclude as follows:

$$\begin{aligned}
&\int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) \cdots \int_s^{t_{2p}} \int_s^{t_{2p+1}} |\xi|^2 a^2(t_{2p+2}) dt_{2p+2} dt_{2p+1} \cdots dt_2 dt_1 \\
&\le \int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) \frac{1}{p!}\Big(\int_s^{t_2} \int_s^{t_3} |\xi|^2 a^2(t_4) dt_4 dt_3 \Big)^p dt_2 dt_1 \\
&= \frac{|\xi|^{2p+2}}{p!} \int_s^t \int_s^{t_1} d_{t_2}^2 \underbrace{\Big[\int_s^{t_2} \int_s^{t_3} a^2(t_4) dt_4 dt_3 \Big]}_{F(t_2)} \Big(\int_s^{t_2} \int_s^{t_3} a^2(t_4) dt_4 dt_3 \Big)^p dt_2 dt_1.
\end{aligned}$$

Taking into consideration that $F(t_2) \ge 0$ and $d_{t_2}F(t_2) \ge 0$ we continue to estimate

$$\begin{aligned}
&= \frac{|\xi|^{2p+2}}{p!} \int_s^t \int_s^{t_1} d_{t_2}^2 F(t_2) \cdot F(t_2)^p dt_2 dt_1 \\
&= \frac{|\xi|^{2p+2}}{p!} \int_s^t \Big[d_{t_2}F(t_2) \cdot F(t_2)^p \Big|_s^{t_1} - \int_s^{t_1} d_{t_2}F(t_2) \cdot d_{t_2}F(t_2)^p \Big] \\
&\le \frac{|\xi|^{2p+2}}{p!} \int_s^t d_{t_1}F(t_1) \cdot F(t_1)^p dt_1 \\
&\le \frac{|\xi|^{2p+2}}{p!} \frac{1}{p+1} F(t)^{p+1} = \frac{1}{(p+1)!}\Big(\int_s^t \int_s^{t_1} |\xi|^2 a^2(t_2) dt_2 dt_1 \Big)^{(p+1)}.
\end{aligned}$$

This we wanted to prove. $\square$

Applying this lemma to

$$|y(t,s,\xi)| \leq 1 + \int_s^t \int_s^\tau a^2(\theta)|\xi|^2 |y(\theta,s,\xi)| d\theta d\tau,$$

where

$$y(t,s,\xi) := \frac{a(s)}{A(s)} \frac{A(t)}{a(t)} E^{(11)}(t,s,\xi),$$

it follows

$$|y(t,s,\xi)| \leq \exp\Big(\int_s^t \int_s^\tau a^2(\theta)|\xi|^2 d\theta d\tau \Big).$$

Now we shall estimate the right-hand side. By using the assumptions for the coefficient $a = a(t)$ and the definition of the pseudo-differential zone it holds

$$\begin{aligned} \int_s^t \int_s^\tau a^2(\theta)|\xi|^2 d\theta d\tau &\leq C \int_s^t \int_s^\tau a'(\theta)A(\theta)|\xi|^2 d\theta d\tau \leq C_N|\xi| \int_s^t \int_s^\tau a'(\theta) d\theta d\tau \\ &= C_N|\xi| \int_s^t (a(\tau) - a(s)) d\tau \leq C_N|\xi| \int_s^t a(\tau) d\tau \leq C_N|\xi|A(t) \leq C_N. \end{aligned}$$

So we may conclude that

$$|E^{(11)}(t,s,\xi)| \leq C \frac{a(t)}{A(t)} \frac{A(s)}{a(s)}.$$

Now we consider

$$E^{(21)}(t,s,\xi) = \frac{i|\xi|^2}{N} \int_s^t A(\tau)a(\tau)E^{(11)}(\tau,s,\xi) d\tau.$$

By using the estimate for $|E^{(11)}(t,s,\xi)|$ we have

$$\begin{aligned} |E^{(21)}(t,s,\xi)| &\leq \frac{|\xi|^2}{N} \int_s^t A(\tau)a(\tau)|E^{(11)}(\tau,s,\xi)| d\tau \leq C \frac{|\xi|^2}{N} \int_s^t A(\tau)a(\tau) \frac{a(\tau)}{A(\tau)} \frac{A(s)}{a(s)} d\tau \\ &= C \frac{|\xi|^2 A(s)}{a(s)} \int_s^t a(\tau)^2 d\tau \leq C \frac{|\xi|^2 A(s)}{a(s)} \int_s^t a'(\tau)A(\tau) d\tau \leq C \frac{|\xi|^2 A(t)A(s)}{a(s)} \int_s^t a'(\tau) d\tau \\ &= C \frac{|\xi|^2 A(t)A(s)}{a(s)} (a(t) - a(s)) \leq C \frac{|\xi|^2 A(t)A(s)}{a(s)} a(t) = C \frac{a(t)}{A(t)} \frac{A(s)}{a(s)} |\xi|^2 A^2(t) \leq CN^2 \frac{a(t)}{A(t)} \frac{A(s)}{a(s)}. \end{aligned}$$

On this way we obtained $|E^{(21)}(t,s,\xi)| \leq C \dfrac{a(t)}{A(t)} \dfrac{A(s)}{a(s)}$.

Next we consider the system

$$\begin{aligned} E^{(12)}(t,s,\xi) &= iN \frac{a(t)}{A(t)} \int_s^t E^{(22)}(\tau,s,\xi) d\tau, \\ E^{(22)}(t,s,\xi) &= 1 + \frac{i|\xi|^2}{N} \int_s^t A(\tau)a(\tau)E^{(12)}(\tau,s,\xi) d\tau. \end{aligned}$$

Then we get

$$\begin{aligned} E^{(12)}(t,s,\xi) &= iN \frac{a(t)}{A(t)} \int_s^t \Big(1 + \frac{i|\xi|^2}{N} \int_s^\tau A(\theta)a(\theta)E^{(12)}(\theta,s,\xi) d\theta \Big) d\tau \\ &= iN \frac{a(t)}{A(t)}(t-s) - \frac{a(t)}{A(t)} |\xi|^2 \int_s^t \int_s^\tau A(\theta)a(\theta)E^{(12)}(\theta,s,\xi) d\theta d\tau. \end{aligned}$$

As in the previous steps we estimate as follows:

$$|E^{(12)}(t,s,\xi)| \le N\frac{a(t)}{A(t)}t + \frac{a(t)}{A(t)}|\xi|^2 \int_s^t \int_s^\tau A(\theta)a(\theta)|E^{(12)}(\theta,s,\xi)|d\theta d\tau,$$

$$\begin{aligned}\Big|\frac{A(t)}{a(t)Nt}E^{(12)}(t,s,\xi)\Big| &\le 1 + \frac{|\xi|^2}{t}\int_s^t \int_s^\tau \theta a(\theta)^2\Big|\frac{A(\theta)}{a(\theta)N\theta}E^{(12)}(\theta,s,\xi)\Big|d\theta d\tau \\ &\le 1 + |\xi|^2 \int_s^t \int_s^\tau a(\theta)^2\Big|\frac{A(\theta)}{a(\theta)N\theta}E^{(12)}(\theta,s,\xi)\Big|d\theta d\tau.\end{aligned}$$

So, we see that after setting $y(t,s,\xi) := \dfrac{A(t)}{a(t)Nt}E^{(12)}(t,s,\xi)$ we are able to apply Lemma 2.1.3. In the same way as we did it for $E^{(11)}(t,s,\xi)$ it follows immediately

$$\Big|\frac{A(t)}{a(t)Nt}E^{(12)}(t,s,\xi)\Big| \le C_N \quad \text{thus } |E^{(12)}(t,s,\xi)| \le C_N\frac{a(t)}{A(t)}t.$$

In a similar way we also get

$$|E^{(22)}(t,s,\xi)| \le C_N\frac{a(t)}{A(t)}t.$$

This completes the proof. □

Now let us come back to

$$U(t,\xi) = E(t,0,\xi)U(0,\xi) \text{ for all } 0 \le t \le t_\xi. \tag{2.1.2}$$

Because of $a(t)|\xi||\hat{u}(t,\xi)| \le N\dfrac{a(t)}{A(t)}|\hat{u}(t,\xi)|$ in $Z_{pd}(N)$ from (2.1.2) and Lemma 2.1.2 the following statement can be concluded:

Corollary 2.1.4. *We have in the pseudo-differential zone $Z_{pd}(N)$ the following estimates for all $0 \le t \le t_\xi$:*

$$a(t)|\xi||\hat{u}(t,\xi)| \le C_N\frac{a(t)}{A(t)}|\hat{u}(0,\xi)| + C_N\frac{a(t)}{A(t)}t|D_t\hat{u}(0,\xi)|,$$

$$|D_t\hat{u}(t,\xi)| \le C_N\frac{a(t)}{A(t)}|\hat{u}(0,\xi)| + C_N\frac{a(t)}{A(t)}t|D_t\hat{u}(0,\xi)|.$$

Considerations in the hyperbolic zone

Here we use the hyperbolic micro-energy $U = (a(t)|\xi|\hat{u}, D_t\hat{u})^T$. Then U satisfies

$$D_tU = A(t,\xi)U := \begin{pmatrix} \frac{D_ta}{a} & a(t)|\xi| \\ a(t)|\xi| & 0 \end{pmatrix}U. \tag{2.1.3}$$

Let us carry out the first step of diagonalization. For this reason we set

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{and } U^{(0)} = M^{-1}U.$$

So $D_tU^{(0)} = \mathcal{D}(t,\xi)U^{(0)} + \mathcal{R}(t)U^{(0)}$, where

$$\mathcal{D}(t,\xi) := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \begin{pmatrix} a(t)|\xi| & 0 \\ 0 & -a(t)|\xi| \end{pmatrix}$$

and

$$\mathcal{R}(t) = \frac{1}{2}\begin{pmatrix} \frac{D_t a}{a} & -\frac{D_t a}{a} \\ -\frac{D_t a}{a} & \frac{D_t a}{a} \end{pmatrix}.$$

Let $F_0(t)$ be the diagonal part of $\mathcal{R}(t)$. Now we carry out the second step of diagonalization procedure. Therefore we introduce the matrices

$$N^{(1)} = \begin{pmatrix} 0 & \frac{R_{12}}{\tau_1 - \tau_2} \\ \frac{R_{21}}{\tau_2 - \tau_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\frac{\frac{a'}{a}}{4a(t)|\xi|} \\ -i\frac{\frac{a'}{a}}{4a(t)|\xi|} & 0 \end{pmatrix}, \quad N_1 = I + N^{(1)}.$$

Because of $\Big(\frac{\frac{a'}{a}}{4a(t)|\xi|}\Big)^2 \sim \Big(\frac{\frac{a}{A}}{4a(t)|\xi|}\Big)^2 \leqslant \frac{C}{N}$ we can choose a sufficiently large N such that the determinant of N_1 is $\det N_1 = 1 - \Big(\frac{\frac{a'}{a}}{4a(t)|\xi|}\Big)^2 \geq \frac{1}{2}$. Hence, the matrix N_1 is invertible. Set

$$B^{(1)} = D_t N^{(1)} - (\mathcal{R} - F_0)N^{(1)} = \begin{pmatrix} \frac{(\frac{a'}{a})^2}{8a(t)|\xi|} & \partial_t \frac{\frac{a'}{a}}{4a(t)|\xi|} \\ -\partial_t \frac{\frac{a'}{a}}{4a(t)|\xi|} & \frac{(\frac{a'}{a})^2}{8a(t)|\xi|} \end{pmatrix} \text{ and } \mathcal{R}_1(t,\xi) = -N_1^{-1}B^{(1)}(t,\xi).$$

To understand our strategy let us define the following classes of symbols with limited smoothness with respect to t ($m_3 \geq 0$):

Definition 2.1.1. *The time-dependent function $c(t,\xi)$ belongs to the symbol class $S_l\{m_1, m_2, m_3\}$ with restricted smoothness l, if it satisfies the following estimates*:

$$S_l\{m_1,m_2,m_3\} = \Big\{c(t,\xi) : |D_\xi^\alpha D_t^k c(t,\xi)| \leqslant C_{\alpha,k}|\xi|^{m_1-|\alpha|}a(t)^{m_2}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k} \text{ in } Z_{hyp}(N) \text{ for all } \alpha \text{ and } k \leq l\Big\}.$$

Lemma 2.1.5. *The family of symbol classes $S_l\{m_1, m_2, m_3\}$ generates a hierarchy of symbol classes having the following properties*:

- $S_l\{m_1, m_2, m_3\}$ *is a vector space.*
- $S_l\{m_1, m_2, m_3\}S_l\{m_1', m_2', m_3'\} \subset S_l\{m_1 + m_1', m_2 + m_2', m_3 + m_3'\}$.
- $D_t^k D_\xi^\alpha S_l\{m_1, m_2, m_3\} \subset S_{l-k}\{m_1 - |\alpha|, m_2, m_3 + k\}$.
- $S_0\{-1, -1, 2\} \subset L_\xi^\infty L_t^1(Z_{hyp}(N))$.

Proof. We only verify the fourth property. If $a(t,\xi) \in S_0\{-1,-1,2\}$, then

$$\int_{t_\xi}^\infty |a(\tau,\xi)|d\tau \leq \int_{t_\xi}^\infty C\frac{1}{|\xi|}\frac{a(\tau)}{A^2(\tau)}d\tau = \frac{1}{A(t_\xi)|\xi|} = C_N$$

due to the definition of the hyperbolic zone. $\square$

Using the above introduced symbol classes and the assumptions for $a = a(t)$ we have $\frac{D_t a}{a} \in S_1\{0,0,1\}$, $B^{(1)}, \mathcal{R}_1 \in S_0\{-1,-1,2\}$. Moreover, we conclude

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{R}(t)\big)N_1(t,\xi)U^{(1)}(t,\xi) = N_1(t,\xi)\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)U^{(1)}(t,\xi).$$

Hence, we can find the solution $U^{(0)}(t,\xi) =: N_1(t,\xi)U^{(1)}(t,\xi)$, where $U^{(1)}(t,\xi)$ is the solution to the system

$$\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)V(t,\xi) = 0.$$

We can write $U^{(1)}(t,\xi) = E_1(t,t_\xi,\xi)U^{(1)}(t_\xi,\xi)$, where $E(t,s,\xi)$ is the fundamental solution, that is, the solution of the system

$$D_tE_1(t,s,\xi) = \big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)E_1(t,s,\xi), \quad E_1(s,s,\xi) = I,\ t \geq s \geq t_\xi.$$

The solution $E_0 = E_0(t,s,\xi)$ of the "principal part" (concerning the hierarchy of symbol classes) fulfils

$$D_tE_0(t,s,\xi) = (\mathcal{D}(t,\xi) + F_0(t))E_0(t,s,\xi), \quad E_0(s,s,\xi) = I,\ t \geq s \geq t_\xi.$$

Thus

$$E_0(t,s,\xi) = \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\begin{pmatrix} \exp\Big(\int_s^t ia(\tau)|\xi|d\tau\Big) & 0 \\ 0 & \exp\Big(-\int_s^t ia(\tau)|\xi|d\tau\Big)\end{pmatrix}.$$

Let us set

$$\begin{aligned}
&\mathcal{R}_2(t,s,\xi) = E_0(t,s,\xi)^{-1}\mathcal{R}_1(t,\xi)E_0(t,s,\xi),\\
&Q(t,s,\xi) = I + \sum_{k=1}^{\infty} i^k \int_s^t \mathcal{R}_2(t_1,s,\xi)\int_s^{t_1}\mathcal{R}_2(t_2,s,\xi)\cdots\int_s^{t_{k-1}}\mathcal{R}_2(t_k,s,\xi)dt_k\cdots dt_2dt_1.
\end{aligned}$$

Then $Q(t,s,\xi)$ solves the Cauchy problem

$$D_tQ(t,s,\xi) = \mathcal{R}_2(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I,\ t \geq s \geq t_\xi.$$

The fundamental solution $E_1 = E_1(t,s,\xi)$ is representable in the form $E_1(t,s,\xi) = E_0(t,s,\xi)Q(t,s,\xi)$. Analogous to the statement of Lemma 2.1.3 we are able to show the following estimate for $Q(t,s,\xi)$:

$$|Q(t,s,\xi)| \leq \exp\Big(\int_s^t |\mathcal{R}_1(\tau,\xi)|d\tau\Big) \leq \exp\Big(\frac{1}{|\xi|}\Big(\frac{-1}{A(\tau)}\Big)\Big|_s^t\Big) \leq C_N.$$

Here we use the fourth statement of Lemma 2.1.5 and $\mathcal{R}_1 \in S_0\{-1,-1,2\}$. The backward transformation yields $U(t,\xi) = MN_1(t,\xi)E_0(t,s,\xi)Q(t,s,\xi)N_1^{-1}(s,\xi)M^{-1}U(s,\xi)$,

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \leq \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\left|\begin{pmatrix} a(s)|\xi|\hat{u}(s,\xi) \\ D_t\hat{u}(s,\xi)\end{pmatrix}\right| \quad \text{for all } t \geq s \geq t_\xi.$$

Corollary 2.1.6. *We have in the hyperbolic zone $Z_{hyp}(N)$ the estimate*

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \leq C\frac{\sqrt{a(t)}}{\sqrt{a(t_\xi)}}\left|\begin{pmatrix} a(t_\xi)|\xi|\hat{u}(t_\xi,\xi) \\ D_t\hat{u}(t_\xi,\xi)\end{pmatrix}\right|$$

for all $t \geq t_\xi$.

Conclusion

From the statements of Corollaries 2.1.4 and 2.1.6 we derive the statement of our theorem.

1.case: $\{|\xi| \geq N\}$

Then the statement of Corollary 2.1.6 implies immediately

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi) \end{pmatrix}\right| \leq C\sqrt{a(t)}\left|\begin{pmatrix} |\xi|\hat{u}(0,\xi) \\ D_t\hat{u}(0,\xi) \end{pmatrix}\right|$$

for all $t \geq 0$.

2.case: $\{|\xi| \leq N\}$

Then the statements of Corollaries 2.1.6 and 2.1.4 give immediately

$$\begin{aligned} a(t)|\xi||\hat{u}(t,\xi)| &\leq C\frac{\sqrt{a(t)}}{\sqrt{a(t_\xi)}}\Big(a(t_\xi)|\xi||\hat{u}(t_\xi,\xi)| + |D_t\hat{u}(t_\xi,\xi)|\Big) \\ &\leq C_N\sqrt{a(t)}\Big(\frac{\sqrt{a(t_\xi)}}{A(t_\xi)}|\hat{u}(0,\xi)| + \frac{\sqrt{a(t_\xi)}}{A(t_\xi)}t_\xi|D_t\hat{u}(0,\xi)|\Big). \end{aligned}$$

This inequality together with assumption $(A4)$ implies

$$a(t)|\xi||\hat{u}(t,\xi)| \leq C\sqrt{a(t)}(|\hat{u}(0,\xi)| + |D_t\hat{u}(0,\xi)|) \text{ for all } t \geq t_\xi.$$

The statements of Corollary 2.1.4 and assumption $(A4)$ yield

$$a(t)|\xi||\hat{u}(t,\xi)| \leq C\sqrt{a(t)}(|\hat{u}(0,\xi)| + |D_t\hat{u}(0,\xi)|) \text{ for all } t \leq t_\xi.$$

Summarizing both cases we have shown

$$a(t)|\xi||\hat{u}(t,\xi)| \leq C\sqrt{a(t)}\big(|\hat{u}(0,\xi)| + |D_t\hat{u}(0,\xi)|\big) \text{ for all } (t,\xi) \in \{t \geq 0\} \times \mathbb{R}^n_\xi.$$

In the same way we prove

$$|D_t\hat{u}(t,\xi)| \leq C\sqrt{a(t)}\big(|\hat{u}(0,\xi)| + |D_t\hat{u}(0,\xi)|\big) \text{ for all } (t,\xi) \in \{t \geq 0\} \times \mathbb{R}^n_\xi.$$

This completes the proof to Theorem 2.1.1. $\square$

Some examples

Typical examples for possible increasing speeds $a = a(t)$ are

$$a(t) = (1+t)^l,\ a(t) = e^t,\ a(t) = (e^{[n]})^t := e^{e^{\cdot^{\cdot^{e^t}}}}.$$

Example 2.1.1. If we choose $a(t) = (1+t)^l$, then all assumptions (A1) to (A4) are satisfied. The solutions to the Cauchy problem for $u_{tt} - (1+t)^{2l}\Delta u = 0$ satisfy the following energy estimates: For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2} \leq C(1+t)^{\frac{l}{2}}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

For the "elastic" energy we have

$$\|(1+t)^l\nabla u(t,\cdot)\|_{L^2} \leq C(1+t)^{\frac{l}{2}}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

Example 2.1.2. If we choose for example $a(t) = e^t$, then all assumptions (A1) to (A4) are satisfied. The solutions to the Cauchy problem for $u_{tt} - e^{2t}\Delta u = 0$ satisfy the following energy estimates:
For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2} \leq Ce^{\frac{1}{2}t}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

For the "elastic" energy we have

$$\|e^t\nabla u(t,\cdot)\|_{L^2} \leq Ce^{\frac{1}{2}t}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

Example 2.1.3. If we choose for example $a(t) = (e^{[n]})^t$, then all assumptions (A1) to (A4) are satisfied. The solutions to the Cauchy problem for $u_{tt} - (e^{[n]})^{2t}\Delta u = 0$ satisfy the following energy estimates:
For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2} \leq C(e^{[n]})^{\frac{t}{2}}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

For the "elastic" energy we have

$$\|(e^{[n]})^t\nabla u(t,\cdot)\|_{L^2} \leq C(e^{[n]})^{\frac{t}{2}}(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

2.2. Critical cases of damped wave models

2.2.1. Some model cases

Increasing speed of potential order

Let us study

$$u_{tt} - (1+t)^{2l}\Delta u + \frac{a(l+1)}{(1+t)}u_t = 0, \;\; u(0,x) = u_1(x), \;\; u_t(0,x) = u_2(x). \tag{2.2.1}$$

We introduce the energy of the solution in L^2:

$$E(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n}\Big(|u_t(t,x)|^2 + |(1+t)^l\nabla u(t,x)|^2\Big)dx.$$

We will look for the behavior of the solution $u(t,\cdot)$ and the energy $E(u)(t)$ as t tends to infinity.
First we reduce (2.2.1) by *special functions to the Bessel equation.* Applying the partial Fourier transformation with respect to x to (2.2.1) we obtain

$$\hat{u}_{tt} + (1+t)^{2l}|\xi|^2\hat{u} + \frac{a(l+1)}{1+t}\hat{u}_t = 0, \; \hat{u}(0,\xi) = \hat{u}_1(\xi), \; \hat{u}_t(0,\xi) = \hat{u}_2(\xi). \tag{2.2.2}$$

Setting $\tau = \dfrac{(1+t)^{l+1}}{1+l}|\xi| = K(t,\xi)$, and $\mu := a + \dfrac{l}{l+1} > 0$ then we conclude for $\hat{u} = \hat{u}(\tau)$

$$\hat{u}_{\tau\tau} + \frac{\mu}{\tau}\hat{u}_\tau + \hat{u} = 0.$$

We will look for a solution in the form $\hat{u}(\tau) = \tau^\rho\omega(\tau)$. Choosing $\rho = \dfrac{1-\mu}{2}$, $\rho < \dfrac{1}{2}$, we come to the Bessel differential equation

$$\tau^2 w_{\tau\tau} + \tau w_\tau + (\tau^2 - \rho^2)w = 0.$$

This equation has two fundamental solutions

$$w_{\pm}(\tau)=\tau^{\rho}H^{\pm}_{\rho}(\tau).$$

So the general solution of (2.2.1) is

$$\hat{u}(t,\xi)=C_1(\xi)w_+(t,\xi)+C_2(\xi)w_-(t,\xi).$$

Here H^+_ρ and H^-_ρ denote the *Hankel functions*. The solution to (2.2.1) can be represented by

$$\hat{u}(t,\xi)=V_1(t,\xi)\hat{u}_1(\xi)+V_2(t,\xi)\hat{u}_2(\xi).$$

Introducing

$$\Psi_{k,\rho,\delta}(t,\xi)=|\xi|^k\begin{vmatrix} H^-_\rho\Big(\frac{|\xi|}{l+1}\Big) & H^-_{\rho+\delta}\Big(\frac{(1+t)^{l+1}|\xi|}{l+1}\Big)\\ H^+_\rho\Big(\frac{|\xi|}{l+1}\Big) & H^+_{\rho+\delta}\Big(\frac{(1+t)^{l+1}|\xi|}{l+1}\Big)\end{vmatrix}$$

we have

$$\begin{aligned} V_1(t,\xi) &= \frac{i\pi}{4(l+1)}(1+t)^{\rho(l+1)}\Psi_{1,\rho-1,1}(t,\xi),\\ V_2(t,\xi) &= -\frac{i\pi}{4(l+1)}(1+t)^{\rho(l+1)}\Psi_{0,\rho,0}(t,\xi),\\ V_{1,t}(t,\xi) &= \frac{i\pi}{4(l+1)}(1+t)^{l+\rho(l+1)}\Psi_{2,\rho-1,0}(t,\xi),\\ V_{2,t}(t,\xi) &= -\frac{i\pi}{4(l+1)}(1+t)^{l+\rho(l+1)}\Psi_{1,\rho,-1}(t,\xi). \end{aligned}$$

We will use the following properties of the Hankel functions to estimate $V_j(t,\xi)$ and $V_{j,t}(t,\xi)$:

- For $\tau\geq N$ (N is a large constant): $|H^{\pm}_\rho(\tau)|\leq C|\tau|^{-\frac12}$.
- For $0<\tau\leq c<1$: $|H^{\pm}_\rho(\tau)|\leq\begin{cases}\tau^{-|\rho|} & \text{when } \rho\neq0,\\ -\log\tau & \text{when } \rho=0.\end{cases}$
- For an integral value n the Weber function $Y_n(\tau)$ satisfies

  $$Y_n(\tau)=\frac{2}{\pi}J_n(\tau)\log\tau+A_n(\tau),$$

  where $\tau^nA_n(\tau)$ is entire and $A_n(0)\neq0$. We also have $J_{-n}(\tau)=(-1)^nJ_n(\tau)$.
- The function $\Lambda_\nu(\tau)=\tau^{-\nu}J_\nu(\tau)$ is entire in ν and τ, furthermore $\Lambda_\nu(0)\neq0$.

We divide the extended phase space into three zones:

- $Z_1=\{|\xi|:|\xi|\geq N\}$:

  $$|\Psi_{k,\rho,\delta}(t,\xi)|\leq C|\xi|^{k-1}(1+t)^{-\frac12(l+1)},$$

- $Z_2=\{|\xi|:|\xi|\leq N\leq K(t,\xi)\}$:

  $$|\Psi_{k,\rho,\delta}(t,\xi)|\leq\begin{cases}(1+t)^{-\frac12(l+1)} & \text{if } k-|\rho|>\frac12,\\ (1+t)^{(|\rho|-k)(l+1)} & \text{if } k-|\rho|\leq\frac12,\ \rho\neq0,\\ (1+t)^{-k(l+1)}\log(e+t) & \text{if } k\leq\frac12,\ \rho=0,\end{cases}$$

- $Z_3 = \{|\xi| : K(t,\xi) \le N\}$:
  In the above relations for $V_1(t,\xi), V_2(t,\xi), V_{1,t}(t,\xi), V_{2,t}(t,\xi)$ we have δ as an integer $0,\ \pm 1$, so ρ and $\rho+\delta$ have the same integral property.

  Let us assume that ρ and $\rho+\delta$ are no integers.

  To evaluate the Hankel functions we write $H^{\pm}_{\rho}(\tau) = J_\rho(\tau) \pm iY_\rho(\tau)$. Then $\Psi_{k,\rho,\delta}(t,\xi) = 2i|\xi|^k \begin{vmatrix} J_\rho(K(0,\xi)) & J_{\rho+\delta}(K(t,\xi)) \\ Y_\rho(K(0,\xi)) & Y_{\rho+\delta}(K(t,\xi)) \end{vmatrix}$. For non-integer ρ the Weber function is determined by the Bessel functions of the first kind

  $$Y_\rho(\tau) = \frac{J_\rho(\tau)\cos(\rho\pi) - J_{-\rho}(\tau)}{\sin(\rho\pi)}.$$

  So the determinant can be substituted by

  $$C(J_\rho(K(0,\xi))J_{-(\rho+\delta)}(K(t,\xi)) - J_{-\rho}(K(0,\xi))J_{\rho+\delta}(K(t,\xi)).$$

  Noting that $J_\rho(\tau) \le C\tau^\rho$ we arrive at

  $$|\Psi_{k,\rho,\delta}(t,\xi)| \le C((1+t)^{l+1}|\xi|)^{k-\delta}(1+t)^{(l+1)(-k-\rho)} + C((1+t)^{l+1}|\xi|)^{k+\delta}(1+t)^{(l+1)(-k+\rho)}.$$

  In our case it holds $k \ge |\delta|, \delta \in \{0, \pm 1\}$. Then we have in Z_3 the estimate

  $$|\Psi_{k,\rho,\delta}(t,\xi)| \le C(1+t)^{(|\rho|-k)(l+1)}.$$

  Let us assume that ρ is an integer.

  We use the third property to get

  $$\begin{aligned} \Psi_{k,\rho,\delta}(t,\xi) &= -\frac{4i}{\pi}|\xi|^k \log((1+t)^{l+1})J_\rho(K(0,\xi))J_{\rho+\delta}(K(t,\xi)) \\ &\quad + 2i|\xi|^k \begin{vmatrix} J_\rho(K(0,\xi)) & J_{\rho+\delta}(K(t,\xi)) \\ A_\rho(K(0,\xi)) & A_{\rho+\delta}(K(t,\xi)) \end{vmatrix}. \end{aligned}$$

  For the first part we will sometimes use $J_{-\rho}$ instead of J_ρ and $J_{-(\rho+\delta)}$ instead of $J_{\rho+\delta}$. Using the fourth property we have that this term can be estimated by

  $$C(1+t)^{(l+1)(-|\rho|-k)}\log(e+t).$$

  The determinant in the second term can be estimated by

  $$\begin{aligned} \le & |(J_\rho(K(0,\xi))|\xi|^{-\rho})(A_{\rho+\delta}(K(t,\xi))K(t,\xi)^{(\rho+\delta)})(1+t)^{-(\delta+\rho)(l+1)}|\xi|^{-\delta}| \\ & + |(A_\rho(K(0,\xi))|\xi|^{\rho})(J_{\rho+\delta}(K(t,\xi))K(t,\xi)^{-(\rho+\delta)})(1+t)^{(\delta+\rho)(l+1)}|\xi|^{\delta}| \\ \le & C(1+t)^{(-\rho-\delta)(l+1)}|\xi|^{-\delta} + C(1+t)^{(\rho+\delta)(l+1)}|\xi|^{\delta}. \end{aligned}$$

  In our case it holds $k \ge |\delta|$. By the definition of the zone we can estimate $|\xi|^{k\pm\delta} \le C(1+t)^{-(k\pm\delta)(l+1)}$. Finally, the second term is estimated by $(1+t)^{(l+1)(|\rho|-k)}$.
  Hence, in Z_3 we have the estimate $|\Psi_{k,\rho,\delta}(t,\xi)| \le \begin{cases} (1+t)^{(l+1)(|\rho|-k)} & \text{if } \rho \ne 0, \\ (1+t)^{-k(l+1)}\log(e+t) & \text{if } \rho = 0. \end{cases}$

Using the above estimates implies

$$|V_1(t,\xi)| \lesssim \begin{cases} (1+t)^{(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ 1 & \text{in } Z_2 \cup Z_3, \end{cases}$$

$$|V_2(t,\xi)| \lesssim \begin{cases} |\xi|^{-1}(1+t)^{(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ 1 & \text{in } Z_2 \cup Z_3,\ \rho<0, \\ \log(e+t) & \text{in } Z_2 \cup Z_3,\ \rho=0, \\ (1+t)^{2\rho(l+1)} & \text{in } Z_2 \cup Z_3,\ \frac{1}{2}>\rho>0, \end{cases}$$

$$|V_{1,t}(t,\xi)| \lesssim \begin{cases} |\xi|(1+t)^{l+(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ (1+t)^{l+\max\{\rho-\frac{1}{2};-1\}(l+1)} & \text{in } Z_2 \cup Z_3, \end{cases}$$

$$|V_{2,t}(t,\xi)| \lesssim \begin{cases} (1+t)^{l+(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ (1+t)^{l+\max\{\rho-\frac{1}{2},-1\}(l+1)} & \text{in } Z_2, \\ (1+t)^{l-(l+1)} & \text{in } Z_3, \rho<0, \\ (1+t)^{l-(l+1)}\log(e+t) & \text{in } Z_3, \rho=0, \\ (1+t)^{l+(2\rho-1)(l+1)} & \text{in } Z_3, 0<\rho<\frac{1}{2}. \end{cases}$$

Combining the last four cases gives

$$|V_{2,t}(t,\xi)| \lesssim \begin{cases} (1+t)^{l+(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ (1+t)^{l+\max\{\rho-\frac{1}{2},-1\}(l+1)} & \text{in } Z_2 \cup Z_3. \end{cases}$$

Proposition 2.2.1. *We have the following estimates for the solution to (2.2.1):*

$$\|u(t,\cdot)\|_{L^2} \le C\|u_1\|_{L^2} + C\|u_2\|_{H^{-1}} \begin{cases} 1 & \textit{if } \rho<0, \\ \log(e+t) & \textit{if } \rho=0, \\ (1+t)^{2\rho(l+1)} & \textit{if } 0<\rho<\frac{1}{2}. \end{cases}$$

For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2} \le C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_1\|_{H^1} + C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_2\|_{L^2}.$$

For the "elastic" energy we have

$$\|(1+t)^l\nabla u(t,\cdot)\|_{L^2} \le C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_1\|_{H^1} + C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_2\|_{L^2}.$$

Consequently, the energy

$$E(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n}\Big(|u_t(t,x)|^2 + \big((1+t)^l|\nabla u(t,x)|\big)^2\Big)dx$$

can be estimated in the form

$$\begin{aligned} E(u)(t) &\le C(1+t)^{2l+(l+1)\max\{2\rho-1,-2\}}\|u_1\|^2_{H^1} + C(1+t)^{2l+(l+1)\max\{2\rho-1,-2\}}\|u_2\|^2_{L^2} \\ &= C(1+t)^{2l+(l+1)\max\{-a-\frac{l}{l+1},-2\}}\|u_1\|^2_{H^1} + C(1+t)^{2l+(l+1)\max\{-a-\frac{l}{l+1},-2\}}\|u_2\|^2_{L^2}. \end{aligned}$$

Proof. Let us begin to prove the estimate for the solution u. We have

$$\|u(t,\cdot)\|_{L^2} = \|\widehat{u}(t,\cdot)\|_{L^2} = \|V_1(t,\xi)\widehat{u}_1(\xi) + V_2(t,\xi)\widehat{u}_2(\xi)\|_{L^2}.$$

Applying Hölder's inequality to the right-hand side gives

$$\|u(t,\cdot)\|_{L^2} \le \|V_1(t,\xi)\|_{L^\infty}\|\widehat{u}_1(\xi)\|_{L^2} + \|\langle\xi\rangle V_2(t,\xi)\|_{L^\infty}\|\langle\xi\rangle^{-1}\widehat{u}_2(\xi)\|_{L^2}.$$

We have

$$\|V_1(t,\xi)\|_{L^\infty} \lesssim \max\big\{(1+t)^{(l+1)(\rho-\frac{1}{2})}, 1\big\},$$

and we can estimate $\|\langle\xi\rangle V_2(t,\xi)\|_{L^\infty}$ by the following estimates:

$$\begin{aligned}
|\langle\xi\rangle V_2(t,\xi)| \quad &\lesssim \begin{cases} \langle\xi\rangle|\xi|^{-1}(1+t)^{(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ \langle\xi\rangle & \text{in } Z_2\cup Z_3,\ \rho<0, \\ \langle\xi\rangle\log(e+t) & \text{in } Z_2\cup Z_3,\ \rho=0, \\ \langle\xi\rangle(1+t)^{2\rho(l+1)} & \text{in } Z_2\cup Z_3,\ \frac{1}{2}>\rho>0, \end{cases} \\
&\lesssim \begin{cases} 1 & \text{in } Z_1, \\ C & \text{in } Z_2\cup Z_3,\ \rho<0, \\ \log(e+t) & \text{in } Z_2\cup Z_3,\ \rho=0, \\ (1+t)^{2\rho(l+1)} & \text{in } Z_2\cup Z_3,\ \frac{1}{2}>\rho>0. \end{cases}
\end{aligned}$$

Here we have used the properties $(1+t)^{(l+1)(\rho-\frac{1}{2})} \le 1$ for $\rho \le \frac{1}{2}$; $\langle\xi\rangle \le N$ (N is a constant) in the zones $Z_2 \cup Z_3$. From the above estimates we can conclude that

$$\|u(t,\cdot)\|_{L^2} \le C\|u_1\|_{L^2} + C\|u_2\|_{H^{-1}} \begin{cases} 1 & \text{if } \rho<0, \\ \log(e+t) & \text{if } \rho=0, \\ (1+t)^{2\rho(l+1)} & \text{if } 0<\rho<\frac{1}{2}. \end{cases}$$

Now let us prove the statement for $(1+t)^l\nabla u(t,\cdot)$. We have in the extended phase space

$$\begin{aligned}
&(1+t)^l|\xi|\hat{u}(t,\xi) = (1+t)^l|\xi|V_1(t,\xi)\hat{u}_1(\xi) + (1+t)^l|\xi|V_2(t,\xi)\hat{u}_2(\xi), \text{ where} \\
&|\xi|V_1(t,\xi) = \frac{i\pi}{4(l+1)}(1+t)^{\rho(l+1)}\Psi_{2,\rho-1,1}(t,\xi),\ |\xi|V_2(t,\xi) = -\frac{i\pi}{4(l+1)}(1+t)^{\rho(l+1)}\Psi_{1,\rho,0}(t,\xi).
\end{aligned}$$

Using the above estimates in different zones we have

$$|\Psi_{2,\rho-1,1}(t,\xi)| \lesssim \begin{cases} |\xi|(1+t)^{-\frac{1}{2}(l+1)} & \text{in } Z_1, \\ (1+t)^{-\frac{1}{2}(l+1)} & \text{in } Z_2 \text{ if } 2-|\rho-1|>\frac{1}{2}, \\ (1+t)^{(|\rho-1|-2)(l+1)} & \text{in } Z_2 \text{ if } 2-|\rho-1|\le\frac{1}{2}, \\ (1+t)^{(|\rho-1|-2)(l+1)} & \text{in } Z_3, \end{cases}$$

and

$$|\Psi_{1,\rho,0}(t,\xi)| \lesssim \begin{cases} (1+t)^{-\frac{1}{2}(l+1)} & \text{in } Z_1, \\ (1+t)^{-\frac{1}{2}(l+1)} & \text{in } Z_2 \text{ if } 1-|\rho|>\frac{1}{2}, \\ (1+t)^{(|\rho|-1)(l+1)} & \text{in } Z_2 \text{ if } 1-|\rho|\le\frac{1}{2}, \\ (1+t)^{(|\rho|-1)(l+1)} & \text{in } Z_3 \text{ if } \rho\ne0, \\ (1+t)^{-(l+1)}\log(e+t) & \text{in } Z_3 \text{ if } \rho=0. \end{cases}$$

Consequently,

$$(1+t)^l|\xi||V_1(t,\xi)| \lesssim \begin{cases} |\xi|(1+t)^{l+(\rho-\frac{1}{2})(l+1)} & \text{in } Z_1, \\ (1+t)^{l+(\rho-\frac{1}{2})(l+1)} & \text{in } Z_2 \text{ if } \rho\in(-\frac{1}{2},\frac{1}{2}), \\ (1+t)^{l-(l+1)} & \text{in } Z_2 \text{ if } \rho\in(-\infty,-\frac{1}{2}], \\ (1+t)^{l-(l+1)} & \text{in } Z_3, \end{cases}$$

and

$$(1+t)^l|\xi||V_2(t,\xi)| \lesssim \begin{cases} (1+t)^{l+(\rho-\frac{1}{2})(l+1)} & \text{in } Z_1, \\ (1+t)^{l+(\rho-\frac{1}{2})(l+1)} & \text{in } Z_2 \text{ if } \rho \in (-\frac{1}{2},\frac{1}{2}), \\ (1+t)^{l-(l+1)} & \text{in } Z_2 \text{ if } \rho \in (-\infty,-\frac{1}{2}], \\ (1+t)^{l+(2\rho-1)(l+1)} & \text{in } Z_3 \text{ if } \rho > 0, \\ (1+t)^{l-(l+1)} & \text{in } Z_3 \text{ if } \rho < 0, \\ (1+t)^{l-(l+1)}\log(e+t) & \text{in } Z_3 \text{ if } \rho = 0. \end{cases}$$

Summarizing we conclude

$$(1+t)^l|\xi||\hat{u}(t,\xi)| \leq C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\big(\langle\xi\rangle|\hat{u}_1(\xi)|+|\hat{u}_2(\xi)|\big).$$

This yields the desired estimate for the elastic energy.

Finally, let us prove the statement for the kinetic energy. We have

$$\|u_t(t,\cdot)\|_{L^2} = \|\widehat{u}_t(t,\cdot)\|_{L^2} = \|V_{1,t}(t,\xi)\widehat{u}_1(\xi)+V_{2,t}(t,\xi)\widehat{u}_2(\xi)\|_{L^2}.$$

Using the estimates

$$\begin{aligned} |V_{1,t}(t,\xi)| &\lesssim \begin{cases} |\xi|(1+t)^{l+(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ (1+t)^{l+\max\{\rho-\frac{1}{2};-1\}(l+1)} & \text{in } Z_2\cup Z_3, \end{cases} \\ &\lesssim \begin{cases} \langle\xi\rangle(1+t)^{l+(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ (1+t)^{l+\max\{\rho-\frac{1}{2};-1\}(l+1)} & \text{in } Z_2\cup Z_3, \end{cases} \end{aligned}$$

and

$$|V_{2,t}(t,\xi)| \lesssim \begin{cases} (1+t)^{l+(l+1)(\rho-\frac{1}{2})} & \text{in } Z_1, \\ (1+t)^{l+\max\{\rho-\frac{1}{2},-1\}(l+1)} & \text{in } Z_2\cup Z_3, \end{cases}$$

then

$$\|u_t(t,\cdot)\|_{L^2} \lesssim C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\big(\|\langle\xi\rangle\hat{u}_1(\xi)\|_{L^2}+\|\hat{u}_2(\xi)\|_{L^2}\big).$$

This completes the proof. □

Remark 2.2.1. The estimate of the solution is determined by the small frequencies from $Z_2 \cup Z_3$. The decay rate with respect to the norm of u_1 comes from the large frequencies from Z_1, and the decay rate with respect to the norm of u_2 comes from the small frequencies in $Z_2 \cup Z_3$.

Remark 2.2.2. The statement of Proposition 2.2.1 gives a decay estimate for $\|\nabla u(t,\cdot)\|_{L^2}$ even for $a > -\frac{l}{l+1}$.

Let us compare the result with the result from [W04]. We start again with the Cauchy problem

$$u_{tt} - (1+t)^{2l}\Delta u + \frac{a(l+1)}{(1+t)}u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x).$$

Setting $\tau = \dfrac{(1+t)^{l+1}}{l+1}$ we get

$$u_{\tau\tau} - \Delta u + \frac{l+a(l+1)}{(1+t)^{l+1}}u_\tau = 0, \quad u_{\tau\tau} - \Delta u + \frac{a+\frac{l}{l+1}}{\tau}u_\tau = 0, \text{ respectively.}$$

Substituting $\tau =: \tilde{\tau} + 1$, then (2.2.1) becomes

$$u_{\tilde{\tau}\tilde{\tau}} - \Delta u + \frac{\mu}{1+\tilde{\tau}} u_{\tilde{\tau}} = 0.$$

This is exactly the case $\mu = a + \dfrac{l}{l+1}$ from [W04]. From [W04] we conclude

$$\|u(\tilde{\tau},\cdot)\|_{L^2} \le C\|u_1\|_{L^2} + C\|u_2\|_{H^{-1}} \begin{cases} (1+\tilde{\tau})^{2\rho} & \text{if } 0<\rho<\frac{1}{2}, \\ \log(e+\tilde{\tau}) & \text{if } \rho = 0, \\ 1 & \text{if } \rho<0, \end{cases}$$

$$\|u_{\tilde{\tau}}(\tilde{\tau},\cdot)\|_{L^2} \le C(1+\tilde{\tau})^{\rho-\frac{1}{2}}\|u_1\|_{H^1} + C(1+\tilde{\tau})^{\max\{\rho-\frac{1}{2},-1\}}\|u_2\|_{L^2}.$$

Transforming back gives

$$\|u(t,\cdot)\|_{L^2} \le C\|u_1\|_{L^2} + C\|u_2\|_{H^{-1}} \begin{cases} (1+t)^{(l+1)(2\rho)} & \text{if } 0<\rho<\frac{1}{2}; \\ \log\left(e + \dfrac{(1+t)^{l+1}}{l+1}\right) \sim \log(e+t) & \text{if } \rho = 0; \\ 1 & \text{if } \rho<0; \end{cases}$$

$$\|(1+t)^{-l}u_t(t,\cdot)\|_{L^2} \le C(1+t)^{(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_1\|_{H^1} + C(1+t)^{(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_2\|_{L^2},$$

$$\|u_t(t,\cdot)\|_{L^2} \le C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_1\|_{H^1} + C(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\|u_2\|_{L^2},$$

respectively. These are the same estimates as in Proposition 2.2.1.

Increasing speed of exponential order

Using the transformations from the previous section we are interested in another model case

$$u_{tt} - \exp(2t)\Delta u + a u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x),\ a>0. \tag{2.2.3}$$

Applying the Fourier transform to (2.2.3) with respect to x and using the change of variables $\tau = e^t|\xi|$ we get

$$\hat{u}_{\tau\tau} + \frac{a+1}{\tau}\hat{u}_\tau + \hat{u} = 0.$$

Similarly to the first case we introduce

$$\Psi_{k,\rho,\delta}(t,\xi) = |\xi|^k \begin{vmatrix} H^-_\rho(|\xi|) & H^-_{\rho+\delta}(e^t|\xi|) \\ H^+_\rho(|\xi|) & H^+_{\rho+\delta}(e^t|\xi|) \end{vmatrix},$$

then

$$V_1(t,\xi) = \frac{i\pi}{4}e^{t\rho}\Psi_{1,\rho-1,1}(t,\xi), \quad V_2(t,\xi) = -\frac{i\pi}{4}e^{t\rho}\Psi_{0,\rho,0}(t,\xi),$$

$$V_{1,t}(t,\xi) = \frac{i\pi}{4}e^{t(\rho+1)}\Psi_{2,\rho-1,0}(t,\xi), \quad V_{2,t}(t,\xi) = -\frac{i\pi}{4}e^{t(\rho+1)}\Psi_{1,\rho,-1}(t,\xi),$$

where $\rho := \dfrac{1-(a+1)}{2} = -\dfrac{a}{2} < 0$. We divide the phase space into three zones and conclude there the following estimates:

- in $Z_1 = \{\xi : |\xi| \ge K\}$:

$$|\Psi_{k,\rho,\delta}(t,\xi)| \le C|\xi|^{k-1}e^{-\frac{t}{2}},$$

- in $Z_2=\{\xi : |\xi|\leq K\leq e^t|\xi|\}$:

$$|\Psi_{k,\rho,\delta}(t,\xi)|\leq C\begin{cases} e^{-\frac{t}{2}} & \text{if } k+\rho>\frac{1}{2},\\ e^{-t(k+\rho)} & \text{if } k+\rho\leq\frac{1}{2}, \rho\neq 0,\\ e^{-tk}\log(e+t) & \text{if } k\leq\frac{1}{2}, \rho=0,\end{cases}$$

- in $Z_3=\{\xi : e^t|\xi|\leq K\}$:

$$|\Psi_{k,\rho,\delta}(t,\xi)|\leq C\begin{cases} e^{-t(k+\rho)} & \text{if } \rho\neq 0,\\ e^{-tk}\log(e+t) & \text{if } \rho=0.\end{cases}$$

These estimates lead to the following estimates for $V_j(t,\xi)$ and $V_{j,t}(t,\xi)$ for $j=1,2$:

$$|V_1(t,\xi)|\lesssim\begin{cases} e^{t(\rho-\frac{1}{2})} & \text{in } Z_1,\\ 1 & \text{in } Z_2\cup Z_3,\end{cases}$$
$$|V_2(t,\xi)|\lesssim\begin{cases} |\xi|^{-1}e^{t(\rho-\frac{1}{2})} & \text{in } Z_1,\\ 1 & \text{in } Z_2\cup Z_3,\end{cases}$$
$$|V_{1,t}(t,\xi)|\lesssim\begin{cases} |\xi|e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_1,\\ e^{t+t\max\{\rho-\frac{1}{2},-1\}} & \text{in } Z_2\cup Z_3,\end{cases}$$
$$|V_{2,t}(t,\xi)|\lesssim\begin{cases} e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_1,\\ e^{t+t\max\{\rho-\frac{1}{2},-1\}} & \text{in } Z_2\cup Z_3.\end{cases}$$

All estimates together allow to prove the following result:

Proposition 2.2.2. *We have the following estimates for the solution to (2.2.3)*:

$$\|u(t,\cdot)\|_{L^2}\leq C\|u_1\|_{L^2}+C\|u_2\|_{H^{-1}}.$$

For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2}\leq Ce^{t+t\max\{\rho-\frac{1}{2},-1\}}\|u_1\|_{H^1}+Ce^{t+t\max\{\rho-\frac{1}{2},-1\}}\|u_2\|_{L^2}.$$

For the "elastic" energy we have

$$\|e^t\nabla u(t,\cdot)\|_{L^2}\leq Ce^{t+t\max\{\rho-\frac{1}{2},-1\}}\|u_1\|_{H^1}+Ce^{t+t\max\{\rho-\frac{1}{2},-1\}}\|u_2\|_{L^2}.$$

Consequently, the energy

$$E(u)(t)=\frac{1}{2}\int_{\mathbb{R}^n}\Big(|u_t(t,x)|^2+(e^t|\nabla u(t,x)|)^2\Big)dx$$

can be estimated in the form

$$\begin{aligned} E(u)(t) &\leq Ce^{2t+t\max\{2\rho-1,-2\}}\|u_1\|^2_{H^1}+Ce^{2t+t\max\{2\rho-1,-2\}}\|u_2\|^2_{L^2}\\ &= Ce^{2t+t\max\{-a-1,-2\}}\|u_1\|^2_{H^1}+Ce^{2t+t\max\{-a-1,-2\}}\|u_2\|^2_{L^2}.\end{aligned}$$

Proof. Let us begin to prove the estimate for the solution u. Applying Hölder's inequality to the right-hand side gives

$$\|u(t,\cdot)\|_{L^2}\leq\|V_1(t,\xi)\|_{L^\infty}\|\widehat{u}_1(\xi)\|_{L^2}+\|\langle\xi\rangle V_2(t,\xi)\|_{L^\infty}\|\langle\xi\rangle^{-1}\widehat{u}_2(\xi)\|_{L^2}.$$

We have
$$\|V_1(t,\xi)\|_{L^\infty} \lesssim \max\left\{e^{t(\rho-\frac{1}{2})},1\right\} \lesssim 1,$$
and we can estimate $\|\langle\xi\rangle V_2(t,\xi)\|_{L^\infty}$ by using the above estimates
$$|\langle\xi\rangle V_2(t,\xi)| \lesssim 1.$$
Here we have used the properties $e^{t(\rho-\frac{1}{2})} \leq 1$ for $\rho \leq 0$; $\langle\xi\rangle \leq N$ (N is a constant) in the zones $Z_2 \cup Z_3$. From the above estimates we can conclude
$$\|u(t,\cdot)\|_{L^2} \lesssim C\|u_1\|_{L^2} + C\|u_2\|_{H^{-1}}.$$
Now let us prove the statement for $e^t\nabla u(t,\cdot)$. In the extended phase space we have
$$\begin{aligned}
&e^t|\xi|\hat{u}(t,\xi) = e^t|\xi|V_1(t,\xi)\hat{u}_1(\xi) + e^t|\xi|V_2(t,\xi)\hat{u}_2(\xi), \text{ where}\\
&|\xi|V_1(t,\xi) = \frac{i\pi}{4}e^t\rho\Psi_{2,\rho-1,1}(t,\xi),\ |\xi|V_2(t,\xi) = -\frac{i\pi}{4}e^t\rho\Psi_{1,\rho,0}(t,\xi).
\end{aligned}$$
Using the above estimates in different zones we have
$$|\Psi_{2,\rho-1,1}(t,\xi)| \lesssim \begin{cases} |\xi|e^{-\frac{t}{2}} & \text{in } Z_1,\\ e^{-\frac{t}{2}} & \text{in } Z_2 \text{ if } 1+\rho > \frac{1}{2},\\ e^{-t(1+\rho)} & \text{in } Z_2 \text{ if } 1+\rho \leq \frac{1}{2},\\ e^{-t(1+\rho)} & \text{in } Z_3,\end{cases}$$
and
$$|\Psi_{1,\rho,0}(t,\xi)| \lesssim \begin{cases} e^{-\frac{t}{2}} & \text{in } Z_1,\\ e^{-\frac{t}{2}} & \text{in } Z_2 \text{ if } 1+\rho > \frac{1}{2},\\ e^{-t(1+\rho)} & \text{in } Z_2 \text{ if } 1+\rho \leq \frac{1}{2},\\ e^{-t(1+\rho)} & \text{in } Z_3.\end{cases}$$
Consequently,
$$e^t|\xi||V_1(t,\xi)| \lesssim \begin{cases} |\xi|e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_1,\\ e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_2 \text{ if } -\frac{1}{2} < \rho < 0,\\ e^{t+(-t)} & \text{in } Z_2 \text{ if } \rho \leq -\frac{1}{2},\\ e^{t+(-t)} & \text{in } Z_3,\end{cases}$$
and
$$e^t|\xi||V_2(t,\xi)| \lesssim \begin{cases} e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_1,\\ e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_2 \text{ if } -\frac{1}{2} < \rho < 0,\\ e^{t+(-t)} & \text{in } Z_2 \text{ if } \rho \leq -\frac{1}{2},\\ e^{t+(-t)} & \text{in } Z_3.\end{cases}$$
Summarizing we conclude
$$\|e^t|\xi|\hat{u}(t,\xi)\|_{L^2} \leq Ce^{t+t\max\{\rho-\frac{1}{2},-1\}}\big(\|\langle\xi\rangle\hat{u}_1(\xi)\|_{L^2} + \|\hat{u}_2(\xi)\|_{L^2}\big).$$
This yields the desired estimate for the elastic energy. Finally, let us prove the statement for the kinetic energy. We have
$$\begin{aligned}
\|u_t(t,\cdot)\|_{L^2} &= \|\widehat{u}_t(t,\cdot)\|_{L^2} = \|V_{1,t}(t,\xi)\widehat{u}_1(\xi) + V_{2,t}(t,\xi)\widehat{u}_2(\xi)\|_{L^2}\\
&\lesssim \|\langle\xi\rangle^{-1}V_{1,t}(t,\xi)\|_{L^\infty}\|\langle\xi\rangle\widehat{u}_1(\xi)\|_{L^2} + \|V_{2,t}(t,\xi)\|_{L^\infty}\|\widehat{u}_2(\xi)\|_{L^2}.
\end{aligned}$$

Using the estimates

$$|V_{1,t}(t,\xi)| \lesssim \begin{cases} |\xi| e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_1, \\ e^{t+\max\{\rho-\frac{1}{2};-1\}} & \text{in } Z_2 \cup Z_3, \end{cases}$$
$$\lesssim \begin{cases} \langle\xi\rangle e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_1, \\ e^{t+\max\{\rho-\frac{1}{2};-1\}} & \text{in } Z_2 \cup Z_3, \end{cases}$$

and

$$|V_{2,t}(t,\xi)| \lesssim \begin{cases} e^{t+t(\rho-\frac{1}{2})} & \text{in } Z_1, \\ e^{t+\max\{\rho-\frac{1}{2};-1\}} & \text{in } Z_2 \cup Z_3, \end{cases}$$

then

$$\|u_t(t,\cdot)\|_{L^2} \lesssim Ce^{t+t\max\{\rho-\frac{1}{2},-1\}}\big(\|\langle\xi\rangle\hat{u}_1(\xi)\|_{L^2} + \|\hat{u}_2(\xi)\|_{L^2}\big).$$

This yields the desired estimate for the kinetic energy. □

Remark 2.2.3. The statement of Proposition 2.2.2 gives a decay estimate for $\|\nabla u(t,\cdot)\|_{L^2}$ even for $a > -1$.

3. Wave models with time dependent propagation speed and dissipation

3.1. Scattering theory

In this section we concern with conditions for $b = b(t)$ that the solutions $u = u(t,x)$ of

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x) \tag{3.1.1}$$

behave asymptotically equal to the solution of the corresponding wave equation with strictly increasing speed of propagation

$$v_{tt} - a^2(t)\Delta v = 0,\ v(0,x) = v_1(x),\ v_t(0,x) = v_2(x) \tag{3.1.2}$$

with some suitable Cauchy data (v_1, v_2). We will use the operator relating (u_1, u_2) to (v_1, v_2) and which is denoted as Møller wave operator which was mentioned in particular in the Lax-Phillips approach [L-P73] or in the lectures of R.B.Melrose [Mel95]. That means we will construct an operator $W_+ u = \lim_{t\to\infty} S_1^{-1}(t)S(t)u$ from the energy space $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for the solution of (3.1.1) to the energy space $\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ for the solution of (3.1.2). The operators are introduced in the proof of our scattering result.

3.1.1. Result in the L^2-scale

Here we introduce the energy space $E(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and we assume $(u_1, u_2) \in E(\mathbb{R}^n)$, this means, $(|D|u_1, u_2) \in L^2(\mathbb{R}^n)$.

Theorem 3.1.1. *We assume that the coefficient $b = b(t)$ satisfies $b \in L^1(\mathbb{R}_+)$. Then there exists the Møller wave operator $W_+ : E \to E$ mapping the Cauchy data $(a(0)u_1, u_2) \in E$ from (3.1.1) to Cauchy data $(a(0)v_1, v_2)$ from (3.1.2) by*

$$(a(0)v_1, v_2)^T = W_+(a(0)u_1, u_2)^T$$

such that the asymptotic equivalence of solutions of the problems (3.1.1) and (3.1.2) holds in the following way:

$$\frac{1}{\sqrt{a(t)}}\Big\|(a(t)u, D_t u) - (a(t)v, D_t v)\Big\|_E \to 0 \tag{3.1.3}$$

while $t \to \infty$. Moreover, we have the decay estimate

$$\frac{1}{\sqrt{a(t)}}\Big\|(a(t)u, D_t u) - (a(t)v, D_t v)\Big\|_E \lesssim \|(u_1, u_2)\|_E \int_t^\infty b(\tau)d\tau \tag{3.1.4}$$

with the convergence rate $\int_t^\infty b(\tau)d\tau$ to 0 as $t \to \infty$.

Proof. Let $U = (a(t)|\xi|\hat{u}, D_t\hat{u})^T$. Then U satisfies

$$D_t U = A(t,\xi)U := \begin{pmatrix} \dfrac{D_t a}{a} & a(t)|\xi| \\ a(t)|\xi| & ib(t) \end{pmatrix} U. \tag{3.1.5}$$

We carry out one step of diagonalization of the principal part by the matrix of eigenvectors M and its inverse M^{-1},

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{and we set} \quad U^{(0)} = M^{-1}U.$$

We get $D_t U^{(0)} = \mathcal{D}(t,\xi)U^{(0)} + \mathcal{R}_a(t)U^{(0)} + \mathcal{R}_b(t)U^{(0)}$, where

$$\mathcal{D}(t,\xi) = \begin{pmatrix} a(t)|\xi| & 0 \\ 0 & -a(t)|\xi| \end{pmatrix}, \quad \mathcal{R}_a(t) = \frac{1}{2}\frac{D_t a}{a}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \mathcal{R}_b(t) = \frac{1}{2} ib(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $E_a = E_a(t,s,\xi)$ be the fundamental solution of the operator $D_t - \mathcal{D}(t,\xi) - \mathcal{R}_a(t)$, that is, E_a satisfies the Cauchy problem $\big(D_t - \mathcal{D}(t,\xi) - \mathcal{R}_a(t)\big)E_a = 0$, $E_a(s,s,\xi) = I$. According to the results from Section 2.1 we have proved that $\|E_a(t,s,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} \lesssim \dfrac{\sqrt{a(t)}}{\sqrt{a(s)}}$. Moreover, using Liouville's formula, see Lemma B.3.4 in section B.3 of Appendix, we obtain

$$\det E_a(t,s,\xi) = \exp\Big(i\int_s^t \operatorname{tr}\big(\mathcal{D}(\tau,\xi) + \mathcal{R}_a(\tau)\big)d\tau\Big) = \frac{a(t)}{a(s)}.$$

The matrix-valued function $ME_a(t,s,\xi)M^{-1}$ generates a Fourier multiplier corresponding to the operator

$$S_1(t,s,D) : (a(s)|D|v(s), D_t v(s))^T \mapsto (a(t)|D|v(t), D_t v(t))^T$$

for solutions v to the Cauchy problem (3.1.2).
Now we construct the fundamental solution to the operator $D_t - \mathcal{D}(t,\xi) - \mathcal{R}_a(t) - \mathcal{R}_b(t)$. Therefore, let us introduce

$$\mathcal{P}(t,s,\xi) := E_a^{-1}(t,s,\xi)\mathcal{R}_b(t)E_a(t,s,\xi).$$

After application of Peano-Baker formula it follows that

$$Q_b(t,s,\xi) = I + \sum_{k=1}^{\infty} i^k \int_s^t \mathcal{P}(t_1,s,\xi)\int_s^{t_1}\mathcal{P}(t_2,s,\xi)\cdots\int_s^{t_{k-1}}\mathcal{P}(t_k,s,\xi)dt_k\cdots dt_2 dt_1 \tag{3.1.6}$$

is the solution to the Cauchy problem

$$D_t Q_b(t,s,\xi) - \mathcal{P}(t,s,\xi)Q_b(t,s,\xi) = 0, \;\; Q_b(s,s,\xi) = I.$$

Let $E_1(t,s,\xi) = E_a(t,s,\xi)Q_b(t,s,\xi)$. Then we derive

$$\begin{aligned} D_t(E_a Q_b) &= (D_t E_a)Q_b + E_a(D_t Q_b) = (\mathcal{D}(t,\xi) + \mathcal{R}_a(t))E_a Q_b + E_a\mathcal{P}Q_b \\ &= \big(\mathcal{D}(t,\xi) + \mathcal{R}_a(t)\big)E_a Q_b + \mathcal{R}_b(t)E_a Q_b = \big(\mathcal{D}(t,\xi) + \mathcal{R}_a(t) + \mathcal{R}_b(t)\big)E_a Q_b \end{aligned}$$

and $E_a(s,s,\xi)Q_b(s,s,\xi) = I$. Thus, $E_1(t,s,\xi)$ is the desired fundamental solution.
Taking account of $\|\mathcal{P}(t,s,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} \le \|\mathcal{R}_b(t)\| \in L^1(\mathbb{R}_+)$ implies the estimate

$$\|Q_b(t,s,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} \le \exp\Big(\int_s^t \|\mathcal{P}(\tau,s,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)}d\tau\Big) \le C.$$

Consequently,

$$\|E_1(t,s,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} \lesssim \|E_a(t,s,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)}\|Q_b(t,s,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(s)}}.$$

Moreover, the matrix-valued function $ME_1(t,s,\xi)M^{-1}$ generates a Fourier multiplier to the operator

$$S(t,s,D) : (a(s)|D|u(s), D_tu(s))^T \mapsto (a(t)|D|u(t), D_tu(t))^T$$

for the solutions u to the Cauchy problem (3.1.1).
Now our aim is to prove that the limit

$$W_+(D) := \lim_{t\to\infty} S_1^{-1}(t,0,D)S(t,0,D)$$

exists in E. To describe the behavior of the operator $S_1^{-1}(t,0,D)S(t,0,D)$ it is sufficient to study in the phase space $ME_a^{-1}(t,0,\xi)E_1(t,0,\xi)M^{-1} = MQ_b(t,0,\xi)M^{-1}$. Thus the question for the existence of the Møller wave operator is connected with the study of the limit

$$\lim_{t\to\infty} Q_b(t,0,\xi).$$

Furthermore, using formula (3.1.6) for large times s,t we consider the difference

$$Q_b(t,0,\xi) - Q_b(s,0,\xi) = \sum_{k=1}^\infty i^k \int_s^t \mathcal{P}(t_1,0,\xi)\int_0^{t_1}\mathcal{P}(t_2,0,\xi)\dots\int_0^{t_{k-1}}\mathcal{P}(t_k,0,\xi)dt_k\dots dt_2dt_1.$$

We obtain from the above considerations the following estimate:

$$\begin{aligned}
\|Q_b(t,0,\xi) - Q_b(s,0,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} &\le \sum_{k=1}^\infty \int_s^t \|\mathcal{P}(t_1,0,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)}\frac{1}{(k-1)!}\\
&\qquad\times\Big(\int_0^{t_1}\|\mathcal{P}(\tau,0,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)}d\tau\Big)^{k-1}dt_1\\
&\le \int_s^t \|\mathcal{P}(t_1,0,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)}\sum_{k=0}^\infty\frac{1}{k!}\Big(\int_0^{t_1}\|\mathcal{P}(\tau,0,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)}d\tau\Big)^k dt_1\\
&\le \int_s^t \|\mathcal{R}_b(t_1)\|e^{\int_0^{t_1}\|\mathcal{R}_b(\tau)\|d\tau}dt_1.
\end{aligned}$$

According to the assumption $b\in L^1$ we have proved $\mathcal{R}_b \in L^1$. This leads to $\|Q_b(t,0,\xi) - Q_b(s,0,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} \to 0$ as $t,s\to\infty$. Therefore, $Q_b(\infty,0,\xi)$ exists in the Banach space $L^\infty(\mathbb{R}^n_\xi)$. We define

$$W_+(\xi) = \lim_{t\to\infty} MQ_b(t,0,\xi)M^{-1} \in L^\infty(\mathbb{R}^n_\xi).$$

The operator $W_+(D)$ has the desired property, that is, (3.1.3) holds by the following considerations:

$$\begin{aligned}
&(a(t)|\xi|\hat{u}, D_t\hat{u})^T - (a(t)|\xi|\hat{v}, D_t\hat{v})^T = ME_aQ_b(t,0,\xi)M^{-1}(a(0)|\xi|\hat{u}_1,\hat{u}_2)^T\\
&-ME_aM^{-1}(a(0)|\xi|\hat{v}_1,\hat{v}_2)^T = ME_aM^{-1}\Big(MQ_b(t,0,\xi)M^{-1} - W_+\Big)(a(0)|\xi|\hat{u}_1,\hat{u}_2)^T \to 0
\end{aligned}$$

as $t\to\infty$. Finally, the estimate (3.1.4) can be immediately concluded from the estimate

$$\|Q_b(t,0,\xi) - Q_b(\infty,0,\xi)\|_{L^\infty(\mathbb{R}^n_\xi)} \lesssim \int_t^\infty \|\mathcal{R}_b(t_1)\|e^{\int_0^{t_1}\|\mathcal{R}_b(\tau)\|d\tau}dt_1 \lesssim \int_t^\infty b(\tau)d\tau,$$

where $Q_b(\infty,0,\xi) = \lim_{t\to\infty} Q_b(t,0,\xi)$. The proof is finished. $\square$

Example 3.1.1. If we choose the special case $b(t) = (1+t)^{-\gamma}$ with $\gamma > 1$, then the assumptions of Theorem 3.1.1 are satisfied. Moreover, the convergence rate is $O(t^{1-\gamma})$.

Example 3.1.2. If we choose the coefficient

$$b(t) = \frac{e^t e^{e^t} ... e^{[n-1]t}(e^{[n-1]t} - 1)}{e^{[n]t}},$$

here $e^{[n]t} = e^{e^{[n-1]t}}$, then in this case we have the convergence rate

$$\int_t^\infty b(\tau)d\tau = \frac{e^{[n-1]t}}{e^{[n]t}}.$$

Example 3.1.3. We can even consider the convergence rate for

$$b(t) = \frac{1}{(e^{[m]} + t)\log(e^{[m]} + t)... \log^{[m-1]}(e^{[m]} + t)\big(\log^{[m]}(e^{[m]} + t)\big)^{\gamma}}, \ \gamma > 1$$

here $e^{[0]} = 1, e^{[m]} = e^{e^{[m-1]}}, \log^{[0]}(\tau) = \tau$ and $\log^{[m]}(\tau) = \log(\log^{[m-1]}(\tau))$. The convergence rate is

$$\int_t^\infty b(\tau)d\tau = \frac{1}{\gamma - 1}\big(\log^{[m]}(e^{[m]} + t)\big)^{1-\gamma}.$$

3.2. Non-effective dissipation

Let us devote to the Cauchy problem

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{3.2.1}$$

Our question is, under which assumptions to the coefficient $b = b(t)$ for a given $a = a(t)$ can we call b a non-effective dissipation? Here non-effective means, that on the one hand we have a dissipation (classical scattering is excluded), but on the other hand the model is hyperbolic like (from the point of view of decay estimates) and not parabolic like. Motivated by the considerations from J. Wirth [W07a] we assume:

(B1) $b(t) > 0,\ b(t) = \mu(t)\dfrac{a(t)}{A(t)},\ b \notin L^1(\mathbb{R}_+)$,

(B2) $|\mu'(t)| \le C_\mu \mu(t)\dfrac{a(t)}{A(t)}$,

(B3) $\limsup_{t\to\infty} \mu(t) < 1$.

Besides the assumption (B3) we introduce another assumption

(B3)' $\liminf_{t\to\infty} \mu(t) > 1$.

We will later need this assumption to understand the influence of a different class of dissipations $b(t)u_t$. Finally, we will assume

(C) $\limsup_{t\to\infty} \big(\mu(t) + \alpha(t)\big) < 2$, where $\alpha(t)$ is defined by

$$\frac{a'(t)}{a(t)} =: \alpha(t)\frac{a(t)}{A(t)}.$$

Theorem 3.2.1. *Let us consider the Cauchy problem (3.2.1) under the assumptions to the coefficient $a = a(t)$ from Theorem 2.1.1. If the coefficient $b(t)$ satisfies conditions (B1) to (B3) or (B3)' and $a(t), b(t)$ also satisfy condition (C), then we have the following estimates for the kinetic and elastic energy*:

$$\|u_t(t,\cdot)\|_{L^2} \leq C\frac{\sqrt{a(t)}}{\lambda(t)}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big),$$
$$\|a(t)\nabla u(t,\cdot)\|_{L^2} \leq C\frac{\sqrt{a(t)}}{\lambda(t)}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Here $\lambda = \lambda(t)$ is defined by

$$\lambda(t) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big). \tag{3.2.2}$$

Proof. Applying partial Fourier transformation we have $\hat{u}_{tt} + a^2(t)|\xi|^2\hat{u} + b(t)\hat{u}_t = 0$. We will later derive estimates for the fundamental solution $E(t,s,\xi)$ of an equivalent system of first order by different approaches in different zones of the extended phase space $(0,\infty)\times\mathbb{R}^n$: in the dissipative zone and the hyperbolic zone. These zones are defined by

- $Z_{hyp}(N) := \{(t,\xi) : t \geq t_\xi\}$,
- $Z_{diss}(N) := \{(t,\xi) : 0 \leq t \leq t_\xi\}$,

where t_ξ satisfies $A(t_\xi)|\xi| = N$.

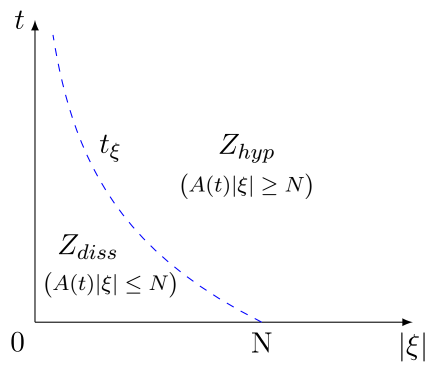


Fig. 3.1.: Description for the definition of zones in the non-effective dissipation

3.2.1. Considerations in the dissipative zone

Let us define the micro-energy $U = (N\delta(t)\hat{u}, D_t\hat{u})^T$, where we denote $\delta(t) = \dfrac{a(t)}{A(t)}$. Then the transformed equation can be written in the form of a system of first order (in D_t)

$$D_tU = A(t,\xi)U,\ A(t,\xi) = \begin{pmatrix} -i\dfrac{d_t\delta(t)}{\delta(t)} & N\delta(t) \\ \dfrac{a^2(t)|\xi|^2}{N\delta(t)} & ib(t) \end{pmatrix}.$$

Thus the solution $U = U(t,\xi)$ can be represented as $U(t,\xi) = E(t,s,\xi)U(s,\xi)$, where $E(t,s,\xi)$ is the fundamental solution, that is, the solution to the system

$$D_t E(t,s,\xi) = A(t,\xi)E(t,s,\xi),\ E(s,s,\xi) = I,\ 0 \le s \le t \le t_\xi.$$

In the further calculations we use the following statement:

Lemma 3.2.2. *1. The assumption (B3) implies with the auxiliary function $\lambda(t)$ the estimate*

$$\int_0^t \frac{a(\tau)}{\lambda^2(\tau)}d\tau \lesssim \frac{A(t)}{\lambda^2(t)}.$$

Moreover, the function $\frac{A(t)}{\lambda^2(t)}$ is monotonously increasing if t tends to infinity.

2. The assumption (B3)' implies $\frac{a(t)}{\lambda^2(t)} \in L^1(\mathbb{R}_+)$ with

$$\int_t^\infty \frac{a(\tau)}{\lambda^2(\tau)}d\tau \lesssim \frac{A(t)}{\lambda^2(t)}.$$

Furthermore, $\frac{A(t)}{\lambda^2(t)}$ is monotonously decreasing for large t.

Proof. to 1. Integration by parts yields

$$\int_0^t \frac{a(\tau)}{\lambda^2(\tau)}d\tau = \frac{A(t)}{\lambda^2(t)} - 1 + \int_0^t \frac{A(\tau)\mu(\tau)\frac{a(\tau)}{A(\tau)}}{\lambda^2(\tau)}d\tau.$$

We conclude for $t \ge t_0$ from the condition $\limsup_{t\to\infty}\mu(t) \le c < 1$ that

$$\int_0^t \frac{a(\tau)\mu(\tau)}{\lambda^2(\tau)}d\tau \le \int_0^{t_0} \frac{a(\tau)\mu(\tau)}{\lambda^2(\tau)}d\tau + c\int_{t_0}^t \frac{a(\tau)d\tau}{\lambda^2(\tau)} \le C + c\int_0^t \frac{a(\tau)}{\lambda^2(\tau)}d\tau.$$

The statement follows from

$$\int_0^t \frac{a(\tau)d\tau}{\lambda^2(\tau)} \le \frac{1}{1-c}\Big(C + \frac{A(t)}{\lambda^2(t)}\Big) \lesssim \frac{A(t)}{\lambda^2(t)}.$$

The monotonic behavior is a consequence of

$$\frac{d}{dt}\frac{A(t)}{\lambda^2(t)} = \frac{a(t)(1-\mu(t))}{\lambda^2(t)} \tag{3.2.3}$$

and $\mu(t) < 1$ for large t.

to 2. From $\liminf_{t\to\infty}\mu(t) > 1$ it follows $\liminf_{t\to\infty}\mu(t) \ge 1+\varepsilon$. So we can conclude

$$\lambda^2(t) = \exp\left(\int_0^t b(\tau)d\tau\right) \gtrsim A(t)^{1+\varepsilon},$$

which implies the integrability of $\dfrac{a(t)}{\lambda^2(t)}$. Furthermore, for large t we have

$$\varepsilon\int_t^\infty \frac{a(\tau)}{\lambda^2(\tau)}d\tau \le \int_t^\infty \frac{A(\tau)b(\tau)-a(\tau)}{\lambda^2(\tau)}d\tau = \frac{A(t)}{\lambda^2(t)}.$$

The monotonic behavior follows from (3.2.3). The statement is proved. $\square$

Lemma 3.2.3. *Assume that the function $a(t)$ satisfies (A1), (A2), and the function $\mu(t)$ satisfies the condition (B3). Then there exists a constant $\delta \in (0,1)$ such that*

$$a(t)^\delta \int_0^t \frac{a(\tau)^{1-\delta}}{\lambda^2(\tau)} d\tau \lesssim \frac{A(t)}{\lambda^2(t)}. \tag{3.2.4}$$

Proof. The statement follows directly after integration of the following inequalities:

$$\begin{aligned} \frac{a(t)^{1-\delta}}{\lambda^2(t)} \quad & \lesssim \left(\frac{A(t)}{a(t)^\delta \lambda^2(t)}\right)' = \frac{a(t)^{1-\delta}}{\lambda^2(t)} - \delta \frac{A(t)a'(t)}{a(t)^{1+\delta}\lambda^2(t)} - \frac{A(t)\mu(t)\dfrac{a(t)}{A(t)}}{a(t)^\delta \lambda^2(t)} \\ & \lesssim (1-c)\frac{a(t)^{1-\delta}}{\lambda^2(t)} - \delta \frac{A(t)a'(t)}{a(t)^{1+\delta}\lambda^2(t)} \text{ for large } t, \end{aligned}$$

where $c < 1$ due to condition (B3). The latter inequality is true if it exits a constant $C > \dfrac{1}{1-c}$ such that

$$A(t)a'(t) \le (1 - c - C^{-1})\delta^{-1}a^2(t).$$

From that we can choose any δ satisfying $\delta < (\limsup_t A(t)a'(t)/a^2(t))^{-1}$. This supremum is finite by condition (A2). □

Denoting by $E^{(jk)}$ the entries of E we get for $k = 1, 2$ the system

$$\begin{aligned} D_t E^{(1k)} &= -i\frac{d_t\delta(t)}{\delta(t)} E^{(1k)} + N\delta(t)E^{(2k)}, \\ D_t E^{(2k)} &= \frac{a^2(t)|\xi|^2}{N\delta(t)} E^{(1k)} + ib(t)E^{(2k)}, \; E^{(jk)}(s,s,\xi) = \delta_{jk}. \end{aligned}$$

Integration yields

$$\left\{ \begin{aligned} E^{(1k)}(t,s,\xi) &= \frac{\delta(t)}{\delta(s)} E^{(1k)}(s,s,\xi) + iN\delta(t)\int_s^t E^{(2k)}(\tau,s,\xi)d\tau, \\ E^{(2k)}(t,s,\xi) &= \frac{\lambda^2(s)}{\lambda^2(t)} E^{(2k)}(s,s,\xi) + \frac{i|\xi|^2}{N\lambda^2(t)} \int_s^t \frac{a^2(\tau)}{\delta(\tau)} \lambda^2(\tau) E^{(1k)}(\tau,s,\xi)d\tau. \end{aligned} \right. \tag{3.2.5}$$

We are going to prove the following lemma:

Lemma 3.2.4. *Let us assume (A1) to (A3) for $a(t)$ and (B3) for $b(t)$. Then we have the following estimates for the entries $E^{(kl)}(t,0,\xi)$ of the fundamental solution $E(t,0,\xi)$ in the dissipative zone :*

$$(|E(t,0,\xi)|) := \begin{pmatrix} |E^{(11)}(t,0,\xi)| & |E^{(12)}(t,0,\xi)| \\ |E^{(21)}(t,0,\xi)| & |E^{(22)}(t,0,\xi)| \end{pmatrix} \lesssim \begin{pmatrix} \dfrac{a(t)}{A(t)} & \dfrac{a(t)^{1-\delta}}{\lambda^2(t)} \\ \dfrac{|\xi|^2 K(t)}{\lambda^2(t)} & \dfrac{a(t)^{1-\delta}}{\lambda^2(t)} \end{pmatrix} \tag{3.2.6}$$

with $K(t) := \int_0^t a^2(\tau)\lambda^2(\tau)d\tau \le \lambda^2(t)a(t)A(t)$.

Proof. Let us consider

$$
\begin{aligned}
E^{(21)}(t,0,\xi) &= \frac{i|\xi|^2}{N\lambda^2(t)}\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)E^{(11)}(\tau,0,\xi)d\tau \\
&= \frac{i|\xi|^2}{N\lambda^2(t)}\Big(\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)\frac{\delta(\tau)}{\delta(0)}d\tau + \int_0^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)iN\delta(\tau)d\tau\int_0^\tau E^{(21)}(\theta,0,\xi)d\theta\Big) \\
&= \frac{i|\xi|^2}{N\delta(0)\lambda^2(t)}K(t) - \frac{|\xi|^2}{\lambda^2(t)}\int_0^t a^2(\tau)\lambda^2(\tau)\int_0^\tau E^{(21)}(\theta,0,\xi)d\theta d\tau \\
&= \frac{i|\xi|^2}{C_N\lambda^2(t)}K(t) - \frac{|\xi|^2}{\lambda^2(t)}\int_0^t \Big(\int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau\Big)E^{(21)}(\theta,0,\xi)d\theta.
\end{aligned}
$$

Rewriting the integral equation gives

$$
\frac{C_N\lambda^2(t)E^{(21)}(t,0,\xi)}{|\xi|^2K(t)} = i + \int_0^t k_1(t,\theta,\xi)\frac{C_N\lambda^2(\theta)E^{(21)}(\theta,0,\xi)}{|\xi|^2K(\theta)}d\theta \tag{3.2.7}
$$

with kernel

$$
k_1(t,\theta,\xi) = -|\xi|^2\frac{K(\theta)}{K(t)\lambda^2(\theta)}\int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau,\ \ \theta\in[0,t]. \tag{3.2.8}
$$

Now we estimate

$$
\begin{aligned}
\int_0^t \sup_{\theta\le\tilde{t}\le t}|k_1(\tilde{t},\theta,\xi)|d\theta \quad &\lesssim |\xi|^2\int_0^{t_\xi}\sup_{\tilde{t}}\frac{K(\theta)}{\lambda^2(\theta)K(\tilde{t})}\big(K(\tilde{t})-K(\theta)\big)d\theta \le |\xi|^2\int_0^{t_\xi}\frac{K(\theta)}{\lambda^2(\theta)}d\theta \\
&\lesssim |\xi|^2\int_0^{t_\xi} a(\theta)A(\theta)d\theta = \frac{1}{2}|\xi|^2A^2(t_\xi)\lesssim 1
\end{aligned}
$$

uniformly in $Z_{diss}(N)$. Therefore, we obtained

$$
|E^{(21)}(t,0,\xi)| \lesssim \frac{|\xi|^2K(t)}{\lambda^2(t)}. \tag{3.2.9}
$$

Substituting this estimate into the first integral equation implies

$$
|E^{(11)}(t,0,\xi)| \le \frac{\delta(t)}{\delta(0)} + N\delta(t)\int_0^t \frac{|\xi|^2K(\tau)}{\lambda^2(\tau)}d\tau \lesssim \delta(t) + |\xi|^2\delta(t)A^2(t) \lesssim \delta(t) = \frac{a(t)}{A(t)}.
$$

Next we consider

$$
\begin{aligned}
E^{(22)}(t,0,\xi) &= \frac{\lambda^2(0)}{\lambda^2(t)} + \frac{i|\xi|^2}{N\lambda^2(t)}\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)E^{(12)}(\tau,0,\xi)d\tau \\
&= \frac{\lambda^2(0)}{\lambda^2(t)} - \frac{|\xi|^2}{\lambda^2(t)}\int_0^t a^2(\tau)\lambda^2(\tau)\int_0^\tau E^{(22)}(\theta,0,\xi)d\theta d\tau, \\
\lambda^2(t)E^{(22)}(t,0,\xi) &= 1 - |\xi|^2\int_0^t\Big(\int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau\Big)E^{(22)}(\theta,0,\xi)d\theta,
\end{aligned}
$$

respectively. Our goal is to show that $|E^{(22)}(t,0,\xi)| \lesssim \dfrac{a(t)^{1-\delta}}{\lambda^2(t)}$. Therefore, we rewrite the integral equation as

$$
\frac{\lambda^2(t)E^{(22)}(t,0,\xi)}{a(t)^{1-\delta}} = \frac{1}{a(t)^{1-\delta}} + \int_0^t k_2(t,\theta,\xi)\frac{\lambda^2(\theta)E^{(22)}(\theta,0,\xi)}{a(\theta)^{1-\delta}}d\theta \tag{3.2.10}
$$

with the kernel

$$k_2(t,\theta,\xi) = -|\xi|^2 \frac{a(\theta)^{1-\delta}}{a(t)^{1-\delta}\lambda^2(\theta)} \int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau,\ \theta \in [0,t]. \tag{3.2.11}$$

The following integral over the kernel satisfies the desired estimate. It holds

$$\begin{aligned}
&\int_0^t \sup_{\theta\le\tilde{t}\le t} |k_2(\tilde{t},\theta,\xi)|d\theta \lesssim |\xi|^2 \int_0^{t_\xi} \sup_{\tilde{t}} \frac{(a(\theta))^{1-\delta}}{(a(\tilde{t})^{1-\delta})\lambda^2(\theta)}\big(K(\tilde{t}) - K(\theta)\big)d\theta \\
&\le |\xi|^2 \int_0^{t_\xi} \sup_{\tilde{t}} \frac{(a(\theta))^{1-\delta}K(\tilde{t})}{(a(\tilde{t}))^{1-\delta}\lambda^2(\theta)}d\theta \le |\xi|^2\lambda^2(t_\xi)A(t_\xi)(a(t_\xi))^\delta \int_0^{t_\xi} \frac{(a(\theta))^{1-\delta}}{\lambda^2(\theta)}d\theta \\
&\lesssim |\xi|^2\lambda^2(t_\xi)A(t_\xi)\frac{A(t_\xi)}{\lambda^2(t_\xi)} \le |\xi|^2A^2(t_\xi) \lesssim 1.
\end{aligned}$$

Here we have used Lemma 3.2.3 and, therefore

$$|E^{(22)}(t,0,\xi)| \lesssim \frac{a(t)^{1-\delta}}{\lambda^2(t)}. \tag{3.2.12}$$

Plugging this estimate into the first integral equation and using Lemma 3.2.3 again we have

$$|E^{(12)}(t,0,\xi)| \lesssim \delta(t)\int_0^t \frac{a(\tau)^{1-\delta}}{\lambda^2(\tau)}d\tau \lesssim \frac{a(t)^{1-\delta}}{A(t)}\frac{A(t)}{\lambda^2(t)} \lesssim \frac{a(t)^{1-\delta}}{\lambda^2(t)}. \tag{3.2.13}$$

This completes the proof. □

Now let us come back to

$$U(t,\xi) = E(t,0,\xi)U(0,\xi) \text{ for all } 0 \le t \le t_\xi. \tag{3.2.14}$$

Because of $a(t)|\xi||\hat{u}(t,\xi)| \le N\frac{a(t)}{A(t)}|\hat{u}(t,\xi)|$ in $Z_{diss}(N)$ the following statement can be concluded from (3.2.14) and Lemma 3.2.4:

Corollary 3.2.5. *We have the following estimates for all $0 \le t \le t_\xi$ (the dissipative zone $Z_{diss}(N)$):*

$$\begin{aligned}
a(t)|\xi||\hat{u}(t,\xi)| &\le C_N\frac{a(t)}{A(t)}|\hat{u}(0,\xi)| + C_N\frac{a(t)^{1-\delta}}{\lambda^2(t)}|D_t\hat{u}(0,\xi)|, \\
|D_t\hat{u}(t,\xi)| &\le C_N\frac{|\xi|^2K(t)}{\lambda^2(t)}|\hat{u}(0,\xi)| + C_N\frac{a(t)^{1-\delta}}{\lambda^2(t)}|D_t\hat{u}(0,\xi)|.
\end{aligned}$$

Lemma 3.2.6. *Let us assume (A1) to (A3) for $a(t)$ and (B3)' for $b(t)$. Then we have the following estimates for the entries $E^{(kl)}(t,0,\xi)$ of the fundamental solution $E(t,0,\xi)$:*

$$\begin{pmatrix} |E^{(11)}(t,0,\xi)| & |E^{(12)}(t,0,\xi)| \\ |E^{(21)}(t,0,\xi)| & |E^{(22)}(t,0,\xi)| \end{pmatrix} \lesssim \begin{pmatrix} \frac{a(t)}{A(t)} & \frac{a(t)}{A(t)} \\ \frac{a(t)}{A(t)} & \frac{a(t)}{A(t)} \end{pmatrix}. \tag{3.2.15}$$

Proof. We start by estimating the first column. Plugging the representation for $E^{(21)}(t,s,\xi)$ into the integral equation for $E^{(11)}(t,s,\xi)$ gives

$$\begin{aligned}
&\frac{\delta(0)}{\delta(t)}E^{(11)}(t,0,\xi) = 1-|\xi|^2\int_0^t\int_0^\tau \frac{\lambda^2(\theta)}{\lambda^2(\tau)}a^2(\theta)\frac{\delta(0)}{\delta(\theta)}E^{(11)}(\theta,0,\xi)d\theta d\tau,\\
&\frac{1}{\delta(t)}|E^{(11)}(t,0,\xi)| \lesssim 1+|\xi|^2\int_0^t\int_0^\tau \underbrace{\frac{\lambda^2(\theta)}{\lambda^2(\tau)}}_{\leq 1}a^2(\theta)\frac{1}{\delta(\theta)}E^{(11)}(\theta,0,\xi)d\theta d\tau,\\
&\frac{1}{\delta(t)}|E^{(11)}(t,0,\xi)| \lesssim \exp\left(|\xi|^2\int_0^t\int_0^\tau a^2(\theta)d\theta d\tau\right) \lesssim \exp\left(|\xi|^2 A^2(t)\right)\lesssim 1,\\
&|E^{(11)}(t,0,\xi)| \lesssim \delta(t) = \frac{a(t)}{A(t)}.
\end{aligned}$$

Here we have used the definition of dissipative zone and assumption (A2) for $a(t)$.
Let us consider $E^{(21)}(t,0,\xi)$. We have

$$\begin{aligned}
|E^{(21)}(t,0,\xi)| &\lesssim \frac{|\xi|^2}{\lambda^2(t)}\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)|E^{(11)}(\tau,0,\xi)|d\tau\\
&\lesssim |\xi|^2\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\underbrace{\frac{\lambda^2(\tau)}{\lambda^2(t)}}_{\leq 1}\delta(\tau)d\tau \lesssim |\xi|^2A(t)a(t)\leq C_N\frac{a(t)}{A(t)}.
\end{aligned}$$

Now we will estimate the entries of the second column. We get

$$\frac{1}{\delta(t)}E^{(12)}(t,0,\xi) = iN\lambda^2(0)\int_0^t\frac{d\tau}{\lambda^2(\tau)} - |\xi|^2\int_0^t\int_0^\tau \underbrace{\frac{\lambda^2(\theta)}{\lambda^2(\tau)}}_{\leq 1}a^2(\theta)\frac{1}{\delta(\theta)}E^{(12)}(\theta,0,\xi)d\theta d\tau.$$

Because the first integral is uniformly bounded by the second statement from Lemma 3.2.2 we can obtain by the above reasoning together with assumption (A1) the desired estimate for $E^{(12)}$. For $E^{(22)}$ we have

$$\begin{aligned}
\frac{1}{\delta(t)}|E^{(22)}(t,0,\xi)| &\lesssim \frac{A(t)}{a(t)\lambda^2(t)}+\frac{|\xi|^2A(t)}{a(t)\lambda^2(t)}\int_0^t a^2(\tau)\lambda^2(\tau)d\tau\\
&\lesssim \frac{A(t)}{a(t)\lambda^2(t)}+\underbrace{\frac{|\xi|^2A(t)}{a(t)}\int_0^t a^2(\tau)d\tau}_{\leq C_N}.
\end{aligned}$$

If we notice $\lambda^2(t)\gtrsim A^{1+\varepsilon}(t)$, then $\dfrac{A(t)}{a(t)\lambda^2(t)}$ is uniformly bounded for large t. This completes the proof. □

3.2.2. Considerations in the hyperbolic zone

Here we use the hyperbolic micro-energy $U=(a(t)|\xi|\hat{u},D_t\hat{u})^T$. Then U satisfies

$$D_tU = A(t,\xi)U := \begin{pmatrix} \dfrac{D_ta}{a} & a(t)|\xi| \\ a(t)|\xi| & ib(t)\end{pmatrix}U. \tag{3.2.16}$$

Let us carry out the first step of diagonalization. For this reason we set

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{and } U^{(0)} := M^{-1}U.$$

So $D_t U^{(0)} = \mathcal{D}(t,\xi)U^{(0)} + \mathcal{R}(t)U^{(0)}$, where

$$\mathcal{D}(t,\xi) := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \begin{pmatrix} a(t)|\xi| & 0 \\ 0 & -a(t)|\xi| \end{pmatrix}$$

and

$$\mathcal{R}(t) = \frac{1}{2}\begin{pmatrix} \frac{D_t a}{a} + ib(t) & -\frac{D_t a}{a} + ib(t) \\ -\frac{D_t a}{a} + ib(t) & \frac{D_t a}{a} + ib(t) \end{pmatrix}.$$

Let $F_0(t)$ be the diagonal part of $\mathcal{R}(t)$. Now we carry out the second step of diagonalization procedure. Therefore we introduce the matrices

$$N^{(1)} = \begin{pmatrix} 0 & \frac{R_{12}}{\tau_1 - \tau_2} \\ \frac{R_{21}}{\tau_2 - \tau_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\frac{\delta_1(t)}{4a(t)|\xi|} \\ -i\frac{\delta_1(t)}{4a(t)|\xi|} & 0 \end{pmatrix}, \quad N_1 = I + N^{(1)}.$$

Here $\delta_1 := \frac{a'}{a} + b$. We have

$$\left(\frac{\frac{a'}{a}(t)}{4a(t)|\xi|}\right)^2 \lesssim \left(\frac{\frac{a}{A}(t)}{4a(t)|\xi|}\right)^2 \lesssim \Big(\frac{1}{A(t)|\xi|}\Big)^2 \leqslant \frac{C}{N^2}.$$

If we use $b(t) = \mu(t)\dfrac{a(t)}{A(t)}$ and the assumptions $(B3)$ or $(B3)'$ and (C), i.e. we have $\limsup_{t\to\infty}\mu(t) \lesssim 1$, then

$$\Big(\frac{b(t)}{4a(t)|\xi|}\Big)^2 = \Big(\frac{\mu(t)a(t)}{4a(t)A(t)|\xi|}\Big)^2 \lesssim \Big(\frac{1}{A(t)|\xi|}\Big)^2 \leq \frac{C}{N^2}.$$

Thus we can choose a sufficiently large N such that the determinant of N_1 is $\det N_1 = 1 - \Big(\frac{\delta_1(t)}{4a(t)|\xi|}\Big)^2 \geq \frac{1}{2}$. Hence, the matrix N_1 is invertible. Set

$$B^{(1)} = D_t N^{(1)} - (\mathcal{R} - F_0)N^{(1)} = \begin{pmatrix} -\frac{\delta_1^2(t)}{8a(t)|\xi|} & i\partial_t \frac{\delta_1(t)}{4a(t)|\xi|} \\ -i\partial_t \frac{\delta_1(t)}{4a(t)|\xi|} & \frac{\delta_1^2(t)}{8a(t)|\xi|} \end{pmatrix} \text{ and } \mathcal{R}_1(t,\xi) = -N_1^{-1}B^{(1)}(t,\xi).$$

We can conclude that

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{R}(t)\big)N_1(t,\xi)U^{(1)}(t,\xi) = N_1(t,\xi)\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)U^{(1)}(t,\xi).$$

Now we shall find the solution $U^{(0)}(t,\xi) := N_1(t,\xi)U^{(1)}(t,\xi)$, where $U^{(1)}(t,\xi)$ is the solution to the system

$$\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)U^{(1)}(t,\xi) = 0.$$

We can write $U^{(1)}(t,\xi) = E_1(t,t_\xi,\xi)U^{(1)}(t_\xi,\xi)$, where $E_1(t,s,\xi)$ is the fundamental solution, that is, the solution of the system

$$\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)E_1(t,s,\xi) = 0, \quad E_1(s,s,\xi) = I,\ t \geq s \geq t_\xi.$$

The solution $E_0 = E_0(t,s,\xi)$ of the "principal diagonal part" fulfils

$$D_t E_0(t,s,\xi) = (\mathcal{D}(t,\xi) + F_0(t))E_0(t,s,\xi), \quad E_0(s,s,\xi) = I,\ t \geq s \geq t_\xi.$$

Thus

$$E_0(t,s,\xi) = \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\frac{\lambda(s)}{\lambda(t)}\begin{pmatrix} \exp\Big(\int_s^t ia(\tau)|\xi|d\tau\Big) & 0 \\ 0 & \exp\Big(-\int_s^t ia(\tau)|\xi|d\tau\Big)\end{pmatrix}.$$

Let us set

$$\mathcal{R}_2(t,s,\xi) = E_0(t,s,\xi)^{-1}\mathcal{R}_1(t,\xi)E_0(t,s,\xi),$$
$$Q(t,s,\xi) = I + \sum_{k=1}^{\infty} i^k \int_s^t \mathcal{R}_2(t_1,s,\xi)\int_s^{t_1}\mathcal{R}_2(t_2,s,\xi)\cdots\int_s^{t_{k-1}}\mathcal{R}_2(t_k,s,\xi)dt_k\cdots dt_2dt_1.$$

Then $Q(t,s,\xi)$ solves the Cauchy problem

$$D_tQ(t,s,\xi) = \mathcal{R}_2(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I,\ t \geq s \geq t_\xi.$$

The fundamental solution $E_1 = E_1(t,s,\xi)$ is representable in the form $E_1(t,s,\xi) = E_0(t,s,\xi)Q(t,s,\xi)$. Analogous to the statement of Lemma 2.1.3 we are able to show the following estimate for $Q(t,s,\xi)$:

$$|Q(t,s,\xi)| \leq \exp\Big(\int_s^t |\mathcal{R}_1(\tau,\xi)|d\tau\Big) \leq \exp\Big(\frac{1}{|\xi|}\Big(\frac{1}{A(\tau)}\Big)\Big|_s^t\Big) \leq C_N.$$

The backward transformation yields $U(t,\xi) = MN_1(t,\xi)E_0(t,s,\xi)Q(t,s,\xi)N_1^{-1}(s,\xi)M^{-1}U(s,\xi)$,

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \leq \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\frac{\lambda(s)}{\lambda(t)}\left|\begin{pmatrix} a(s)|\xi|\hat{u}(s,\xi) \\ D_t\hat{u}(s,\xi)\end{pmatrix}\right| \quad \text{for all } t \geq s \geq t_\xi.$$

Corollary 3.2.7. *We have in the hyperbolic zone $Z_{hyp}(N)$ the estimate*

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \leq C\frac{\sqrt{a(t)}}{\sqrt{a(t_\xi)}}\frac{\lambda(t_\xi)}{\lambda(t)}\left|\begin{pmatrix} a(t_\xi)|\xi|\hat{u}(t_\xi,\xi) \\ D_t\hat{u}(t_\xi,\xi)\end{pmatrix}\right|$$

for all $t \geq t_\xi$.

3.2.3. Conclusion

We have the following lemma.

Lemma 3.2.8. *Assume that the functions $\mu = \mu(t)$ and $\alpha = \alpha(t)$ satisfy the assumption*

$$\limsup_{t\to\infty}\big(\mu(t) + \alpha(t)\big) < 2.$$

Then the following inequality holds:

$$\frac{\lambda(t)\sqrt{a(t)}}{A(t)} \leq C.$$

Proof. We have from the definition of λ and α

$$\begin{aligned}\frac{\lambda(t)\sqrt{a(t)}}{A(t)} &\lesssim \frac{\exp\Big(\frac{1}{2}\int_0^t \mu(s)\frac{a(s)}{A(s)}ds\Big)\exp\Big(\frac{1}{2}\int_0^t \alpha(s)\frac{a(s)}{A(s)}ds\Big)}{\exp\Big(\int_0^t \frac{a(s)}{A(s)}ds\Big)}\\ &= \exp\Big(\frac{1}{2}\int_0^t (\mu(s)+\alpha(s)-2)\frac{a(s)}{A(s)}ds\Big).\end{aligned}$$

According to the assumption (C) it holds $\mu(t)+\alpha(t)-2\le 0$ for $t\ge t_0$ with a suitable t_0. From that we may conclude

$$\frac{\lambda(t)\sqrt{a(t)}}{A(t)} \lesssim \exp\Big(\int_0^{t_0} (\mu(s)+\alpha(s)-2)\frac{a(s)}{A(s)}ds\Big) \le C(t_0). \tag{3.2.17}$$

This completes the proof. □

From the statements of Corollaries 3.2.5 and 3.2.7 we derive the statement of our theorem.
1.case $\{|\xi|\ge N\}$:
Then the statement of Corollary 3.2.7 implies immediately

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \le C\frac{\sqrt{a(t)}}{\lambda(t)}\left|\begin{pmatrix} |\xi|\hat{u}(0,\xi) \\ D_t\hat{u}(0,\xi)\end{pmatrix}\right|$$

for all $t\ge 0$.
2.case $\{|\xi|\le N\}$ *and* $\{t\ge t_\xi\}$:
Then the statements of Corollary 3.2.7 imply immediately

$$\begin{aligned}a(t)|\xi||\hat{u}(t,\xi)|+|D_t\hat{u}(t,\xi)| &\le C\frac{\sqrt{a(t)}}{\sqrt{a(t_\xi)}}\frac{\lambda(t_\xi)}{\lambda(t)}\Big(a(t_\xi)|\xi||\hat{u}(t_\xi,\xi)|+|D_t\hat{u}(t_\xi,\xi)|\Big)\\ &\le C_N\frac{\sqrt{a(t)}}{\lambda(t)}\Big(\frac{\lambda(t_\xi)}{\sqrt{a(t_\xi)}}a(t_\xi)|\xi||\hat{u}(t_\xi,\xi)|+\frac{\lambda(t_\xi)}{\sqrt{a(t_\xi)}}|D_t\hat{u}(t_\xi,\xi)|\Big).\end{aligned}$$

From Corollary 3.2.5 we have for $t=t_\xi$

$$a(t_\xi)|\xi||\hat{u}(t_\xi,\xi)|+|D_t\hat{u}(t_\xi,\xi)| \le C\frac{a(t_\xi)}{A(t_\xi)}|\hat{u}(0,\xi)|+C\frac{a(t_\xi)^{1-\delta}}{\lambda^2(t_\xi)}|D_t\hat{u}(0,\xi)|.$$

Summarizing we get

$$a(t)|\xi||\hat{u}(t,\xi)|+|D_t\hat{u}(t_\xi,\xi)| \le C\frac{\sqrt{a(t)}}{\lambda(t)}\Big(\frac{\sqrt{a(t_\xi)}\lambda(t_\xi)}{A(t_\xi)}|\hat{u}(0,\xi)|+C\frac{a(t_\xi)^{\frac{1}{2}-\delta}}{\lambda(t_\xi)}|D_t\hat{u}(0,\xi)|\Big)$$

for all admissible (t,ξ). If we choose $\delta\ge\dfrac{1}{2}$ and apply Lemma 3.2.8, then we may conclude

$$a(t)|\xi||\hat{u}(t,\xi)|+|D_t\hat{u}(t,\xi)| \le C_N\frac{\sqrt{a(t)}}{\lambda(t)}(|\hat{u}(0,\xi)|+|D_t\hat{u}(0,\xi)|) \text{ for all admissible } (t,\xi).$$

3.case $\{|\xi|\le N\}$ *and* $\{t\le t_\xi\}$:

Then the statements of Corollary 3.2.5 imply immediately

$$a(t)|\xi||\hat{u}(t,\xi)| + |D_t\hat{u}(t,\xi)| \le C_N \frac{a(t)}{A(t)}|\hat{u}(0,\xi)| + C_N \frac{a(t)^{1-\delta}}{\lambda^2(t)}|D_t\hat{u}(0,\xi)|.$$

If we choose $\delta \ge \dfrac{1}{2}$ and apply Lemma 3.2.8, then we may conclude

$$a(t)|\xi||\hat{u}(t,\xi)| + |D_t\hat{u}(t,\xi)| \le C_N \frac{\sqrt{a(t)}}{\lambda(t)}(|\hat{u}(0,\xi)| + |D_t\hat{u}(0,\xi)|) \text{ for all admissible } (t,\xi).$$

This completes the proof to Theorem 3.2.1. $\square$

Example 3.2.1. Let $\mu \in (0,1)$ or $\mu \in (1, 1+1/(l+1)]$. We choose

$$a(t) = (1+t)^l,\ A(t) = \frac{1}{l+1}(1+t)^{l+1},\ b(t) = \frac{\mu(l+1)}{1+t}.$$

These coefficients satisfy the assumptions of Theorem 3.2.1. Taking in to consideration $\lambda(t) = (1+t)^{\frac{\mu(l+1)}{2}}$ we may conclude

$$\big\|\big((1+t)^l\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim (1+t)^{\frac{l}{2}-\frac{\mu(l+1)}{2}}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Example 3.2.2. Let $\mu \in (0,1)$. We choose

$$a(t) = e^t,\ A(t) = e^t,\ b(t) = \mu.$$

These coefficients satisfy the assumptions of Theorem 3.2.1. Taking into consideration $\lambda(t) = e^{\frac{\mu}{2}t}$ we may conclude

$$\big\|\big(e^t\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim e^{\frac{t}{2}-\frac{\mu}{2}t}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Example 3.2.3. Let $\mu > 0$ and $m \ge 1$. If we choose

$$a(t) = (e^{[m]}+t)^l,\ \ \mu(t) = \frac{\mu}{(l+1)\log(e^{[m]}+t)...\log^{[m]}(e^{[m]}+t)},$$

then we have

$$A(t) = \frac{1}{l+1}(e^{[m]}+t)^{l+1},\ \ b(t) = \frac{\mu}{(e^{[m]}+t)\log(e^{[m]}+t)...\log^{[m]}(e^{[m]}+t)}.$$

These coefficients satisfy the assumptions of Theorem 3.2.1. Thus, we obtain $\lambda(t) = \big(\log^{[m]}(e^{[m]} + t)\big)^{\frac{\mu}{2}}$. So, we may conclude

$$\big\|\big((e^{[m]}+t)^l\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim \frac{(e^{[m]}+t)^{\frac{l}{2}}}{\big(\log^{[m]}(e^{[m]}+t)\big)^{\frac{\mu}{2}}}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Example 3.2.4. Let $\mu \in (0,1)$, and $m \ge 1$. If we choose

$$a(t) = e^t e^{e^t}...e^{[m]^t},\ \mu(t) = \mu,$$

then we have

$$A(t) = e^{[m]^t},\ b(t) = \mu e^t e^{e^t}...e^{[m-1]^t}.$$

These coefficients satisfy the assumptions of Theorem 3.2.1. Taking into consideration $\lambda(t) = e^{\frac{\mu}{2}[m]^t}$ we may conclude

$$\big\|\big(e^t e^{e^t}...e^{[m]^t}\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim \frac{e^{t/2}e^{e^t/2}...e^{1/2[m]^t}}{e^{\frac{\mu}{2}[m]^t}}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

3.3. Effective dissipation

We consider the following Cauchy problem

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0, \ \ u(0,x) = u_1(x), \ \ u_t(0,x) = u_2(x). \tag{3.3.1}$$

In the previous chapters we have concerned with the influence of the dissipation term $b(t)u_t$ for a given $a(t)$ such that the equation (3.3.1) is from the point of view of long time behavior of solutions and its energies in some sense close to the *wave equation with increasing speed of propagation* (2.1.1). We have studied *scattering* in Section 3.1 and *non-effective dissipation* in Section 3.2. In this section we want to understand the so-called *effective dissipation*. This notion hints to relations to *parabolic models* from the point of view of long time behavior of solutions and its energies.
We will apply a transformation of the damped wave equation from (3.3.1) to a wave equation with time-dependent speed of propagation and potential. Thus, we define the new function

$$v(t,x) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big)u(t,x).$$

After some calculations we get

$$v_{tt} - a^2(t)\Delta v - \Big(\frac{1}{4}b^2(t) + \frac{1}{2}b'(t)\Big)v = 0.$$

Applying Fourier transformation we have

$$\hat{v}_{tt} + m(t,\xi)\hat{v} = 0, \tag{3.3.2}$$

here

$$m(t,\xi) := a^2(t)|\xi|^2 - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t). \tag{3.3.3}$$

To study the interacting between $a(t)$ and $b(t)$ we assume:

(B'1) $b(t) > 0$, $b(t) = \mu(t)\dfrac{a(t)}{A(t)}$,

(B'2) $\big|d_t^k\mu(t)\big| \leq C_k\mu(t)\left(\dfrac{a(t)}{A(t)}\right)^k$ for $k = 1,2$,

(B'3) $\mu(t)/A(t)$ is monotonic and $\mu(t) \to \infty$ as $t \to \infty$,

(B'4) $a^2(t)/b(t) = a(t)A(t)/\mu(t) \notin L^1(\mathbb{R}_+)$.

Using assumption (B'1) we can rewrite the formula (3.3.3) by the following formula:

$$m(t,\xi) = a^2(t)|\xi|^2 - \frac{1}{4}\mu^2(t)\frac{a^2(t)}{A^2(t)} - \frac{1}{2}\Big(\mu(t)\frac{a(t)}{A(t)}\Big)'.$$

Assumptions (B'2) and (B'3) show that $b'(t)$ is a negligible term in comparison with $b^2(t)$, this means $|b'(t)| = o(b^2(t))$ as $t \to \infty$. Indeed, we have

$$\begin{aligned}\frac{|b'(t)|}{b^2(t)} &= \frac{\Big|\Big(\mu(t)\frac{a(t)}{A(t)}\Big)'\Big|}{\mu^2(t)\frac{a^2(t)}{A^2(t)}} = \frac{\Big|\mu'(t)\frac{a(t)}{A(t)} + \mu(t)\frac{a^2(t)-a'(t)A(t)}{A^2(t)}\Big|}{\mu^2(t)\frac{a^2(t)}{A^2(t)}} \leq \frac{|\mu'(t)|\frac{a(t)}{A(t)} + C_1\mu(t)\frac{a^2(t)}{A^2(t)}}{\mu^2(t)\frac{a^2(t)}{A^2(t)}}\\ &\leq \frac{C\mu(t)\frac{a^2(t)}{A^2(t)} + C_1\mu(t)\frac{a^2(t)}{A^2(t)}}{\mu^2(t)\frac{a^2(t)}{A^2(t)}} \lesssim \frac{1}{\mu(t)} \to 0 \ \text{ for } t \to \infty.\end{aligned}$$

We introduce the auxiliary symbol

$$\langle\xi\rangle_{b(t)} := \sqrt{\Big|a^2(t)|\xi|^2 - \frac{b^2(t)}{4}\Big|} = \sqrt{\Big|a^2(t)|\xi|^2 - \frac{\mu^2(t)}{4}\frac{a^2(t)}{A^2(t)}\Big|}. \tag{3.3.4}$$

3.3.1. Regions and zones

We define the separating curve $t_\xi = t(|\xi|)$ by

$$\Gamma = \Big\{(t,\xi) : |\xi| = \frac{1}{2}\frac{\mu(t)}{A(t)}\Big\},$$

and introduce the following regions in the extended phase space $(0,\infty)\times\mathbb{R}^n_\xi$:

the hyperbolic region: $\Pi_{hyp} = \Big\{(t,\xi) : |\xi| > \frac{1}{2}\frac{\mu(t)}{A(t)}\Big\}$,

the elliptic region: $\Pi_{ell} = \Big\{(t,\xi) : |\xi| < \frac{1}{2}\frac{\mu(t)}{A(t)}\Big\}$.

The auxiliary symbol $\langle\xi\rangle_{b(t)}$ is differentiable in these regions and satisfies

$$\partial_t\langle\xi\rangle_{b(t)} = \pm\frac{a'(t)a(t)|\xi|^2 - \frac{\mu(t)a(t)}{2A(t)}\Big(\frac{\mu(t)a(t)}{2A(t)}\Big)'}{\langle\xi\rangle_{b(t)}}, \quad \partial_{|\xi|}\langle\xi\rangle_{b(t)} = \pm\frac{a^2(t)|\xi|}{\langle\xi\rangle_{b(t)}}, \tag{3.3.5}$$

where the upper sign is taken in the hyperbolic region.

We will also divide both regions of the extended phase space into zones. For this reason we define

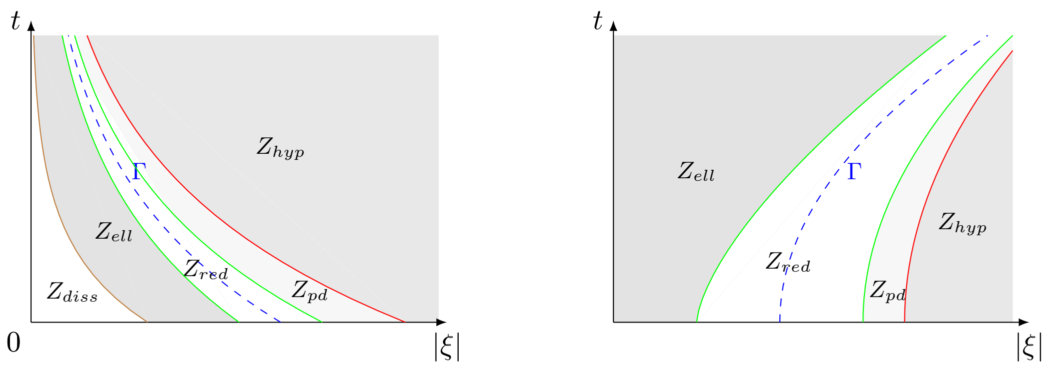
$$\text{the hyperbolic zone}: \ Z_{hyp}(N) = \Big\{(t,\xi) : \langle\xi\rangle_{b(t)} \geq N\mu(t)\frac{a(t)}{2A(t)}\Big\}\cap\Pi_{hyp},$$

$$\text{the pseudo-differential zone}: \ Z_{pd}(N,\varepsilon) = \Big\{(t,\xi) : \varepsilon\frac{\mu(t)a(t)}{2A(t)} \leq \langle\xi\rangle_{b(t)} \leq N\frac{\mu(t)a(t)}{2A(t)}\Big\}\cap\Pi_{hyp},$$

$$\text{the dissipative zone}: \ Z_{diss}(c_0) = \Big\{(t,\xi) : |\xi| \leq c_0\frac{1}{A(t)}\Big\}\cap\Pi_{ell},$$

$$\text{the elliptic zone}: Z_{ell}(c_0,\varepsilon) = \Big\{(t,\xi) : |\xi| \geq c_0\frac{1}{A(t)}\Big\}\cap\Big\{\langle\xi\rangle_{b(t)} \geq \varepsilon\mu(t)\frac{a(t)}{2A(t)}\Big\}\cap\Pi_{ell},$$

$$\text{the reduced zone}: \ Z_{red}(\varepsilon) = \Big\{(t,\xi) : \langle\xi\rangle_{b(t)} \leq \varepsilon\mu(t)\frac{a(t)}{2A(t)}\Big\}.$$



a. The case $\mu(t)/A(t)$ is decreasing b. The case $\mu(t)/A(t)$ is increasing

Fig. 3.2.: Sketch of zones are used in our approach

Remark 3.3.1. The dissipative zone can be skipped if we assume the further assumption

(S1) $\dfrac{a^2(t)}{b(t)A^2(t)} \in L^1(\mathbb{R}_+)$.

Under this assumption we define $Z_{ell}(\varepsilon) := Z_{ell}(0,\varepsilon)$.

3.3.2. The hyperbolic region

Symbols in Π_{hyp}.

Definition 3.3.1. *Let us define the following classes of symbols in the hyperbolic zone*:

$$\begin{aligned} S_l\{m_1,m_2,m_3\} = \Big\{ c = c(t,\xi) : |D_\xi^\alpha D_t^k c(t,\xi)| \leqslant C_{\alpha,k} \langle\xi\rangle_{b(t)}^{m_1-|\alpha|} a(t)^{m_2+|\alpha|} \Big(\frac{a(t)}{A(t)}\Big)^{m_3+k} \\ \text{for all } (t,\xi)\in Z_{hyp}(N),\ \alpha \ \text{ and } k \le l \Big\}. \end{aligned}$$

Lemma 3.3.1. *The family of symbol classes $S_l\{m_1,m_2,m_3\}$ generates a hierarchy of symbol classes having the following properties*:

- $S_l\{m_1,m_2,m_3\}$ *is a vector space,*
- $S_l\{m_1,m_2,m_3\}S_l\{m_1',m_2',m_3'\} \subset S_l\{m_1+m_1',m_2+m_2',m_3+m_3'\}$,
- $D_t^k D_\xi^\alpha S_l\{m_1,m_2,m_3\} \subset S_{l-k}\{m_1-|\alpha|, m_2+|\alpha|, m_3+k\}$,
- $S_0\{-1,0,2\} \subset L_\xi^\infty L_t^1(Z_{hyp}(N))$.

Proof. We only verify the fourth property. Indeed, if $c = c(t,\xi) \in S_0\{-1,0,2\}$, then

$$\int_{t_\xi}^\infty |c(\tau,\xi)|d\tau \lesssim \int_{t_\xi}^\infty \frac{a^2(\tau)}{\langle\xi\rangle_{b(\tau)}A^2(\tau)}d\tau \sim \int_{t_\xi}^\infty \frac{a(\tau)}{|\xi|A^2(\tau)}d\tau \le \frac{C}{A(t_\xi)|\xi|} \le \frac{C}{N\mu(t_\xi)} < \infty$$

due to the definition of the hyperbolic zone and assumption (B'3). Remark, that here we used

$$\langle\xi\rangle_{b(t)} \sim a(t)|\xi| \ \text{ uniformly on } Z_{hyp}(N) \tag{3.3.6}$$

to conclude what we wanted to have. $\square$

Consideration in the hyperbolic zone

Proposition 3.3.2. *Let us assume (B'1),(B'2) and (B'3). Then $b(t)$, $\langle\xi\rangle_{b(t)} \in S_2\{1,0,0\}$.*

Proof. Applying assumptions (B'1), (B'2) and the definition of the hyperbolic region we have for

$l = 1, 2$

$$
\begin{aligned}
b(t) &= \mu(t)\frac{a(t)}{A(t)} \le a(t)|\xi| \sim \langle\xi\rangle_{b(t)}, && (3.3.7)\\
|b(t)'| &\le |\mu(t)'| \underbrace{\frac{a(t)}{A(t)}}_{\in S_2\{0,0,1\}} + \mu(t)\underbrace{\Big|\Big(\frac{a(t)}{A(t)}\Big)'\Big|}_{\in S_1\{0,0,2\}} \lesssim \mu(t)\Big(\frac{a(t)}{A(t)}\Big)^2 \\
&\lesssim a(t)|\xi|\frac{a(t)}{A(t)} \sim \langle\xi\rangle_{b(t)}\frac{a(t)}{A(t)}, && (3.3.8)\\
|b(t)''| &\le |\mu(t)''| \underbrace{\frac{a(t)}{A(t)}}_{\in S_2\{0,0,1\}} + 2|\mu(t)'|\underbrace{\Big|\Big(\frac{a(t)}{A(t)}\Big)'\Big|}_{\in S_1\{0,0,2\}} + \mu(t)\underbrace{\Big|\Big(\frac{a(t)}{A(t)}\Big)''\Big|}_{\in S_0\{0,0,3\}} \\
&\lesssim \mu(t)\Big(\frac{a(t)}{A(t)}\Big)^3 \lesssim a(t)|\xi|\Big(\frac{a(t)}{A(t)}\Big)^2 \sim \langle\xi\rangle_{b(t)}\Big(\frac{a(t)}{A(t)}\Big)^2. && (3.3.9)
\end{aligned}
$$

Thanks to (3.3.7), (3.3.8) and (3.3.9) we obtain $b(t) \in S_2\{1,0,0\}$. Next, let us prove $\langle\xi\rangle_{b(t)} \in S_2\{1,0,0\}$. By the definition of $\langle\xi\rangle_{b(t)}$ we have

$$\langle\xi\rangle^2_{b(t)} = \underbrace{a^2(t)|\xi|^2}_{\in S_2\{2,0,0\}} - \underbrace{\frac{b^2(t)}{4}}_{\in S_2\{2,0,0\}} \in S_2\{2,0,0\}. \tag{3.3.10}$$

We assert that

$$\big|D^\alpha_\xi\langle\xi\rangle_{b(t)}\big| \lesssim \langle\xi\rangle^{1-|\alpha|}_{b(t)} a(t)^{|\alpha|} \tag{3.3.11}$$

for all multi-indices α with $|\alpha| > 0$. We apply the principle of induction with respect to $|\alpha|$. For $|\alpha| = 1$ we have

$$D^\alpha_\xi\langle\xi\rangle^2_{b(t)} = 2D^\alpha_\xi\langle\xi\rangle_{b(t)}\langle\xi\rangle_{b(t)}. \tag{3.3.12}$$

Since (3.3.10) implies $\big|D^\alpha_\xi\langle\xi\rangle^2_{b(t)}\big| \lesssim \langle\xi\rangle_{b(t)}a(t)$ we conclude from (3.3.12) that

$$\big|D^\alpha_\xi\langle\xi\rangle_{b(t)}\big| \lesssim a(t) \text{ for all } |\alpha| = 1.$$

Let us assume that the inequality (3.3.11) is valid for all $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \le k-1$. Then we prove this inequality for $|\alpha| = k$. For convenience we introduce the notation $\langle\xi\rangle_{b(t)} := g(t,|\xi|)$. After applying Faà di Bruno's formula (see Appendix: Lemma B.3.6) and performing straight-forward calculations we get for $|\alpha| \le 2$ (higher derivatives vanish)

$$
\begin{aligned}
D^\alpha_\xi g^2(t,|\xi|) &= \sum_{j=1}^{|\alpha|} \sum_{\substack{\beta_1+\ldots+\beta_j=\alpha \\ |\beta_j|\ge 1}} C_{\beta_1,\ldots,\beta_j} g^2(t,|\xi|)^{(j)} \prod_{i=1}^{j} D^{\beta_i}_\xi g(t,|\xi|) \\
&= 2C_{\beta_1} g(t,|\xi|) D^\alpha_\xi g(t,|\xi|) + 2\sum_{\substack{\beta_1+\beta_2=\alpha \\ |\beta_1|,|\beta_2|\ge 1}} C_{\beta_1,\beta_2} D^{\beta_1}_\xi g(t,|\xi|) D^{\beta_2}_\xi g(t,|\xi|). && (3.3.13)
\end{aligned}
$$

Owing to (3.3.10) we deduce $\big|D^\alpha_\xi g(t,|\xi|)^2\big| \lesssim g(t,|\xi|)^{2-|\alpha|}a(t)^{|\alpha|}$ and taking account of (3.3.11) we have $\big|D_\xi^{\beta_j} g(t,|\xi|)\big| \lesssim g(t,|\xi|)^{1-|\beta_j|}a(t)^{|\beta_j|}$, $j=1,2$. Returning to (3.3.13) we therefore obtain

$$\begin{aligned} g(t,|\xi|)\big|D_\xi^\alpha g(t,|\xi|)\big| &\lesssim \big|D_\xi^\alpha g^2(t,|\xi|)\big| + \sum_{\substack{\beta_1+\beta_2=\alpha\\ |\beta_1|,|\beta_2|\ge 1}} \big|D_\xi^{\beta_1} g(t,|\xi|)\big|\big|D_\xi^{\beta_2} g(t,|\xi|)\big| \\ &\lesssim g(t,|\xi|)^{2-|\alpha|}a(t)^{|\alpha|} + \sum_{\substack{\beta_1+\beta_2=\alpha\\ |\beta_1|,|\beta_2|\ge 1}} g(t,|\xi|)^{1-|\beta_1|}a(t)^{|\beta_1|}g(t,|\xi|)^{1-|\beta_2|}a(t)^{|\beta_2|} \\ &\lesssim g(t,|\xi|)^{2-|\alpha|}a(t)^{|\alpha|} + g(t,|\xi|)^{2-|\alpha|}a(t)^{|\alpha|}. \end{aligned} \tag{3.3.14}$$

Thanks to (3.3.14) we conclude our desired estimate (3.3.11). Next, since $D_t\langle\xi\rangle_{b(t)}^2 = 2D_t\langle\xi\rangle_{b(t)}\langle\xi\rangle_{b(t)}$ and $D_t^2\langle\xi\rangle_{b(t)}^2 = 2D_t^2\langle\xi\rangle_{b(t)}\langle\xi\rangle_{b(t)} + 2\big(D_t\langle\xi\rangle_{b(t)}\big)^2$ we can easy conclude

$$\big|D_t\langle\xi\rangle_{b(t)}\big| \lesssim \langle\xi\rangle_{b(t)}\frac{a(t)}{A(t)},\ \big|D_t^2\langle\xi\rangle_{b(t)}\big| \lesssim \langle\xi\rangle_{b(t)}\Big(\frac{a(t)}{A(t)}\Big)^2. \tag{3.3.15}$$

Finally, consequently formula (3.3.13), using Leibniz's rule we obtain for $k=1,2$

$$\begin{aligned} D_t^k D_\xi^\alpha g^2(t,|\xi|) &= \sum_{k_1+k_2=k} 2C_{\beta_1,k_1,k_2} D_t^{k_1} g(t,|\xi|) D_t^{k_2} D_\xi^\alpha g(t,|\xi|) \\ &\quad +2\sum_{k_1+k_2=k}\ \sum_{\substack{\beta_1+\beta_2=\alpha\\ |\beta_1|,|\beta_2|\ge 1}} C_{\beta_1,\beta_2,k_1,k_2} D_t^{k_1} D_\xi^{\beta_1} g(t,|\xi|) D_t^{k_1} D_\xi^{\beta_2} g(t,|\xi|). \end{aligned} \tag{3.3.16}$$

Due to the formula (3.3.16) and by induction we can prove

$$\big|D_t^k D_\xi^\alpha \langle\xi\rangle_{b(t)}\big| \lesssim \langle\xi\rangle_{b(t)}^{1-|\alpha|} a(t)^\alpha \Big(\frac{a(t)}{A(t)}\Big)^k. \tag{3.3.17}$$

This completes our proof. $\square$

Now we consider the micro-energy

$$V = (\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v})^T. \tag{3.3.18}$$

Then it holds

$$D_tV = \begin{pmatrix} 0 & \langle\xi\rangle_{b(t)} \\ \langle\xi\rangle_{b(t)} & 0 \end{pmatrix} V + \begin{pmatrix} \frac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0 \\ -\frac{\left(\mu(t)\frac{a(t)}{A(t)}\right)'}{2\langle\xi\rangle_{b(t)}} & 0 \end{pmatrix} V. \tag{3.3.19}$$

Lemma 3.3.3. *Assuming (B'1), (B'2) and (B'3). Then the following estimate holds for the fundamental solution $E_V(t,s,\xi)$, with $(s,\xi),(t,\xi)\in Z_{hyp}(N)$, $s\le t$:*

$$|E_V(t,s,\xi)| \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(s)}}.$$

Proof. Let us carry out the first step of diagonalization. For this reason we set

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\quad M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\quad \text{and } V^{(0)} := M^{-1}V.$$

Hence,

$$D_tV^{(0)} = \mathcal{D}(t,\xi)V^{(0)} + \mathcal{R}(t,\xi)V^{(0)}, \tag{3.3.20}$$

where

$$\mathcal{D}(t,\xi)=\begin{pmatrix}\langle\xi\rangle_{b(t)} & 0\\ 0 & -\langle\xi\rangle_{b(t)}\end{pmatrix}\in S_2\{1,0,0\}, \tag{3.3.21}$$

$$\mathcal{R}(t,\xi)=\begin{pmatrix}\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}-\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}+\frac{b'(t)}{4\langle\xi\rangle_{b(t)}}\\ -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}-\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & \frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}+\frac{b'(t)}{4\langle\xi\rangle_{b(t)}}\end{pmatrix}\in S_1\{0,0,1\}. \tag{3.3.22}$$

Let $F_0(t,\xi)$ be the diagonal part of $\mathcal{R}(t,\xi)$. Now we carry out the second step of diagonalization procedure. Therefore we introduce the matrices

$$N^{(1)}=\begin{pmatrix}0 & \frac{\mathcal{R}_{12}}{\tau_1-\tau_2}\\ \frac{\mathcal{R}_{21}}{\tau_2-\tau_1} & 0\end{pmatrix}=\begin{pmatrix}0 & -\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}}-\frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}}\\ -\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}}+\frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}} & 0\end{pmatrix}\in S_1\{-1,0,1\},$$

$N_1(t,\xi)=I+N^{(1)}(t,\xi)\in S_1\{0,0,0\}$. For sufficiently large time $t_0=t_0(\varepsilon)$ the matrix $N_1(t,\xi)$ is invertible with uniformly bounded inverse $N_1^{-1}(t,\xi)$ for $t\geq t_0$ in $Z_{hyp}(N)$ (see Remark 3.3.3). Now we can follow the usual procedure to diagonalize. Let

$$\begin{aligned}B^{(1)}(t,\xi)&=D_tN^{(1)}(t,\xi)-\big(\mathcal{R}(t,\xi)-F_0(t,\xi)\big)N^{(1)}(t,\xi)\in S_0\{-1,0,2\},\\ \mathcal{R}_1(t,\xi)&=-N_1^{-1}(t,\xi)B^{(1)}(t,\xi)\in S_0\{-1,0,2\}.\end{aligned}$$

Then we can conclude

$$\big(D_t-\mathcal{D}(t,\xi)-\mathcal{R}(t,\xi)\big)N_1(t,\xi)V^{(1)}(t,\xi)=N_1(t,\xi)\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)V^{(1)}(t,\xi).$$

Now we shall find the solution $V^{(0)}(t,\xi)=:N_1(t,\xi)V^{(1)}(t,\xi)$, where $V^{(1)}(t,\xi)$ is the solution to the system

$$\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)V(t,\xi)=0.$$

We can write $V^{(1)}(t,\xi)=E_{V,1}(t,t_\xi,\xi)V^{(1)}(t_\xi,\xi)$. Here $E_V(t,s,\xi)$ is the fundamental solution to the following system

$$\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)E_{V,1}(t,s,\xi)=0,\quad E_{V,1}(s,s,\xi)=I,\ t\geq s\geq t_\xi.$$

The solution $E_0=E_0(t,s,\xi)$ of the "principal diagonal part" fulfils

$$D_tE_0(t,s,\xi)=\big(\mathcal{D}(t,\xi)+F_0(t,\xi)\big)E_0(t,s,\xi),\quad E_0(s,s,\xi)=I,\ t\geq s\geq t_\xi.$$

Thus

$$E_0(t,s,\xi)=\exp\left(i\int_s^t\big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right),$$

and we can get

$$|E_0(t,s,\xi)|\lesssim\exp\left(\int_s^t\frac{\partial_t\langle\xi\rangle_{b(\tau)}}{2\langle\xi\rangle_{b(\tau)}}d\tau\right)=\frac{\sqrt{\langle\xi\rangle_{b(t)}}}{\sqrt{\langle\xi\rangle_{b(s)}}}\sim\frac{\sqrt{a(t)|\xi|}}{\sqrt{a(s)|\xi|}}\sim\frac{\sqrt{a(t)}}{\sqrt{a(s)}}.$$

Let us set

$$\begin{aligned}\mathcal{R}_2(t,s,\xi)&=E_0^{-1}(t,s,\xi)\mathcal{R}_1(t,\xi)E_0(t,s,\xi),\\ Q(t,s,\xi)&=I+\sum_{k=1}^\infty i^k\int_s^t\mathcal{R}_2(t_1,s,\xi)\int_s^{t_1}\mathcal{R}_2(t_2,s,\xi)\cdots\int_s^{t_{k-1}}\mathcal{R}_2(t_k,s,\xi)dt_k\cdots dt_2dt_1.\end{aligned}$$

Then $Q(t,s,\xi)$ solves the Cauchy problem

$$D_tQ(t,s,\xi) = \mathcal{R}_2(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I, \ t \geq s \geq t_\xi.$$

The fundamental solution $E_{V,1}(t,s,\xi)$ is representable in the form $E_{V,1}(t,s,\xi) = E_0(t,s,\xi)Q(t,s,\xi)$. Furthermore, applying the fourth property from Lemma 3.3.1 to $\mathcal{R}_1 \in S_0\{-1,0,2\} \subset L_\xi^\infty L_t^1(Z_{hyp})$ we see that

$$|Q(t,s,\xi)| \leq \exp\Big(\int_s^t |\mathcal{R}_1(\tau,\xi)|d\tau\Big) \leq C_N.$$

This completes the proof. □

Remark 3.3.2. Transforming back we obtain the following representation for the micro-energy $V = (\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v})^T$:

$$V(t,\xi) = MN_1(t,\xi)E_0(t,s,\xi)Q(t,s,\xi)N_1^{-1}(s,\xi)M^{-1}V(s,\xi).$$

Remark 3.3.3. The large constant N guarantees the invertibility of N_1 in the whole hyperbolic zone. The remaining problem consists in the understanding of invertibility in the pseudo-differential zone. For $t \geq t_0(\varepsilon)$ this zone can be skipped after the choice $N = \varepsilon$. The other set $\{t \in (0, t_0(\varepsilon)] : (t,\xi) \in Z_{pd}(N,\varepsilon)\}$ is compact.

3.3.3. The elliptic region

Symbols in Π_{ell}.

The symbols in the elliptic zone are constructed in a similar manner as in the hyperbolic zone with a little change for the auxiliary symbol $\langle\xi\rangle_{b(t)}$ which can be estimated

$$\langle\xi\rangle_{b(t)} \sim \frac{b(t)}{2} \sim \mu(t)\frac{a(t)}{2A(t)} \qquad \text{uniformly on } Z_{ell}(c_0,\varepsilon). \tag{3.3.23}$$

Definition 3.3.2. *Let us define the following classes of symbols in the elliptic zone*:

$$S_l\{m_1,m_2,m_3\} = \Big\{c = c(t,\xi) : |D_\xi^\alpha D_t^k c(t,\xi)| \leqslant C_{\alpha,k}\langle\xi\rangle_{b(t)}^{m_1-|\alpha|}a(t)^{m_2+|\alpha|}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k}$$
$$\text{for all } (t,\xi) \in Z_{ell}(c_0,\varepsilon),\ \alpha \ \text{ and } k \leq l\Big\}.$$

Lemma 3.3.4. *The family of symbol classes $S_l\{m_1,m_2,m_3\}$ generates a hierarchy of symbol classes having the following properties*:

- $S_l\{m_1,m_2,m_3\}$ *is a vector space,*
- $S_l\{m_1,m_2,m_3\}S_l\{m_1',m_2',m_3'\} \subset S_l\{m_1+m_1',m_2+m_2',m_3+m_3'\}$,
- $D_t^kD_\xi^\alpha S_l\{m_1,m_2,m_3\} \subset S_{l-k}\{m_1-|\alpha|,m_2+|\alpha|,m_3+k\}$,
- $S_0\{-1,0,2\} \subset L_\xi^\infty L_t^1(Z_{ell}(c_0,\varepsilon))$.

Proof. We only verify the fourth property. Indeed, if $c \in S_0\{-1,0,2\}$, then

$$\begin{aligned}\int_{t_{\xi_1}}^{t_{\xi_2}} |c(\tau,\xi)|d\tau &\lesssim \int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a^2(\tau)}{\langle\xi\rangle_{b(t)}A^2(\tau)}d\tau \sim \int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a(\tau)}{\mu(\tau)A(\tau)}d\tau \lesssim \int_{t_{\xi_1}}^{t_{\xi_2}} \frac{\sqrt{1-\varepsilon^2}a(\tau)}{|\xi|A^2(\tau)}d\tau \\ &\lesssim \frac{1}{|\xi|A(t_{\xi_1})} \lesssim C(\varepsilon,c_0),\end{aligned}$$

where t_{ξ_1}, t_{ξ_2} denotes the lower, upper boundary of the elliptic zone, respectively. From the definitions of the elliptic zone and dissipative zone we have $\mu(t) \geq \frac{2|\xi|A(t)}{\sqrt{1-\varepsilon^2}}$ for all $t \in [t_{\xi_1}, t_{\xi_2}]$ and $|\xi|A(t_{\xi_1}) \sim 1$. □

Consideration in the elliptic zone

In this region we introduce again the micro-energy

$$V = (\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v})^T.$$

Then we can get the system of differential equations

$$D_tV = \left[\begin{pmatrix} 0 & \langle\xi\rangle_{b(t)} \\ -\langle\xi\rangle_{b(t)} & 0 \end{pmatrix} + \begin{pmatrix} \frac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0 \\ -\frac{b'(t)}{2\langle\xi\rangle_{b(t)}} & 0 \end{pmatrix}\right]V. \tag{3.3.24}$$

In a first step we use the diagonalizer of the first matrix, which is defined as follows:

$$M = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}, \quad \text{and } V^{(0)} := M^{-1}V.$$

Hence,

$$D_tV^{(0)} = \mathcal{D}(t,\xi)V^{(0)} + \mathcal{R}(t,\xi)V^{(0)}, \tag{3.3.25}$$

where

$$\mathcal{D}(t,\xi) = \begin{pmatrix} -i\langle\xi\rangle_{b(t)} & 0 \\ 0 & i\langle\xi\rangle_{b(t)} \end{pmatrix} \in S_2\{1,0,0\}, \tag{3.3.26}$$

$$\mathcal{R}(t,\xi) = \frac{1}{2}\begin{pmatrix} \frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}} - i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}} + i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} \\ -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}} - i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & \frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}} + i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} \end{pmatrix} \in S_1\{0,0,1\}. \tag{3.3.27}$$

Let $F_0 = F_0(t,\xi)$ be the diagonal part of $\mathcal{R} = \mathcal{R}(t,\xi)$. Now we carry out the second step of diagonalization procedure. Thus, we consider the difference δ of the entries of $\mathcal{D}(t,\xi) + F_0(t,\xi)$. We have

$$i\delta(t,\xi) = 2\langle\xi\rangle_{b(t)} + \frac{b'(t)}{2\langle\xi\rangle_{b(t)}} \sim 2\langle\xi\rangle_{b(t)} + \frac{o(b^2(t))}{2\langle\xi\rangle_{b(t)}} \sim \langle\xi\rangle_{b(t)} \tag{3.3.28}$$

for $t \geq t_0$ with a sufficiently large $t_0 = t_0(\varepsilon)$ by using $|b'(t)| = o(b^2(t))$. Now we can follow the usual procedure of diagonalization. Therefore we introduce the matrices

$$N^{(1)} = \begin{pmatrix} 0 & -\frac{\mathcal{R}_{12}}{\delta} \\ \frac{\mathcal{R}_{21}}{\delta} & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & i\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}} - \frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}} \\ i\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}} + \frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}} & 0 \end{pmatrix} \in S_1\{-1,0,1\},$$

$N_1(t,\xi) = I + N^{(1)}(t,\xi) \in S_1\{0,0,0\}$. For a sufficiently large time $t \geq t_0$ the matrix $N_1(t,\xi)$ is invertible with uniformly bounded inverse $N_1^{-1}(t,\xi)$ in $Z_{ell}(c_0,\varepsilon)$. Let

$$B^{(1)}(t,\xi) = D_tN^{(1)}(t,\xi) - \big(\mathcal{R}(t,\xi) - F_0(t,\xi)\big)N^{(1)}(t,\xi) \in S_0\{-1,0,2\},$$
$$\mathcal{R}_1(t,\xi) = -N_1^{-1}(t,\xi)B^{(1)}(t,\xi) \in S_0\{-1,0,2\}.$$

We can conclude that

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{R}(t,\xi)\big)N_1(t,\xi)V^{(1)}(t,\xi) = N_1(t,\xi)\big(D_t - \mathcal{D}(t,\xi) - F_0(t,\xi) - \mathcal{R}_1(t,\xi)\big)V^{(1)}(t,\xi).$$

Now we shall find the solution $V^{(1)}(t,\xi) := N_1^{-1}(t,\xi)V^{(0)}(t,\xi)$, where $V^{(1)}(t,\xi)$ is the solution to the system

$$\big(D_t - \mathcal{D}(t,\xi) - F_0(t,\xi) - \mathcal{R}_1(t,\xi)\big)V^{(1)}(t,\xi) = 0.$$

We can write $V^{(1)}(t,\xi) = E_{V,1}(t,s,\xi)V^{(1)}(s,\xi)$. Here $E_{V,1}(t,s,\xi)$ is the fundamental solution, that is, the solution to the following system:

$$\big(D_t - \mathcal{D}(t,\xi) - F_0(t,\xi) - \mathcal{R}_1(t,\xi)\big)E(t,s,\xi) = 0, \quad E(s,s,\xi) = I,\ t \geq s \geq t_\xi.$$

We transform the system for $E_{V,1}(t,s,\xi)$ to an integral equation for a new matrix-valued function $Q_{ell}(t,s,\xi)$ by considering

$$\exp\left(i\int_s^t \big(\mathcal{D}(\tau,\xi) + F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(t,s,\xi).$$

Using this ansatz we have after differentiation

$$\begin{aligned}
&D_t\left(\exp\left(i\int_s^t \big(\mathcal{D}(\tau,\xi) + F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(t,s,\xi)\right)\\
&= -\big(\mathcal{D}(t,\xi) + F_0(t,\xi)\big)\exp\left(i\int_s^t \big(\mathcal{D}(\tau,\xi) + F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(t,s,\xi)\\
&\quad + \exp\left(i\int_s^t \big(\mathcal{D}(\tau,\xi) + F_0(\tau,\xi)\big)d\tau\right)\big(\mathcal{D}(t,\xi) + F_0(t,\xi) + \mathcal{R}_1(t,\xi)\big)E_{V,1}(t,s,\xi)\\
&= \exp\left(i\int_s^t \big(\mathcal{D}(\tau,\xi) + F_0(\tau,\xi)\big)d\tau\right)\mathcal{R}_1(t,\xi)E_{V,1}(t,s,\xi).
\end{aligned}$$

Consequently,

$$\begin{aligned}
E_{V,1}(t,s,\xi) &= \exp\left(i\int_s^t \big(\mathcal{D}(\tau,\xi) + F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(s,s,\xi)\\
&\quad -i\int_s^t \exp\Big(i\int_\theta^t \big(\mathcal{D}(\tau,\xi) + F_0(\tau,\xi)\big)d\tau\Big)\mathcal{R}_1(\theta,\xi)E_{V,1}(\theta,s,\xi)d\theta.
\end{aligned}$$

We introduce an unknown weight factor to represent $Q_{ell,1}$ in the following way:

$$Q_{ell,1}(t,s,\xi) = \exp\Big(-\int_s^t w(\tau,\xi)d\tau\Big)E_{V,1}(t,s,\xi).$$

Then we get

$$\begin{aligned}
Q_{ell,1}(t,s,\xi) &= \exp\left(\int_s^t \big(i\mathcal{D}(\tau,\xi) + iF_0(\tau,\xi) - w(\tau,\xi)I\big)d\tau\right)\\
&\quad + \int_s^t \exp\left(\int_\theta^t \big(i\mathcal{D}(\tau,\xi) + iF_0(\tau,\xi) - w(\tau,\xi)I\big)d\tau\right)\mathcal{R}_1(\theta,\xi)Q_{ell,1}(\theta,s,\xi)d\theta.
\end{aligned}$$

The main entries of the diagonal matrix $i\mathcal{D}(t,\xi) + iF_0(t,\xi)$ are given by

$$\begin{aligned}
(I) &= \langle\xi\rangle_{b(t)} + \frac{\partial_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}} + \frac{b'(t)}{4\langle\xi\rangle_{b(t)}},\\
(II) &= -\langle\xi\rangle_{b(t)} + \frac{\partial_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}} - \frac{b'(t)}{4\langle\xi\rangle_{b(t)}}.
\end{aligned}$$

For the difference (II)-(I) we get

$$(II)-(I)=-2\langle\xi\rangle_{b(t)}-\frac{b'(t)}{2\langle\xi\rangle_{b(t)}}=-\frac{b^2(t)+b'(t)-4a^2(t)|\xi|^2}{2\langle\xi\rangle_{b(t)}}\le 0$$

in $Z_{ell}(c_0,\varepsilon)$ for $t\ge t_0$ by using $|b'(t)|=o(b^2(t))$. We are choosing the weight $w(t,\xi)=(I)$. By this choice the matrix

$$\begin{aligned}H(t,s,\xi)&=\exp\Big(\int_s^t\Big(i\mathcal{D}(\tau,\xi)+iF_0(\tau,\xi)-w(\tau,\xi)I\Big)d\tau\Big)\\&=\operatorname{diag}\Big(1,\exp\Big(\int_s^t\Big(-2\langle\xi\rangle_{b(\tau)}-\frac{b'(\tau)}{2\langle\xi\rangle_{b(\tau)}}\Big)d\tau\Big)\Big)\to\begin{pmatrix}1&0\\0&0\end{pmatrix}\end{aligned}$$

as $t\to\infty$ with a fixed s. Hence, the matrix $H(t,s,\xi)$ is uniformly bounded for (s,ξ), $(t,\xi)\in Z_{ell}(c_0,\varepsilon)$. Taking account of $\mathcal{R}_1\in S_0\{-1,0,2\}$ is uniformly integrable over the elliptic zone the matrix which can be represented by Neumann series

$$\begin{aligned}Q_{ell,1}(t,s,\xi)=H(t,s,\xi)+\sum_{k=1}^{\infty}i^k\int_s^t H(t,t_1,\xi)\mathcal{R}_1(t_1,s,\xi)\int_s^{t_1}H(t_1,t_2,\xi)\mathcal{R}_1(t_2,s,\xi)\\ \cdots\int_s^{t_{k-1}}H(t_{k-1},t_k,\xi)\mathcal{R}_1(t_k,s,\xi)dt_k\cdots dt_2dt_1\end{aligned}$$

is uniformly bounded in $Z_{ell}(c_0,\epsilon)$. From the last considerations we can conclude

$$\begin{aligned}E_{V,1}(t,s,\xi)&=\exp\Big(\int_s^t w(\tau,\xi)d\tau\Big)Q_{ell,1}(t,s,\xi)\\&=\exp\Big(\int_s^t\Big(\langle\xi\rangle_{b(\tau)}+\frac{\partial_\tau\langle\xi\rangle_{b(\tau)}}{2\langle\xi\rangle_{b(\tau)}}+\frac{b'(\tau)}{4\langle\xi\rangle_{b(\tau)}}\Big)d\tau\Big)Q_{ell,1}(t,s,\xi)\\&\sim\exp\Big(\int_s^t\Big(\langle\xi\rangle_{b(\tau)}+\frac{\partial_\tau\langle\xi\rangle_{b(\tau)}}{2\langle\xi\rangle_{b(\tau)}}+\frac{b'(\tau)}{2b(\tau)}\Big)d\tau\Big)Q_{ell,1}(t,s,\xi)\\&\sim\frac{\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(s)}}\exp\Big(\int_s^t\langle\xi\rangle_{b(\tau)}d\tau\Big)Q_{ell,1}(t,s,\xi).\end{aligned}$$

Summarizing the considerations of this section we have proved the following lemma:

Lemma 3.3.5. *Under the assumptions (B'1), (B'2) and (B'3) the fundamental solution $E_V(t,s,\xi)$ to the operator $D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)$ with $(t,\xi),(s,\xi)\in Z_{ell}(c_0,\varepsilon)\cap\{t\ge t_0(\varepsilon)\}, s\le t$ has the following behavior*:

$$E_{V,1}(t,s,\xi)\sim\frac{\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(s)}}\exp\Big(\int_s^t\langle\xi\rangle_{b(\tau)}d\tau\Big)Q_{ell,1}(t,s,\xi).$$

3.3.4. The reduced zone

In this zone we can estimate $\langle\xi\rangle_{b(t)}$ by $\varepsilon\frac{b(t)}{2}$. Thus, we consider the micro-energy

$$V=\Big(\varepsilon\frac{b(t)}{2}\hat{v},D_t\hat{v}\Big)^T. \tag{3.3.29}$$

We get the following system of first order

$$D_tV = \begin{pmatrix} \frac{D_tb(t)}{b(t)} & \varepsilon\frac{b(t)}{2} \\ \frac{a^2(t)|\xi|^2-\frac{1}{4}b^2(t)-\frac{1}{2}b'(t)}{\varepsilon\frac{1}{2}b(t)} & \end{pmatrix} V. \tag{3.3.30}$$

To estimate the entries of this matrix we will use:

- $|b'(t)| = o(b^2(t))$,
- $\langle\xi\rangle_{b(t)} \lesssim \varepsilon\frac{b(t)}{2}$,
- consequently, $\frac{a^2(t)|\xi|^2-\frac{1}{4}b^2(t)-\frac{1}{2}b'(t)}{\varepsilon\frac{1}{2}b(t)} \lesssim \varepsilon\frac{b(t)}{2} - \frac{b'(t)}{\varepsilon b(t)} \lesssim \varepsilon b(t)$.

Thus, we can estimate the norm of the coefficient matrix by $\varepsilon b(t)$ for sufficiently large t. Summarizing the following statement holds:

Lemma 3.3.6. *If we assume (B'1) to (B'3), then the fundamental solution $E_V(t,s,\xi)$ to (3.3.30) can be estimated by*

$$|E_V(t,s,\xi)| \lesssim \exp\left(\varepsilon\int_s^t b(\tau)d\tau\right)$$

for $t_0 \le s \le t$ with sufficiently large $t_0 = t_0(\varepsilon)$ and $(t,\xi),(s,\xi) \in Z_{red}(\varepsilon)$.

Remark 3.3.4. We can make the reduced zone as small as we want by the control of the constant ε.

The dissipative zone

Let us assume that the assumption $(S1)$ does not hold. This means, that $\mu(t)$ is "very close" to 1. Thus, we introduced the dissipative zone to ensure integrability of $S_0\{-1,0,2\}$ over the elliptic zone $Z_{ell}(c_0,\epsilon)$. In this case we can apply directly Lemma 3.2.6 to estimate the fundamental solution $E(t,s,\xi)$ related to a system of first order for the micro-energy $U = (\frac{a(t)}{A(t)}\hat{u}, D_t\hat{u})^T$, and relate this estimate to the fundamental solution $E_V(t,s,\xi)$ related to a system of first order for $V = (\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v})^T$.

3.3.5. Estimates for the fundamental solution

We want to obtain estimates for the energy of the solution to our original Cauchy problem. For this reason we need to transform back to estimate the fundamental solution $E(t,s,\xi)$ which is related to a system of first order for the micro-energy $(a(t)|\xi|\hat{u}, D_t\hat{u})$.
Outside the reduced zone it holds

$$E(t,s,\xi) = T(t,\xi)E_V(t,s,\xi)T^{-1}(s,\xi), \tag{3.3.31}$$

where the matrix $T(t,\xi)$ is defined in the following way:

$$\begin{pmatrix} h(t,\xi)\hat{u} \\ D_t\hat{u} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{h(t,\xi)}{\lambda(t)\langle\xi\rangle_{b(t)}} & 0 \\ i\frac{b(t)}{2\lambda(t)\langle\xi\rangle_{b(t)}} & \frac{1}{\lambda(t)} \end{pmatrix}}_{T(t,\xi)} \begin{pmatrix} \langle\xi\rangle_{b(t)}\hat{v} \\ D_t\hat{v} \end{pmatrix} \tag{3.3.32}$$

with the inverse matrix

$$T^{-1}(t,\xi) = \begin{pmatrix} \frac{\lambda(t)\langle\xi\rangle_{b(t)}}{h(t,\xi)} & 0 \\ -i\frac{b(t)\lambda(t)}{2h(t,\xi)} & \lambda(t) \end{pmatrix}. \tag{3.3.33}$$

Recall that outside the dissipative zone we have $h(t,\xi) = a(t)|\xi|$ and especially in the dissipative zone we use $h(t,\xi) = \dfrac{a(t)}{A(t)}$. Inside the reduced zone we have estimated $\langle\xi\rangle_{b(t)}$ by $\varepsilon\frac{b(t)}{2}$. Therefore, we change the definition of the matrix $T(t,\xi)$ by

$$\begin{pmatrix} \frac{2h(t,\xi)}{\varepsilon\lambda(t)b(t)} & 0 \\ i\frac{1}{\lambda(t)} & \frac{1}{\lambda(t)} \end{pmatrix}, \qquad |T(t,\xi)| \sim \lambda^{-1}(t) \tag{3.3.34}$$

for all $(t,\xi) \in Z_{red}(\varepsilon)$.

Remark 3.3.5. We may conclude that in the hyperbolic and reduced zones the fundamental solution to our original Cauchy problem in the extended phase space can be estimated by

$$(|E(t,s,\xi)|) \lesssim \frac{\lambda(s)}{\lambda(t)}(|E_V(t,s,\xi)|).$$

Some auxiliary estimates. We continue with some auxiliary estimates which are used to obtain energy estimates later on.

Lemma 3.3.7. *Let us suppose (B'1) to (B'3) and let* $\lambda(t) = \exp\left(\frac{1}{2}\int_0^t b(\tau)d\tau\right)$. *Then the following holds*:

1. *In the elliptic zone it holds* $\langle\xi\rangle_{b(t)} - \dfrac{b(t)}{2} \leq -\dfrac{a^2(t)|\xi|^2}{b(t)}$.

2. $\dfrac{\lambda(s)}{\lambda(t)}\exp\left(\int_s^t \langle\xi\rangle_{b(\tau)}d\tau\right) \leq \exp\left(-|\xi|^2\int\limits_s^t \dfrac{a^2(\tau)}{b(\tau)}d\tau\right)$.

3. *With* $A(t_{\xi_1})|\xi| \sim 1$ *(separating line between dissipative and elliptic zone) it holds*
$$\exp\left(-|\xi|^2\int_0^{t_{\xi_1}}\frac{a^2(\tau)}{b(\tau)}d\tau\right) \sim 1.$$

4. *With* $a(t_{\xi_3})|\xi| = \sqrt{1+\varepsilon^2}b(t_{\xi_3})/2$ *(separating line between reduced zone and pseudo-differential zone) it holds*
$$|d_{|\xi|}t_{\xi_3}| \gtrsim \frac{\mu(t_{\xi_3})}{|\xi|b(t_{\xi_3})}.$$

Proof. The first statement is equivalent to the following inequality

$$\sqrt{\frac{b^2(t)}{4} - a^2(t)|\xi|^2} - \frac{b(t)}{2} \leq -\frac{a^2(t)|\xi|^2}{b(t)}.$$

The second statement follows directly from the first one together with the definition of $\lambda(t)$. The third statement can be directly obtained from the following estimate:

$$|\xi|^2\int_0^{t_{\xi_1}}\frac{a^2(\tau)}{b(\tau)}d\tau = |\xi|^2\int_0^{t_{\xi_1}}\frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \leq |\xi|^2\int_0^{t_{\xi_1}}\frac{a(\tau)A(\tau)}{\mu_0}d\tau \lesssim |\xi|^2A^2(t_{\xi_1}) \lesssim 1.$$

The last statement follows directly after straight-forward calculations by using the definition of t_{ξ_3}. We have

$$
\begin{aligned}
&|\xi| \approx \frac{b(t_{\xi_3})}{a(t_{\xi_3})} = \frac{\mu(t_{\xi_3})}{A(t_{\xi_3})} \Rightarrow \left(d_{|\xi|}t_{\xi_3}\right)^{-1} \approx \left(\frac{\mu(t_{\xi_3})}{A(t_{\xi_3})}\right)' \\
&\Rightarrow |d_{|\xi|}t_{\xi_3}|^{-1} \lesssim \left|\frac{\mu' A - \mu a}{A^2}\right| \lesssim \frac{\mu \frac{a}{A} A + \mu a}{A^2} \lesssim \frac{\mu}{A}\frac{a}{A} \approx |\xi| \frac{b(t_{\xi_3})}{\mu(t_{\xi_3})}.
\end{aligned}
$$

The proof is complete. □

A refined estimate for the fundamental solution in the elliptic zone.
Inside the elliptic zone we have

$$
|E_V(t,s,\xi)| \lesssim \frac{b(t)}{b(s)} \exp\left(\int_s^t \langle\xi\rangle_{b(\tau)} d\tau\right).
$$

This yields in combination with (3.3.31) the estimate

$$
\begin{aligned}
(|E(t,s,\xi)|) &\lesssim \begin{pmatrix} \frac{a(t)|\xi|}{b(t)} & 0 \\ & b(t) \end{pmatrix} \exp\left(\int_s^t \left(\langle\xi\rangle_{b(\tau)} - \frac{b(\tau)}{2}\right) d\tau\right) \begin{pmatrix} \frac{1}{a(s)|\xi|} & 0 \\ \frac{1}{a(s)|\xi|} & \frac{1}{b(s)} \end{pmatrix} \\
&\lesssim \exp\left(-|\xi|^2 \int_s^t \frac{a^2(\tau)}{b(\tau)} d\tau\right) \begin{pmatrix} \frac{a(t)}{a(s)} & \frac{a(t)|\xi|}{b(s)} \\ \frac{b(t)}{a(s)|\xi|} & \frac{b(t)}{b(s)} \end{pmatrix}.
\end{aligned} \tag{3.3.35}
$$

Here we have used the first statement from Lemma 3.3.7. The estimate for the first row seems to be optimal while the estimate for the second row is not optimal, because at least for increasing coefficient functions $b(t)$ for fixed ξ this estimate is increasing in t. This estimate contradicts to our a-priori knowledge that the wave type energy itself is decreasing. For this reason we need a refined estimate which will be presented in the following steps. If $\Phi^k(t,s,\xi)$, $k=1,2$, are solutions to the equation $\Phi_{tt} + a^2(t)|\xi|^2\Phi + b(t)\Phi_t = 0$ with initial values $\Phi^k(s,s,\xi) = \delta_{1k},\ \partial_t\Phi^k(s,s,\xi) = \delta_{2k}$, then we have

$$
\begin{pmatrix} a(t)|\xi|v(t,\xi) \\ D_t v(t,\xi) \end{pmatrix} = \begin{pmatrix} \dfrac{a(t)}{a(s)}\Phi_1(t,s,\xi) & ia(t)|\xi|\Phi_2(t,s,\xi) \\ \dfrac{D_t\Phi_1(t,s,\xi)}{a(s)|\xi|} & iD_t\Phi_2(t,s,\xi) \end{pmatrix} \begin{pmatrix} a(s)|\xi|v(s,\xi) \\ D_t v(s,\xi) \end{pmatrix}.
$$

Hence, if we compare with the estimate (3.3.35), then

$$
|\Phi^1(t,s,\xi)| \lesssim \exp\left(-|\xi|^2 \int_s^t \frac{a^2(\tau)}{b(\tau)} d\tau\right), \tag{3.3.36}
$$

$$
|\Phi^2(t,s,\xi)| \lesssim \frac{1}{b(s)} \exp\left(-|\xi|^2 \int_s^t \frac{a^2(\tau)}{b(\tau)} d\tau\right), \tag{3.3.37}
$$

$$
|\partial_t\Phi^1(t,s,\xi)| \lesssim b(t) \exp\left(-|\xi|^2 \int_s^t \frac{a^2(\tau)}{b(\tau)} d\tau\right), \tag{3.3.38}
$$

$$
|\partial_t\Phi^2(t,s,\xi)| \lesssim \frac{b(t)}{b(s)} \exp\left(-|\xi|^2 \int_s^t \frac{a^2(\tau)}{b(\tau)} d\tau\right). \tag{3.3.39}
$$

Let $\Psi_k(t,s,\xi) = \partial_t\Phi^k(t,s,\xi)$, $k=1,2$. Then we obtain the equations of first order

$$
\partial_t\Psi_k + b(t)\Psi_k = -a^2(t)|\xi|^2\Phi^k(t,s,\xi), \qquad \Psi_k(s,s,\xi) = \delta_{k2}.
$$

Applying Duhamel's principle we get

$$\begin{aligned}
\Psi_1(t,s,\xi) &= -|\xi|^2\int_s^t a^2(\tau)\frac{\lambda^2(\tau)}{\lambda^2(t)}\Phi_1(\tau,s,\xi)d\tau,\\
|\Psi_1(t,s,\xi)| &\lesssim \frac{|\xi|^2}{\lambda^2(t)}\int_s^t a^2(\tau)\lambda^2(\tau)\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau\\
&\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)}\int_s^t \lambda^2(\tau)\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau\\
&\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)}\int_s^t \underbrace{b(\tau)\lambda^2(\tau)}_{\partial_\tau\lambda^2(\tau)}\frac{1}{b(\tau)}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau\\
&\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)}\Big(\lambda^2(\tau)\frac{1}{b(\tau)}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)\Big)\Big|_s^t\\
&\quad+\frac{a^2(t)|\xi|^2}{\lambda^2(t)}\int_s^t\lambda^2(\tau)\underbrace{\Big(\frac{|\xi|^2a^2(\tau)}{b^2(\tau)}+\frac{b'(\tau)}{b^2(\tau)}\Big)}_{\lesssim C(\tau)<1}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau\\
&\lesssim \frac{a^2(t)|\xi|^2}{b(t)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)-\frac{a^2(t)|\xi|^2}{b(s)}\frac{\lambda^2(s)}{\lambda^2(t)}.
\end{aligned}$$

Here we have used $a^2(t)|\xi|^2/b^2(t)\le 1/2$ from the definition of the elliptic zone and $\frac{b'(t)}{b^2(t)}=o(1)$. We see that the second summand is subordinate to the first one because

$$\frac{b(s)}{b(t)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\frac{\lambda^2(t)}{\lambda^2(s)}=\exp\Big(\int_s^t\underbrace{\Big(b(\tau)-\frac{a^2(\tau)|\xi|^2}{b(\tau)}-\frac{b'(\tau)}{b(\tau)}\Big)}_{>0,\ \text{if}\,\tau\ge t_0}d\tau\Big)$$

for $t_0\le s\le t$ with t_0 sufficiently large. Thus, we get

$$\big|\Phi_t^1(t,s,\xi)\big|\lesssim\frac{a^2(t)|\xi|^2}{b(t)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big). \tag{3.3.40}$$

Similarly, we can represent Ψ_2 in the following way:

$$\begin{aligned}
\Psi_2(t,s,\xi) &= \frac{\lambda^2(s)}{\lambda^2(t)}-|\xi|^2\int_s^t a^2(\tau)\frac{\lambda^2(\tau)}{\lambda^2(t)}\Phi_2(\tau,s,\xi)d\tau, && (3.3.41)\\
|\Psi_2(t,s,\xi)| &\lesssim \frac{\lambda^2(s)}{\lambda^2(t)}+\frac{|\xi|^2}{\lambda^2(t)}\int_s^t a^2(\tau)\lambda^2(\tau)\frac{1}{b(s)}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau\\
&\lesssim \frac{\lambda^2(s)}{\lambda^2(t)}+\frac{a^2(t)|\xi|^2}{b(t)b(s)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big). && (3.3.42)
\end{aligned}$$

Thus, we have proved the following lemma:

Lemma 3.3.8. *Let $(s,\xi),(t,\xi)\in Z_{ell}(c_0,\varepsilon)$ with $s\le t$. Then the fundamental solution $E(t,s,\xi)$ can be estimated in the following way:*

$$(|E(t,s,\xi)|)\lesssim\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\begin{pmatrix}\frac{a(t)}{a(s)} & \frac{a(t)|\xi|}{b(s)}\\ \frac{a^2(t)|\xi|}{a(s)b(t)} & \frac{a^2(t)|\xi|^2}{b(s)b(t)}\end{pmatrix}+\frac{\lambda^2(s)}{\lambda^2(t)}\begin{pmatrix}0&0\\0&1\end{pmatrix}. \tag{3.3.43}$$

Remark 3.3.6. Let us choose a fixed s. Then the second summand in (3.3.43) is dominated by the first one. If we set $s = t_{\xi_2}$, then in the two cases $\Big(\dfrac{\mu(t)}{A(t)}$ is increasing or decreasing$\Big)$ we can use $a(t_{\xi_2})|\xi| \sim b(t_{\xi_2})$ to get the following estimate:

$$(|E(t,t_{\xi_2},\xi)|) \lesssim \exp\left(-|\xi|^2\int_{t_{\xi_2}}^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix} \frac{a(t)}{a(t_{\xi_2})} & \frac{a(t)}{a(t_{\xi_2})} \\ \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)} & \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)} \end{pmatrix}. \tag{3.3.44}$$

3.3.6. Gluing procedure

Case 1: the function $\dfrac{\mu(t)}{A(t)}$ *is monotonously decreasing*

In the previous sections we have considered the fundamental solution in different zones. Now we have to glue the estimates from Lemmas 3.2.6, 3.3.8, 3.3.6 and 3.3.3. We obtain for the part of the hyperbolic zone which contains large frequencies $\{\xi : |\xi| \geqslant c > 0\}$ the following estimate for the fundamental solution:

$$(|E(t,0,\xi)|) \lesssim \sqrt{a(t)}\exp\left(-\frac{1}{2}\int_0^t b(\tau)d\tau\right)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

to our original problem in the extended phase space, cf. Lemma 3.3.3 and Remark 3.3.5. It remains to consider the influence of the dissipative zone, the elliptic zone, the reduced zone and the hyperbolic zone for small frequencies. We denote by $t_{\xi_k}, k = 1,2,3$, the separating lines between the dissipative zone and the elliptic zone ($k = 1$), between the elliptic zone and the reduced zone ($k = 2$) and between the reduced zone and the hyperbolic zone ($k = 3$).

Case 1.1: $t \leq t_{\xi_1}$

In this case we follow directly Lemma 3.2.6.

Case 1.2: $t_{\xi_1} \leq t \leq t_{\xi_2}$

Now we have to glue the estimates from Lemmas 3.2.6 and 3.3.6. We have the following corollary:

Corollary 3.3.9. *The following estimates hold for all* $t \in [t_{\xi_1}, t_{\xi_2}]$:

$$\begin{aligned}(|E(t,0,\xi)|) \;\lesssim\;& \exp\left(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix} a(t)|\xi| & a(t)|\xi| \\ \frac{a^2(t)|\xi|^2}{b(t)} & \frac{a^2(t)|\xi|^2}{b(t)} \end{pmatrix} \\ &+\exp\left(-\int_{t_{\xi_1}}^t b(\tau)d\tau\right)a(t_{\xi_1})|\xi|\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.\end{aligned}$$

Proof. The fundamental solution $E(t,0,\xi)$ can be represented as $E(t,t_{\xi_1},\xi)E(t_{\xi_1},0,\xi)$. This yields for all $(t,\xi) \in Z_{ell}(c_0,\varepsilon)$

$$\begin{aligned}
(|E(t,0,\xi)|) &\lesssim (|E(t,t_{\xi_1},\xi)|)(|E(t_{\xi_1},0,\xi)|)\\
&\lesssim \left(\exp\left(-|\xi|^2\int_{t_{\xi_1}}^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix} \frac{a(t)}{a(t_{\xi_1})} & \frac{a(t)|\xi|}{b(t_{\xi_1})} \\ \frac{a^2(t)|\xi|}{a(t_{\xi_1})b(t)} & \frac{a^2(t)|\xi|^2}{b(t_{\xi_1})b(t)} \end{pmatrix} + \frac{\lambda^2(t_{\xi_1})}{\lambda^2(t)}\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right)\\
&\quad\times \frac{a(t_{\xi_1})}{A(t_{\xi_1})}\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}\\
&\lesssim \exp\left(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix} a(t)|\xi| & a(t)|\xi| \\ \frac{a^2(t)|\xi|^2}{b(t)} & \frac{a^2(t)|\xi|^2}{b(t)}\end{pmatrix}\\
&\quad + \exp\left(-\int_{t_{\xi_1}}^t b(\tau)d\tau\right) a(t_{\xi_1})|\xi| \begin{pmatrix} 0 & 0\\ 1 & 1\end{pmatrix}.
\end{aligned}$$

Here we used $a(t_{\xi_1})|\xi| \lesssim b(t_{\xi_1})$, $|\xi| \sim \dfrac{c_0}{A(t_{\xi_1})}$ together with the third statement from Lemma 3.3.7 to extend the above integral. This completes the proof. □

Case 1.3: $t_{\xi_2} \le t \le t_{\xi_3}$

Now we will glue the estimates from Lemma 3.3.6 and Corollary 3.3.9.

Corollary 3.3.10. *The following estimate holds for all* $t \in [t_{\xi_2}, t_{\xi_3}]$:

$$(|E(t,0,\xi)|) \lesssim \exp\left(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right) a(t)|\xi| \begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

Proof. From Lemma 3.3.6 and Remark 3.3.5 we have the following estimate:

$$(|E(t,t_{\xi_2},\xi)|) \lesssim \frac{\lambda(t_{\xi_2})}{\lambda(t)} \exp\left(\varepsilon\int_{t_{\xi_2}}^t b(\tau)d\tau\right)\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

Taking account of the representation of the fundamental solution $E(t,0,\xi)$ as $E(t,t_{\xi_2},\xi)E(t_{\xi_2},0,\xi)$ gives after application of Corollary 3.3.9 the following estimate:

$$\begin{aligned}
(|E(t,0,\xi)|) &\lesssim (|E(t,t_{\xi_2},\xi)|)(|E(t_{\xi_2},0,\xi)|)\\
&\lesssim \exp\left(\Big(\varepsilon - \frac{1}{2}\Big)\int_{t_{\xi_2}}^t b(\tau)d\tau\right)\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}\left[\exp\left(-\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau\right) a(t_{\xi_1})|\xi|\begin{pmatrix} 0 & 0\\ 1 & 1\end{pmatrix}\right.\\
&\quad \left. + \exp\left(-|\xi|^2\int_0^{t_{\xi_2}}\frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix} a(t_{\xi_2})|\xi| & a(t_{\xi_2})|\xi| \\ \frac{a^2(t_{\xi_2})|\xi|^2}{b(t_{\xi_2})} & \frac{a^2(t_{\xi_2})|\xi|^2}{b(t_{\xi_2})}\end{pmatrix}\right]\\
&\lesssim \left[\exp\left(\Big(\varepsilon - \frac{1}{2}\Big)\int_{t_{\xi_2}}^t b(\tau)d\tau\right)\exp\left(-|\xi|^2\int_0^{t_{\xi_2}}\frac{a^2(\tau)}{b(\tau)}d\tau\right)\left(a(t_{\xi_2})|\xi| + \frac{a^2(t_{\xi_2})|\xi|^2}{b(t_{\xi_2})}\right)\right.\\
&\quad \left. + \exp\left(\Big(\varepsilon - \frac{1}{2}\Big)\int_{t_{\xi_2}}^t b(\tau)d\tau\right)\exp\left(-\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau\right)a(t_{\xi_1})|\xi|\right]\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.
\end{aligned}$$

From the definition of $Z_{red}(\varepsilon)$ with a sufficiently small ε we have

$$a^2(t)|\xi|^2 \le \Big(\frac{1}{2} - \varepsilon\Big)b^2(t).$$

For $t \leq t_{\xi_2}$ we use

$$a(t)|\xi| \lesssim b(t).$$

Hence, the integral

$$\exp\left(-\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau\right)$$

can be estimated by

$$\exp\left(-|\xi|^2\int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a^2(\tau)}{b(\tau)}d\tau\right),$$

and the last integral can be extended up to $t=0$. Using $t \geq t_{\xi_2}$ and the increasing behavior of a we conclude from the last estimates the desired statement. □

Case 1.4: $t_{\xi_3} \leq t < \infty$

From Lemma 3.3.3 and Remark 3.3.5 we obtain the following statement:

Corollary 3.3.11. *The following estimate holds for all $t \in [t_{\xi_3}, \infty)$:*

$$(|E(t,t_{\xi_3},\xi)|) \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(t_{\xi_3})}}\exp\left(-\frac{1}{2}\int_{t_{\xi_3}}^{t} b(\tau)d\tau\right)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Finally, we have to glue the estimates from Corollaries 3.3.10 and 3.3.11.

Corollary 3.3.12. *The following estimate holds for all $t \in [t_{\xi_3}, \infty)$:*

$$(|E(t,0,\xi)|) \lesssim \exp\left(-|\xi|^2\int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)}d\tau\right)\exp\left(-\frac{1}{2}\int_{t_{\xi_3}}^{t} b(\tau)d\tau\right)\sqrt{a(t)}\sqrt{a(t_{\xi_3})}|\xi|\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Case 2: *the function* $\dfrac{\mu(t)}{A(t)}$ *is monotonously increasing*

The elliptic part lies on the top of the hyperbolic part in this case. For small frequencies the set $\{\xi : |\xi| \leq c_0\}$ lies completely inside the elliptic zone. For this reason we can use the estimates from the elliptic zone and obtain immediately

$$(|E(t,0,\xi)|) \lesssim \exp\left(-|\xi|^2\int_0^{t} \frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix} \frac{a(t)}{a(0)} & \frac{a(t)}{a(0)} \\ \frac{a^2(t)|\xi|}{a(0)b(t)} & \frac{a^2(t)|\xi|}{a(0)b(t)} \end{pmatrix}. \tag{3.3.45}$$

It remains to consider the influence of the elliptic zone, the reduced zone and the hyperbolic zone for large frequencies. We denote by $t_{\xi_k}, k=1,2$, the separating lines between the hyperbolic zone and the reduced zone ($k=1$) and between the reduced zone and the elliptic zone ($k=2$).

Case 2.1: $t \leq t_{\xi_1}$

In this case we conclude directly from Lemma 3.3.3 and Remark 3.3.5

$$(|E(t,0,\xi)|) \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(0)}}\exp\left(-\frac{1}{2}\int_0^{t} b(\tau)d\tau\right)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.3.46}$$

Case 2.2: $t_{\xi_1} \leq t \leq t_{\xi_2}$

In this case we need to glue the estimate in the hyperbolic zone and the estimate in the reduced zone. We have

$$(|E(t,0,\xi)|) \lesssim \exp\left(\left(\varepsilon - \frac{1}{2}\right)\int_{t_{\xi_1}}^{t} b(\tau)d\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)d\tau\right)\frac{\sqrt{a(t_{\xi_1})}}{\sqrt{a(0)}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Case 2.3: $t_{\xi_2} \leq t$

In this case we need to glue the estimate in the elliptic zone and the estimate in the reduced zone. Summarizing yields the following corollary:

Corollary 3.3.13. *The following estimate holds for all* $t \in [t_{\xi_2}, \infty)$:

$$\begin{aligned}(|E(t,0,\xi)|) \quad &\lesssim \quad \exp\left(-|\xi|^2\int_{t_{\xi_2}}^{t}\frac{a^2(\tau)}{b(\tau)}d\tau + \left(\varepsilon - \frac{1}{2}\right)\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)d\tau\right) \\ &\times \quad \frac{\sqrt{a(t_{\xi_1})}}{\sqrt{a(0)}}\begin{pmatrix} \frac{a(t)}{a(t_{\xi_2})} & \frac{a(t)}{a(t_{\xi_2})} \\ \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)} & \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)} \end{pmatrix}.\end{aligned}$$

$L^2 - L^2$ estimates

Theorem 3.3.14. *Assume (A1) to (A3) and (B'1) to (B'3). Then we have the following* $L^2 - L^2$ *estimates*:
For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2} \lesssim a(t)\left(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}}\left(\|u_1\|_{H^1} + \|u_2\|_{L^2}\right).$$

For the "elastic" energy we have

$$\|a(t)\nabla u(t,\cdot)\|_{L^2} \lesssim a(t)\left(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}}\left(\|u_1\|_{H^1} + \|u_2\|_{L^2}\right).$$

Proof. Case 1: *the function* $\dfrac{\mu(t)}{A(t)}$ *is monotonously decreasing*
In the case $t \in [0, t_{\xi_1}]$ we have from Lemma 3.2.6 the estimate

$$|E(t,0,\xi)| \lesssim \frac{a(t)}{A(t)} \lesssim \frac{a(t)}{\sqrt{1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau}}.$$

This follows directly from

$$\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau = \int_0^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \lesssim \int_0^t a(\tau)A(\tau)d\tau \lesssim A^2(t)$$

for large t.
In the case $t \in [t_{\xi_1}, t_{\xi_2}]$ we will estimate separately each row in the estimate from Corollary 3.3.9. Let us consider the first row. It holds

$$a(t)|\xi|\exp\left(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right) \lesssim a(t)\left(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}},$$

therefore we get the desired decay estimate. Using the monotonicity of a for the entries of the second row we can estimate them by those from the first one

$$\frac{a^2(t)}{b(t)}|\xi|^2 = a(t)|\xi|\frac{a(t)|\xi|}{b(t)} \lesssim a(t)|\xi|,$$
$$a(t_{\xi_1})\exp\left(-\int_0^t b(\tau)d\tau\right) \lesssim a(t)\exp\left(-|\xi|^2\int_{t_{\xi_1}}^t \frac{a^2(\tau)}{b(\tau)}d\tau\right).$$

In the last inequality we used the third statement from Lemma 3.3.7.
In the case $t\in[t_{\xi_2},t_{\xi_3}]$ from Corollary 3.3.10 we can estimate like in the case $t\in[t_{\xi_1},t_{\xi_2}]$.
To derive the corresponding estimates from Corollary 3.3.12 we have in the case $t\in[t_{\xi_3},\infty)$ to estimate the term

$$S(t,|\xi|) := |\xi|\exp\left(-|\xi|^2\int_0^{t_{\xi_3}}\frac{a^2(\tau)}{b(\tau)}d\tau\right)\exp\left(-\frac{1}{2}\int_{t_{\xi_3}}^t b(\tau)d\tau\right).$$

This term glues phase function from different zones.

Lemma 3.3.15. *For any fixed $t\geq t_{\xi_3}$ the function $S(t,|\xi|)$ can be estimated as follows:*

$$S(t,|\xi|) \leq \max_{\xi\in\mathbb{R}^n}\left\{|\xi|\exp\left(-|\xi|^2\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau\right)\right\}.$$

Proof. To estimate the function $S(t,|\xi|)$ it is important that we will prove that the first partial derivative $\partial_{|\xi|}S(t,|\xi|)$ is negative for $|\xi|$ small. This follows from

$$\begin{aligned}
\partial_{|\xi|}S(t,|\xi|) &= S(t,|\xi|)\left(\frac{1}{|\xi|}-2|\xi|\int_0^{t_{\xi_3}}\frac{a^2(\tau)}{b(\tau)}d\tau-\frac{a^2(t_{\xi_3})|\xi|^2}{b(t_{\xi_3})}d_{|\xi|}t_{\xi_3}+\frac{b(t_{\xi_3})}{2}d_{|\xi|}t_{\xi_3}\right)\\
&< S(t,|\xi|)\left(\frac{1}{|\xi|}+\left(\frac{b(t_{\xi_3})}{2}-\frac{a^2(t_{\xi_3})|\xi|^2}{b(t_{\xi_3})}\right)d_{|\xi|}t_{\xi_3}\right)\\
&< S(t,|\xi|)\left(\frac{1}{|\xi|}+\frac{(1-\varepsilon^2)b(t_{\xi_3})}{4}d_{|\xi|}t_{\xi_3}\right),
\end{aligned}$$

here we have used
$$\frac{a^2(t_{\xi_3})|\xi|^2}{b(t_{\xi_3})}=\frac{(1+\varepsilon^2)b(t_{\xi_3})}{4}.$$

Hence, a sufficiently small ε guarantees $\frac{(1-\varepsilon^2)b(t_{\xi_3})}{4}>0$. Taking account of $d_{|\xi|}t_{\xi_3}<0$, $|d_{|\xi|}t_{\xi_3}|\geq\frac{\mu(t_{\xi_3})}{|\xi|b(t_{\xi_3})}$ (this property comes from the fourth statement of Lemma 3.3.7) and $\mu(t_{\xi_3})\to\infty$ for $|\xi|\to 0$ we have the desired decreasing behavior of the function $S(t,|\xi|)$ in $|\xi|$. Now let us fix $t>0$. Then the function $S(t,|\xi|)$ takes its maximum for $|\tilde{\xi}|$ satisfying $t=t_{\tilde{\xi}_3}$, that is, the second integral vanishes in $S(t,|\xi|)$. This completes the proof. □

Consequently,

$$\begin{aligned}
\sqrt{a(t)}\sqrt{a(t_{\xi_3})}|\xi|S(t,|\xi|) &\leq \sqrt{a(t)}\sqrt{a(t_{\tilde{\xi}_3})}|\tilde{\xi}|S(t_{\tilde{\xi}_3},\tilde{\xi}) = a(t)|\tilde{\xi}|\exp\left(-|\tilde{\xi}|^2\int_0^{t_{\tilde{\xi}_3}}\frac{a^2(\tau)}{b(\tau)}d\tau\right)\\
&\leq \max_{\xi\in\mathbb{R}^n}\left\{a(t)|\xi|\exp\left(-|\xi|^2\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau\right)\right\}\lesssim\frac{a(t)}{\sqrt{1+\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau}}.
\end{aligned}$$

Corollary 3.3.12 and Lemma 3.3.15 yield the following result:

$$|E(t,0,\xi)| \lesssim a(t)\left(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}} \quad \text{for } t\in[t_{\xi_3},\infty).$$

In this way all statements are proved.

Case 2: the function $\dfrac{\mu(t)}{A(t)}$ *is monotonously increasing*

For small frequencies $\{\xi : |\xi| \leq c_0\}$ we can apply the estimate in (3.3.45). Here we use that $\dfrac{A(t)}{\mu(t)}$ is monotonously decreasing. For large frequencies $\{\xi : |\xi| \geq c_0\}$ we consider the estimates from Corollary 3.3.13, that is, we have

$$\exp\left(-|\xi|^2\int_{t_{\xi_2}}^t \frac{a^2(\tau)}{b(\tau)}d\tau + \Big(\varepsilon-\frac{1}{2}\Big)\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)d\tau\right) \lesssim \exp\left(-c_0^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right).$$

Here we use for ε sufficiently small the inequality

$$\Big(\frac{1}{2}-\varepsilon\Big)\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau \geq |\xi|^2\int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a^2(\tau)}{b(\tau)}d\tau.$$

Moreover, the following estimate holds for $c_0 < \dfrac{1}{\sqrt{2}}\dfrac{\mu(0)}{A(0)}$:

$$\frac{b(t)}{2} \geq c_0^2\frac{a^2(t)}{b(t)} \Leftrightarrow \frac{b^2(t)}{2} \geq c_0^2 a^2(t) \Leftrightarrow \frac{1}{2}\frac{\mu^2(t)}{A^2(t)} \geq c_0^2.$$

We can see that the first row in the estimate from Corollary 3.3.13 has its maximum for large t inside $\{\xi : |\xi| \leq c_0\}$. From that the theorem is completely proved. $\square$

Examples. We will give some examples for special coefficients.

Example 3.3.1. Let $a(t) = (1+t)^l$, $b(t) = C(1+t)^k$, $k\in(-1,2l+1)$. Then we have

$$\|\big((1+t)^l\nabla u(t,\cdot), u_t(t,\cdot)\big)\|_{L^2} \lesssim (1+t)^{\frac{k-1}{2}}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big).$$

Example 3.3.2. Let $a(t) = e^t$, $b(t) = e^{\beta t}$, $\beta\in(0,2)$. Then we have

$$\|\big(e^t\nabla u(t,\cdot), u_t(t,\cdot)\big)\|_{L^2} \lesssim e^{\frac{\beta}{2}t}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big).$$

Example 3.3.3. Let $a(t) = e^t e^{e^t}$, $b(t) = e^t e^{\beta e^t}$, $\beta\in(0,2)$. Then we have

$$\|\big(e^t e^{e^t}\nabla u(t,\cdot), u_t(t,\cdot)\big)\|_{L^2} \lesssim e^t e^{\frac{\beta}{2}e^t}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big).$$

3.3.7. Comparison of results

Let us compare some results for the scale-invariant case from Section 2.2.1 with results for the cases of non-effective dissipation from Section 3.2 and of effective dissipation from Section 3.3.

Speed of potential order

We start again with the Cauchy problem

$$u_{tt} - (1+t)^{2l}\Delta u + \frac{\mu(l+1)}{(1+t)}u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x) \tag{3.3.47}$$

for $l > 0$. Using the notations from Section 3.2 we have $\alpha(t) = \frac{l}{l+1}$.

Case 1 : non-effective dissipation $\big(\max\{\rho - \frac{1}{2}, -1\} = \rho - \frac{1}{2}\big)$

With $\mu \neq 1$ we can see that $a(t) = (1+t)^{1+l}$, $b(t) = \dfrac{\mu(l+1)}{(1+t)}$ satisfy all assumptions from Theorem 3.2.1. Otherwise, from the definition of ρ and the condition $\max\{\rho - \frac{1}{2}, -1\} = \rho - \frac{1}{2}$ we obtain $\mu + \frac{l}{l+1} < 2$, i.e., this condition satisfies the condition (C): $\limsup_{t\to\infty}\mu(t) + \alpha(t) < 2$.
Applying Theorem 3.2.1 in the case of non-effective dissipation the asymptotic profile for the kinetic energy $\|u_t(t,\cdot)\|_{L^2}$ and for the "elastic energy" $\|(1+t)^l\nabla u(t,\cdot)\|_{L^2}$ is determined by

$$\frac{\sqrt{a(t)}}{\lambda(t)} = \frac{(1+t)^{\frac{l}{2}}}{e^{\frac{1}{2}\int_0^t \frac{\mu(l+1)}{1+s}ds}} = (1+t)^{\frac{l}{2} - \frac{\mu(l+1)}{2}}.$$

This profile coincides with the profiles from the estimates in Proposition 2.2.1.

Case 2 : effective dissipation $\big(\max\{\rho - \frac{1}{2}, -1\} = -1\big)$

From the definition of ρ we can see that the above condition implies $\mu + \frac{l}{l+1} \geq 2$. Thus, the condition (C) is not satisfied. Applying Theorem 3.3.14 for the case of effective dissipation the asymptotic profile of the kinetic energy $\|u_t(t,\cdot)\|_{L^2}$ and for the the "elastic energy" $\|(1+t)^l\nabla u(t,\cdot)\|_{L^2}$ is determined by

$$a(t)\left(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}} = \frac{(1+t)^l}{\sqrt{1 + \int_0^t \frac{(1+\tau)^{2l+1}}{\mu(l+1)}d\tau}} \sim \frac{1}{1+t}.$$

Due to assumption (B'3) it is not allowed to apply Theorem 3.3.14 directly to the Cauchy problem (3.3.47). But, if we formally do it for $\mu \geq 2 - \frac{l}{l+1}$, then this profile coincides with the profiles from the estimates of Proposition 2.2.1. For the case $\mu = 0$ some $L^p - L^q$ estimates on the conjugate line are proposed in M. Reissig [Rei97].

Speed of exponential order

Now we consider another model case to compare with the general results of Theorem 3.2.1 for non-effective dissipation and of Theorem 3.3.14 for effective dissipation. We devote to the Cauchy problem

$$u_{tt} - e^{2t}\Delta u + \mu u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{3.3.48}$$

Using the notations from Section 3.2 we have $\alpha(t) \equiv 1$, $\rho = -\frac{\mu}{2}$.

Case 1 : non-effective dissipation $\big(\max\{\rho - \frac{1}{2}, -1\} = \rho - \frac{1}{2}\big)$

The assumptions from Theorem 3.2.1 are satisfied for $\mu \neq 1$. Keep in mind that $\rho - \frac{1}{2} > -1 \Leftrightarrow -\frac{\mu}{2} - \frac{1}{2} > -1$, this condition implies $\mu + 1 < 2$, i.e., it satisfies the condition (C): $\limsup_{t\to\infty}\mu(t) + \alpha(t) < 2$.

Applying Theorem 3.2.1 in the case of non-effective dissipations the asymptotic profile for the kinetic energy $\|u_t(t,\cdot)\|_{L^2}$ and for the "elastic energy" $\|e^t\nabla u(t,\cdot)\|_{L^2}$ is determined by

$$\frac{\sqrt{a(t)}}{\lambda(t)} = \frac{e^{\frac{t}{2}}}{e^{\frac{1}{2}\int_0^t \mu ds}} = e^{\frac{t}{2}-\frac{\mu t}{2}}.$$

This profile coincides with the profiles from the estimates from Proposition 2.2.2.

Case 2 : *effective dissipation* $\left(\max\{\rho - \frac{1}{2}, -1\} = -1\right)$

From the definition of ρ we can see that the above condition implies $\mu+1 \geq 2$. Hence, this condition does not satisfy the condition (C). Applying Theorem 3.3.14 in the case of effective dissipations the asymptotic profile of the kinetic energy $\|u_t(t,\cdot)\|_{L^2}$ and of the "elastic energy" $\|e^t\nabla u(t,\cdot)\|_{L^2}$ is determined by

$$a(t)\left(1+\int_0^t \frac{a^2(s)}{b(s)}ds\right)^{-\frac{1}{2}} = \frac{e^t}{\sqrt{1+\int_0^t \frac{e^{2s}}{\mu}ds}} \sim C.$$

Due to assumption (B'3) it is not allowed to apply Theorem 3.3.14 to the Cauchy problem (3.3.48). But if we formally do it for $\mu \geq 1$, then this profile coincides with the profiles from the estimates of Proposition 2.2.2. For the case $\mu = 0$ some $L^p - L^q$ estimates on the conjugate line are proposed in A. Galstian [Gal03].

3.4. Over-damping

We consider now "large" coefficients $b = b(t)$ in the damping term. For this reason we may assume

$$\textbf{(OD)} \qquad \int_0^\infty \frac{a^2(\tau)}{b(\tau)}d\tau < \infty.$$

Then the formal application of Theorem 3.3.14 implies among other things

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

The following result shows that no more can be expected in this case of so-called over-damping.

Theorem 3.4.1. *Assume (A1) to (A3), (B'1) to (B'3) and (OD). Then for* $(u_1,u_2) \in L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$ *the limit*

$$u(\infty,x) = \lim_{t\to\infty} u(t,x)$$

exists in $L^2(\mathbb{R}^n)$ *and is different from zero for non-zero data. Furthermore, under the regularity assumption* $(u_1,u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ *it holds*

$$\|u(t,\cdot) - u(\infty,\cdot)\|_{L^2} = \mathcal{O}\left(\int_t^\infty \frac{a^2(\tau)}{b(\tau)}d\tau\right).$$

Proof. The proof is based on the representation of solutions which has been introduced in the hyperbolic region and elliptic region together with the following statement.

Lemma 3.4.2. *Let us assume the conditions (B'1), (B'2) (B'3) and (OD). Then the limit*

$$S(s,\xi) = (1,0)^T \lim_{t\to\infty} \frac{1}{\lambda(t)\langle\xi\rangle_{b(t)}} E_V(t,s,\xi)$$

exists uniformly on compact sets in ξ *and it is different from zero.*

Proof. The goal of this lemma is to extract $\hat{u}(t,\xi)$ from $V = (\langle\xi\rangle_{b(t)}\lambda(t)\hat{u}, D_t(\lambda(t)\hat{u}))^T$. From the over-damping condition we can conclude that $\dfrac{\mu(t)}{A(t)}$ is a monotonous increasing function because we assume monotonicity of this function. Thus, it suffices to consider the elliptic region of the extended phase space and we can use the representation of the fundamental solution in this region. We have from Lemma 3.3.5 after backward transformation

$$\frac{1}{\lambda(t)\langle\xi\rangle_{b(t)}}E_V(t,s,\xi) \sim \frac{1}{\lambda(s)\langle\xi\rangle_{b(s)}}\exp\left(\int_s^t\Big(\langle\xi\rangle_{b(\tau)} - \frac{b(\tau)}{2}\Big)d\tau\right)Q_{ell,0}(t,s,\xi).$$

We see that the exponential term converges uniformly on compact sets in ξ as $t\to\infty$. Moreover, $Q_{ell,0}(t,s,\xi)$ converges to $Q_{ell,0}(\infty,s,\xi)$ for t to ∞, and the (11)-entry of this matrix is non-zero. Therefore, at least the first element of the row $S(s,\xi)$ is non-zero for large s. If we note the relation $S(s,\xi) = S(s_1,\xi)E_V(s_1,s,\xi)$, then from the invertibility of $E_V(s_1,s,\xi)$ we can conclude that $S(s_1,\xi)$ can never be zero for any choice of s_1 and ξ. □

In the case $s=0$ the multiplier $S(0,\xi)$ takes the Cauchy data in the form

$$V(0,\xi) := \Big(\langle\xi\rangle_{b(0)}\hat{u}_1, \hat{u}_2 - i\frac{1}{2}b(0)\hat{u}_1\Big)^T$$

and maps it to the asymptotic state $\hat{u}(\infty,\xi)$, that is,

$$\hat{u}(\infty,\xi) = S(0,\xi)V(0,\xi). \tag{3.4.1}$$

The convergence follows at least for data having compact support on the Fourier level, therefore on a dense subset of L^2 space. Together with an a-priori bound of the solution we can conclude that the limit exists for all data from $L^2(\mathbb{R}^n)\times H^{-1}(\mathbb{R}^n)$. This a-priori bound can be obtained in a similar way we have proven the L^2-L^2 estimate for energy solutions.
Now let us assume the regularity assumption on the data $(u_1,u_2)\in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. From formula (3.3.45) by taking out $|\xi|$ from the last matrix (this needs the better regularity of the data) we get $\|\hat{u}_t(t,\cdot)\|_{L^2} \le \mathcal{O}\left(\dfrac{a^2(t)}{b(t)}\right)$.

Lemma 3.4.3. *The second time derivative of $\hat{u}$ satisfies the following estimate*:

$$\|\hat{u}_{tt}(t,\cdot)\|_{L^2} \lesssim \Big(\frac{a^2(t)}{\mu(t)} + \frac{a(t)\sqrt{a(t)}}{\lambda(t)}\Big)\big(\|u_1\|_{H^2} + \|u_2\|_{H^1}\big). \tag{3.4.2}$$

Proof. In order to separate the extended phase space we will use a smooth cut-off function $\psi\in C^\infty(\mathbb{R}_+)$ such that $\psi(r)=1$ for $r\le 1/2$, $\psi(r)=0$ for $r\ge 1$ and $\psi'(r)\le 0$. Then we define functions ψ_1, ψ_2 and ψ_3 as follows:

$$\begin{aligned}
\psi_1(\xi) &= \psi\Big(\frac{|\xi|}{c_0}\Big),\\
\psi_2(t,\xi) &= \left(1-\psi\Big(\frac{|\xi|}{c_0}\Big)\right)\psi\Big(\frac{A(t)|\xi|}{\mu(t)}\frac{2}{\sqrt{1-\varepsilon^2}}\Big),\\
\psi_3(t,\xi) &= \left(1-\psi\Big(\frac{|\xi|}{c_0}\Big)\right)\left(1-\psi\Big(\frac{A(t)|\xi|}{\mu(t)}\frac{2}{\sqrt{1-\varepsilon^2}}\Big)\right)
\end{aligned}$$

such that $\psi_1(\xi)+\psi_2(t,\xi)+\psi_3(t,\xi)=1$.
Firstly, we devote to the elliptic zone. In order to prove the desired estimate for $\hat{u}_{tt}$ we carry out one derivative with respect to t in the equation

$$\Phi^k_{tt} + a^2(t)|\xi|^2\Phi^k + b(t)\Phi^k_t = 0$$

with initial values $\Phi^k(s,s,\xi)=\delta_{1k},\ \partial_t\Phi^k(s,s,\xi)=\delta_{2k},\ k=1,2$. It yields

$$\partial_t\Phi^k_{tt}+b(t)\Phi^k_{tt}=-2a(t)a'(t)|\xi|^2\Phi^k-a^2(t)|\xi|^2\Phi^k_t-b'(t)\Phi^k_t.$$

Applying Duhamel's principle we get

$$\Phi^k_{tt}(t,s,\xi)=\frac{\lambda^2(s)}{\lambda^2(t)}\Phi^k_{tt}(s,s,\xi)-\int_s^t\frac{\lambda^2(\tau)}{\lambda^2(t)}\Big(2a(\tau)a'(\tau)|\xi|^2\Phi^k(\tau,s,\xi)+(b'(\tau)+a^2(\tau)|\xi|^2)\Phi^k_\tau(\tau,s,\xi)\Big)d\tau.$$

Using estimates (3.3.36), (3.3.37), (3.3.40) and (3.3.42) it holds

$$\begin{aligned}|\Phi^k_{tt}(t,s,\xi)| \lesssim\ & \frac{\lambda^2(s)}{\lambda^2(t)}\Phi^k_{tt}(s,s,\xi)+\underbrace{\frac{1}{\lambda^2(t)}\int_s^t\lambda^2(\tau)a(\tau)a'(\tau)\frac{|\xi|^2}{b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau}_{(A)}\\
&+\underbrace{\frac{1}{\lambda^2(t)}\int_s^t\lambda^2(\tau)|b'(\tau)|\frac{a^2(\tau)|\xi|^2}{b(\tau)b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau}_{(B)}\\
&+\underbrace{\frac{1}{\lambda^2(t)}\int_s^t\lambda^2(\tau)\frac{a^4(\tau)|\xi|^4}{b(\tau)b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau}_{(C)}.\end{aligned}$$

We notice that $\Phi^1_{tt}(s,s,\xi)=-a^2(s)|\xi|^2$ and $\Phi^2_{tt}(s,s,\xi)=-b(s)$.
Applying the assumptions $(A1)$ and $(A2)$ for $a(t)$ we have

$$\begin{aligned}(A)\ &\lesssim \frac{a^3(t)|\xi|^2}{\lambda^2(t)b(s)^{k-1}}\int_s^t\underbrace{b(\tau)\lambda^2(\tau)}_{\partial_\tau\lambda^2(\tau)}\frac{1}{A(\tau)b(\tau)}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau\\
&\lesssim \frac{a^3(t)|\xi|^2}{\lambda^2(t)b(s)^{k-1}}\Big(\lambda^2(\tau)\frac{1}{b(\tau)A(\tau)}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)\Big)\Big|_s^t\\
&\quad+\frac{a^3(t)|\xi|^2}{\lambda^2(t)b(s)^{k-1}}\int_s^t\lambda^2(\tau)\underbrace{\Big(\frac{|\xi|^2a^2}{b^2A}+\frac{|b'|}{b^2A}+\frac{a}{bA^2}\Big)}_{\leq(1/2+\varepsilon)A^{-1}(\tau)}\exp\Big(-|\xi|^2\int_s^\tau\frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau\\
&\lesssim \frac{a^3(t)|\xi|^2}{b(t)b(s)^{k-1}A(t)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)-\frac{a^3(t)|\xi|^2}{b(s)^kA(s)}\frac{\lambda^2(s)}{\lambda^2(t)}.\end{aligned}$$

Here we have used $a^2(t)|\xi|^2/b^2(t)\leq 1/2$ (from the definition of the elliptic zone), $\dfrac{|b'(t)|}{b^2(t)}\lesssim\dfrac{1}{\mu(t)}=o(1)$ and $\mu(t)\to\infty$ as $t\to\infty$. We see also that the second summand is subordinate to the first one because

$$\frac{b(s)}{b(t)}\frac{A(s)}{A(t)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\frac{\lambda^2(t)}{\lambda^2(s)}=\exp\Big(\int_s^t\underbrace{\Big(b(\tau)-\frac{a^2(\tau)|\xi|^2}{b(\tau)}-\frac{b'(\tau)}{b(\tau)}-\frac{a(\tau)}{A(\tau)}\Big)}_{>0,\ \text{if}\,\tau\geq t_0}d\tau\Big)$$

for $t_0\leq s\leq t$ with t_0 sufficiently large. Thus, we get

$$(A)\lesssim\frac{a^3(t)|\xi|^2}{b(t)b(s)^{k-1}A(t)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)=\frac{a^2(t)|\xi|^2}{\mu(t)b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big).\quad(3.4.3)$$

Similarly, we can estimate (B) as follows:

$$\begin{aligned}
(B) &\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)b(s)^{k-1}}\int_s^t \lambda^2(\tau)\frac{|b'(\tau)|}{b(\tau)}\exp\Big(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\Big)\,d\tau\\
&\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)b(s)^{k-1}}\int_s^t \underbrace{b(\tau)\lambda^2(\tau)}_{\partial_\tau\lambda^2(\tau)}\frac{1}{\mu(\tau)}\exp\Big(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\Big)\,d\tau\\
&\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)b(s)^{k-1}}\Big(\lambda^2(\tau)\frac{1}{\mu(\tau)}\exp\Big(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\Big)\Big)\Big|_s^t\\
&\quad+\frac{a^2(t)|\xi|^2}{\lambda^2(t)b(s)^{k-1}}\int_s^t \lambda^2(\tau)\underbrace{\Big(\frac{|\xi|^2a^2(\tau)}{b(\tau)\mu(\tau)}+\frac{|\mu'(\tau)|}{\mu^2(\tau)}\Big)}_{\le(1/2+\varepsilon)b(\tau)/\mu(\tau)}\exp\Big(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\Big)\,d\tau\\
&\lesssim \frac{a^2(t)|\xi|^2}{\mu(t)b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)-\frac{a^2(t)|\xi|^2}{\mu(s)b(s)^{k-1}}\frac{\lambda^2(s)}{\lambda^2(t)}.
\end{aligned}$$

Here we have also used $a^2(t)|\xi|^2/b^2(t)\le 1/2$, $\frac{|b'(t)|}{b^2(t)}\lesssim\frac{1}{\mu(t)}=o(1)$, $|\mu'(t)|\lesssim\mu(t)\frac{a(t)}{A(t)}$ and $\mu(t)\to\infty$ as $t\to\infty$. Obviously we see that the second summand is subordinate to the first one because

$$\frac{\mu(s)}{\mu(t)}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\frac{\lambda^2(t)}{\lambda^2(s)}=\exp\Big(\int_s^t\underbrace{\Big(b(\tau)-\frac{a^2(\tau)|\xi|^2}{b(\tau)}-\frac{\mu'(\tau)}{\mu(\tau)}\Big)}_{>0,\ \text{if}\,\tau\ge t_0}d\tau\Big)$$

for $t_0\le s\le t$ with t_0 sufficiently large. Thus, we get

$$(B)\lesssim\frac{a^2(t)|\xi|^2}{\mu(t)b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big). \tag{3.4.4}$$

Analogously, it is not to difficult to get the estimate for (C) as follows:

$$(C)\lesssim\frac{a^4(t)|\xi|^4}{b^2(t)b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big). \tag{3.4.5}$$

From the estimates (3.4.3), (3.4.4) and (3.4.5) we obtain

$$|\Phi_{tt}^k(t,s,\xi)|\lesssim\Big(\frac{a^2(t)|\xi|^2}{\mu(t)}+\frac{a^4(t)|\xi|^4}{b^2(t)}\Big)\frac{1}{b(s)^{k-1}}\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big),\ k=1,2, \tag{3.4.6}$$

for all (s,ξ) and $(t,\xi)\in Z_{ell}(0,\varepsilon)$.
Now we devote to the hyperbolic zone. Using the result from Remark 3.3.2 we get

$$D_tV(t,\xi)=D_t\Big(MN_1(t,\xi)E_0(t,0,\xi)Q(t,0,\xi)N_1^{-1}(0,\xi)M^{-1}V(0,\xi)\Big),$$

with $V=(\langle\xi\rangle_{b(t)}\hat{v},D_t\hat{v})^T$. We keep in mind that

$$N_1(t,\xi)\in S_1\{0,0,0\}\Rightarrow D_tN_1(t,\xi)\in S_0\{0,0,1\},$$

$$\begin{aligned}
D_tQ(t,0,\xi) &= \mathcal{R}_2(t,0,\xi)Q(t,0,\xi)\\
\Rightarrow|D_tQ(t,0,\xi)| &= |\mathcal{R}_2(t,0,\xi)||Q(t,0,\xi)|=|\mathcal{R}_1(t,\xi)|\lesssim\frac{a(t)}{A(t)},\\
D_tE_0(t,0,\xi) &= (\underbrace{\mathcal{D}(t,\xi)}_{\in S_2\{1,0,0\}}+\underbrace{F_0(t,\xi)}_{\in S_1\{0,0,1\}})E_0(t,0,\xi)\\
\Rightarrow|D_tE_0(t,0,\xi)| &\lesssim \sqrt{a(t)}\langle\xi\rangle_{b(t)}\sim\sqrt{a(t)}a(t)|\xi|.
\end{aligned}$$

Thus

$$\|v_{tt}(t,\cdot)\|_{L^2} \lesssim \sqrt{a(t)}a(t)\big(\|v_0\|_{H^2} + \|v_1\|_{H^1}\big). \tag{3.4.7}$$

Now we transform back to $u_{tt}(t,x)$. Using

$$u_{tt} = \frac{v_{tt} - b'/2v + b^2/4v - bv_t}{\lambda(t)}$$

and (3.4.7) we get

$$\left\| \left(1 - \psi\Big(\frac{A(t)\|\xi|}{\mu(t)} \frac{2}{\sqrt{1-\varepsilon^2}}\Big)\right) \hat{u}_{tt}(t,\cdot) \right\|_{L^2} \lesssim \frac{\sqrt{a(t)}a(t)}{\lambda(t)} \big(\|u_0\|_{H^2} + \|u_1\|_{H^1}\big). \tag{3.4.8}$$

Now we shall derive the final estimate (3.4.2).

Small frequencies: $|\xi| \leq c_0$

In this case this part of the extended phase space lies completely inside the elliptic zone. For this reason, it follows from (3.4.6) that

$$\begin{aligned} \|\psi_1(\xi)\hat{u}_{tt}(t,\cdot)\|_{L^2} &\lesssim \left(\frac{a^2(t)}{\mu(t)} + \frac{a^4(t)}{b^2(t)}\right) \big(\|u_0\|_{L^2} + \|u_1\|_{L^2}\big) \\ &\lesssim \frac{a^2(t)}{\mu(t)} \big(\|u_0\|_{L^2} + \|u_1\|_{L^2}\big). \end{aligned} \tag{3.4.9}$$

Here, we use the following proposition:

Proposition 3.4.4. *Assume (B'1) and (OD). Then it holds*

$$\mu(t) \gtrsim A^2(t). \tag{3.4.10}$$

Proof. Due to Assumption (OD) we see that, at least, $A(t)/\mu(t)$ is decreasing and for large t we obtain

$$C \geq \int_0^t \frac{a(\tau)A(\tau)}{\mu(\tau)} d\tau \geq \frac{A(t)}{\mu(t)} \int_0^t a(\tau) d\tau \geq \frac{A(t)}{\mu(t)} (A(t) - A(0)) \gtrsim \frac{A^2(t)}{\mu(t)} \Rightarrow \mu(t) \gtrsim A^2(t).$$

This we wanted to prove. □

Large frequencies : $|\xi| \geq c_0$

Actually, according to our calculations in the reduced zone this zone does not influence our desired estimates. Thus, we can glue this zone to the hyperbolic zone and call the new zone *the hyperbolic part.* Let t_ξ be the separating line between the hyperbolic part and the elliptic zone.

Case 1: $t \leq t_\xi$

In this case we use directly the result (3.4.8).

Case 2: $t \geq t_\xi$

In this case the elliptic zone lies on the top of the hyperbolic part. For this reason we have

$$\hat{u}_{tt}(t,\xi) = \underbrace{\Phi^1_{tt}(t,t_\xi,\xi)\hat{u}(t_\xi,\xi)}_{(1)} + \underbrace{\Phi^2_{tt}(t,t_\xi,\xi)\hat{u}_t(t_\xi,\xi)}_{(2)}.$$

Using the estimates (3.4.6) for $\Phi^k_{tt}(t,t_\xi,\xi)$ and the estimates (3.3.46) for $\hat{u}(t_\xi,\xi), \hat{u}_t(t_\xi,\xi)$ we obtain

$$\begin{aligned}
(1) &\lesssim \Big(\frac{a^2(t)|\xi|^2}{\mu(t)} + \frac{a^4(t)|\xi|^4}{b^2(t)}\Big) \exp\Big(-|\xi|^2\int_{t_\xi}^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\Big(\frac{|\hat{u_0}(\xi)|}{\sqrt{a(t_\xi)}\lambda(t_\xi)} + \frac{|\hat{u_1}(\xi)|}{\sqrt{a(t_\xi)}|\xi|\lambda(t_\xi)}\Big)\\
&\lesssim \Big(\frac{a^2(t)}{\mu(t)} + \frac{a^4(t)|\xi|^2}{b^2(t)}\Big) \exp\Big(-|\xi|^2\int_{t_\xi}^t \frac{a^2(\tau)}{b(\tau)}d\tau - \frac{1}{2}\int_0^{t_\xi}\Big(\frac{a(\tau)}{A(\tau)} + b(\tau)\Big)d\tau\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\big(|\xi|^2|\hat{u_0}(\xi)| + |\xi||\hat{u_1}(\xi)|\big).
\end{aligned}$$

This implies

$$\begin{aligned}
\Big\|F^{-1}\Big(\psi_2(t,\xi)\Phi^1_{tt}(t,t_\xi,\xi)\hat{u}(t_\xi,\xi)\Big)\Big\| &\lesssim \Big(\frac{a^2(t)}{\mu(t)} + \frac{a^4(t)}{b^2(t)}\Big(\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-1}\Big)\big(\|u_0\|_{H^2} + \|u_1\|_{H^1}\big)\\
&\lesssim \frac{a^2(t)}{\mu(t)}\big(\|u_0\|_{H^2} + \|u_1\|_{H^1}\big).
\end{aligned}$$

Here, we used the following proposition:

Proposition 3.4.5. *Assume (B'1) and (OD). Then it holds*

$$\frac{b(t)}{a(t)}\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau \gtrsim A(t). \tag{3.4.11}$$

Proof. We have

$$\frac{b(t)}{a(t)}\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau = \frac{\mu(t)}{A(t)}\int_0^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \geq \frac{\mu(t)}{A(t)}\frac{A(t)}{\mu(t)}(A(t) - A(0)) \gtrsim A(t),$$

the statement is proved. □

Analogously, we can prove the following estimate for (2):

$$\Big\|F^{-1}\Big(\psi_2(t,\xi)\Phi^2_{tt}(t,t_\xi,\xi)\hat{u}_t(t_\xi,\xi)\Big)\Big\| \lesssim \frac{a^2(t)}{\mu(t)}\big(\|u_0\|_{H^2} + \|u_1\|_{H^1}\big).$$

Summarizing we have

$$\|\psi_2(t,\xi)\hat{u}_{tt}(t,\cdot)\|_{L^2} \lesssim \frac{a^2(t)}{\mu(t)}\big(\|u_0\|_{H^2} + \|u_1\|_{H^1}\big). \tag{3.4.12}$$

From the estimates (3.4.8), (3.4.9) and (3.4.12) we can conclude our desired estimate. This completes the proof. □

From Lemma 3.4.3 we obtain $\frac{1}{a^2(t)}\hat{u}_{tt} \to 0$ in $L^2(\mathbb{R}^n)$ under the (H^2,H^1) regularity for the data. Now we consider the following differential equation

$$\hat{u}_{tt} + a^2(t)|\xi|^2\hat{u} + b(t)\hat{u}_t = 0 \tag{3.4.13}$$

for all t. Taking into consideration the existence of $\hat{u}(t,\xi)$ when $t \to \infty$ we can see that

$$\lim_{t\to\infty} \frac{b(t)}{a^2(t)} u_t(t,x) = -\Delta u(\infty,x) \tag{3.4.14}$$

converges in $L^2(\mathbb{R}^n)$. Furthermore,

$$\|u(\infty,\cdot) - u(t,\cdot)\|_{L^2} \le \int_t^\infty \|u_t(\tau,x)\|_{L^2} d\tau \le C\big(\|u_1\|_{H^2} + \|u_2\|_{H^1}\big) \int_t^\infty \frac{a^2(\tau)}{b(\tau)} d\tau.$$

This completes the proof. $\square$

4. $L^p - L^q$ estimates on the conjugate line

As we did before we will divide our considerations into two cases, the case of non-effective dissipation in Section 4.1 and the case of effective dissipation in Section 4.2.

4.1. The case of non-effective dissipation

In Section 4.1.1 we present the WKB analysis to get representations of solutions by Fourier multipliers. Here we shall use symbol classes with infinite smoothness with respect to the time variable, too. However, in Section 4.1.4 we will also study these Fourier multipliers by using stationary phase method. The principal ideas were introduced by K. Yagdjian, [Yag97], and M. Reissig - K. Yagdjian, [R-Y00a].

4.1.1. Higher diagonalization modulo $S\{-p,-p,p+1\}$

Step 1. We will use again symbol classes which are defined in Definition 2.1.1, but now with infinite smoothness with respect to phase and time variable ($l=\infty$) as well. We will denote them by $S\{m_1,m_2,m_3\}$. The family of symbol classes $S\{m_1,m_2,m_3\}$ satisfies corresponding properties to those from Lemma 2.1.5. Starting the diagonalization procedure we recall the treatment in Section 3.2.2. There we have already performed the first step of diagonalization. By using the new symbol classes we get

$$\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - R_1(t,\xi)\big)U^{(1)}(t,\xi) = 0,$$

where $\mathcal{D} \in S\{1,1,0\}$, $F_0 \in S\{0,0,1\}$, and $R_1 \in S\{0,0,1\}$. This step of diagonalization scheme we call as the *diagonalization mod* $S\{0,0,1\}$. To prove $L^p - L^q$ estimates we need *diagonalization mod* $S\{-p,-p,p+1\}$, where p is a suitable chosen (large) number.

Step $p+1$. To carry out further steps of diagonalization we propose the following new conditions:

(A3)$^\infty$ $|a^{(k)}(t)| \lesssim a(t)\left(\dfrac{a(t)}{A(t)}\right)^k$, $k=1,2,\cdots$,

(B2)$^\infty$ $|\mu^{(k)}(t)| \lesssim \mu(t)\left(\dfrac{a(t)}{A(t)}\right)^k$, $k=1,2,\cdots$.

We have the following lemma:

Lemma 4.1.1. *Assume (A1), (A2), (A3)$^\infty$, (B1), (B2)$^\infty$, (B3) or (B3)'. Then there exist matrix-valued functions*

- $N_p(t,\xi) \in S\{0,0,0\}$ *are invertible for all* $(t,\xi) \in Z_{hyp}(N)$ *and* $N_p^{-1}(t,\xi) \in S\{0,0,0\}$,
- $F_{p-1}(t,\xi) \in S\{0,0,1\}$ *are diagonal with* $F_{p-1}(t,\xi) + i\frac{a'(t)}{2a(t)}I - i\frac{b(t)}{2}I \in S\{-1,-1,2\}$,
- $R_p(t,\xi) \in S\{-p,-p,p+1\}$

such that the following operator-valued identity holds:

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{R}(t,\xi)\big)N_p(t,\xi) = N_p(t,\xi)\big(D_t - \mathcal{D}(t,\xi) - F_{p-1}(t,\xi) - R_p(t,\xi)\big) \tag{4.1.1}$$

for all $(t,\xi) \in Z_{hyp}(N)$.

Proof. We introduce for $N_p(t,\xi)$ and $F_p(t,\xi)$ the following representations:

$$N_p(t,\xi) = \sum_{k=0}^{p} N^{(k)}(t,\xi), \;\; F_p(t,\xi) = \sum_{k=0}^{p} F^{(k)}(t,\xi),$$

where $N^{(0)} = I, B^{(0)} = \mathcal{R}(t,\xi)$ and $F^{(0)} = \operatorname{diag} B^{(0)} = F_0(t,\xi)$. Then we propose the following scheme:

$$\begin{aligned}
&F^{(k)} := \operatorname{diag}(B^{(k)}),\\
&N^{(k+1)} := \begin{pmatrix} 0 & -B_{12}^{(k)}/(2a(t)|\xi|) \\ B_{21}^{(k)}/(2a(t)|\xi|) & 0 \end{pmatrix},\\
&B^{(k+1)} := (D_t - \mathcal{D} - \mathcal{R})N_{p+1} - N_{p+1}(D_t - \mathcal{D} - F_p).
\end{aligned}$$

Now we will prove by induction that $N^{(k)} \in S\{-k,-k,k\}$ and $B^{(k)} \in S\{-k,-k,k+1\}$. For $k=0$ we have already

$$F^{(0)} \in S\{0,0,1\}, \;\; N^{(1)} \in S\{-1,-1,1\}, \;\; B^{(1)} \in S\{-1,-1,2\}.$$

For $k \geq 1$ we apply an inductive argument, we assume $B^{(k)} \in S\{-k,-k,k+1\}$. Thus, by definition of $N^{(k+1)}$ we have $F^{(k)} \in S\{-k,-k,k+1\}$ and $N^{(k+1)} \in S\{-k-1,-k-1,k+1\}$ (from $a(t)|\xi| \in S\{1,1,0\}$). Furthermore,

$$\begin{aligned}
B^{(k+1)} &= (D_t - \mathcal{D} - \mathcal{R})\Big(\sum_{j=0}^{k+1} N^{(j)}\Big) - \Big(\sum_{j=0}^{k+1} N^{(j)}\Big)(D_t - \mathcal{D} - \sum_{j=0}^{k} F^{(j)})\\
&= B^{(k)} + [N^{(k+1)}, \mathcal{D}] - F^{(k)} + D_t N^{(k+1)} + \mathcal{R}N^{(k+1)}\\
&\quad + N^{(k+1)}\sum_{j=0}^{k} F^{(j)} - \Big(\sum_{j=0}^{k+1} N^{(j)}\Big)F^{(k)}.
\end{aligned}$$

Due to the construction scheme $B^{(k)} + [N^{(k+1)}, \mathcal{D}] - F^{(k)} = 0$ for all k, the sum of remaining terms belongs to the symbol class $S\{-k-1,-k-1,k+2\}$. Thus, $B^{(k+1)} \in S\{-k-1,-k-1,k+2\}$. Consequently, the definition of $B^{(p)}$ implies the operator-valued identity

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{R}(t,\xi)\big)N_p(t,\xi) = N_p(t,\xi)\big(D_t - \mathcal{D}(t,\xi) - F_{p-1}(t,\xi) - R_p(t,\xi)\big) \tag{4.1.2}$$

mod $S\{-p,-p,p+1\}$, where we used the notation $R_p(t,\xi) = N_p^{-1}(t,\xi)B^{(p)}(t,\xi) \in S\{-p,-p,p+1\}$. From the construction scheme we have $N^{(k)} \in S\{-k,-k,k\}$. Due to properties of the symbol class $S\{-k,-k,k\}$ we conclude $\left|N_{ij}^{(k)}\right| \leq C_k/N^k$. Hence, for N large enough $\|N_p - I\| \leq 1/2$ in $Z_{hyp}(N)$. This implies the invertibility of N_p. The lemma is proved. $\square$

4.1.2. Some auxiliary estimates

In order to get estimates for the fundamental solution we need the following propositions:

Proposition 4.1.2. *If a function $g = g(|\xi|)$ satisfies for all $\xi \in \mathbb{R}^n$ and all multi-indices α the estimates $\big|d_{|\xi|}^{|\alpha|} g(|\xi|)\big| \lesssim |\xi|^{-|\alpha|}$, then it holds $|D_\xi^\alpha g(|\xi|)| \lesssim |\xi|^{-|\alpha|}$.*

Proof. Applying Faà di Bruno's formula for a multivariate version, see Lemma B.3.6 in section B.3 of Appendix, and performing straight-forward calculations we get

$$\begin{aligned}
\big|D_\xi^\alpha g(|\xi|)\big| &= \Big|\sum_{j=1}^{|\alpha|} \sum_{\substack{\beta_1+\ldots+\beta_j=\alpha \\ |\beta_j|\geq 1}} C_{\beta_1,\ldots,\beta_j} g^{(j)}(|\xi|) \prod_{i=1}^{j} D_\xi^{\beta_i}|\xi|\Big| \\
&\leq \sum_{j=1}^{|\alpha|} \sum_{\substack{\beta_1+\ldots+\beta_j=\alpha \\ |\beta_j|\geq 1}} |C_{\beta_1,\ldots,\beta_j}||g^{(j)}(|\xi|)|\Big|\prod_{i=1}^{j} D_\xi^{\beta_i}|\xi|\Big| \\
&\leq \sum_{j=1}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_j=\alpha} |C_{\beta_1,\ldots,\beta_j}||\xi|^{-j}|\xi|^{-(|\beta_1|+\ldots+|\beta_j|)+j} \leq C_\alpha |\xi|^{-|\alpha|},
\end{aligned}$$

the desired estimate we wanted to prove. □

Due to this proposition, hereinafter we will replace proofs for statements for derivatives in ξ by proofs for statements by derivatives in $|\xi|$.

Proposition 4.1.3. *Assume that $t_{|\xi|}$ is the separated line between the hyperbolic zone and the dissipative zone. Then we have the following estimates:*

$$\big|D_\xi^\alpha t_{|\xi|}\big| \lesssim \frac{1}{a(t_{|\xi|})}|\xi|^{-1-|\alpha|} \tag{4.1.3}$$

for all multi-indices α with $|\alpha| > 0$.

Proof. We will apply the principle of induction. Let us consider the first derivative with respect to $|\xi|$ of $t_{|\xi|}$ that can be obtained directly from the following calculations:

$$A(t_{|\xi|})|\xi| = N \Rightarrow d_{|\xi|}t_{|\xi|}a(t_{|\xi|})|\xi| + A(t_{|\xi|}) = 0 \Rightarrow d_{|\xi|}t_{|\xi|} = -\frac{A(t_{|\xi|})}{a(t_{|\xi|})|\xi|} = -\frac{N}{a(t_{|\xi|})|\xi|^2}. \tag{4.1.4}$$

Hence, we obtain

$$d_{|\xi|}t_{|\xi|} = -\frac{N}{a(t_{|\xi|})|\xi|^2}. \tag{4.1.5}$$

Now, let us assume that

$$|D_{|\xi|}^k t_{|\xi|}| \lesssim \frac{1}{a(t_{|\xi|})}|\xi|^{-1-k} \tag{4.1.6}$$

holds for all $1 \leq k \leq p$. Multiplying (4.1.5) by $a(t_{|\xi|})$ and taking p derivatives with respect to $|\xi|$ gives

$$\begin{aligned}
&d_{|\xi|}^p \left(d_{|\xi|}t_{|\xi|}a(t_{|\xi|})\right) = -d_{|\xi|}^p\left(\frac{N}{|\xi|^2}\right) \Rightarrow \sum_{k=0}^{p} C_p^k d_{|\xi|}^{k+1}t_{|\xi|}d_{|\xi|}^{p-k}a(t_{|\xi|}) = (-1)^{p+1}(p+1)!\frac{N}{|\xi|^{p+2}} \\
&\Rightarrow |d_{|\xi|}^{p+1}t_{|\xi|}a(t_{|\xi|})| \lesssim \frac{1}{|\xi|^{p+2}} + \sum_{k=0}^{p-1}\big|d_{|\xi|}^{k+1}t_{|\xi|}d_{|\xi|}^{p-k}a(t_{|\xi|})\big|.
\end{aligned} \tag{4.1.7}$$

Applying Faà di Bruno's formula (see Appendix: Lemma B.3.5) we have

$$d^n_{|\xi|}a(t_{|\xi|}) = \sum \frac{n!}{m_1!1!^{m_1}m_2!2!^{m_2}\dots m_n!n!^{m_n}}.a^{(m_1+m_2+\dots+m_n)}(t_{|\xi|})\prod_{j=1}^{n}\Big(d^j_{|\xi|}t_{|\xi|}\Big)^{m_j},$$

where the sum is taken over all n-tuples of nonnegative integers $(m_1, m_2, ..., m_n)$ satisfying the constraint

$$1\cdot m_1 + 2\cdot m_2 + \dots + n\cdot m_n = n.$$

The assumption $(A3)^\infty$ together with (4.1.6) yields the estimate

$$\begin{aligned}
|d^n_{|\xi|}a(t_{|\xi|})| &\lesssim \sum a(t_{|\xi|})\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^{m_1+m_2+\dots+m_n}\prod_{j=1}^{n}\left(\frac{1}{a(t_{|\xi|})|\xi|^{j+1}}\right)^{m_j}\\
&\lesssim \sum a(t_{|\xi|})\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^{m_1+m_2+\dots+m_n}\frac{1}{(a(t_{|\xi|})|\xi|)^{m_1+m_2+\dots+m_n}|\xi|^{m_1+2m_2+\dots+nm_n}}\\
&\lesssim \frac{a(t_{|\xi|})}{|\xi|^n}. \qquad (4.1.8)
\end{aligned}$$

Combining (4.1.7) and (4.1.8) we obtain

$$\begin{aligned}
&|d^{p+1}_{|\xi|}t_{|\xi|}a(t_{|\xi|})| \lesssim \frac{1}{|\xi|^{p+2}} + \sum_{k=0}^{p-1}\frac{1}{a(t_{|\xi|})|\xi|^{k+2}}\frac{a(t_{|\xi|})}{|\xi|^{p-k}} \lesssim \frac{1}{|\xi|^{p+2}}\\
&\Rightarrow |d^{p+1}_{|\xi|}t_{|\xi|}| \lesssim \frac{1}{a(t_{|\xi|})|\xi|^{p+2}}.
\end{aligned}$$

This completes the proof. □

Proposition 4.1.4. *Let us assume that $g = g(|\xi|)$ with $\xi \in \mathbb{R}^n \setminus \{0\}$ is an infinitely differentiable function. Then it holds*

$$d^m_{|\xi|}e^{g(|\xi|)} = C(k_1, k_2, \dots, k_j, j)e^{g(|\xi|)}\sum_{j=1}^{m}\sum_{\substack{k_1+\dots+k_j=m\\ |k_j|\geq 1}}\prod_{i=1}^{j}d^{k_i}_{|\xi|}g(|\xi|) \qquad (4.1.9)$$

with $k_i \geq 1$.

Proof. This proposition will be proved by induction with respect to m. For $m = 1$ we have

$$d_{|\xi|}e^{g(|\xi|)} = g'(|\xi|)e^{g(|\xi|)}.$$

Now let us suppose that the equality (4.1.9) is valid for all $m \leq k$ with $k \geq 1$. We shall prove this equality for $m = k + 1$. We carry out straight-forward calculations to obtain

$$\begin{aligned}
d^{k+1}_{|\xi|}e^{g(|\xi|)} &= d_{|\xi|}d^k_{|\xi|}e^{g(|\xi|)} = d_{|\xi|}\Big(C(k_1, k_2, \dots, k_j, j)e^{g(|\xi|)}\sum_{j=1}^{k}\sum_{k_1+\dots+k_j=k}d_{|\xi|}\prod_{i=1}^{j}d^{k_i}_{|\xi|}g(|\xi|)\Big)\\
&= e^{g(|\xi|)}\bar{C}(k_1, k_2, \dots, k_j, j)\Big(\sum_{j=1}^{k}\sum_{k_1+\dots+k_j=k}\Big(g'(|\xi|)\prod_{i=1}^{j}d^{k_i}_{|\xi|}g(|\xi|) + d_{|\xi|}\prod_{i=1}^{j}d^{k_i}_{|\xi|}g(|\xi|)\Big)\Big)\\
&= C'(k_1, k_2, \dots, k_j, j)e^{g(|\xi|)}\sum_{j=1}^{k+1}\sum_{l_1+\dots+l_j=k+1}\prod_{i=1}^{j}d^{l_i}_{|\xi|}g(|\xi|).
\end{aligned}$$

The proof is completed. □

Proposition 4.1.5. *Let us introduce $\tilde{A}(t) = \int_0^t a(\tau)d\tau$. Then the following estimates hold :*

1. $|D_\xi^\alpha\big(\tilde{A}(t_{|\xi|})|\xi|\big)| \le C_\alpha|\xi|^{-|\alpha|}$.
2. $|D_t^k\tilde{A}(t)^m| \le C_{k,m}a(t)^kA(t)^{m-k}$.
3. $|D_t^k e^{\pm i\tilde{A}(t)|\xi|}| \le C_k(a(t)|\xi|)^k$ *in the hyperbolic zone.*

Proof. to 1. In order to prove the first statement we start with the following identity:

$$|D_\xi^\alpha\big(\tilde{A}(t_{|\xi|})|\xi|\big)| = \Big|\sum_{\beta_1+\beta_2=\alpha} C_{\beta_1,\beta_2}D_\xi^{\beta_1}\tilde{A}(t_{|\xi|})D_\xi^{\beta_2}|\xi|\Big|.$$

Thus, we need to estimate $|D_\xi^{\beta_1}\tilde{A}(t_{|\xi|})|$. Moreover, from the definition of $t_{|\xi|}$ we have $\tilde{A}(t_{|\xi|}) = \frac{N}{|\xi|}-1$. It clues $|D_\xi^{\beta_1}\tilde{A}(t_{|\xi|})| \lesssim |\xi|^{-|\beta_1|-1}$. This estimate helps us to conclude the desired inequality.
to 2. We will prove this statement by the induction principle with respect to two parameters k and m. For fixed m and $k=1$ and for fixed k and $m=1$ we can check our statement. Let us assume that it holds for all $k \le p$ with $p \ge 1$ and for all $m \le q$ with $q \ge 1$. We show our statement for $k=p+1, m=q+1$. Indeed,

$$\begin{aligned}\big|D_t^{p+1}\tilde{A}(t)^{q+1}\big| &\lesssim \big|D_t^p\big(a(t)\tilde{A}(t)^q\big)\big| = \Big|\sum_{i+j=p}\tilde{C}_{i,j}D_t^ia(t)D_t^j\tilde{A}(t)^q\Big| \\ &\le C_{p+1,q+1}\sum_{i+j=p}a(t)\Big(\frac{a(t)}{A(t)}\Big)^i a(t)^jA(t)^{q-j} \le C_{p+1,q+1}a(t)^{p+1}A(t)^{q+1-(p+1)}.\end{aligned}$$

This completes the proof of the second statement.
to 3. In order to prove the third statement we apply directly Proposition 4.1.4 with respect to the variable t. We get

$$\begin{aligned}\big|D_t^k e^{\pm i\tilde{A}(t)|\xi|}\big| &\le C(k)\Big|e^{\pm i\tilde{A}(t)|\xi|}\sum_{j=1}^k\sum_{k_1+\cdots+k_j=k}\prod_{i=1}^j D_t^{k_i}(\tilde{A}(t)|\xi|)\Big| \\ &\le \tilde{C}(k)\sum_{j=1}^k\sum_{k_1+\cdots+k_j=k}\prod_{i=1}^j a(t)\Big(\frac{a(t)}{A(t)}\Big)^{k_i-1}|\xi| \le C_ka(t)^k|\xi|^k.\end{aligned}$$

The proof is completed. □

Definition 4.1.1. *The time-dependent function $c(t,\xi)$ belongs to the symbol class $S^{l_1,l_2}\{m_1,m_2,m_3\}$ with restricted smoothness l_1, l_2, if it satisfies the following estimates:*

$$S^{l_1,l_2}\{m_1,m_2,m_3\} = \Big\{c(t,\xi) : |D_\xi^\alpha D_t^k c(t,\xi)| \le C_{\alpha,k}|\xi|^{m_1-|\alpha|}a(t)^{m_2}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k} \text{ in } Z_{hyp}(N) \\ \text{for all } |\alpha| \le l_2 \text{ and } k \le l_1\Big\}.$$

Obviously, it holds

$$S^{l_1,l_2}\{m_1,m_2,m_3\} \subset S^{l_1',l_2'}\{m_1,m_2,m_3\} \qquad \text{for all } l_1' \le l_1, l_2' \le l_2.$$

Using the definition of hyperbolic zone we have

$$S^{l_1,l_2}\{m_1-k,m_2-k,m_3+k\} \subset S^{l_1,l_2}\{m_1,m_2,m_3\} \qquad \text{for all } k \ge 0.$$

This property will be essentially used in the diagonalization scheme. We have also corresponding properties to those ones from Lemma 2.1.5.

Proposition 4.1.6. *The family of symbol classes $S^{l_1,l_2}\{m_1,m_2,m_3\}$ generates a hierarchy of symbol classes having the following properties:*

- $S^{l_1,l_2}\{m_1,m_2,m_3\}$ *is a vector space.*
- $S^{l_1,l_2}\{m_1,m_2,m_3\}S^{l_1,l_2}\{m_1',m_2',m_3'\} \subset S^{l_1,l_2}\{m_1+m_1',m_2+m_2',m_3+m_3'\}$.
- $D_t^k D_\xi^\alpha S^{l_1,l_2}\{m_1,m_2,m_3\} \subset S^{l_1-k,l_2-|\alpha|}\{m_1-|\alpha|,m_2,m_3+k\}$.
- $S^{0,0}\{-1,-1,2\} \subset L_\xi^\infty L_t^1(Z_{hyp}(N))$.

Proposition 4.1.7. *The following relations hold for all m_1,m_2,m_3 :*

1. $e^{\pm i\tilde{A}(t_{|\xi|})|\xi|}S\{m_1,m_2,m_3\} \hookrightarrow S\{m_1,m_2,m_3\}$.
2. $e^{\pm i\tilde{A}(t)|\xi|}S^{l_1,l_2}\{m_1,m_2,m_3\} \hookrightarrow S^{l_1,l_2}\{m_1+l,m_2+l,m_3-l\}$.
   Here $l = l_1+l_2$.

Proof. to 1. In order to prove the first statement we choose $c = c(t,\xi) \in S\{m_1,m_2,m_3\}$. Then it holds

$$\left|D_t^k D_\xi^\alpha\left(e^{\pm i\tilde{A}(t_{|\xi|})|\xi|}c(t,\xi)\right)\right| = \Big|\sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} D_\xi^{\alpha_1} e^{\pm i\tilde{A}(t_{|\xi|})|\xi|} D_t^k D_\xi^{\alpha_2} c(t,\xi)\Big|$$

by the aid of Proposition 4.1.2, it is enough to consider the following estimates:

$$\begin{aligned}
&\Big|\sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} D_{|\xi|}^{|\alpha_1|} e^{\pm i\tilde{A}(t_{|\xi|})|\xi|} D_t^k D_\xi^{\alpha_2} c(t,\xi)\Big| \\
&\le \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2}\Big(C'(\alpha_1)\Big|e^{\pm i\tilde{A}(t_{|\xi|})|\xi|}\sum_{j=1}^{|\alpha_1|}\sum_{l_1+\ldots+l_j=|\alpha_1|}\prod_{i=1}^{j} d_{|\xi|}^{l_i}\big(\tilde{A}(t_{|\xi|})|\xi|\big)\Big|\Big)\big|D_t^k D_\xi^{\alpha_2} c(t,\xi)\big| \\
&\le \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2}\Big(C'(\alpha_1)\sum_{j=1}^{|\alpha_1|}\sum_{l_1+\ldots+l_j=|\alpha_1|}\prod_{i=1}^{j}|\xi|^{-l_i}\Big)\big|D_t^k D_\xi^{\alpha_2} c(t,\xi)\big| \\
&\le \sum_{\alpha_1+\alpha_2=\alpha} \tilde{C}_{\alpha_1,\alpha_2}|\xi|^{-|\alpha_1|}|\xi|^{m_1-|\alpha_2|}a(t)^{m_2}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k} = C_\alpha|\xi|^{m_1-|\alpha|}a(t)^{m_2}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k}.
\end{aligned}$$

In the third line we have used the first statement of Proposition 4.1.5.

to 2. If $c(t,\xi) \in S^{l_1,l_2}\{m_1,m_2,m_3\}$, then it holds

$$\Big|D_t^k D_\xi^\alpha\Big(e^{\pm i\tilde{A}(t)|\xi|}c(t,\xi)\Big)\Big| = \Big|\sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C_{k_1,k_2,\alpha_1,\alpha_2} D_t^{k_1} D_\xi^{\alpha_1}\Big(e^{\pm i\tilde{A}(t)|\xi|}\Big) D_t^{k_2} D_\xi^{\alpha_2} c(t,\xi)\Big|$$

We take account of the following estimates:

$$
\begin{aligned}
&\Big|\sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C_{k_1,k_2,\alpha_1,\alpha_2} D_t^{k_1} D_{|\xi|}^{\alpha_1}\Big(e^{\pm i\tilde{A}(t)|\xi|}\Big) D_t^{k_2} D_\xi^{\alpha_2} c(t,\xi)\Big|\\
&\le \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha}\sum_{i+j=k_1} |C_{k_1,k_2,\alpha_1,\alpha_2,i,j}||a(t)^i A(t)^{|\alpha_1|-i} D_t^j e^{\pm iA(t)|\xi|}||D_t^{k_2} D_\xi^{\alpha_2} c(t,\xi)|\\
&\le \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha}\sum_{i+j=k_1} |C_{k_1,k_2,\alpha_1,\alpha_2,i,j}|a(t)^i A(t)^{|\alpha_1|-i} (a(t)|\xi|)^j \big|D_t^{k_2} D_\xi^{\alpha_2} c(t,\xi)\big|\\
&\le \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C'_{k_1,k_2,\alpha_1,\alpha_2} a(t)^{k_1}|\xi|^{k_1} A(t)^{|\alpha_1|}|\xi|^{m_1-|\alpha_2|} a(t)^{m_2}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k_2}\\
&= \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C'_{k_1,k_2,\alpha_1,\alpha_2} |\xi|^{m_1+k-k_2-|\alpha_2|} a(t)^{m_2+k+|\alpha|-k_2-|\alpha_2|}\Big(\frac{a(t)}{A(t)}\Big)^{m_3-|\alpha|+|\alpha_2|+k_2}\\
&\le C'_{k,\alpha}|\xi|^{m_1+(k+|\alpha|)-|\alpha|} a(t)^{m_2+(k+|\alpha|)}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k-(k+|\alpha|)}\\
&\le C_{k,\alpha}|\xi|^{m_1+l-|\alpha|} a(t)^{m_2+l}\Big(\frac{a(t)}{A(t)}\Big)^{m_3-l+k}.
\end{aligned}
$$

Here we have used the property $A(t)|\xi| \ge N$, it comes from the definition of the hyperbolic zone. In the first line and in the third line the second and the third statement of Proposition 4.1.5 is applied, correspondingly. This completes the proof of this proposition. □

Remark 4.1.1. The first statement of Proposition 4.1.7 tells us that it is allowed to extent the phase function $\pm\int_{t_{|\xi|}}^t a(\tau)d\tau|\xi|$ in $\exp\Big(\pm i\int_{t_{|\xi|}}^t a(\tau)d\tau|\xi|\Big)$ which we use later to get $L^p - L^q$ estimates to the phase function $\pm\int_0^t a(\tau)d\tau|\xi|$ in $\exp\Big(\pm i\int_0^t a(\tau)d\tau|\xi|\Big)$. Here we use that the remainder term $\exp\Big(\pm i\int_0^{t_{|\xi|}} a(\tau)d\tau|\xi|\Big)$ satisfies

$$
\Big|D_{|\xi|}^\alpha e^{\pm i|\xi|\tilde{A}(t_{|\xi|})}\Big| \lesssim |\xi|^{-|\alpha|}. \tag{4.1.10}
$$

Thus, we can put the term $e^{\pm i|\xi|\tilde{A}(t_{|\xi|})}$ into the amplitude.

4.1.3. Construction of a fundamental solution

Now we want to construct the fundamental solution as the solution of the following system

$$
\big(D_t - \mathcal{D}(t,\xi) - F_{p-1}(t,\xi) - R_p(t,\xi)\big)E_p(t,s,\xi) = 0,\ \ E_p(s,s,\xi) = I. \tag{4.1.11}
$$

Let $E_0(t,s,\xi)$ be the fundamental solution to $D_t - \mathcal{D}(t,\xi)$. We have

$$
E_0(t,s,\xi) = \begin{pmatrix} e^{i\int_s^t a(\tau)d\tau|\xi|} & 0\\ 0 & e^{-i\int_s^t a(\tau)d\tau|\xi|}\end{pmatrix}. \tag{4.1.12}
$$

Let us define $\tilde{E}_0(t,s,\xi) = \dfrac{\sqrt{a(t)}}{\lambda(t)}\dfrac{\lambda(s)}{\sqrt{a(s)}}E_0(t,s,\xi)$. We can see that this matrix-valued function is the fundamental solution to

$$
D_t - \mathcal{D}(t,\xi) - F^{(0)}(t,\xi). \tag{4.1.13}
$$

We define

$$\begin{aligned}\mathcal{R}_p(t,s,\xi) &= \tilde{E}_0^{-1}(t,s,\xi)\Big(F_{p-1}(t,\xi)+R_p(t,\xi)-F^{(0)}(t,\xi)\Big)\tilde{E}_0(t,s,\xi)\\ &= F_{p-1}(t,\xi)-F^{(0)}(t,\xi)+E_0^{-1}(t,s,\xi)R_p(t,\xi)E_0(t,s,\xi). \end{aligned}\tag{4.1.14}$$

Here we recall that the matrix-valued function $Q_p(t,s,\xi)$ is the solution to

$$\big(D_t-\mathcal{R}_p(t,s,\xi)\big)Q_p(t,s,\xi)=0,\ Q_p(s,s,\xi)=I. \tag{4.1.15}$$

From the Peano-Baker formula we have that the solution to (4.1.15) can be represented as

$$Q_p(t,s,\xi)=I+\sum_{j=1}^{\infty} i^j\int_s^t \mathcal{R}_p(t_1,s,\xi)\int_s^{t_1}\mathcal{R}_p(t_2,s,\xi)\cdots\int_s^{t_{j-1}}\mathcal{R}_p(t_j,s,\xi)dt_j\dots dt_1. \tag{4.1.16}$$

The matrix $E_p(t,s,\xi)$ can be represented as

$$E_p(t,s,\xi)=\tilde{E}_0(t,s,\xi)Q_p(t,s,\xi)=\frac{\sqrt{a(t)}}{\lambda(t)}\frac{\lambda(s)}{\sqrt{a(s)}}E_0(t,s,\xi)Q_p(t,s,\xi). \tag{4.1.17}$$

Lemma 4.1.8. *The matrix-valued function $\mathcal{R}_p(t,t_{|\xi|},\xi)$ satisfies*

$$\mathcal{R}_p(t,t_{|\xi|},\xi)\in S^{l_1,l_2}\{-p+l,-p+l,p+1-l\}\hookrightarrow S^{l_1,l_2}\{-1,-1,2\} \tag{4.1.18}$$

for all $l_1+l_2=l\le p-1$.

Proof. Due to the representation of $\mathcal{R}_p(t,s,\xi)$ in (4.1.14) we have

$$\mathcal{R}_p(t,t_{|\xi|},\xi)=F_{p-1}(t,\xi)-F^{(0)}(t,\xi)+E_0^{-1}(t,t_{|\xi|},\xi)R_p(t,\xi)E_0(t,t_{|\xi|},\xi).$$

As a result of Lemma 4.1.1 we obtain

$$F_{p-1}(t,\xi)-F^{(0)}(t,\xi)\in S\{-1,-1,2\},$$

thus, now only the remainder term

$$\tilde{R}_p(t,t_{|\xi|},\xi):=E_0^{-1}(t,t_{|\xi|},\xi)R_p(t,\xi)E_0(t,t_{|\xi|},\xi)=\begin{pmatrix} r_{11}^p & r_{12}^p e^{2i\int_{t_{|\xi|}}^t a(\tau)d\tau|\xi|}\\ r_{21}^p e^{-2i\int_{t_{|\xi|}}^t a(\tau)d\tau|\xi|} & r_{22}^p\end{pmatrix}$$

should be considered. Applying Lemma 4.1.1, Proposition 4.1.7 and Remark 4.1.1 we deduce $\tilde{R}_p(t,t_{|\xi|},\xi)\in S^{l_1,l_2}\{-p+l,-p+l,p+1-l\}\hookrightarrow S^{l_1,l_2}\{-1,-1,2\}$ for all $l\le p-1$. □

Lemma 4.1.9. *Assume (A1), (A2), (A3)$^\infty$, (B1), (B2)$^\infty$, (B3) or (B3)'. Then the fundamental solution $E_p(t,s,\xi)$ to (4.1.11) can be represented as*

$$E_p(t,s,\xi)=\frac{\sqrt{a(t)}}{\sqrt{a(s)}}\frac{\lambda(s)}{\lambda(t)}E_0(t,s,\xi)Q_p(t,s,\xi)$$

for all $t,s\ge t_{|\xi|}$ with an amplitude $Q_p(t,s,\xi)$ satisfying the following estimates:

$$\|D_\xi^\alpha Q_p(t,t_{|\xi|},\xi)\|\le C_{p,\alpha,N}|\xi|^{-|\alpha|},\ t\ge t_{|\xi|}$$

for all multi-indices α satisfying $|\alpha|\le p-1$.

Proof. Let us consider the first statement. The Cauchy condition is obviously satisfied. Furthermore, we have

$$\begin{aligned} D_t E_p &= D_t \tilde{E}_0 Q_p + \tilde{E}_0 D_t Q_p = (\mathcal{D} + F^{(0)}) \tilde{E}_0 Q_p + \tilde{E}_0 \mathcal{R}_p Q_p \\ &= (\mathcal{D} + F^{(0)} + F_{p-1} + R_p - F^{(0)}) \tilde{E}_0(t,s,\xi) Q_p \\ &= (\mathcal{D} + F_{p-1} + R_p) E_p. \end{aligned}$$

The first step of proof of the second statement arises from the unitary behavior of $E_0(t,s,\xi)$. It follows

$$\|\mathcal{R}_p(t,s,\xi)\| \lesssim \|(F_{p-1} - F^{(0)})(t,\xi)\| + \|R_p(t,\xi)\|, \text{ where } R_p(t,\xi) \in S\{-p,-p,p+1\} \subset S\{-1,-1,2\}.$$

Applying the fourth statement of Lemma 2.1.5, that is,

$$S\{-1,-1,2\} \subset L^\infty_\xi L^1_t(Z_{hyp}(N)),$$

we obtain

$$\int_{t_{|\xi|}}^t \|\mathcal{R}_p(\tau, t_{|\xi|}, \xi)\| d\tau \le C_N$$

for all $t \ge t_{|\xi|}$. Hence,

$$\|Q_p(t, t_{|\xi|}, \xi)\| \lesssim \exp\Big(\int_{t_{|\xi|}}^t \|\mathcal{R}_p(\tau, s, \xi)\| d\tau \Big) \lesssim 1.$$

Now let us take α derivatives with respect to ξ in the representation formula for $Q_p(t, t_{|\xi|}, \xi)$ in (4.1.16). Then

$$D^\alpha_\xi Q_p(t,t_{|\xi|},\xi) = \sum_{j=1}^\infty i^j D^\alpha_\xi \Big(\int_{t_{|\xi|}}^t \mathcal{R}_p(t_1, t_{|\xi|}, \xi) \int_{t_{|\xi|}}^{t_1} \mathcal{R}_p(t_2, t_{|\xi|}, \xi) \cdots \int_{t_{|\xi|}}^{t_{j-1}} \mathcal{R}_p(t_j, t_{|\xi|}, \xi) dt_j \dots dt_1 \Big).$$

Let us consider terms of the form

$$\int_{t_{|\xi|}}^t \mathcal{D}^{\alpha_1}_\xi \mathcal{R}_p(t_1, t_{|\xi|}, \xi) \int_{t_{|\xi|}}^{t_1} D^{\alpha_2}_\xi \mathcal{R}_p(t_2, t_{|\xi|}, \xi) \cdots \int_{t_{|\xi|}}^{t_{j-1}} D^{\alpha_j}_\xi \mathcal{R}_p(t_j, t_{|\xi|}, \xi) dt_j \dots dt_1$$

with $\sum_{k=1}^j \alpha_k = |\alpha|$. Using Lemma 4.1.8 the norm of these terms can be estimated by

$$\begin{aligned} & C'(\alpha,p,N) \int_{t_{|\xi|}}^t \Big(|\xi|^{-1-|\alpha_1|} a(t_1)^{-1} \Big(\frac{a(t_1)}{A(t_1)}\Big)^2 \Big) \int_{t_{|\xi|}}^{t_1} \Big(|\xi|^{-1-|\alpha_2|} a(t_2)^{-1} \Big(\frac{a(t_2)}{A(t_2)}\Big)^2 \Big) \\ & \times \cdots \times \int_{t_{|\xi|}}^{t_{j-1}} \Big(|\xi|^{-1-|\alpha_j|} a(t_j)^{-1} \Big(\frac{a(t_j)}{A(t_j)}\Big)^2 \Big) dt_j \dots dt_1 \\ & \le C(\alpha,p,N) |\xi|^{-|\alpha|}. \end{aligned}$$

Here the assumption $|\alpha| \le p-1$ guarantees that $\alpha_k \le p-1$, which is necessary to apply Lemma 4.1.8. Accordingly, we only have to care for derivatives of the lower integral bound $t_{|\xi|}$. Then there arise terms as $D^{\alpha-\beta}_\xi \big(\mathcal{R}_p(t_{|\xi|}, t_{|\xi|}, \xi) D^\beta_\xi t_{|\xi|} \big)$ for $|\beta| = 1$ which can be estimated as follows:

$$\begin{aligned} \Big| D^{\alpha-\beta}_\xi \big(\mathcal{R}_p(t_{|\xi|}, t_{|\xi|}, \xi) d_{|\xi|} t_{|\xi|} \big) \Big| &= \Big| \sum_{|\alpha_1|+|\alpha_2|=|\alpha|-1} C_{\alpha_1,\alpha_2} D^{\alpha_1}_\xi \mathcal{R}_p(t_{|\xi|}, t_{|\xi|}, \xi) D^{\alpha_2+\beta}_\xi t_{|\xi|} \Big| \\ \le \quad C(\alpha) & \sum_{|\alpha_1|+|\alpha_2|=|\alpha|-1} |\xi|^{-|\alpha_1|-1} a(t_{|\xi|})^{-1} \Big(\frac{a(t_{|\xi|})}{A(t_{|\xi|})} \Big)^2 a(t_{|\xi|})^{-1} |\xi|^{-|\alpha_2|-2} \le C_{\alpha,N} |\xi|^{-|\alpha|}. \end{aligned}$$

To estimate the terms $\left|D_\xi^{\alpha_2+\beta} t_{|\xi|}\right|$ we have used the result of Proposition 4.1.3. For the other terms $\left|D_\xi^{\alpha_1}\mathcal{R}_p(t_{|\xi|}, t_{|\xi|}, \xi)\right|$ we use the following statement:

If all the necessary derivatives are defined, then we have

$$\left|D_\xi^\alpha \mathcal{R}_p(t_{|\xi|}, t_{|\xi|}, \xi)\right| \lesssim |\xi|^{-1-|\alpha|} a(t_{|\xi|})^{-1}\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^2. \tag{4.1.19}$$

Indeed, due to the representation of $\mathcal{R}_p(t,s,\xi)$ in (4.1.14) we have

$$\mathcal{R}_p(t_{|\xi|}, t_{|\xi|}, \xi) = F_{p-1}(t_{|\xi|}, \xi) - F^{(0)}(t_{|\xi|}, \xi) + R_p(t_{|\xi|}, \xi) := G(t_{|\xi|}, \xi).$$

Applying the generalized version of Faa di Bruno's formula (See Appendix: Lemma B.3.7) we obtain for the case $|\alpha| = n$

$$D_\xi^\alpha G(t_{|\xi|}, \xi) = \sum_0 \sum_1 \cdots \sum_n C(n, k_i, q_{ij}) \frac{\partial^\kappa G}{\partial_{t_{|\xi|}}^{p_1} \partial_\xi^{\alpha_2}}(t_{|\xi|}, \xi) \prod_{i=1,\, |\alpha_i|=i}^{n} \left(D_\xi^{\alpha_i} t_{|\xi|}\right)^{q_{i1}} \left(D_\xi^{\alpha_i}\xi\right)^{q_{i2}}, \tag{4.1.20}$$

where the respective sums are taken over all non-negative integer solutions of the Diophantine equations as follows:

$$\begin{aligned} &\sum_0 \to k_1 + 2k_2 + \cdots + nk_n = n, \\ &\sum_1 \to q_{11} + q_{12} = k_1, \\ &\vdots \\ &\sum_n \to q_{n1} + q_{n2} = k_n, \end{aligned}$$

and

$$\begin{aligned} &p_1 = \sum_{i=1}^n q_{ij},\ |\alpha_2| = \sum_{i=1}^n q_{i2}, \\ &|\kappa| = k_1 + k_2 + \ldots + k_n = p_1 + |\alpha_2|. \end{aligned}$$

By virtue of $\partial_{\xi_k}\xi_l = \delta_{kl}$ we may conclude that $q_{i2} = 0$, for all $i \geq 2$ and $|\alpha_2| = q_{12}$. This yields the estimate

$$\begin{aligned} |D_\xi^\alpha G(t_{|\xi|}, \xi)| &\lesssim \sum_0 \sum_1 \cdots \sum_n |\xi|^{-1-|\alpha_2|} a(t_{|\xi|})^{-1}\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^{2+p_1} \prod_{i=1}^n a(t_{|\xi|})^{-q_{i1}} |\xi|^{-(|\alpha_i|+1)q_{i1}} \\ &= \sum_0 \sum_1 \cdots \sum_n |\xi|^{-1-q_{12}} a(t_{|\xi|})^{-1}\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^{2+p_1} a(t_{|\xi|})^{-p_1} |\xi|^{-p_1 - iq_{i1}} \\ &= \sum_0 \sum_1 \cdots \sum_n |\xi|^{-1-q_{12}} a(t_{|\xi|})^{-1}\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^{2+p_1} a(t_{|\xi|})^{-p_1} |\xi|^{-p_1 - n + q_{12}} \\ &= \sum_0 \sum_1 \cdots \sum_n |\xi|^{-1-q_{12}-p_1+q_{12}-n} a(t_{|\xi|})^{-1-p_1}\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^{2+p_1} \\ &\lesssim |\xi|^{-1-n} a(t_{|\xi|})^{-1}\left(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}\right)^2. \end{aligned}$$

Therefore $\|D_\xi^\alpha Q_p(t, t_{|\xi|}, \xi)\|$ can be estimated by $C_{\alpha,p,N}|\xi|^{-|\alpha|}$, where we use $C_{\alpha,p,N}$ as a universal constant depending on α, p, N. This completes the proof of this lemma. $\square$

Remark 4.1.2. From Lemma 4.1.1 and Lemma 4.1.9 we have the following representation of the micro-energy $U(t,\xi) = (a(t)|\xi|\hat{u}, D_t\hat{u})^T$:

$$U(t,\xi) = \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\frac{\lambda(s)}{\lambda(t)} M N_p(t,\xi) E_0(t,s,\xi) Q_p(t,s,\xi) N_p^{-1}(s,\xi) M^{-1} U(s,\xi).$$

4.1.4. $L^p - L^q$ estimates

Theorem 4.1.10. *If the conditions (A1), (A2), (A3)$^\infty$, (B1), (B2)$^\infty$, (B3) or (B3)' and (C) hold, then we have the following $L^p - L^q$ estimates for the kinetic and the "elastic" energy:*

$$\|u_t(t,\cdot), a(t)\nabla u(t,\cdot)\|_{L_q} \lesssim \frac{1}{\lambda(t)}\sqrt{a(t)} A(t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_1\|_{L^{p,r+1}} + \|u_2\|_{L^{p,r}}\right)$$

with regularity $r = n\left(\frac{1}{p} - \frac{1}{q}\right)$, $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u_0(\cdot), u_1(\cdot) \in S(\mathbb{R}^n)$. Here $A(t) = 1 + \int_0^t a(\tau)d\tau$.

Proof. The strategy of proof is based on the decomposition of the extended phase space into the following three parts:

1. the dissipative part: $Z_{diss} := \{(t,\xi) : A(t)|\xi| \le N\}$,
2. the hyperbolic part with small frequencies: $Z_{hyp} := \{(t,\xi) : A(t)|\xi| \ge \frac{N}{2} \cap |\xi| \le N\}$,
3. the hyperbolic part with large frequencies: $Z_{hyp} := \{(t,\xi) : |\xi| \ge \frac{N}{2}\}$.

In order to separate the extended phase space we will use a smooth cut-off function $\psi \in C^\infty(\mathbb{R}_+)$ such that $\psi(r) = 1$ for $r \le 1/2$, $\psi(r) = 0$ for $r \ge 1$ and $\psi'(r) \le 0$. Then we define functions ψ_1, ψ_2 and ψ_3 as follows:

$$\begin{aligned}
\psi_1(t,\xi) &= \psi\left(\frac{|\xi|}{N}\right)\psi\left(\frac{A(t)|\xi|}{N}\right), \\
\psi_2(t,\xi) &= \psi\left(\frac{|\xi|}{N}\right)\left(1 - \psi\left(\frac{A(t)|\xi|}{N}\right)\right), \\
\psi_3(\xi) &= 1 - \psi\left(\frac{|\xi|}{N}\right)
\end{aligned}$$

such that $\psi_1(t,\xi) + \psi_2(t,\xi) + \psi_3(\xi) = 1$. Thus this decomposition corresponds to the definition of the three parts which we have introduced before.

Estimates for the dissipative part

Let us come back to the micro-energy $U(t,\xi) = \left(\frac{a(t)}{A(t)}\hat{u}, D_t\hat{u}\right)$ in the dissipative zone which can be represented by

$$U(t,\xi) = \psi_1(t,\xi) E(t,0,\xi) U(0,\xi).$$

Thus, it is reasonable to consider estimates for

$$\left\|F^{-1}\Big(\psi_1(t,\xi) E_{k,l}(t,0,\xi) F(v)\Big)\right\|_{L^q},$$

where $v \in S$, $E_{k,l}(t,0,\xi)$, $k=1,2$, $l=1,2$ are the entries of the fundamental solution $E(t,0,\xi)$. It holds

$$\begin{aligned}&\|F^{-1}\left(\psi_1(t,\xi)E_{k,l}(t,0,\xi)F(v)\right)\|_{L^q} \le \|\psi_1(t,\xi)E_{k,l}(t,0,\xi)F(v)\|_{L^p}\\ &\le \|\psi_1(t,\xi)\|_{L^{\frac{pq}{q-p}}}\|E_{k,l}(t,0,\xi)\|_{L^\infty}\|F(v)\|_{L^q} \le CA(t)^{-n(\frac{1}{p}-\frac{1}{q})}\|E_{k,l}(t,0,\xi)\|_{L^\infty}\|v(.)\|_{L^p}.\end{aligned}$$

By Lemma 3.2.4 and Lemma 3.2.6 we get

$$\|E_{k,l}(t,0,\xi)\|_{L^\infty} \lesssim \max\Big\{\frac{a(t)^{1-\delta}}{\lambda^2(t)}, \frac{a(t)}{A(t)}\Big\}$$

for all $k,l=1,2$. Summarizing we have shown

$$\begin{aligned}&\|F^{-1}(\psi_1(t,\xi)\hat{u}_t(t,\cdot))\|_{L^q} + \|F^{-1}(\psi_1(t,\xi)a(t)|\xi|\hat{u}(t,\cdot))\|_{L^q}\\ &\lesssim \max\Big\{\tfrac{a(t)^{1-\delta}}{\lambda^2(t)}, \tfrac{a(t)}{A(t)}\Big\}A(t)^{-n(\frac{1}{p}-\frac{1}{q})}\big(\|u_1(\cdot)\|_{L^p} + \|u_2(\cdot)\|_{L^p}\big).\end{aligned} \tag{4.1.21}$$

Estimates for the hyperbolic part

Due to the Remark 4.1.2 the micro-energy $U(t,\xi) = (a(t)|\xi|\hat{u}, D_t\hat{u})^T$ can be represented as

$$U(t,\xi) = \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\frac{\lambda(s)}{\lambda(t)}MN_p(t,\xi)E_0(t,s,\xi)Q_p(t,s,\xi)N_p^{-1}(s,\xi)M^{-1}U(s,\xi).$$

For this reason we will investigate the following Fourier multipliers depending on the parameter t:

$$F^{-1}\Big(e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}\frac{\lambda(s)}{\sqrt{a(s)}}b(t,\xi)|\xi|^{-r}F(v)\Big),$$

here we recall that $\tilde{A}(t) = \int_0^t a(s)ds$, r is a real number and $v \in S$.

An auxiliary result

The key tool to prove in this part is we use the following result, see more detail the Lemma B.2.2 in section B.2 of Appendix.

Lemma 4.1.11. *Let us assume that $K(t)$ is a real-valued function and $d(t,\xi) \in C_0^\infty(\mathbb{R}^n_\xi)$. Then there exists a positive integer M such that*

$$\|F^{-1}\left(e^{iK(t)|\xi|}d(t,\xi)\right)\|_{L^\infty} \le C(1+K(t))^{-\frac{n-1}{2}}\sum_{|\alpha|\le M}\|D_\xi^\alpha d(t,\xi)\|_{L^\infty}$$

with a constant C which is independent of t and ξ.

Estimates in the hyperbolic part for large frequencies

In this part we shall study the following Fourier multiplier:

$$F^{-1}\Big(\psi_3(\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}b(t,\xi)|\xi|^{-r}F(v)\Big),$$

here the amplitude $b = b(t,\xi)$ satisfies for all $|\alpha| \le p-1$ the estimates

$$|D_\xi^\alpha b(t,\xi)| \le C_{\alpha,N}|\xi|^{-|\alpha|}.$$

Here we use $N_p(t,\xi) \in S\{0,0,0\}$ and $|D_\xi^\alpha Q_p(t,0,\xi)| \le C_{\alpha,N}|\xi|^{-|\alpha|}$ for $|\alpha| \le p-1$. Now let us choose a non-negative function $\phi = \phi(r) \in C_0^\infty(\mathbb{R}_+)$ with $\operatorname{supp}\phi \subseteq [\frac{1}{2},1]$ such that $\sum_{j=-\infty}^{\infty}\phi(2^{-j}r) = 1$, $r \ne 0$. Furthermore, we define

$$\phi_j(|\xi|) = \phi\Big(2^{-j}\frac{|\xi|}{N}\Big),\ j \in \mathbb{Z}.$$

The strategy of proof in this part is to obtain an $L^p - L^q$ estimate by interpolating $L^1 - L^\infty$ and $L^2 - L^2$ estimates with Riesz-Thorin interpolation theorem. We introduce

$$\begin{aligned} I_j &= \Big\|F^{-1}\Big(\psi_3(t,\xi)\phi_j(|\xi|)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}b(t,\xi)\Big)\Big\|_{L^\infty},\\ \tilde{I}_j &= \Big\|\psi_3(\xi)\phi_j(|\xi|)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}b(t,\xi)\Big\|_{L^\infty}. \end{aligned}$$

$L^1 - L^\infty$*estimates.* For all $j<0$ we have $I_j = \tilde{I}_j = 0$. For $j \ge 0$ we perform the change of variables $\xi = 2^jN\eta$ and conclude as follows, where $\psi_3(\xi) \equiv 1$ for $j \ge 2$:

$$\begin{aligned} I_j &\le C2^{j(n-r)}\Big\|F^{-1}\Big(\phi(|\eta|)e^{\pm i2^jN\tilde{A}(t)|\eta|}\frac{\sqrt{a(t)}}{\lambda(t)}|\eta|^{-r}b(t,2^jN\eta)\Big)\Big\|_{L^\infty}\\ &\le C2^{j(n-r)}\big(1+2^jN\tilde{A}(t)\big)^{-\frac{n-1}{2}}\frac{\sqrt{a(t)}}{\lambda(t)}\sum_{|\alpha|\le M}\Big\|D_\eta^\alpha\Big(\phi(|\eta|)|\eta|^{-r}b(t,2^jN\eta)\Big)\Big\|_{L^\infty}\\ &\le C2^{j(n-r)}\big(1+2^jN\tilde{A}(t)\big)^{-\frac{n-1}{2}}\frac{\sqrt{a(t)}}{\lambda(t)}\sum_{|\alpha+\beta|\le M}\sup_{1/2\le|\eta|\le 2}|\eta|^{-r-|\alpha|}(2^jN)^{|\beta|}(2^jN|\eta|)^{-|\beta|}\\ &\le C2^{j(n-r)}A(t)^{-\frac{n-1}{2}}\frac{\sqrt{a(t)}}{\lambda(t)}. \end{aligned}$$

Here we have used in the second estimate the auxiliary Lemma 4.1.11 with a suitably positive constant M. The constant M determines the necessary steps of diagonalization. Additional, we take advantage of $1+A(t) \lesssim 1+2^jNA(t)$ for all $j \ge 0$ and N sufficiently large. Summarizing gives

$$\begin{aligned} &\Big\|F^{-1}\Big(\psi_3(\xi)\phi_j(|\xi|)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}b(t,\xi)F(v)(\xi)\Big)\Big\|_{L^\infty}\\ &\lesssim 2^{j(n-r)}\frac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}}\|v(\cdot)\|_{L^1}. \end{aligned}$$

$L^2 - L^2$ *estimates.* In order to derive an $L^2 - L^2$ estimate we shall estimate $\tilde{I}_j$. We have

$$\tilde{I}_j \le C\sup_{1/2\le|\eta|\le 2}\phi(|\eta|)\frac{\sqrt{a(t)}}{\lambda(t)}(2^jN|\eta|)^{-r}|b(t,2^jN\eta)| \lesssim \frac{\sqrt{a(t)}}{\lambda(t)}2^{-jr}$$

for $j \ge 0$. Consequently, we arrive at the following estimate:

$$\Big\|F^{-1}\Big(\psi_3(\xi)\phi_j(|\xi|)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}b(t,\xi)F(v)(\cdot)\Big)\Big\|_{L^2} \lesssim \frac{\sqrt{a(t)}}{\lambda(t)}2^{-jr}\|v(\cdot)\|_{L^2}.$$

$L^p - L^q$ estimates. Applying the above mentioned interpolation argument yields

$$\begin{aligned}&\Big\|F^{-1}\Big(\psi_3(\xi)\phi_j(|\xi|)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}b(t,\xi)F(v)(\cdot)\Big)\Big\|_{L^q}\\ &\lesssim 2^{j\left(n\left(\frac{1}{p}-\frac{1}{q}\right)-r\right)}\frac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v(\cdot)\|_{L^p}.\end{aligned}$$

Finally, we conclude after fixing the regularity $r = n\big(\frac{1}{p}-\frac{1}{q}\big)$ and applying Brenner's lemma for $p \in (1,2]$ the following estimate

$$\Big\|F^{-1}\Big(\psi_3(\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}b(t,\xi)|\xi|^{-r}F(v)\Big)\Big\|_{L^q} \lesssim \frac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v(\cdot)\|_{L^p}.$$

Therefore, we have

$$\begin{aligned}&\Big\|F^{-1}\Big(\psi_3(\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}b(t,\xi)|\xi|^{-r}F\Big(a(0)|D|^{r+1}u(0,\cdot)+|D|^{r}u_t(0,\cdot)\Big)\Big)\Big\|_{L^q}\\ &\lesssim \frac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_1(\cdot)\|_{L^{p,r+1}}+\|u_2(\cdot)\|_{L^{p,r}}\big).\end{aligned}$$

Summarizing all estimates which we derived in this part of the extended phase space leads to

$$\begin{aligned}&\|F^{-1}(\psi_3(\xi)\hat{u}_t(t,\cdot))\|_{L^q}+\|F^{-1}(\psi_3(\xi)a(t)|\xi|\hat{u}(t,\cdot))\|_{L^q}\\ &\lesssim \tfrac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_1(\cdot)\|_{L^{p,r+1}}+\|u_2(\cdot)\|_{L^{p,r}}\big).\end{aligned}\tag{4.1.22}$$

Estimates in the hyperbolic part for small frequencies

In this part of the extended phase space we deal with bounded frequencies $|\xi| \le N$ and unbounded $A(t)|\xi| \ge \frac{N}{2}$. Let us consider the Fourier multiplier

$$F^{-1}\Big(\psi_2(t,\xi)e^{\pm i\int_{t_{|\xi|}}^{t}a(s)ds|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}\frac{\lambda(t_{|\xi|})}{\sqrt{a(t_{|\xi|})}}b_1(t,\xi)E_{kl}(t_{|\xi|},0,\xi)|\xi|^{-r}F(v)\Big),$$

where $v \in S$, $E_{kl}(t_{|\xi|},0,\xi)$, $k,l=1,2$, are the entries of the fundamental solution $E(t_{|\xi|},0,\xi)$. We can rewrite this Fourier multiplier in the following form

$$F^{-1}\Big(\psi_2(t,\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}\tilde{b}(t,\xi)|\xi|^{-r}F(v)\Big),$$

where we introduced the amplitude $\tilde{b}(t,\xi)$ which is defined by

$$\tilde{b}(t,\xi) := \exp\Big(\mp i\int_0^{t_{|\xi|}}a(s)ds|\xi|\Big)\frac{\lambda(t_{|\xi|})}{\sqrt{a(t_{|\xi|})}}b_1(t,\xi)$$

and which satisfies the estimates

$$|D_\xi^\alpha\tilde{b}(t,\xi)| \le C_{\alpha,N}|\xi|^{-|\alpha|}$$

for all $|\alpha| \le p-1$. Here we use $N_p(t,\xi) \in S\{0,0,0\}$, $|D_\xi^\alpha Q_p(t,t_{|\xi|},\xi)| \le C_{\alpha,N}|\xi|^{-|\alpha|}$ for all $|\alpha| \le p-1$ and Remark 4.1.1. We use again a dyadic decomposition by defining

$$\phi_j(t,|\xi|) = \phi\left(2^{-j}\frac{A(t)|\xi|}{N}\right),\ j \in \mathbb{Z},$$

with $\phi \in C_0^\infty$ being introduced as above. Then for all $j < 0$ the product $\psi_2(t,\xi)\phi_j(t,\xi)$ vanishes. Thus, for $j \ge 0$ we will estimate the following L^∞ norms:

$$\begin{aligned} I_j &= \Big\|F^{-1}\Big(\psi_2(t,\xi)\phi_j(t,|\xi|)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}\psi(|\xi|N^{-1})\tilde{b}(t,\xi)\Big)\Big\|_{L^\infty}, \\ \tilde{I}_j &= \Big\|\psi_2(t,\xi)\phi_j(t,|\xi|)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}\psi(|\xi|N^{-1})\tilde{b}(t,\xi)\Big\|_{L^\infty}. \end{aligned}$$

We perform the change of variables $A(t)\xi = 2^jN\eta$ and estimate as follows:

$$\begin{aligned} I_j &\le C2^{j(n-r)}A(t)^{(r-n)}\Big\|F^{-1}\Big(\phi(|\eta|)e^{\pm i2^jN|\eta|}\frac{\sqrt{a(t)}}{\lambda(t)}|\eta|^{-r}\psi\Big(\frac{2^j|\eta|}{A(t)}\Big)\tilde{b}\Big(t,\frac{2^jN\eta}{A(t)}\Big)\Big)\Big\|_{L^\infty} \\ &\le C2^{j(n-r)}(1+2^jN)^{-\frac{n-1}{2}}A(t)^{(r-n)}\frac{\sqrt{a(t)}}{\lambda(t)}\sum_{|\alpha|\le M}\Big\|D_\eta^\alpha\phi(|\eta|)|\eta|^{-r}\psi\Big(\frac{2^j|\eta|}{A(t)}\Big)\tilde{b}\Big(t,\frac{2^jN\eta}{A(t)}\Big)\Big\|_{L^\infty} \\ &\le C2^{j(\frac{n+1}{2}-r)}A(t)^{(r-n)}\frac{\sqrt{a(t)}}{\lambda(t)}\sum_{|\alpha+\beta|\le M}\sup_{1/2\le|\eta|\le 2}|\eta|^{-r-|\alpha|}\Big(\frac{2^jN}{A(t)}\Big)^{|\beta|}\Big(\frac{2^jN}{A(t)}\Big)^{-|\beta|} \\ &\le C2^{j(\frac{n+1}{2}-r)}A(t)^{(r-n)}\frac{\sqrt{a(t)}}{\lambda(t)}. \end{aligned}$$

Here we have used in the second estimate the auxiliary Lemma 4.1.11 with a suitably chosen positive constant M. Thus, we have

$$\Big\|F^{-1}\Big(\psi_2(t,\xi)\phi_j(t,\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}\tilde{b}(t,\xi)|\xi|^{-r}F(v)\Big)\Big\|_{L^\infty} \lesssim 2^{j(\frac{n+1}{2}-r)}A(t)^{(r-n)}\frac{\sqrt{a(t)}}{\lambda(t)}\|v(\cdot)\|_{L^1}.$$

$L^2 - L^2$ estimates. Now we shall estimate $\tilde{I}_j$. We have

$$\tilde{I}_j \le C\sup_{1/2\le|\eta|\le 2}\phi(|\eta|)\frac{\sqrt{a(t)}}{\lambda(t)}\Big(\frac{2^jN|\eta|}{A(t)}\Big)^{-r}\Big|\psi\Big(\frac{2^j|\eta|}{A(t)}\Big)\Big|\Big|\tilde{b}\Big(t,\frac{2^jN\eta}{A(t)}\Big)\Big| \lesssim \frac{\sqrt{a(t)}}{\lambda(t)}2^{-jr}A(t)^r$$

for $j \ge 0$. This implies immediately

$$\begin{aligned} &\Big\|F^{-1}\Big(\psi_2(t,\xi)\phi_j(t,\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}\tilde{b}(t,\xi)F(v)(\cdot)\Big)\Big\|_{L^2} \\ &\lesssim \frac{\sqrt{a(t)}}{\lambda(t)}2^{-jr}A(t)^r\|v(\cdot)\|_{L^2}. \end{aligned}$$

$L^p - L^q$ estimates. Applying again the interpolation argument we get

$$\begin{aligned} &\Big\|F^{-1}\Big(\psi_2(t,\xi)\phi_j(t,\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}|\xi|^{-r}\psi(|\xi|N^{-1})\tilde{b}(t,\xi)F(v)(\cdot)\Big)\Big\|_{L^q} \\ &\lesssim 2^{j\left(\frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-r\right)}\frac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{r-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|v(\cdot)\|_{L^p}. \end{aligned}$$

Therefore, we can conclude after the choice $r = \frac{n+1}{2}\big(\frac{1}{p} - \frac{1}{q}\big)$ and Brenner's lemma for $p \in (1, 2]$ the following estimate uniformly for all $j \geq 0$:

$$\Big\|F^{-1}\Big(\psi_2(t,\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}\tilde{b}(t,\xi)|\xi|^{-r}F(v)\Big)\Big\|_{L^q} \lesssim \frac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v(\cdot)\|_{L^p}.$$

From the last estimate it follows

$$\begin{aligned}&\Big\|F^{-1}\Big(\psi_2(t,\xi)e^{\pm i\tilde{A}(t)|\xi|}\frac{\sqrt{a(t)}}{\lambda(t)}\tilde{b}(t,\xi)|\xi|^{-r}F\Big(|D|^r\Big(\frac{a(0)}{A(0)}u(0,\cdot)+u_t(0,\cdot)\Big)\Big)\Big)\Big\|_{L^q}\\ &\lesssim \frac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_1(\cdot)\|_{L^p}+\|u_2(\cdot)\|_{L^p}\big).\end{aligned}$$

Summarizing all estimates which we derived in this part of the extended phase space we may conclude

$$\begin{aligned}&\|F^{-1}(\psi_2(t,\xi)\hat{u}_t(t,\cdot))\|_{L^q}+\|F^{-1}(\psi_2(t,\xi)a(t)|\xi|\hat{u}(t,\cdot))\|_{L^q}\\ &\lesssim \tfrac{\sqrt{a(t)}}{\lambda(t)}\big(1+A(t)\big)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_1(\cdot)\|_{L^p}+\|u_2(\cdot)\|_{L^p}\big).\end{aligned} \tag{4.1.23}$$

Combining and comparing the decay estimates from the three different parts of the extended phase space we see that the decay in the dissipative part is better than those in the hyperbolic parts. The desired regularity comes from the hyperbolic part with large frequencies and the desired decay comes from the the hyperbolic part with small frequencies. In this way the proof is completed. □

4.2. The case of effective dissipation

We recall our model of interest:

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x), \tag{4.2.1}$$

where the propagation speed term $a = a(t)$ satisfies the assumptions (A1) to (A3) and the dissipative term $b = b(t)$ satisfies the assumptions (B'1) to (B'4). At the beginning we prove the following auxiliary lemma for large t.

Lemma 4.2.1. *1. Under the assumption $(B'1)$ the following inequality holds :*

$$\Big\||\xi|^s\exp\Big(-|\xi|^2\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\Big\|_{L^p} \lesssim \Big(\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{s}{2}-\frac{n}{2p}}.$$

2. If we suppose the assumptions $(B'1)$, $(B'3)$ and $\dfrac{\mu(t)}{A(t)}$ is a monotonic decreasing function, then the function

$$\frac{b^2(t)}{a^2(t)}\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau$$

tends to ∞ for t to ∞.

3. If we suppose the assumptions $(B'1)$ and $(B'3)$ and if we choose $\alpha \in \mathbb{R}$, then the following function is monotonously increasing for large t:

$$\Big(1+\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{\alpha}\lambda(t)$$

Proof. Direct calculations will prove the first statement. Indeed, we have

$$
\begin{aligned}
&\Big(\Big\||\xi|^s \exp\Big(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\Big\|_{L^p}\Big)^p \lesssim \int_0^\infty \Big(|\xi|^s \exp\Big(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\Big)^p|\xi|^{n-1}d|\xi| \\
&= \int_0^\infty |\xi|^{sp+n-1}\exp\Big(-p|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)d|\xi| \\
&= \frac{1}{2}\int_0^\infty \zeta^{\frac{n+sp}{2}-1}e^{-p\zeta}d\zeta\Big(\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+sp}{2}} \\
&\lesssim \Gamma\Big(n+\frac{sp}{2}\Big)\Big(\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+sp}{2}} \lesssim \Big(\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{sp}{2}-\frac{n}{2}}.
\end{aligned}
$$

Here we used the change of variables

$$
\zeta = |\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau,\ \ d\zeta = 2|\xi|d|\xi|\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau.
$$

To prove the second statement we conclude in the following way:

$$
\begin{aligned}
\frac{b^2(t)}{a^2(t)}\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau = \frac{\mu^2(t)}{A^2(t)}\int_0^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau &= \frac{\mu^2(t)}{A^2(t)}\Big(\frac{A^2(t)}{2\mu(t)}-\frac{1}{2\mu(0)}+\int_0^t \frac{\mu'(\tau)A^2(\tau)}{2\mu^2(\tau)}d\tau\Big) \\
&= \frac{\mu(t)}{2}-\frac{\mu^2(t)}{A^2(t)}\frac{1}{2\mu(0)}+\frac{\mu^2(t)}{A^2(t)}\int_0^t \frac{\mu'(\tau)A^2(\tau)}{2\mu^2(\tau)}d\tau \\
&\geq \frac{\mu(t)}{2}-\frac{\mu^2(t)}{A^2(t)}\frac{1}{2\mu(0)}-c\frac{\mu^2(t)}{A^2(t)}\int_0^t \frac{a(\tau)A(\tau)}{2\mu(\tau)}d\tau,
\end{aligned}
$$

here we used the Assumption (B'2): $|\mu'(t)| \leq c\mu(t)a(t)/A(t)$. From the last estimate we obtain

$$
\frac{\mu^2(t)}{A^2(t)}\int_s^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \geq \frac{1}{2+c}\big(\mu(t)-\frac{\mu^2(t)}{A^2(t)}\frac{1}{\mu(0)}\big) \gtrsim \mu(t).
$$

We note that $\frac{\mu(t)}{A(t)}$ is decreasing and $\mu(t)\to\infty$ as $t\to\infty$.
To prove the last statement we study the derivative

$$
\begin{aligned}
&\partial_t\Big(\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^\alpha\lambda(t)\Big) \\
&\quad = \alpha\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{\alpha-1}\frac{a^2(t)}{b(t)}\lambda(t)+\frac{b(t)}{2}\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^\alpha\lambda(t) \\
&\quad = \frac{a^2(t)}{2b(t)}\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{\alpha-1}\lambda(t)\Big(2\alpha+\frac{b^2(t)}{a^2(t)}+\frac{b^2(t)}{a^2(t)}\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big).
\end{aligned}
$$

Combining with the second statement for α is fixed, we can find a sufficiently large time $t_0 = t_0(\alpha)$ such that the expression is positive for all $t \geq t_0$. The lemma is proved. □

The strategy for getting $L^p - L^q$ estimates on the conjugate line for effective dissipations is to estimate the L^1-norm of the Fourier multiplier to get a $L^1 - L^\infty$ estimate and apply the Riesz-Thorin interpolation theorem with the previously obtained $L^2 - L^2$ estimates. We have the following theorem.

Theorem 4.2.2. *Assume the conditions (B'1) to (B'4). Then for all times t we have the $L^p - L^q$ decay estimate*

$$\big\|(u_t(t,\cdot), a(t)\nabla_x u(t,\cdot))\big\|_{L^q} \lesssim a(t)\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_1\|_{L^{p,r+1}}+\|u_2\|_{L^{p,r}}\big),$$

where $r > n\Big(\dfrac{1}{p}-\dfrac{1}{q}\Big)$ *with* $1<p\le 2$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Proof. As we mentioned before it remains to derive a $L^1 - L^\infty$ estimate. For this reason we devote to estimate the following L^1-norm:

$$\|\langle\xi\rangle^{-r}E(t,0,\xi)\|_{L^1}.$$

We will investigate the localized L^1-norm in different zones separately.

Small frequencies

Dissipative zone: $|\xi| \lesssim \dfrac{1}{A(t)}$. Inside this zone all entries of $E(t,0,\xi)$ can be estimated by $\dfrac{a(t)}{A(t)}$. Consequently, the desired L^1-norm can be estimated by

$$\int_0^{\xi_{t_1}} \frac{a(t)}{A(t)}|\xi|^{n-1}d|\xi| \lesssim a(t)A(t)^{-1-n} \tag{4.2.2}$$

with ξ_{t_1} is the inverse of t_{ξ_1}.

Region $\Pi_{ell}\cap\{|\xi|\le c_0\}$. In this region we have $\langle\xi\rangle\sim 1$ and from Corollary 3.3.10 we get that all components of $E(t,0,\xi)$ can be estimated by

$$\exp\Big(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)a(t)|\xi|.$$

After application of the first statement in Lemma 4.2.1 we can estimate the desired L^1-norm by

$$\|E(t,0,\xi)\|_{L^1(\Pi_{ell}\cap\{|\xi|\le c_0\})} \lesssim a(t)\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{1}{2}-\frac{n}{2}}. \tag{4.2.3}$$

Region $\Pi_{hyp}\cap\{|\xi|\le c_0\}$. Let us distinguish two cases. If $\dfrac{\mu(t)}{A(t)}$ increases and tends to ∞ as $t\to\infty$, then the elliptic region lies on top of the hyperbolic region which is away from frequency 0. In this case, the small frequencies lie completely in the elliptic zone.

Now let us consider the case $\dfrac{\mu(t)}{A(t)}$ decreases and tends to 0 as $t\to\infty$. Using Corollary 3.3.12 we have for large t the following estimates, where ξ_{t_3} denotes the inverse of t_{ξ_3}:

$$\begin{aligned}
\|E(t,0,\xi)\|_{L^1(\Pi_{hyp}\cap\{|\xi|\le c_0\})} &\lesssim \int_{\xi_{t_3}}^{c_0} |\xi|^n \exp\Big(-|\xi|^2\int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\frac{\lambda(t_{\xi_3})}{\lambda(t)}\sqrt{a(t)}\sqrt{a(t_{\xi_3})}d|\xi| \\
&\lesssim \int_{\xi_{t_3}}^{c_0} \Big(|\xi|^2\int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{\frac{n}{2}}\exp\Big(-|\xi|^2\int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)}d\tau\Big) \\
&\qquad\times\Big(\int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+1}{2}}\frac{\lambda(t_{\xi_3})}{\lambda(t)}a(t)d\Big(|\xi|\Big(\int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{\frac{1}{2}}\Big) \\
&\lesssim a(t)\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+1}{2}}.
\end{aligned}$$

Here we have used the monotonic increasing behavior of the function

$$\Big(\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+1}{2}}\lambda(t)$$

from the third statement of Lemma 4.2.1.

Large frequencies $|\xi| \geq c$

In this case the L^1-norm of the multiplier can be estimated with $r > n$ as follows:

$$\|\langle\xi\rangle^{-r}E(t,0,\xi)\|_{L^1} \lesssim \|\langle\xi\rangle^{-r}\|_{L^1}\|E(t,0,\xi)\|_{L^\infty} \lesssim \|E(t,0,\xi)\|_{L^\infty}.$$

$\Pi_{hyp} \cap \{|\xi| \geq c\}$: From Lemma 3.3.3 and Remark 3.3.5 we have that the fundamental solution can be uniformly estimated by

$$|E(t,0,\xi)| \lesssim \frac{\sqrt{a(t)}}{\lambda(t)}. \tag{4.2.4}$$

$\Pi_{ell} \cap \{|\xi| \geq c\}$: In this case we have the estimate

$$|E(t,0,\xi)| \lesssim a(t)\exp\Big(-c_0^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big). \tag{4.2.5}$$

The decay behaviors in (4.2.2), (4.2.4) and (4.2.5) are not slower than the decay behavior $a(t)\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+1}{2}}$ which we obtained from our considerations for small frequencies. Indeed, in order to compare the decay behavior in (4.2.2) with the above one we use the following calculation:

$$\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau = \int_0^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \lesssim \int_0^t \frac{A(\tau)}{\mu_0}d\left(A(\tau)\right) \lesssim A(t)^2,$$

here, due to $\mu(t) \to \infty$ as $t \to \infty$ it exists a constant μ_0 such that $\mu(t) \geq \mu_0$ for large t.
Comparing (4.2.4) with $a(t)\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+1}{2}}$ by the aid of the monotonously increasing property of $\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{\alpha}\lambda(t)$ we conclude the desired dominance (with the choice $\alpha = -(n+1)/2$). The decay behavior from (4.2.5) is of course faster than the behavior we are interested in because the exponential function is more dominant than any power function. Thus, the decay behavior appearing in the $L^1 - L^\infty$ estimate uniformly for all frequencies is

$$a(t)\Big(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{-\frac{n+1}{2}}.$$

Combining with the estimate of $L^2 - L^2$ in Theorem 3.3.14 by the aid of Riesz-Thorin interpolation theorem we will get the desired estimate. In this way the theorem is proved. $\square$

5. Global existence of small data solutions to semi-linear dissipative wave models

5.1. Semi-linear models with non-effective dissipation

We are interested in the study of global existence (in time) of small data solutions to the following semi-linear Cauchy problem

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = u_t^2 - a(t)^2|\nabla_x u|^2,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x) \tag{5.1.1}$$

with data $u_1(x)$ and $u_2(x)$ belonging to $C_0^\infty(\mathbb{R}^n)$. If we apply Nirenberg's transformation

$$v(t,x) = 1 - \exp(-u(t,x)), \tag{5.1.2}$$

then the Cauchy problem (5.1.1) can be rewritten as the following linear Cauchy problem

$$v_{tt} - a(t)^2\Delta v + b(t)v_t = 0,$$
$$v(0,x) = v_1(x) = 1 - \exp(-u_1(x)),\ \ v_t(0,x) = v_2(x) = u_2(x)\exp(-u_1(x))$$

with data $v_1(x)$ and $v_2(x)$ belonging to $C_0^\infty(\mathbb{R}^n)$ and with the constrain condition

$$v(t,x) < 1.$$

This constrain condition follows from the transformation (5.1.2) and the goal to get global (in time) solutions. Let us recall the assumptions to the coefficient $a = a(t)$:

- (A1) $a(t) > 0,\ a'(t) > 0$ for $t \in [0,\infty)$,
- (A2) $a_0 \le \alpha(t) \le a_1,\ a_0, a_1 > 0,\ \dfrac{a'(t)}{a(t)} := \alpha(t)\dfrac{a(t)}{A(t)}$,
- (A3) $|a''(t)| \le a_2 a(t)\Big(\dfrac{a(t)}{A(t)}\Big)^2,\ \ a_2 \ge 0$,

and $b = b(t) := \mu(t)\dfrac{a(t)}{A(t)}$ with a little change to assumption (B3) as follows:

(B1) $b(t) > 0$, $b \notin L^1(\mathbb{R}_+)$,

(B2) $|\mu'(t)| \le C_\mu \mu(t)\dfrac{a(t)}{A(t)}$,

(B3) $\limsup_{t\to\infty}\mu(t) < \max\big\{\limsup_{t\to\infty}\alpha(t), 1\big\}$.

Then the following statement is true for the case $n > 1$.

Theorem 5.1.1. *Suppose that the coefficients $a = a(t)$ and $b = b(t)$ satisfy the assumptions $(A1)$ to $(A3)$ and $(B1)$ to $(B3)$, respectively. Then the solution $v = v(t,x)$ satisfies the following a-priori estimate*:

$$\|v(t,\cdot)\|_\infty \le C_\varepsilon(N)\big(\|\langle D\rangle^{n+\varepsilon}v_0\|_1 + \|\langle D\rangle^{n-1+\varepsilon}v_1\|_1\big) \tag{5.1.3}$$

for all $\varepsilon > 0$ and $n > 1$.

Proof. We will only sketch the proof. Applying partial Fourier transformation we get the equation $\hat{v}_{tt} + a^2(t)|\xi|^2\hat{v} + b(t)\hat{v}_t = 0$. We carry out the proof in two zones of the extended phase space $(0,\infty)\times\mathbb{R}^n$. These zones are defined as in Section 3.2 as

- $Z_{hyp}(N) := \{(t,\xi) : t \ge t_{|\xi|}\}$,
- $Z_{diss}(N) := \{(t,\xi) : 0 \le t \le t_{|\xi|}\}$,

where $t_{|\xi|}$ satisfies $A(t_{|\xi|})|\xi| = N$. Let $V = (a(t)|\xi|\hat{v}, D_t\hat{v})^T$. Then V satisfies

$$D_tV = A(t,\xi)V := \begin{pmatrix} \dfrac{D_ta}{a} & a(t)|\xi| \\ a(t)|\xi| & ib(t) \end{pmatrix} V. \tag{5.1.4}$$

We will derive estimates for the fundamental solution $E = E(t,s,\xi)$ of this first order system with data $E(s,s,\xi) = I$. Thus,

$$V(t,\xi) = E(t,s,\xi)V(s,\xi).$$

In the dissipative zone $Z_{diss}(N)$ straight-forward calculations and the definition of this zone give

$$\begin{aligned} \|E(t,0,\xi)\| &\le \exp\Big(\int_0^t \|A(\tau,\xi)\|d\tau\Big) \\ &\le \exp\Big(\max\Big\{\sup_{(t,\xi)\in Z_{diss}}\int_0^t\Big(\frac{a'(\tau)}{a(\tau)} + a(\tau)|\xi|\Big)d\tau, \sup_{(t,\xi)\in Z_{diss}}\int_0^t(b(\tau)+a(\tau)|\xi|)d\tau\Big\}\Big) \\ &\le C_N\exp\Big(\max\Big\{\sup_{t\in(0,\infty)}\int_0^t\frac{a'(\tau)}{a(\tau)}d\tau, \sup_{t\in(0,\infty)}\int_0^t\mu(\tau)\frac{a(\tau)}{A(\tau)}d\tau\Big\}\Big) \\ &\le C_Na(t). \end{aligned}$$

Here we used $\mu(t)\frac{a(t)}{A(t)} = \frac{\mu(t)}{\alpha(t)}\frac{a'(t)}{a(t)}$ and assumption (B3). Moreover, from the representation of $V(t,\xi)$ we obtain

$$a(t)|\xi|\hat{v}(t,\xi) = E_{11}(t,0,\xi)a(0)|\xi|\hat{v}_0 - iE_{12}(t,0,\xi)\hat{v}_1,$$

thus,

$$|\hat{v}(t,\xi)| = \Big|\frac{a(0)}{a(t)}E_{11}(t,0,\xi)\hat{v}_0 - i\frac{1}{a(t)|\xi|}E_{12}(t,0,\xi)\hat{v}_1\Big| \le C_N|\hat{v}_0| + C_N|\xi|^{-1}|\hat{v}_1|. \tag{5.1.5}$$

In the hyperbolic zone $Z_{hyp}(N)$ we carry out again two steps of diagonalization as in Section 3.2.2. Then we obtain the following estimate:

$$\|E(t,t_{|\xi|},\xi)\| \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(t_{|\xi|})}}\frac{\sqrt{\lambda(t_{|\xi|})}}{\sqrt{\lambda(t)}} \tag{5.1.6}$$

for all $t \geq t_{|\xi|}$. On the other hand we have the following representation of solution

$$\begin{pmatrix} a(t)|\xi|\hat{v}(t,\xi) \\ D_t\hat{v}(t,\xi) \end{pmatrix} = E(t,t_{|\xi|},\xi)E(t_{|\xi|},0,\xi)\begin{pmatrix} a(0)|\xi|\hat{v}_0(\xi) \\ -i\hat{v}_1(\xi) \end{pmatrix}. \tag{5.1.7}$$

Combining the formula (5.1.7) with the estimate (5.1.6) and the estimate in the dissipative zone we get

$$\begin{aligned} a(t)|\xi||\hat{v}(t,\xi)| &\leq \frac{\sqrt{a(t)}}{\sqrt{a(t_{|\xi|})}}\frac{\sqrt{\lambda(t_{|\xi|})}}{\sqrt{\lambda(t)}}C_N a(t_\xi)\big(a(0)|\xi||\hat{v}_0| + |\hat{v}_1|\big), \\ |\hat{v}(t,\xi)| &\leq C_N \underbrace{\frac{\sqrt{a(t_\xi)}}{\sqrt{a(t)}}}_{\lesssim 1}\underbrace{\frac{\sqrt{\lambda(t_{|\xi|})}}{\sqrt{\lambda(t)}}}_{\lesssim 1}\big(a(0)|\hat{v}_0| + |\xi|^{-1}|\hat{v}_1|\big) \\ &\lesssim C_N|\hat{v}_0| + C_N|\xi|^{-1}|\hat{v}_1|. \end{aligned} \tag{5.1.8}$$

Combining both formula (5.1.5) and (5.1.8) we get the final estimate

$$\begin{aligned} \|v(t,\cdot)\|_\infty &\leq \|\hat{v}(t,\cdot)\|_1 \leq C_N\|\hat{v}_0\|_1 + C_N\||\xi|^{-1}\hat{v}_1\|_1 \\ &\leq C_N\big(\|\langle\xi\rangle^{-n-\epsilon}\|_1\|\langle\xi\rangle^{n+\epsilon}\hat{v}_0\|_\infty + \|\langle\xi\rangle^{1-n-\epsilon}|\xi|^{-1}\|_1\|\langle\xi\rangle^{n-1+\epsilon}\hat{v}_1\|_\infty\big) \\ &\leq C_N\big(\|\langle D\rangle^{n+\epsilon}v_0\|_1 + \|\langle D\rangle^{n-1+\epsilon}v_1\|_1\big) \end{aligned}$$

for all $n > 1$ and $v_i \in C_0^\infty(\mathbb{R}^n)$, $i = 1,2$. This completes the proof. □

We can immediately conclude the global existence of small data solutions for our semi-linear problem in the case $n > 1$.

Corollary 5.1.2. *Under the assumptions of Theorem 5.1.1 there exists a unique global (in time) classical solution $u = u(t,x)$ to*

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = u_t^2 - a(t)^2|\nabla_x u|^2,\;\; u(0,x) = \epsilon u_1(x),\;\; u_t(0,x) = \epsilon u_2(x)$$

for given $u_0, u_1 \in C_0^\infty(\mathbb{R}^n), n > 1$, and all $\epsilon \in [0,\epsilon^*)$ with an in general suitable positive and small ϵ^*.

Now we will formulate the statement for the case of dimension $n = 1$. In this formulation we use again assumptions for $a(t)$ and $b(t)$ which were recalled at the beginning of this section. There is a small modification of assumption (B3) in the following way: There exists a constant $\varepsilon_0 > 0$ small enough such that the following condition

(B3)' $1 - \delta\alpha(t) - \varepsilon_0 \leq \liminf_{t\to\infty}\mu(t) \leq \limsup_{t\to\infty}\mu(t) < 1$

holds with an arbitrary δ satisfying $\delta < \liminf_{t\to\infty}\alpha(t)^{-1}$.

Theorem 5.1.3. *Let us suppose that the coefficients $a = a(t)$ and $b = b(t)$ satisfy the assumptions (A1) to (A3), (B1), (B2) and (B3)', respectively. Then the solution $v = v(t,x)$ satisfies the following a-priori estimate:*

$$\|v(t,\cdot)\|_\infty \leq C(s,N)\|\langle D\rangle^s v_0\|_{L^1} + C(\varepsilon_0,N)\|v_1\|_{L^1} + C(s,N)\|\langle D\rangle^{s-1}v_1\|_1 \tag{5.1.9}$$

for all $s > 1$.

Proof. In the further calculations we will use the following statement:

Proposition 5.1.4. *The assumption* $(B3)'$ *implies that*

$$\frac{A(t)a(0)^\delta}{a(t)^\delta\lambda^2(t)} \leq A(t)^{\varepsilon_0}.$$

Proof. We have

$$\begin{aligned}\frac{A(t)a(0)^\delta}{a(t)^\delta\lambda^2(t)} &= \exp\Big(\int_0^t \Big(\frac{a(s)}{A(s)} - \delta\frac{a'(s)}{a(s)} - b(s)\Big)ds\Big)\\ &= \exp\Big(\int_0^t \big(1-\delta\alpha(s)-\mu(s)\big)\frac{a(s)}{A(s)}ds\Big) \leq A(t)^{\varepsilon_0}.\end{aligned}$$

This gives the desired statement. □

We also divide the extended phase space into the dissipative zone $Z_{diss}(N)$ and the hyperbolic zone $Z_{hyp}(N)$. In order to prove our statement we distinguish between two cases:

1.case large frequencies $\{|\xi| \geq N\}$:

This part of the extended phase space belongs completely to $Z_{hyp}(N)$. Let us choose a smooth function $\chi(r) = 1$ for $r \leq 1/2$, $\chi(r) = 0$ for $r \geq 1$. According to the statement of Corollary 3.2.7 in Chapter 3 we have

$$a(t)|\xi||\hat{v}(t,\xi)| \leq C\frac{\sqrt{a(t)}}{\lambda(t)}\Big(1-\chi\Big(\frac{|\xi|}{N}\Big)\Big)\big(a(0)|\xi||\hat{v}_0| + |\hat{v}_1|\big)$$

for all $t \geq 0$. Thus

$$\begin{aligned}\Big\|\Big(1-\chi\Big(\frac{|D|}{N}\Big)\Big)v(t,\cdot)\Big\|_\infty &\leq \Big\|\Big(1-\chi\Big(\frac{|\xi|}{N}\Big)\Big)\hat{v}(t,\cdot)\Big\|_{L^1} &(5.1.10)\\ &\leq \frac{C}{\sqrt{a(t)}\lambda(t)}\Big(\Big\|\Big(1-\chi\Big(\frac{|\xi|}{N}\Big)\Big)\hat{v}_0\Big\|_{L^1} + \Big\|\Big(1-\chi\Big(\frac{|\xi|}{N}\Big)\Big)|\xi|^{-1}\hat{v}_1\Big\|_{L^1}\Big)\\ &\leq \frac{C}{\sqrt{a(t)}\lambda(t)}\Big(\Big\|\Big(1-\chi\Big(\frac{|\xi|}{N}\Big)\Big)\frac{1}{\langle\xi\rangle^s}\Big\|_{L^1}\|\langle\xi\rangle^s\hat{v}_0\|_{L^\infty}\\ &\qquad + \Big\|\frac{1}{\langle\xi\rangle^s}\Big\|_{L^1}\Big\|\Big(1-\chi\Big(\frac{|\xi|}{N}\Big)\Big)\langle\xi\rangle^s|\xi|^{-1}\hat{v}_1\Big\|_{L^\infty}\Big)\\ &\leq C\big(\|\langle D\rangle^s v_0\|_{L^1} + \|\langle D\rangle^{s-1}v_1\|_{L^1}\big) &(5.1.11)\end{aligned}$$

for all $s > 1$.

2.case small frequencies $\{|\xi| \leq N\}$:

Here we have to divide our considerations in those in the dissipative zone and in the hyperbolic zone.

$Z_{diss}(N)$: We will follow the reasoning in Section 3.2.1. Therefore we define $V = \Big(N\dfrac{a(t)}{A(t)}\hat{v}, D_t\hat{v}\Big)^T$. Thus V satisfies

$$D_tV = A(t,\xi)V,\ A(t,\xi) = \begin{pmatrix} -i\dfrac{d_t\delta(t)}{\delta(t)} & N\delta(t)\\ \dfrac{a^2(t)|\xi|^2}{N\delta(t)} & ib(t)\end{pmatrix}, \tag{5.1.12}$$

here, $\delta(t) := \dfrac{a(t)}{A(t)}$. The solution $V = V(t,\xi)$ can be represented as $V(t,\xi) = E(t,s,\xi)V(s,\xi)$, where $E(t,s,\xi)$ is the fundamental solution, that is, the solution to the system

$$D_t E(t,s,\xi) = A(t,\xi)E(t,s,\xi),\ \ E(s,s,\xi) = I,\ \ 0 \le s \le t \le t_\xi.$$

According to Lemma 3.2.4 from Section 3.2.1 we have the following estimate:

$$\big(|E(t,0,\xi)|\big) \lesssim \begin{pmatrix} \dfrac{a(t)}{A(t)} & \dfrac{a(t)^{1-\delta}}{\lambda^2(t)} \\ \dfrac{a(t)}{A(t)} & \dfrac{a(t)^{1-\delta}}{\lambda^2(t)} \end{pmatrix}. \tag{5.1.13}$$

Using the above representation for $V(t,\xi)$ we get

$$\frac{a(t)}{A(t)}\hat{v}(t,\xi) = E_{11}(t,0,\xi)\frac{a(0)}{A(0)}\chi(|\xi|/N)\chi(t/t_{|\xi|})\hat{v}_0(\xi) - iE_{12}(t,0,\xi)\chi(|\xi|/N)\chi(t/t_{|\xi|})\hat{v}_1(\xi).$$

This gives the estimate

$$|\hat{v}(t,\xi)| \lesssim |E_{11}(t,0,\xi)|\frac{A(t)}{a(t)}\chi\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)|\hat{v}_0(\xi)| + |E_{12}(t,0,\xi)|\frac{A(t)}{a(t)}\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)|\hat{v}_1(\xi)|. \tag{5.1.14}$$

Thanks to the estimates (5.1.13) and (5.1.14) we obtain the following estimate:

$$|\hat{v}(t,\xi)| \le C\chi\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)|\hat{v}_0(\xi)| + \frac{A(t)}{a(t)^\delta\lambda^2(t)}\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)|\hat{v}_1(\xi)|.$$

Summarizing the application of Proposition 5.1.4 implies

$$\begin{aligned}
\big\|\chi(|D|/N)\chi(t/t_{|D|})v(t,\cdot)\big\|_\infty &\le \Big\|\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)\hat{v}(t,\cdot)\Big\|_{L^1} \\
&\le C\Big\|\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)\hat{v}_0\Big\|_{L^1} + \Big\|\frac{A(t)}{a(t)^\delta\lambda^2(t)}\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)\hat{v}_1\Big\|_{L^1} \\
&\le C\Big\|\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big\|_{L^1}\|\hat{v}_0\|_\infty + \Big\|A(t)^{\varepsilon_0}\Big(\frac{|\xi|}{N}\Big)\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big\|_{L^1}\|\hat{v}_1\|_\infty \\
&\le C(N)\|v_0\|_{L^1} + N\||\xi|^{-\varepsilon_0}\chi(|\xi|/N)\|_{L^1}\|v_1\|_{L^1} \\
&\le C(N)\|v_0\|_{L^1} + C(\varepsilon_0,N)\|v_1\|_{L^1}.
\end{aligned} \tag{5.1.15}$$

$Z_{hyp}(N)$: Applying again Corollary 3.2.7 from Chapter 3 we get

$$\begin{aligned}
a(t)|\xi||\hat{v}(t,\xi)| &\le C\frac{\sqrt{a(t)}}{\lambda(t)}\frac{\lambda(t_{|\xi|})}{\sqrt{a(t_{|\xi|})}}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\big(a(t_{|\xi|})|\xi||\hat{v}(t_{|\xi|},\xi)| + |\hat{v}_t(t_{|\xi|},\xi)|\big) \\
&\lesssim \frac{\sqrt{a(t)}}{\lambda(t)}\frac{\lambda(t_{|\xi|})}{\sqrt{a(t_{|\xi|})}}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\Big(\frac{a(t_{|\xi|})}{A(t_\xi)}|\hat{v}(t_{|\xi|},\xi)| + |\hat{v}_t(t_{|\xi|},\xi)|\Big) \\
&\lesssim \frac{\sqrt{a(t)}}{\sqrt{a(t_{|\xi|})}}\frac{\lambda(t_{|\xi|})}{\lambda(t)}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\Big(\frac{a(t_{|\xi|})}{A(t_{|\xi|})}|\hat{v}_0(\xi)| + \frac{a(t_{|\xi|})^{1-\delta}}{\lambda^2(t_{|\xi|})}|\hat{v}_1(\xi)|\Big).
\end{aligned}$$

Thus

$$
\begin{aligned}
|\hat{v}(t,\xi)| &\lesssim \underbrace{\frac{\sqrt{a(t_{|\xi|})}}{\sqrt{a(t)}}}_{\leq 1}\underbrace{\frac{\lambda(t_{|\xi|})}{\lambda(t)}}_{\leq 1}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\Big(\frac{1}{A(t_{|\xi|})|\xi|}|\hat{v}_0(\xi)|+\frac{1}{a(t_{|\xi|})^{\delta}\lambda^2(t_{|\xi|})|\xi|}|\hat{v}_1(\xi)|\Big)\\
&\leq \frac{1}{N}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)|\hat{v}_0(\xi)|+\frac{A(t_{|\xi|})}{Na(t_{|\xi|})^{\delta}\lambda^2(t_{|\xi|})}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)|\hat{v}_1(\xi)|.
\end{aligned}
$$

In this way we conclude with Proposition 5.1.4 the final estimate

$$
\begin{aligned}
&\Big\|\chi\Big(\frac{|D|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|D|}}\Big)\Big)v(t,\cdot)\Big\|_{\infty}\leq \Big\|\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\hat{v}(t,\cdot)\Big\|_{L^1}\\
&\leq \frac{1}{N}\Big\|\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\hat{v}_0\Big\|_{L^1}+\Big\|\frac{A(t_{|\xi|})^{\varepsilon_0}}{N}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\hat{v}_1\Big\|_{L^1}\\
&\leq \frac{1}{N}\Big\|\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\Big\|_{L^1}\|\hat{v}_0\|_{\infty}+\Big\|\frac{1}{N^{1-\varepsilon_0}|\xi|^{\varepsilon_0}}\chi\Big(\frac{|\xi|}{N}\Big)\Big(1-\chi\Big(\frac{t}{t_{|\xi|}}\Big)\Big)\Big\|_{L^1}\|\hat{v}_1\|_{\infty}\\
&\leq C(N)\|v_0\|_{L^1}+C(\varepsilon_0,N)\|v_1\|_{L^1}. \qquad (5.1.16)
\end{aligned}
$$

Combining (5.1.10), (5.1.15) and (5.1.16) we obtain for all $s>1$ the estimate

$$
\|v(t,\cdot)\|_{\infty}\leq C(s,N)\|\langle D\rangle^{s}v_0\|_{L^1}+C(\varepsilon_0,N)\|v_1\|_{L^1}+C(s,N)\|\langle D\rangle^{s-1}v_1\|_{L^1}.
$$

This completes the proof. □

An immediate consequence of Theorem 5.1.3 is the following statement for our semi-linear Cauchy problem in the one-dimensional case.

Corollary 5.1.5. *Under the assumptions of Theorem 5.1.3 there exists a unique global (in time) classical solution $u=u(t,x)$ to*

$$
u_{tt}-a(t)^2u_{xx}+b(t)u_t=u_t^2-a(t)^2u_x^2,\;\; u(0,x)=\epsilon u_1(x),\;\; u_t(0,x)=\epsilon u_2(x)
$$

for given $u_0,u_1\in C_0^{\infty}(\mathbb{R})$ and all $\epsilon\in[0,\epsilon^*)$ with an in general suitable positive and small ϵ^*.

Let us compare our results with the results from Ebert-Reissig [E-R11] for the case $b(t)=0$. For this case, the assumptions $(B1)$ and $(B2)$ disappear and the assumption $(B3)'$ is automatically satisfied for any fixed $\varepsilon_0>0$. It turns out that we only need the assumptions $(A1)$ to $(A3)$ in Theorem 5.1.3. Whereas, the a-priori estimate (5.1.9) in Theorem 5.1.3 coincides with a-priori estimate (15) in Theorem 2.1, [E-R11]. The difference between both theorems is that we herein use the lower bound

$$
a_0\frac{a(t)}{A(t)}\leq\frac{a'(t)}{a(t)}
$$

for large t in the assumption $(A2)$. In their theorem they only use the upper bound. Moreover, the additional assumption

$$
t_{\xi}=A^{-1}(N/|\xi|)\in L^1(-1,1)
$$

was used there.

5.2. Semi-linear models with effective dissipation

In this section we will consider the following semi-linear Cauchy problem

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = |u|^p,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x), \tag{5.2.1}$$

where the coefficient $a(t)$ satisfies the assumptions (A1) to (A3) from the previous section. Let us recall the assumptions for the time-dependent damping term $b(t)u_t$ in the case it is called effective dissipation term. We suppose

(B'1) $b(t) > 0,\ b(t) = \mu(t)\dfrac{a(t)}{A(t)}$,

(B'2) $\left|d_t^k\mu(t)\right| \leq C_k\mu(t)\left(\dfrac{a(t)}{A(t)}\right)^k$ for $k = 1, 2$,

(B'3) $\dfrac{\mu(t)}{A(t)}$ is monotone, and $\mu(t) \to \infty$ as $t \to \infty$,

(B'4) $\dfrac{a^2(t)}{b(t)} = \dfrac{a(t)A(t)}{\mu(t)} \notin L^1(\mathbb{R}_+)$,

(B'5) $\dfrac{a^2(t)}{A^2(t)b(t)} \in L^1(\mathbb{R}_+)$.

5.2.1. Matsumura-type estimates for parameter-dependent linear Cauchy problems

In order to prove results for the semi-linear model we shall derive estimates for solutions to the following family of parameter-dependent Cauchy problems with suitable initial data $(0, g(s,x))$:

$$v_{tt} - a(t)^2\Delta v + b(t)v_t = 0,\ v(s,x) = 0,\ v_t(s,x) = g(s,x), \tag{5.2.2}$$

where $t \in [s,\infty)$, and $s \geq 0$.

Definition 5.2.1. *We denote by $B_a(s,t)$ the primitive of $a^2(t)/b(t)$ which vanishes at $t = s$ and which is defined by*

$$B_a(s,t) := \int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau = B_a(0,t) - B_a(0,s). \tag{5.2.3}$$

We have the following result for estimates of solutions to (5.2.2):

Theorem 5.2.1. *We assume that $a(t)$ satisfies the assumptions (A1) to (A3), $b(t)$ satisfies the assumptions (B'1) to (B'5) and $g(s,\cdot) \in L^m \cap L^2$ for some $m \in [1,2]$. Then the solution $v(t,x)$ to (5.2.2) satisfies the following Matsumura-type estimates for $t \geq s$:*

$$\|v(t,\cdot)\|_{L^2} \leq C\frac{1}{b(s)}\big(1 + B_a(s,t)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|g(s,\cdot)\|_{L^m\cap L^2}, \tag{5.2.4}$$

$$\|\nabla v(t,\cdot)\|_{L^2} \leq C\frac{1}{b(s)}\big(1 + B_a(s,t)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\|g(s,\cdot)\|_{L^m\cap L^2}, \tag{5.2.5}$$

$$\|v_t(t,\cdot)\|_{L^2} \leq C\frac{a^2(t)}{b(s)b(t)}\big(1 + B_a(s,t)\big)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1}\|g(s,\cdot)\|_{L^m\cap L^2}. \tag{5.2.6}$$

The non-negative constant C is independent of s.

Proof. We apply the partial Fourier transformation to (5.2.2) and use the change of variables

$$w(t,\xi)=\frac{\lambda(t)}{\lambda(s)}\hat{v}(t,\xi),\ \ \lambda(t):=\exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big). \tag{5.2.7}$$

Then we get the Cauchy problem

$$w''+m(t,\xi)w=0,\ \ w(s,\xi)=0, w'(s,\xi)=\hat{g}(s,\xi), \tag{5.2.8}$$

where

$$\begin{aligned} m(t,\xi) &= a^2(t)|\xi|^2-\frac{b^2(t)}{4}-\frac{b'(t)}{2} \\ &= a^2(t)|\xi|^2-\frac{\mu^2(t)a^2(t)}{4A^2(t)}-\Big(\frac{\mu(t)a(t)}{2A(t)}\Big)'. \end{aligned}$$

Let us introduce $\eta(t):=\frac{b(t)}{2a(t)}=\frac{\mu(t)}{2A(t)}$ and $\langle\xi\rangle_{b(t)}:=\sqrt{a^2(t)|\xi|^2-b^2(t)/4}$. Similar to the considerations from Section 3.3 we also divide the extended phase space $[s,\infty)\times\mathbb{R}^n$ into four zones: *the hyperbolic, pseudo-differential, reduced and elliptic zone.* We denote

$$\eta_\infty:=\lim_{t\to\infty}\eta(t)\in[0,\infty].$$

This limit exists because of the monotonic behavior of $\eta(t)$.
We define

$$h(t,\xi)=\chi\Big(\frac{\langle\xi\rangle_{b(t)}}{\epsilon a(t)\eta(t)}\Big)\epsilon a(t)\eta(t)+\Big(1-\chi\Big(\frac{\langle\xi\rangle_{b(t)}}{\epsilon a(t)\eta(t)}\Big)\Big)\sqrt{|m(t,\xi)|},$$

where $\chi\in\mathcal{C}^\infty[0,+\infty)$ localizes: $\chi(\rho)=1$ if $0\le\rho\le 1/2$ and $\chi(\rho)=0$ if $\rho\ge 1$. By the definition of $Z_{red}(\epsilon)$, for any $(t,\xi)\notin Z_{red}(\epsilon)$, it implies that $|m(t,\xi)|\ge C\epsilon^2a^2(t)\eta^2(t)$, thus $h(t,\xi)\ge C_1\epsilon a(t)\eta(t)$. Let $W(t,\xi)=(h(t,\xi)w(t,\xi),D_tw(t,\xi))^T$. Then we get

$$D_tW(t,\xi)=\underbrace{\begin{pmatrix} D_th(t,\xi)/h(t,\xi) & h(t,\xi)\\ m(t,\xi)/h(t,\xi) & 0\end{pmatrix}}_{A(t,\xi)}W(t,\xi),\ \ W(s,\xi)=(0,-i\hat{g}(s,\xi))^T. \tag{5.2.9}$$

We denote by $E^W(t,t_1,\xi)$ the fundamental solution to (5.2.9) for any $t\ge t_1\ge s$, i.e., the solution to

$$D_tE^W(t,t_1,\xi)=A(t,\xi)E^W(t,t_1,\xi),\ \ E^W(t_1,t_1,\xi)=I. \tag{5.2.10}$$

It is clear that $W(t,\xi)=E^W(t,s,\xi)(0,-i\hat{g}(s,\xi))^T$ and that $E^W(t,t_1,\xi)=E^W(t,t_2,\xi)E^W(t_2,t_1,\xi)$ for any $t\ge t_2\ge t_1\ge s$. For $t_1\le t_2$ and $(t_1,\xi),(t_2,\xi)\in Z_{hyp}(N,\epsilon)$ we will introduce $E^W(t_2,t_1,\xi)=E^W_{hyp}(t_2,t_1,\xi)$ and we introduce corresponding notations in the other zones.

Remark 5.2.1. In Section 3.3 we used another energy $V(t,\xi)=(\tilde{h}(t,\xi)v(t,\xi),D_tv(t,\xi))$ with

$$\tilde{h}(t,\xi):=\chi\Big(\frac{\langle\xi\rangle_{b(t)}}{\epsilon a(t)\eta(t)}\Big)\epsilon a(t)\eta(t)+\Big(1-\chi\Big(\frac{\langle\xi\rangle_{b(t)}}{\epsilon a(t)\eta(t)}\Big)\Big)\langle\xi\rangle_{b(t)},$$

and we have considered the fundamental solution $E_V(t,t_1,\xi)$ as the solution to

$$D_tE_V(t,t_1,\xi)=\begin{pmatrix} D_t\tilde{h}(t,\xi)/\tilde{h}(t,\xi) & \tilde{h}(t,\xi)\\ m(t,\xi)/\tilde{h}(t,\xi) & 0\end{pmatrix}E_V(t,t_1,\xi),\ \ E_V(t_1,t_1,\xi)=I. \tag{5.2.11}$$

Since $h(t,\xi)\sim\sqrt{m(t,\xi)}\sim\langle\xi\rangle_{b(t)}\sim\tilde{h}(t,\xi)$ for all $(t,\xi)\notin Z_{red}(\epsilon)$ and $h(t,\xi)=\epsilon a(t)\eta(t)=\tilde{h}(t,\xi)$ for all $(t,\xi)\in Z_{red}(\epsilon)$ we can use for $E^W(t,t_1,\xi)$ all results which we have proved in Section 3.3 for $E_V(t,t_1,\xi)$.

Estimates in the zones

Applying in the hyperbolic zone $Z_{hyp}(N)$ the results from Lemma 3.3.3 leads to

$$\|E^W_{hyp}(t_2,t_1,\xi)\| \leq C_N \frac{\sqrt{a(t_2)}}{\sqrt{a(t_1)}}. \tag{5.2.12}$$

Following Remark 3.3.3 the pseudo-differential zone $Z_{p.d}(N,\epsilon)$ can be skipped (in one part we can apply the result from the hyperbolic zone, the other part is compact).
In the reduced zone $Z_{red}(\epsilon)$ we use the result from Lemma 3.3.6. It implies

$$\|E^W_{red}(t_2,t_1,\xi)\| \leq \exp\Big(C\epsilon\int_{t_1}^{t_2} b(\tau)d\tau\Big) = \frac{\lambda(t_2)^{2C\epsilon}}{\lambda(t_1)^{2C\epsilon}}.$$

In the elliptic zone $Z_{ell}(\epsilon)$ by the result from Lemma 3.3.5 it follows

$$\|E^W_{ell}(t_2,t_1,\xi)\| \leq C\frac{b(t_2)}{b(t_1)}\exp\Big(\int_{t_1}^{t_2} b(\tau)d\tau\Big).$$

Representation of the solution

Now we return to our problem (5.2.2). Let us assume that

$$w(t,s,\xi) = \Psi(t,s,\xi)\hat{g}(s,\xi)$$

is the solution to (5.2.8). Together with the representation of the fundamental solution $E^W(t_2,t_1,s)$ in (5.2.10) we obtain

$$\begin{pmatrix} 0 & \Psi \\ 0 & D_t\Psi \end{pmatrix}(0,\hat{g}(s,\xi))^T = \operatorname{diag}(1/h(t,\xi),1)E^W(t,s,\xi)(0,-i\hat{g}(s,\xi))^T,$$

that is,

$$\Psi(t,s,\xi) = -iE^W_{12}(t,s,\xi)/h(t,\xi), \quad \Psi_t(t,s,\xi) = E^W_{22}(t,s,\xi).$$

On the other hand we write the Fourier transform of the solution to (5.2.2) as

$$\hat{v}(t,\xi) = \hat{\Phi}(t,s,\xi)\hat{g}(s,\xi).$$

Recalling (5.2.7) we have

$$\begin{aligned} \hat{\Phi}(t,s,\xi) &= \frac{\lambda(s)}{\lambda(t)}\Psi(t,s,\xi) = -i\frac{\lambda(s)}{\lambda(t)}\frac{1}{h(t,\xi)}E^W_{12}(t,s,\xi), \\ \hat{\Phi}_t(t,s,\xi) &= \frac{\lambda(s)}{\lambda(t)}\Big(\Psi_t(t,s,\xi) - \frac{b(t)}{2}\Psi(t,s,\xi)\Big) \\ &= \frac{\lambda(s)}{\lambda(t)}\Big(E^W_{22}(t,s,\xi) + i\frac{b(t)}{2h(t,\xi)}E^W_{12}(t,s,\xi)\Big). \end{aligned} \tag{5.2.13}$$

In our proof we will distinguish into four cases: $\eta(t) \searrow 0$, $\eta(t) \searrow \eta_\infty > 0$, $\eta(t) \nearrow \eta_\infty > 0$, and $\eta(t) \nearrow \infty$. In order to get better overview we prefer to use illustrated figures for these cases.
Case 1. $\eta(t) \searrow 0$

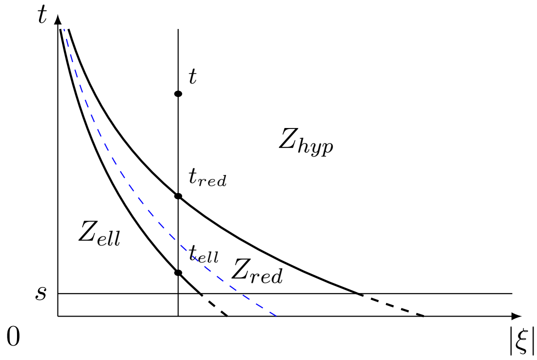


Fig. 5.1.: An illustrated picture for the case $\eta(t) \searrow 0$.

Case 2. $\eta(t) \searrow \eta_\infty > 0$,

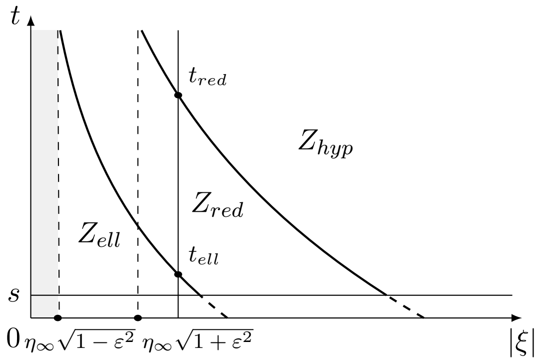


Fig. 5.2.: An illustrated picture for the case $\eta(t) \searrow \eta_\infty > 0$.

Case 3. $\eta(t) \nearrow \eta_\infty > 0$,

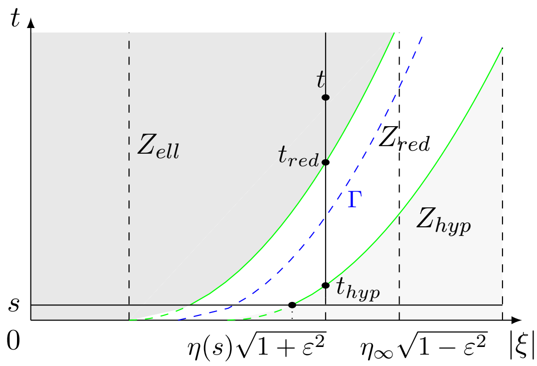


Fig. 5.3.: An illustrated picture for the case $\eta(t) \nearrow \eta_\infty > 0$.

Case 4. $\eta(t) \nearrow \infty$,

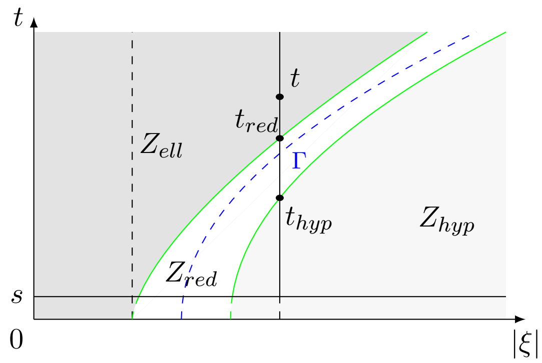


Fig. 5.4.: An illustrated picture for the case $\eta(t) \nearrow \infty$.

We first consider the case that $\eta(t)$ is decreasing, that is $\eta(t) \searrow \eta_\infty$, and that $(s,\xi) \in Z_{ell}(\epsilon)$, i.e., $|\xi| \leq \eta(s)\sqrt{1-\epsilon^2}$.

- If $|\xi| > \eta_\infty\sqrt{1+\epsilon^2}$, then for all s, t with $s \leq t_{ell} < t_{red} \leq t$ we have the representation

$$E^W(t,s,\xi) = E^W_{hyp}(t,t_{red},\xi)E^W_{red}(t_{red},t_{ell},\xi)E^W_{ell}(t_{ell},s,\xi).$$

  In the case $\eta_\infty = 0$ this relation is valid for any frequency $\xi \neq 0$.

- If $\eta_\infty\sqrt{1-\epsilon^2} \leq |\xi| \leq \eta_\infty\sqrt{1+\epsilon^2}$, then for all s, t with $s \leq t_{ell} \leq t$ it follows that

$$E^W(t,s,\xi) = E^W_{red}(t,t_{ell},\xi)E^W_{ell}(t_{ell},s,\xi).$$

- If $|\xi| \leq \eta_\infty\sqrt{1-\epsilon^2}$, then $E^W(t,s,\xi) = E^W_{ell}(t,s,\xi)$.

In the other case $|\xi| \geq \eta(s)\sqrt{1+\epsilon^2}$ we get $E^W(t,s,\xi) = E^W_{hyp}(t,s,\xi)$ for any $t \in [s,\infty)$. The intermediate cases are similar.
If we consider the case of $\eta(t) \nearrow \eta_\infty$ with $\eta_\infty \in (0,+\infty]$, then the situation is reversed, that is

$$E^W(t,s,\xi) = E^W_{ell}(t,t_{red},\xi)E^W_{red}(t_{red},t_{hyp},\xi)E^W_{hyp}(t_{hyp},s,\xi),$$

for the case $\eta(s)\sqrt{1+\epsilon^2} \leq |\xi| \leq \eta_\infty\sqrt{1-\epsilon^2}$ (if this set is not empty).

Estimates for the multipliers

In order to estimate our solution we need to estimate $|\hat{\Phi}(t,s,\xi)|$ in each zone of the extended phase space. The estimates for $|\hat{\Phi}_t(t,s,\xi)|$ will be obtained by a more refined approach.
In $Z_{hyp}(N)$ it holds $h(t,\xi) \sim a(t)|\xi|$. Thus,

$$|\hat{\Phi}_{hyp}(t_2,t_1,\xi)| \lesssim \frac{\lambda(t_1)}{\lambda(t_2)}\frac{1}{a(t_2)|\xi|}\frac{\sqrt{a(t_2)}}{\sqrt{a(t_1)}} \lesssim \frac{1}{|\xi|}\frac{\lambda(t_1)}{\lambda(t_2)}\frac{1}{\sqrt{a(t_1)}\sqrt{a(t_2)}}, \tag{5.2.14}$$

whereas in $Z_{red}(\epsilon)$ it holds $h(t,\xi) \sim a(t)\eta(t) \sim a(t)|\xi|$. Therefore, we obtain

$$|\hat{\Phi}_{red}(t_2,t_1,\xi)| \lesssim \frac{1}{a(t_2)|\xi|}\frac{\lambda(t_1)}{\lambda(t_2)}\exp\Big(C\epsilon\int_{t_1}^{t_2} b(\tau)d\tau\Big) \lesssim \frac{1}{a(t_2)|\xi|}\Big(\frac{\lambda(t_1)}{\lambda(t_2)}\Big)^{1-2\delta}, \tag{5.2.15}$$

where we denote $\delta := C\epsilon$. It is not difficult to prove that we can uniformly estimate $|\hat{\Phi}_{hyp}(t_2,t_1,\xi)|$ in (5.2.14) by the upper bound from (5.2.15). Indeed, this statement can be directly obtained by the following estimate:

$$\frac{\sqrt{a(t_2)}}{\lambda(t_2)^{2\delta}} \lesssim \frac{\sqrt{a(t_1)}}{\lambda(t_1)^{2\delta}}. \tag{5.2.16}$$

In order to prove this estimate we consider the monotonic behavior of the function

$$f(t) = \frac{a(t)}{\lambda(t)^{4\delta}}$$

for large t. We have

$$f'(t) = \frac{a'(t) - 2\delta a(t)b(t)}{\lambda(t)^{4\delta}} = \frac{a'(t) - 2\delta a(t)\mu(t)\frac{a^2(t)}{A(t)}}{\lambda(t)^{4\delta}}.$$

Due to the assumption (A2) we get $a'(t)A(t)/a^2(t) \le a_1$ for all t and due to assumption (B'3) we have $\mu(t) \to \infty$ as $t \to \infty$. Therefore, the function $f'(t)$ satisfies $f'(t) \le 0$ for all t large. This hint clues us to define

$$\Pi_{hyp}(\epsilon) = Z_{red}(\epsilon) \cup Z_{hyp}(N),$$

and we denote by $t_{|\xi|}$ the separating curve between $Z_{ell}(\epsilon)$ and $\Pi_{hyp}(\epsilon)$. This curve is given by

$$\eta^2(t_{|\xi|}) - |\xi|^2 = \epsilon^2\eta^2(t_{|\xi|}).$$

We will consider the following four cases:

Small frequencies $|\xi| \le \eta(s)\sqrt{1-\epsilon^2}$:

- *Case 1,* $t \le t_{|\xi|}$: in this case $(t,\xi),(s,\xi)$ belong to $Z_{ell}(\epsilon)$, therefore we can use the estimate (3.3.37). It holds
$$|\hat{\Phi}(t,s,\xi)| \lesssim \frac{1}{b(s)} \exp\big(-C|\xi|^2 B_a(s,t)\big). \tag{5.2.17}$$

- *Case 2,* $t \ge t_{|\xi|}$: it holds
$$|\hat{\Phi}(t,s,\xi)| \lesssim \frac{a(t_{|\xi|})}{a(t)b(s)} \exp\big(-C|\xi|^2 B_a(s,t_{|\xi|})\big)\Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{1-2\delta} \tag{5.2.18}$$
by using the definition of $t_{|\xi|}$.

We remark that there is no separating line if $|\xi| \le \eta_\infty\sqrt{1-\epsilon^2}$ (in particular, this is also true if $\eta(t)$ is increasing).

Large frequencies $|\xi| \ge \eta(s)\sqrt{1-\epsilon^2}$:

- *Case 3,* $t \le t_{|\xi|}$: in this case $(t,\xi),(s,\xi)$ belong to $\Pi_{hyp}(\epsilon)$. Therefore it holds
$$|\hat{\Phi}(t,s,\xi)| \lesssim \frac{1}{a(t)|\xi|}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta}. \tag{5.2.19}$$

- *Case 4,* $t \ge t_{|\xi|}$: it holds
$$|\hat{\Phi}(t,s,\xi)| \lesssim \frac{1}{a(t_{|\xi|})|\xi|} \exp\big(-C|\xi|^2 B_a(t_{|\xi|},t)\big)\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta}. \tag{5.2.20}$$

We remark that there is no separating line if $|\xi| \geq \eta_\infty\sqrt{1-\epsilon^2}$ (in particular, this is also true if $\eta(t)$ is decreasing).

Estimates for the time derivative of the multipliers

In this part we derive estimates for $\hat{\Phi}_t(t,s,\xi)$ in all zones. In $\Pi_{hyp}(\epsilon)$ we directly use the representation (5.2.13) together with $b(t) \lesssim h(t,\xi)$. Thus, in the *Case 3* we can estimate

$$|\hat{\Phi}_t(t,s,\xi)| \lesssim \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta}, \tag{5.2.21}$$

whereas in the *Case 2* we get

$$|\hat{\Phi}_t(t,s,\xi)| \lesssim |\hat{\Phi}_t(t_{|\xi|},s,\xi)|\Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{1-2\delta}. \tag{5.2.22}$$

We can treat the *Case 1* analogously to the proof of Lemma 3.3.8. Thus, we obtain the following estimate

$$|\hat{\Phi}_t(t,s,\xi)| \lesssim \frac{a^2(t)|\xi|^2}{b(s)b(t)}\exp\big(-C|\xi|^2 B_a(s,t)\big). \tag{5.2.23}$$

Plugging this estimate into (5.2.22) we get the following estimate in the *Case 2*:

$$|\hat{\Phi}_t(t,s,\xi)| \lesssim \frac{a(t_{|\xi|})|\xi|}{b(s)}\exp\big(-C|\xi|^2 B_a(s,t_{|\xi|})\big)\Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{1-2\delta}. \tag{5.2.24}$$

Here we used $b(t_{|\xi|}) \sim a(t_{|\xi|})|\xi|$.
In the *Case 4* we put $\Psi(t,s,\xi) = \hat{\Phi}_t(t,s,\xi)$. Then we obtain the equation of first order

$$\Psi_t + b(t)\Psi = -a^2(t)|\xi|^2\hat{\Phi}(t,s,\xi), \qquad \Psi(t_{|\xi|},s,\xi) = \hat{\Phi}_t(t_{|\xi|},s,\xi).$$

By using (5.2.20) for $|\hat{\Phi}(t,s,\xi)|$ and the estimate (5.2.21) for $|\hat{\Phi}_t(t_{|\xi|},s,\xi)|$ we derive for $t \geq t_{|\xi|}$ the following estimate:

$$\begin{aligned}
&|\hat{\Phi}_t(t,s,\xi)| \\
&\quad\lesssim \frac{\lambda^2(t_{|\xi|})}{\lambda^2(t)}\left(\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta} + \int_{t_{|\xi|}}^t \frac{\lambda^2(\tau)}{\lambda^2(t_{|\xi|})}\frac{|\xi|^2a^2(\tau)}{a(t_{|\xi|})|\xi|}\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta} e^{-C|\xi|^2B_a(t_{|\xi|},\tau)}d\tau\right) \\
&\quad\lesssim \Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta}\left(\frac{\lambda^2(t_{|\xi|})}{\lambda^2(t)} + \frac{|\xi|a^2(t)}{a(t_{|\xi|})}\int_{t_{|\xi|}}^t \Big(\frac{\lambda^2(\tau)}{\lambda^2(t)}b(\tau)\Big)\frac{1}{b(\tau)}e^{-C|\xi|^2B_a(t_{|\xi|},\tau)}d\tau\right).
\end{aligned}$$

We can now follow the proof for Lemma 3.3.8 in the case of effective dissipation. Consequently, we obtain the following estimate:

$$|\hat{\Phi}_t(t,s,\xi)| \lesssim \frac{a^2(t)|\xi|}{a(t_{|\xi|})b(t)}\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta}\exp\big(-C|\xi|^2B_a(t_{|\xi|},t)\big). \tag{5.2.25}$$

Final estimates

Lemma 5.2.2. *Let us define*

$$\Theta(s,t) := \max\{\eta(s),\eta(t)\}\sqrt{1-\epsilon^2}$$

for any $t \geq s$ and for any $s \in [0,\infty)$. If $|\xi| \geq \Theta(s,t)$, that is, $(t,\xi) \in \Pi_{hyp}(\epsilon)$ for any $t \geq s$, then $|\hat{\Phi}(t,s,\xi)|$ satisfies the estimate (5.2.19), while $|\hat{\Phi}_t(t,s,\xi)|$ satisfies the estimate (5.2.21). If $|\xi| \leq \Theta(s,t)$, then the corresponding estimates are as follows:

$$|\hat{\Phi}(t,s,\xi)| \lesssim \frac{1}{b(s)} \exp\big(-C'|\xi|^2 B_a(s,t)\big), \tag{5.2.26}$$

$$|\hat{\Phi}_t(t,s,\xi)| \lesssim \frac{a^2(t)|\xi|^2}{b(s)b(t)} \exp\big(-C'|\xi|^2 B_a(s,t)\big). \tag{5.2.27}$$

Proof. In order to prove (5.2.26) and (5.2.27) for $|\xi| \leq \Theta(s,t)$ we consider three cases:
(A) $|\xi| \leq \min\{\eta(s),\eta(t)\}\sqrt{1-\epsilon^2}$;
(B) η is decreasing and $\eta(t)\sqrt{1-\epsilon^2} \leq |\xi| \leq \eta(s)\sqrt{1-\epsilon^2}$;
(C) η is increasing and $\eta(s)\sqrt{1-\epsilon^2} \leq |\xi| \leq \eta(t)\sqrt{1-\epsilon^2}$.
In the case (A) we can easily check that $(t,\xi) \in Z_{ell}(\epsilon)$. Then the two estimates (5.2.26), (5.2.27) follow directly from (5.2.17) and (5.2.23). Now let us consider the case (B). Introducing

$$S(t,|\xi|) := \exp\big(-C_1|\xi|^2 B_a(s,t_{|\xi|})\big)\Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{2C_2}$$

the application of Lemma 3.3.15 implies

$$S(t,|\xi|) \leq \exp\big(-\min\{C_1,C_2\}|\xi|^2 B_a(s,t)\big).$$

In this way (5.2.26) follows from (5.2.18) by using $\frac{a(t_{|\xi|})}{a(t)b(s)} \lesssim \frac{1}{b(s)}$ for any $t \geq t_{|\xi|} \geq s$, and (5.2.27) follows from (5.2.24) by using $\frac{a(t_{|\xi|})|\xi|}{b(s)} \lesssim \frac{a^2(t)|\xi|^2}{b(s)b(t)}$.
Analogously, in the case (C) we have the following estimate, too:

$$\exp\big(-C_1|\xi|^2 B_a(t_{|\xi|},t)\big)\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{2C_2} \leq \exp\big(-\min\{C_1,C_2\}|\xi|^2 B_a(s,t)\big).$$

Hence, (5.2.26) follows from (5.2.20) by using

$$\frac{1}{a(t_{|\xi|})|\xi|} \lesssim \frac{1}{b(s)}$$

for any $t \geq t_{|\xi|}$, and (5.2.27) follows from (5.2.25) by using

$$\frac{a^2(t)|\xi|}{a(t_{|\xi|})b(t)} \lesssim \frac{a^2(t)|\xi|^2}{b(s)b(t)}.$$

This completes the proof. □

Matsumura-type estimates

Lemma 5.2.3. *The following estimates hold for large frequencies $|\xi| \geq \Theta(s,t)$:*

$$\||\xi|^{|\alpha|}\partial_t^l \widehat{\Phi}(t,s,\xi)\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\geq\Theta\}}} \lesssim \frac{1}{b(s)}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta} \|g(s,\cdot)\|_{H^{[|\alpha|+l-1]^+}} \tag{5.2.28}$$

for $l=0,1$ and for any $|\alpha| \geq 0$.

Proof. If $|\alpha|+l\geq 1$, then we can estimate

$$\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\xi)\|_{L^2_{\{|\xi|\geq\Theta\}}}\leq\||\xi|^{1-l}\partial_t^l\widehat{\Phi}(t,s,\xi)\|_{L^\infty_{\{|\xi|\geq\Theta\}}}\||\xi|^{|\alpha|+l-1}\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\geq\Theta\}}}.$$

Since $|\xi|^{|\alpha|+l-1}\leq\langle\xi\rangle^{|\alpha|+l-1}$ for any $|\alpha|+l\geq 1$, then the second term can be estimated by $\|g(s,\cdot)\|_{H^{|\alpha|+l-1}}$. Thanks to the estimates (5.2.19) and (5.2.21) we obtain

$$|\partial_t^l\widehat{\Phi}(t,s,\xi)|\lesssim a(t)^{-1+l}|\xi|^{-1+l}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta}$$

which has a decay uniformly for $|\xi|\geq\Theta(s,t)$.
Let us choose $|\alpha|=l=0$. In the case $\eta_\infty>0$ it holds $\Theta(s,t)\geq C>0$ for any $s\leq t$. Thus, it is reasonable to use $|\xi|^{-1}\sim\langle\xi\rangle^{-1}$ uniformly on the set $\{|\xi|\geq C\}$. Otherwise, when $\eta_\infty=0$ we use $b(s)\sim a(s)\eta(s)\lesssim a(t)|\xi|$ for large frequencies. Therefore, we can estimate

$$|\widehat{\Phi}(t,s,\xi)|_{L^2_{\{|\xi|\geq\Theta\}}}\lesssim\frac{1}{b(s)}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta}\|g(s,\cdot)\|_{L^2}.$$

This completes the proof. □

Lemma 5.2.4. *The following estimates hold for small frequencies $|\xi|\leq\Theta(s,t)$:*

$$\begin{aligned}&\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\xi)\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\leq\Theta\}}}\\&\quad\lesssim\frac{a^{2l}(t)}{b(s)b(t)^l}(B_a(s,t))^{-l}(B_a(s,t))^{-\frac{|\alpha|}{2}-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|g(s,\cdot)\|_{L^m}\end{aligned}\tag{5.2.29}$$

for $l=0,1$ and for any $|\alpha|\geq 0$.

Proof. Let us define m' and p by $1/m+1/m'=1$ and $1/p+1/m'=1/2$, that is, $1/p=1/m-1/2$. Then we have the following estimate:

$$\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\cdot)\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\leq\Theta\}}}\leq\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\cdot)\|_{L^p_{\{|\xi|\leq\Theta\}}}\|\widehat{g}(s,\cdot)\|_{L^{m'}_{\{|\xi|\leq\Theta\}}}.$$

We can estimate $\|\widehat{g}(s,\cdot)\|_{L^{m'}}$ by $\|g(s,\cdot)\|_{L^m}$. Therefore, we have only to control the L^p norm of the multiplier. Thanks to (5.2.26) and (5.2.27) we have the following estimate:

$$\begin{aligned}&\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\cdot)\|_{L^p_{\{|\xi|\leq\Theta\}}}\\&\quad\lesssim\frac{a^{2l}(t)}{b(s)b(t)^l}\Big(\int_{|\xi|\leq\Theta}|\xi|^{p(|\alpha|+2l)}\exp\big(-Cp|\xi|^2B_a(s,t)\big)d\xi\Big)^{\frac{1}{p}}.\end{aligned}$$

After using a change of variables $r=Cp|\xi|^2B_a(s,t)$ we conclude

$$\int_{|\xi|\leq\Theta}|\xi|^{p(|\alpha|+2l)}\exp\big(-Cp|\xi|^2B_a(s,t)\big)d\xi\lesssim(B_a(s,t))^{(-p(|\alpha|+2l)+n)/2}\int_0^\infty r^{p(|\alpha|+2l)+n-1}e^{-r}dr.$$

The integral on the right-hand side is bounded and we have a decay which is given by

$$\frac{a^{2l}(t)}{b(s)b(t)^l}(B_a(s,t))^{-\frac{|\alpha|}{2}-l-\frac{n}{2p}}=\frac{a^{2l}(t)}{b(s)b(t)^l}(B_a(s,t))^{-l}(B_a(s,t))^{-\frac{|\alpha|}{2}-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}.$$

This completes the proof. □

Now we show that the decay function given in (5.2.29) is worse than the one from (5.2.28). For this reason we compare

$$\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta} = \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{2C_1} = \exp\Big(-C_1\int_s^t b(\tau)d\tau\Big)$$

with

$$\big(B_a(s,t)\big)^{-\frac{|\alpha|}{2}-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} = \big(B_a(s,t)\big)^{C_{|\alpha|}}.$$

We distinguish two cases:

- If $\frac{\mu(t)}{A(t)}$ is increasing, then there exists a constant t_0 such that $c_0 = \frac{\mu(t_0)}{A(t_0)} \leq \frac{\mu(t)}{A(t)}$ uniformly for all $t \geq t_0$. Thus, we have

$$a^2(t)\Big(\frac{\mu(t)}{A(t)}\Big)^2 \geq c_0^2 a^2(t) \Rightarrow b(t) \geq c_0^2\frac{a^2(t)}{b(t)}$$

  for all $t \geq t_0$. Then (5.2.29) is of potential order decay in $b(t)$ while (5.2.28) is of exponential order decay in $b(t)$. This brings the desired dominance.

- If $\frac{\mu(t)}{A(t)}$ is decreasing, then we show the monotonicity of the function

$$\big(B_a(s,t)\big)^{C_{|\alpha|}}\Big(\frac{\lambda(t)}{\lambda(s)}\Big)^{2C_1}.$$

  We form the derivative of this function

$$\begin{aligned}
&\partial_t\Big[\Big(\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{C_{|\alpha|}}\Big(\frac{\lambda(t)}{\lambda(s)}\Big)^{2C_1}\Big]\\
&= C_{|\alpha|}\Big(\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{C_{|\alpha|}-1}\frac{a^2(t)}{b(t)}\Big(\frac{\lambda(t)}{\lambda(s)}\Big)^{2C_1} + 2C_1 b(t)\Big(\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{C_{|\alpha|}}\Big(\frac{\lambda(t)}{\lambda(s)}\Big)^{C_1}\\
&= \frac{a^2(t)}{b(t)}\Big(\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)^{C_{|\alpha|}-1}\Big(\frac{\lambda(t)}{\lambda(s)}\Big)^{C_1}\Big(C_{|\alpha|} + 2C_1\frac{b^2(t)}{a^2(t)}\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big). \qquad (5.2.30)
\end{aligned}$$

  Now we can see that the second term in the bracket tends to ∞ as t tends to ∞. Indeed, we have

$$\begin{aligned}
\frac{b^2(t)}{a^2(t)}\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau &= \frac{\mu^2(t)}{A^2(t)}\int_s^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau\\
&= \frac{\mu^2(t)}{A^2(t)}\Big(\frac{A^2(t)}{2\mu(t)} - \frac{A^2(s)}{2\mu(s)} + \int_s^t \frac{\mu'(\tau)A^2(\tau)}{2\mu^2(\tau)}\Big)d\tau\\
&\geq \frac{\mu(t)}{2} - \frac{\mu^2(t)}{2A^2(t)}\frac{A^2(s)}{\mu(s)} - \frac{\mu^2(t)}{2A^2(t)}\int_s^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau,
\end{aligned}$$

  here we used the Assumption (B'2), that is, $|\mu'(t)| \leq c\mu(t)a(t)/A(t)$. From the last estimate we obtain

$$\frac{\mu^2(t)}{A^2(t)}\int_s^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \geq \frac{1}{2+c}\Big(\mu(t) - \frac{\mu(t)}{A(t)}A(s)\Big) \gtrsim \mu(t).$$

  Here we used that the function $\frac{\mu(t)}{A(t)}$ is decreasing and the function $A(t)$ is increasing. Therefore for α is fixed, we can find a sufficiently large time $t_0 = t_0(\alpha)$ such that the expression (5.2.30) is positive for all $t \geq t_0$. This yields our expected comparison.

Gluing together (5.2.28) and (5.2.29) we conclude the desired estimates. This completes the proof of Theorem 5.2.1. $\square$

5.2.2. Properties of the decay function $B_a(s,t)$

Now let us turn back to the Cauchy problem for the semi-linear wave equation (5.2.1). In contemplation the proof of the next theorem we shall use the following properties of the function $B_a(s,t)$:

Proposition 5.2.5. *Let us assume that $b(t)$ satisfies the assumptions (B'1) to (B'5) and the additional assumption*

(R1) There exists a constant $\gamma \in [0,2)$ such that

$$\mu'(t) \leq \gamma \mu(t) \frac{a(t)}{A(t)} \ \textit{for}\ t \geq 0. \tag{5.2.31}$$

Then the following statements hold:

1. *For any $s \in [0,t]$ we have*

$$B_a(s,t) = \int_s^t \frac{a^2(\tau)}{b(\tau)} d\tau \approx \frac{A^2(t)}{\mu(t)} - \frac{A^2(s)}{\mu(s)}. \tag{5.2.32}$$

2. *For any $s \in [0,\lambda t]$ with λ is fixed and $\lambda \in (0,1)$ we have*

$$B_a(s,t) \approx B_a(0,t). \tag{5.2.33}$$

3. *For any $s \in [\lambda t, t]$ there exists a constant $M \geq 0$ such that*

$$\Big(\frac{A(\lambda t)}{A(t)} \Big)^{2+M} B_a(0,t) \leq B_a(0,s) \leq B_a(0,t). \tag{5.2.34}$$

Remark 5.2.2. The Assumption (R1) is stronger than the upper bound of Assumption (B'2) for $k = 1$ and this assumption also implies the Assumption (B'4).

Example 5.2.1. Let us choose $a(t) = (1+t)^l$ with $l > 0$. Therefore, we obtain

$$A(t) \approx (1+t)^{l+1} \text{ and } b(t) = \frac{\mu(t)}{1+t}.$$

If we consider functions $\mu(t)$ satisfying

$$(\log(c+t))^\kappa \leq \mu(t) \leq (1+t)^{2(l+1)} \text{ for } \kappa > 1,$$

then assumptions (B'4) and (B'5) hold, whereas the assumption (R1) holds if we choose

$$\mu(t) = (\log(c+t))^\kappa \ \text{ or } \ \mu(t) = (1+t)^{\gamma(l+1)} \text{ for } \kappa > 1, 0 < \gamma < 2.$$

Example 5.2.2. Let us choose $a(t) = e^t$. Thus, we obtain

$$A(t) = e^t \ \text{ and } \ b(t) = \mu(t).$$

Let us choose functions $\mu(t)$ satisfying

$$(1+t)^k (\log(c+t))^\kappa \leq \mu(t) \leq e^{2t} \ \text{ for } \ k \geq 1,\ k+\kappa > 2.$$

Then the assumptions (B'4) and (B'5) hold, whereas the assumption (R1) holds if we choose

$$\mu(t) = (1+t)^k (\log(c+t))^\kappa \ \text{ or } \ \mu(t) = e^{\beta t}, \text{ for } k \geq 1,\ k+\kappa > 2, 0 < \beta < 2.$$

Proof. 1. It follows from the Assumption $(R1)$ that

$$\begin{aligned}\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau &= \int_s^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau = \frac{A^2(t)}{2\mu(t)} - \frac{A^2(s)}{2\mu(s)} + \int_s^t \frac{\mu'(\tau)A^2(\tau)}{2\mu^2(\tau)}d\tau \\ &\leq \frac{A^2(t)}{2\mu(t)} - \frac{A^2(s)}{2\mu(s)} + \int_s^t \frac{\gamma a(\tau)A(\tau)}{2\mu(\tau)}d\tau.\end{aligned}$$

Hence,

$$(2-\gamma)\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau \leq \frac{A^2(t)}{\mu(t)} - \frac{A^2(s)}{\mu(s)}.$$

Furthermore, there exists a constant $M \geq 0$ such that $|\mu'(t)| \leq M\mu(t)a(t)/A(t)$ (Assumption (B'2)). Consequently,

$$\int_s^t \frac{\mu'(\tau)A^2(\tau)}{2\mu^2(\tau)}d\tau \geq -\int_s^t \frac{Ma(\tau)A(\tau)}{2\mu(\tau)}d\tau,$$

thus

$$(2+M)\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau \geq \frac{A^2(t)}{\mu(t)} - \frac{A^2(s)}{\mu(s)}.$$

2. By integrating (5.2.31) over $[s,t]$ we derive

$$\frac{\mu(t)}{\mu(s)} \leq \Big(\frac{A(t)}{A(s)}\Big)^\gamma \tag{5.2.35}$$

for any $s \geq 0$ and $t \geq s$. Taking into consideration (5.2.32) we get

$$\begin{aligned}B_a(0,t) \geq B_a(s,t) \approx \frac{A^2(t)}{\mu(t)} - \frac{A^2(s)}{\mu(s)} &\geq \frac{A^2(t)}{\mu(t)} - \frac{A^2(t)}{\mu(t)}\Big(\frac{A(s)}{A(t)}\Big)^{2-\gamma} \\ &= \bigg(1 - \Big(\frac{A(s)}{A(t)}\Big)^{2-\gamma}\bigg)\frac{A^2(t)}{\mu(t)} \geq c_{\lambda,\gamma}\frac{A^2(t)}{\mu(t)} = c_{\lambda,\gamma}B_a(0,t),\end{aligned}$$

where after application of l'Hospital we may put $c_{\lambda,\gamma} = \liminf_{t\to\infty}\bigg(1 - \Big(\frac{A(\lambda t)}{A(t)}\Big)^{2-\gamma}\bigg) > 0$ since $s \in [0,\lambda t]$ with a fixed $\lambda \in (0,1)$ and $\gamma \in [0,2)$.

3. Using the Assumption (B'2) for $k=1$ we conclude

$$\frac{\mu'(t)}{\mu(t)} \geq -M\frac{a(t)}{A(t)}. \tag{5.2.36}$$

It is clear that when $\mu(t)$ is increasing we can take $M=0$. By integrating (5.2.36) over $[s,t]$ we derive

$$\frac{\mu(t)}{\mu(s)} \geq \Big(\frac{A(t)}{A(s)}\Big)^{-M}$$

for any $s \geq 0$ and $t \geq s$. For any fixed $\lambda \in (0,1)$ it holds

$$B_a(0,t) \geq B_a(0,s) \approx \frac{A^2(s)}{\mu(s)} \geq \frac{A^2(t)}{\mu(t)}\Big(\frac{A(s)}{A(t)}\Big)^{2+M} \approx \Big(\frac{A(s)}{A(t)}\Big)^{2+M} B_a(0,t).$$

This completes the proof of this statement. □

Now we introduce an additional assumption which will explain a restriction of damping terms depending on the term of increasing speed of propagation.

(R2) The function $\theta(t) := \dfrac{A^2(t)}{\mu(t)}$ satisfies the following conditions:

- $\theta(t)$ is increasing,
- $\theta'(t) \leq \beta\theta(t)\dfrac{1}{\log A(t)}\dfrac{a(t)}{A(t)}$, $\beta > 0$.

Example 5.2.3. Let us choose $a(t) = e^t e^{e^t}$. Thus, we obtain

$$A(t) = e^{e^t} \text{ and } b(t) = \mu(t)e^t.$$

We consider functions $\mu(t)$ satisfying

$$e^t(1+t)^\kappa \leq \mu(t) \leq e^{2e^t} \text{ for } \kappa > 1.$$

Then the assumptions (B'4) and (B'5) hold, whereas the assumption (R2) holds if we choose, for example,

$$\mu(t) = e^{2e^t}/e^{\beta t} \text{ or } \mu(t) = e^{2e^t}/\log(c+t) \text{ for } \beta, c > 0.$$

We have the following proposition:

Proposition 5.2.6. *Let us assume that $b(t)$ satisfies the assumptions (B'1) to (B'5) and the additional assumption (R2). Then the following statements hold:*

1. *For any $s \in [0,t]$ we have*
$$B_a(s,t) = \int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau \approx \log\big(A(t)\big)\theta(t) - \log\big(A(s)\big)\theta(s). \tag{5.2.37}$$

2. *For any $s \in [0,\lambda t]$ with λ is fixed and $\lambda \in (0,1)$ we have*
$$B_a(s,t) \approx B_a(0,t). \tag{5.2.38}$$

Proof. 1. It follows from the Assumption $(R2)$ that

$$\begin{aligned}
\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau &= \int_s^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau = \int_s^t \frac{a(\tau)}{A(\tau)}\theta(\tau)d\tau = \theta(\tau)\log A(\tau)\big|_s^t - \int_s^t \theta'(\tau)\log A(\tau)d\tau \\
&\geq \theta(t)\log A(t) - \theta(s)\log A(s) - \beta\int_s^t \frac{a(\tau)}{A(\tau)}\theta(\tau)d\tau.
\end{aligned}$$

Hence,

$$(2+\beta)\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau \geq \theta(t)\log A(t) - \theta(s)\log A(s).$$

Otherwise, we have $\theta(t)$ is increasing. Thus

$$\begin{aligned}
\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau &= \int_s^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau = \int_s^t \frac{a(\tau)}{A(\tau)}\theta(\tau)d\tau = \theta(\tau)\log A(\tau)\big|_s^t - \int_s^t \underbrace{\theta'(\tau)}_{\geq 0}\log A(\tau)d\tau \\
&\leq \log A(t)\theta(t) - \log A(s)\theta(s).
\end{aligned}$$

2. Taking into consideration (5.2.37) we get for any $s \geq 0$ and $t \geq s$

$$\begin{aligned} B_a(0,t) \geq B_a(s,t) &\approx \theta(t)\log A(t) - \theta(s)\log A(s) \geq \theta(t)\log A(t) - \theta(t)\log A(s) \\ &= \Big(1 - \frac{\log A(s)}{\log A(t)}\Big)\theta(t)\log A(t) \geq \bar{c}_{\lambda,\gamma}\theta(t)\log A(t) \approx \bar{c}_{\lambda,\gamma}B_a(0,t), \end{aligned}$$

where after application of l'Hospital we may put $\bar{c}_{\lambda,\gamma} = \liminf_{t\to\infty}\Big(1 - \frac{\log A(\lambda t)}{\log A(t)}\Big) > 0$ since $s \in [0,\lambda t]$ with a fixed $\lambda \in (0,1)$. This completes the proof. □

In order to classify the sub-exponential case and super-exponential case we introduce

$$\nu(\lambda,t) := \frac{a(t)}{A(t)}\frac{A(\lambda t)}{a(\lambda t)} \quad \text{and} \quad \nu(\lambda) := \limsup_{t\to\infty}\frac{a(t)}{A(t)}\frac{A(\lambda t)}{a(\lambda t)}. \tag{5.2.39}$$

Then we classify:

1. *Sub-exponential case :* $\nu(\lambda) \lesssim 1$,
2. *Super-exponential case :* $\nu(\lambda) = \infty$.

5.2.3. Global existence of small data solutions for wave models with sub-exponential propagation speed

We define the following parameters:

$$\begin{aligned} p_{Fuj} &:= 1 + \frac{2}{n} \quad \text{for } n \geq 1, \\ \bar{p}_1 &:= 1 + \Big(1 - \frac{2a_0}{2+M}\Big)\frac{2}{n} \quad \text{for } n \geq 1, \\ \bar{p}_2 &:= 1 + \Big(1 - \frac{\lambda}{\nu(\lambda)}\frac{2a_0}{2+M}\Big)\frac{2}{n} \quad \text{for } n \geq 1, \\ \bar{p}_3 &:= \frac{1}{2} + \frac{1}{2\Big(1 - \frac{2+M}{2-\gamma} + \frac{\lambda}{\nu(\lambda)}\frac{2+M}{2-\gamma}\Big)} + \left(\frac{1}{1 - \frac{2+M}{2-\gamma} + \frac{\lambda}{\nu(\lambda)}\frac{2+M}{2-\gamma}} - \frac{\frac{\lambda}{\nu(\lambda)}\frac{2a_0}{2+M}}{1 - \frac{2+M}{2-\gamma} + \frac{\lambda}{\nu(\lambda)}\frac{2+M}{2-\gamma}}\right)\frac{2}{n} \\ &\qquad \text{for } n \geq 1, \\ p_{GN} &:= 1 + \frac{2}{n-2} \quad \text{for } n \geq 3. \end{aligned}$$

In the following we use the notations

$$\begin{aligned} \mathcal{A}_{1,1} &:= (L^1 \cap H^1) \times (L^1 \cap L^2), \\ \|(u,v)\|_{\mathcal{A}_{1,1}} &:= \|u\|_{L^1} + \|u\|_{H^1} + \|v\|_{L^1} + \|v\|_{L^2}. \end{aligned}$$

Theorem 5.2.7. ***(Sub-exponential order case)*** *We assume the Hypotheses (A1) to (A3), (B'1) to (B'5) and (R1). Let us assume $n \leq 4$ and*

$$\begin{cases} p > \bar{p} \ \text{ and } \ p \geq 2 & \text{if } n=1,2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if } n=3, \\ p = 2 = p_{GN}(4) & \text{if } n=4. \end{cases} \tag{5.2.40}$$

Here $\bar{p}$ is defined as

$$\bar{p} := \max\{\bar{p}_1; \bar{p}_2; \bar{p}_3\}, \tag{5.2.41}$$

where $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_3$ are defined as above. Moreover, we need the assumption

$$\frac{\lambda}{\nu(\lambda)} > \frac{(2+M)n}{8a_0}, \tag{5.2.42}$$

where a_0, M and $\nu(\lambda)$ are defined in (A1), (5.2.36) and (5.2.39), respectively.
Then there exists a constant $\epsilon_0 > 0$ such that data with

$$\|(u_1,u_2)\|_{\mathcal{A}_{1,1}} \le \epsilon_0,$$

imply the existence of a unique solution to (5.2.1) in $\mathcal{C}([0,\infty),H^1)\cap\mathcal{C}^1([0,\infty),L^2)$. Furthermore, there exists a constant $C>0$ such that this solution satisfies the estimates

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\le C \quad \|(u_1,u_2)\|_{\mathcal{A}_{1,1}}(1+B_a(0,t))^{-\frac{n}{4}}, && (5.2.43)\\
\|\nabla u(t,\cdot)\|_{L^2} &\le C \quad \|(u_1,u_2)\|_{\mathcal{A}_{1,1}}(1+B_a(0,t))^{-\frac{n}{4}-\frac{1}{2}}, && (5.2.44)\\
\|u_t(t,\cdot)\|_{L^2} &\le C \quad \|(u_1,u_2)\|_{\mathcal{A}_{1,1}}(1+B_a(0,t))^{-\frac{n}{4}-1}a^2(t)(b(t))^{-1}. && (5.2.45)
\end{aligned}$$

Proof. We introduce the space

$$X(t) = \mathcal{C}([0,t],H^1)\cap\mathcal{C}^1([0,t],L^2) \tag{5.2.46}$$

with the norm

$$\begin{aligned}
\|u\|_{X(t)} := \sup_{0\le\tau\le t}\Big((1+B_a(0,\tau))^{n/4}\|u(\tau,\cdot)\|_{L^2} + (1+B_a(0,\tau))^{n/4+1/2}\|\nabla u(\tau,\cdot)\|_{L^2}\\
+ (1+B_a(0,\tau))^{n/4+1}b(\tau)/a^2(\tau)\|u_t(\tau,\cdot)\|_{L^2}\Big).
\end{aligned}$$

We define the operator N in the form

$$Nu(t,x) = E_1(t,0,x)*_x u_1(x) + E_2(t,0,x)*_x u_2(x) + \int_0^t E_2(t,s,x)*_x f\big(u(s,x)\big)ds.$$

Our goal is to prove that

$$\begin{aligned}
\|Nu\|_{X(t)} &\le C\|(u_1,u_2)\|_{\mathcal{A}_{1,1}} + C\|u\|^p_{X(t)}, && (5.2.47)\\
\|Nu-Nv\|_{X(t)} &\le C\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big) && (5.2.48)
\end{aligned}$$

uniformly with respect to $t\in[0,\infty)$.
We shall even establish the stronger inequalities than (5.2.47) and (5.2.48), namely,

$$\begin{aligned}
\|Nu\|_{X(t)} &\le C\|(u_1,u_2)\|_{\mathcal{A}_{1,1}} + C\|u\|^p_{X_0(t)}, && (5.2.49)\\
\|Nu-Nv\|_{X(t)} &\le C\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big), && (5.2.50)
\end{aligned}$$

where

$$\|u\|_{X_0(t)} := \sup_{0\le\tau\le t}\Big((1+B_a(0,\tau))^{n/4}\|u(\tau,\cdot)\|_{L^2} + (1+B_a(0,\tau))^{n/4+1/2}\|\nabla u(\tau,\cdot)\|_{L^2}\Big).$$

The completion of the proof of Theorem 5.2.7 follows from the next proposition:

Proposition 5.2.8. *Let us assume that the power p and the dimension n satisfy (5.2.40). Let $(u_1,u_2)\in\mathcal{A}_{1,1}$ and $u\in X(t)$. Then we have for $j+l=0,1$ the following estimates:*

$$\Big(\frac{b(t)}{a^2(t)}\Big)^l\big(1+B_a(0,t)\big)^{(n/4+j/2+l)}\|\nabla^j\partial_t^l Nu(t,\cdot)\|_{L^2}\leq C\|(u_1,u_2)\|_{\mathcal{A}_{1,1}}+C\|u\|_{X_0(t)}^p, \tag{5.2.51}$$

$$\begin{aligned}\Big(\frac{b(t)}{a^2(t)}\Big)^l\big(1+B_a(0,t)\big)^{(n/4+j/2+l)}\big\|\nabla^j\partial_t^l\big(Nu(t,\cdot)-Nv(t,\cdot)\big)\big\|_{L^2}&\\ \leq C\|u-v\|_{X_0(t)}\big(\|u\|_{X_0(t)}^{p-1}+\|v\|_{X_0(t)}^{p-1}\big).&\end{aligned}\tag{5.2.52}$$

Proof. From the Matsumura type estimates for the linear models we get

$$\begin{aligned}\|\nabla^j\partial_t^l Nu(t,\cdot)\|_{L^2}\ \leq\ & C\Big(\frac{a^2(t)}{b(t)}\Big)^l\big(1+B_a(0,t)\big)^{-(n/4+j/2+l)}\|(u_1,u_2)\|_{\mathcal{A}_{1,1}}\\ &+C\int_0^{\lambda t}(b(s))^{-1}\Big(\frac{a^2(t)}{b(t)}\Big)^l\big(1+B_a(s,t)\big)^{-(n/4+j/2+l)}\|f(u(s,\cdot))\|_{L^1\cap L^2}ds\\ &+C\int_{\lambda t}^t(b(s))^{-1}\Big(\frac{a^2(t)}{b(t)}\Big)^l\big(1+B_a(s,t)\big)^{-(j/2+l)}\|f(u(s,\cdot))\|_{L^2}ds\end{aligned}\tag{5.2.53}$$

for $j+l=0,1$. Taking account of $f(u)=|u|^p$ brings

$$\|f(u(s,\cdot))\|_{L^1\cap L^2}\lesssim\|u(s,\cdot)\|_{L^p}^p+\|u(s,\cdot)\|_{L^{2p}}^p,$$

and

$$\|f(u(s,\cdot))\|_{L^2}\lesssim\|u(s,\cdot)\|_{L^{2p}}^p.$$

Applying Gagliardo-Nirenberg inequality we have

$$\|u(s,\cdot)\|_{L^p}^p\lesssim\|u(s,\cdot)\|_{L^2}^{p(1-\theta(p))}\|\nabla u(s,\cdot)\|_{L^2}^{p\,\theta(p)}, \tag{5.2.54}$$

$$\|u(s,\cdot)\|_{L^{2p}}^p\lesssim\|u(s,\cdot)\|_{L^2}^{p(1-\theta(2p))}\|\nabla u(s,\cdot)\|_{L^2}^{p\,\theta(2p)}, \tag{5.2.55}$$

where

$$\theta(p)=\frac{n}{2}\frac{p-2}{p},\qquad \theta(2p)=\frac{n}{2}\frac{p-1}{p}.$$

The restriction $\theta(p)\geq 0$ implies that $p\geq 2$, while the restriction $\theta(2p)\leq 1$ implies that $p\leq p_{GN}(n)$ if $n\geq 3$. By using (5.2.54), (5.2.55) and the definition of the function space $X(t)$ we have the following estimate for $\|f(u(s,\cdot))\|_{L^1\cap L^2}$:

$$\|f(u(s,\cdot))\|_{L^1\cap L^2}\lesssim\|u\|_{X_0(s)}^p\big(1+B_a(0,s)\big)^{-p(n/4+\theta(p)/2)}=\|u\|_{X_0(s)}^p\big(1+B_a(0,s)\big)^{-(p-1)n/2}, \tag{5.2.56}$$

here we have used $\theta(p)<\theta(2p)$, whereas the following estimate is obtained for $\|f(u(s,\cdot))\|_{L^2}$:

$$\|f(u(s,\cdot))\|_{L^2}\lesssim\|u\|_{X_0(s)}^p\big(1+B_a(0,s)\big)^{-p(n/4+\theta(2p)/2)}=\|u\|_{X_0(s)}^p\big(1+B_a(0,s)\big)^{-(2p-1)n/4}. \tag{5.2.57}$$

Plugging (5.2.56) and (5.2.57) into (5.2.53) we get

$$\begin{aligned}\|\nabla^j\partial_t^l Nu(t,\cdot)\|_{L^2} &\leq C\left(\frac{a^2(t)}{b(t)}\right)^l\big(1+B_a(0,t)\big)^{-(n/4+j/2+l)}\epsilon\\ &+C\|u\|^p_{X_0(t)}\underbrace{\int_0^{\lambda t}(b(s))^{-1}\left(\frac{a^2(t)}{b(t)}\right)^l\big(1+B_a(s,t)\big)^{-(n/4+j/2+l)}\big(1+B_a(0,s)\big)^{-(p-1)n/2}ds}_{\mathbf{A}}\\ &+C\|u\|^p_{X_0(t)}\underbrace{\int_{\lambda t}^{t}(b(s))^{-1}\left(\frac{a^2(t)}{b(t)}\right)^l\big(1+B_a(s,t)\big)^{-(j/2+l)}\big(1+B_a(0,s)\big)^{-(2p-1)n/4}ds}_{\mathbf{B}}\,.\end{aligned}$$

Let us derive for $j+l=0,1$ estimates for the case $s\in[0,\lambda t]$. We have

$$\mathbf{A}\approx\left(\frac{a^2(t)}{b(t)}\right)^l\big(1+B_a(0,t)\big)^{-(n/4+j/2+l)}\underbrace{\int_0^{\lambda t}(b(s))^{-1}\big(1+B_a(0,s)\big)^{-(p-1)n/2}ds}_{A_1}\,.$$

Here, we have used (5.2.33) and we denote $\alpha_1(p):=(p-1)n/2$. Now we try to find a condition for the power p which guarantees that the integral term A_1 remains bounded.

Case 1: $\alpha_1(p)>1$ *(i.e.* $p>p_{Fuj}$*)*

We have for $j+l=0,1$

$$A_1=\int_0^{\lambda t}(b(s))^{-1}\big(1+B_a(0,s)\big)^{-(p-1)n/2}ds\leq\frac{1}{a^2(0)}\int_0^{B_a(0,\lambda t)}(1+r)^{-(p-1)n/2}dr\leq C.$$

Here we have used the change of variables $r=B_a(0,s)$ and the condition $p>p_{Fuj}(n)$ guarantees that the integral term is bounded.

Case 2: $\alpha_1(p)<1$

We perform the following straight-forward calculations:

$$\begin{aligned}A_1&=\int_0^{\lambda t}\big(1+B_a(0,s)\big)^{-\alpha_1(p)}d\big(1+B_a(0,s)\big)\frac{1}{a^2(s)}\\ &=\frac{(1+B_a(0,s))^{-\alpha_1(p)+1}}{(1-\alpha_1(p))a^2(s)}\Bigg|_0^{\lambda t}+\frac{2}{1-\alpha_1(p)}\int_0^{\lambda t}\big(1+B_a(0,s)\big)^{-\alpha_1(p)+1}\frac{a'(s)}{a^3(s)}ds\\ &\geq\frac{(1+B_a(0,s))^{-\alpha_1(p)+1}}{(1-\alpha_1(p))a^2(s)}\Bigg|_0^{\lambda t}+\frac{2a_0}{1-\alpha_1(p)}\int_0^{\lambda t}\frac{(1+B_a(0,s))^{-\alpha_1(p)}}{b(s)}\frac{b(s)\big(1+B_a(0,s)\big)}{a(s)A(s)}ds\\ &\geq\frac{(1+B_a(0,s))^{-\alpha_1(p)+1}}{(1-\alpha_1(p))a^2(s)}\Bigg|_0^{\lambda t}+\frac{2a_0}{(1-\alpha_1(p))(2+M)}\int_0^{\lambda t}\frac{(1+B_a(0,s))^{-\alpha_1(p)}}{b(s)}\frac{\mu(s)a(s)A^2(s)}{\mu(s)a(s)A^2(s)}ds\\ &\geq\frac{(1+B_a(0,s))^{-\alpha_1(p)+1}}{(1-\alpha_1(p))a^2(s)}\Bigg|_0^{\lambda t}+\frac{2a_0}{(1-\alpha_1(p))(2+M)}\underbrace{\int_0^{\lambda t}b(s)^{-1}(1+B_a(0,s))^{-\alpha_1(p)}ds}_{A_1}\,.\end{aligned}$$

From the last estimate we obtain

$$\left(\frac{2a_0}{(1-\alpha_1(p))(2+M)}-1\right)A_1\leq\frac{1}{1-\alpha_1(p)}\Big(\frac{1}{a^2(0)}-\frac{(1+B_a(0,\lambda t))^{-\alpha_1(p)+1}}{a^2(\lambda t)}\Big).\tag{5.2.58}$$

Since $p > \bar{p}_1$ we may conclude $\frac{2a_0}{(1-\alpha_1(p))(2+M)} - 1 > 0$. Otherwise $\mathcal{G}_1(\lambda t) := \frac{(1+B_a(0,\lambda t))^{-\alpha_1(p)+1}}{a^2(\lambda t)}$ is strictly decreasing. Indeed, taking the derivative of function $\mathcal{G}_1(t)$ we get

$$\begin{aligned}
\mathcal{G}_1'(t) &= \frac{\Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)}a^2(t) - 2a'(t)a(t)\big(1+B_a(0,t)\big)\Big)\big(1+B_a(0,t)\big)^{-\alpha_1(p)}}{a^4(t)} \\
&\leq \frac{\Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)} - 2\frac{a'(t)}{a(t)}B_a(0,t)\Big)\big(1+B_a(0,t)\big)^{-\alpha_1(p)}}{a^2(t)} \\
&\leq \frac{\Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)} - \frac{2a_0}{2+M}\frac{a(t)}{A(t)}\frac{A^2(t)}{\mu(t)}\Big)\big(1+B_a(0,t)\big)^{-\alpha_1(p)}}{a^2(t)}.
\end{aligned}$$

The condition $p > \bar{p}_1$ implies $\mathcal{G}_1'(t) < 0$. This gives the boundedness of the right-hand side of (5.2.58). Therefore, we can obtain our desired estimate for A_1.

Case 3: $\alpha_1(p) = 1$

We have

$$A_1 = \int_0^{\lambda t} \frac{1}{b(s)\big(1+B_a(0,s)\big)}ds \approx \int_0^{\lambda t} \frac{\mu(s)}{A^2(s)}\frac{A(s)}{a(s)\mu(s)}ds \approx \int_0^{\lambda t} \frac{ds}{a(s)A(s)} \leq \int_0^{\lambda t} \frac{dA(s)}{A(s)^{1+2a_0}} \leq C.$$

Summarizing, for all $p > \bar{p}_1$ we have

$$\mathbf{A} \lesssim \Big(\frac{a^2(t)}{b(t)}\Big)^l \big(1+B_a(0,t)\big)^{-(n/4+j/2+l)}. \tag{5.2.59}$$

Now let us carry out necessary estimates for the case $s \in [\lambda t, t]$. We have

$$\begin{aligned}
\mathbf{B} &= \Big(\frac{a^2(t)}{b(t)}\Big)^l \int_{\lambda t}^t (b(s))^{-1}\big(1+B_a(s,t)\big)^{-(j/2+l)}\Big(1+B_a(0,s)\Big)^{-(2p-1)n/4}ds \\
&= -\Big(\frac{a^2(t)}{b(t)}\Big)^l \int_{\lambda t}^t \big(1+B_a(s,t)\big)^{-(j/2+l)}\frac{\big(1+B_a(0,s)\big)^{-\alpha_2(p)}}{a^2(s)}d\big(1+B_a(s,t)\big).
\end{aligned}$$

Here $\alpha_2(p) := (2p-1)n/4$. We consider the integral term in the last equality. For $j=0$, $l=0$ and $\alpha_2(p) \neq 1$ we get

$$\begin{aligned}
\mathbf{B} &= \int_{\lambda t}^t (b(s))^{-1}\Big(1+B_a(0,s)\Big)^{-\alpha_2(p)}ds = \int_{\lambda t}^t \big(1+B_a(0,s)\big)^{-\alpha_2(p)}\frac{1}{a^2(s)}d\big(1+B_a(0,s)\big) \\
&= \frac{\big(1+B_a(0,s)\big)^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Bigg|_{\lambda t}^t + \frac{2}{1-\alpha_2(p)}\int_{\lambda t}^t (1+B_a(0,s))^{-\alpha_2(p)+1}\frac{a'(s)}{a^3(s)}ds \\
&= \frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Bigg|_{\lambda t}^t + \frac{2}{1-\alpha_2(p)}\int_{\lambda t}^t \frac{\big(1+B_a(0,s)\big)^{-\alpha_2(p)}}{b(s)}\frac{b(s)\big(1+B_a(0,s)\big)a'(s)}{a^3(s)}ds.
\end{aligned} \tag{5.2.60}$$

Case 1: $\alpha_2(p) > 1$

Then from the equality (5.2.60) we have

$$\mathbf{B}+\frac{2}{\alpha_2(p)-1}\int_{\lambda t}^{t}\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{b(s)}\frac{b(s)\big(1+B_a(0,s)\big)a'(s)}{a^3(s)}ds=\frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Big|_{\lambda t}^{t},$$

$$\mathbf{B}+\frac{2a_0}{(\alpha_2(p)-1)(2+M)}\underbrace{\int_{\lambda t}^{t}\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{b(s)}}_{\mathbf{B}}\frac{\mu(s)a(s)A^2(s)}{A(s)\mu(s)a(s)A(s)}ds\leq\frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Big|_{\lambda t}^{t},$$

$$\Big(1+\frac{2a_0}{(\alpha_2(p)-1)(2+M)}\Big)\mathbf{B}\lesssim\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{a^2(\lambda t)}-\frac{(1+B_a(0,t))^{-\alpha_2(p)+1}}{a^2(t)}.$$

It implies that

$$\mathbf{B}\lesssim\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{a^2(\lambda t)}-\frac{(1+B_a(0,t))^{-\alpha_2(p)+1}}{a^2(t)}.$$

Case 2: $\alpha_2(p)<1$

Then from the equality (5.2.60) we get

$$\begin{aligned}\mathbf{B}&\geq\frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Big|_{\lambda t}^{t}+\frac{2a_0}{(1-\alpha_2(p))(2+M)}\int_{\lambda t}^{t}\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{b(s)}\frac{\mu(s)a(s)A^2(s)}{A(s)\mu(s)a(s)A(s)}ds\\&\geq\frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Big|_{\lambda t}^{t}+\frac{2a_0}{(1-\alpha_2(p))(2+M)}\int_{\lambda t}^{t}\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{b(s)}ds.\end{aligned}$$

From the last inequality we get

$$\Big(\frac{2a_0}{(1-\alpha_2(p))(2+M)}-1\Big)\mathbf{B}\lesssim\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{a^2(\lambda t)}-\frac{(1+B_a(0,t))^{-\alpha_2(p)+1}}{a^2(t)}.$$

Since $\alpha_2(p)=\alpha_1(p)+n/4$ we have

$$\frac{2a_0}{(1-\alpha_2(p))(2+M)}-1>\frac{2a_0}{(1-\alpha_1(p))(2+M)}-1.$$

Moreover, by using $p>\bar{p}_1$ we obtain $\frac{2a_0}{(1-\alpha_1(p))(2+M)}-1>0$. Thus, we have

$$\frac{2a_0}{(1-\alpha_2(p))(2+M)}-1>0.$$

It also implies that

$$\mathbf{B}\lesssim\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{a^2(\lambda t)}-\frac{(1+B_a(0,t))^{-\alpha_2(p)+1}}{a^2(t)}.$$

In order to get our desired estimate for $\mathbf{B}$ in both cases it suffices to show the following estimate:

$$\begin{aligned}(1+B_a(0,t))^{n/4}\mathbf{B}&\lesssim\frac{(1+B_a(0,t))^{n/4}(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{a^2(\lambda t)}-\frac{(1+B_a(0,t))^{-\alpha_2(p)+1+n/4}}{a^2(t)}\\&\lesssim\frac{(1+B_a(0,t))^{n/4}(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{a^2(\lambda t)}-\frac{(1+B_a(0,t))^{-\alpha_1(p)+1}}{a^2(t)}\\&\lesssim\frac{(1+B_a(0,t))^{n/4+1-\alpha_2(p)}(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{(1+B_a(0,t))^{-\alpha_2(p)+1}a^2(\lambda t)}-\frac{(1+B_a(0,t))^{-\alpha_1(p)+1}}{a^2(t)}\lesssim 1.\end{aligned}$$

By the aid of the decreasing behavior of $\mathcal{G}_1(t)$ for all $p > \bar{p}_1$ the second term in the last inequality is bounded. Therefore, we only consider the boundedness of the first term for both cases $1-\alpha_2(p) > 0$ and $1-\alpha_2(p) < 0$. In the case $1-\alpha_2(p) > 0$ we have

$$\frac{(1+B_a(0,t))^{n/4+1-\alpha_2(p)}(1+B_a(0,\lambda t))^{-\alpha_2(p)+1}}{(1+B_a(0,t))^{-\alpha_2(p)+1}a^2(\lambda t)} \lesssim \frac{(1+B_a(0,t))^{-\alpha_1(p)+1}}{a^2(\lambda t)} := \mathcal{G}_2(t).$$

We form the derivative of this function

$$\begin{aligned}
\mathcal{G}_2'(t) &= \Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)} - 2\lambda\frac{a'(\lambda t)}{a(\lambda t)}(1+B_a(0,t))\Big)\big(1+B_a(0,t)\big)^{-\alpha_1(p)}a(\lambda t)^{-2}\\
&\leq \Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)} - 2\lambda a_0\frac{a(\lambda t)}{A(\lambda t)}\frac{1}{2+M}\frac{A^2(t)}{\mu(t)}\Big)\big(1+B_a(0,t)\big)^{-\alpha_1(p)}a(\lambda t)^{-2}\\
&\leq \Big((1-\alpha_1(p))\frac{a(t)A(\lambda t)}{A(t)a(\lambda t)} - \frac{2\lambda a_0}{2+M}\Big)\big(1+B_a(0,t)\big)^{-\alpha_1(p)}a(\lambda t)^{-2}\\
&\leq \Big((1-\alpha_1(p))\nu(\lambda,t) - \frac{2\lambda a_0}{2+M}\Big)\frac{a(\lambda t)A^2(t)}{A(\lambda t)\mu(t)}\big(1+B_a(0,t)\big)^{-\alpha_1(p)}a(\lambda t)^{-2}.
\end{aligned}$$

Since $p > \bar{p}_2$ we have for large times t

$$\big(1-\alpha_1(p)\big)\nu(\lambda) - \frac{2\lambda a_0}{2+M} < 0.$$

This implies $\mathcal{G}_2'(t) < 0$.
In the other case $1-\alpha_2(p) < 0$, our desired estimate can be obtained directly from the case $1-\alpha_2(p) > 0$. Therefore, for all $p > \bar{p}_2$ we have

$$\mathbf{B} \lesssim \big(1+B_a(0,t)\big)^{-n/4}. \tag{5.2.61}$$

Case 3: $\alpha_2(p) = 1$
We have

$$\begin{aligned}
\mathbf{B} &= \int_{\lambda t}^t \big(1+B_a(0,s)\big)^{-1}b(s)^{-1}ds \approx \int_{\lambda t}^t \frac{\mu(s)}{A^2(s)}\frac{A(s)}{a(s)\mu(s)}ds = \int_{\lambda t}^t \frac{ds}{a(s)A(s)}\\
&\leq \int_{\lambda t}^t \frac{dA(s)}{A(s)^{1+2a_0}} = \frac{1}{2a_0A(\lambda t)^{2a_0}} - \frac{1}{2a_0A(t)^{2a_0}}.
\end{aligned}$$

It implies

$$\big(1+B_a(0,t)\big)^{n/4}\mathbf{B} \lesssim \frac{(1+B_a(0,t))^{n/4}}{A(\lambda t)^{2a_0}} - \frac{(1+B_a(0,t))^{n/4}}{A(t)^{2a_0}}.$$

Due to (5.2.42) the functions $\frac{(1+B_a(0,t))^{n/4}}{A(\lambda t)^{2a_0}}$ and $\frac{(1+B_a(0,t))^{n/4}}{A(t)^{2a_0}}$ are strictly decreasing. Therefore, we have our desired estimate for $\mathbf{B}$.
For $j = 1$ and $l = 0$ we obtain

$$\begin{aligned}
\mathbf{B} &= \int_{\lambda t}^t (b(s))^{-1}\big(1+B_a(s,t)\big)^{-1/2}\big(1+B_a(0,s)\big)^{-\alpha_2(p)}ds\\
&= -\int_{\lambda t}^t \big(1+B_a(s,t)\big)^{-1/2}\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{a^2(s)}d(1+B_a(s,t))\\
&= -\frac{2(1+B_a(s,t))^{1/2}(1+B_a(0,s))^{-\alpha_2(p)}}{a^2(s)}\Big|_{\lambda t}^t + 2\int_{\lambda t}^t (1+B_a(s,t))^{1/2}d\Big(\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{a^2(s)}\Big)\\
&\approx -\frac{2(1+B_a(s,t))^{1/2}(1+B_a(0,s))^{-\alpha_2(p)}}{a^2(s)}\Big|_{\lambda t}^t + 2\int_{\lambda t}^t \frac{(1+B_a(s,t))^{1/2}}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}.
\end{aligned}$$

Here we have used

$$\begin{aligned}
d\Big(\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{a^2(s)}\Big) &\approx \frac{1}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{a'(s)}{a(s)}ds\right)\\
&\approx \frac{1}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{a(s)}{A(s)}ds\right)\\
&\approx \frac{1}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{a(s)A(s)}{\mu(s)}\frac{\mu(s)}{A^2(s)}ds\right)\\
&\approx \frac{1}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}\right)\\
&\approx \frac{1}{2(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}.
\end{aligned}$$

It implies

$$\begin{aligned}
\big(1+B_a(0,t)\big)^{n/4+1/2}\mathbf{B} &\approx \frac{(1+B_a(0,t))^{-\alpha_2(p)}(1+B_a(0,t))^{n/4+1/2}}{a^2(t)}\\
&+\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)}(1+B_a(0,t))^{n/4+1/2}(1+B_a(\lambda t,t))^{1/2}}{a^2(\lambda t)}\\
&+(1+B_a(0,t))^{n/4+1/2}\int_{\lambda t}^{t}\frac{(1+B_a(s,t))^{1/2}}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}.
\end{aligned}$$

Thus

$$\begin{aligned}
(1+B_a(0,t))^{n/4+1/2}\mathbf{B} \lesssim &\underbrace{\frac{(1+B_a(0,t))^{-\alpha_2(p)}(1+B_a(0,t))^{n/4+1/2}}{a^2(t)}}_{B_1}\\
&+\underbrace{\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)}(1+B_a(0,t))^{n/4+1}}{a^2(\lambda t)}}_{B_2}\\
&+\underbrace{\int_{\lambda t}^{t}\frac{(1+B_a(0,t))^{n/4+1}}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}}_{B_3}.
\end{aligned}$$

Using the definition of $\alpha_2(p)=\alpha_1(p)+n/4$ and $p>\bar{p}_1$ it follows

$$B_1 \lesssim \big(1+B_a(0,t)\big)^{-1/2} < 1.$$

In order to prove that B_2 and B_3 are bounded to above by a constant it suffices to show that

$$B_2 \lesssim \big(1+B_a(0,\lambda t)\big)^{-\varepsilon}$$

with a small positive constant ε. Thus, we need to show that the following function is bounded

$$\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)+\varepsilon}(1+B_a(0,t))^{\frac{n}{4}+1}}{a^2(\lambda t)}.$$

Taking account of (5.2.34) we obtain

$$\frac{\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)+\varepsilon}(1+B_a(0,t))^{\frac{n}{4}+1}}{a^2(\lambda t)} \leq \frac{\big(1+B_a(0,t)\big)^{\frac{n}{4}+1-\alpha_2(p)+\varepsilon}}{a^2(\lambda t)\Big(\frac{A(lambdat)}{A(t)}\Big)^{(2+M)(\alpha_2(p)-\varepsilon)}} := \mathcal{G}_3(t).$$

Let us consider the monotonicity of $\mathcal{G}_3(t)$ by taking the derivative of this function. It holds

$$\begin{aligned}
\mathcal{G}_3'(t) &= \bigg(\Big(\frac{n}{4}+1-\alpha_2(p)+\varepsilon\Big)\frac{a(t)A(t)}{\mu(t)}\frac{A(t)}{A(\lambda t)} + (2+M)(\alpha_2(p)-\varepsilon)\Big(\frac{a(t)}{A(\lambda t)} - \lambda\frac{a(\lambda t)A(t)}{A^2(\lambda t)}\Big) \\
&\qquad \times\big(1+B_a(0,t)\big) - 2\lambda\frac{a'(\lambda t)}{a(\lambda t)}\big(1+B_a(0,t)\big)\frac{A(t)}{A(\lambda t)}\bigg)\,\big(1+B_a(0,t)\big)^{n/4+\varepsilon-\alpha_2(p)} \\
&\qquad \times\Big(\frac{A(t)}{A(\lambda t)}\Big)^{(2+M)(\alpha_2(p)-\varepsilon)-1} a(\lambda t)^{-2} \\
&\leq \bigg(\Big(\frac{n}{4}+1-\alpha_2(p)+\varepsilon\Big)\frac{a(t)A^2(t)}{\mu(t)A(\lambda t)} + (2+M)(\alpha_2(p)-\varepsilon)\Big(\frac{a(t)}{A(\lambda t)} - \lambda\frac{a(\lambda t)A(t)}{A^2(\lambda t)}\Big) \\
&\qquad \times\frac{1}{2-\gamma}\frac{A^2(t)}{\mu(t)} - 2\lambda a_0\frac{a(\lambda t)}{A(\lambda t)}\frac{1}{2+M}\frac{A^2(t)}{\mu(t)}\frac{A(t)}{A(\lambda t)}\bigg)\,\big(1+B_a(0,t)\big)^{n/4+\varepsilon-\alpha_2(p)} \\
&\qquad \times\Big(\frac{A(t)}{A(\lambda t)}\Big)^{(2+M)(\alpha_2(p)-\varepsilon)-1} a(\lambda t)^{-2} \\
&\leq \bigg(\Big(\frac{n}{4}+1-\alpha_2(p)+\varepsilon\Big)\frac{a(t)A(\lambda t)}{A(t)a(\lambda t)} + \frac{2+M}{2-\gamma}(\alpha_2(p)-\varepsilon)\Big(\frac{a(t)A(\lambda t)}{A(t)a(\lambda t)} - \lambda\Big) - 2\lambda a_0\frac{1}{2+M}\bigg) \\
&\qquad \times\frac{a(\lambda t)A^3(t)}{A^2(\lambda t)\mu(t)}\big(1+B_a(0,t)\big)^{n/4+\varepsilon-\alpha_2(p)}\Big(\frac{A(t)}{A(\lambda t)}\Big)^{(2+M)(\alpha_2(p)-\varepsilon)-1} a(\lambda t)^{-2} \\
&\leq \bigg(\Big(\frac{n}{4}+1-\alpha_2(p)+\varepsilon\Big)\nu(\lambda,t) + \frac{2+M}{2-\gamma}(\alpha_2(p)-\varepsilon)\big(\nu(\lambda,t)-\lambda\big) - \frac{2\lambda a_0}{2+M}\bigg) \\
&\qquad \times\frac{a(\lambda t)A^3(t)}{A^2(\lambda t)\mu(t)}\big(1+B_a(0,t)\big)^{n/4+\varepsilon-\alpha_2(p)}\Big(\frac{A(t)}{A(\lambda t)}\Big)^{(2+M)(\alpha_2(p)-\varepsilon)-1} a(\lambda t)^{-2}.
\end{aligned}$$

The condition $p > \bar{p}_3$ implies for large times t

$$\Big(\frac{n}{4}+1-\alpha_2(p)+\varepsilon\Big)\nu(\lambda) + \frac{2+M}{2-\gamma}(\alpha_2(p)-\varepsilon)\big(\nu(\lambda)-\lambda\big) - \frac{2\lambda a_0}{2+M} < 0.$$

Thus $\mathcal{G}_3'(t) < 0$. Therefore, for all $p > \bar{p}_3$ we have

$$\mathbf{B} \lesssim \big(1+B_a(0,t)\big)^{-n/4-1/2}. \tag{5.2.62}$$

Analogously, for $j=0$ and $l=1$ we can prove

$$\begin{aligned}
\frac{b(t)}{a^2(t)}\big(1+B_a(0,t)\big)^{n/4+1}\mathbf{B} &\lesssim \underbrace{\frac{\log\big(1+B_a(0,t)\big)(1+B_a(0,\lambda t))^{-\alpha_2(p)}(1+B_a(0,t))^{n/4+1}}{a^2(\lambda t)}}_{\bar{B}_2} \\
&\quad + \underbrace{\int_{\lambda t}^{t}\frac{\log\big(1+B_a(0,t)\big)(1+B_a(0,t))^{n/4+1}}{(1+B_a(0,s))^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}}_{\bar{B}_3}.
\end{aligned}$$

By using l'Hospital we have

$$\limsup_{t\to\infty} \frac{\log\big(1+B_a(0,t)\big)}{\big(1+B_a(0,\lambda t)\big)^{\varepsilon}} \approx \limsup_{t\to\infty} \frac{\nu(\lambda)}{\varepsilon\lambda\big(1+B_a(0,\lambda t)\big)^{\varepsilon}} < C. \tag{5.2.63}$$

Moreover, since $p > \bar{p}_3$ we obtain

$$\bar{B}_2 = \frac{\log\big(1+B_a(0,t)\big)}{\big(1+B_a(0,\lambda t)\big)^{\varepsilon}} \mathcal{G}_3(t) \lesssim \frac{\log\big(1+B_a(0,t)\big)}{\big(1+B_a(0,\lambda t)\big)^{\varepsilon}} < C$$

for large t. Finally, after using the change of variables $r = 1+B_a(0,s)$ we get

$$\bar{B}_3 \lesssim \log\big(1+B_a(0,t)\big) \int_{1+B_a(0,\lambda t)}^{1+B_a(0,t)} \frac{dr}{r^{1+\varepsilon}} = \frac{\log\big(1+B_a(0,t)\big)}{\varepsilon\big(1+B_a(0,\lambda t)\big)^{\varepsilon}} - \frac{\log\big(1+B_a(0,t)\big)}{\varepsilon\big(1+B_a(0,t)\big)^{\varepsilon}} < C.$$

Here we use that $\bar{B}_3$ is related to $\bar{B}_2$ and (5.2.63). This implies that

$$\mathbf{B} \lesssim \frac{a^2(t)}{b(t)}\big(1+B_a(0,t)\big)^{-(n/4+1)}. \tag{5.2.64}$$

From (5.2.61), (5.2.62) and (5.2.64) we obtain

$$\mathbf{B} \lesssim \Big(\frac{a^2(t)}{b(t)}\Big)^{l}\big(1+B_a(0,t)\big)^{-(n/4+j/2+l)}. \tag{5.2.65}$$

Thanks to (5.2.59) and (5.2.65) we can conclude the statement (5.2.51). □

Now let us prove (5.2.52). We remark that

$$\|Nu - Nv\|_{X(t)} = \Big\| \int_0^t E_1(t,s,x) *_x \big(f(u(s,x)) - f(v(s,x))\big) ds \Big\|_{X(t)}.$$

Thanks to (5.2.4), (5.2.5) and (5.2.6) we have the following estimates

$$\begin{aligned} &\|\nabla^j \partial_t^l E_1(t,s,x) *_x \big(f(u(s,x)) - f(v(s,x))\big)\|_{L^2} \\ &\quad \lesssim \begin{cases} b(s)^{-1}\Big(\frac{a^2(t)}{b(t)}\Big)^{l}\big(1+B_a(s,t)\big)^{-\frac{n}{4}-\frac{j}{2}-l}\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^1\cap L^2}, & s\in[0,\lambda t], \\ b(s)^{-1}\Big(\frac{a^2(t)}{b(t)}\Big)^{l}\big(1+B_a(s,t)\big)^{-\frac{j}{2}-l}\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2}, & s\in[\lambda t,t], \end{cases} \end{aligned}$$

for $j+l=0,1$. By using Hölder's inequality we obtain

$$\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^1} \lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^p}\Big(\|u(s,\cdot)\|_{L^p}^{p-1} + \|v(s,\cdot)\|_{L^p}^{p-1}\Big),$$
$$\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2} \lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}}\Big(\|u(s,\cdot)\|_{L^{2p}}^{p-1} + \|v(s,\cdot)\|_{L^{2p}}^{p-1}\Big).$$

We apply Gagliardo-Nirenberg inequality to the following terms:

$$\|u(s,\cdot) - v(s,\cdot)\|_{L^q}, \qquad \|u(s,\cdot)\|_{L^q}, \qquad \|v(s,\cdot)\|_{L^q},$$

with $q = p$ and $q = 2p$ and, analogously, to the proof of the statement (5.2.51) we can conclude the proof of the statement (5.2.52) by the aid of assumption $p > \bar{p}$. In this way our Theorem 5.2.7 is proved. □

Example 5.2.4. Let us choose formally $a(t) \equiv 1$. Then we have $A(t) = 1+t$. Let $\mu(t)$ be an arbitrary function satisfying the assumptions (B'1) - (B'5) and (R1). Then we have $a_0 = 0$ and $\nu(\lambda) = \lambda$. Thus, applying Theorem 5.2.7 formally we have

$$\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = 1 + \frac{2}{n} = p_{Fuj}.$$

These results coincide with results from the paper of D'Abbicco, Lucente and Reissig (see [D-R13]).

Example 5.2.5. If we choose $a(t) = (l+1)(1+t)^l$, $l > 0$, then we have $A(t) = (1+t)^{l+1}$ and $\nu(\lambda) = \lambda$. Let us choose $\mu(t)$ as in the Example 5.2.1, that is,

$$\mu(t) = (\log(c+t))^\kappa \text{ or } \mu(t) = (1+t)^{\gamma(l+1)} \text{ for } \kappa > 1, 0 \leq \gamma < 2.$$

In the case $\mu(t) = (\log(c+t))^\kappa$, $\kappa > 1$, we obtain $M = 0$ and $\gamma = \varepsilon$ with an arbitrary small ε. Thus, applying Theorem 5.2.7 we see that the condition (5.2.42) for the case $\alpha_2(p) = \frac{(2p-1)n}{4} = 1$ is satisfied if only if

$$n = 1 \Rightarrow l > \frac{1}{3} \text{ and } p = \frac{5}{2};\ n = 2 \Rightarrow l > 1 \text{ and } p = \frac{3}{2};\ n = 3 \Rightarrow l > 3 \text{ and } p = \frac{7}{6}.$$

Whereas, the critical exponent is

$$\bar{p} = 1 + \frac{2}{(l+1)n}.$$

So, we have global existence of small data solutions for

$$\begin{cases} \bar{p} < p & \text{if n} = 1 \text{ and } l \in (1/3, 1], \\ 2 \leq p & \text{if n} = 1 \text{ and } l \in (1, \infty) \text{ or if n} = 2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if n} = 3, \\ p = 2 = p_{GN}(4) & \text{if n} = 4. \end{cases} \tag{5.2.66}$$

In the other case $\mu(t) = (1+t)^{\gamma(l+1)}$, $0 \leq \gamma < 2$, we obtain $M = 0$, $\gamma \in [0, 2)$. Thus, applying Theorem 5.2.7 we also get

$$\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = 1 + \frac{2}{(l+1)n} = \bar{p}.$$

So, we have global existence of small data solutions under the same conditions for critical exponent as in (5.2.66). We remark that the statement of Theorem 5.2.7 holds uniformly for all damped wave models, where the propagation speed and dissipation satisfy the assumptions. If we focus to the special case $a(t) = (l+1)(1+t)^l$ and $\mu(t) = (1+t)^{\gamma(l+1)}$, $0 \leq \gamma < 2$, with fixed l and γ, then the abstract condition (5.2.42) can be replaced by

$$\frac{l}{l+1} > \frac{(2-\gamma)n}{8}.$$

If we take $\gamma \to 0$, then it implies the condition (5.2.42). This condition is also valid for $\mu(t) = (\log(c+t))^\kappa$, $\kappa > 1$.

Example 5.2.6. If we choose $a(t) = e^t$, then we have $A(t) = e^t$. So, $a_0 = 1$ and $\nu(\lambda) = 1$. Let us choose $\mu(t)$ as in the Example 5.2.2, that is,

$$\mu(t) = (1+t)^k(\log(e+t))^\kappa \text{ or } \mu(t) = e^{\gamma t}, \text{ for } k \geq 1,\ k + \kappa > 2,\ 0 \leq \gamma < 2.$$

In the case $\mu(t) = (1+t)^k(\log(e+t))^\kappa$ $k \geq 1$, $k+\kappa > 2$, we obtain $M = 0$ and $\gamma = \varepsilon$ with an arbitrary small ε. Thus, applying Theorem 5.2.7 we see that the condition (5.2.42) is satisfied if only if

$$\lambda > \frac{1}{4}.$$

So we have to choose such a λ. Moreover, we get

$$\bar{p}_1 = 1 < \bar{p}_2 \to \frac{1}{2} + \frac{1}{2\lambda} < \bar{p}_3 \to \frac{1}{2} + \frac{1}{2\lambda} + \Big(\frac{1}{\lambda} - 1\Big)\frac{2}{n} < \bar{p}.$$

Choosing λ close to 1 gives $\bar{p} > 1$. So, we have global existence of small data solutions for

$$\begin{cases} 2 \leq p & \text{if n} = 1,\, 2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if n} = 3, \\ p = 2 = p_{GN}(4) & \text{if n} = 4. \end{cases} \tag{5.2.67}$$

In the other case $\mu(t) = e^{\gamma t}$, $0 \leq \gamma < 2$, we obtain $M = 0$, $\gamma \in [0,2)$. Thus, applying Theorem 5.2.7 we get

$$\begin{aligned} \bar{p}_1 &= 1,\ \bar{p}_2 = \frac{1}{2} + \frac{1}{2\Big(1 - \frac{2}{2-\gamma}(1-\lambda)\Big)} + \Bigg(1 - \frac{\lambda}{1 - \frac{2}{2-\gamma}(1-\lambda)}\Bigg)\frac{2}{n}, \\ \bar{p}_3 &= \frac{1}{2} + \frac{1}{2\Big(1 - \frac{2}{2-\gamma}(1-\lambda)\Big)} + \frac{1-\lambda}{1 - \frac{2}{2-\gamma}(1-\lambda)}\frac{2}{n}. \end{aligned}$$

Choosing λ close to 1 gives $\bar{p} > 1$. So, we have global existence of small data solutions under the same conditions for critical exponent as in (5.2.67).

Example 5.2.7. If we choose $a(t) = mt^{m-1}e^{t^m}$, $m > 0$, then we have $A(t) = e^{t^m}$. So, $a_0 = 1$ and $\nu(\lambda) = 1/\lambda^{m-1}$. Let us choose functions $\mu(t)$ satisfying the assumption $(R2)$, for example,

$$\mu(t) = (1+t)^k(\log(e+t))^\kappa \ \text{ or } \ \mu(t) = e^{\gamma t^m}, \text{ for } k \geq m,\, k+\kappa > m+1,\, 0 \leq \gamma < 2.$$

In the case $\mu(t) = (1+t)^k(\log(e+t))^\kappa$, $k \geq m$, $k+\kappa > m+1$, we obtain $M = 0$ and $\gamma = \varepsilon$ with an arbitrary small ε. Thus, applying Theorem 5.2.7 we see that the condition (5.2.42) is satisfied if only if

$$\lambda > \Big(\frac{1}{4}\Big)^{1/m}.$$

So we have to choose such a λ. Moreover, we get

$$\bar{p}_1 = 1 < \bar{p}_2 \to \frac{1}{2} + \frac{1}{2\lambda^m} < \bar{p}_3 \to \frac{1}{2} + \frac{1}{2\lambda^m} + \Big(\frac{1}{\lambda^m} - 1\Big)\frac{2}{n} = \bar{p}.$$

Choosing λ close to 1 gives $\bar{p} > 1$. So, we have global existence of small data solutions for

$$\begin{cases} 2 \leq p & \text{if n} = 1,\, 2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if n} = 3, \\ p = 2 = p_{GN}(4) & \text{if n} = 4. \end{cases} \tag{5.2.68}$$

In the other case $\mu(t) = e^{\gamma t^m}$, $0 \leq \gamma < 2$, we obtain $M = 0$, $\gamma \in [0,2)$. Thus, applying Theorem 5.2.7 we get

$$\begin{aligned} \bar{p}_1 &= 1,\ \bar{p}_2 = \frac{1}{2} + \frac{1}{2\Big(1 - \frac{2}{2-\gamma}(1-\lambda^m)\Big)} + \Bigg(1 - \frac{\lambda^m}{1 - \frac{2}{2-\gamma}(1-\lambda^m)}\Bigg)\frac{2}{n}, \\ \bar{p}_3 &= \frac{1}{2} + \frac{1}{2\Big(1 - \frac{2}{2-\gamma}(1-\lambda^m)\Big)} + \frac{1-\lambda^m}{1 - \frac{2}{2-\gamma}(1-\lambda^m)}\frac{2}{n}. \end{aligned}$$

Choosing λ close to 1 gives $\bar{p} > 1$. So, we have global existence of small data solutions under the same conditions for the critical exponent as in (5.2.68).

5.2.4. Global existence of small data solutions for wave models with super-exponential propagation speed

Now let us devote to the super-exponential order case $\nu(\lambda) = \infty$.

Theorem 5.2.9. ***(Super-exponential order case)*** *We assume the Hypotheses (A1) to (A3), (B'1) to (B'5) and (R2). Moreover, we assume a growth condition for the function $\nu(\lambda, t)$:*

$$\nu(\lambda, t) = \mathcal{O}(\log A(t)) . \tag{5.2.69}$$

Let us assume $n \leq 4$ and

$$\begin{cases} 2 \leq p & \text{if } n=1,2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if } n=3, \\ p = 2 = p_{GN}(4) & \text{if } n=4. \end{cases} \tag{5.2.70}$$

Then there exists a constant $\epsilon_0 > 0$ such that data with

$$\|(u_1, u_2)\|_{\mathcal{A}_{1,1}} \leq \epsilon_0$$

imply the existence of a unique solution to (5.2.1) in $\mathcal{C}\big([0,\infty), H^1\big) \cap \mathcal{C}^1\big([0,\infty), L^2\big)$. Furthermore, there exists a constant $C > 0$ such that this solution satisfies the estimates

$$\|u(t,\cdot)\|_{L^2} \leq C \quad \|(u_1, u_2)\|_{\mathcal{A}_{1,1}} (1 + B_a(0,t))^{-\frac{n}{4}}, \tag{5.2.71}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C \quad \|(u_1, u_2)\|_{\mathcal{A}_{1,1}} (1 + B_a(0,t))^{-\frac{n}{4}-\frac{1}{2}}, \tag{5.2.72}$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C \quad \|(u_1, u_2)\|_{\mathcal{A}_{1,1}} (1 + B_a(0,t))^{-\frac{n}{4}-1} a^2(t) (b(t))^{-1}. \tag{5.2.73}$$

Proof. The scheme to prove this theorem is almost the same which we presented in the proof of Theorem 5.2.7. The only change is to show that Proposition 5.2.8 remains true, in particular, we will show the estimates (5.2.51), under the new assumption (R2).

In our proof we need the following result: Let us consider for $j + l = 0, 1$ estimates in the case $s \in [0, \lambda t]$. We have again

$$\mathbf{A} \approx \left(\frac{a^2(t)}{b(t)}\right)^l \big(1 + B_a(0,t)\big)^{-(n/4+j/2+l)} \underbrace{\int_0^{\lambda t} (b(s))^{-1} \big(1 + B_a(0,s)\big)^{-(p-1)n/2} ds}_{\bar{A}_1} .$$

Here, we have used (5.2.38). We now try to find a condition for the power p which guarantees that the integral term $\bar{A}_1$ remains uniformly bounded for $t \to \infty$. We define $\alpha_1(p) := (p-1)n/2$.

In the case $\alpha_1(p) \neq 1$ to handle the integral $\bar{A}_1$ we will use integration by parts and the assumption

$(A2)$. Thus

$$
\begin{aligned}
\bar{A}_1 &= \int_0^{\lambda t} \big(1+B_a(0,s)\big)^{-\alpha_1(p)} \frac{1}{a^2(s)} d\big(1+B_a(0,s)\big)\\
&= \frac{\big(1+B_a(0,s)\big)^{-\alpha_1(p)+1}}{(1-\alpha_1(p))a^2(s)}\Bigg|_0^{\lambda t} + \frac{2}{1-\alpha_1(p)} \int_0^{\lambda t} \big(1+B_a(0,s)\big)^{-\alpha_1(p)+1} \frac{a'(s)}{a^3(s)} ds\\
&\approx \frac{(1+B_a(0,s))^{-\alpha_1(p)+1}}{(1-\alpha_1(p))a^2(s)}\Bigg|_0^{\lambda t} + \frac{2}{1-\alpha_1(p)} \int_0^{\lambda t} \frac{(1+B_a(0,s))^{-\alpha_1(p)+1}}{a^2(s)} \frac{a(s)}{A(s)} ds\\
&\approx \frac{1}{1-\alpha_1(p)} \left(\frac{1}{a^2(0)} - \frac{(1+B_a(0,\lambda t))^{-\alpha_1(p)+1}}{a^2(\lambda t)} + 2\int_0^{\lambda t} \frac{(1+B_a(0,s))^{-\alpha_1(p)+1}}{a^2(s)} \frac{a(s)}{A(s)} ds \right).
\end{aligned}
$$

In order to prove that for a fixed $\lambda \in (0,1)$ the integral $\bar{A}_1$ is uniformly upper bounded by a constant for $t \to \infty$ it suffices to show that with an arbitrary small positive ε we have

$$
\frac{\big(1+B_a(0,t)\big)^{-\alpha_1(p)+1}}{a^2(t)} \lesssim A(t)^{-\varepsilon}.
$$

Let us consider

$$
\bar{\mathcal{G}}_1(t) := \frac{\big(1+B_a(0,t)\big)^{-\alpha_1(p)+1} A(t)^{\varepsilon}}{a^2(t)}.
$$

Performing the first derivative of $\bar{\mathcal{G}}_1(t)$ we get

$$
\begin{aligned}
\bar{\mathcal{G}}_1'(t) &= \frac{\Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)} + \varepsilon \frac{a(t)}{A(t)}\big(1+B_a(0,t)\big) - 2\frac{a'(t)}{a(t)}\big(1+B_a(0,t)\big)\Big)A(t)^{\varepsilon}}{a^2(t)\big(1+B_a(0,t)\big)^{\alpha_1(p)}}\\
&\lesssim \frac{\Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)} + \varepsilon \frac{a(t)}{A(t)}\frac{A^2(t)}{\mu(t)} \log A(t) - 2a_0 \frac{a(t)}{A(t)}\frac{A^2(t)}{\mu(t)} \log A(t)\Big)A(t)^{\varepsilon}}{a^2(t)\big(1+B_a(0,t)\big)^{\alpha_1(p)}}\\
&\lesssim \frac{\Big((1-\alpha_1(p))\frac{a(t)A(t)}{\mu(t)} + \varepsilon \frac{a(t)A(t)}{\mu(t)} \log A(t) - 2a_0 \frac{a(t)A(t)}{\mu(t)} \log A(t)\Big)A(t)^{\varepsilon}}{a^2(t)\big(1+B_a(0,t)\big)^{-\alpha_1(p)}}\\
&\le \frac{\Big((1-\alpha_1(p)) + \big(\varepsilon - 2a_0\big)\log A(t)\Big)\frac{a(t)A(t)}{\mu(t)} A(t)^{\varepsilon}}{a^2(t)\big(1+B_a(0,t)\big)^{\alpha_1(p)}}.
\end{aligned}
$$

The last inequality show us that $\bar{\mathcal{G}}_1'(t) < 0$ after choosing ε small enough and for large t. Hence, $\bar{\mathcal{G}}_1(t)$ is bounded. Therefore, we can obtain our desired estimate for $\bar{A}_1$.
The case $\alpha_1(p) = 1$ can be treated directly. We have

$$
\begin{aligned}
\bar{A}_1 &= \int_0^{\lambda t} \frac{ds}{b(s)\big(1+B_a(0,s)\big)} \approx \int_0^{\lambda t} \frac{\mu(s)}{A^2(s)\log A(s)} \frac{A(s)}{a(s)\mu(s)} ds = \int_0^{\lambda t} \frac{ds}{\log A(s) a(s) A(s)}\\
&\le \int_0^{\lambda t} \frac{dA(s)}{a^2(s)A(s)} \le \int_0^{\lambda t} \frac{dA(s)}{A(s)^{1+2a_0}} \le C.
\end{aligned}
$$

Summarizing, we can conclude

$$
\mathbf{A} \lesssim \left(\frac{a^2(t)}{b(t)}\right)^l \big(1+B_a(0,t)\big)^{-(n/4+j/2+l)}. \tag{5.2.74}
$$

Now let us estimate $\mathbf{B}$. We start with the case $j=0$ and $l=0$.

Case 1: $\alpha_2(p) \neq 1$

$$\begin{aligned}
\mathbf{B} &= \int_{\lambda t}^{t} (b(s))^{-1}\big(1+B_a(0,s)\big)^{-\alpha_2(p)}ds = \int_{\lambda t}^{t} \big(1+B_a(0,s)\big)^{-\alpha_2(p)} \frac{1}{a^2(s)} d\big(1+B_a(0,s)\big) \\
&= \frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Bigg|_{\lambda t}^{t} + \frac{2}{1-\alpha_2(p)}\int_{\lambda t}^{t} (1+B_a(0,s))^{-\alpha_2(p)+1}\frac{a'(s)}{a^3(s)}ds \\
&\approx \frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{(1-\alpha_2(p))a^2(s)}\Bigg|_{\lambda t}^{t} + \frac{2}{1-\alpha_2(p)}\int_{\lambda t}^{t} \frac{(1+B_a(0,s))^{-\alpha_2(p)+1}}{a^2(s)}\frac{a(s)}{A(s)}ds. \qquad (5.2.75)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(1+B_a(0,t))^{n/4}\mathbf{B} \lesssim\; & \underbrace{\frac{\big(1+B_a(0,t)\big)^{-\alpha_2(p)+1}\big(1+B_a(0,t)\big)^{n/4}}{a^2(t)}}_{B_1} \\
&+ \underbrace{\frac{\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)+1}\big(1+B_a(0,t)\big)^{n/4}}{a^2(\lambda t)}}_{B_2} \\
&+ \underbrace{\int_{\lambda t}^{t} \frac{\big(1+B_a(0,s)\big)^{-\alpha_2(p)+1}\big(1+B_a(0,t)\big)^{n/4}}{a^2(s)}\frac{a(s)}{A(s)}ds}_{B_3}.
\end{aligned}$$

Since $\alpha_2(p) = \alpha_1(p) + n/4$ we may conclude for large t

$$B_1 = \bar{\mathcal{G}}_1(t)A(t)^{-\varepsilon} \lesssim A(t)^{-\varepsilon} \lesssim 1.$$

In order to prove that B_2 and B_3 are both uniformly upper bounded by a constant it suffices to show that for a small positive ε we have

$$B_2 \lesssim \big(1+B_a(0,\lambda t)\big)^{-\varepsilon}.$$

Thus, we use the monotonicity behavior of the following function:

$$\bar{\mathcal{G}}_2(t) := \frac{\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)+1}\big(1+B_a(0,t)\big)^{n/4}A(t)^{\varepsilon}}{a^2(\lambda t)}$$

by the aid of the strictly decreasing behavior of $\frac{a(t)}{A(t)}\frac{A(\lambda t)}{a(\lambda t)}\frac{1}{\log A(t)}$. After performing the derivative $\bar{\mathcal{G}}_2'(t)$ we can see that $\bar{\mathcal{G}}_2'(t) < 0$ for large t. It implies that

$$B_3 \lesssim \int_{\lambda t}^{t} \frac{1}{A(s)^{\varepsilon}}\frac{a(s)}{A(s)}ds \lesssim 1$$

for large t. Summarizing for $j = l = 0$ all the estimates we may conclude

$$\mathbf{B} \lesssim \big(1+B_a(0,t)\big)^{-n/4}. \qquad (5.2.76)$$

Case 2: $\alpha_2(p) = 1$

Here we proceed in the usual way by the aid of the supposed decreasing behavior of the function $\frac{\nu(\lambda,t)}{\log A(t)}$. We have

$$
\begin{aligned}
\mathbf{B} &= \int_{\lambda t}^{t} \big(1+B_a(0,s)\big)^{-1} b(s)^{-1} ds \int_{\lambda t}^{t} d\Big(\log\big(1+B_a(0,s)\big)\Big)\frac{1}{a^2(s)} \\
&= \frac{\log\big(1+B_a(0,s)\big)}{a^2(s)}\Big|_{\lambda t}^{t} + \int_{\lambda t}^{t} \log\big(1+B_a(0,s)\big)\frac{a'(s)}{a^3(s)}ds \\
&\lesssim \frac{\log\big(1+B_a(0,t)\big)}{a^2(t)} - \frac{\log\big(1+B_a(0,\lambda t)\big)}{a^2(\lambda t)} + a_1\int_{\lambda t}^{t} \frac{\log\big(1+B_a(0,s)\big)}{a^2(s)}\frac{a(s)}{A(s)}ds.
\end{aligned}
$$

It implies

$$
\begin{aligned}
\big(1+B_a(0,t)\big)^{n/4}\mathbf{B} \lesssim{}& \frac{\log\big(1+B_a(0,t)\big)(1+B_a(0,t))^{n/4}}{a^2(t)} - \frac{\log\big(1+B_a(0,\lambda t)\big)(1+B_a(0,t))^{n/4}}{a^2(\lambda t)} \\
&+ a_1\int_{\lambda t}^{t} \frac{\log\big(1+B_a(0,s)\big)(1+B_a(0,t))^{n/4}}{a^2(s)}\frac{a(s)}{A(s)}ds.
\end{aligned}
$$

Analogously, in order to prove $(1+B_a(0,t))^{n/4}\mathbf{B}$ is bounded, it is enough to prove that for small positive ε we have

$$
\frac{\log\big(1+B_a(0,\lambda t)\big)(1+B_a(0,t))^{n/4}}{a^2(\lambda t)} \lesssim \frac{1}{A(\lambda t)^{\varepsilon}}.
$$

Thus, for arbitrary small $\delta>0$ we have

$$
\begin{aligned}
&\frac{\log\big(1+B_a(0,\lambda t)\big)(1+B_a(0,t))^{n/4}A(\lambda t)^{\varepsilon}}{a^2(\lambda t)} \\
&\qquad\lesssim \frac{\log\big(1+B_a(0,\lambda t)\big)}{(1+B_a(0,\lambda t))^{\delta}}\frac{(1+B_a(0,\lambda t))^{\delta}(1+B_a(0,t))^{n/4}A(\lambda t)^{\varepsilon}}{a^2(\lambda t)} \\
&\qquad\lesssim \frac{(1+B_a(0,\lambda t))^{\delta}(1+B_a(0,t))^{n/4}A(\lambda t)^{\varepsilon}}{a^2(\lambda t)}.
\end{aligned}
$$

Taking account of the monotonic behavior of $\bar{\mathcal{G}}_2$ we obtain our desired estimate for $\mathbf{B}$.
For $j=1$ and $l=0$ we obtain

$$
\begin{aligned}
\mathbf{B} &= \int_{\lambda t}^{t} (b(s))^{-1}\big(1+B_a(s,t)\big)^{-1/2}\big(1+B_a(0,s)\big)^{-\alpha_2(p)}ds \\
&= -\int_{\lambda t}^{t} \big(1+B_a(s,t)\big)^{-1/2}\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{a^2(s)}d\big(1+B_a(s,t)\big) \\
&= -\frac{2\big(1+B_a(s,t)\big)^{1/2}\big(1+B_a(0,s)\big)^{-\alpha_2(p)}}{a^2(s)}\Big|_{\lambda t}^{t} + 2\int_{\lambda t}^{t}\big(1+B_a(s,t)\big)^{1/2}d\Big(\frac{(1+B_a(0,s))^{-\alpha_2(p)}}{a^2(s)}\Big) \\
&\approx \frac{\big(1+B_a(s,t)\big)^{1/2}\big(1+B_a(0,s)\big)^{-\alpha_2(p)}}{2a^2(s)}\Big|_{\lambda t}^{t} + \int_{\lambda t}^{t}\frac{\big(1+B_a(s,t)\big)^{1/2}}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}.
\end{aligned}
$$

Here we have used

$$\begin{aligned}
d\Big(\frac{\big(1+B_a(0,s)\big)^{-\alpha_2(p)}}{a^2(s)}\Big) &\approx \frac{1}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{a'(s)}{a(s)}ds\right)\\
&\approx \frac{1}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{a(s)}{A(s)}ds\right)\\
&\approx \frac{1}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{a(s)A(s)}{\mu(s)}\frac{\mu(s)}{A^2(s)}ds\right)\\
&\approx \frac{1}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\left(\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}+\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}\right)\\
&\approx \frac{1}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}.
\end{aligned}$$

It implies

$$\begin{aligned}
\big(1+B_a(0,t)\big)^{n/4+1/2}\mathbf{B} &\approx \frac{\big(1+B_a(0,t)\big)^{-\alpha_2(p)}\big(1+B_a(0,t)\big)^{n/4+1/2}}{a^2(t)}\\
&+\frac{(1+B_a(0,\lambda t))^{-\alpha_2(p)}\big(1+B_a(0,t)\big)^{n/4+1/2}\big(1+B_a(\lambda t,t)\big)^{1/2}}{a^2(\lambda t)}\\
&+\big(1+B_a(0,t)\big)^{n/4+1/2}\int_{\lambda t}^t\frac{\big(1+B_a(s,t)\big)^{1/2}}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\big(1+B_a(0,t)\big)^{n/4+1/2}\mathbf{B} &\lesssim \underbrace{\frac{\big(1+B_a(0,t)\big)^{-\alpha_2(p)}\big(1+B_a(0,t)\big)^{n/4+1/2}}{a^2(t)}}_{\bar{B}_1}\\
&+\underbrace{\frac{\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)}\big(1+B_a(0,t)\big)^{n/4+1}}{a^2(\lambda t)}}_{\bar{B}_2}\\
&+\underbrace{\int_{\lambda t}^t\frac{\big(1+B_a(0,t)\big)^{n/4+1}}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}}_{\bar{B}_3}.
\end{aligned}$$

Since $\alpha_2(p)=\alpha_1(p)+n/4$ we may conclude that

$$\bar{B}_1=\bar{\mathcal{G}}_1(t)\big(1+B_a(0,t)\big)^{-1/2}A(t)^{-\varepsilon}\lesssim\big(1+B_a(0,t)\big)^{-1/2}A(t)^{-\varepsilon}\lesssim 1.$$

In order to prove that $\bar{B}_2$ and $\bar{B}_3$ are uniformly (with respect to $t\to\infty$) upper bounded by a constant it suffices to show that for $r>1$ we have

$$\bar{B}_2\lesssim\big(1+B_a(0,\lambda t)\big)^{-r}.$$

For this reason we consider the monotonicity of the following function:

$$\bar{\mathcal{G}}_2(t) := \frac{\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)+r}\big(1+B_a(0,t)\big)^{n/4+1}}{a^2(\lambda t)}.$$

We form the first derivative of this function and take account of $\mu(t) = A^2(t)/\theta(t)$ to obtain

$$\begin{aligned}
\bar{\mathcal{G}}_2'(t) &= \Bigg(\Big(\frac{n}{4}+1\Big)\frac{a(t)A(t)}{\mu(t)}\big(1+B_a(0,\lambda t)\big) - (\alpha_2(p)-r)\lambda\big(1+B_a(0,t)\big)\frac{a(\lambda t)A(\lambda t)}{\mu(\lambda t)} \\
&\quad -2\lambda\frac{a'(\lambda t)}{a(\lambda t)}\big(1+B_a(0,t)\big)\big(1+B_a(0,\lambda t)\big)\Bigg)\frac{\big(1+B_a(0,t)\big)^{n/4}\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)+r-1}}{a(\lambda t)^2} \\
&\le \Bigg(\Big(\frac{n}{4}+1\Big)\frac{a(t)\theta(t)}{A(t)}\theta(\lambda t)\log A(\lambda t) - \frac{(\alpha_2(p)-r)\lambda}{2+\beta}\frac{a(\lambda t)\theta(\lambda t)}{A(\lambda t)}\theta(t)\log A(t) - \frac{2\lambda a_0}{(2+\beta)^2} \\
&\quad \times\frac{a(\lambda t)}{A(\lambda t)}\big(\theta(t)\log A(t)\big)\big(\theta(\lambda t)\log A(\lambda t)\big)\Bigg)\frac{\big(1+B_a(0,t)\big)^{n/4}\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)+r-1}}{a(\lambda t)^2} \\
&\le \Bigg(\Big(\frac{n}{4}+1\Big)\frac{a(t)}{A(t)}\frac{A(\lambda t)}{a(\lambda t)}\frac{1}{\log A(t)} - \frac{(\alpha_2(p)-r)\lambda}{2+\beta}\frac{1}{\log A(\lambda t)} - \frac{2\lambda a_0}{(2+\beta)^2}\Bigg)\frac{a(\lambda t)}{A(\lambda t)}\theta(t)\theta(\lambda t) \\
&\qquad \times \log A(t)\log A(\lambda t)\big(1+B_a(0,t)\big)^{n/4}\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)+r-1}a(\lambda t)^{-2}.
\end{aligned}$$

Now, the supposed strictly decreasing behavior of the function $\frac{a(t)}{A(t)}\frac{A(\lambda t)}{a(\lambda t)}\frac{1}{\log A(t)}$ implies $\bar{\mathcal{G}}_2'(t) < 0$ for large t. Summarizing we have shown for $j=1$, $l=0$ the desired inequality

$$\mathbf{B} \lesssim \big(1+B_a(0,t)\big)^{-n/4-1/2}. \tag{5.2.77}$$

Analogously, for $j=0$ and $l=1$ we can prove

$$\begin{aligned}
\frac{b(t)}{a^2(t)}\big(1+B_a(0,t)\big)^{n/4+1}\mathbf{B} &\lesssim \underbrace{\frac{\log\big(1+B_a(0,t)\big)\big(1+B_a(0,\lambda t)\big)^{-\alpha_2(p)}\big(1+B_a(0,t)\big)^{n/4+1}}{a^2(\lambda t)}}_{D_2} \\
&\quad + \underbrace{\int_{\lambda t}^{t}\frac{\log\big(1+B_a(0,t)\big)\big(1+B_a(0,t)\big)^{n/4+1}}{\big(1+B_a(0,s)\big)^{\alpha_2(p)}a^2(s)}\frac{d\big(1+B_a(0,s)\big)}{1+B_a(0,s)}}_{D_3}.
\end{aligned}$$

After applying the rule of l'Hospital it follows

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\log\big(1+B_a(0,t)\big)}{\big(1+B_a(0,\lambda t)\big)^r} &= \limsup_{t\to\infty}\frac{\frac{a(t)}{A(t)}\theta(t)}{r\lambda\big(1+B_a(0,t)\big)\frac{a(\lambda t)}{A(\lambda t)}\theta(\lambda t)\big(1+B_a(0,\lambda t)\big)^{r-1}} \\
&\approx \limsup_{t\to\infty}\frac{\frac{a(t)}{A(t)}\theta(t)}{r\lambda\big(\theta(t)\log A(t)\big)\frac{a(\lambda t)}{A(\lambda t)}\theta(\lambda t)\big(1+B_a(0,\lambda t)\big)^{r-1}} \\
&\approx \limsup_{t\to\infty}\frac{a(t)}{A(t)}\frac{A(\lambda t)}{a(\lambda t)}\frac{1}{\theta(\lambda t)\log A(t)}\frac{1}{\big(1+B_a(0,\lambda t)\big)^{r-1}} \lesssim 1.
\end{aligned} \tag{5.2.78}$$

Thus, we get

$$D_2 = \frac{\log\big(1+B_a(0,t)\big)}{\big(1+B_a(0,\lambda t)\big)^r}\bar{\mathcal{G}}_2(t) \lesssim \frac{\log\big(1+B_a(0,t)\big)}{\big(1+B_a(0,\lambda t)\big)^r} \lesssim 1$$

for large t. Finally, after using the change of variables $y = 1 + B_a(0,s)$ it follows

$$D_3 \lesssim \log\big(1 + B_a(0,t)\big)\int_{1+B_a(0,\lambda t)}^{1+B_a(0,t)} \frac{dy}{y^{1+r}} = \frac{\log\big(1 + B_a(0,t)\big)}{r\big(1 + B_a(0,\lambda t)\big)^r} - \frac{\log\big(1 + B_a(0,t)\big)}{r\big(1 + B_a(0,t)\big)^r} \lesssim 1.$$

Here we use that $\bar{B}_3$ contains $\bar{B}_2$ and (5.2.78). This implies

$$\mathbf{B} \lesssim \frac{a^2(t)}{b(t)}\big(1 + B_a(0,t)\big)^{-(n/4+1)}. \tag{5.2.79}$$

From (5.2.76), (5.2.77) and (5.2.79) we obtain

$$\mathbf{B} \lesssim \Big(\frac{a^2(t)}{b(t)}\Big)^l \big(1 + B_a(0,t)\big)^{-(n/4+j/2+l)}. \tag{5.2.80}$$

Thanks to (5.2.74) and (5.2.80) we can conclude the statement (5.2.51). In this way our Theorem 5.2.9 is proved. $\square$

Example 5.2.8. If we choose $a(t) = e^t e^{e^t}$, then we have $A(t) = e^{e^t}$. So, $a_0 = 1$ and $\nu(\lambda) = \infty$. Let us choose $\mu(t)$ as in the Example 5.2.3, that is,

$$\mu(t) = e^{2e^t}/e^{\beta t} \ \text{ or } \ e^{2e^t}/\log(e+t), \ \text{for } \beta > 0.$$

Thus, applying Theorem 5.2.9 we see that the condition (5.2.69) is satisfied. Indeed,

$$\frac{\nu(\lambda,t)}{\log A(t)} = \frac{e^t e^{e^t}}{e^{e^t}}\frac{e^{e^{\lambda t}}}{e^{\lambda t}e^{e^{\lambda t}}}\frac{1}{e^t} = \frac{1}{e^{\lambda t}}.$$

So, we have global existence of small data solutions for

$$\begin{cases} 2 \le p & \text{if n = 1, 2,} \\ 2 \le p \le 3 = p_{GN}(3) & \text{if n = 3,} \\ p = 2 = p_{GN}(4) & \text{if n = 4.} \end{cases} \tag{5.2.81}$$

Example 5.2.9. If we choose $a(t) = e^t e^{e^t} e^{e^{e^t}}$, then we have $A(t) = e^{e^{e^t}}$. So, $a_0 = 1$ and $\nu(\lambda) = \infty$. Let us choose functions $\mu(t)$ satisfying the assumption $(R2)$, for example,

$$\mu(t) = e^{2e^{e^t}}/e^{\beta e^t} \ \text{ or } \ e^{2e^{e^t}}/\log(e+t), \ \text{for } \beta > 0.$$

Thus, applying Theorem 5.2.9 we see that the condition (5.2.69) is satisfied. Indeed,

$$\frac{\nu(\lambda,t)}{\log A(t)} = \frac{e^t e^{e^t} e^{e^{e^t}}}{e^{e^{e^t}}}\frac{e^{e^{e^{\lambda t}}}}{e^{\lambda t}e^{e^{\lambda t}}e^{e^{e^{\lambda t}}}}\frac{1}{e^{e^t}} = \frac{e^t}{e^{\lambda t}e^{e^{\lambda t}}}.$$

So, we have global existence of small data solutions for

$$\begin{cases} 2 \le p & \text{if n = 1, 2,} \\ 2 \le p \le 3 = p_{GN}(3) & \text{if n = 3,} \\ p = 2 = p_{GN}(4) & \text{if n = 4.} \end{cases} \tag{5.2.82}$$

6. Concluding remarks and open questions

6.1. Linear theory

6.1.1. Modified Scattering

We try to establish if there exists a relation between the solution $u = u(t, x)$ of

$$u_{tt} - a(t)^2 \Delta u + b(t) u_t = 0,\ u(0, \cdot) = u_1,\ u_t(0, \cdot) = u_2 \tag{6.1.1}$$

and the solution $v = v(t, x)$ of

$$v_{tt} - a(t)^2 \Delta v = 0,\ v(0, \cdot) = v_1,\ v_t(0, \cdot) = v_2 \tag{6.1.2}$$

not only in the case of *scattering dissipation* $b(t)u_t$ with $b(t) \in L^1(\mathbb{R}_+)$ but also in the case of *non-effective dissipation*. Our basic idea is that we will construct a Møller wave operator which relates (u_1, u_2) to (v_1, v_2). The key idea is to multiply the representation $E(t, \xi)$ which respect to (6.1.1) and the representation $E_a(t, \xi)$ with respect to (6.1.2) by the decay rate $\frac{\lambda(t)}{\sqrt{a(t)}}$.

Theorem 6.1.1 (Conjecture). *Assume (A_1), (A_2), $(A_3)^l$ with $l \geq 1$ to $a(t)$ and (B_1), (B_2), (B_3) or $(B_3)'$ to $b(t)$ and (C). Then the operator*

$$W_+(D) = \lim_{t \to \infty} \frac{\lambda(t)}{\sqrt{a(t)}} \big(E_a(t, D)\big)^{-1} E(t, D)$$

exists in $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and its symbol satisfies

$$W_+(\xi) = \frac{1}{\sqrt{a(t_\xi)}} \big(E_a(t_\xi, \xi)\big)^{-1} M Q_k(\infty, t_\xi, \xi) N_k^{-1}(t_\xi, \xi) M^{-1} \lambda(t_\xi) E(t_\xi, \xi)$$

for all $1 \leq k \leq l$, such that the asymptotic of solutions of the problem (6.1.1) and (6.1.2) satisfy

$$\frac{1}{\sqrt{a(t)}} \| E_a(t, \xi) \big(\langle \xi \rangle v_1, v_2\big)^T - E(t, \xi) \big(\langle \xi \rangle u_1, u_2\big)^T \|_{L^2} \to 0$$

as $t \to \infty$.

6.1.2. Energy estimates of higher order

In the thesis we did not study estimates for energies of higher order. Here we propose $L^p - L^q$-estimates for derivatives of the solutions for the non-effective dissipation case.

Theorem 6.1.2 (Conjecture). *Assume $(A1)$ to $(A3)^l$, $(B1)$, $B(2)^l$, $(B3)$ or $(B3)'$ together with (C). Then the $L^p - L^q$ estimate*

$$\| D_t^l D_x^\alpha u(t, \cdot) \|_{L^q} \lesssim \frac{\sqrt{a(t)}}{\lambda(t)} A(t)^{-\frac{n-1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \Big(\|u_1\|_{L^{p, l+r+|\alpha|}} + \|u_2\|_{L^{p, l+r+|\alpha|-1}} \Big)$$

holds for dual indices $pq = p + q$, $p \in (1, 2]$ and regularity $r = n\Big(\frac{1}{p} - \frac{1}{q}\Big)$.

For the effective dissipation case, there are two conjectured theorems with respect two cases of monotonic function $\mu(t)/A(t)$:

Theorem 6.1.3 (Conjecture). *Assume* $(A1)$ *to* $(A3)^l$, $(B'1)$, $(B'2)^l$, $(B'3)$ *together with* $\frac{\mu(t)}{A(t)}$ *is decreasing function. Then the* $L^2 - L^2$ *estimate*

$$\|D_t^l D_x^\alpha u(t,\cdot)\| \lesssim \Big(\frac{a^2(t)}{b(t)}\Big)^l \Big(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}\Big)^{-\frac{|\alpha|}{2}-l} \Big(\|u_1\|_{H^{l+|\alpha|}} + \|u_2\|_{H^{l+|\alpha|-1}}\Big)$$

holds for all $k \geq l$.

Theorem 6.1.4 (Conjecture). *Assume* $(A1)$ *to* $(A3)^l$, $(B'1)$, $(B'2)^l$, $(B'3)$ *together with* $\frac{\mu(t)}{A(t)}$ *is increasing function. Then the* $L^2 - L^2$ *estimate*

$$\|D_t^l D_x^\alpha u(t,\cdot)\| \lesssim \frac{a^2(t)}{b(t)} \Big(\frac{a(t)}{A(t)}\Big)^{l-1} \Big(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}\Big)^{-\frac{|\alpha|+l}{2}} \Big(\|u_1\|_{H^{l+|\alpha|}} + \|u_2\|_{H^{l+|\alpha|-1}}\Big)$$

holds for all $k \geq l \geq 1$.

6.2. Non-linear theory

6.2.1. Blow-up behavior of semi-linear models with effective dissipation

Still open is the question for the blow-up behavior of solutions to the Cauchy problem for damped waves with time dependent speed of propagation and dissipation in the general case

$$u_{tt} + a^2(t)\Delta u + b(t)u_t = f(u),\ u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1. \tag{6.2.1}$$

An effort has been done for proving not only the global existence for (6.2.1) but also the blow-up behavior for this model. Todorova and Yordanov [T-Y01] and Zhang [Zha01], have handled the case $a(t) = b(t) \equiv 1, f(u) \approx |u|^p$. These authors proved that the Fujta exponent $p_c = 1 + 2/n$ is the critical exponent. The modified test function method is an effective tool to prove blow-up in the parabolic like case $a(t) \equiv 1, f(u) \approx |u|^p$ and $b(t) = b_0(1+t)^\beta$, with $|\beta| < 1$, the exponent $p_c = 1 + 2/n$ is still critical exponent, see J. Lin-K. Nishihara-J. Zhai [LNZ12]. For more general cases $a(t), b(t)$ we refer to the paper D'Abbicco and Lucente [D-L12] for blow-up results.

6.3. Abstract problems

The main idea is to understand qualitative properties of solutions to the abstract Cauchy problem

$$u_{tt} + a(t)Au + b(t)A^\sigma u_t = 0,\ u(0) = u_0,\ u_t(0) = u_1,$$

where A is a self-adjoint operator on a Hilbert space X, strictly positive, with dense domain $D(A)$. The term $A^\sigma u_t$ describes a class of damping terms, if $\sigma = 0$, it describes the external damping, if $\sigma \in (0,1]$, it describes the structural damping, the case $\sigma = 1$ is called visco-elastic damping. The coefficient $a = a(t)$ describes in some cases a propagation speed of waves and $b(t)A^\sigma u_t$ describes a damping effect. In connection with the long time behavior of $a(t)$ up to ∞, here decreasing or increasing or oscillating behaviors are of interest. The case $a = b \equiv 1$ was studied in a paper of Chen and Russel [C-R82]. In particular, the case $\sigma = 0$ was motivated by Matsumura [Mat76], while in the case $\sigma = 1$ we can cite, for example, the papers of Ponce [Pon85], Shibata [Shi00], Ikehata-Todorova-Yordanov [ITY13b], Ikehata-Natsume [I-N12]. For the case $a \equiv 1$ abstract scattering results and the diffusion phenomenon are well understood in a series of papers K. Yagdjian [Ya05], [Ya06] and J. Wirth [W07b]. In a recent paper, X. Lu-M. Reissig [L-R09], energy estimates are derived for solutions to the Cauchy problem

$$u_{tt} - \Delta u + b(t)(-\Delta)^{\sigma} u_t = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

6.4. Damped wave models with decreasing speed of propagation

In the future we are interested in the following Cauchy problem:

$$u_{tt} + a^2(t)\Delta u + b(t)u_t = f(u),\ u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1,$$

where $a(t)$ is a decreasing function. Our studies are motivated by the papers Galstian [Gal03] for the special linear model with $a(t) = (1+t)^{-l}$, $a(t) = e^{-t}$ and the paper X. Gang, Y. Huicheng [G-H13].

6.5. Wave models with time-dependent and spatial-dependent coefficients

It seems to be reasonable to attack the Cauchy problem with the dissipation term depending on time and spatial variables

$$u_{tt} + a^2(t)\Delta u + b(t,x)u_t = f(u),\ u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1. \tag{6.5.1}$$

A main goal of this issue could be to find suitable conditions for coefficients a and b, suitable function spaces for the right-hand side and spaces for initial data (u_1, u_2) such that we can obtain the critical exponent power p_{crit}. Let us introduce here papers which contain some notable approaches to reach our goal. Firstly, we start with the Cauchy problem (6.5.1) without right-hand side and $a \equiv 1$. An approach to handle coefficients depending on t and x bases on so-called weighted energy inequalities and was used in the papers of Matsumura, [Mat77], Hirosawa and Nakazawa, [H-N03]. These results provide $L^2 - L^2$-estimates under the assumption

$$b(t,x) \geq b_0 > 0$$

for large values of $|x|$. In the papers of Mochizuki-Nakazawa, [M-N96], Uesaka, [Ues80], they discussed energy decay estimates with the weaker effective assumption for large x

$$b(t,x) \geq b_0(1+t+|x|)^{-1} > 0.$$

Recently, Todorova-Yordanov [T-Y09] treated the case $b(t,x) = b_0(1+|x|)^{-\alpha}$ with $\alpha \in [0,1)$. After that, Ikehata-Todorova-Yordanov [ITY13a] have the optimal decay estimate for the critical case $b(t,x) = b_0(1+|x|)^{-1}$ $(\alpha = 1)$.
Now, let us turn back to the case of right-hand side $f(u(t,x)) = |u(t,x)|^p$. When the coefficient $b(t,x)$ is constant, in two papers of Todorova-Yordanov [T-Y00, T-Y01], the authors have shown that the critical exponent is $p_{crit} = p_c(N) = 1 + 2/N$, where N is the dimension. For the case $b(t,x) > 0$ and $b(t,x) \sim b_0(1+|x|)^{-\alpha}$ for large values of $|x|$ and $\alpha \in [0,1)$ in the paper of Ikehata-Todorova-Yordanov [ITY09] they have proved that the critical exponent is

$$p_{crit} = p_c(N,\alpha) = 1 + \frac{2}{N-\alpha}.$$

For the case $\alpha \geq 1$ this problem is still open.

Appendices

A. Notation

A.1. General notation in thesis

We will introduce here some notions that appear throughout our thesis. We use C to denote an arbitrary constant. The exact value denoted by C may change from line to line in a given computation. The big advantage is that our calculations will be simpler looking, since we continually absorb "extraneous" factors into the term C. In formulas, the brackets [, (, { are used without special meaning, $\{\cdot\}$ is also used to denote sets. Bracket symbols with special meaning are

$\langle\cdot\rangle$	this Japanese bracket stands for $\langle x\rangle = \sqrt{1+\lvert x\rvert^2}$,
$\lceil\cdot\rceil$	denotes the smallest integer larger then a given number, $\lceil x\rceil = \min\{m \in \mathbb{Z},\ \text{s.t. } x \le m\}$,
$\lvert\cdot\rvert$	denotes the absolute value of a scalar expression,
$\lVert\cdot\rVert$	stands for a vector or a matrix norm, in our thesis we prefer to use the row sum norm,
$(\lvert\cdot\rvert)$	denotes for a matrix the matrix of the absolute values of its entries,
$\lVert\cdot\rVert_p$	stands for $\lVert\cdot\rVert_{L^p}$,
$\lVert\cdot\rVert_{p,r}$	stands for $\lVert\cdot\rVert_{L_{p,r}}$,
$\operatorname{tr} A$	denotes the trace of matrix A, i.e, the sum of the diagonal entries of the diagonal matrix A.

For the derivatives we use the following notations:

D_{x_i}	stands for $-\imath\partial_{x_i}$, with $i = 1,\dots,n$, for $x \in \mathbb{R}^n$,
D_t	stands for $-\imath\partial_t$, $t \in \mathbb{R}_+$,
D	stands for $-\imath\nabla_x = -\imath\big(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}\big)^T$ for $x \in \mathbb{R}^n$,
∂_x^α	stands for $\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\dots\partial_{x_n}^{\alpha_n}$ with a multi-index $\alpha = \big(\alpha_1, \alpha_2, \dots, \alpha_n\big)^T$, where each component α_i is a non-negative integer and $\lvert\alpha\rvert = \alpha_1 + \alpha_2 + \dots + \alpha_n$,
Δ	denotes the Laplace operator with respect to $x \in \mathbb{R}^n$: $\Delta_x = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$.

Furthermore, we use the following asymptotic relations

$f \lesssim g$	if there exists a positive constant C_1 such that $f \le C_1 g$ for all arguments,
$f \gtrsim g$	if there exists a positive constant C_2 such that $f \ge C_2 g$ for all arguments,
$f \sim g$	if $f \lesssim g$ and $f \gtrsim g$,
$f = o(g)$	denotes that $\limsup_{x\to\infty} \left\lvert \frac{f(x)}{g(x)} \right\rvert = 0$,
$f = \mathcal{O}(g)$	denotes that $\limsup_{x\to\infty} \left\lvert \frac{f(x)}{g(x)} \right\rvert < \infty$.

A.2. Function spaces

We introduce here function spaces which are frequently used within thesis with a short definition:

$C^k(\mathbb{R}^n)$	space of k-times continuously differentiable functions,
$C^\infty(\mathbb{R}^n)$	space of infinitely continuously differentiable functions,
$C_0^\infty(\mathbb{R}^n)$	space of functions belonging to $C^\infty(\mathbb{R}^n)$ with compact support,
$\mathcal{D}'(\mathbb{R}^n)$	space of distributions, continuous dual space to $C_0^\infty(\mathbb{R}^n)$,
$L^p(\mathbb{R}^n)$	Lebesgue spaces with $1 \le p \le \infty$,
$L^p_{loc}(\mathbb{R}^n)$	$L^p_{loc}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} : u \in L^p(U)$ for each $U \subset\subset \mathbb{R}^n\}$,
$L^{p,r}(\mathbb{R}^n)$	Bessel potential space, $L^{p,r}(\mathbb{R}^n) := \langle D\rangle^{-r} L^p(\mathbb{R}^n)$, $1 \le p < \infty$, $r \in \mathbb{R}$,
$\dot{L}^{p,r}(\mathbb{R}^n)$	Riesz potential space, $\dot{L}^{p,r}(\mathbb{R}^n) := \vert D\vert^{-r} L^p(\mathbb{R}^n)$, $1 \le p < \infty$, $r \in \mathbb{R}$,
$H^s(\mathbb{R}^n)$	Sobolev space based on $L^2(\mathbb{R}^n)$, $H^s(\mathbb{R}^n) = L^{2,s}(\mathbb{R}^n)$,
$H^s(\mathbb{R}^n)$	special Riesz potential space, $\dot{H}^s(\mathbb{R}^n) = \dot{L}^{2,s}(\mathbb{R}^n)$,
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space of rapidly decreasing functions, $\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} \vert x^\alpha \partial_x^\beta f(x)\vert < \infty, \forall \alpha, \beta\}$,
$\mathcal{S}_0(\mathbb{R}^n)$	space of Schwartz functions satisfying $\mathcal{S}_0(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : D_\xi^\alpha \hat{f}(0) = 0, \forall \alpha\}$,
$W^{p,r}(\mathbb{R}^n)$	Sobolev space based on $L^p(\mathbb{R}^n)$ with parameters $r \ge 0$,
$M_q^p(\mathbb{R}^n)$	space of multipliers with parameters $1 \le p, q \le \infty$ correspond to $L^p \to L^q$.

A.3. Symbols used throughout the thesis

In order to make a convenience for the readers we pick and choose some of symbols are often used in our thesis. The following list we will introduced some most important definitions and symbols in a short way. If the symbols are related to a particular chapter, we give also the corresponding reference.

Assumptions are used in this thesis:

(A1)	$a(t) > 0$, $a'(t) > 0$, for $t \in [0, \infty)$
(A2)	$a_0 \dfrac{a(t)}{A(t)} \le \frac{a'(t)}{a(t)} \le a_1 \dfrac{a(t)}{A(t)}$, $a_0, a_1 > 0$
(A3)	$\vert a''(t)\vert \le a_2 a(t) \Big(\frac{a(t)}{A(t)}\Big)^2$, $a_2 \ge 0$
(A4)	$t + C/\sqrt{a(t)}$ is strictly increasing with a positive constant C and for large t,
$(A)^k$	$\vert a^{(j)}(t)\vert \le a_j a(t) \Big(\frac{a(t)}{A(t)}\Big)^j$, $a_j \ge 0$, $j = 1, 2, \dots, k$
$(A)^\infty$	$(A)^\infty := (A)^k$ as $k \to \infty$
(B1)	$b(t) > 0$, $\quad b \notin L^1(\mathbb{R}_+)$
(B2)	$\vert\mu'(t)\vert \le C_\mu \mu(t) \frac{a(t)}{A(t)}$
$(B2)^\infty$	$\vert\mu'(t)\vert \le C_k \mu(t) \Big(\frac{a(t)}{A(t)}\Big)^k$, $k = 1, 2, \dots$
(B3)	$\limsup_{t\to\infty} \mu(t) < 1$
(B3)'	$\liminf_{t\to\infty} \mu(t) > 1$
(C)	$\limsup_{t\to\infty} \big(\mu(t) + \alpha(t)\big) < 2$
(B'1)	$b(t) > 0$

(B'2)	$\left\lvert d_t^k \mu(t) \right\rvert \le C_k \mu(t) \left(\frac{a(t)}{A(t)} \right)^k$ for $k = 1, 2$
(B'3)	$\mu(t)/A(t)$ is monotonic and $\mu(t) \to \infty$ as $t \to \infty$
(B'4)	$a^2(t)/b(t) = a(t)A(t)/\mu(t) \notin L^1(\mathbb{R}_+)$.
(B'5)	$\frac{a^2(t)}{A^2(t)b(t)} \in L^1(\mathbb{R}_+)$
(OD)	$\int_0^\infty \frac{a^2(\tau)}{b(\tau)} d\tau < \infty$
(S1)	$\frac{a^2(t)}{b(t)A^2(t)} \in L^1(\mathbb{R}_+)$

Tab. A.1.: Summarizing assumptions in thesis

Symbols are used throughout our thesis:

$\alpha(t)$	$\frac{a'(t)}{a(t)} =: \alpha(t) \frac{a(t)}{A(t)}$,
$A(t)$	$A(t) = 1 + \int_0^t a(\tau) \mathrm{d}\tau$,
$\lambda(t)$	$\lambda(t) = \exp\left(\frac{1}{2} \int_0^t b(\tau) d\tau \right)$,
$\delta(t)$	$\delta(t) = a(t)/A(t)$,
$\eta(t)$	$\eta(t) = \mu(t)/A(t)$, $\eta_\infty = \lim_{t\to\infty} \eta(t)$,
$\nu(\lambda, t)$	$\nu(\lambda, t) = \frac{a(t)}{A(t)} \frac{A(\lambda t)}{a(\lambda t)}$, $\nu(\lambda) = \lim_{t\to\infty} \nu(\lambda, t)$,
$\langle \xi \rangle_{b(t)}$	$\langle \xi \rangle_{b(t)} = \sqrt{\left\lvert \lvert\xi\rvert^2 - b^2(t)/4 \right\rvert}$ (Section 3.3),
$E_c(u)(t)$	the free wave energy, $E_c(u)(t) = \frac{1}{2} \int \left(c^2 \lvert \nabla u(t,x) \rvert^2 + \lvert u_t(t,x) \rvert^2 \right) dx$,
$E(u)(t)$	the energy, $E(u)(t) = \frac{1}{2} \int \left(a(t)^2 \lvert \nabla u(t,x) \rvert^2 + \lvert u_t(t,x) \rvert^2 \right) dx$,
$U(t, \xi)$	micro-energy, $U = \left(N\delta(t)\hat{u}, D_t \hat{u} \right)^T$. We use this definition for both zones:Z_{pd} in Chapter 2 and Z_{diss} in Chapter 3. $U = \left(a(t)\lvert\xi\rvert\hat{u}, D_t\hat{u} \right)^T$ for all Z_{hyp} in these Chapters. Moreover, $D_t U = A(t, \xi) U$,
$E(t, s, \xi)$	fundamental solution to $D_t U = A(t, \xi) U$, i.e. the matrix-valued solution to $D_t E = A(t, \xi) E$, $E(s, s, \xi) = I$,
$E_0(t, s, \xi)$	fundamental solution to, $D_t E_0 = (\mathcal{D}(t, \xi) + F_0(t)) E_0$, $\quad E_0(s, s, \xi) = I$,
$E_a(t, s, \xi)$	fundamental solution to, $D_t E_a = (\mathcal{D}(t, \xi) + \mathcal{R}_a(t)) E_a$, $\quad E_a(s, s, \xi) = I$,
$E_k(t, s, \xi)$	fundamental solution of the system after k steps of diagonalization, $k \ge 1$, is used to obtain $L^p - L^q$-estimates in Section 4.1,
$v(t, x)$	$v(t, x) = \lambda(t) u(t, x)$, is used in Section 3.3
$V(t, \xi)$	micro-energy, $V(t, \xi) = \left(\langle \xi \rangle_{b(t)} \hat{v}, D_t \hat{v} \right)^T$,
$E_V(t, s, \xi)$	fundamental solution to the system $D_t V = A(t, \xi) V$ Section 3.3,
$E_{V,1}(t, s, \xi)$	fundamental solution of the system after the first step of diagonalization in the elliptic zone, Subsection 3.3.3.

Tab. A.2.: Summarizing symbols are used in thesis

B. Basic tools

B.1. Bessel functions

In part 2 of Chapter 2 we have transformed our partial differential equation to Bessel's equation in order to represent solutions explicitly. For this reason we introduce here some formulae used throughout the calculations in this thesis. Bessel functions are the canonical solutions $y(x)$ of Bessel's differential equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{B.1.1}$$

for an arbitrary complex number α (the order of the Bessel function). There are several ways to define the Bessel functions. We introduce firstly Bessel functions of the first kind J_α:

$$J_\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \Big(\frac{x}{2}\Big)^{2k+\alpha}. \tag{B.1.2}$$

On the one hand, for non-integer α, the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent, and are therefore two linear independent solutions of the differential equation. On the other hand, for integer order α, the following relationship is valid:

$$J_{-n}(x) = (-1)^n J_n(x).$$

The Bessel functions of the second kind, denoted by $Y_\alpha(x)$, occasionally denoted instead by $N_\alpha(x)$, are solutions of the Bessel differential equation that have a singularity at the origin ($x = 0$). These are sometimes called Weber functions due to Heinrich Martin Weber. One defines for non-integer α, $Y_\alpha(x)$ is related to J_α by:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}. \tag{B.1.3}$$

The Bessel functions of third kind or *Hankel functions* which are defined due to N. Nielsen as

$$H_\alpha^\pm(x) = J_\alpha(x) \pm \imath Y_\alpha(x). \tag{B.1.4}$$

We collect here some important properties:

$$\begin{aligned}
H^\pm_{\alpha-1}(x) + H^\pm_{\alpha+1}(x) &= \frac{2\alpha}{x} H^\pm_\alpha(x), && \text{(B.1.5)}\\
H^\pm_{\alpha-1}(x) - H^\pm_{\alpha+1}(x) &= 2\big(H^\pm_{\alpha+1}\big)'(x), && \text{(B.1.6)}\\
\alpha H^\pm_\alpha(x) + x\big(H^\pm_\alpha\big)'(x) &= xH^\pm_{\alpha-1}(x), && \text{(B.1.7)}\\
\alpha H^\pm_\alpha(x) - x\big(H^\pm_\alpha\big)'(x) &= xH^\pm_{\alpha+1}(x). && \text{(B.1.8)}
\end{aligned}$$

All this definitions and properties one can see in [Wat22] or in many other books about Bessel functions.

B.2. Bessel potential spaces and multiplier spaces

The Bessel potential space $L^{p,r}(\mathbb{R}^n)$, $1 \le p < \infty$, $r \in \mathbb{R}$, can be defined as the space of functions (or distributions) u such that $\big(I-\Delta\big)^{r/2}u$ belongs to the Lebesgue space $L^p(\mathbb{R}^n)$ and is endowed with the corresponding Lebesgue norm. The operator $\big(I-\Delta\big)^{r/2}$, which for $r > 0$ is a kind of fractional differentiation, is most easily defined by means of the Fourier transform. In fact, it corresponds to multiplication of the Fourier transform of f by $(1+|\xi|^2)^{r/2}$. For the sake of clarity, we now define the Bessel potential space [AH96] as follows:

$$L^{p,r}(\mathbb{R}^n) := \Big\{f : f = (I-\Delta)^{-r/2} * g,\ g \in L^p(\mathbb{R}^n)\}. \tag{B.2.1}$$

In various of lecture books it is shown that $C_0^\infty(\mathcal{D})$ and $\mathcal{S}$ are dense in $L^{p,r}$. It is a theorem of A.P. Calderón [Cal61] that for positive integers r and $1 < p < \infty$ the space $L^{p,r}$ coincides (with equivalence of norms) with the Sobolev space $W^{p,r}$, that is

Lemma B.2.1 (A. P. Calderón). *For $r \in \mathbb{N}$, $1 < p < \infty$ we have $W^{p,r}(\mathbb{R}^n) = L^{p,r}(\mathbb{R}^n)$ with equivalence of norms, i.e., there is a constant C such that for all f*

$$C^{-1}\|f\|_{L^{p,r}} \le \|f\|_{W^{p,r}} C\|f\|_{L^{p,r}}.$$

In order to obtain $L^p - L^q$-estimates in this thesis we have used a dyadic decomposition and stationary phase method. The basic idea here is the following version of Littman's lemma taken from the paper of P. Brenner and Pecher, [Bre75, Pec76]. We can conclude the following estimate.

Lemma B.2.2. *Let us assume that $K = K(t)$ is a real-valued function and Let P be a real and smooth function in the neighborhood of* $\operatorname{supp}\phi(t,\xi)$, $\phi(t,\xi) \in C_0^\infty(\mathbb{R}^n_\xi)$. *Assume further, that the rank of the Hessian $H_P(\xi) = (\partial P/\partial_{\xi_i}\partial_{\xi_i})$ is at least ρ on* $\operatorname{supp}\phi$. *Then, there exists an integer M, depending on the space dimension, and a constant $C > 0$, depending on bounds of derivatives of P on* $\operatorname{supp}\phi$, *such that*

$$\|F^{-1}\Big(e^{\imath K(t)P(\xi)}\phi(t,\xi)\Big)\|_\infty \le C\big(1+K(t)\big)^{-\frac{n-1}{2}} \sum_{|\alpha|\le M} \|D_\xi^\alpha \phi(t,\xi)\|_\infty$$

holds with a constant C independent of t and ξ.

On the other hand, to handle the $L^p - L^q$-estimates we often use the Riesz-Thorin interpolation theorem, see e.g. in the book of E.M. Stein on singular integrals, [Ste70] or in the lectures of Racke, [Rac92].

Lemma B.2.3. *Let the linear operator T satisfy*

$$\begin{aligned} T &: W^{n,1} \longrightarrow L^\infty,\ \textit{bounded with norm } X_0,\\ T &: L^2 \longrightarrow L^2,\ \textit{bounded with norm } X_1. \end{aligned}$$

There exists a constant $C = C(p,n)$ such that

$$T : W^{N,p} \longrightarrow L^q,\ \textit{bounded with norm } X \le C X_0^{1-\theta} X_1^\theta$$

with $1 < p < 2$, $pq = p + q$, $\theta = 2/q$, $N > n(1-\theta)$ and $N \in \mathbb{N}$.

Next, we introduce lemmas which help us to estimate the dyadic components. We refer to Lemma 3 in the work by Brenner, [Bre75]. Therefore, we can list up here some important properties which are related to interpolation theorems.

Lemma B.2.4. *Let us assume $\phi \in L^1$.*

1. *If $\|F^{-1}(\phi)\|_\infty \leq C_0$, then $\|F^{-1}(\phi F(u))\|_\infty \leq C_0\|u\|_1$.*
2. *If $\|\phi\|_\infty \leq C_1$, then $\|F^{-1}(\phi F(u))\|_2 \leq C_0\|u\|_2$.*
3. *If $\|F^{-1}(\phi)\|_\infty \leq C_0$ and $\|\phi\|_\infty \leq C_1$, then $\|F^{-1}(\phi F(u))\|_q \leq CC_0^{1-\theta}C_1^\theta\|u\|_p$ with $1 \leq p \leq 2$, $pq = p + q$ and $\theta = 2/q$.*

Now, we can summarize Lemma 1 and Lemma 2 in Brenner's paper, [Bre75]. Therefore, let us assume $\chi \in C_0^\infty$ to be non-negative with support contained in $[1/2, 2]$ and

$$\sum_{j=-\infty}^{\infty} \chi(2^j r) = 1, \qquad r \neq 0.$$

This functions form a so-called dyadic decomposition. The basic idea of the proof are embedding relations between Lebesgue and Besov spaces.

Lemma B.2.5. *Let $\phi \in L^\infty(\mathbb{R}^n)$ and assume that*

$$\|F^{-1}(\phi(\xi)\chi_j(\xi)\hat{v})\|_q \leq C\|v\|_p$$

holds uniform for all j and $p \in (1, 2]$, $pq = p+q$. Then for a constant M independent of ϕ it follows

$$\|F^{-1}(\phi(\xi)\hat{v})\|_q \leq MC\|v\|_p.$$

For completeness of this section we just introduce very briefly the multiplier spaces M_p^q which were treated in the paper of L. Hörmander, [Hör60]. The multiplier space M_p^q is defined as the set of all Fourier transforms $F(f)$ of distributions $f \in L_p^q$, and the elements $F(f) \in M_p^q$ are called multipliers. Here L_p^q is the set of all distributions $f \in \mathcal{S}'$ with

$$\|f * u\|_q \leq C\|u\|_p$$

for all $u \in \mathcal{S}$.

B.3. Further lemmas and useful calculations

Gronwall's inequality. There are two forms of the lemma, a differential form and an integral form. The differential form was proven by Grönwall in 1919, [Gro19]. The integral form was proved by Richard Bellman in 1943, [Bel43]. In our thesis, Gronwall's inequality is a useful tool for energy estimates.

Lemma B.3.1 (Integral form)**.** *Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let α, β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded sub-interval of I.*

(a) If β is non-negative and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\,\mathrm{d}s, \qquad \forall t \in I,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\Big(\int_s^t \beta(r)\,\mathrm{d}r\Big)\mathrm{d}s, \qquad t \in I.$$

(b) If, in addition, the function α is non-decreasing, then

$$u(t) \le \alpha(t) \exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right), \qquad t \in I.$$

Furthermore, in our thesis we often have to handle homogeneous linear systems of ordinary differential equations

$$D_t U = A(t)U \tag{B.3.1}$$

with $t \in \mathbb{R}_+$. It is well-known, that the Peano-Baker formula can be used for the representation of the fundamental solution. This approach goes back to Peano, [Pea88], and was further developed by Baker, [Bak05]. The explicit integral expansions (Peano-Baker series) one can see in Sec. 16.5 of the book of E.L. Ince, [Inc56].

Lemma B.3.2. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution $E(t,s)$ to $\partial_t - A(t)$ is given by the Peano-Baker formula*

$$E(t,s) = I + \sum_{k=1}^{\infty} \int_s^t A(t_1) \int_s^{t_1} A(t_2) \cdots \int_s^{t_{k-1}} A(t_k)\mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1.$$

The proof follows by differentiating the series term by term.

Corollary B.3.3. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental matrix $E(t,s)$ satisfies*

$$\|E(t,s)\| \le \exp\left\{\int_s^t \|A(\tau)\|\mathrm{d}\tau\right\}.$$

In order to guarantee the invertibility of the fundamental solution which arises from estimates in scattering results or statements about asymptotic behavior of fundamental solution it is convenient to use the Liouville's formula in the following form.

Lemma B.3.4. *Let us assume that $E(t,s)$ is a matrix-valued solution of the system (B.3.1). Then*

$$\det E(t,s) = detE(s,s) \exp\left(i \int_s^t \operatorname{tr} A(\tau)\mathrm{d}\tau\right)$$

for $0 \le s \le t$.

A proof for this lemma one may found in standard text-books on differential equation, for instance, V. I. Arnold, [Arn01], or Chicone, [Chi06].
Faà di Bruno's formula. Perhaps the most well-known form of Faà di Bruno's formula, F. d. Bruno [Bru55, Bru57], says the following:

Lemma B.3.5. *Let $f(g(x)) = (f \circ g)(x)$ with $x \in \mathbb{R}$. Then we have*

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{m_1!1!^{m_1} m_2!2!^{m_2} \cdots m_n!n!^{m_n}} \cdot f^{(m_1+\cdots m_n)}(g(x)) \cdot \prod_{j=1}^{n} \left(g^{(j)}(x)\right)^{m_j}, \tag{B.3.2}$$

where the sum is taken over all n-tuples of non-negative integers $(m_1, \ldots, m_n)$ satisfying the constraint

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots n \cdots m_n = n.$$

A multivariate version of Faà di Bruno's formula, Constantine-Savits [C-S96], Leipnik-Pearce [L-P06], is given in the next statement.

Lemma B.3.6. *Let $y = g(x_1, \dots, x_n)$. Then the following identity holds regardless of whether the n variables are all distinct, or all identical, or partitioned into several distinguishable classes of indistinguishable variables*

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} f(y) = \sum_{\pi \in \Pi} f^{(|\pi|)}(y) \cdot \prod_{B \in \pi} \frac{\partial^{|B|} y}{\prod_{j \in B} \partial x_j}, \tag{B.3.3}$$

where

- *π runs through the set Π of all partitions of the set $\{1, \dots, n\}$,*
- *$B \in \pi$ means the variable B runs through the list of all of the "blocks" of the partition π, and*
- *$|A|$ denotes the cardinality of the set A (so that $|\pi|$ is the number of blocks in the partition π and $|B|$ is the size of the block B).*

Let us give some generalizations of the formula of Faà di Bruno for a composite function with a vector argument, see Mishkhov [Mis00].

Lemma B.3.7. *If f and t are scalars, $\boldsymbol{x}(t) = [x_1(t), x_2(t), \dots, x_r(t)]^T$ is an r-vector and $f(\boldsymbol{x}(t))$ is a composite function for which all the necessary derivatives are defined. Then*

$$D^n f(\boldsymbol{x}(t)) = \sum_0 \sum_1 \cdots \sum_n C(n, k_i, q_{ij}) \frac{\partial^k f}{\partial {x_1}^{p_1} \partial {x_2}^{p_2} \cdots \partial {x_r}^{p_r}} \prod_{i=1}^{n} \left(x_1^i\right)^{q_{i1}} \left(x_2^i\right)^{q_{i2}} \cdots \left(x_r^i\right)^{q_{ir}}, \tag{B.3.4}$$

where the respective sums are taken over all non-negative integer solutions of the Diophantine equations as follows:

$$\begin{aligned}
&\sum_0 \to k_1 + 2k_2 + \dots n k_n = n, \\
&\sum_1 \to q_{11} + q_{12} + \cdots + q_{1r} = k_1, \\
&\vdots \\
&\sum_n \to q_{n1} + q_{n2} + \cdots + q_{nr} = k_n,
\end{aligned}$$

and the differential operator $D = d/dt$, p_j-the order of the partial derivative with respect to x_j, k-the order of the partial derivative are

$$\begin{aligned}
p_j &= q_{1j} + q_{2j} + \cdots + q_{nj}, \ j = 1, 2, \dots, r, \\
k &= p_1 + p_2 + \cdots + p_r = k_1 + k_2 + \dots + k_n.
\end{aligned}$$

Gagliardo-Nirenberg inequality. The Gagliardo–Nirenberg inequality is a result in the theory of Sobolev spaces that estimates the weak derivatives of a function. The meaning of this inequality is the estimates are in terms of L^q norms of the function can be estimated by its derivatives, and these one interpolates among various values of q and orders of differentiation. See Part 1 in A. Friedman [Fri76]

Lemma B.3.8. *The inequality concerns functions $u : \mathbb{R}^n \longrightarrow \mathbb{R}$. Fix $1 \leq p, r \leq \infty$ and a natural number m. Suppose also that a real number a and a natural number j are such that*

$$\frac{1}{q} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right) a + \frac{1-a}{p}$$

and

$$\frac{j}{m} \leq \alpha \leq 1.$$

Then

1. *every function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ that lies in $L^p(\mathbb{R}^n)$ with m^{th} derivative in $L^r(\mathbb{R}^n)$ also has j^{th} derivative in $L^q(\mathbb{R}^n)$,*

2. *and, furthermore, there exists a constant C depending only on m, n, j, q, r and a such that*

$$\|\mathrm{D}^j u\|_{L^q} \leq C \|\mathrm{D}^m u\|_{L^r}^a \|u\|_{L^p}^{1-a}.$$

The result has two exceptional cases:

1. *If $j = 0, mr < n$ and $p = \infty$, then it is necessary to make the additional assumption that either u tends to zero at infinity or that u lies in L^s for some finite $s > 0$.*

2. *If $1 < r < \infty$ and $m - j - n/r$ is a non-negative integer, then it is necessary to assume also that $a \neq 1$.*

If we choose $j = 0, m = 1$ and $r = p = 2$, then we obtain

$$\|u\|_{L^q} \leq C \|\nabla u\|_{L^2}^{\theta(q)} \|u\|_{L^2}^{1-\theta(q)}.$$

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