

The University of Resources. Since 1765.



Fuzzy Bilevel Optimization

By the Faculty of Mathematik und Informatik
of the Technische Universität Bergakademie Freiberg

approved

Thesis

to attain the academic degree of

doctor rerum naturalium
(Dr.-rer. nat.)

submitted by **Dipl.-math. Alina Ruziyeva**

born on the 8. Mai 1986 in Almaty (Kazakhstan)

Assessor: Prof. Dr. Stephan Dempe
Prof. Dr. Andreas Fischer

Date of the award: Freiberg, 13th February 2013

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

Die Hilfe eines Promotionsberaters habe ich nicht in Anspruch genommen. Weitere Personen haben von mir keine geldwerten Leistungen für Arbeiten erhalten, die nicht als solche kenntlich gemacht worden sind. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

13. Februar 2013

Dipl.-math. Alina Ruziyeva

Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such.

I did not seek the help of a professional doctorate-consultant. Only those persons identified as having done so received any financial payment from me for any work done for me. This thesis has not previously been published in the same or a similar form in Germany or abroad.

13th February 2013

Dipl.-math. Alina Ruziyeva

Acknowledgement

First of all I would like to express my deepest gratitude to my supervisor Prof. Dr. Stephan Dempe for giving me the opportunity to pursue my PhD studies at TU Bergakademie Freiberg and for his continuing support over the years.

All the colleagues and staff members of the Department of Mathematics and Computer Science of TU Bergakademie Freiberg deserve my special thanks for providing me with a pleasant working environment.

Last but not least, I would like to sincerely thank my fiancé Michael and my mother Raisa for their support and trust.

Contents

1	Introduction	1
1.1	Why optimization	1
1.2	Fuzziness as a concept	2
1.3	Bilevel problems	6
2	Preliminaries	11
2.1	Fuzzy sets and fuzzy numbers	11
2.2	Operations	15
2.3	Fuzzy order	16
2.4	Fuzzy functions	17
3	Optimization problem with fuzzy objective	19
3.1	Formulation	19
3.2	Solution method	20
3.3	Local optimality	24
3.4	Existence of an optimal solution	25
4	Linear optimization with fuzzy objective	27
4.1	Main approach	28
4.2	Example	30
4.3	Optimality conditions	33
4.4	Membership function value	34
4.4.1	Special case of triangular fuzzy numbers	36
4.4.2	Example	39
5	Optimality conditions	47
5.1	Differentiable fuzzy optimization problem	48
5.1.1	Basic notions	48
5.1.2	Necessary optimality conditions	49
5.1.3	Sufficient optimality conditions	49
5.2	Nondifferentiable fuzzy optimization problem	51
5.2.1	Basic notions	51
5.2.2	Necessary optimality conditions	52
5.2.3	Sufficient optimality conditions	54
5.2.4	Example	55
6	Fuzzy linear optimization problem over fuzzy polytope	59
6.1	Basic notions	60
6.2	The fuzzy polytope	63

6.3	Formulation and solution method	65
6.4	Example	69
7	Bilevel optimization with fuzzy objectives	73
7.1	General formulation	74
7.2	Solution approaches	74
7.3	Yager index approach	76
7.4	Algorithm I	77
7.5	Membership function approach	78
7.6	Algorithm II	80
7.7	Example	81
8	Linear fuzzy bilevel optimization (with fuzzy objectives and constraints)	87
8.1	Formulation	87
8.2	Solution approach	88
8.3	Algorithm	90
8.4	Example	91
9	Conclusions	95
	Bibliography	97

1 Introduction

1.1 Why optimization

The mathematical area of optimization plays a very important role in our modern life. In our age of scarce natural resources such as oil and gas, it is particularly important not to waste. This requires an optimal use of these supplies. The same applies to other limited resources as well as time. This strive for optimal solutions is not limited to industrial applications but has long reached our daily life.

Our quest is complicated by the fact that we - more often than not - have to make decisions without being able to rely on precise information. We have to deal with an amount of uncertainty every single day. Moreover, our understanding of linguistic variables often depends on e.g. mood. Let us consider the phrase "The shop is near the house". In terms of fuzzy logic we understand word "near" differently. It strongly depends on the age of the decision making person (or decision-maker for short). Thus, the young decision-maker can say that the phrase is true, even if the distance from the house to the shop exceeds 2 km. For the old person, the phrase holds true only if the distance is less than, say, 800 m. Of course, in this example we have to define what words "young" and "old" mean. However, it becomes now clear that different persons define usual things differently. Thus, it makes perfect sense to speak about fuzzy optimization problems from a vague predicate approach, as it is understood that this vagueness arises from the way we express the decision-makers' (i.e. our) knowledge and not from any random event. In short, it is supposed that the nature of the data defining the problem is *fuzzy*.

In practical situations, the problem of optimization is even more complicated, since it involves conflict resolution. Each partly involved decision-makers try to maximize their own benefits. Such an optimization problem can be illustrated with the following example of gas use: The government tries to maximize profit with its tax-policy for private gas companies. In turn, those companies try to maximize their profit by setting a price for gas. Consumers decide on which company to choose by comparing prices. They try to minimize costs by choosing the company that offers a smallest gas price. The easiest way for the government is to prescribe infinitely large taxes. But with such a policy it will lose all the companies and a region (e.g. a city) can be left without any gas. Therefore, the government has to establish taxes wisely. Each company, in turn, has to fix prices which are acceptable for the clients. Otherwise, the clients choose another corporation. Multilevel optimization problems are important for decentralized organizations and systems, where each unit (or department) seeks its own interest. Carefully defined multilevel mathematical programming problems can also serve as useful tools in modelling structured economic units.

In the present work we focus our attention on a special case of multilevel optimization, namely *bilevel optimization*. We discuss hierarchical problems of two decision makers,

in which one - the so-called leader - has the first choice and the other one - the so-called follower - reacts optimally on the leader's selection. It is important to note, that each decision-maker maximizes his / her own benefits independently, but is affected by actions of the other decision-maker (through externalities). The formulation of the bilevel programming problem for crisp (i.e. with exactly known and fixed) data can be found e.g. in the book of Dempe (2002).

1.2 Fuzziness as a concept

In crisp optimization problems it is assumed that the decision-maker has exact and full information on the data entering the problem. Even when this is the case, the decision-maker usually finds it more convenient to express his / her knowledge in linguistic terms, i.e. through conventional linguistic variables (see e.g. Zadeh (1975a,b,c)), rather than by using high precision numerical data.

One commonly used approach to deal with these problems is to model them as fuzzy optimization problems, see e.g. Zadeh (1965). This approach proved to be very useful in many applied sciences, such as engineering, economics, applied mathematics, physics, as well as in other disciplines: Buckley and Feuring (2000); Chanas and Kuchta (1996a); Jiménez et al. (2006); Kasperski and Zieliński (2006); Peidro et al. (2010); Weber et al. (1990); Wu and Xu (2008); Wu et al. (1997); Zimmermann (1978); Zhang et al. (2010); Zimmermann (1976). Among linear programming problems, the so-called transportation problem is very popular, see e.g. Chanas and Kuchta (1996a); Shih and Lee (1999). The model which customarily has been referred to as transportation problem represents not only delivery planning problem with given supplies and demands and with a criterion of minimizing the total transportation cost. Many other decision-making problems, whose motivation is quite different from that of the delivery planning, have the same mathematical structure. For example, this is the case for the periodical production planning problem with given demands for the product in consecutive periods and with the criterion of minimizing the total production and storage cost.

There exist effective algorithms solving the transportation problem in the case when all coefficients in the model, i.e. supply and demand values as well as the unit transportation costs, are given in a crisp way. In practice, however, this condition may not be fulfilled. For example, the unit transportation costs are rarely constant and predictable. Therefore, the ability to define and to determine the optimal solution of the transportation problem with fuzzy costs coefficients is important. This is exactly the topic of most examples presented in the present work.

In the thesis the following (nonlinear) fuzzy optimization problem is investigated, where the objective function has fuzzy values and the constraint function is a crisp one, i.e.:

$$\begin{aligned} \tilde{f}(x) &\rightarrow \min \\ g(x) &\leq 0. \end{aligned} \tag{1.1}$$

Here $g = (g_1, \dots, g_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a crisp function and $\tilde{f} : \mathbb{R}^n \rightarrow \mathfrak{F}$ is a fuzzy function, where \mathfrak{F} is a set of fuzzy numbers over \mathbb{R} and $1 \leq k < \infty$, $1 \leq n < \infty$. The investigated problem can be transformed into a more general problem where the constraint function

is fuzzy. The formulation of fuzzy optimization problems with crisp objective and fuzzy constraints can be found in Delgado et al. (1989) and Tanaka et al. (1984).

An early approach for solving a fuzzy optimization problem is the extension principle of Bellman and Zadeh (1970). Even nowadays many authors base their solution algorithms on this approach (see e.g. Ekel et al. (1998)). In the present work fuzzy optimization problem (1.1) is solved with modern solution algorithms based on the minimization of a certain α -cut on the feasible set (see e.g. Chanas and Kuchta (1994, 1996b); Dempe and Ruziyeva (2011); Rommelfanger et al. (1989); Zimmermann (1991)).

In this approach fuzzy optimization problem (1.1) is reformulated into an interval optimization problem

$$\begin{aligned} [f_L(x, \alpha), f_R(x, \alpha)] &\rightarrow \min \\ g(x) &\leq 0 \end{aligned} \quad (1.2)$$

for a certain level-cut α ($0 \leq \alpha \leq 1$) of the fuzzy function $\tilde{f}(x)$. Through the agency of a special order for the intervals defined later, both of the left- and right-side functions $f_L(x, \alpha)$ and $f_R(x, \alpha)$ have to be minimized simultaneously.

Thus, a crisp biobjective optimization problem arises

$$\begin{aligned} f_L(x, \alpha) &\rightarrow \min \\ f_R(x, \alpha) &\rightarrow \min \\ g(x) &\leq 0, \end{aligned} \quad (1.3)$$

that is solved, in turn, with application of methods of the multiobjective optimization problem's scalarization technique (see e.g. Ehrgott (2005)). Elements of the Pareto set of each biobjective optimization problem are interpreted as solutions of the initial fuzzy optimization problem on a certain level-cut. Thus, we reflect incomparability (and variability) of the solutions of the fuzzy optimization problem. This discussion is presented for the general case in Chapter 3.

The problems usually considered in optimization are mathematical models and, thus, idealizations of real world problems. Therefore, the classical "achieve the best value of the objective function" approach may be too restrictive. Often a set of alternative solutions is more valuable to the decision-maker.

Many authors (see e.g. Jiménez et al. (2006); Verdegay (1982)) try to find a *single* best solution of the fuzzy optimization problem in the linear case

$$\begin{aligned} \tilde{c}^\top x &\rightarrow \min \\ Ax &\leq b \\ x &\geq 0, \end{aligned} \quad (1.4)$$

where the fuzzy vector \tilde{c} , the constraint matrix $A \in \mathbb{R}^{m \times n}$ and the right-hand side vector $b \in \mathbb{R}^m$ are given. These approaches are based on the extension principle of Bellman and Zadeh (1970). We suggest to reflect the uncertainty in fuzzy optimization problems through (the existence of) a set of optimal solutions, i.e. a set of Pareto optimal solutions of corresponding biobjective optimization problem. Under the assumption that this set consists of more than one element, the decision-maker can improve the choice relying on some criteria that are not a priori considered in the optimization problem.

Therefore, it is natural to consider the solutions of a fuzzy optimization problem as fuzzy. Hence, a criterion for comparing the elements of the fuzzy set of optimal solutions is required. As soon as this fuzzy set has membership function, the possible criteria for comparison the elements of the fuzzy solution can be the values of the membership function. For this it is necessary to compute the values of the membership function *exactly*. An approach to calculate such membership function values is suggested by Chanas and Kuchta (1994, 1996a,b) and further developed by Dempe and Ruziyeva (2012).

The approach to determine the membership function values is based on calculating the sum of lengths of certain intervals. One of the purposes of the present work is to realize this idea based on modern solution algorithms (see e.g. Cadenas and Verdegay (2009); Chanas (1983); Chanas and Kuchta (1994, 1996b); Jiménez et al. (2006); Rommelfanger et al. (1989); Zimmermann (1978)). Our solution approach for fuzzy linear optimization problem (1.4) is based on the reformulation of the well-known optimality conditions for the crisp linear optimization problem (see e.g. Bertsimas and Tsitsiklis (1997)).

With this innovative approach, published in Dempe and Ruziyeva (2012), the decision-maker obtains a collection of some basic solutions, each accompanied by a measure of the extent to which it is the optimal solution of fuzzy optimization problem (1.4). It is up to the decision-maker to make the final choice - the decision-maker can restrict himself / herself to the solutions which are equal to a degree greater than a fixed value, or the solutions which have membership function values greater than a fixed value, or equal to one, etc. In any case our fuzzy solution will constitute an important support and source of information for the decision-maker.

In Chapter 4 we discuss the fuzzy linear optimization problem and derive explicit formulas for the calculation the membership function value of the elements of the fuzzy solution in the case of triangular fuzzy numbers and show that only one certain interval needs to be considered.

Generalizing our approach to *nonlinear* fuzzy optimization problem (1.1), the question of optimality of a feasible solution arises, which is very important and essential in optimization. In the nonlinear case of the fuzzy optimization problem, using basic methods of convex multiobjective optimization, necessary and sufficient conditions for an optimal solution of the *differentiable* fuzzy optimization problem are derived e.g. in a form of Karush-Kuhn-Tucker optimality conditions.

Wu (2004) gave sufficient optimality conditions for a solution of fuzzy optimization problem (1.1) under convexity assumptions. In Wu (2008) integrals in the Karush-Kuhn-Tucker conditions were used for sufficient optimality conditions of fuzzy optimization problem (1.1). This means, the author used a certain average value of the level sets of the fuzzy objective function. In distinction, in Wu (2004) only one α -cut was used. Left- and right-hand side functions were used by Wu (2004, 2008) to describe the level-cuts of the fuzzy objective function which then appear in the Karush-Kuhn-Tucker optimality conditions.

For the differentiable case of fuzzy optimization problem (1.1) we present sufficient optimality conditions by using the Karush-Kuhn-Tucker conditions. It turns out that these conditions are similar to the sufficient optimality conditions used in the works of Wu (2004, 2008). The distinction is the use of weighting coefficients in the objective. Further, we derive necessary optimality conditions.

In the differentiable fuzzy case necessary and sufficient optimality conditions are given by Dempe and Ruziyeva (2011) and in the nondifferentiable crisp case by Bomze et al. (2010). If the fuzzy objective function $\tilde{f}(x)$ is *nondifferentiable*, it requires some modifications in the standard approach.

Adapting the notions of the tangent cone, the directional derivative and Hadamard derivatives to the fuzzy case permits us to derive necessary and sufficient optimality conditions for a (global / local) optimal solution of the nondifferentiable fuzzy optimization problem.

As soon as we define a set of optimal solutions of fuzzy optimization problem (1.1) on some fixed α -cut through the set of Pareto optimal solutions of biobjective optimization problem (1.3), we can derive necessary and sufficient optimality conditions (for both differentiable and nondifferentiable fuzzy optimization problems) to guarantee that a feasible point belongs to the fuzzy solution set. This result generalizes one obtained in Panigrahi et al. (2008); Wu (2007) in four important aspects:

1. The derivative of the fuzzy function $\tilde{f}(x)$ is defined as a pair of functions which need not to be an interval as it was supposed in the paper of Panigrahi et al. (2008). This assumption is unnecessary restrictive.
2. Not only sufficient but also necessary optimality conditions are derived.
3. An optimality condition which is valid for all level-cuts at the same time is derived.
4. Nondifferentiable (and nonconvex) problems are discussed.

Optimality conditions of the nonlinear fuzzy optimization problem are examined in Chapter 5 in detail.

The next interesting question is the generalization of the fuzzy optimization problem to a fuzzy optimization problem with fuzzy constraints. An evolutionary algorithm based on multi-objective approach was presented by Jiménez et al. (2006). A so-called interactive approach is developed by Ammar (2000). A shortcoming of this approach is predetermined level-cut.

A fuzzy optimization problem with fuzzy constraints have been examined for the linear case e.g. by Chanas (1983); Ekel et al. (1998); Werners (1987) using the min-max approach and Buckley (1995) using the possibilistic approach.

We present a solution algorithm for the linear case of fuzzy optimization problem defined as

$$\begin{aligned} F(\tilde{c}, x) = d^\top \tilde{c} &\rightarrow \min \\ \text{s.t. } \tilde{c} &\in \mathfrak{P}, \end{aligned} \quad (1.5)$$

where \mathfrak{P} is a fuzzy polytope, d is a known crisp vector and \tilde{c} is a fuzzy variable.

Because of the vagueness, the decision-maker prefers to have not just one solution but a set of them, so that the most suitable solution can be applied according to his / her judgement.

Fuzzy optimization problem (1.5) is solved by taking level-cuts of the fuzzy polytope \mathfrak{P} for all $\alpha \in [0, 1]$. Each α -cut, in turn, provides two crisp optimization problems

$$\begin{aligned} F(c_L(\alpha)) = d^\top c_L(\alpha) &\rightarrow \min \\ \text{s.t. } c_L(\alpha) &\in \mathfrak{P}_L(\alpha) \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} F(c_R(\alpha)) &= d^\top c_R(\alpha) \rightarrow \min \\ \text{s.t. } c_R(\alpha) &\in \mathfrak{F}_R(\alpha), \end{aligned} \quad (1.7)$$

where $[c_L(\alpha), c_R(\alpha)]$ denotes the α -cut of fuzzy variable \tilde{c} . Let us denote the solution sets of problems (1.6) and (1.7) are $c_L^*(\alpha)$ and $c_R^*(\alpha)$, respectively. Then, an optimal solution of the fuzzy optimization problem on the fixed α -cut is a convex hull of $c_L^*(\alpha)$ and $c_R^*(\alpha)$.

The fuzzy solution of initial problem (1.5) is the union of these convex hulls for all level-cuts, i.e.

$$\tilde{c}^* = \bigcup_{\alpha \in [0,1]} (\text{conv}\{c_L^*(\alpha), c^*(1)\} \cup \text{conv}\{c_R^*(\alpha), c^*(1)\}), \quad (1.8)$$

where $c^*(1)$ is an optimal solution of problem

$$\begin{aligned} F(c_1) &= d^\top c_1 \rightarrow \min \\ \text{s.t. } c_1 &\in \mathfrak{F}(1) \end{aligned} \quad (1.9)$$

for $\alpha = 1$ under assumption that we operate with triangular fuzzy numbers. Ideas are investigated in Chapter 6.

1.3 Bilevel problems

Bilevel programming problems are challenging problems of mathematical optimization, which are interesting from the theoretical point-of-view (as special case in nonsmooth optimization) and have a variety of applications. Problems with a predominantly hierarchical structure are often found in government policy, economic systems, finance and are especially suitable for conflict resolutions.

Since its first formulation by Heinrich von Stackelberg (1934) in market economy (in the context of unbalanced economic markets), bilevel optimization has successfully been applied to many real world problems: Bard et al. (1998); Bjørndal and Jørnsten (2005); Camacho (2006); Candler et al. (1981); Cassidy et al. (1971); Dempe (2002); Fortuny-Amat and McCarl (1981); Hobbs and Nelson (1992); Marcotte and Savard (2001); Parraga (1981). For the past twenty years transportation problems have been benefiting from the formulation of advances in bilevel programming: Ben-Ayed (1988); Ben-Ayed et al. (1992); Dempe et al. (2009); Kim and Suh (1988); Labbè et al. (1998); Migdalas (1995), which cover issues like network design, revenue management and other traffic control problems (where the transportation problem is on the lower level, depending on the parameter selected from the upper level).

Considering the inherently difficult nature of bilevel problems due to their nonconvexity, nonsmoothness and implicitly determined feasible set, it is difficult to design convergent algorithms, and the few algorithms that converge appear to be very slow most of the time. Even in the simplest case, i.e. when the upper and lower level problems are crisp and linear, the bilevel programming problem has been shown to be \mathcal{NP} -hard (see Ben-Ayed and Blair (1990); Blair (1992)).

One approach to solving bilevel optimization problems in the crisp case is based on its transformation into a one-level optimization problem using e.g. Karush-Kuhn-Tucker

optimality conditions. S. Dempe and co-workers investigated this problem as well as its optimality conditions, see e.g. Bialas et al. (1980); Dempe (1987, 2000, 2002); Dempe et al. (2006). However, this approach can not be shown to give a global optimal solution.

To solve crisp multilevel linear programming problems (bilevel programming problems are a special case) Shih et al. (1996) presented the so-called fuzzy approach. The authors allege that the solution of a multilevel linear optimization problem over a polytope is not necessarily situated at a vertex. This approach is based on predetermined tolerance limits as well as Bellman and Zadeh (1970) max-min approach. But this method has the vice that for some artificially introduced membership function the hierarchical order can become redundant, i.e. the inclusion of other levels into the system will not affect the solution. Later Sinha (2003) showed that another technique based on fuzzy mathematical programming gives better solutions than that proposed by Shih et al. (1996). In this algorithm the author uses a payoff matrix consisting in ideal solutions and the same *tolerance limits* as in Shih et al. (1996). However, the question of the existence of a feasible solution that is located in such a strongly restricted interval is not addressed.

Recently, numerous algorithmic approaches have been proposed for the special case of the linear bilevel optimization problem (also known as two-level linear source control problem) by Bard and Moore (1990); Bialas and Karwan (1984); Dempe (1987). Using duality theory Shi et al. (2007) applied the k -th best algorithm to linear bilevel optimization problem in the case of multiple followers.

Several works tried to establish a relationship between multilevel and multicriterion programming problems (see e.g. Bard (1983); Wen and Hsu (1989)). Bialas and Karwan (1978); Marcotte and Savard (1991) have shown that Pareto and bilevel optimality are distinct concepts. Further evidence can be found in e.g. Candler (1988); Haurie et al. (1990). It should be noted that bilevel and bicriterial problems are often confused in literature (see e.g. Arora and Gupta (2009)).

All the aforesaid can be combined to the more complicated problem of *fuzzy bilevel optimization* (also called fuzzy bilevel decision making in the literature), if the data involved in the bilevel optimization problem are only approximately known. While very important for number of applications, this problem is poorly investigated. A number of fuzzy bilevel programming problems can be found in Dempe et al. (2009); Dempe and Starostina (2006) and references therein. While some convergent algorithms for crisp bilevel problems already exist in the literature (see e.g. Bard (1982); Bard and Moore (1990); Dempe (1987); Ishizuka and Aiyoshi (1992); Önal (1993); Tuy and Ghannadan (1998); Wen and Huang (1996); White and Anandalingam (1993); Wu et al. (1998)), solution strategies for fuzzy bilevel programming problems are an emerging new field with a wide range of practical applicability.

In the case of crisp bilevel optimization problems many authors consider that the optimal solution is over polytope (see e.g. Calvete et al. (2011)), analogously we assume that we have fuzzy bilevel optimization problem over fuzzy polytope. Thus, we suppose that the optimal solution (and later we would be interested in best optimal solution) is located in one of the extreme points of the corresponding polytope. Strict definitions are given further in the work. At the moment the fuzzy bilevel optimization problem with fuzzy objective function is formulated as follows.

The follower seeks to minimize his / her fuzzy objective function $f(\tilde{c}, x) = \tilde{c}^\top x$ with

respect to x over a crisp polytope X . The leader, in turn, minimizes his / her objective function $F(\tilde{c}, x)$ over the given fuzzy polytope \tilde{C} . We assume that the fuzzy function $F(\tilde{c}, x)$ is also bilinear. More formally, the fuzzy bilevel optimization problem can be formulated as

$$\begin{aligned} F(\tilde{c}, x) &\rightarrow \min_{\tilde{c} \in \tilde{C}} \\ \text{s.t. } x &\in \arg \min_x \{f(\tilde{c}, x) : x \in X\}. \end{aligned} \quad (1.10)$$

Let us denote the set of optimal solutions of lower-level optimization problem through $\Psi(\tilde{c})$; that is to say

$$\Psi(\tilde{c}) = \arg \min_x \{f(\tilde{c}, x) : x \in X\}. \quad (1.11)$$

In general, the problem of determining the best solution \tilde{c}^* for the leader can be described as that of finding a vector of parameters for the fuzzy parametric optimization problem, which together with the response of the follower $x(\tilde{c}) \in \Psi(\tilde{c})$ proves to give the best possible function value for the upper level objective function $F(\tilde{c}, x)$. That is

$$” \min ”_{\tilde{c} \in \tilde{C}} \{F(\tilde{c}, x) : x \in \Psi(\tilde{c})\}. \quad (1.12)$$

Strictly speaking, this definition of the fuzzy bilevel programming problem is valid only in the case of a uniquely determined lower level solution for each possible \tilde{c} . The quotation marks in (1.12) have been used to express this uncertainty in case of non-uniquely set of optimal solutions on the lower level. From now onwards those marks would be dropped.

As soon as we apply the approach presented in Chapter 4 to the lower level problem

$$f(\tilde{c}, x) \rightarrow \min_{x \in X}, \quad (1.13)$$

more than one optimal solution can be obtained, i.e. in this case there can be multiple optima. That means, that $\Psi(\tilde{c})$ not necessarily is a singleton for some permissible \tilde{c} . If the upper level objective function is sensitive to different values of $x \in \Psi(\tilde{c})$, it is necessary to give a rule of selection of an optimal $x^* \in \Psi(\tilde{c})$ in order to evaluate $F(\tilde{c}, x)$.

Notice that there is no reason why both decision-makers should collaborate, i.e. it is not certain that the upper level decision-maker can enforce a particular choice of the lower level decision-maker.

There exist only few possibilities to deal with this problem of non-uniqueness. Namely,

1. Assume that the follower always selects the optimal decisions which give the worst values of $F(\tilde{c}, x)$. This is the *pessimistic* or *strong approach* (Lohse (2011)), which is used when the leader is not able to influence the follower and is forced to choose an approach bounding the damage resulting from an unfavourable selection by the follower. The resulting problem is:

$$\min_{\tilde{c} \in \tilde{C}} \phi_p(\tilde{c}),$$

$$\text{where } \phi_p(\tilde{c}) = \max_{x \in \Psi(\tilde{c})} F(\tilde{c}, x).$$

2. Assume that the leader is able to influence the follower, so that the last one always selects the variables x to provide the best value of $F(\tilde{c}, x)$. This results in the

so-called *optimistic* or *weak approach* (Dempe and Starostina (2007)), where the resulting problem is stated as

$$\min_{\tilde{c} \in \tilde{C}} \phi_o(\tilde{c}),$$

where $\phi_o(\tilde{c}) = \min_{x \in \Psi(\tilde{c})} F(\tilde{c}, x)$.

3. In the case when none of the assumptions described earlier can be conceded, selection function approach (Dempe and Starostina (2006)) can be applied to the initial fuzzy bilevel optimization problem.

The selection approach 3. is quite new and it seems to be more appropriate for our needs. The aim of this approach is to compute a selection function $x(\tilde{c}) \in \Psi(\tilde{c})$.

In the present work two different algorithms for the solution of fuzzy bilevel optimization problem (1.10) are presented in Chapter 7. Those algorithms are based on the selection function approach and a switch between upper- and lower-level problems.

One possibility to compute a selection function $x(\tilde{c}) \in \Psi(\tilde{c})$ is to use Yager ranking indices to avoid the incomparability of the fuzzy vectors. As shown by Liu and Kao (2004), the Yager ranking indices approach can be very useful in solving (single level) fuzzy optimization problems. Then, the fuzzy bilevel optimization problem can easily be reformulated into a crisp bilevel optimization problem (see Yager (1981)). According to Liu and Kao (2004), an optimal solution for fixed parameter is then taken as an optimal solution of the initial follower's fuzzy optimization problem. This approach can easily be extended to nonlinear convex fuzzy optimization problems.

Another approach is based on the calculation of membership functions values of the elements of the fuzzy solution on the lower level. The preferable optimal solution is supposed to have a maximal membership function value, i.e. according to Chanas and Kuchta (1994), the solution has the highest potential being realized by the follower. An algorithm for computation of a value of the membership function of the elements of the fuzzy solution is presented in Dempe and Ruziyeva (2012).

Using of stability regions and a switch between two problems it is possible to find an optimal vector of coefficients at the upper level of the fuzzy bilevel optimization problem such that the chosen solution on the lower level stays optimal in the fuzzy bilevel programming problem. By examining all possible regions of stability, we obtain the global optimal solution.

The fuzzy bilevel optimization problem with fuzzy objectives and fuzzy constraints leads to further increase in complexity. The aim is to investigate this problem for the linear case. The optimization problem is formulated as follows.

On the lower level the follower attempts to solve a fuzzy optimization problem

$$f(\tilde{c}, x) = p_1^\top \tilde{c} + p_2^\top x \rightarrow \min_x \quad (1.14)$$

subject to $(\tilde{c}, x) \in \mathbf{P} := \mathfrak{P} \times P$ for the fixed fuzzy vector \tilde{c} . Here \mathfrak{P} is a fuzzy polytope in the space of fuzzy vectors \mathfrak{F}^n and P is a polytope in the space of real vectors \mathbb{R}^m . Crisp coefficients p_1 and p_2 are supposed to be known. Here the set of optimal solutions of fuzzy optimization problem (1.14) is

$$\Psi(\tilde{c}) = \arg \min_x \{f(\tilde{c}, x) : (\tilde{c}, x) \in \mathbf{P}\}. \quad (1.15)$$

In turn, the leader minimizes his / her own linear objective function

$$F(\tilde{c}, x) = d_1^\top \tilde{c} + d_2^\top x \rightarrow \min_{\tilde{c}} \quad (1.16)$$

subject to $(\tilde{c}, x) \in \mathbf{P}$ with known coefficients d_1 and d_2 . Then, a fuzzy bilevel optimization problem is formulated as

$$\begin{aligned} F(\tilde{c}, x) &= d_1^\top \tilde{c} + d_2^\top x \rightarrow \min_{\tilde{c}} \\ \text{s.t. } &(\tilde{c}, x) \in \mathbf{P} \\ &x \in \Psi(\tilde{c}). \end{aligned} \quad (1.17)$$

For the *linear* case of the fuzzy bilevel optimization problems solution procedures have been proposed by Zhang et al. (2006) and Zhang and Lu (2010). The authors assumed that we are dealing with the fuzzy variables and fuzzy constants that have trapezoidal membership functions. The authors reformulate fuzzy bilevel optimization into a crisp optimization problem for the fixed level-cut using the proposition that the weights of all the left- and right-hand side functions are given. This conflicts with idea of bicriteria optimization. The same is done with the constraints on the both levels. After all branch and bound and k -th best algorithms are applied to the crisp bilevel optimization problem.

Dempe and Starostina (2007) considered the fuzzy bilevel optimization problem with linear constraints. The authors followed Zimmermann (1978) and reformulated the fuzzy bilevel optimization problem into a crisp bilevel optimization problem, where the optimization task of both decision-makers is replaced by maximizing the minimum value of all the membership functions in the respective problems, i.e. the max-min approach was used.

Zhang et al. (2008) reformulated the fuzzy linear bilevel optimization problem into a linear bilevel biobjective optimization problem and then applied the Karush-Kuhn-Tucker approach to this crisp bilevel optimization problem.

In the thesis the selection function approach is applied to solve linear fuzzy bilevel optimization problem (1.17). The adopted k -th best algorithm is based on taking α -cuts of the fuzzy polytope \mathfrak{P} and applying methods described in Chapter 6. It is proved that with the solution algorithm the global optimal solution is obtained. Moreover, membership function values are calculated. Thus, it is possible to provide the leader with quantitative information concerning the superiority of one optimal solution over others. These derivations can be found in Chapter 8.

2 Preliminaries

In this Chapter we give definitions of fundamental concepts such as fuzzy set and its level-cut, fuzzy number and its level-cut, the space of fuzzy numbers and fuzzy vectors. This is presented in Section 2.1. Further we define operations with fuzzy (and crisp) numbers in Section 2.2. We introduce a fuzzy order in Section 2.3, respectively. Definition of a fuzzy function is given in Section 2.4. These are the prerequisites required to discuss fuzzy (bilevel) optimization problems.

2.1 Fuzzy sets and fuzzy numbers

Let us begin with a general definition of a fuzzy set. Fuzzy sets are sets whose elements have degrees of membership. The concept was introduced by Lotfi A. Zadeh (1965) as an extension of the classical notion of a set. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set. This is described with the aid of a membership function valued in the real unit interval $[0, 1]$. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, which only take values 0 or 1. In fuzzy set theory classical bivalent sets are usually called *crisp sets*.

Definition 2.1. A fuzzy set \tilde{C} is defined as a pair $(C, \mu_{\tilde{C}})$, where C is a crisp set ($C \subset \mathbb{R}^n$) and $\mu_{\tilde{C}} : C \rightarrow [0, 1]$ is the membership function of the fuzzy set \tilde{C} . For each element $x \in C$, the value of $\mu_{\tilde{C}}(x)$ is called the grade of membership of x in \tilde{C} .

Corollary 2.1. The empty fuzzy set \tilde{D} is defined with its membership function $\mu_{\tilde{D}}(x) \equiv 0$ for all $x \in D$, i.e. is the same as a crisp empty set.

Definition 2.2. The α -level set \tilde{C}_α of the fuzzy set \tilde{C} is defined for a fixed $\alpha \in [0, 1]$ as the crisp set for which the degree of membership function exceeds or is equal to the level α :

$$\tilde{C}_\alpha = \{x \mid \mu_{\tilde{C}}(x) \geq \alpha\}.$$

Corollary 2.2. It is obvious that for $\alpha, \alpha' \in [0, 1]$ such that $\alpha \leq \alpha'$, an inclusion $\tilde{C}_\alpha \supseteq \tilde{C}_{\alpha'}$ holds true.

Definition 2.3. A fuzzy set \tilde{C} is convex if and only if (iff) for all $x, y \in C$ and for all $\lambda \in [0, 1]$ the following inequality holds true:

$$\mu_{\tilde{C}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\tilde{C}}(x), \mu_{\tilde{C}}(y)\}.$$

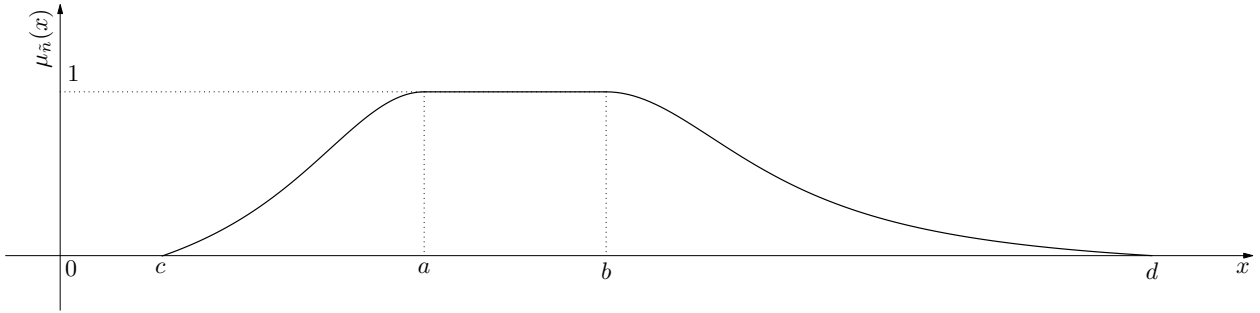


Fig. 2.1: A membership function of fuzzy number \tilde{n} .

A fuzzy number is an extension of a regular number in the sense that it does not represent one single value but rather a connected set of possible values, where each possible value has its own "weight" between 0 and 1. This "weight" is called the membership function. A fuzzy number is thus a special case of a convex fuzzy set. A fuzzy number \tilde{c} is an element of the nonempty fuzzy set \tilde{C} enriched with the nontrivial membership function $\mu_{\tilde{c}}$. In other words, a real fuzzy number is a convex continuous fuzzy subset of a real line. More precisely, a fuzzy number was defined by Dubois and Prade (1978) as follows:

Definition 2.4. A real fuzzy number \tilde{n} is a convex continuous fuzzy subset of the real line, whose membership function $\mu_{\tilde{n}}$ is

- a continuous mapping from \mathbb{R} to the closed interval $[0, 1]$;
- constant on $(-\infty, c]$: $\mu_{\tilde{n}}(x) = 0 \forall x \in (-\infty, c]$;
- strictly increasing on $[c, a]$;
- constant on $[a, b]$: $\mu_{\tilde{n}}(x) = 1 \forall x \in [a, b]$;
- strictly decreasing on $[b, d]$;
- constant on $[d, +\infty)$: $\mu_{\tilde{n}}(x) = 0 \forall x \in [d, +\infty)$.

Here a, b, c and d are real numbers.

Within this Definition we say that $\mu_{\tilde{n}}(x)$ is the so-called truth value of the assertion "the value of \tilde{n} is x ". The membership function $\mu_{\tilde{n}}(x)$ can be seen in Fig. 2.1.

Remark 2.1. In general we can have $c = -\infty$, or $a = b$, or $c = a$, or $b = d$, or $d = +\infty$.

Remark 2.2. If $a = c = b = d$, it is a crisp real number. If $a = c$ and $b = d$, it is a representation of the tolerance interval $[a, b]$ of the measurement of a quantity. If $a = b$, it is a representation of a fuzzy number, the value of which is "approximately a ".

Example 2.1. A linguistic variable can be described as a fuzzy number. For instance, we say "about 3". That means, that the number is not well-defined. Thus, it can be represented as a fuzzy number $\tilde{3}$ with a membership function equal to

$$\mu_{\tilde{3}}(x) = \frac{1}{(x - 3)^2 + 1}. \quad (2.1)$$

This membership function is illustrated in Fig. 2.2.

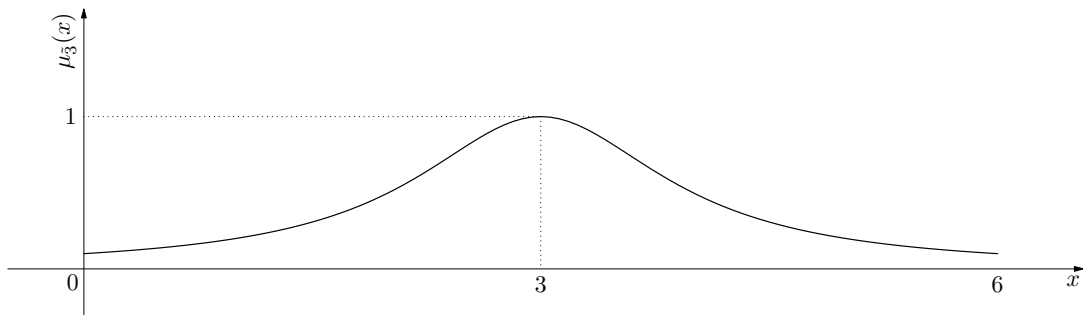


Fig. 2.2: A possible membership function of fuzzy number $\tilde{3}$.

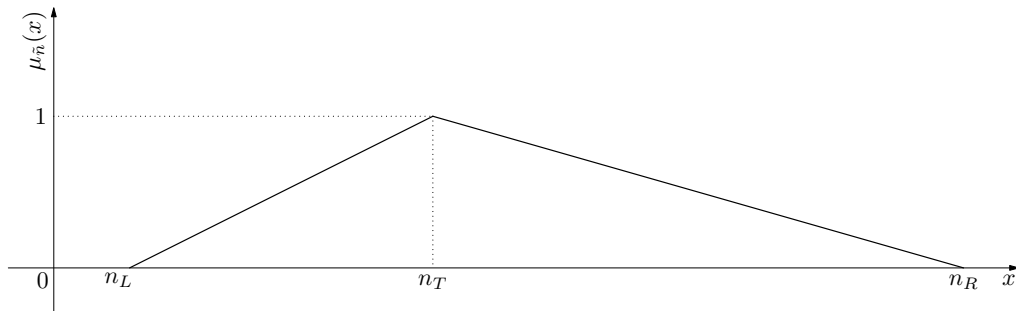


Fig. 2.3: A membership function of triangular fuzzy number \tilde{n} .

The simplest way to define a fuzzy number \tilde{n} is presented by e.g. Buckley (1995) as follows.

Definition 2.5. A continuous triangular fuzzy number \tilde{n} is represented with a triple (n_L, n_T, n_R) , where $n_L < n_T < n_R$ and the membership function $\mu_{\tilde{n}}$ is piecewise-linear.

A possible membership function of triangular fuzzy number \tilde{n} can be seen in Fig. 2.3.

Example 2.2. The fuzzy number $\tilde{2}$ can be represented as a triple $(0, 2, 6)$, i.e. its membership function is equal to

$$\mu_{\tilde{2}}(x) = \begin{cases} \frac{1}{2}x, & x \in [0, 2]; \\ -\frac{1}{4}x + \frac{3}{2}, & x \in [2, 6]. \end{cases} \quad (2.2)$$

In the majority of examples presented in the present work we use triangular fuzzy numbers. The main reason is the straight forward interpretation of the triple (n_L, n_T, n_R) : n_T is the best estimate, n_L is the minimum possible and n_R is the maximum possible values. There exist a generalization to Definition 2.5. It was shown by Dubois and Prade (1978), that a convenient representation for a fuzzy number \tilde{m} is another triple (m, β, γ) of parameters of its membership function $\mu_{\tilde{m}}$.

Definition 2.6. A fuzzy number \tilde{m} of LR type is defined with the following membership

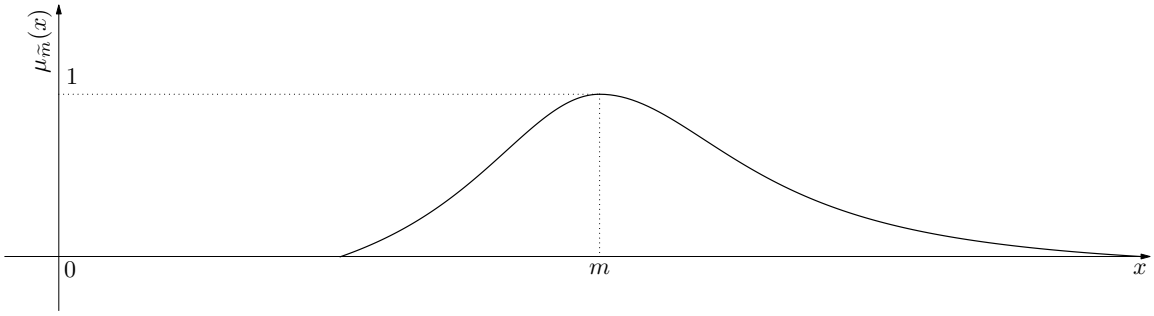


Fig. 2.4: A membership function of the fuzzy LR-number \tilde{m} .

function

$$\mu(x) = \begin{cases} L\left(\frac{m-x}{\beta}\right), & \text{if } x \leq m; \\ R\left(\frac{x-m}{\gamma}\right), & \text{if } x \geq m. \end{cases}$$

Here L and R are symmetric bell-shaped functions such that

$$L(0) = R(0) = 1.$$

m is called the mean value; β and γ are respectively named left and right spreads.

The example for a membership function of the fuzzy LR-number \tilde{m} can be seen in Fig. 2.4.

Definition 2.7. A normalized fuzzy number \tilde{c} is a fuzzy number with a membership function, that reaches value equal to 1:

$$\max \mu_{\tilde{c}}(x) = 1 \quad \forall x \in \mathbb{R}.$$

In the present work it is assumed that all fuzzy numbers are normalized.

Definition 2.8 (Sakawa and Yano (1989)). The α -cut of a fuzzy number \tilde{a} is defined as the ordinary set \tilde{a}_α for which its degree of membership function exceeds the level α ($\alpha \in [0, 1]$):

$$\tilde{a}_\alpha = \{a \mid \mu_{\tilde{a}}(a) \geq \alpha\}. \quad (2.3)$$

An alternative definition is the following.

Definition 2.9. A level-cut (also called α -cut, α -level) of a fuzzy number \tilde{c} is a special threshold described as an interval $[c_L(\alpha), c_R(\alpha)] \subset \mathbb{R}$ for some fixed $\alpha \in [0, 1]$ (see Fig. 2.5). Here $c_L(\alpha)$ and $c_R(\alpha)$ represent left- and right-hand side bounds of the fuzzy number \tilde{c} on this certain α -cut.

Assumption 2.1. Without loss of generality, from now on we assume that $\tilde{0} = 0$, i.e.

$$\mu_{\tilde{0}}(x) = \begin{cases} 1, & x = 0; \\ 0, & x \neq 0. \end{cases}$$

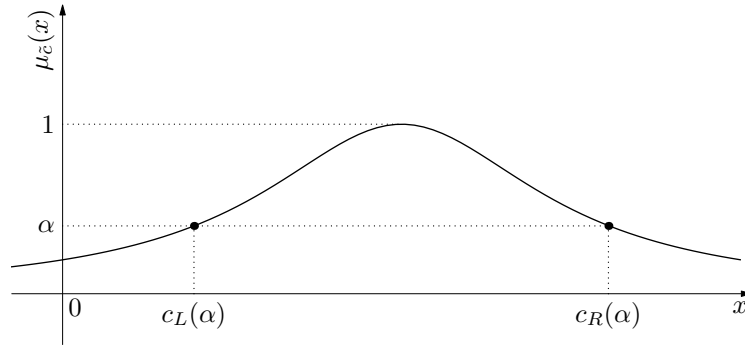


Fig. 2.5: α -cut of fuzzy number \tilde{c} .

Let us extend the concept of a fuzzy set to a space of fuzzy numbers and a fuzzy number to a fuzzy vector.

Definition 2.10. A space of fuzzy vectors \mathfrak{F}^n is defined as a pair $(\mathbb{R}^n, \mu_{\mathfrak{F}^n})$, where $\mu_{\mathfrak{F}^n} : \mathbb{R}^n \rightarrow [0, 1]$ is a membership function of the elements of the fuzzy space \mathfrak{F}^n .

Remark 2.3. In the thesis we denote a space of fuzzy numbers through \mathfrak{F} , where $\mu_{\mathfrak{F}} : \mathbb{R} \rightarrow [0, 1]$.

Definition 2.11. A fuzzy vector \tilde{c} is an element of the fuzzy space \mathfrak{F}^n enriched with a nontrivial membership function $\mu_{\tilde{c}}$.

2.2 Operations

Now we give some propositions concerning operations on fuzzy vectors. Operations on fuzzy numbers obviously are corollaries.

Proposition 2.1. Two fuzzy vectors are equal iff they have the same membership functions, i.e. for $\tilde{a}, \tilde{b} \in \mathfrak{F}^n$

$$\tilde{a} = \tilde{b} \text{ iff } \mu_{\tilde{a}}(x) = \mu_{\tilde{b}}(x) \quad \forall x \in \mathbb{R}^n.$$

Corollary 2.3. A fuzzy vector $\tilde{a} \in \mathfrak{F}^n$ is equal to a crisp vector $a \in \mathbb{R}^n$ iff \tilde{a} has the following membership function:

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & x = a; \\ 0, & x \neq a. \end{cases}$$

Definition 2.12. Let $\tilde{a}, \tilde{b} \in \mathfrak{F}^n$. Then the sum of two fuzzy vectors $\tilde{a} + \tilde{b}$ is defined as a fuzzy vector $\tilde{d} \in \mathfrak{F}^n$ with the following property

$$d_L(\alpha) = a_L(\alpha) + b_L(\alpha) \text{ and } d_R(\alpha) = a_R(\alpha) + b_R(\alpha) \text{ for all } \alpha \in [0, 1].$$

Here $a_L(\alpha)$ and $a_R(\alpha)$ are left- and right-side values of the fuzzy vector \tilde{a} on a certain α -cut. The same notation is used for the fuzzy vectors \tilde{b} and \tilde{d} .

Let A and B be two compact and convex subsets of \mathbb{R}^n . If there exists a compact and convex subset C of \mathbb{R}^n , such that $A = B + C = \{b + c : b \in B \text{ and } c \in C\}$, then C is called the *Hakuhara difference* of A and B . According to Banks and Jacobs (1970) we also write $C = A \ominus_H B$. Inspired by this concept, Wu (2008, 2009) defined the Hakuhara difference between two fuzzy vectors as following.

Definition 2.13. *Let \tilde{a} and \tilde{b} be two fuzzy vectors in \mathfrak{F}^n . If there exists a fuzzy vector $\tilde{c} \in \mathfrak{F}^n$, such that $\tilde{c} + \tilde{b} = \tilde{a}$ (note that fuzzy addition is commutative) and \tilde{c} is unique, then \tilde{c} is called the Hakuhara difference of \tilde{a} and \tilde{b} and is denoted by $\tilde{a} \ominus_H \tilde{b}$.*

The following proposition follows from Definition 2.12 immediately, and is very useful for the future considerations of the differentiation of fuzzy-valued functions.

Proposition 2.2. *Let $\tilde{a}, \tilde{b} \in \mathfrak{F}^n$. If the Hakuhara difference $\tilde{c} = \tilde{a} \ominus_H \tilde{b} \in \mathfrak{F}^n$ exists, then*

$$c_L(\alpha) = a_L(\alpha) - b_L(\alpha) \text{ and } c_R(\alpha) = a_R(\alpha) - b_R(\alpha) \text{ for all } \alpha \in [0, 1].$$

Here $[c_L(\alpha), c_R(\alpha)]$ is an α -cut of the fuzzy number \tilde{c} .

A sum and a difference of a fuzzy vector and a crisp vector are corollaries of Definition 2.12 and Proposition 2.2.

Corollary 2.4. *Let $\tilde{a} \in \mathfrak{F}^n$ and $b \in \mathbb{R}^n$. Then the sum $\tilde{a} + b (= b + \tilde{a})$ is defined as a fuzzy vector $\tilde{d} \in \mathfrak{F}^n$*

$$d_L(\alpha) = a_L(\alpha) + b \text{ and } d_R(\alpha) = a_R(\alpha) + b \text{ for all } \alpha \in [0, 1].$$

Corollary 2.5. *Let $\tilde{a} \in \mathfrak{F}^n$ and $b \in \mathbb{R}^n$. If the Hakuhara difference $\tilde{c} = \tilde{a} \ominus_H b \in \mathfrak{F}^n$ exists, then*

$$c_L(\alpha) = a_L(\alpha) - b \text{ and } c_R(\alpha) = a_R(\alpha) - b \text{ for all } \alpha \in [0, 1].$$

If there exist vector $\tilde{d} \in \mathfrak{F}^n$ with the following property

$$d_L(\alpha) = b - a_L(\alpha) \text{ and } d_R(\alpha) = b - a_R(\alpha) \text{ for all } \alpha \in [0, 1],$$

the we say that \tilde{d} is the Hakuhara difference $\tilde{d} = b \ominus_H \tilde{a}$.

2.3 Fuzzy order

With aforesaid notions we are ready to talk about an order on fuzzy sets. An order for fuzzy numbers can not be properly defined, if we talk about full order (that exist for real numbers). Thus, we suggest to use the following order relations between two fuzzy numbers \tilde{a} and \tilde{b} :

Proposition 2.3. *A fuzzy number \tilde{a} is smaller than a fuzzy number \tilde{b} iff for all $\alpha \in [0, 1]$ the α -levels of \tilde{a} are smaller than the α -levels of \tilde{b} , i.e.*

$$\tilde{a} \prec \tilde{b} \Leftrightarrow [a_L(\alpha), a_R(\alpha)] \prec [b_L(\alpha), b_R(\alpha)] \forall \alpha \in [0, 1].$$

For the comparison of two intervals $[a, b]$ and $[c, d]$ in \mathbb{R} the definition of Chanas and Kuchta (1996b) is adopted:

Definition 2.14. *An interval $[a, b]$ is smaller than an interval $[c, d]$, i.e. $[a, b] \prec [c, d]$, if $a \leq c$ and $b \leq d$ (with at least one strong inequality) for $a, b, c, d \in \mathbb{R}$.*

For two fuzzy numbers \tilde{a} and \tilde{b} and two crisp numbers a and b the following two corollaries hold true.

Corollary 2.6. $\tilde{a} \prec b \Leftrightarrow a_R(0) < b$.

Proof. When at the right side of the fuzzy relation is a crisp number b we obviously have that

$$\tilde{a} \prec b \Leftrightarrow a_L(\alpha) < b \text{ and } a_R(\alpha) < b \forall \alpha \in [0, 1].$$

Thus, as soon as $a_L(\alpha) < a_R(\alpha)$ for all $\alpha \in [0, 1]$, it is enough to write only one inequality at the right side

$$\tilde{a} \prec b \Leftrightarrow a_R(\alpha) < b \forall \alpha \in [0, 1].$$

According to Definition 2.4, a maximum of $a_R(\alpha)$ is obtained for $\alpha = 0$. Thus, the corollary is proved. \square

Corollary 2.7. $a \prec \tilde{b} \Leftrightarrow a < b_L(0)$.

Proof. The case when at the left side of the fuzzy relation is a crisp number a is proved analogously. \square

Analogously, for the non-strong relations we obtain the following.

Proposition 2.4. *A fuzzy number \tilde{a} is not greater than a fuzzy number \tilde{b} iff for all $\alpha \in [0, 1]$ the α -levels of \tilde{a} do not exceed the α -levels of \tilde{b} , i.e.*

$$\tilde{a} \preceq \tilde{b} \Leftrightarrow [a_L(\alpha), a_R(\alpha)] \preceq [b_L(\alpha), b_R(\alpha)] \forall \alpha \in [0, 1].$$

Definition 2.15. *The relation $[a, b] \preceq [c, d]$ holds true if $a \leq c$ and $b \leq d$ for $a, b, c, d \in \mathbb{R}$.*

For fuzzy numbers \tilde{a} and \tilde{b} and crisp numbers a and b the Corollaries 2.6 and 2.7 are transformed into

Corollary 2.8. $\tilde{a} \preceq b \Leftrightarrow a_R(0) \leq b$.

Corollary 2.9. $a \preceq \tilde{b} \Leftrightarrow a \leq b_L(0)$.

2.4 Fuzzy functions

To investigate fuzzy nonlinear problems we have to define the notions of a fuzzy function and its α -cut.

Definition 2.16. *A fuzzy function \tilde{f} is an image, such that for every real number / vector x_0 from its domain $D(\tilde{f})$ it sets to conformity a fuzzy number / vector $\tilde{f}(x_0) \in \mathfrak{F} / \mathfrak{F}^n$.*

Definition 2.17. An α -cut of a fuzzy function $\tilde{f}(x)$ is defined as an interval $\tilde{f}(x)[\alpha] := [f_L(x, \alpha), f_R(x, \alpha)]$.

Thus, the fuzzy function $\tilde{f}(x)$ is fully described using the functions $f_L(x, \alpha)$ and $f_R(x, \alpha)$, which are called the left- and right-hand side functions for the certain α -level of the fuzzy function $\tilde{f}(x)$. Following Panigrahi et al. (2008) it is assumed that $f_L(x, \alpha)$ is a bounded increasing and $f_R(x, \alpha)$ is a bounded decreasing function of α . Moreover, it is obvious that $f_L(x^0, \alpha) \leq f_R(x^0, \alpha)$ for all $\alpha \in [0, 1]$ and fixed $x^0 \in D(\tilde{f})$.

Definition 2.18 (Wu (2007)). The fuzzy function $\tilde{f}(x)$ is called convex iff for all $\alpha \in [0, 1]$ the functions $f_L(\cdot, \alpha)$ and $f_R(\cdot, \alpha)$ are convex.

Recall that

Definition 2.19. A crisp function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is called convex on \mathbb{R}^n if for all $x, y \in \mathbb{R}^n$ and all $\gamma \in [0, 1]$ we have

$$\varphi(\gamma x + (1 - \gamma)y) \leq \gamma\varphi(x) + (1 - \gamma)\varphi(y).$$

Continuity and differentiability of the fuzzy function $\tilde{f}(x)$ can also be defined through continuity and differentiability of the left- and right-hand side functions for fixed aspiration level α .

In the whole dissertation we assume that the fuzzy function is continuous and its membership function is properly defined.

3 Optimization problem with fuzzy objective

In the dissertation the optimization problem with fuzzy objective function is solved by its reformulation into the biobjective optimization problem and application of methods of the scalarization technique, where minimization of the α -cut on the feasible set is used Dempe and Ruziyeva (2011). Elements of the Pareto set of each biobjective optimization problem are interpreted as optimal solutions of the fuzzy optimization problem on certain level-cuts. The solution algorithm and the problem itself are well-described for the linear case e.g. in Chanas (1983); Chanas and Kuchta (1996b); Rommelfanger et al. (1989); Zimmermann (1978). For nonlinear fuzzy optimization problems the interested reader is referred to Panigrahi et al. (2008); Wu (2004, 2007, 2008). The problem itself is stated in Section 3.1.

In Section 3.2 using some convexity assumptions it is obtained that an optimal solution of the fuzzy optimization problem is an optimal solution of some nonlinear optimization problem obtained via scalarization of the objective functions of the corresponding bicriterial optimization problem.

Section 3.3 is devoted to a question of local optimality of the optimal solution of fuzzy optimization problem.

In Section 3.4 a question of existence of an optimal solution is considered.

3.1 Formulation

In many situations optimization problems with unknown or only approximately known data need to be solved, e.g. the fuzzy flow problem (compute optimal flows in a traffic network with fuzzy costs for passing streets, see e.g. Dempe et al. (2009)) or problems of optimal planning, see e.g. Orlovski (1985). It seems reasonable to approach these problems within the framework of fuzzy set theory because, according to Zadeh (1965), continuous fuzzy numbers are particularly suited for describing such kind of ambiguities.

In this Chapter we investigate the nonlinear fuzzy optimization problem

$$\begin{aligned} \tilde{f}(x) &\rightarrow \min \\ g(x) &\leq 0. \end{aligned} \tag{3.1}$$

Here

- $\tilde{f} : \mathbb{R}^n \rightarrow \mathfrak{F}$ is a fuzzy function,
- $g = (g_1, \dots, g_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a crisp vector-valued function and
- \mathfrak{F} is a set of fuzzy numbers over \mathbb{R} .

It is assumed that the functions $\tilde{f}(x)$ and $g(x)$ are always convex and continuous.

Let us denote through

$$X := \{x : g(x) \leq 0\} \quad (3.2)$$

the feasible set of fuzzy optimization problem (3.1). Let us assume henceforth that the set X is convex and closed.

Fuzzy optimization problem (3.1) is solved, according to Dempe and Ruziyeva (2011): The α -cuts are used to describe the objective function and it is assumed that its left- and right-hand sides values are given by functions $f_L(x, \alpha)$ and $f_R(x, \alpha)$ for $\alpha \in [0, 1]$. Using a suitable ordering of the intervals $\tilde{f}(x)[\alpha] = [f_L(x, \alpha), f_R(x, \alpha)]$ for the fixed α the task of the fuzzy function minimization over a feasible set X is transformed into a bicriterial optimization problem. Application of methods of the scalarization technique allows to solve such a problem with conflicting objectives (see e.g. Ehrgott (2005)). This solution approach is described in the present Chapter.

If more than one α is used at the same time as e.g. in Rommelfanger et al. (1989) this would lead to a multiobjective optimization problem. The generalization to this case is straightforward.

3.2 Solution method

In this Section fuzzy convex optimization problem

$$\tilde{f}(x) \rightarrow \min_{x \in X} \quad (3.3)$$

is considered. Here X is the feasible set defined by (3.2) and $\tilde{f} : \mathbb{R}^n \rightarrow \mathfrak{F}$ is a fuzzy function.

This problem (3.3) is replaced with the minimization of its α -cut (for some fixed $\alpha \in [0, 1]$) $\tilde{f}(x)[\alpha] = [f_L(x, \alpha), f_R(x, \alpha)]$ on the feasible set X as it was done for the linear case (see e.g. Chanas and Kuchta (1996b); Rommelfanger et al. (1989); Zimmermann (1991)).

The interval optimization problem

$$\tilde{f}(x)[\alpha] = [f_L(x, \alpha), f_R(x, \alpha)] \rightarrow \min_{x \in X} \quad (3.4)$$

is obtained. Assume that $f_L(x, \alpha)$ and $f_R(x, \alpha)$ are continuous convex functions with finite values. To find an optimal solution of problem (3.4) it is necessary to compare intervals in the objective for different values of x for fixed $\alpha \in (0, 1)$.

Applying Definition 2.14 to interval optimization problem (3.4) with the fixed α it is easy to see that $\tilde{f}(x_1)[\alpha] \prec \tilde{f}(x_2)[\alpha]$, i.e.

$$[f_L(x_1, \alpha), f_R(x_1, \alpha)] \prec [f_L(x_2, \alpha), f_R(x_2, \alpha)]$$

iff $f_L(x_1, \alpha) \leq f_L(x_2, \alpha)$ and $f_R(x_1, \alpha) \leq f_R(x_2, \alpha)$ (with at least one strong inequality) for $x_1, x_2 \in X$. Then, a value of the fuzzy function $\tilde{f}(x)$ at this α -level at the point x_1 is smaller than at x_2 .

Using this ordering of intervals, the task of finding an optimal solution of interval optimization problem (3.4) reduces to finding a solution of the following biobjective optimization problem with the fixed α -cut:

$$\begin{aligned} f_L(x, \alpha) &\rightarrow \min \\ f_R(x, \alpha) &\rightarrow \min \\ x &\in X. \end{aligned} \tag{3.5}$$

It is well-known that in multiobjective optimization problems objective functions often conflict with each other. In general, no single solution will simultaneously minimize all scalar objective functions. Therefore, solutions of problem (3.5) are defined by means of the Pareto optimality concept (Ehrgott (2005)).

Definition 3.1. *A feasible point $\hat{x} \in X$ is called Pareto optimal for biobjective optimization problem (3.5) for some $\alpha \in [0, 1]$ if there does not exist another feasible point $\tilde{x} \in X$ such that $f_L(\tilde{x}, \alpha) \leq f_L(\hat{x}, \alpha)$ and $f_R(\tilde{x}, \alpha) \leq f_R(\hat{x}, \alpha)$ with at least one strong inequality.*

Definition 3.2. *A feasible point $\hat{x} \in X$ is called weakly Pareto optimal if there is no $\tilde{x} \in X$ such that $f_L(\tilde{x}, \alpha) < f_L(\hat{x}, \alpha)$ and $f_R(\tilde{x}, \alpha) < f_R(\hat{x}, \alpha)$.*

Let us denote the set of weakly Pareto optimal solutions of problem (3.5) through $\Psi_w(\alpha)$.

To tie fuzzy optimization problem (3.3) and biobjective optimization problem (3.5) together, we use the following definition.

Definition 3.3 (Dempe and Ruziyeva (2011)). *A feasible solution \hat{x} is optimal for fuzzy optimization problem (3.3) if there exist some α -cut such that \hat{x} is a Pareto optimal solution for corresponding biobjective optimization problem (3.5).*

Let $\Psi(\alpha)$ denote the set of Pareto optimal solutions (for short, the Pareto set) of problem (3.5) for a fixed α -cut. Now it is possible to rewrite Definition 3.3 as

Definition 3.4. *A point $\hat{x} \in X$ is called an optimal solution of fuzzy optimization problem (3.3) provided that for some α -cut $\hat{x} \in \Psi(\alpha)$.*

Note that, in general, using this approach an optimal solution of the fuzzy optimization problem turns out ambiguity since the Pareto optimal solutions of biobjective optimization problem (3.5) form a certain set $\Psi(\alpha)$ in \mathbb{R}^n . This is related to the idea of Chanas and Kuchta (1994, 1996b). One general approach to compute the set of Pareto optimal solutions of biobjective optimization problem (3.5) is to replace this problem with the following scalarized optimization problem (Ehrgott (2005); Zadeh (1963))

$$f(x, \lambda)[\alpha] := \lambda f_L(x, \alpha) + (1 - \lambda) f_R(x, \alpha) \rightarrow \min_{x \in X} \tag{3.6}$$

and to compute optimal solutions for this problem for $0 \leq \lambda \leq 1$, where λ is the so-called *coefficient of scalarization*.

Optimal points of this problem for the fixed $\alpha \in [0, 1]$ form the sets of optimal solutions $\Psi_\alpha(\lambda)$. Observe that a point $x \in \Psi_\alpha(0)$ (or $x \in \Psi_\alpha(1)$) is a Pareto optimal solution

provided that this set reduces to a singleton (see e.g. Ehrgott (2005)). In general, it can only be shown that this set contains at least one Pareto optimal solution if it is bounded.

Ideas to compute the set of Pareto optimal solutions of the biobjective optimization problem can also be found e.g. in Audet et al. (2008); Fliege (2004); Li et al. (2003).

The following two theorems, connecting sets of optimal solutions of biobjective optimization problem (3.5) and its scalarized problem (3.6), are well-known (see Ehrgott (2005)).

Theorem 3.1. *Let X be a convex set and $f_L(\cdot, \alpha)$ and $f_R(\cdot, \alpha)$ are convex functions. Then the following statements hold.*

1. *For each $x \in \Psi(\alpha)$ there exists $0 \leq \lambda \leq 1$ such that $x \in \Psi_\alpha(\lambda)$.*
2. *For each $x \in \Psi_w(\alpha)$ there exists $0 \leq \lambda \leq 1$ such that $x \in \Psi_\alpha(\lambda)$.*

This Theorem says that for the fixed $\alpha \in [0, 1]$ every optimal solution of biobjective optimization problem (3.5) is obligatorily an optimal solution of scalarized optimization problem (3.6) for the same α and some $\lambda \in [0, 1]$. For the weakly Pareto optimal solution λ can be chosen from closed interval $[0, 1]$.

For the next Remark we recall

Definition 3.5 (Libor et al. (2003)). *A real function h defined on \mathbb{R}^n (or, more generally, on a convex subset of \mathbb{R}^n) is called a d.c.-function if it is a difference of two convex functions.*

In the literature such functions are sometimes labeled as δ -convex, Δ -convex (or delta-convex) functions.

Remark 3.1. *Some authors use other ordering (see e.g. Ishibuchi and Tanaka (1990); Sengupta and Pal (2000, 2006); Wu (2007)) distinct from ours (see Definition 2.14). The mid-point and half-width ordering is not necessarily convex (in this case mid-point function is convex, however half-width is a d.c.-function). Therefore, in general, Theorem 3.1 does not hold true. Then we recommend using Benson's approach or the parametric approach (Ehrgott (2005)) to derive an optimization problem for computing all Pareto optimal solutions of biobjective optimization problem (3.5).*

Converse of Theorem 3.1 is formulated for the fixed α -cut as

Theorem 3.2. *Let $x \in \Psi_\alpha(\lambda)$. The following statements hold.*

1. *If $0 < \lambda < 1$ then $x \in \Psi(\alpha)$.*
2. *If $0 \leq \lambda \leq 1$ then $x \in \Psi_w(\alpha)$.*

That means, that for $\lambda \in (0, 1)$ every solution of scalarized optimization problem (3.6) is necessarily an optimal solution of corresponding biobjective optimization problem (3.5). And for $\lambda \in [0, 1]$ an optimal solution of problem (3.6) is only weakly Pareto optimal of corresponding biobjective optimization problem.

Remark 3.2. *Note that Theorem 3.2 holds true without convexity assumption and the solution x is a global optimal solution.*

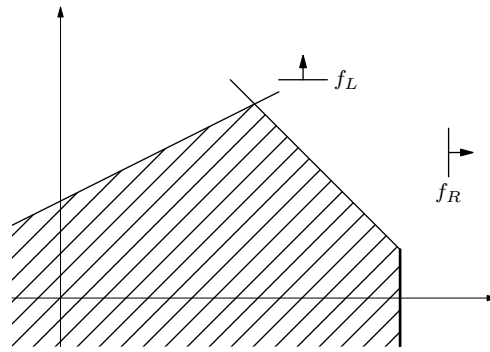


Fig. 3.1: Solutions are weak Pareto.

Remark 3.3. Note that problem (3.6) was used by several authors to compute solutions of the fuzzy optimization problem, see e.g. Chanas and Kuchta (1996b) for fuzzy linear optimization problems. This approach is also called determination of a compromise solution by means of a compromise objective function. However, other authors as e.g. Rommelfanger et al. (1989) consider also the following problem

$$\max\{f_L(x, \alpha), f_R(x, \alpha)\} \rightarrow \min_{x \in X}. \quad (3.7)$$

This approach is called determination of a compromise solution by progressive reduction. Using this approach just a weak Pareto optimal solution of problem (3.5) could be computed.

It is clear, that according to the min-max approach, one function has unreasonable preference over other. Fig. 3.1 demonstrates this: If maximum between left- and right-hand side functions is f_R , then left-hand side function f_L is irrelevant. Thus, in min-max approach the decision-maker loses his availability to choose, because the choice is already done. Moreover, in general case if one function is more valuable for the decision-maker, the min-max approach cannot reflect the importance of one function over the other.

Example 3.1. Let us consider following fuzzy optimization problem

$$\tilde{f}(x) = (x + \tilde{1})^2 \rightarrow \min_{x \in [-4, 0]}, \quad (3.8)$$

where $\tilde{1} = (0, 1, 2)$. Taking an α -cut for $\alpha = 0.5$, after all derivations, we obtain this biobjective optimization problem

$$\begin{aligned} f_L(x, \alpha) &= (x + 1/2)^2 \rightarrow \min \\ f_R(x, \alpha) &= (x + 3/2)^2 \rightarrow \min. \end{aligned} \quad (3.9)$$

With scalarization approach the set of Pareto optimal solutions

$$\Psi(\alpha) = [-3/2, -1/2]$$

is obtained. However the min-max approach provides us with a single solution $x_0 = -1$ and thus, the fuzzy nature in formulation of problem (3.8) cannot be reflected.

3.3 Local optimality

In this section we introduce definitions of local optimality for future consideration. A local optimum of an optimization problem is a solution that is optimal (either maximal or minimal) within a neighbouring set of solutions. Let S be a subset of \mathbb{R}^n .

Definition 3.6. \hat{x} is a local minimizer of a crisp function $f(x) : S \rightarrow \mathbb{R}$, if there exist $\delta > 0$ such that $B_\delta(\hat{x}) \subset S$,

$$f(\hat{x}) \leq f(x) \quad \forall x \in B_\delta(\hat{x}), \quad (3.10)$$

where $B_\delta(\hat{x}) = \{x \mid |x - \hat{x}| \leq \delta\}$.

If

$$f(\hat{x}) < f(x) \quad \forall x \in B_\delta(\hat{x}), \quad x \neq \hat{x}, \quad (3.11)$$

then \hat{x} is a strict local minimizer of function f .

This is in contrast to a global optimum, which is the optimal solution among all possible solutions.

Definition 3.7. If (3.10) holds for all $x \in S$, then \hat{x} is a global minimizer of function f on S .

If (3.11) holds for all $x \in S$, $x \neq \hat{x}$, then \hat{x} is a strict global minimizer of function f on S .

We adopt this concept for fuzzy optimization problem (3.3) in following definitions.

Definition 3.8. \hat{x} is a local minimizer of a fuzzy function $\tilde{f} : S \rightarrow \mathfrak{F}$, if there exist $\delta > 0$ such that and there does not exist $x \in B_\delta(\hat{x}) \cap S$:

$$\tilde{f}(x) \prec \tilde{f}(\hat{x}), \quad (3.12)$$

where $B_\delta(\hat{x}) = \{x \mid |x - \hat{x}| \leq \delta\}$.

Global optimal solution of the fuzzy optimization problem is defined earlier in Definition 3.3.

For biobjective optimization problem (3.5) we formulate following

Definition 3.9. A feasible point \hat{x} is called a local Pareto optimal solution for problem (3.5) for some fixed $\alpha \in [0, 1]$ if there exist $\delta > 0$ such that there does not exist $x \in B_\delta(\hat{x}) \cap S$:

$$f_L(x, \alpha) \leq f_L(\hat{x}, \alpha) \text{ and } f_R(x, \alpha) \leq f_R(\hat{x}, \alpha) \quad (3.13)$$

with at least one strong inequality.

With the use of Section 2.3, Definition 3.8 can be equivalently rewritten:

Corollary 3.1. \hat{x} is local minimizer of fuzzy optimization problem (3.3) iff there exist $\alpha \in [0, 1]$ such that \hat{x} is a local Pareto optimal solution for problem (3.5).

Let us formulate a relationship between sets of local optimal solutions of problems (3.5) and (3.6) for the some fixed α -cut.

Theorem 3.3. *Let S be a convex set and $f_L(\cdot, \alpha)$ and $f_R(\cdot, \alpha)$ are convex functions. Then for each local Pareto optimal solution \hat{x} of problem (3.5) there exists $0 \leq \lambda \leq 1$ such that \hat{x} is a local optimal solution of problem (3.6).*

The converse of Theorem 3.3 is formulated for the fixed α -cut as follows.

Theorem 3.4. *Let \hat{x} be a local optimal solution of problem (3.6) for some $0 < \lambda < 1$. Then \hat{x} is a local optimal solution of problem (3.5).*

Proofs of the last two Theorems are based on Separation Theorem (see Ehrgott (2005)).

3.4 Existence of an optimal solution

Let us consider fuzzy optimization problem

$$\begin{aligned} \tilde{f}(x) \rightarrow \min \\ x \in X \end{aligned} \tag{3.14}$$

Definition 3.10. *A fuzzy function $\tilde{f}(x)$ has finite value, if functions $|f_L(x, \alpha)| < \infty$ and $|f_R(x, \alpha)| < \infty$ simultaneously for all $\alpha \in [0, 1]$.*

Theorem 3.5. *An optimal solution of problem (3.14) exists if $X \neq \emptyset$, X is a compact and $\tilde{f}(x)$ is continuous fuzzy function with finite values.*

Proof. Existence of an optimal solution of problem (3.14) means, that for some $\alpha \in [0, 1]$ there exist Pareto optimal solution $\hat{x} \in X$ of problem (3.5) (see Definition 3.3). Let us take $\lambda \in (0, 1)$ and solve scalarized problem (3.6). Due to continuity of $f_L(\cdot, \alpha)$ and $f_R(\cdot, \alpha)$, this is the problem of minimizing continuous function over a compact set. Weierstrass Theorem guaranties the existence of the optimal solution \hat{x} of problem (3.6). Due to Theorem 3.2 \hat{x} is Pareto optimal solution of problem (3.5). According to Definition 3.3, \hat{x} is an optimal solution of fuzzy optimization problem (3.14). \square

4 Linear optimization with fuzzy objective

In Chapter 3 we have considered a solution of the fuzzy optimization problem as an element of the Pareto set of the corresponding biobjective optimization problem. However, when the fuzzy optimization problem is solved it is natural to consider its solutions to be fuzzy. The fuzzy solution consist of all solutions, given in Definition 3.3.

Hence, a criterion for comparison the elements of the fuzzy optimal solution is required. As soon as the fuzzy solution has a membership function, a possible opportunity to choose an appropriate for the decision-maker element of the fuzzy solution - the so-called best solution - is to compare values of the membership function of these elements. The criteria allows to see a correlation among all the solutions and quantitatively measure the advantage of one choice over others.

The main aim of this Chapter is to find the best realization of this idea based on modern solution algorithms Cadenas and Verdegay (2009); Chanas and Kuchta (1994, 1996b); Dempe and Ruziyeva (2012); Jiménez et al. (2006).

Let us consider the fuzzy linear optimization problem, where a fuzzy objective function is $\tilde{f}(x) = \tilde{c}x$ and a feasible set is X . It is assumed that left- and right-hand side values of the objective function coefficients are given by continuous functions $c_L(\alpha)$ and $c_R(\alpha)$ for all $\alpha \in [0, 1]$. (Remember that $c_L(\alpha)$ is increasing and $c_R(\alpha)$ is decreasing functions on α). Using a suitable ordering of the intervals $[c_L(\alpha)x, c_R(\alpha)x]$ for fixed level-cut, the task of the fuzzy function minimization over the set X is transformed into a bicriterial optimization problem, which is solved by means of the scalarization technique. This is described in Section 4.1.

As soon as a solution of the scalarized problem (with a fixed α -cut) depends on a parameter λ , a variation of $\lambda \in [0, 1]$ gives the optimal solution points. The set of those points then represents a subset of a Pareto set. This is explained with the illustrative example, given in Section 4.2.

Clearly, the approach of partial ordering the intervals may lead to situations of indecisiveness (Sengupta and Pal (2000)). This reflects the incomparability of the elements of the set of Pareto optimal solutions of the biobjective optimization problem. The computation of all such solutions is the basis of our approach to compute the membership function values of the solutions of the initial fuzzy optimization problem. For this, we have to compute the membership function values for all feasible points with the presented in this Chapter method.

As soon as our method is based on optimality conditions for the fuzzy linear optimization problem, we derive them in Section 4.3.

A procedure of computation the membership function value of each element of the fuzzy solution is based on these optimality conditions and is described in Section 4.4. For brevity, the discussions are limited to one element of the set of fuzzy optimal solutions,

but can easily be extended.

The procedure is numerically solved for the wide class of triangular fuzzy numbers (that can be extended to the class of LR-numbers) in Subsection 4.4.1.

This Chapter is concluded with a numerical example where the crucial elements of the fuzzy optimal solution are compared with respect to the membership function values and to objective function value, as well. Of course, the method provides the decision-maker with important quantitative information. These discussions are presented in Subsection 4.4.2.

4.1 Main approach

The linear case of fuzzy optimization problem (3.3) is represented by the following programming problem with fuzzy coefficients in the objective function:

$$\begin{aligned} \tilde{c}^\top x &\rightarrow \min \\ Ax &\leq b \\ x &\geq 0 \end{aligned} \tag{4.1}$$

with an n -dimensional vector of decision variables x .

Here

- $\tilde{c} \in \mathfrak{F}^n$ is a vector of fuzzy numbers;
- $A \in \mathbb{R}^{m \times n}$ is the constraint matrix;
- $b \in \mathbb{R}^m$ is the right-hand side vector.

For simplicity instead of problem (4.1) we investigate the fuzzy linear optimization problem

$$\begin{aligned} \tilde{c}^\top x &\rightarrow \min \\ Ax &= b \\ x &\geq 0. \end{aligned} \tag{4.2}$$

Adding slack variables, the investigation of this special problem is of no loss of generality.

This problem is replaced with the minimization of the α -cut on the feasible set (see e.g. Chanas and Kuchta (1994, 1996b); Rommelfanger et al. (1989); Zimmermann (1991)).

Thereby, the interval optimization problem is obtained:

$$\begin{aligned} [c_L^\top(\alpha)x, c_R^\top(\alpha)x] &\rightarrow \min \\ Ax &= b \\ x &\geq 0. \end{aligned} \tag{4.3}$$

Using Definition 2.14, the task of finding an optimal solution of interval optimization problem (4.3) reduces to the search of a solution of the following biobjective optimization problem with a fixed α -cut:

$$\begin{aligned} c_L^\top(\alpha)x &\rightarrow \min \\ c_R^\top(\alpha)x &\rightarrow \min \\ Ax &= b \\ x &\geq 0. \end{aligned} \tag{4.4}$$

Optimal solutions of problem (4.4) are defined by means of the Pareto optimality concept:

Definition 4.1. A feasible point $\hat{x} \in X := \{x : Ax = b, x \geq 0\}$ is called Pareto optimal for linear biobjective optimization problem (4.4) if there does not exist another feasible point $\check{x} \in X$ such that $c_L^\top(\alpha)\check{x} \leq c_L^\top(\alpha)\hat{x}$ and $c_R^\top(\alpha)\check{x} \leq c_R^\top(\alpha)\hat{x}$ with at least one strong inequality.

Let $\Psi(\alpha)$ denote the set of Pareto optimal solutions of biobjective optimization problem (4.4) for a certain α -cut.

Such an approach gives no unique optimal solution of the fuzzy optimization problem, since the Pareto optimal solutions of problem (4.4) form a certain set in \mathbb{R}^n . According to Ehrgott (2005) and Zadeh (1963), to compute all Pareto optimal solutions of linear biobjective optimization problem (4.4) it is sufficient to compute all optimal points of the linear scalarized problem

$$\begin{aligned} \lambda c_L^\top(\alpha)x + (1 - \lambda)c_R^\top(\alpha)x &\rightarrow \min \\ Ax &= b \\ x &\geq 0 \end{aligned} \quad (4.5)$$

with $0 < \lambda < 1$. For a fixed α -cut the optimal points of problem (4.5) form the sets of optimal solutions $\Psi_\alpha(\lambda)$.

This λ can be explained referring to the decision rule of Hurwicz. The optimism / pessimism parameter λ reflects the attitude of the decision-maker. Because of this restriction for λ , the weighted sum $\lambda c_L^\top(\alpha)x + (1 - \lambda)c_R^\top(\alpha)x$ is a convex combination of the objective functions $c_L^\top(\alpha)x$ and $c_R^\top(\alpha)x$. Therefore, the weighting factor λ can be interpreted as the relative importance between two objective functions of biobjective optimization problem (4.4).

In general, each single optimization problem (4.5) for some fixed α and λ determines an optimal solution set. The weighted sum method changes weights among the objective functions $c_L^\top(\alpha)x$ and $c_R^\top(\alpha)x$ systematically (e.g. by a predetermined step size in the hyper-ellipse approximation of Fadel and Li (2002); Li et al. (2003).) The weight on each single objective function may be adaptively determined (see, for instance, the adaptive weighted sum method of Kim and Weck (2005)). Benson's approach is shown to be effective in constructing approximation to the set of Pareto optimal solutions (Löhne (2011)). It is based on idea, that it is sufficient to know the upper and lower images (in the objective space), which are fully determined by finitely generated solutions. The algorithm can be understood as primal-dual method. However, a dual variant of Benson's algorithm also exist. With the use of geometric duality, the result would be the same, as obtained with Benson's algorithm.

Here we assume that Pareto optimal solutions are already determined.

As stated in Theorem 3.2: Any optimal solution of scalarized optimization problem (4.5) for $0 < \lambda < 1$ is Pareto optimal for biobjective optimization problem (4.4). Hence, solutions of problem (4.5) for different weight combinations produce a subset of the Pareto solutions. Vice-versa, for each Pareto optimal solution \hat{x} of problem (4.4), there exists $0 < \lambda < 1$ such that $\hat{x} \in \Psi_\alpha(\lambda)$, i.e.

$$\Psi(\alpha) = \bigcup_{\lambda \in (0,1)} \Psi_\alpha(\lambda). \quad (4.6)$$

Note that if a vertex x_1 is a Pareto optimal solution of biobjective optimization problem (4.4) for $\alpha \in [\underline{\alpha}_1, \bar{\alpha}_1]$ and an adjacent vertex x_2 is also optimal for problem (4.4) for

$\alpha \in [\underline{\alpha}_2, \bar{\alpha}_2]$, then each point x on the face in between these two optimal solutions is also Pareto optimal for this problem (4.4) for $\alpha \in [\underline{\alpha}_1, \bar{\alpha}_1] \cap [\underline{\alpha}_2, \bar{\alpha}_2]$.

It is essential to understand, that if for some α there exists λ such that x_1 is optimal for scalarized optimization problem (4.5) and there exist no λ for x_2 , then the face in between these two vertices would not be a subset of Pareto optimal points.

Thereby it can be assumed from now onwards that we have accurate results of the set of Pareto optimal solutions for our future discussion.

4.2 Example

To show illustratively the Rule of Hurwicz let us consider the traffic problem with fuzzy cost coefficients.

Consider a road system as a traffic network $G = (V, E)$ consisting of a node set V for junctions and an edge set E containing all streets connecting the junctions. These streets may have different capacities.

Assume that v units of a certain good should be transported with minimal overall costs from the origin $s \in V$ to the destination $d \in V$ through the traffic network G . The problem is to compute optimal amounts of transported goods on the streets of the network. The travel costs for traversing a street usually are not known exactly, that motivates us to assume that all travel costs have fuzzy values.

Let x_{kl} denote the amount of transported units over the edge $(k, l) \in E$, that connects two vertices $k, l \in V$. Let O_k (I_k) denote the set of all edges leaving (entering) the node k , i.e. such designations are connected to the words OUT and IN.

Assume that the flow x_{kl} on the edge (k, l) is bounded by the capacity u_{kl} . This is expressed in inequality (4.8) given below. Constraint (4.9) is used to guarantee that the total incoming flow is equal to the total outgoing flow. Moreover, the outgoing flow in the origin equals to v (see equation (4.10)).

$$\tilde{f}(x) = \sum_{k,l \in V} \tilde{c}_{kl} x_{kl} \rightarrow \min \quad (4.7)$$

$$x_{kl} \leq u_{kl} \quad \forall k, l \in V, \quad (4.8)$$

$$\sum_{k \in I_l} x_{kl} - \sum_{i \in O_l} x_{li} = 0, \quad \forall l \in V \setminus \{s, d\}, \quad (4.9)$$

$$\sum_{k \in I_s} x_{ks} - \sum_{i \in O_s} x_{si} = -v, \quad (4.10)$$

$$x_{kl} \geq 0 \quad (4.11)$$

This problem is a special case of fuzzy linear optimization problem (4.2), where the objective function in (4.7) reflects the total fuzzy cost and $Ax = b$ is an abbreviation of constraints (4.8) - (4.10). Inequality (4.11) insures non-negativity of the flow.

Description of a numerical example is following.

Let $\tilde{f}(x)$ be the total fuzzy flow that we have to minimize:

$$\tilde{f}(x) = \tilde{3}x_{12} + \tilde{8}x_{13} + \tilde{6}x_{14} + \tilde{7}x_{23} + \tilde{7}x_{25} + \tilde{3}x_{35} + \tilde{4}x_{45} \rightarrow \min$$

$$\begin{array}{l}
x_{s1} + x_{s2} = 90 \\
x_{4d} + x_{5d} = 90 \\
x_{s1} = x_{12} + x_{13} + x_{14} \\
\text{with given demands } x_{s2} + x_{12} = x_{23} + x_{25} \\
x_{13} + x_{23} = x_{35} \\
x_{14} = x_{45} + x_{4d} \\
x_{25} + x_{35} + x_{45} = x_{5d}
\end{array}
\quad
\begin{array}{l}
\text{and capacities} \\
0 \leq x_{s1} \leq 90 \\
0 \leq x_{s2} \leq 90 \\
0 \leq x_{12} \leq 90 \\
0 \leq x_{13} \leq 55 \\
0 \leq x_{14} \leq 35 \\
0 \leq x_{23} \leq 60 \\
0 \leq x_{25} \leq 30 \\
0 \leq x_{35} \leq 90 \\
0 \leq x_{45} \leq 35 \\
0 \leq x_{4d} \leq 35 \\
0 \leq x_{5d} \leq 90
\end{array}$$

The numerical example of the fuzzy optimization problem is illustrated in Fig. 4.1.

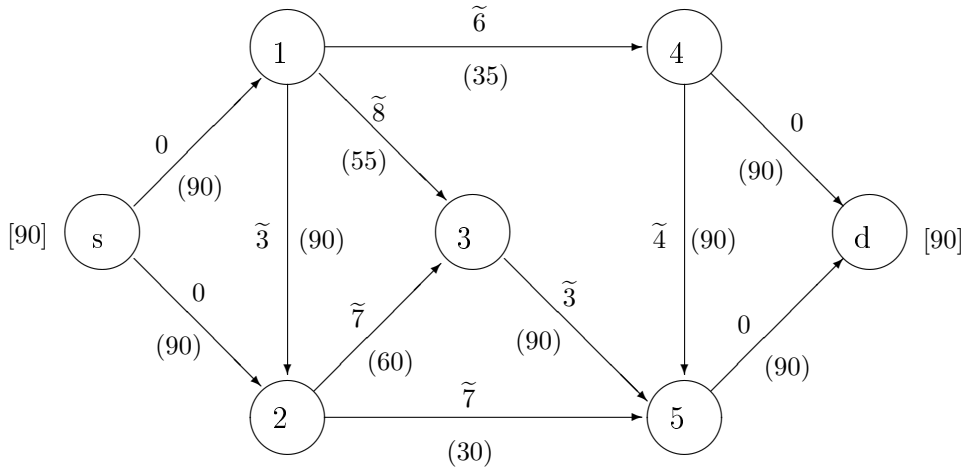


Fig. 4.1: The example of the traffic network.

Suppose that fuzzy numbers \tilde{c} are defined according to Definition 2.5 as continuous triangular fuzzy numbers (c_L, c_T, c_R) . Let take an α -cut for $\alpha = 0.5$ and write the left- and right-side bounds of the fuzzy numbers as intervals $[c_L, c_R]_{0.5}$:

$$\begin{array}{l}
\tilde{c} = (c_L, c_T, c_R) \quad [c_L, c_R]_{0.5} \\
\tilde{3} = (1, 3, 5) \quad [2, 4] \\
\tilde{4} = (2, 4, 6) \quad [3, 5] \\
\tilde{6} = (0, 6, 22) \quad [3, 14] \\
\tilde{7} = (5, 7, 9) \quad [6, 8] \\
\tilde{8} = (0, 8, 16) \quad [4, 12]
\end{array}$$

For $\alpha = 0.5$ the interval optimization problem could be analogously to problem (4.3) written as

$$\begin{aligned}
\tilde{f}(x)[0.5] = & [2, 4]x_{12} + [4, 12]x_{13} + [3, 14]x_{14} + [6, 8]x_{23} + \\
& + [6, 8]x_{25} + [2, 4]x_{35} + [3, 5]x_{45} \rightarrow \min
\end{aligned}$$

with the same capacities and demands.

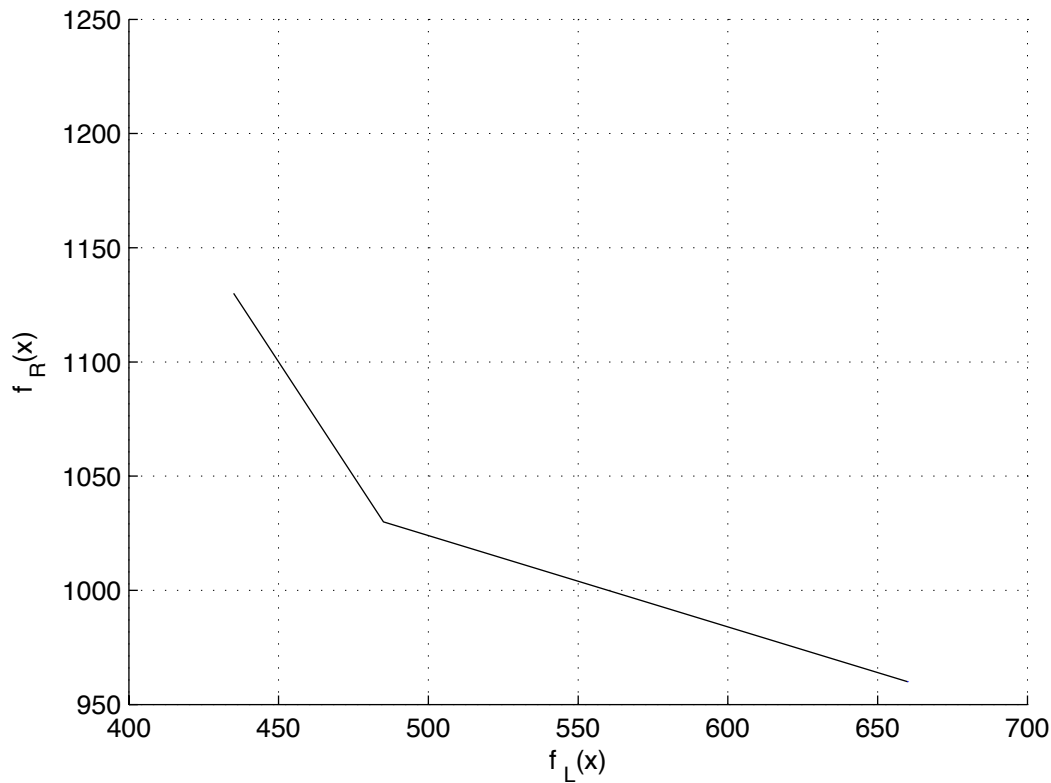


Fig. 4.2: The Pareto front.

Then it is possible to reformulate this problem to the biobjective optimization problem on the level-cut $\alpha = 0.5$:

$$f_L^{0.5}(x) = 2x_{12} + 4x_{13} + 3x_{14} + 6x_{23} + 6x_{25} + 2x_{35} + 3x_{45} \rightarrow \min$$

$$f_R^{0.5}(x) = 4x_{12} + 12x_{13} + 14x_{14} + 8x_{23} + 8x_{25} + 4x_{35} + 5x_{45} \rightarrow \min$$

with above defined capacities and demands.

Now it is possible to use some of mathematical programming software tools to approximate the Pareto set of optimal solutions of the scalarized optimization problem. Our choice is MATLAB, by means of that a plot of the Pareto front - the so-called image of the set of all Pareto optimal solutions in the objective space - is also built. For the computation the step size equal to 0.01 for steps with respect to the coefficient of scalarization λ is chosen.

For weighting factor $\lambda = 1$, when the estimation of decision-maker is relatively optimistic that all costs for traversing traffic network G are minimal, the optimal solution is $x^L = (0, 25, 35, 0, 30, 25, 0)$ and the total function value is $f_L = 435$.

On the contrary, the pessimistic estimation, which is represented by $\lambda = 0$, has the optimal solution $x^R = (0, 0, 0, 60, 30, 60, 0)$ and the total function value is $f_R = 960$.

For the illustrative point it is important to focus on Fig. 4.2.

It is obvious that any attempt for decreasing one value of the functions $f_L(x)$ and $f_R(x)$ has as a consequence increasing the value of the other. As a final decision the decision-maker has to select one of the Pareto optimal solutions (i.e. one of the optimal solutions

of the initial fuzzy optimization problem) according to an additional rule. This rule can be based e.g. on the graphical interpretation.

4.3 Optimality conditions

In this Section a fuzzy linear optimization problem

$$\begin{aligned} \tilde{c}^\top x \rightarrow \min \\ x \in X \end{aligned} \quad (4.12)$$

and its optimality conditions are discussed. The idea resembles the approach to optimality conditions for a crisp linear optimization problem

$$\begin{aligned} c^\top x \rightarrow \min \\ x \in X \end{aligned} \quad (4.13)$$

Here

- x is an n -dimensional vector of decision variables,
- \tilde{c} is a vector in the space of fuzzy numbers \mathfrak{F}^n
- c is a vector of crisp numbers in \mathbb{R}^n .

The feasible set in a decision space $X = \{x : Ax = b, x \geq 0\}$ is defined by the $m \times n$ constraint matrix A and the right-hand side vector $b \in \mathbb{R}^m$. Assume that $\text{rank}(A) = m$ and $b \geq 0$.

Definition 4.2 (Bertsimas and Tsitsiklis (1997)). *A nonsingular $m \times m$ submatrix A_B of A is called basic matrix, where B is a set of the columns of the matrix A defining A_B . The set B is called a basis. Let $N := \{1, \dots, n\} \setminus B$ be a set of nonbasic column indices. A variable x_i and an index i are called basic if $i \in B$, nonbasic otherwise.*

With the notion of a basis it is possible to split A , x , c and \tilde{c} into basis and nonbasis parts, using B and N as index sets. Let us write $A = (A_B, A_N)$, $x = (x_B, x_N)$, $c^\top = (c_B^\top, c_N^\top)$ and $\tilde{c}^\top = (\tilde{c}_B^\top, \tilde{c}_N^\top)$.

Using those notations, consider the feasible set in the decision space

$$\begin{aligned} X = \{x : Ax = b, x \geq 0\} &= \{x : A_B x_B + A_N x_N = b, x \geq 0\} = \\ &= \{x = (x_B, x_N)^\top : x_B = A_B^{-1}b - A_B^{-1}A_N x_N \geq 0, x_N \geq 0\} \end{aligned} \quad (4.14)$$

under invertibility assumption of the matrix A_B .

Setting $\bar{x}_N = 0$ (and, therefore, $\bar{x}_B = A_B^{-1}b$), a *basic solution* can be obtained as $\bar{x} = (A_B^{-1}b, 0)$. If in addition $\bar{x}_B \geq 0$, it is called a *basic feasible solution*. Then, the basis B is also called *feasible*.

Each single basic feasible solution, i.e. each vertex of the convex polytope X , determines a corresponding matrix A_B , with A_B being nonsingular.

Please note, that the above-stated reasoning are not dependent on the objective function. So, result (4.14) can be used also in the case when the objective function is fuzzy.

The well-known optimality condition for crisp linear optimization problem (4.13) reads as follows [Bertsimas and Tsitsiklis (1997)]:

$$c_B^\top A_B^{-1} A - c^\top \leq 0. \quad (4.15)$$

Adaptation of the optimality condition for fuzzy linear optimization problem (4.12) is the following

$$\tilde{c}_B^\top A_B^{-1} A - \tilde{c}^\top \leq 0. \quad (4.16)$$

Optimality condition (4.16) can be reformulated as optimality condition for interval optimization problem (4.3) with a fixed level-cut $\alpha \in [0, 1]$

$$\tilde{c}_B^\top[\alpha] A_B^{-1} A - \tilde{c}^\top[\alpha] \leq 0. \quad (4.17)$$

Then, taking into account all derivations from Section 4.1 of the reduction of interval optimization problem (4.3) to scalarized optimization problem (4.5), the optimality condition for problem (4.5) can be defined using the auxiliary function

$$h(\alpha, \lambda) = (\lambda c_L(\alpha) + (1 - \lambda)c_R(\alpha))^\top A_B^{-1} A - (\lambda c_L(\alpha) + (1 - \lambda)c_R(\alpha))^\top \quad (4.18)$$

as

$$h(\alpha, \lambda) \leq 0. \quad (4.19)$$

It is easy to see that the vector function $h(\alpha, \lambda) := (h_1(\alpha, \lambda), \dots, h_n(\alpha, \lambda))^\top$ is linear with respect to the weighting factor λ .

Remark 4.1. *In the crisp case the following fact is well-known. If we have a linear minimization problem with the upper bounds and the solution \hat{x} for some nonbasic index i obtains the upper bound, i.e. \hat{x} has a saturated variable $[\hat{x}]_i$, the optimality condition reads as*

$$[c_B^\top A_B^{-1} A - c^\top]_i \geq 0.$$

See Dempe and Schreier (2006) for details.

Let us adopt this optimality condition for the fuzzy case of linear optimization problem with the upper bounds. If we assume, that some of the solutions of such fuzzy optimization problem \hat{x} has a saturated variable $[\hat{x}]_i$ for some nonbasic index i , optimality condition for this index is

$$h_i(\alpha, \lambda) \geq 0, \quad (4.20)$$

where $h(\alpha, \lambda)$ is defined in formula (4.18).

4.4 Membership function value

Consider a fuzzy linear programming problem with fuzzy coefficients in the objective function

$$\begin{aligned} & \tilde{c}^\top x \rightarrow \min \\ & x \in X = \{x : Ax = b, x \geq 0\} \end{aligned} \quad (4.21)$$

where $\tilde{c} \in \mathfrak{F}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

As soon as a solution of some problem cannot have better degree of exactness, we assume that the solution \bar{x} of a fuzzy optimization problem is also fuzzy, i.e. $\bar{x} = (\chi, \mu_{\bar{x}})$. We

have already obtained for each level cut $\alpha \in [0, 1]$ the crisp set $\chi = \Psi(\alpha)$. The rest is to compute the membership function $\mu_{\bar{x}}$. The shape of this function can be very complicated. And actually, to compare the elements of the set χ , we do not need the whole function, but only its degree.

Knowledge of the membership function values of the elements of the fuzzy optimal solution enables the decision-maker to make an educated choice between these solutions. Moreover, using our approach, a decision-maker can see a correlation among solutions and quantitatively measure the advantage of his / her choice over other solutions.

An optimal solution of fuzzy optimization problem (4.21) is defined by Chanas and Kuchta (1994) as a fuzzy set in the set of feasible solutions with the membership function equal to the geometric measure of the set of all $\alpha \in [0, 1]$ such that this solution is optimal of interval optimization problem (4.3).

In a view of the fact that interval optimization problem (4.3) transforms into bicriterial parametric optimization problem (4.4) it is possible to determine for each basic solution, which is efficient for at least one value of α , the whole set of the α -cuts, for which it remains efficient. This set can be composed of a certain number of subintervals $[\alpha_{s-1}, \alpha_s] \subseteq [0, 1]$ for $s = 1, \dots, k$ (where k is fixed) for a fixed solution \bar{x} . Then, according to Chanas and Kuchta (1994), the membership function value can be obtained as $\mu(\bar{x}) = \sum_{s=1}^k [\alpha_s - \alpha_{s-1}]$.

Or more precisely:

Definition 4.3 (Chanas and Kuchta (1994)). *The solution \bar{x} of fuzzy linear programming problem (4.12) (with fuzzy coefficients in the objective function) is a fuzzy set in the set of feasible solutions with the following membership function*

$$\mu_{\bar{x}}(x) = |\{\alpha \mid x \in \Psi_{\alpha}, \alpha \in (0, 1]\}|, \quad (4.22)$$

where Ψ_{α} is the set of optimal solutions of interval optimization problem (4.3) and $|\cdot|$ stands for the geometric measure of the set.

The main approach, described in Section 4.1 provides us the following. Using described above order relation, we obtain that the set of optimal solutions of interval optimization problem (4.3) are equal to the set of Pareto optimal solutions of problem (4.4) for the same α -cut.

Now, using the linearization method, described e.g. in Ehrgott (2005), the determination of the set of all α such that $\hat{x} \in \Psi(\alpha)$ involves determining the set of all α such that there exists $\lambda = \lambda(\alpha) \in [0, 1] : \hat{x}$ is an optimal solution for scalarized optimization problem (4.5). As soon as among the vertices of the polytope of the feasible set X are the basic solutions, let us compute them using the Simplex-method or other appropriate method (see e.g. Avis and Fukuda (1992); Dyer and Proll (1977)). For the future discussions, let us assume, that \hat{x} is a nondegenerate solution.

For all $i = 1, \dots, n$ let us solve equation $h_i(\alpha, \lambda) = 0$ for a fixed α -cut. This means that for the basic solutions there are n equations instead of the same number of inequalities (4.19), i.e.

$$h_i(\alpha, \lambda) = [(\lambda c_L(\alpha) + (1 - \lambda)c_R(\alpha))_B^{\top} A_B^{-1} A - (\lambda c_L(\alpha) + (1 - \lambda)c_R(\alpha))_i^{\top}] = 0. \quad (4.23)$$

For defining signs of $h_i(\alpha, \lambda)$ for all $i = 1, \dots, n$, let us reorder the functions h_i such that h_1, \dots, h_t are decreasing and h_{t+1}, \dots, h_n are increasing functions with respect to λ . For this it is enough to calculate the signs of the derivatives

$$h'_i(\alpha, \cdot) = [(c_L(\alpha) - c_R(\alpha))_B^T A_B^{-1} A - (c_L(\alpha) - c_R(\alpha))_i^T]_i \quad (4.24)$$

for all $i = 1, \dots, n$.

For the fixed α let us denote by λ_i the root of the function $h_i(\alpha, \lambda)$. Knowing all roots $\lambda_1, \dots, \lambda_n$, it is easy to compute important for the future discussion

$$l.h.s.(\alpha) := \max\{\lambda_1, \dots, \lambda_t\} \text{ and } r.h.s.(\alpha) := \min\{\lambda_{t+1}, \dots, \lambda_n\}$$

and, thus, to define an interval $I(\alpha) := [l.h.s.(\alpha), r.h.s.(\alpha)]$ for which inequality (4.19) holds. This interval is moving on the λ -axis when α is changing. The interval $I(\alpha)$ is presented in Fig. 4.3 for a fixed α so that the feasible solution \hat{x} is Pareto optimal. In Fig. 4.3, k ($1 \leq k \leq t$) is an index of a decreasing function $h_k(\alpha, \lambda)$ such that $l.h.s.(\alpha) = \lambda_k$ and p ($t+1 \leq p \leq n$) is an index of an increasing function $h_p(\alpha, \lambda)$ such that $r.h.s.(\alpha) = \lambda_p$.

Note, that emptiness of the interval $I(\alpha)$ means that the chosen solution \hat{x} is nonoptimal for scalarized problem (4.5), i.e. there is no $\lambda \in [0, 1]$ such that $\bar{x} \in \Psi_\alpha(\lambda)$. Therefore, for this α the set of Pareto optimal solutions for biobjective optimization problem (4.4) $\Psi(\alpha)$ does not include \hat{x} . This means, that in this certain α -cut the set of optimal solutions of fuzzy optimization problem (4.21) does not contain \hat{x} .

With this notation it is possible to rewrite optimality condition (4.19) as

$$|I(\alpha)| > 0 \quad (4.25)$$

and to compute a membership function value of the solution \hat{x} :

$$\mu_{\bar{x}}(\hat{x}) = |\{\alpha : \bar{x} \in \Psi(\alpha)\}| = |\{\alpha : |I(\alpha)| > 0\}| = |\{\alpha : r.h.s.(\alpha) - l.h.s.(\alpha) > 0\}|. \quad (4.26)$$

Remark 4.2. *In the case, when for some solution x_0 $r.h.s.(\alpha) \leq l.h.s.(\alpha)$ for all $\alpha \in [0, 1]$, the value $\mu_{\bar{x}}(x_0)$ is equal to zero.*

This provides the decision-maker with all the basic solutions of initial problem (4.21) of which it can reasonably be said that to a positive extent they are optimal solutions of the problem. A value expressing this extension (between 0 and 1) is also supposed to be given each time. Thus, it is up to the decision-maker to eliminate those basic solutions for which, up to him / her, the measure of optimality is too small and to choose the final solution from among the others.

For the future discussions let us make following

Definition 4.4. *A solution of the fuzzy optimization problem \hat{x} is the best solution provided that $\mu(\hat{x}) \geq \mu(x_i)$ for all other fuzzy solutions x_i ($i \in B$).*

4.4.1 Special case of triangular fuzzy numbers

Consider now a subclass of LR-numbers - the continuous triangular fuzzy numbers that are represented according to Definition 2.5 as a triple (c_L, c_T, c_R) . In this case it is possible to write

$$c_L(\alpha) = (c_T - c_L)\alpha + c_L \text{ and } c_R(\alpha) = (c_T - c_R)\alpha + c_R. \quad (4.27)$$

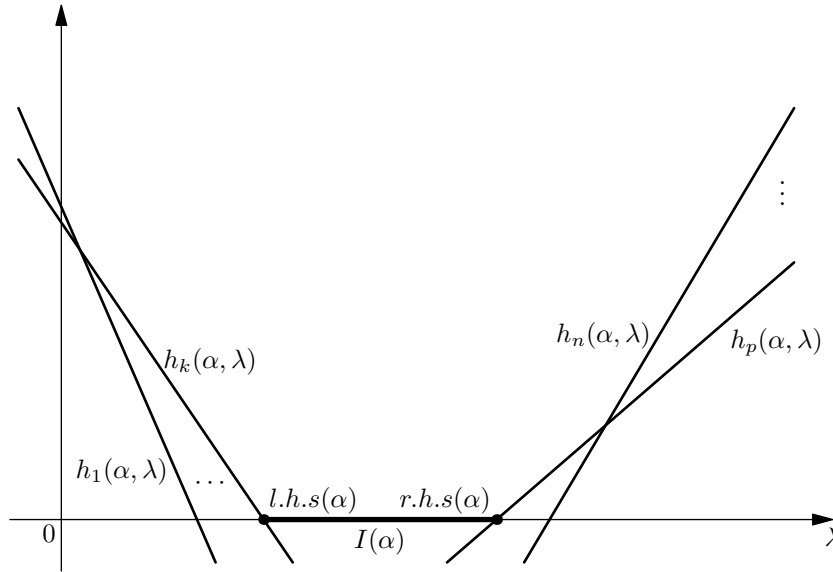


Fig. 4.3: Interval of satisfied optimality conditions (α is fixed).

Assume that the solution \bar{x} is optimal for fuzzy linear optimization problem (4.21). Let us compute its membership function value.

Consider optimality condition (4.19) for fuzzy numbers defined as (4.27). Note that in this case function $h(\alpha, \lambda)$ is linear with respect to α . According to equation (4.23), as soon as all components of the vector function $h(\alpha, \lambda) = (h_1(\alpha, \lambda), \dots, h_n(\alpha, \lambda))^T$ are equal to zero for basic indices $i \in B$, it makes sense to check optimality condition (4.19) only for nonbasic indices $i \in N$. Let us rewrite condition (4.23) in terms of α componentwise.

$$h_i(\alpha, \lambda) = [(\lambda[(c_T - c_L)\alpha + c_L] + (1 - \lambda)[(c_T - c_R)\alpha + c_R])_B^T A_B^{-1} A - (\lambda[(c_T - c_L)\alpha + c_L] + (1 - \lambda)[(c_T - c_R)\alpha + c_R])^T]_i = 0. \quad (4.28)$$

Let us denote $c(\lambda) := \lambda c_L + (1 - \lambda)c_R$. Now equation (4.28) can equivalently be reformulated as

$$h_i(\alpha, \lambda) = [\alpha(c_T - c(\lambda))_B^T A_B^{-1} A + c_B^T(\lambda) A_B^{-1} A - \alpha(c_T - c(\lambda))^T - c^T(\lambda)]_i = 0. \quad (4.29)$$

Denoting a numerator through

$$Num_i(\lambda) := [c^T(\lambda)]_i - [c_B^T(\lambda) A_B^{-1} A]_i \quad (4.30)$$

and a denominator through

$$Den_i(\lambda) := [(c_T^T - c^T(\lambda))_B A_B^{-1} A]_i - [c_T^T]_i + [c^T(\lambda)]_i, \quad (4.31)$$

optimality condition (4.19) results in

$$z_i^-(\lambda) := \frac{Num_i(\lambda)}{Den_i(\lambda)} \leq \alpha \text{ for } Den_i(\lambda) < 0 \text{ and } i \in N \quad (4.32)$$

and

$$z_i^+(\lambda) := \frac{Num_i(\lambda)}{Den_i(\lambda)} \geq \alpha \text{ for } Den_i(\lambda) > 0 \text{ and } i \in N. \quad (4.33)$$

Since $0 \leq \alpha \leq 1$, it is obvious that it only makes sense to compute $z_i^-(\lambda)$ and $z_i^+(\lambda)$ if $Den_i(\lambda)$ and $Num_i(\lambda)$ have the same signs, i.e.

$$Den_i(\lambda)Num_i(\lambda) > 0.$$

Let us start the computation of the interval $I(\alpha)$ with the lower bound. We solve a following optimization problem for each $i \in N$:

$$\begin{aligned} z_i^-(\lambda) &\rightarrow \min \\ \lambda &\in \{\lambda : Den_i(\lambda) < 0\} \end{aligned} \quad (4.34)$$

and denote an optimal function value through z_i^{*-} .

This means that the solution \bar{x} belongs to the set $\Psi_\alpha(\lambda)$ of the optimal solutions of scalarized optimization problem (4.5) if there exists $\lambda \in (0, 1) : Den_i(\lambda) < 0$ and some $0 \leq \alpha \leq 1$ such that the inequality $\alpha \geq z_i^{*-}$ holds true for all $i \in N$.

The inverse problem for computing the upper bound of the interval for α is given by analogy as

$$\begin{aligned} z_i^+(\lambda) &\rightarrow \max \\ \lambda &\in \{\lambda : Den_i(\lambda) > 0\}. \end{aligned} \quad (4.35)$$

Let the optimal function value of this problem be z_i^{*+} for each $i \in N$.

Using similar discussions, \bar{x} is an optimal solution of scalarized optimization problem (4.5) if there exists $\lambda \in (0, 1)$ such that $Den_i(\lambda) > 0$ and $\alpha \in [0, 1]$ such that the inequality $\alpha \leq z_i^{*+}$ holds true for all $i \in N$.

Recalling equation (4.6), \bar{x} has to be an optimal solution of biobjective optimization problem (4.4).

Using (4.26) and combining $z_i^{*-} \leq \alpha$ and $z_i^{*+} \geq \alpha$ it is easy to derive the following result, that guarantees that \bar{x} is an optimal solution of biobjective optimization problem (4.4) for some $0 \leq \alpha \leq 1$:

$$\bar{x} \in \Psi(\alpha) \Leftrightarrow \max_{i \in N} \{0, z_i^{*-}\} \leq \alpha \leq \min_{i \in N} \{z_i^{*+}, 1\}. \quad (4.36)$$

Thus, the membership function value of the element of the fuzzy solution \bar{x} can be obtained as a set of all α , such that (4.36) holds true, i.e.

$$\mu(\bar{x}) = \min_{i \in N} \{z_i^{*+}, 1\} - \max_{i \in N} \{0, z_i^{*-}\}. \quad (4.37)$$

Thus, we have obtained a very important formula and shown that the membership function value can *exactly* be computed with the use of optimality conditions. The formula (4.37) is also used for future discussions and algorithms.

Remark 4.3. *Applying the same discussions to the problem mentioned in Remark 4.1, for nonbasic index i \hat{x} has a saturated variable $[\hat{x}]_i$, optimality condition (4.20) results in*

$$z_i^-(\lambda) \leq \alpha \text{ for } Den_i(\lambda) > 0 \text{ and } i \in N \quad (4.38)$$

and

$$z_i^+(\lambda) \geq \alpha \text{ for } Den_i(\lambda) < 0 \text{ and } i \in N. \quad (4.39)$$

Note, that the rest of discussions (including formula (4.37)) stays unchanged.

There is a special case of the solution, when its membership function value achieves the maximal value equal to 1, as in the normalized case (see Definition 2.7).

Definition 4.5. A solution \bar{x} is called a strongest solution if $\mu(\bar{x}) = 1$.

Foregoing reasoning can be extended to a more complicated problem statement, i.e. on fuzzy numbers of LR -type.

It is clear that for the known type of the fuzzy numbers it is possible to compute a membership function value of the optimal solution and thus, define how much this optimal solution is better than others.

This we demonstrate with the following example.

4.4.2 Example

Let us consider the traffic problem with fuzzy cost coefficients.

In the following a numerical example is considered. Let $\tilde{f}(x)$ be the total fuzzy cost that we have to minimize:

$$\tilde{f}(x) = \tilde{3}x_{12} + \tilde{8}x_{13} + \tilde{7}x_{23} + \tilde{7}x_{24} + \tilde{3}x_{34} \rightarrow \min$$

$$\begin{array}{ll} x_{12} + x_{13} = 90 & 0 \leq x_{12} \leq 90 \\ x_{24} + x_{34} = 90 & 0 \leq x_{13} \leq 90 \\ \text{with demand} & \text{and capacity } 0 \leq x_{23} \leq 60 \\ x_{12} = x_{23} + x_{24} & 0 \leq x_{24} \leq 30 \\ x_{13} + x_{23} = x_{34} & 0 \leq x_{34} \leq 90 \end{array}$$

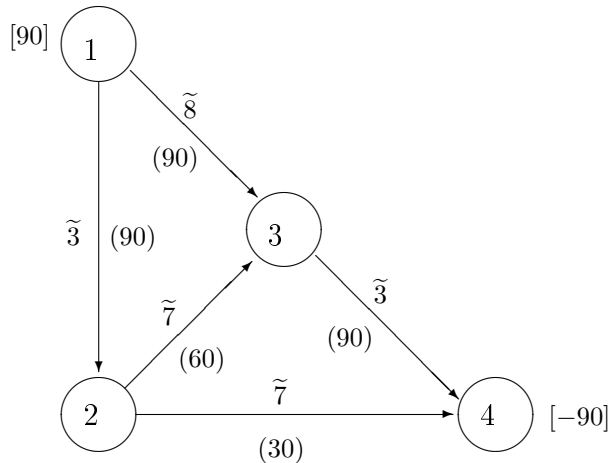


Fig. 4.4: The example of the traffic network.

Suppose that continuous triangular fuzzy numbers (c_L, c_T, c_R) are used:

$$\tilde{3} = (1, 3, 5), \tilde{7} = (5, 7, 9) \text{ and } \tilde{8} = (0, 8, 16).$$

A schematic illustration for the traffic network is given in Fig. 4.4.

Let us now compose the constraint matrix

$$A = \begin{array}{c} x_{12} \ x_{13} \ x_{23} \ x_{24} \ x_{34} \\ \left[\begin{array}{ccccc} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{array} \right] \end{array}.$$

Note that one superfluous row is dropped to make rank of matrix A be equal to amount of demand-equations minus one.

This optimization problem with fuzzy cost coefficients in the objective has three different basic solutions: $x_1 = (30, 60, 0, 30, 60)$, $x_2 = (0, 90, 0, 0, 90)$ and $x_3 = (90, 0, 60, 30, 60)$. Let us now compute the membership function value for each basic solution with the method described in Section 4.4.1.

First of all we compose two vectors

$$c_T^\top = (3, 8, 7, 7, 3)$$

and

$$c^\top(\lambda) = (5 - 4\lambda, 16 - 16\lambda, 9 - 4\lambda, 9 - 4\lambda, 5 - 4\lambda).$$

Consider now the solution x_1 . The path is given on Fig. 4.5 as a firm line. A dotted line here shows unused path. The amount of the transporting goods is indicated in brackets.

The basis $B = \{1, 2, 5\}$, then the basic matrix is

$$A_B = \begin{array}{c} x_{12} \ x_{13} \ x_{34} \\ \left[\begin{array}{ccc} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{array} \right] \end{array}$$

and $N = \{3, 4\}$. An inverse matrix can easily be computed:

$$A_B^{-1} = \begin{array}{c} x_{12} \ x_{13} \ x_{34} \\ \left[\begin{array}{ccc} 0 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{array} \right] \end{array}$$

Using derived formulas (4.30) and (4.31), let us perform the computations for nonbasic indices $i = 3, 4$:

$$Den_3(\lambda) = 8\lambda - 4 \text{ and } Num_3(\lambda) = 8\lambda - 2;$$

$$Den_4(\lambda) = 12\lambda - 6 \text{ and } Num_4(\lambda) = 12\lambda - 7.$$

Further, $Den_3 > 0$ for $\lambda > 1/2$ and $Num_3 > 0$ for $\lambda \geq 1/4$. According to (4.33), $z_3^+(\lambda) = \frac{8\lambda - 2}{8\lambda - 4}$ has to be considered for $\lambda \in (1/2, 1]$. So, the optimization problem, obtained due to (4.35),

$$z_3^+(\lambda) \rightarrow \max_{\lambda \in (1/2, 1]} \quad (4.40)$$

has to be solved. The optimal function value of this problem is $z_3^{*+} = \infty$.

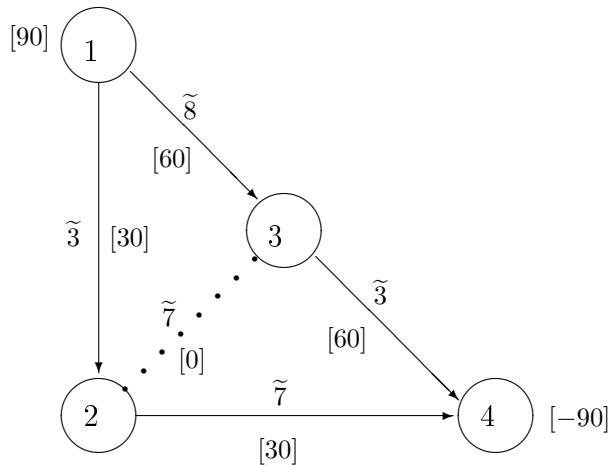


Fig. 4.5: The path corresponding to the solution x_1 .

The lower bound can be calculated with the help of function $z_3^-(\lambda) = \frac{8\lambda - 2}{8\lambda - 4}$. According to (4.34) we have to solve the following optimization problem

$$\begin{aligned} z_3^-(\lambda) &\rightarrow \min \\ \lambda &\in [0, 1/4]. \end{aligned} \quad (4.41)$$

The optimal function value of this problem is $z_3^{*-} = 0$.

For the next nonbasic index $i = 4$ it is easy to see that solution x_1 has a saturated variable. Thus, according to Remark 4.3 we make the following calculations.

Using formula (4.39) we compute the upper bound with the use of function $z_4^+(\lambda) = \frac{12\lambda - 7}{12\lambda - 6}$ for $Den_4 < 0$ and $Num_4 \leq 0$:

$$\begin{aligned} z_4^+(\lambda) &\rightarrow \max \\ \lambda &\in [0, 1/2). \end{aligned} \quad (4.42)$$

The optimal function value of this problem is $z_4^{*+} = \infty$.

For the lower bound, according to formula (4.38), we solve the following problem

$$\begin{aligned} z_4^-(\lambda) &\rightarrow \min \\ \lambda &\in [7/12, 1], \end{aligned} \quad (4.43)$$

where $z_4^-(\lambda) = \frac{12\lambda - 7}{12\lambda - 6}$. The optimal function value of this problem is $z_4^{*-} = 0$.

Let $z^{*+} := \min \{z_3^{*+}, z_4^{*+}, 1\} = \min \{\infty, \infty, 1\} = 1$ and $z^{*-} := \max \{z_3^{*-}, z_4^{*-}, 0\} = \max \{0, 0, 0\} = 0$.

According to (4.37) the membership function value of the optimal solution $x_1 = (30, 60, 0, 30, 60)$ is equal to

$$\mu(x_1) = z^{*+} - z^{*-} = 1 - 0 = 1.$$

For all level-cuts this solution is an optimal one. Moreover, its membership function value achieves the maximal value equal to 1, i.e. x_1 is a strongest solution (see Definition 4.5).

Let us analyse other solutions to see which of them make a competitiveness to this solution.

For the second solution $x_2 = (0, 90, 0, 0, 90)$ let us pay our attention on Fig. 4.6. The basis of x_2 is $B = \{2, 4, 5\}$, i.e.

$$A_B = \begin{matrix} & x_{13} & x_{23} & x_{24} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \end{matrix}$$

and $N = \{1, 3\}$. Thus, above-stated method gives the following result:

$$\begin{aligned} Den_1(\lambda) &= 12\lambda - 6 \text{ and } Num_1(\lambda) = 12\lambda - 7; \\ Den_3(\lambda) &= 2 - 4\lambda \text{ and } Num_3(\lambda) = 5 - 4\lambda. \end{aligned}$$

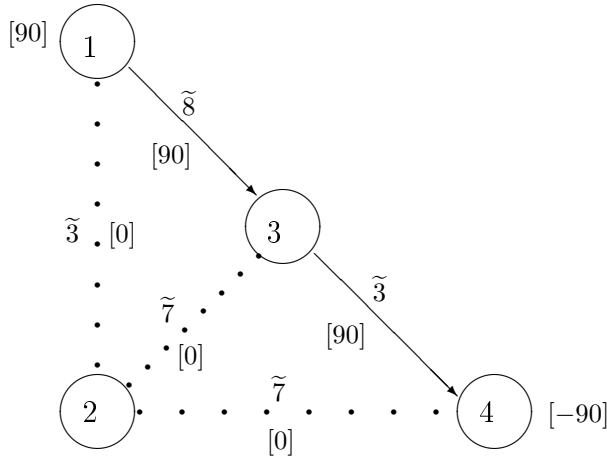


Fig. 4.6: The path corresponding to the solution x_2 .

The denominator $Den_1 > 0$ for $\lambda > 1/2$ and the nominator $Num_1 \geq 0$ when $\lambda \geq 7/12$. Thus, according to (4.33), $z_1^+(\lambda) = \frac{12\lambda - 7}{12\lambda - 6}$ and we consider it for $\lambda \in [7/12, 1]$. Due to (4.35) the following problem is obtained

$$z_1^+(\lambda) \rightarrow \max_{\lambda \in [7/12, 1]} \tag{4.44}$$

The optimal function value of this problem is $z_1^{*+} = 5/6$.

According to (4.34) the lower bound we start to compute with solving problem

$$z_1^-(\lambda) \rightarrow \min_{\lambda \in [0, 1/2)} \tag{4.45}$$

where $z_1^-(\lambda) = \frac{8\lambda - 2}{8\lambda - 4}$. The optimal function value here is $z_1^{*-} = -\infty$.

Most interesting reasoning of computations are connected with $i = 5$. Here it can easily be noted that $Num_5 \geq 0$ for all $\lambda \in [0, 1]$ and $Den_1 > 0$ is only for $\lambda < 1/2$. Thus, according to (4.33), $z_3^+(\lambda) = \frac{5 - 4\lambda}{2 - 4\lambda}$. Due to (4.35) the following problem is obtained

$$z_3^+(\lambda) \rightarrow \max_{\lambda \in [0, 1/2)}. \quad (4.46)$$

The optimal function value of this problem is $z_3^{*+} = \infty$ and $z_3^{*-} := 0$.

There we compute a membership function value of the second solution according to (4.37) as

$$\mu(x_2) = \min \{z_1^{*+}, z_5^{*+}, 1\} - \max \{z_1^{*-}, z_5^{*-}, 0\} = \min \{5/6, \infty, 1\} - \max \{0, -\infty, 0\} = 5/6.$$

The membership function value of this solution has value "almost one". Thus, this solution is a good alternative to solution x_1 examined earlier. Solution x_3 can also have membership function value equal to one. Then solution x_3 can be even better alternative to solution x_1 than x_2 . Let us calculate membership function value of x_3 properly.

For the third solution $x_3 = (90, 0, 60, 30, 60)$ the basic indices are $B = \{1, 3, 5\}$, i.e.

$$A_B = \begin{matrix} & x_{12} & x_{23} & x_{34} \\ \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{matrix}.$$

The pass, corresponding to x_3 can be seen on Fig. 4.7.

Using formulas (4.30) and (4.31), let us perform the computations of denominator and numerator for nonbasic indices $N = \{2, 4\}$:

$$Den_2(\lambda) = 8\lambda - 4 \text{ and } Num_2(\lambda) = 8\lambda - 2;$$

$$Den_4(\lambda) = 4\lambda - 2 \text{ and } Num_4(\lambda) = 4\lambda - 5.$$

Further, $Den_2 > 0$ for $\lambda > 1/2$ and $Num_2 \geq 0$ for $\lambda \geq 1/4$. According to (4.33), $z_2^+(\lambda) = \frac{8\lambda - 2}{8\lambda - 4}$ has to be considered for $\lambda \in (1/2, 1]$. So, the optimization problem, obtained due to (4.35)

$$z_2^+(\lambda) \rightarrow \max_{\lambda \in (1/2, 1]} \quad (4.47)$$

has to be solved. The optimal function value of this problem is $z_2^{*+} = \infty$.

The lower bound can be calculated with the help of function $z_2^-(\lambda) = \frac{8\lambda - 2}{8\lambda - 4}$. According to (4.34) we have to solve the following optimization problem

$$z_2^-(\lambda) \rightarrow \min_{\lambda \in [0, 1/4]}. \quad (4.48)$$

The optimal function value of this problem is $z_2^{*-} = 1/2$.

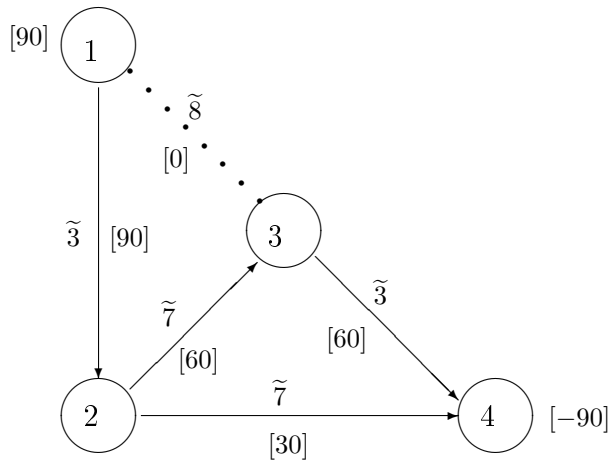


Fig. 4.7: The path corresponding to the solution x_3 .

For the next nonbasic index $i = 4$ x_3 has a saturated variable. Thus, according to Remark 4.5 the calculations are the following.

It is easy to see that $Num_4 < 0$ for all $\lambda \in [0, 1]$, but $Den_4 < 0$ is only for $\lambda < 1/2$. Thus, we compute only upper bound and $z_4^{*-} := 0$. According to (4.39) we have the following problem to solve:

$$z_4^+(\lambda) \rightarrow \max_{\lambda \in [0, 1/2)}, \quad (4.49)$$

where $z_4^+(\lambda) = \frac{4\lambda - 5}{4\lambda - 2}$. The optimal function value of this problem is $z_4^{*+} = \infty$.

According to (4.37) the membership function for the optimal solution x_3 is equal to

$$\mu(x_3) = \min \{z_2^{*+}, z_4^{*+}, 1\} - \max \{z_2^{*-}, z_4^{*-}, 0\} = \min \{\infty, \infty, 1\} - \max \{1/2, 0, 0\} = 1/2.$$

Clear, that for some α -cuts (to be exact, for $\alpha \in [1/2, 1]$) this solution is not optimal.

Thus, we can reason, that solution x_1 is more realizable (as soon as it has maximal membership function value equal to 1 and thus, is the best solution). The second preference has solution x_2 with membership function value equal to $5/6$. The least solution that the decision-maker can choose is x_3 . However, in the case when decision-maker looks for all solutions that have membership function value greater than 0.4, x_3 can be also alternative solution.

On the other hand, let us consider initial fuzzy optimization problem for $\alpha = 1$

$$f(1)(x) = 3x_{12} + 8x_{13} + 7x_{23} + 7x_{24} + 3x_{34} \rightarrow \min$$

with the same demand and capacity as in the initial fuzzy optimization problem. This function obtains minimal value at x_1 . Namely, $f(1)(x_1) = 960$. Objective function $f(1)(x)$ in x_2 is equal to 990. Finally, in x_3 the objective function is equal to 1080.

Thus, it makes solution x_1 more attractive to the decision-maker with respect to the both membership and objective functions values.

Evaluated solution x_3 by two parameters (membership function value and objective function value), is least preferred one.

A final decision meets the decision-maker on the basis of aforesaid. It is up to his / her choice to attach a particular importance to the value of either the membership or the objective function.

5 Optimality conditions

In this Chapter we investigate the following nonlinear fuzzy optimization problem

$$\begin{aligned} \tilde{f}(x) &\rightarrow \min \\ g(x) &\leq 0 \end{aligned} \quad (5.1)$$

and derive its optimality conditions.

Here $g = (g_1, \dots, g_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a crisp vector-valued function and $\tilde{f} : \mathbb{R}^n \rightarrow \mathfrak{F}$ is a fuzzy function.

As above, the α -cuts are used to describe the fuzzy objective function. The approach is based on results from Chapter 3: it is assumed that the left- and right-hand side values of fuzzy function $\tilde{f}(x)$ are given by functions $f_L(x, \alpha)$ and $f_R(x, \alpha)$ for $\alpha \in [0, 1]$, respectively. Then, using a suitable ordering of the intervals $\tilde{f}(x)[\alpha] := [f_L(x, \alpha), f_R(x, \alpha)]$ with the fixed α , the task minimization of the fuzzy function over a convex feasible set

$$X := \{x : g(x) \leq 0\}$$

can be transformed into a bicriterial optimization problem:

$$\begin{aligned} f_L(x, \alpha) &\rightarrow \min \\ f_R(x, \alpha) &\rightarrow \min \\ x &\in X. \end{aligned} \quad (5.2)$$

If more than one α is used at the same time as e.g. in Rommelfanger et al. (1989), this leads to a multiobjective optimization problem. This generalization does not mar the structure and the method of solution stay the same as we have proposed in Chapter 3 for the single α -cut, i.e. the solutions of the fuzzy optimization problem are still defined as Pareto optimal solutions of the corresponding multiobjective optimization problem.

Applying scalarization technique to biobjective optimization problem (5.2), the following problem is obtained:

$$\begin{aligned} f(x, \lambda)[\alpha] := \lambda f_L(x, \alpha) + (1 - \lambda) f_R(x, \alpha) &\rightarrow \min \\ x &\in X. \end{aligned} \quad (5.3)$$

This enables us to derive (necessary and sufficient) conditions to guarantee that a feasible point is an optimal solution. These conditions have e.g. the form of Karush-Kuhn-Tucker optimality conditions.

This result generalizes one obtained by Panigrahi et al. (2008) in four important directions:

Firstly, we define the derivative of a fuzzy function as a pair of functions which need not to be an interval as Panigrahi et al. supposed.

Secondly, we derive also *necessary* and not only sufficient conditions.

Thirdly, we find conditions for all elements of the fuzzy solution. Finally, nondifferentiable and nonconvex problems are discussed.

When in fuzzy optimization problem (5.1) both the objective function $\tilde{f}(x)$ and the constraint function $g(x)$ are differentiable, the whole problem we call *differentiable*. In the differentiable fuzzy case necessary and sufficient optimality conditions are presented in Section 5.1.

When in fuzzy optimization problem (5.1) one of the functions either in the objective $\tilde{f}(x)$ or in the constraint $g(x)$ is nondifferentiable, we call problem (5.1) *nondifferentiable*. We consider here that nondifferentiability appears in the objective. Necessary and sufficient optimality conditions for the feasible solution of the nondifferentiable fuzzy optimization problem are investigated in Section 5.2.

5.1 Differentiable fuzzy optimization problem

Necessary optimality conditions for fuzzy differentiable optimization problem (5.1) are given by using the Karush-Kuhn-Tucker conditions. This is explained in Subsection 5.1.2.

Sufficient optimality conditions which endow this Section with originality are investigated in Subsection 5.1.3.

5.1.1 Basic notions

Definition 5.1. Let $\tilde{f}(x) : \mathbb{R} \rightarrow \mathfrak{F}$ be a convex fuzzy function with finite values and assume that partial derivatives of the left- and right-hand side functions $f_L(\cdot, \alpha)$ and $f_R(\cdot, \alpha)$ for fixed $\alpha \in [0, 1]$ exist and are denoted by $f'_L(x, \alpha)$ and $f'_R(x, \alpha)$, respectively. Then the derivative of the fuzzy function $\tilde{f}(x)$ in $x_0 \in \mathbb{R}$ for the fixed $\alpha \in [0, 1]$ is a pair

$$\tilde{f}'(x_0)[\alpha] = (f'_L(x_0, \alpha), f'_R(x_0, \alpha)).$$

Please note that it is not assumed that $f'_L(x_0, \alpha) \leq f'_R(x_0, \alpha)$ (this unfounded assumption was done by Panigrahi et al. (2008); Wu (2007)).

Definition 5.2. The fuzzy convex function $\tilde{f}(x)$ on \mathbb{R} with finite values is differentiable in x_0 , if its derivative $\tilde{f}'(x_0)[\alpha]$ exist and is finite for all $\alpha \in [0, 1]$.

Definition 5.3. The fuzzy convex function $\tilde{f}(x)$ with finite values is differentiable on \mathbb{R} , if it is differentiable for all points $x \in \mathbb{R}$.

Remember that

Definition 5.4. A differentiable crisp function is a function whose derivative exists and is finite at each point in its domain.

Similarly, for the real-valued convex fuzzy function $\tilde{f}(\cdot)$ mapping \mathbb{R}^n to the space of fuzzy numbers \mathfrak{F} , its gradient is defined through the gradients of the left- and right-hand functions on the certain α -cut.

Definition 5.5. Let $\tilde{f}(x) : \mathbb{R}^n \rightarrow \mathfrak{F}$ be a convex fuzzy function with finite values and assume that all partial derivatives of the functions $f_L(x, \alpha)$ and $f_R(x, \alpha)$ in $x_0 \in \mathbb{R}^n$ exist for a given α . Then the gradient of $\tilde{f}(x)$ in x_0 on this α -cut is the matrix of the pairs of the gradients

$$\nabla \tilde{f}(x_0)[\alpha] = (\nabla f_L(x_0, \alpha), \nabla f_R(x_0, \alpha)) = \begin{pmatrix} \frac{\partial f_L(x_0, \alpha)}{\partial x_1}, & \frac{\partial f_R(x_0, \alpha)}{\partial x_1} \\ \dots & \dots \\ \frac{\partial f_L(x_0, \alpha)}{\partial x_n}, & \frac{\partial f_R(x_0, \alpha)}{\partial x_n} \end{pmatrix}.$$

Definition 5.6. The fuzzy convex function $\tilde{f}(x) : \mathbb{R}^n \rightarrow \mathfrak{F}$ with finite values is differentiable in x_0 , if its gradient $\nabla \tilde{f}(x_0)[\alpha]$ exist and is finite for all $\alpha \in [0, 1]$.

Definition 5.7. The fuzzy convex function $\tilde{f}(x)$ is differentiable on \mathbb{R}^n , if it is differentiable for all $x \in \mathbb{R}^n$.

5.1.2 Necessary optimality conditions

Theorem 5.1. Let \hat{x} be an optimal solution of fuzzy optimization problem (5.1) and assume that all the functions $\tilde{f}(x)$ and $g_i(x)$, $i = 1, \dots, k$ are convex and differentiable. Suppose also that Slater's constraint qualification is satisfied:

$$\exists x^0 \in X : g_i(x^0) < 0 \quad \forall i = 1, \dots, k.$$

Then there exist $\alpha \in [0, 1]$, $\lambda \in [0, 1]$ and $\mu \in \mathbb{R}^n$, $\mu \geq 0$ such that Karush-Kuhn-Tucker optimality conditions for problem (5.3)

$$\begin{aligned} \lambda \nabla f_L(\hat{x}, \alpha) + (1 - \lambda) \nabla f_R(\hat{x}, \alpha) + \mu^\top \nabla g(\hat{x}) &= 0 \\ \mu^\top g(\hat{x}) &= 0 \\ g(\hat{x}) &\leq 0 \end{aligned} \tag{5.4}$$

are satisfied.

Proof. If \hat{x} is an optimal solution of fuzzy optimization problem (5.1) then, according to Definition 3.3, there exists an α -cut, such that it is a Pareto optimal solution of biobjective optimization problem (5.2). Then, using Theorem 3.1, it is possible to find $0 \leq \lambda \leq 1$ such that \hat{x} is an optimal solution of scalarized optimization problem (5.3).

Now the result follows from the theory of necessary optimality conditions for differentiable convex optimization problems, see e.g. Ruszczyński (2006). \square

5.1.3 Sufficient optimality conditions

Theorem 5.2. Consider fuzzy optimization problem (5.1) and assume that the functions $\tilde{f}(x)$ and $g_i(x)$, $i = 1, \dots, k$ are convex and differentiable. Let \hat{x} be a feasible solution of problem (5.1) and assume that there exist $\alpha \in [0, 1]$, $\lambda \in (0, 1)$ and $\mu \geq 0$ such that Karush-Kuhn-Tucker optimality conditions (5.4) are satisfied. Then \hat{x} is an optimal solution of initial fuzzy optimization problem (5.1).

Proof. According to Ruszczyński (2006), if the assumptions of the theorem are satisfied, the point \hat{x} is an optimal solution of problem (5.3). Then Theorem 3.2 implies that the point \hat{x} is Pareto optimal for biobjective optimization problem (5.2) and, hence, by Definition 3.3, is optimal for fuzzy optimization problem (5.1). \square

5.2 Nondifferentiable fuzzy optimization problem

In the present Section nondifferentiable and nonconvex problems are discussed.

If the fuzzy function $\tilde{f}(x)$ is nondifferentiable, it requires some modifications in the standard approach. This we explain in this Section, which is structured as follows.

In Subsection 5.2.1 basic definitions of nondifferentiable fuzzy functions are given.

Subsection 5.2.4 contains an illustrative example.

5.2.1 Basic notions

For the future discussions we have to make the following basic definitions.

Definition 5.8. Let $X \subset \mathbb{R}^n$ and $x_0 \in X$. The tangent cone of X at x_0 is defined as

$$T_X(x_0) := \left\{ h \in \mathbb{R}^n \mid \exists \{\tau_k\} \downarrow 0, \{x^k\} \subset X \text{ s.t. } \{x^k\} \rightarrow x_0 \Rightarrow h = \lim_{k \rightarrow \infty} \frac{1}{\tau_k} (x^k - x_0) \right\}.$$

Let $\tilde{f}(x)$ be a convex nondifferentiable fuzzy function. Further, let $\alpha \in [0, 1]$, $x_0 \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ be fixed.

Definition 5.9. The (one-sided) directional α -derivative of the fuzzy function $\tilde{f}(x)$ in x_0 for some α -cut in a direction h is defined through the directional α -derivatives of the left- and right-hand functions $f_L(x_0, \alpha)$ and $f_R(x_0, \alpha)$ in x_0 in the direction h as

$$\begin{aligned} \tilde{f}'(x_0, h)[\alpha] &:= \lim_{\tau \downarrow 0} \frac{\tilde{f}(x_0 + \tau h)[\alpha] - \tilde{f}(x_0)[\alpha]}{\tau} = \\ &= \left(\lim_{\tau \downarrow 0} \frac{f_L(x_0 + \tau h, \alpha) - f_L(x_0, \alpha)}{\tau}, \lim_{\tau \downarrow 0} \frac{f_R(x_0 + \tau h, \alpha) - f_R(x_0, \alpha)}{\tau} \right) = \\ &=: (f'_L(x_0, h)[\alpha], f'_R(x_0, h)[\alpha]). \end{aligned}$$

Definition 5.10. If $\tilde{f}'(x_0, h)[\alpha]$ exists for all α -cuts then $\tilde{f}(x)$ is said to be differentiable in x_0 in the direction h .

Suppose that for each direction h in x_0 the fuzzy function $\tilde{f}(x)$ admits for some α in x_0 the directional α -derivative $\tilde{f}'(x_0, h)[\alpha]$.

Definition 5.11. The generalized gradient of this convex nondifferentiable fuzzy function $\tilde{f}(x)$ on the α -cut is defined through its subdifferential as a pair of subdifferentials of left- and right-hand side functions on this α -cut:

$$\partial \tilde{f}(x_0)[\alpha] = (\partial f_L(x_0, \alpha), \partial f_R(x_0, \alpha)),$$

where $\partial f(x_0, \alpha) = \{v : v^T h \leq f'(x_0, h)[\alpha] \forall h\}$.

The Hadamard upper (lower) directional derivative of the fuzzy function $\tilde{f}(x)$ can be also defined as a pair of Hadamard upper (lower) directional derivatives of the left- and right-hand functions.

Definition 5.12. Formally $\widetilde{f}_H^\uparrow(x_0, h)[\alpha]$ is called the Hadamard upper directional α -derivative of the fuzzy function $\widetilde{f}(x)$ on the α -cut in x_0 in the direction h and defined as

$$\begin{aligned} \widetilde{f}_H^\uparrow(x_0, h)[\alpha] &:= \limsup_{\tau \downarrow 0, h' \rightarrow h} \frac{\widetilde{f}(x_0 + \tau h')[\alpha] - \widetilde{f}(x_0)[\alpha]}{\tau} = \\ &= \left(\limsup_{\tau \downarrow 0, h' \rightarrow h} \frac{f_L(x_0 + \tau h', \alpha) - f_L(x_0, \alpha)}{\tau}, \limsup_{\tau \downarrow 0, h' \rightarrow h} \frac{f_R(x_0 + \tau h', \alpha) - f_R(x_0, \alpha)}{\tau} \right) =: \\ &=: \left(f_{HL}^{\uparrow\alpha}(x_0, h), f_{HR}^{\uparrow\alpha}(x_0, h) \right). \end{aligned}$$

Definition 5.13. The Hadamard lower directional α -derivative of the fuzzy function $\widetilde{f}(x)$ on the certain α -cut in x_0 in the direction h is analogously defined as

$$\begin{aligned} \widetilde{f}_H^\downarrow(x_0, h)[\alpha] &:= \liminf_{\tau \downarrow 0, h' \rightarrow h} \frac{\widetilde{f}(x_0 + \tau h')[\alpha] - \widetilde{f}(x_0)[\alpha]}{\tau} = \\ &= \left(\liminf_{\tau \downarrow 0, h' \rightarrow h} \frac{f_L(x_0 + \tau h', \alpha) - f_L(x_0, \alpha)}{\tau}, \liminf_{\tau \downarrow 0, h' \rightarrow h} \frac{f_R(x_0 + \tau h', \alpha) - f_R(x_0, \alpha)}{\tau} \right) =: \\ &=: \left(f_{HL}^{\downarrow\alpha}(x_0, h), f_{HR}^{\downarrow\alpha}(x_0, h) \right). \end{aligned}$$

Proposition 5.1. If the fuzzy function $\widetilde{f}(x)$ is continuous and convex in a point x_0 from interior of the feasible set X , i.e. $x_0 \in \text{int}(X)$, then

$$\widetilde{f}'(x_0, h)[\alpha] = \widetilde{f}_H^\uparrow(x_0, h)[\alpha] = \widetilde{f}_H^\downarrow(x_0, h)[\alpha] \quad \forall \alpha \in [0, 1]$$

in the direction h .

Proof. Continuity and convexity of the fuzzy function $\widetilde{f}(x)$ in $x_0 \in \text{int}(X)$ on some fixed α -cut is defined through continuity and convexity of the left- and right-hand side functions $f_L(x, \alpha)$ and $f_R(x, \alpha)$ in $x_0 \in \text{int}(X)$ (see Section 2.4). The rest follows from the classical convex analysis (see Rockafellar (1970)). \square

5.2.2 Necessary optimality conditions

Theorem 5.3. Let \widehat{x} solve nondifferentiable optimization problem (5.1). Assume that functions $\widetilde{f}(x)$ and $g_i(x)$, $i = 1, \dots, k$ are convex on \mathbb{R}^n .

Then there exist $\alpha \in [0, 1]$, $\eta, \mu \in \mathbb{R}^n$, $\eta, \mu \geq 0$ such that

$$\begin{aligned} 0 \in \partial_x \left(\eta^\top [\lambda f_L(\widehat{x}, \alpha) + (1 - \lambda) f_R(\widehat{x}, \alpha)] + \mu^\top g(\widehat{x}) \right) \\ \mu^\top g(\widehat{x}) = 0. \end{aligned} \quad (5.5)$$

Proof. The result follows from the theory of necessary optimality conditions for nondifferentiable convex optimization problem (see Lagrange Multiplier Rule in Clarke (1983)). \square

The necessary optimality conditions of Theorem 5.3 can be viewed as being degenerate when the multiplier η vanishes, since then the function $f(x, \lambda)[\alpha]$ is not involved. Various supplementary conditions have been proposed under which it is possible to assert that Lagrange Multiplier Rule holds in normal form with $\eta = 1$. We formulate the next theorem under one of possible constraint qualifications.

Theorem 5.4. *Let $\hat{x} \in X$ be an optimal solution of nondifferentiable fuzzy optimization problem (5.1) and assume that all the functions $\tilde{f}(x)$ and $g_i(x)$, $i = 1, \dots, k$ are convex on \mathbb{R}^n . Suppose also that Slater's constraint qualification is satisfied:*

$$\exists x^0 \in X : g_i(x^0) < 0 \quad \forall i = 1, \dots, k.$$

Then there exist $\alpha \in [0, 1]$, $\lambda \in [0, 1]$ and $\mu \in \mathbb{R}^n$, $\mu \geq 0$ such that

$$\begin{aligned} 0 &\in \partial_x L_\alpha(\hat{x}, \lambda, \mu) \\ \mu^\top g(\hat{x}) &= 0, \end{aligned} \tag{5.6}$$

where $L_\alpha(\hat{x}, \lambda, \mu) = \lambda f_L(\hat{x}, \alpha) + (1 - \lambda)f_R(\hat{x}, \alpha) + \mu^\top g(\hat{x})$.

Proof. If \hat{x} is an optimal solution of fuzzy optimization problem (5.1) then, according to Definition 3.3, there exist some α -cut such that \hat{x} is Pareto optimal for biobjective optimization problem (5.2). Then, from Theorem 3.1, it follows, that there exists $0 \leq \lambda \leq 1$ such that \hat{x} is an optimal solution of problem (5.3). The rest follows from Theorem 5.3. \square

Proposition 5.2. *For the further reasoning let us denote for some fixed α -cut different derivatives of the function $f(x, \lambda)[\alpha]$ in x_0 in the direction h*

- *the Hadamard upper directional derivative as $f_{H\lambda}^\uparrow(x_0, h)[\alpha]$;*
- *the Hadamard lower directional derivative as $f_{H\lambda}^\downarrow(x_0, h)[\alpha]$;*
- *the directional derivative as $f'_\lambda(x_0, h)[\alpha]$.*

Remember that

Definition 5.14. *The Hadamard upper directional derivative of the crisp function $f(x)$ in x_0 is defined as*

$$f_{H\lambda}^\uparrow(x_0, h) = \limsup_{\tau \downarrow 0, h' \rightarrow h} \frac{f(x_0 + \tau h')[\alpha] - f(x_0)[\alpha]}{\tau}.$$

The Hadamard lower directional derivative of the crisp function $f(x)$ in \hat{x} is defined as

$$f_{H\lambda}^\downarrow(x_0, h) = \liminf_{\tau \downarrow 0, h' \rightarrow h} \frac{f(x_0 + \tau h')[\alpha] - f(x_0)[\alpha]}{\tau}.$$

Theorem 5.5. *Let $\hat{x} \in X$ be a local optimal solution of nondifferentiable optimization problem (5.1). Then there exist $\alpha \in (0, 1)$ and $\lambda \in [0, 1]$ such that the Hadamard upper directional derivative of the function $f(x, \lambda)[\alpha]$ in \hat{x} is nonnegative for all directions h of the tangent cone $T_X(\hat{x})$:*

$$f_{H\lambda}^\uparrow(\hat{x}, h)[\alpha] \geq 0 \quad \forall h \in T_X(\hat{x}).$$

Proof. Assume the opposite, i.e. that $f_{H\lambda}^\uparrow(\hat{x}, h)[\alpha] < 0$. If $h \in T_X(\hat{x})$, then there exists a sequence $\{h_k\}$ such that for $k \rightarrow \infty$ $\{h_k\} \rightarrow h$ and $\{\tau_k\} \downarrow 0$ with $x^k := \hat{x} + \tau_k h_k \in X \forall k$. Thus, we have the following

$$\begin{aligned} 0 > f_{H\lambda}^\uparrow(\hat{x}, h)[\alpha] &= \\ &= \limsup_{\tau_k \downarrow 0, h_k \rightarrow h} \left[\lambda \frac{f_L(\hat{x} + \tau_k h_k, \alpha) - f_L(\hat{x}, \alpha)}{\tau_k} + (1 - \lambda) \frac{f_R(\hat{x} + \tau_k h_k, \alpha) - f_R(\hat{x}, \alpha)}{\tau_k} \right] \geq \\ &\geq \lim_{k \rightarrow \infty} \left[\lambda \frac{f_L(x^k, \alpha) - f_L(\hat{x}, \alpha)}{\tau_k} + (1 - \lambda) \frac{f_R(x^k, \alpha) - f_R(\hat{x}, \alpha)}{\tau_k} \right] = \\ &= \lim_{k \rightarrow \infty} \frac{f(x^k, \lambda)[\alpha] - f(\hat{x}, \lambda)[\alpha]}{\tau_k} \geq 0, \end{aligned}$$

since \hat{x} is a local minimum. This is a contradiction. \square

Corollary 5.1. *Assume that $\hat{x} \in X$ is an optimal solution of nondifferentiable fuzzy optimization problem (5.1) and functions $\tilde{f}(x), g_i(x), i = 1, \dots, k$ are convex. Then there exist $\alpha \in (0, 1)$ and $\lambda \in [0, 1]$ such that the directional derivative of the function $f(x, \lambda)[\alpha]$ in \hat{x} is nonnegative for all directions h from tangent cone $T_X(\hat{x})$:*

$$f'_\lambda(\hat{x}, h)[\alpha] \geq 0 \quad \forall h \in T_X(\hat{x}).$$

Proof. Convexity assumptions, Definition 3.3 and Theorem 3.1 provide an existence of $\alpha \in (0, 1)$ and $\lambda \in (0, 1)$ such that \hat{x} is an optimal solution of problem (5.3). According to Theorem 5.5, with $f_{H\lambda}^\uparrow(\hat{x}, h)[\alpha]$ replaced by $f'_\lambda(\hat{x}, h)[\alpha]$ (see e.g. Rockafellar (1970)), the Corollary is proved. \square

5.2.3 Sufficient optimality conditions

Theorem 5.6. *Consider nondifferentiable fuzzy optimization problem (5.1) and assume that the functions $\tilde{f}(x)$ and $g_i(x), i = 1, \dots, k$ are convex. Let $\hat{x} \in X$ be feasible and assume that there exist $\alpha \in [0, 1], \lambda \in (0, 1)$ and $\mu \in \mathbb{R}^n, \mu \geq 0$ such that conditions (5.6) are satisfied for the fixed $\alpha \in [0, 1]$. Then \hat{x} is an optimal solution of problem (5.1).*

Proof. If the assumptions of the theorem are satisfied, according to Definition 3.3 the point \hat{x} is an optimal solution of problem (5.3) for some fixed α . From Theorem 3.2 it follows that the point \hat{x} is Pareto optimal for biobjective optimization problem (5.2) for this α . Definition 3.3 implies that \hat{x} is optimal for nondifferentiable fuzzy optimization problem (5.1) on the fixed α -cut. \square

Theorem 5.7. *Assume that the Hadamard lower directional α -derivative of the fuzzy function $\tilde{f}(x)$ in \hat{x} for some fixed $\alpha \in (0, 1)$ exists and is positive for all directions h in tangent cone $T_X(\hat{x})$ ($h \neq 0$), i.e. there exists $\lambda \in (0, 1)$ such that*

$$f_{H\lambda}^\downarrow(\hat{x}, h)[\alpha] > 0 \quad \forall h \in T_X(\hat{x}).$$

Then $\hat{x} \in X$ is a local minimum of nondifferentiable fuzzy optimization problem (5.1).

Proof. Let us show that \hat{x} is a local minimum of problem (5.3) for some fixed α . Assume the opposite, i.e. that the point \hat{x} is not a strict local minimizer. Under this assumption there exists a sequence $\{x^k\} : x^k \in X$ for all k such that for $k \rightarrow \infty$ $\{x^k\} \rightarrow \hat{x}$ and $f(x^k, \lambda)[\alpha] \leq f(\hat{x}, \lambda)[\alpha]$.

Let

$$\tau_k := \|x^k - \hat{x}\| \quad \text{and} \quad h_k := \frac{x^k - \hat{x}}{\|x^k - \hat{x}\|}.$$

Then $h_k = \frac{1}{\tau_k}(x^k - \hat{x})$ and thus, $x^k = \hat{x} + \tau_k h_k$. Since $\|h_k\| = 1 \forall k$ there exists an accumulation point h of the sequence $\{h_k\}$. Then $h \in T_X(\hat{x})$ and $h \neq 0$.

This yields a following contradiction

$$\begin{aligned} 0 &< f_{H\lambda}^\downarrow(\hat{x}, h)[\alpha] = \\ &= \liminf_{\tau_k \downarrow 0, h_k \rightarrow h} \left[\lambda \frac{f_L(\hat{x} + \tau_k h_k, \alpha) - f_L(\hat{x}, \alpha)}{\tau_k} + (1 - \lambda) \frac{f_R(\hat{x} + \tau_k h_k, \alpha) - f_R(\hat{x}, \alpha)}{\tau_k} \right] \leq \\ &\leq \lim_{k \rightarrow \infty} \left[\lambda \frac{f_L(x^k, \alpha) - f_L(\hat{x}, \alpha)}{\tau_k} + (1 - \lambda) \frac{f_R(x^k, \alpha) - f_R(\hat{x}, \alpha)}{\tau_k} \right] = \\ &= \lim_{k \rightarrow \infty} \frac{f(x^k, \lambda)[\alpha] - f(\hat{x}, \lambda)[\alpha]}{\tau_k} \leq 0. \end{aligned}$$

Hence, \hat{x} is a local minimum of problem (5.3), that means due to Theorem 3.4 that \hat{x} is a local optimal solution of biobjective optimization problem (5.2). In turn with Corollary 3.1 that means that there exist an α -cut such that \hat{x} is a local optimal solution for initial fuzzy optimization problem (5.1). \square

5.2.4 Example

To complete our discussion it is interesting to explain the results by giving an example with all required calculations. Assume that we have the following nondifferentiable convex fuzzy optimization problem:

$$\tilde{f}(x) = \max\{\tilde{f}_1(x), \tilde{f}_2(x)\} \rightarrow \min_{x \in \mathbb{R}}. \quad (5.7)$$

Here

$$\tilde{f}_1(x) := -4x \ominus_H \tilde{8} \quad \text{and} \quad \tilde{f}_2(x) := \tilde{2}x + 1.$$

Definitions of the fuzzy sum and the Hukuhara difference can be found as Definition 2.12 and Proposition 2.2 in Chapter 1.

The continuous triangular fuzzy numbers are defined as triples:

$$\tilde{2} = (0, 2, 4), \tilde{4} = (1, 4, 8) \quad \text{and} \quad \tilde{8} = (2, 8, 12).$$

Let $\alpha = 0.5$, for this level-cut the left- and right-hand side bounds of the fuzzy numbers are

$$\tilde{2}_{0.5} = [1, 3], \tilde{4}_{0.5} = [2.5, 6] \quad \text{and} \quad \tilde{8}_{0.5} = [5, 10].$$

Now we obtain that the left- and right-hand side functions of fuzzy functions $\tilde{f}_1(x)$ and $\tilde{f}_2(x)$ respectively are

$$\begin{aligned} f_L^1(x, 0.5) &= -2.5x - 5, \\ f_R^1(x, 0.5) &= -6x - 10 \end{aligned}$$

and

$$\begin{aligned} f_L^2(x, 0.5) &= x + 1, \\ f_R^2(x, 0.5) &= 3x + 1. \end{aligned}$$

Then the left- and right-hand side functions of the fuzzy function $\tilde{f}(x)$ are

$$\begin{aligned} f_L(x, 0.5) &= \max\{f_L^1(x, 0.5), f_L^2(x, 0.5)\} = \max\{-2.5x - 5, x + 1\}, \\ f_R(x, 0.5) &= \max\{f_R^1(x, 0.5), f_R^2(x, 0.5)\} = \max\{-6x - 10, 3x + 1\}. \end{aligned}$$

Thus, analogous to the form of problem (5.2) we have a following biobjective optimization problem:

$$\begin{aligned} \max\{-2.5x - 5, x + 1\} &\rightarrow \min \\ \max\{-6x - 10, 3x + 1\} &\rightarrow \min \\ x &\in \mathbb{R}. \end{aligned} \tag{5.8}$$

This problem has two optimal solutions $\hat{x}_1 = -\frac{12}{7}$ and $\hat{x}_2 = -\frac{11}{9}$ (see Fig. 5.1). Now let us check necessary and sufficient optimality conditions for both solutions.

According to Definition 5.11 we calculate the subdifferential of the fuzzy function $\tilde{f}(\hat{x})$ in \hat{x}_1 and \hat{x}_2 at $\alpha = 0.5$:

$$\partial\tilde{f}(\hat{x}_1)[0.5] = (\partial f_L(\hat{x}_1, 0.5), \partial f_R(\hat{x}_1, 0.5)) = ([-2.5, 1], -6) \tag{5.9}$$

and

$$\partial\tilde{f}(\hat{x}_2)[0.5] = (\partial f_L(\hat{x}_2, 0.5), \partial f_R(\hat{x}_2, 0.5)) = (1, [-6, 3]). \tag{5.10}$$

For Theorem 5.4 it is necessary to show that there exist $0 \leq \lambda \leq 1$ such that

$$0 \in \partial_x L_{0.5}(\hat{x}_i, \lambda) = \lambda \partial f_L(\hat{x}_i, 0.5) + (1 - \lambda) \partial f_R(\hat{x}_i, 0.5), \tag{5.11}$$

where $i = 1, 2$. As soon as $\partial f_L(\hat{x}_i, 0.5)$ and $\partial f_R(\hat{x}_i, 0.5)$ are known from (5.9) and (5.10), it is easy to see, that

$$0 \in \partial_x L_{0.5}(\hat{x}_1, \lambda) = \lambda[-2.5, 1] + (1 - \lambda)(-6) \text{ for all } \lambda \in [6/7, 1]$$

and

$$0 \in \partial_x L_{0.5}(\hat{x}_2, \lambda) = \lambda + (1 - \lambda)[-6, 3] \text{ for all } \lambda \in [0, 6/7].$$

Thus, \hat{x}_1 and \hat{x}_2 are optimal solutions of problem

$$\lambda \max\{-2.5x - 5, x + 1\} + (1 - \lambda) \max\{-6x - 10, 3x + 1\} \rightarrow \min_{x \in \mathbb{R}} \tag{5.12}$$

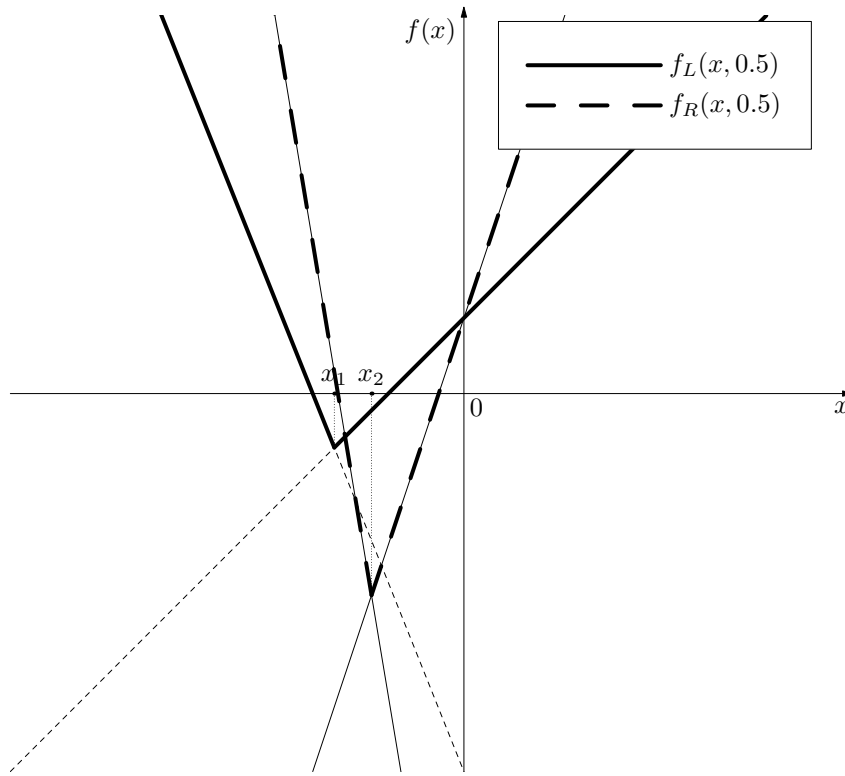


Fig. 5.1: The minimum of the function is obtained in $\hat{x} = -1$.

Then Theorem 3.2 implies that these points are Pareto optimal for biobjective optimization problem (5.8) and, hence, according to Definition 3.3 are optimal for fuzzy optimization problem (5.7). This is in accordance with Theorem 5.6.

For Theorem 5.5, i.e. due to convexity assumption for Corollary 5.1, it is enough to demonstrate that there exists $\lambda \in (0, 1)$ such that

$$f'_\lambda(\hat{x}_i, h)[0.5] \geq 0 \quad \forall h \in T_X(\hat{x}_i) = \mathbb{R}$$

as soon as

$$f'_L(\hat{x}_1, h)[0.5] = \max\{-2.5h, h\}, \quad f'_R(\hat{x}_1, h)[0.5] = -6h,$$

$$f'_L(\hat{x}_2, h)[0.5] = h, \quad f'_R(\hat{x}_1, h)[0.5] = \max\{-6h, 3h\}$$

for all $h \in \mathbb{R}$, it holds true.

It is not complicated procedure to compute directional derivatives of function $f_\lambda(\hat{x}_i, h)[0.5]$ for solutions x_1 and x_2 and to see that

$$f'_\lambda(\hat{x}_1, h)[0.5] = \lambda \max\{-2.5h, h\} + (1 - \lambda)(-6h) \geq 0 \quad \forall h \in \mathbb{R}$$

for all $\lambda \in [6/7, 1]$ and

$$f'_\lambda(\hat{x}_2, h)[0.5] = \lambda h + (1 - \lambda) \max\{-6h, 3h\} \geq 0 \quad \forall h \in \mathbb{R}$$

for all $\lambda \in [0, 6/7]$.

That is in accordance with Theorem 5.7, i.e. \hat{x}_i is a local minimum of nondifferentiable fuzzy optimization problem (5.7).

6 Fuzzy linear optimization problem over fuzzy polytope

In the previous discussions we have considered optimization problems only with fuzzy objective functions. In many cases, however, the constraints can also be fuzzy. The main target of this Chapter is to present an idea of solution for fuzzy optimization problem with fuzzy objectives and fuzzy constraints in a linear case.

A main difficulty in formulation of this problem consist in definition of fuzzy polytope. This problem is similar to a parametric optimization problem (where parameters are involved in the constraints).

As soon as we consider fuzzy *linear* optimization problem, we suppose, that a solution can be found in one of the vertices of the fuzzy polytope. However, the membership function value of the polytope in a certain vertex can differ from the membership function value of the solution.

This Chapter is organized as follows. To clarify a notion about fuzzy polytope, significant definitions as respects to fuzzy line and intersections of fuzzy lines are given in Section 6.1.

The solution method of the fuzzy optimization problem is based on taking level-cuts of the fuzzy polytope. Thus, for a fixed α -cut a pair of crisp polytopes is obtained and the fuzzy optimization problem can be splitted into two crisp optimization problems.

In previous discussions we have already derived that a solution of the problem, that has amount of uncertainty, cannot be exact. Thus, a solution of the fuzzy optimization problem considered to be fuzzy. Thus, we take into account the inherently uncertain nature of the fuzzy optimization problem and consider all the solutions of the corresponding crisp problems simultaneously. Finally, we describe a fuzzy solution as a union of each crisp optimization problem for all level-cuts. Of course, a fuzzy solution has to be enriched with its membership function. Again, we are not interested in computation of the membership function itself, but only its values on some crucial points are a matter of interest. Taking into consideration fuzzy nature of the feasible set, they can easily be computed. When the membership function values of the elements of the set of fuzzy optimal solutions are known, it enables the decision-maker to make an educated choice. Our approach equips the decision-maker with a correlation among all significant solutions and quantitatively measure the advantage of his / her choice over other.

The solution method is presented in Section 6.3.

In Section 6.4 an illustrative example is given.

6.1 Basic notions

To identify points that belong to the same object (e.g. fuzzy line), it is also necessary to define the concept of fuzzy connectedness.

Definition 6.1 (Rosenfeld (1984)). *Given a fuzzy set of points \mathfrak{F}^2 (over \mathbb{R}^2). The degree of fuzzy connectedness of two points p and q within \mathfrak{F}^2 is defined as*

$$C_{\mathfrak{F}^2}(p, q) = \max[\min \mu_{\mathfrak{F}^2}(r)],$$

where the maximum is taken over all paths connecting these points and the minimum is taken over all points r on each path.

Definition 6.2 (Pham (2001)). *Two points p and q are said to be fuzzily connected in \mathfrak{F}^2 if*

$$C_{\mathfrak{F}^2}(p, q) \geq \min[\mu_{\mathfrak{F}^2}(p), \mu_{\mathfrak{F}^2}(q)].$$

In other words, two points are connected in a fuzzy set of points if there exists a path between them which is composed of only points which also belong to this fuzzy set.

This definition is also consistent with the concept of connectedness of two points within a crisp set whose membership values are all equal to 1.

Let us consider two fuzzy points \tilde{p} and \tilde{q} in \mathfrak{F}^2 with membership functions $\mu_{\tilde{p}}(p)$ and $\mu_{\tilde{q}}(q)$, respectively.

Definition 6.3 (Pham (2001)). *A fuzzy line $\tilde{p}\tilde{q}$ which connects two points \tilde{p} and \tilde{q} is defined as a fuzzy set each of whose members is a linear combination of a pair of points p and q with a membership function defined as*

$$\mu_{\tilde{p}\tilde{q}}(p, q) = \min \{ \mu_{\tilde{p}}(p), \mu_{\tilde{q}}(q) \} \quad (6.1)$$

A fuzzy line may be visualised as a centre line with a variable thickness (see Fig. 6.1). This thin area of space (or thin volume of space in \mathfrak{F}^3 (over \mathbb{R}^3)) bounds a family of crisp lines which are formed by pairs of endpoints belonging to the two fuzzy sets of endpoints.

A *fuzzy plane*, which is an extension of a fuzzy line, is a thin planar shell with variable thickness. This shell encloses a family of crisp planes which is an extension of the family of crisp lines representing the fuzzy line. These concepts of fuzzy lines and planes encapsulate exact lines and exact planes as special cases.

Definition 6.4 (Pham (2001)). *The intersection of two fuzzy lines $\tilde{p}\tilde{q}$ and $\tilde{r}\tilde{s}$ is a fuzzy point \tilde{t} which is represented by a fuzzy set $\tilde{t} = (t, \mu_{\tilde{t}}(t))$, where*

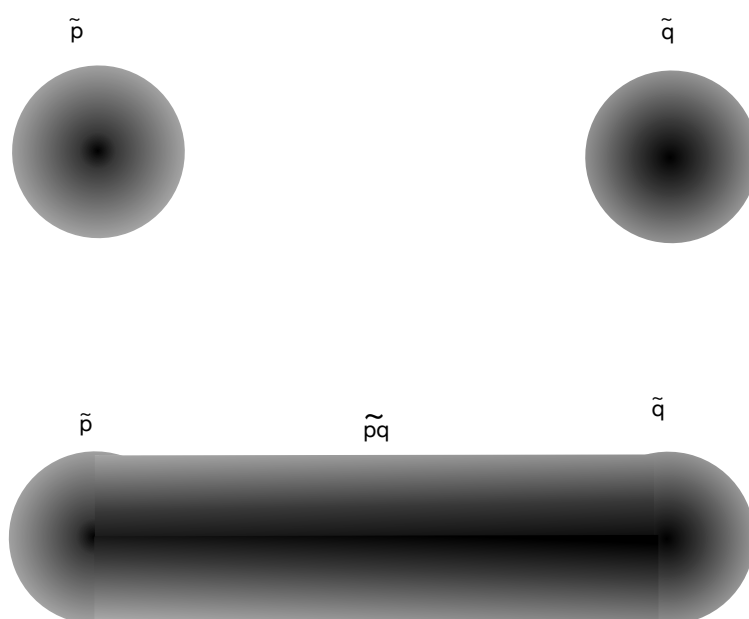
$$\mu_{\tilde{t}}(t) = \min \{ \mu_{\tilde{p}\tilde{q}}(t), \mu_{\tilde{r}\tilde{s}}(t) \}.$$

This Definition is illustrated on Fig. 6.2.

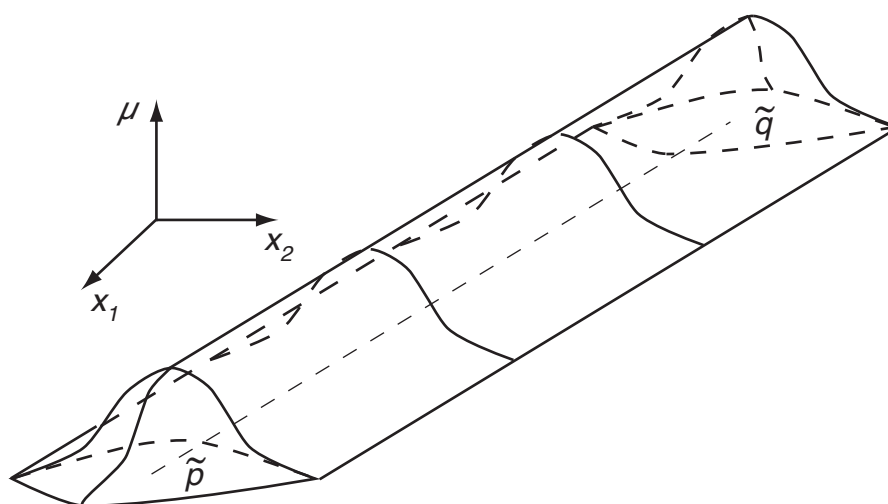
We can extend all aforesaid concepts to cover the intersection of a fuzzy line and a crisp plane, or of a fuzzy line and a fuzzy plane, or of two fuzzy planes, or of two fuzzy surfaces. Thus, the intersection of these geometry entities can be performed as two separate tasks:

- The first task is to compute the intersection of pairs of crisp geometry entities (which belongs to the two families of fuzzy entities) in the same way as in crisp geometry.
- The second task is to compute the membership value for each resulting entity.

Remark 6.1. *There is no reason why all the definitions have to be formulated especially in \mathfrak{F}^2 . Thus, we extend them to \mathfrak{F}^n .*



(a) Density plot of the membership functions.



(b) The membership functions.

Fig. 6.1: The fuzzy line \tilde{pq} connects two points \tilde{p} and \tilde{q} .

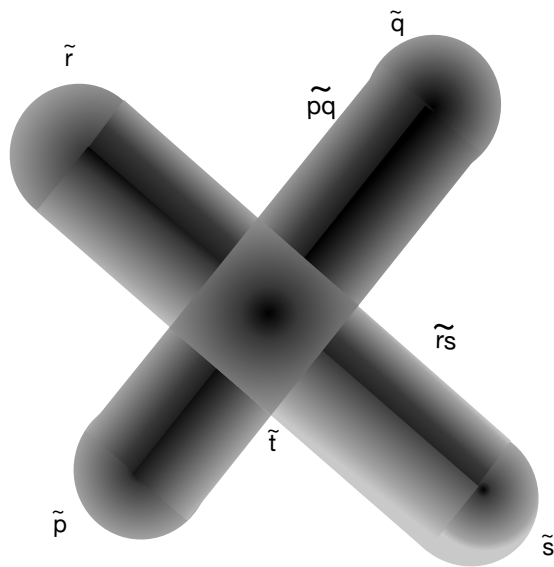


Fig. 6.2: The intersection of two fuzzy lines \tilde{pq} and \tilde{rs} .

6.2 The fuzzy polytope

A *fuzzy traverse* is composed of fuzzy vertices and fuzzy edges. Thus, it may be visualised as having edges of variable thickness, as described for fuzzy lines.

Definition 6.5. *Given a finite set of fuzzy points $\{\tilde{p}^1, \dots, \tilde{p}^n\}$. A fuzzy traverse is a closed shape $\tilde{P} = \text{conv}\{\tilde{p}^1, \dots, \tilde{p}^n\} \setminus \text{int}(\text{conv}\{\tilde{p}^1, \dots, \tilde{p}^n\})$ composed of a finite sequence of fuzzy line / hyperplane segments (faces).*

An illustration of the fuzzy traverse is presented on Fig. 6.3(a).

Remark 6.2. *Membership functions of fuzzy lines $\widetilde{p^1 p^2}, \widetilde{p^2 p^3}, \dots, \widetilde{p^{n-1} p^n}, \widetilde{p^n p^1}$ can be calculated using formula (6.1). In high dimensional space, membership functions of fuzzy hyperplanes are composed using convex linear combinations of the vertices belonging to this face.*

Obviously, the fuzzy traverse can be given by a finite number of linear equations.

This concept is readily extended to that of a fuzzy polytope. The difference between the fuzzy traverse and the fuzzy polytope is that the fuzzy polytope includes its interior. By analogy to Carathéodory's theorem we can state the following

Definition 6.6. *For any given finite set of fuzzy points $\{\tilde{p}^1, \dots, \tilde{p}^n\}$, a fuzzy polytope is defined as $\mathfrak{P} = \text{conv}\{\tilde{p}^1, \dots, \tilde{p}^n\}$. It can be composed into a fuzzy traverse $\tilde{P} = \mathfrak{P} \setminus \text{int}(\mathfrak{P})$ and its interior $\text{int}(\mathfrak{P})$. The membership function $\mu_{\text{int}(\mathfrak{P})}$ of every point p of $\text{int}(\mathfrak{P})$ is equal to*

$$\mu_{\text{int}(\mathfrak{P})}(p) = \sup_{p = \sum_{i=1}^n \omega_i p_i} \min\{\mu_{\tilde{p}^1}(p_1), \dots, \mu_{\tilde{p}^n}(p_n)\},$$

where $\omega_i > 0$ for $i = 1, \dots, n$, $\sum_{i=1}^n \omega_i = 1$ and n is a number of the vertices of the fuzzy polytope.

The illustration is given on Fig. 6.3(b). The fuzzy polytope can be given by a finite number of linear inequalities.

Definition 6.7. *Extreme points of the fuzzy polytope are the fuzzy points on its boundary.*

Since the fuzzy traverse \tilde{P} can be considered as a fuzzy set, we can take its level-cut $P(\alpha)$ for some fixed $\alpha \in [0, 1]$. That means, that we take the same α -cut of all fuzzy vertices $\tilde{p}^1, \dots, \tilde{p}^n$, namely, $[p_L^1(\alpha), p_R^1(\alpha)], \dots, [p_L^n(\alpha), p_R^n(\alpha)]$.

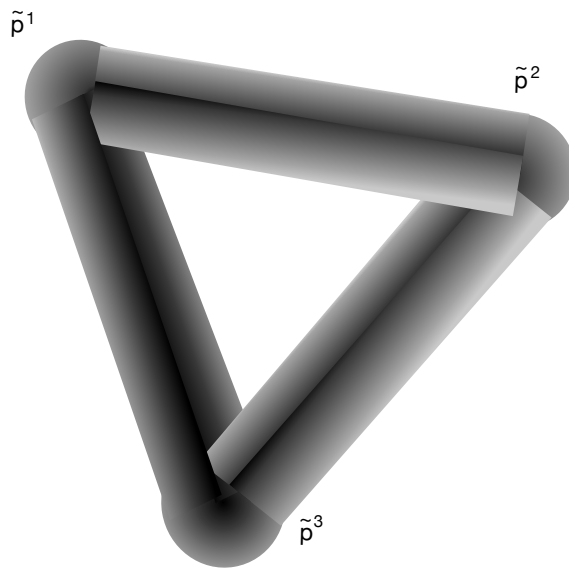
Remark 6.3. *The number of vertices in the level-cuts of $P(\alpha)$ can be different for different α , i.e. we denote through p_L^k and p_R^k ($k = 1, \dots, n$) the sets of the vertices that belong to one fuzzy vertex \tilde{p}^k . This is illustrated in Section 6.4.*

Thus, we obtain two crisp traverses $P_L(\alpha)$ and $P_R(\alpha)$. The left-hand side traverse $P_L(\alpha)$ is a closed figure, composed with the use of left-hand side bounds of all fuzzy vertices in the same order as fuzzy traverse \tilde{P} :

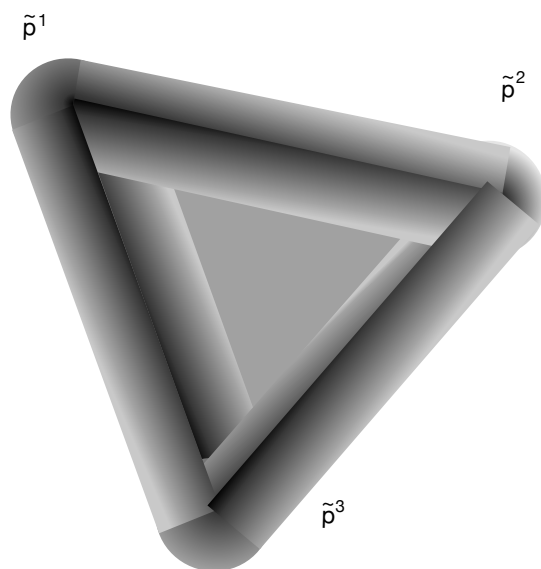
$$P_L(\alpha) = \text{conv}\{p_L^1(\alpha), \dots, p_L^n(\alpha)\} \setminus \text{int}(\text{conv}\{p_L^1(\alpha), \dots, p_L^n(\alpha)\}). \quad (6.2)$$

Analogously, the right-hand side traverse is also a closed figure. It can be written as

$$P_R(\alpha) = \text{conv}\{p_R^1(\alpha), \dots, p_R^n(\alpha)\} \setminus \text{int}(\text{conv}\{p_R^1(\alpha), \dots, p_R^n(\alpha)\}). \quad (6.3)$$



(a) The traverse.



(b) The polytope.

Fig. 6.3: The fuzzy traverse and fuzzy polytope.

Definition 6.8. A level-cut of the fuzzy traverse \tilde{P} for some fixed $\alpha \in [0, 1]$ is defined as

$$P(\alpha) = \text{conv}\{P_R(\alpha)\} \setminus \text{int}(\text{conv}\{P_L(\alpha)\}).$$

Unlike a fuzzy traverse, a fuzzy polytope includes its interior. Thus, under the assumption that taking an α -cut we obtain a left- and a right-hand side polytope, denoted as $\mathfrak{P}_L(\alpha)$ and $\mathfrak{P}_R(\alpha)$, respectively, we can write the following:

Definition 6.9. A level-cut of the fuzzy polytope \mathfrak{P} for some fixed $\alpha \in [0, 1]$ is given by

$$\mathfrak{P}(\alpha) = \mathfrak{P}_R(\alpha) \setminus \text{int}(\mathfrak{P}_L(\alpha)),$$

where, by analogy to formulas (6.2) and (6.3), $\mathfrak{P}_L(\alpha) = \text{conv}\{p_L^k\}$ and $\mathfrak{P}_R(\alpha) = \text{conv}\{p_R^k\}$ ($k = 1, \dots, n$).

The α -cut of a fuzzy polytope is illustrated in Fig. 6.4 .

Remark 6.4. Under the assumption that all fuzzy vectors are defined by fuzzy components with triangular (or at least bell-shaped) membership function, $\mathfrak{P}(1) = \mathfrak{P}_L(1) = \mathfrak{P}_R(1)$ is a crisp traverse.

Remark 6.5. $\mathfrak{P}_L(\alpha) \subset \mathfrak{P}_R(\alpha)$ for all $\alpha \in [0, 1]$.

Proposition 6.1. Let $0 \leq \alpha_1 < \alpha_2 \leq 1$. Then $\mathfrak{P}(\alpha_2) \subset \mathfrak{P}(\alpha_1)$.

For further discussion let us make

Assumption 6.1. Assume that $p_L^k(\alpha) \rightarrow p^k(1)$ and $p_R^k(\alpha) \rightarrow p^k(1)$ for $\alpha \uparrow 1$ (see Fig. 6.5).

That means that we assume that all the vertices that belong to one fuzzy vertex \tilde{p}^k converge to $p^k(1)$.

6.3 Formulation and solution method

Let us consider the fuzzy optimization problem

$$\begin{aligned} F(\tilde{c}) = d^\top \tilde{c} &\rightarrow \min \\ \text{s.t. } \tilde{c} &\in \mathfrak{P}, \end{aligned} \tag{6.4}$$

where

- $\mathfrak{P} \subset \mathfrak{F}^n$ is a fuzzy polytope;
- $d \neq 0$ is a known crisp vector in \mathbb{R}^n ;
- \tilde{c} is a fuzzy variable in \mathfrak{F}^n .

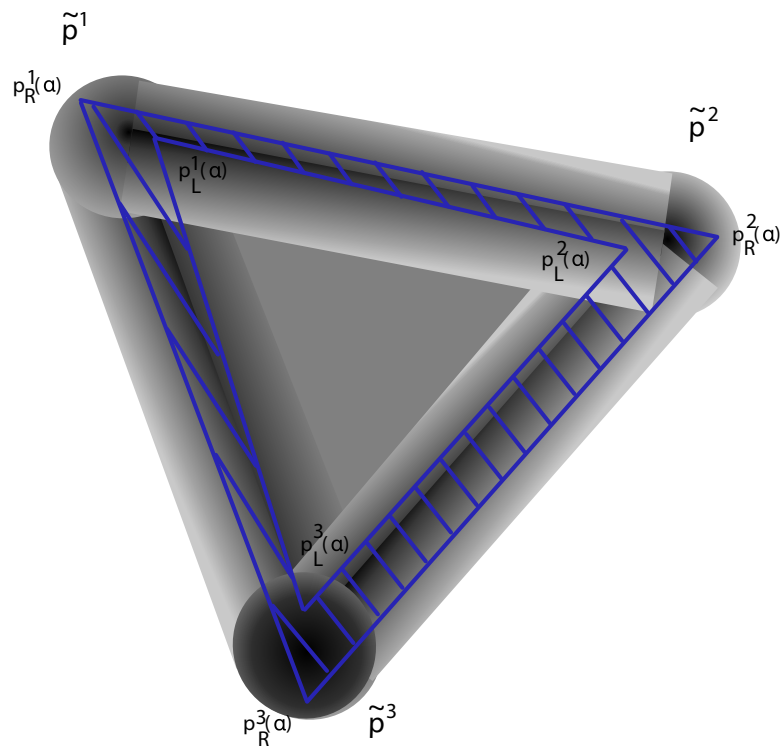


Fig. 6.4: The α -cut of the fuzzy polytope is determined by two crisp polytopes.

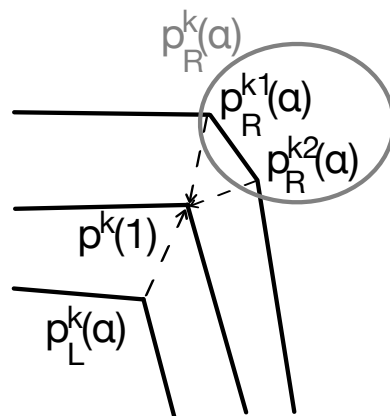


Fig. 6.5: Convergence of the vertices.

The fuzzy polytope \mathfrak{P} is defined according to Definition 6.6 by a given set of fuzzy points $\{\tilde{p}^1, \dots, \tilde{p}^n\}$.

For the solution of linear optimization problems with fuzzy objective function and fuzzy constraints we use the notions introduced in Section 6.2. Let us consider problem (6.4) on some $\alpha \in [0, 1]$:

$$\begin{aligned} F(c_\alpha) = d^\top c_\alpha &\rightarrow \min \\ \text{s.t. } c_\alpha &\in \mathfrak{P}(\alpha) \end{aligned} \quad (6.5)$$

Let us denote the set of optimal solutions of problem (6.5) through

$$\Psi(\alpha) = \arg \min \{d^\top c_\alpha^* : c_\alpha^* \in \mathfrak{P}_\alpha\}.$$

$\Psi(\alpha)$ is a point-to-set mapping which maps $\alpha \in [0, 1]$ to the set of global optimal solutions of problem (6.5).

As we know, the level-cut of fuzzy polytope $\mathfrak{P}(\alpha)$ can be decomposed into two crisp polytopes

$$\mathfrak{P}_L(\alpha) = \text{conv}\{p_L^1(\alpha), \dots, p_L^n(\alpha)\} \text{ and } \mathfrak{P}_R(\alpha) = \text{conv}\{p_R^1(\alpha), \dots, p_R^n(\alpha)\}.$$

For these polytopes let us consider two optimization problems and then take the convex hull to obtain all solutions of problem (6.5) on the α -cut.

$$\begin{aligned} F(c_L(\alpha)) = d^\top c_L(\alpha) &\rightarrow \min \\ \text{s.t. } c_L(\alpha) &\in \mathfrak{P}_L(\alpha) \end{aligned} \quad (6.6)$$

and

$$\begin{aligned} F(c_R(\alpha)) = d^\top c_R(\alpha) &\rightarrow \min \\ \text{s.t. } c_R(\alpha) &\in \mathfrak{P}_R(\alpha), \end{aligned} \quad (6.7)$$

where $c_L(\alpha)$ and $c_R(\alpha)$ denotes the left- and right-hand side bounds of the α -cut of the fuzzy variable \tilde{c} .

These problems, in turn, are crisp linear optimization problems, which can be solved using any of the standard methods (see e.g. Bertsimas and Tsitsiklis (1997); Unger and Dempe (2010)). Let the sets of optimal solutions of problems (6.6) and (6.7) be denoted as $c_L^*(\alpha)$ and $c_R^*(\alpha)$, respectively.

The fact that any optimal solution of these crisp optimization problems is necessarily a vertex of the corresponding polytope is well-known (see e.g. Dyer and Proll (1977)). This means, that optimal solutions of problems (6.6) and (6.7) $c_L^*(\alpha)$ and $c_R^*(\alpha)$ correspond to elements of the vertices $p_L^k(\alpha)$ and $p_R^k(\alpha)$ for some $k = 1, \dots, n$.

Let us consider problem (6.5) for $\alpha = 1$:

$$\begin{aligned} F(c_1) = d^\top c_1 &\rightarrow \min \\ \text{s.t. } c_1 &\in \mathfrak{P}(1), \end{aligned} \quad (6.8)$$

and let $c^*(1)$ be its optimal solution.

Theorem 6.1. $c_L^*(\alpha) \rightarrow c^*(1)$ and $c_R^*(\alpha) \rightarrow c^*(1)$ for $\alpha \uparrow 1$ (see Fig. 6.6).

Proof. The convergence in the formulation of the theorem is caused by nothing but the following fact: $\mathfrak{P}_L(\alpha) \rightarrow \mathfrak{P}(1)$ and $\mathfrak{P}_R(\alpha) \rightarrow \mathfrak{P}(1)$ for $\alpha \uparrow 1$ (see Fig. 6.7) and Assumption 6.1. \square

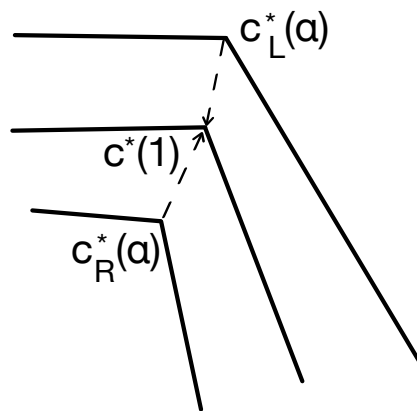


Fig. 6.6: The convergence of the solutions.

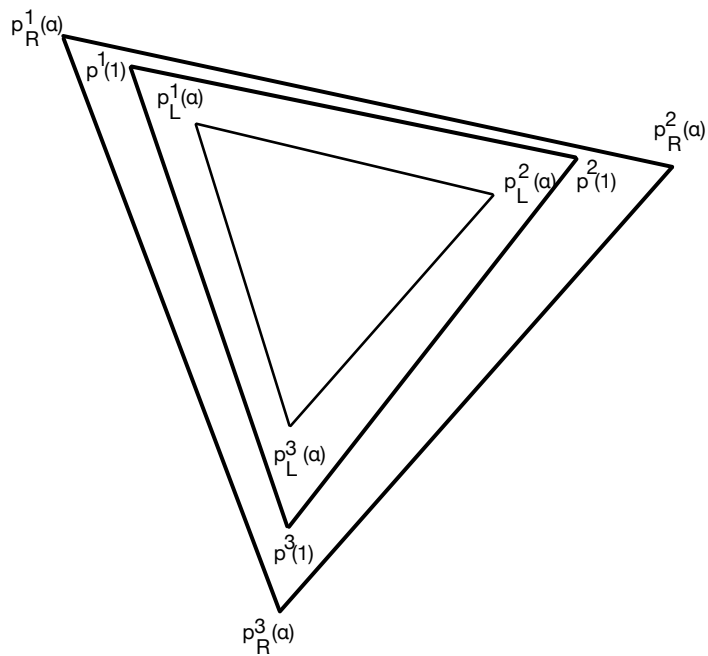


Fig. 6.7: The level-cuts of the fuzzy polytope for α and 1.

Within Definition 6.9 and Theorem 6.1 we obtain

Definition 6.10. *The set of optimal solutions of optimization problem (6.5) is*

$$c_{\alpha}^* = \text{conv}\{c_L^*(\alpha), c^*(1)\} \cup \text{conv}\{c_R^*(\alpha), c^*(1)\}. \quad (6.9)$$

Definition 6.11. *A fuzzy optimal solution of problem (6.4) is a union of the optimal solutions of problem (6.5) for all α -cuts:*

$$\tilde{c}^* = \bigcup_{\alpha \in [0,1]} (\text{conv}\{c_L^*(\alpha), c^*(1)\} \cup \text{conv}\{c_R^*(\alpha), c^*(1)\}). \quad (6.10)$$

By analogy to linear crisp optimization, solution \tilde{c}^* of the fuzzy optimization problem (6.4) is a subset of a vertex of the fuzzy polytope or the subset of a segment of a fuzzy hyperplane defined by its vertices.

Hence, we write

Theorem 6.2. *A fuzzy optimal solution of fuzzy linear optimization problem (6.4) is a subset of the set of extreme points of \mathfrak{P} . In general, $\tilde{c}^* \subset \text{conv}\{\tilde{p}^k\}$ ($k = 1, \dots, n$) and the number of \tilde{p}^k can not exceed $n - 1$, where n is a number of the vertices of the fuzzy polytope.*

Proof. The proof is straightforward and follows directly from the convergence Theorem 6.1 and considerations for the crisp case. \square

The fuzzy polytope \mathfrak{P} has its own membership function $\mu_{\mathfrak{P}}$. Thus, for each component of the fuzzy solution we obtain its membership function value by straight forward calculation using

Definition 6.12. *The membership function value of an element c^* of fuzzy optimal solution \tilde{c}^* of problem (6.4) can be calculated as*

$$\mu_{\tilde{c}^*}(c^*) = |\{\alpha \mid c^* \in \text{conv}\{c_L^*(\alpha), c_R^*(\alpha)\}, \alpha \in [0, 1]\}| \quad (6.11)$$

where $|\cdot|$ stands for the geometric measure of the set.

Then, the final choice can be such an element c^* of the fuzzy solution that has a maximal membership function value (or according to Definition 4.4 - the best solution). However, other solutions can exist. This we can better explain in the next Section.

6.4 Example

Let us consider the fuzzy optimization problem

$$\begin{aligned} F(\tilde{c}) = \tilde{c}_1 + 2\tilde{c}_2 &\rightarrow \max \\ \text{s.t. } \tilde{c}_1 + \tilde{c}_2 &\preceq \tilde{7} \\ 0 &\preceq \tilde{c}_1 \preceq 3 \\ 0 &\preceq \tilde{c}_2 \preceq 5, \end{aligned} \quad (6.12)$$

where $\tilde{7} = (6, 7, 9)$. These inequalities define the fuzzy polytope \mathfrak{P} . The number of vertices of the crisp polytopes $\mathfrak{P}_L(\alpha)$ and $\mathfrak{P}_R(\alpha)$ obtained for different α -cuts depends on the value of α .

To discuss this situation seriatim, let us consider problem (6.12) for two different level-cuts: for $\alpha = 0.3$ and $\alpha = 0.7$. Solutions for these case are visualized in Fig. 6.8.

$$\begin{aligned} F(c_L(0.3)) &= c_{1L}(0.3) + 2c_{2L}(0.3) \rightarrow \max \\ \text{s.t. } c_{1L}(0.3) + c_{2L}(0.3) &\leq 6.3 \\ 0 \leq c_{1L}(0.3) &\leq 3 \\ 0 \leq c_{2L}(0.3) &\leq 5 \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} F(c_R(0.3)) &= c_{1L}(0.3) + 2c_{2L}(0.3) \rightarrow \max \\ \text{s.t. } c_{1R}(0.3) + c_{2R}(0.3) &\leq 8.4 \\ 0 \leq c_{1R}(0.3) &\leq 3 \\ 0 \leq c_{2R}(0.3) &\leq 5. \end{aligned} \tag{6.14}$$

By analogy, for $\alpha = 0.7$ we obtain

$$\begin{aligned} F(c_L(0.7)) &= c_{1L}(0.7) + 2c_{2L}(0.7) \rightarrow \max \\ \text{s.t. } c_{1L}(0.7) + c_{2L}(0.7) &\leq 6.7 \\ 0 \leq c_{1L}(0.7) &\leq 3 \\ 0 \leq c_{2L}(0.7) &\leq 5 \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} F(c_R(0.7)) &= c_{1L}(0.7) + 2c_{2L}(0.7) \rightarrow \max \\ \text{s.t. } c_{1R}(0.7) + c_{2R}(0.7) &\leq 7.6 \\ 0 \leq c_{1R}(0.7) &\leq 3 \\ 0 \leq c_{2R}(0.7) &\leq 5. \end{aligned} \tag{6.16}$$

To compute the fuzzy optimal solution \tilde{c}^* of problem (6.12) we have to take a convex hull for all α -cuts. And we obtain that $\tilde{c}^* = \text{conv}\{(1, 5), (3, 5)\}$. And it is easy to see that for different α -cuts we have different Pareto optimal solutions. It is easy to see that solution $c_1^* = (2, 5)$ (that is obtained for $\alpha = 1$) does not change for all level-cuts and solution $c_2^* = (3, 5)$ stays optimal for all $\alpha \in [0, 0.5]$.

Remark 6.6. *In this example it is easy to see that the maximal convex hull can be obtained for $\alpha = 0$. However, it is also clear that it is not sufficient to consider $\alpha = 0$ only.*

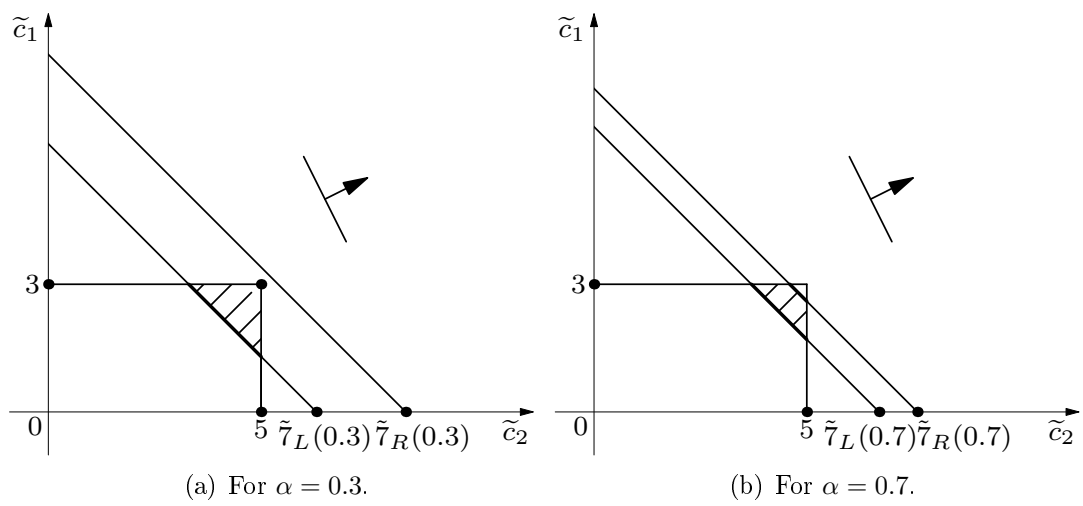


Fig. 6.8: Different solutions are obtained for different α -cuts.

7 Bilevel optimization with fuzzy objectives

To deal with more complicated problem - the bilevel optimization problem with fuzzy objectives and crisp constraints, we combine here the main results from the aforesaid. The purpose of the present Chapter consists in describing an effective algorithm for the bilevel bilinear optimization problem with fuzzy objective functions and crisp constraints. The investigated fuzzy bilevel optimization problem is well-stated in Section 7.1 for a general case.

There exist only few possibilities to deal with this problem of non-uniqueness. Namely,

- optimistic approach (Dempe and Starostina (2007));
- pessimistic approach (Lohse (2011));
- selection function approach (Dempe and Starostina (2006)).

These main approaches and their difference are presented in Section 7.2.

One opportunity to solve such a fuzzy bilevel optimization problem is to use Yager ranking indices to avoid the incomparability of the fuzzy vectors involved in the problem (Ruziyeva and Dempe (2012)). Then, the fuzzy bilevel optimization problem can easily be reformulated into the crisp bilevel optimization problem. This idea is presented in Section 7.3. Algorithm I is presented in Section 7.4 to demonstrate the solution technique of our first approach to solve the fuzzy bilevel optimization problem.

However, there exist another opportunity to deal with the bilevel bilinear optimization problem. In this Chapter is presented completely new idea, where for each selection of the leader the optimal solution of the fuzzy optimization problem on the lower level is considered to be fuzzy.

Then, the fuzzy optimal solution of the lower level problem is described as the set of Pareto optimal solutions of a corresponding multiobjective optimization problem. According to Chanas and Kuchta (1994), the preferable optimal solution is supposed to have a maximal membership function value, i.e. this solution has the highest potential being realized by the follower. The idea of computation of a value of the membership function of the optimal solution is based on optimality conditions and is presented in Chapter 4. Then the solution with the maximal membership function value is chosen for future considerations: For this best solution its region of stability is found. A stability region is a fuzzy polyhedron. Thus, the fuzzy optimization problem on the upper level is solved with respect to the solution method described in Chapter 6. That technique is described in Section 7.5. This naturally results in Algorithm II presented in Section 7.6.

The Chapter concluded with an illustrative example (the fuzzy bilevel optimization problem with fuzzy flow problem on the lower level). Thus, two algorithms are compared in Section 7.7.

7.1 General formulation

As soon as bilevel programming has been proposed for modelling hierarchical decision processes with two decision makers, let, first of all, the leader select a fuzzy solution $\tilde{c} \in \tilde{C}$. Then, the follower's task is to solve the problem

$$\min_x \{f(\tilde{c}, x) : g(x) \leq 0\}, \quad (7.1)$$

where

- $f(\tilde{c}, x) : \tilde{C} \times \mathbb{R}^n \mapsto \mathfrak{F}$ is a fuzzy objective function;
- $g(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$ is a crisp constraint vector-valued function;
- \tilde{C} is a fuzzy set in \mathfrak{F}^n ;
- $X := \{x : g(x) \leq 0\}$ is a crisp set.

Let $\Psi(\tilde{c})$ denote the set of optimal solutions of fuzzy optimization problem (7.1), i.e.

$$\Psi(\tilde{c}) = \arg \min_{x \in X} \{f(\tilde{c}, x) : g(x) \leq 0\}.$$

Then, the leader's aim is to minimize the fuzzy function $F(\tilde{c}, x)$ subject to both $\tilde{c} \in \tilde{C}$ and $x \in \Psi(\tilde{c})$.

If $\Psi(\tilde{c})$ consists of one element for each $\tilde{c} \in \tilde{C}$, i.e. $|\Psi(\tilde{c})| = 1$, then the fuzzy bilevel programming problem can be formulated as

$$\begin{aligned} F(\tilde{c}, x) &\rightarrow \min_{\tilde{c} \in \tilde{C}} \\ \text{s.t. } f(\tilde{c}, x) &\rightarrow \min_{x \in X}, \end{aligned} \quad (7.2)$$

where the upper level objective function $F(\tilde{c}, x)$ is a fuzzy function.

7.2 Solution approaches

The assumption that the set of optimal solutions $\Psi(\tilde{c})$ of problem (7.1) reduces to a singleton is often not satisfied even in the crisp case of bilevel optimization problems. There exist few possibilities to deal with such a kind of problems under the assumption that the follower would choose a certain solution. Namely:

- The optimistic approach can be used in the cooperative case under the assumption that the follower takes the best solution for the leader. Then, the optimal value function is

$$\phi_o(\tilde{c}) = \min_{x \in \Psi(\tilde{c})} \{F(\tilde{c}, x)\} \rightarrow \min_{\tilde{c} \in \tilde{C}}$$

- The pessimistic approach can be used in non-cooperative case when the leader has to bound a damage of the follower's choice. Then, we obtain the following

$$\phi_p(\tilde{c}) = \max_{x \in \Psi(\tilde{c})} \{F(\tilde{c}, x)\} \rightarrow \min_{\tilde{c} \in \tilde{C}}$$

- When a type of behaviour of the players is not known exactly, it is possible to assume that the follower can select the solution according to a selection function. This quite new approach we call the selection function approach and describe it in the present work.

Let us denote some element of $\Psi(\tilde{c})$ by $x(\tilde{c})$ and assume, that this choice is a fixed selection function for all possible $\tilde{c} \in \tilde{C}$ (see Dempe and Starostina (2006)). The vector of parameters \tilde{c} describes the "environmental data" for fuzzy lower level problem (7.1).

The problem of determining optimal solution \tilde{c}^* for the leader can be described as that of finding a vector of parameters for fuzzy parametric optimization problem (7.1), which together with the response of the follower $x(\tilde{c}) \in \Psi(\tilde{c})$ to the leader's decision proves to give the minimal possible function value for the upper level objective function $F(\tilde{c}, x)$. That means, that the aim of the fuzzy bilevel programming problem is then to select \tilde{c} such that it is an optimal one in the following optimization problem:

$$F(\tilde{c}, x(\tilde{c})) \rightarrow \min_{\tilde{c} \in \tilde{C}}.$$

To compare these three approaches let us consider the following example:

Example 7.1. Consider the convex parametric optimization problem

$$\Psi(\tilde{c}) = \arg \min_x \{-\tilde{c}x : 0 \leq x \leq 1\}$$

and the bilevel optimization problem

$$\min_{\tilde{c}} \{F(\tilde{c}, x) = x^2 + \tilde{c}^2 : x \in \Psi(\tilde{c}), -1 \preceq \tilde{c} \preceq 1\}.$$

It is easy to compute that

$$\Psi(\tilde{c}) = \begin{cases} \{0\}, & \tilde{c} \prec 0; \\ [0, 1], & \tilde{c} = 0; \\ \{1\}, & \tilde{c} \succ 0. \end{cases} \quad (7.3)$$

$$F(\tilde{c}, x(\tilde{c})) = \begin{cases} \tilde{c}^2, & \tilde{c} \prec 0; \\ [0, 1], & \tilde{c} = 0; \\ \tilde{c}^2 + 1, & \tilde{c} \succ 0. \end{cases} \quad (7.4)$$

Here it can be seen that the upper level function value is unclear unless the follower has announced his / her selection to the leader.

The upper level problem is solvable only in case when the follower selects $x(0) = 0 \in \Psi(\tilde{c})$. Thus, the notion of an optimal value is not clear: There are choices for \tilde{c} leading to the upper level objective function values sufficiently close to zero, but it is not clear whether the value zero can be attained.

As soon as the leader has to meet his / her choice first, it is very important to consider all options.

In optimistic case we obtain the optimal function value as

$$\phi_o(\tilde{c}) = \begin{cases} \tilde{c}^2, & \tilde{c} \preceq 0; \\ \tilde{c}^2 + 1, & \tilde{c} \succ 0. \end{cases} \quad (7.5)$$

Using optimistic approach, we have to minimize the function $\phi_o(\tilde{c})$ with respect to \tilde{c} , i.e.

$$\min_y \{\phi_o(\tilde{c}), -1 \preceq \tilde{c} \preceq 1\}.$$

This solution $\tilde{c}_o^* = 0$ is optimistic solution.

In pessimistic case we have

$$\phi_p(\tilde{c}) = \begin{cases} \tilde{c}^2, & \tilde{c} \prec 0; \\ \tilde{c}^2 + 1, & \tilde{c} \succeq 0. \end{cases} \quad (7.6)$$

And solving the following problem

$$\min_y \{\phi_p(\tilde{c}), -1 \preceq \tilde{c} \preceq 1\}$$

we obtain the pessimistic solution $\tilde{c}_p^* = 1$.

It is clear that using either optimistic or pessimistic approaches we exclude a whole continuum of the possible decisions of the leader $(0, 1)$.

Assume that follower's provides selection function

$$x(\tilde{c}) = \begin{cases} 0, & \tilde{c} \prec 0; \\ 1/2, & \tilde{c} = 0; \\ 1, & \tilde{c} \succ 0. \end{cases} \quad (7.7)$$

The leader can now choose either solution $\tilde{c}^* = 0$ or solution near to \tilde{c}^* .

7.3 Yager index approach

In the present Section we suggest to compute the selection function $x(\tilde{c})$ using the Yager ranking indices technique. The classical definition in area compensation is defined as follows:

Definition 7.1 (Liu and Kao (2004)). For a fuzzy number $\tilde{d} \in \tilde{C}$ the Yager index is computed as

$$I(\tilde{d}) = \frac{1}{2} \int_0^1 [d_L(\alpha) + d_R(\alpha)] d\alpha, \quad (7.8)$$

where $[d^L(\alpha), d^R(\alpha)]$ is an α -cut of the fuzzy number \tilde{d} .

Let us define for a fuzzy vector $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_n)$ the Yager index as a vector $I(\tilde{c}) = (I(\tilde{c}_1), \dots, I(\tilde{c}_n))$. Hence, the function $I(\tilde{c})$ also possesses the properties of linearity and additivity. Here we adopt this method for ranking the objective function values.

For realizing this idea we investigate fuzzy bilevel optimization problem

$$\begin{aligned} F(\tilde{c}, x) &\rightarrow \min_{\tilde{c} \in \tilde{C}} \\ \text{s.t. } x &\in \Psi(\tilde{c}) \end{aligned} \quad (7.9)$$

with an n -dimensional vector of decision variables x . Here

- $F(\tilde{c}, x)$ is a fuzzy function;
- $\tilde{C} = \mathfrak{F}^n$;
- $\Psi(\tilde{c}) = \arg \min_x \{\tilde{c}^\top x : x \in X\}$ is the set of optimal solutions of the lower level problem;
- $X = \{x : Ax = b, x \geq 0\}$;
 - $A \in \mathbb{R}^{m \times n}$ is the constraint matrix;
 - $b \in \mathbb{R}^m$ is the right-hand side vector.

That means that for some fixed $\tilde{c}^\times \in \tilde{C}$ the lower level fuzzy optimization problem is stated as

$$\tilde{c}^{\times \top} x \rightarrow \min_{x \in X} \quad (7.10)$$

According to Definition 7.1 as a reformulation of follower's fuzzy optimization problem (7.10) we have the following optimization problem on the lower level:

$$I(\tilde{c}^\times)^\top x \rightarrow \min_{x \in X} \quad (7.11)$$

Thus, $\Psi_I(\tilde{c}^\times) = \arg \min_x \{I(\tilde{c}^\times)x : x \in X\}$.

Suppose that fuzzy vectors are defined through (normalized) continuous triangular fuzzy numbers $\tilde{c} = (c_L, c_T, c_R)$, where $c_L, c_T, c_R \in \mathbb{R}^n$. Then, it is easy to calculate

$$c_L(\alpha) = (c_T - c_L)\alpha + c_L \text{ and } c_R(\alpha) = (c_T - c_R)\alpha + c_R.$$

Using Definition 7.1, the Yager index for this particular case is defined as

$$I(\tilde{c}) = \frac{1}{2} \left(c_T + \frac{1}{2}[c_L + c_R] \right). \quad (7.12)$$

This simplifies problem (7.11), that is already a crisp optimization problem. And now initial fuzzy bilevel optimization problem (7.9) is transformed into

$$\begin{aligned} F(I(\tilde{c}), x) &\rightarrow \min_{I(\tilde{c}) \in \mathbb{R}^n} \\ \text{s.t. } I(\tilde{c})^\top x &\rightarrow \min_{x \in X}. \end{aligned} \quad (7.13)$$

7.4 Algorithm I

To solve fuzzy optimization problem (7.9) we have to do the following.

For the fixed fuzzy vector \tilde{c}^\times compute the Yager index $I(\tilde{c}^\times)$.

STEP 1 Find an optimal solution $x^*(I(\tilde{c}^\times)) \in \Psi_I(\tilde{c}^\times)$ of lower level problem (7.11).

If the optimal solution of lower level problem (7.11) is not unique, take the best one with respect to the upper level function $F(I(\tilde{c}^\times), x)$.

STEP 2 Fix this solution $x^* := x^*(I(\tilde{c}^x))$ of the follower's problem (7.11) and solve the upper level problem

$$F(I(\tilde{c}), x^*) \rightarrow \min_{I(\tilde{c}) \in \mathbb{R}^n} \quad (7.14)$$

STEP 3 Fix the optimal solution $I^*(\tilde{c})$ of problem (7.14) and go to STEP 1. Repeat until the solution stops changing.

Then the pair $(I^*(\tilde{c}), x^*)$ is an optimal solution of bilevel programming problem (7.13) in sense of Dempe (1987).

STEP 4 Now the inverse function to $I^*(\tilde{c})$ has to be found. For the case of triangular fuzzy vectors, according to formula (7.12), we choose an optimal $\tilde{c}^* = (c_L^*, c_T^*, c_R^*)$ such that

$$c_T^* = I^*(\tilde{c}), \quad c_L^* + c_R^* = 2I^*(\tilde{c}) \quad \text{and} \quad c_L^* \leq c_T^* \leq c_R^*.$$

With such a triple (c_L^*, c_T^*, c_R^*) the fuzzy vector \tilde{c}^* can be defined by a symmetrical membership function for which

$$c_T^* - c_L^* = c_R^* - c_T^*$$

holds true. This \tilde{c}^* has the Yager index equal to $I^*(\tilde{c})$.

Then the pair (\tilde{c}^*, x^*) is an optimal solution of initial fuzzy bilevel optimization problem (7.9).

The algorithm is demonstrated later in Section 7.7 on illustrative example.

7.5 Membership function approach

In this Section we investigate the fuzzy bilevel optimization problem

$$\begin{aligned} F(\tilde{c}, x) &\rightarrow \min_{\tilde{c} \in \tilde{C}} \\ \text{s.t. } \tilde{c}^\top x &\rightarrow \min_{x \in X} \end{aligned} \quad (7.15)$$

with an n -dimensional vector of decision variables x under the assumption that

- The upper level objective function $F(\tilde{c}, x)$ is bilinear;
- The leader's feasible set \tilde{C} is properly defined fuzzy polytope;
- The follower's feasible set $X = \{x : Ax = b, x \geq 0\}$ is a crisp polytope, where
 - $A \in \mathbb{R}^{m \times n}$ is the constraint matrix,
 - $b \in \mathbb{R}^m$ is the right-hand side vector.

Such strong restrictions we use for simplicity: Bilevel optimization problem is shown to be \mathcal{NP} -hard by Ben-Ayed and Blair (1990); Blair (1992) even in a crisp linear case.

That means that the lower level fuzzy optimization problem is stated for the fixed vector of coefficients $\tilde{c}^x \in \tilde{C}$ in a form of fuzzy linear optimization problem (4.1), namely

$$\begin{aligned} \tilde{c}^{x\top} x &\rightarrow \min \\ x &\in X. \end{aligned} \tag{7.16}$$

This problem is solved with the approach based on minimization of the α -cut on the feasible set. This is described in details in Section 4.1.

In this Section we describe a solution algorithm for fuzzy bilevel optimization problem (7.15) that supposes, that the membership function of the solution of lower level fuzzy linear optimization problem (7.16) can be found (see Chapter 4) and as soon as the feasible set is a polytope, a solution can be found on one of the vertices of this set.

The feasible set X can be presented as the Minkowski sum of a convex hull of all vertices and a convex cone (see e.g. Minkowski's Theorem 4.8. in Nemhauser and Wolsey (1988)).

Henceforth, we assume that the membership functions for all vertices x_i of this polytope - feasible set X - can be computed (for non-optimal solutions the membership function is zero.) As soon as the solution of problem (7.16) is a fuzzy set of feasible points, elements of this set with the largest membership function values should be selected, since these have the largest potential of being realized. Thus, we accept a best solution for further analysis, (an element such that its membership function has maximal value).

On the basis of aforesaid we suppose that the optimal solution of fuzzy linear optimization problem (7.16) $\hat{x} := \hat{x}(\tilde{c}^x) \in \Psi(\tilde{c}^x)$ for the fixed $\tilde{c}^x \in \tilde{C}$ possess the maximal membership function value $\mu(\hat{x})$.

Definition 7.2. *A region of stability of the fuzzy solution \hat{x} is*

$$R(\hat{x}) = \{\tilde{c} : \tilde{c}\hat{x} \preceq \tilde{c}x_i \forall i \in B, \tilde{c} \in \tilde{C}\}$$

where B is a set of basic indices of lower level fuzzy linear optimization problem (7.16).

Theorem 7.1. *The region of stability is a fuzzy polytope.*

Proof. In Definition 7.2 region of stability is defined as an intersection of a finite number of inequalities. Under the additional assumption that the feasible set \tilde{C} is a fuzzy polytope the region of stability is also a fuzzy polytope. \square

We recall now

Definition 7.3. *A connected set is a topological set that cannot be represented as a union of two or more disjoint nonempty open subsets.*

Corollary 7.1. *A region of stability is a convex connected set.*

Remark 7.1. *A region of stability is not a single point since the rows of the matrix A are linear independent. If a region of stability of some solution $x_0 \in X$ is a point, then the solution x_0 is nonstable.*

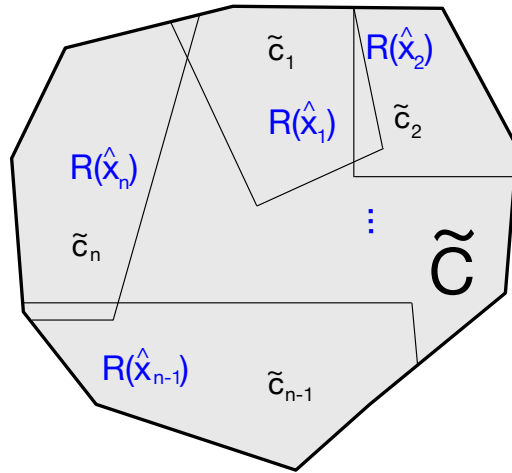


Fig. 7.1: n regions of stability cover the feasible set of the leader.

7.6 Algorithm II

In this Section we describe a solution algorithm for fuzzy bilinear bilevel optimization problem (7.15).

The idea of the algorithm is enumerative technique: we cover the feasible set of the leader with the regions of stability for the best lower level solutions under assumption, that regions of stability can be exactly calculated.

Algorithm.

Set $k := 1$. Fix a random fuzzy vector $\tilde{c}_k \in \tilde{C}$ and a level-cut $\alpha \in (0, 1)$.

STEP 1 For the fixed \tilde{c}_k and α compute a best solution $\hat{x}_k := \hat{x}(\tilde{c}_k)$ of the fuzzy lower level problem (7.16).

STEP 2 Find the region of stability $R(\hat{x}_k)$ using Definition 7.2.

STEP 3 Solve the upper level fuzzy optimization problem

$$F(\tilde{c}, \hat{x}_k) \rightarrow \min_{\tilde{c} \in R(\hat{x}_k)}. \quad (7.17)$$

Denote an optimal solution of problem (7.17) through \tilde{c}_k^* . Retain the pair $(\tilde{c}_k^*, \hat{x}_k)$.

STEP 4 $R := \tilde{C} \setminus R(\hat{x}_k)$. If $R = \emptyset$, then STOP. Else fix $\tilde{c}_{k+1} \in \text{int}(R)$ and go to STEP 1 with $k := k + 1$ (see Fig. 7.1).

STEP 5 Compare the pairs $(\tilde{c}_1^*, \hat{x}_1), \dots, (\tilde{c}_n^*, \hat{x}_n)$, where n is number of iterations, and choose (\tilde{c}^*, \hat{x}) - the best pair with respect to the upper level function $F(\tilde{c}, x)$.

If there exist two different pairs $(\tilde{c}_i^*, \hat{x}_i)$ and $(\tilde{c}_j^*, \hat{x}_j)$ such that $(\tilde{c}_i^*, \hat{x}_i) \neq (\tilde{c}_j^*, \hat{x}_j)$ with the same upper level function value $F(\tilde{c}_i^*, \hat{x}_i) = F(\tilde{c}_j^*, \hat{x}_j)$, we suggest to choose such a pair that has a best second component, i.e. if $\mu(\hat{x}_i) > \mu(\hat{x}_j)$, then the pair $(\tilde{c}_i^*, \hat{x}_i)$ is better (more preferable) than $(\tilde{c}_j^*, \hat{x}_j)$.

Remark 7.2. *As soon as the region of stability is a fuzzy polytope and the function $F(\tilde{c}, \hat{x}_k)$ with the fixed \hat{x}_k is linear on \tilde{c} , upper level optimization problem (7.17) can be solved using the solution method as described in Section 6.3.*

Theorem 7.2. *The algorithm is convergent.*

Proof. Consider lower level problem (7.16). The feasible set X has a finite number of vertices, and we are interested only in the basic solutions. Let the total number of basic solutions be N . It is clear that $N < \infty$. Thus, for each best solution on the lower level we obtain its region of stability. The total number of the regions of stability cannot exceed N . Taking into consideration Corollary 7.1 and Remark 7.1, the theorem is proved. \square

Theorem 7.3. *The pair (\tilde{c}^*, \hat{x}) is a global optimal solution of fuzzy bilevel programming problem (7.15).*

Proof. The proof of this fact is obvious with the rule of contraries. Suppose that there exist other global optimum, e.g. the pair (\tilde{c}_0, x_0) such that $F(\tilde{c}_0, x_0) < F(\tilde{c}^*, \hat{x})$. In consideration of STEP 4 and its STOP criteria the region of stability $R(x_0)$ of the solution x_0 of the lower level problem is considered in one of the iterations. But the assumption of the theorem is following. Since after all the comparisons at the STEP 5 of the algorithm the best solution is (\tilde{c}^*, \hat{x}) , consequently $F(\tilde{c}^*, \hat{x}) < F(\tilde{c}_0, x_0)$. This is contradiction to the assumption that (\tilde{c}_0, x_0) is a global optimal solution. \square

As soon as the fuzzy numbers are non-comparable, the way of the computation of the region of stability according to Definition 7.2 in some cases can be too complicated.

There exist few heuristic ways to overcome this problem, we suggest to compute the region of stability using Yager index within the following formula:

$$R_I(\hat{x}) = \{\tilde{c} : I(\tilde{c})\hat{x} \leq I(\tilde{c})x_i \forall i \in B, \tilde{c} \in \tilde{C}\}. \quad (7.18)$$

For the simplest case, that is presented in the next Section, we assume that the fuzzy numbers are presented by their triangular membership functions.

7.7 Example

To accomplish the discussion it is interesting to explain the results by giving a special example - the traffic problem discussed in Section 4.4.2 with additional crisp tolls in some paths and one more level for a new decision-maker (the leader).

Let the upper level objective function $F(c^{toll}, x) = (c^{toll})^T x$ measure the collected money through a traffic $G = (V, E)$, where $c^{toll} \in C_{toll}$ is a vector of the crisp toll charges and $x \in X$ is the traffic flow. We define $C_{toll} = \{c^{toll} : c^{toll} \in \mathbb{Z}_+^{|T|}, c^{toll} \leq \bar{c}\}$, where \bar{c} is a given upper bound and $T \subset E$ is a set of all toll roads.

Let the lower level objective function $\tilde{f}(c^{toll}, x)$ measure the quality of the flow x , which depends on the toll charges c^{toll} and the usual user's fuzzy costs \tilde{c} such as e.g. fuel. A possible realization could be $\tilde{f}(c^{toll}, x) = (c^{toll} + \tilde{c})^T x$.

Computing the system optimum in the traffic network means that we maximize the collected money for the leader as the upper level objective function value $F(c^{toll}, x)$ and minimize the total costs for the travel of all users (that we describe as one follower) in

the network as the lower level objective function value $\tilde{f}(c^{toll}, x)$. Clearly these costs do not only depend on the fuzzy costs \tilde{c} for traversing the edges but also on the collected toll charges

$$c_e^{toll} = \begin{cases} c_p^{toll}, & \text{if } p \in T \\ 0, & \text{if } p \notin T, \end{cases}$$

where $T \subset E$ is a set of all toll roads. Assume that in the network G there exists at least one toll-free path (i.e. C^{toll} is bounded).

Using this problem a user equilibrium traffic flow can be computed provided that the costs \tilde{c}_e for traversing the edge $e \in E$ are approximately known, i.e. have fuzzy values, and depend on the flow over this edge. Using fuzzy travel costs, the computation of the traffic flow reduces to a fuzzy network flow problem.

Let $x_e = x_{kl}$ denote the amount of transported units over the edge $e = (k, l) \in E$, that connects two vertices k and l ($k, l \in V$). Let O_k (I_k) denote the set of all edges leaving (entering) the node k .

Thus, we have a following bilevel fuzzy optimization problem:

$$F(c^{toll}, x) = \sum_{e \in T} c_e^{toll} x_e \rightarrow \max_{c_e^{toll} \in C^{toll}} \quad (7.19)$$

$$\text{s.t. } \tilde{f}(c^{toll}, x) = \sum_{e \in E} \sum_{e \in T} c_e^{toll} x_e + \tilde{c}_e x_e \rightarrow \min_x \quad (7.20)$$

$$x_e \leq u_e \quad \forall e \in E \quad (7.21)$$

$$\sum_{k \in I_l} x_{kl} - \sum_{i \in O_l} x_{li} = 0, \quad \forall l \in V \setminus \{s, d\} \quad (7.22)$$

$$\sum_{k \in I_s} x_{ks} - \sum_{i \in O_s} x_{si} = -v, \quad (7.23)$$

$$x_e \geq 0 \quad (7.24)$$

To pose (7.19)-(7.24) in a form of (7.16) and $\Psi(\tilde{c})$ is the optimal solution set of problem (7.20)-(7.24).

Description of a numerical example is the following:

$$F(c^{toll}, x) = c_{13}^{toll} x_{13} + c_{23}^{toll} x_{23} + c_{34}^{toll} x_{34} \rightarrow \max_{C=[0,5]^3}$$

$$\text{s.t. } \tilde{f}(c^{toll}, x) = \tilde{3}x_{12} + (\tilde{3} + c_{13}^{toll})x_{13} + (\tilde{3} + c_{23}^{toll})x_{23} + \tilde{7}x_{24} + (\tilde{3} + c_{34}^{toll})x_{34} \rightarrow \min_x$$

$$\begin{array}{ll} x_{12} + x_{13} = 90 & 0 \leq x_{12} \leq 90 \\ x_{24} + x_{34} = 90 & 0 \leq x_{13} \leq 90 \\ \text{with demand} & \text{and capacity } 0 \leq x_{23} \leq 60 \\ x_{12} = x_{23} + x_{24} & 0 \leq x_{24} \leq 30 \\ x_{13} + x_{23} = x_{34} & 0 \leq x_{34} \leq 90 \end{array}$$

The corresponding traffic network is illustrated in Fig. 7.2.

Let here fuzzy numbers be defined as continuous triangular fuzzy numbers: $\tilde{3} = (1, 3, 5)$ and $\tilde{7} = (5, 7, 9)$.

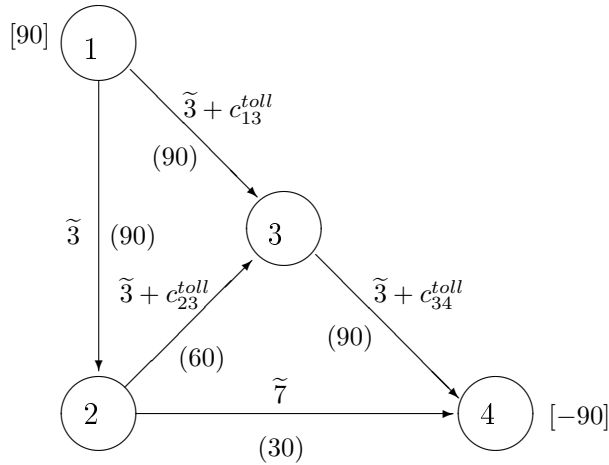


Fig. 7.2: The example of the traffic network.

Now let us implement **Algorithm I** for this example.

STEP 1. Fix the vector of toll parameters $c^1 := (c_{13}^{1toll}, c_{23}^{1toll}, c_{34}^{1toll}) = (5, 4, 0)$ and compute an optimal solution of the lower level problem with the following objective function

$$\tilde{f}(c^1, x) = \tilde{3}x_{12} + \tilde{8}x_{13} + \tilde{7}x_{23} + \tilde{7}x_{24} + \tilde{3}x_{34}, \quad (7.25)$$

where $\tilde{8} = (6, 8, 10)$.

Using Definition 7.1 or property from Yager (1981) for normal symmetric fuzzy numbers we compute Yager-indices: $I(\tilde{3}) = 3$, $I(\tilde{7}) = 7$ and $I(\tilde{8}) = 8$. Now the fuzzy lower level problem (7.25) states as a crisp linear optimization problem

$$\tilde{f}(c^1, x) = 3x_{12} + 8x_{13} + 7x_{23} + 7x_{24} + 3x_{34} \quad (7.26)$$

with the same demands and constraints, that has as an optimal solution a vector $x^* = (30, 60, 0, 30, 60)$.

STEP 2. Now with the fixed solution x^* of the lower level problem (7.26) we solve the following upper level problem:

$$F(c^{toll}, x^*) = c_{13}^{toll}60 + c_{23}^{toll}0 + c_{34}^{toll}60 \rightarrow \max_{c^{toll} \in [0, 5]^3}, \quad (7.27)$$

where an optimal solution is $c^* = (5, 4, 5)$ ($c_{13}^* := c_{13}^{1toll}$).

STEP 3. Now solving again for the lower level problem with fixed c^*

$$\tilde{f}(c^*, x) = 3x_{12} + 8x_{13} + 8x_{23} + 7x_{24} + 8x_{34}.$$

It is easy to see that as soon as a solution x^* remains to be optimal, the pair (c^*, x^*) is an optimal solution of the initial fuzzy bilevel programming problem.

Let us solve this traffic assignment problem according to the **Algorithm II**. As we will see later, the investigated problem has on the lower level has three basic solutions, namely, $\hat{x}_1 = (30, 60, 0, 30, 60)$, $\hat{x}_2 = (0, 90, 0, 0, 90)$ and $\hat{x}_3 = (90, 0, 60, 30, 60)$.

Fix some toll parameters $c_1^{toll} = (c_{13}^{toll}, c_{23}^{toll}, c_{34}^{toll}) := (5, 4, 0) \in C^{toll}$.

STEP 1 (Iteration 1). For this fixed parameter we compute an optimal solution of the lower level problem with the following objective function

$$\tilde{f}(c_1^{toll}, x) = \tilde{3}x_{12} + \tilde{8}x_{13} + \tilde{7}x_{23} + \tilde{7}x_{24} + \tilde{3}x_{34},$$

where $\tilde{8} = (6, 8, 10)$. As soon as the solution \hat{x}_1 has a maximal membership function value, we take it for the further consideration.

STEP 2. For this solution we compute its region of stability. The solution \hat{x}_1 is an optimal one until

$$\begin{aligned} c^{toll} \in R(\hat{x}_1) &= \{c^{toll} : c^{toll} \in C^{toll}, \tilde{f}(c^{toll}, \hat{x}_1) \preceq \tilde{f}(c^{toll}, \hat{x}_2), \tilde{f}(c^{toll}, \hat{x}_1) \preceq \tilde{f}(c^{toll}, \hat{x}_3)\} = \\ &= \{c^{toll} : c^{toll} \in C^{toll}, \tilde{3} \cdot 30 + (\tilde{3} + c_{13}^{toll})60 + \tilde{7} \cdot 30 + (\tilde{3} + c_{34}^{toll})60 \preceq (\tilde{3} + c_{13}^{toll})90 + (\tilde{3} + c_{34}^{toll})90, \\ &\tilde{3} \cdot 30 + (\tilde{3} + c_{13}^{toll})60 + \tilde{7} \cdot 30 + (\tilde{3} + c_{34}^{toll})60 \preceq \tilde{3} \cdot 90 + (\tilde{3} + c_{23}^{toll})60 + \tilde{7} \cdot 30 + (\tilde{3} + c_{34}^{toll})60\} = \\ &= \{c^{toll} : c^{toll} \in C^{toll}, \tilde{7} - \tilde{3} \preceq c_{34}^{toll} + c_{13}^{toll}, c_{13}^{toll} \preceq \tilde{3} + c_{23}^{toll}\} = \\ &= \{c^{toll} : c^{toll} \in C^{toll}, 4 - c_{34}^{toll} \leq c_{13}^{toll} \leq \tilde{3} + c_{23}^{toll}\}. \end{aligned}$$

STEP 3. Now we solve the following upper level problem:

$$F(c^{toll}, \hat{x}_1) = c_{13}^{toll}60 + c_{23}^{toll}0 + c_{34}^{toll}60 \rightarrow \max_{c^{toll} \in R(\hat{x}_1)},$$

where the optimal solution is $\hat{c}_1^{toll} = (5, 4, 5)$.

The pair $(\hat{c}_1^{toll}, \hat{x}_1)$ is a local optimal solution of the initial fuzzy bilevel programming problem. That means, that this solution is optimal within a neighbouring set of solutions, namely in $R(\hat{x}_1) \times X$ with the upper level function value equal to 600.

STEP 4. According to the algorithm, as soon as $R := C^{toll} \setminus R(\hat{x}_1) = \{c^{toll} : c^{toll} \in C^{toll}, 4 - c_{34}^{toll} \geq c_{13}^{toll}\} \cup \{c^{toll} : c^{toll} \in C^{toll}, c_{13}^{toll} \succeq \tilde{3} + c_{23}^{toll}\} \neq \emptyset$, we select another vector in R , e.g. $c_2^{toll} := (1, 5, 2)$ and go to STEP 1.

STEP 1 (Iteration 2). For this c_2^{toll} we have a different lower level problem with the following objective function

$$\tilde{f}(c_2^{toll}, x) = \tilde{3}x_{12} + \tilde{4}x_{13} + \tilde{8}x_{23} + \tilde{7}x_{24} + \tilde{5}x_{34},$$

where $\tilde{4} = (2, 4, 6)$ and $\tilde{5} = (3, 5, 7)$. A solution with a maximal membership function value is $\hat{x}_2 = (0, 90, 0, 0, 90)$.

STEP 2. A region of stability of this solution is

$$R(\hat{x}_2) = \{c^{toll} : c^{toll} \in C^{toll}, 4 - c_{34}^{toll} \geq c_{13}^{toll}\} \cap \{c^{toll} : c^{toll} \in C^{toll}, \tilde{3} + \tilde{7} + 2c_{23}^{toll} \succeq 3c_{13}^{toll} + c_{34}^{toll}\}.$$

STEP 3. Solve now the following upper level problem:

$$F(c^{toll}, \hat{x}_2) = c_{13}^{toll}90 + c_{23}^{toll}0 + c_{34}^{toll}90 \rightarrow \max_{c^{toll} \in R(\hat{x}_2)},$$

that has an optimal solution $\hat{c}_2^{toll} = (1, 5, 4)$.

The pair $(\hat{c}_2^{toll}, \hat{x}_2)$ is a local optimal solution of the initial fuzzy bilevel programming problem with the upper level function value equal to 450.

STEP 4. As soon as $R := R \setminus R(\hat{x}_2) \neq \emptyset$, we select another vector in R , e.g. $c_3^{toll} := (5, 0, 3)$ and go to STEP 1.

STEP 1 (Iteration 3). Then the objective function on the lower level is

$$\tilde{f}(c_3^{toll}, x) = \tilde{3}x_{12} + \tilde{8}x_{13} + \tilde{3}x_{23} + \tilde{7}x_{24} + \tilde{6}x_{34},$$

where $\tilde{6} = (4, 6, 8)$. A solution $\hat{x}_3 = (90, 0, 60, 30, 60)$ has a maximal membership function value.

STEP 2. Analysing sensitivity of \hat{x}_3 we obtain that the region of stability for this solution is

$$R(\hat{x}_3) = \{c^{toll} : c^{toll} \in C^{toll}, c_{13}^{toll} \succeq \tilde{3} + c_{23}^{toll}\} \cap \{c^{toll} : c^{toll} \in C^{toll}, \tilde{3} + \tilde{7} + 2c_{23}^{toll} \preceq 3c_{13}^{toll} + c_{34}^{toll}\}.$$

STEP 3. It is easy to obtain that the problem on the upper level

$$F(c^{toll}, \hat{x}_3) = c_{13}^{toll}0 + c_{23}^{toll}60 + c_{34}^{toll}60 \rightarrow \max_{c^{toll} \in R(\hat{x}_3)}$$

has an optimal solution $\hat{c}_3^{toll} = (5, 0, 5)$.

The pair $(\hat{c}_3^{toll}, \hat{x}_3)$ is a local optimal solution of the initial fuzzy bilevel programming problem with the upper level function value equal to 300.

STEP 4. As soon as now we covered all the leader's feasible set with the regions of stability, i.e. $R := R \setminus R(\hat{x}_3) = \emptyset$, we STOP.

Now we have to compare the pairs $(\hat{c}_1^{toll}, \hat{x}_1)$, $(\hat{c}_2^{toll}, \hat{x}_2)$ and $(\hat{c}_3^{toll}, \hat{x}_3)$ with respect to the upper level function $F(c^{toll}, x)$. Thus, the pair $(\hat{c}_1^{toll}, \hat{x}_1)$ with the upper level function value equal to 600 is a global optimal solution of the initial problem.

It is easy to see that optimal solution x^* , obtained using the Yager index approach, matches with global optimal solution x_1 , obtained with the use of the membership function approach and, of course, upper level function values coincide (i.e. $F(c^*, x^*) = F(\hat{c}_1^{toll}, \hat{x}_1)$). This is coincidence: If another first vector of toll parameters c^1 is chosen, application of Yager index approach may lead to any local optimal solution, e.g. $(\hat{c}_2^{toll}, \hat{x}_2)$ or $(\hat{c}_3^{toll}, \hat{x}_3)$.

Comparing two algorithms it is interesting to note, that Algorithm I is noncomplicated in its implementation. However, it provides us with only local optimal solution. Algorithm II has to run as many times, as many basic solutions has the lower-level problem. But at the end a global optimal solution is obtained. To be more precise, all crucial solutions are calculated and the global optimal solution is chosen to provide the leader with the best solution.

There exist an opportunity to improve the Algorithm I with the same idea, used in Algorithm II. On STEP 3, using formula (7.28), it is possible calculate the region of stability of solution x^* as

$$R_I(x^*) = \{\tilde{c} : I(\tilde{c})\hat{x} \leq I(\tilde{c})x_i \forall i, \tilde{c} \in \mathfrak{F}^n\}, \quad (7.28)$$

and then repeat the Algorithm for some $I(\tilde{c}) \in \mathfrak{F}^n \setminus R_I(x^*)$ and so on, until we cover all the feasible set \mathfrak{F}^n .

8 Linear fuzzy bilevel optimization (with fuzzy objectives and constraints)

The natural extension of a bilevel optimization problem with fuzzy objective functions and crisp constraints is a bilevel optimization problem with fuzzy objective functions and fuzzy constraints. In this Chapter we focus our attention on the last problem for the linear case.

We use the selection function approach and a modified version of k -th best algorithm to solve the linear fuzzy bilevel optimization problem. The optimal solution is obtained as a subset of a vertex of the decision space \mathbf{P} (see below). For the auxiliary fuzzy linear optimization problem the solution approach proposed in Chapter 6 can be applied. It is easily seen that an optimal solution of this fuzzy linear optimization problem is also an optimal solution of two linear optimization problems obtained via application of the α -cut method.

This Chapter is organized as follows: In Section 8.1 we give a formulation of the fuzzy bilevel optimization problem.

Section 8.2 deals with the solution approach for fuzzy linear optimization problem over a fuzzy polytope.

The solution algorithm for the fuzzy bilevel optimization problem is described in Section 8.3.

Finally, in Section 8.4 an example is given.

8.1 Formulation

Let \mathfrak{F}^n be a space of fuzzy vectors over \mathbb{R}^n and \mathfrak{P} be a fuzzy polytope in \mathfrak{F}^n . Let us denote through P a crisp polytope in \mathbb{R}^m . And let \mathbf{P} be a Cartesian product of these two fuzzy and crisp polytopes \mathfrak{P} and P , respectively, in $\mathfrak{F}^n \times \mathbb{R}^m$. Thus, the decision space \mathbf{P} can be considered as a fuzzy polytope.

Bilevel programming involves two optimization problems where the constraint region of the upper level problem is implicitly determined by another optimization problem on the lower level.

Let the leader make a first choice - select a fuzzy solution $\tilde{c} \in \mathfrak{P}$. The follower optimizes his / her objective function based on the parameters prescribed by the leader, i.e. the follower's task is to solve the problem

$$\begin{aligned} f(\tilde{c}, x) = p_1^\top \tilde{c} + p_2^\top x &\rightarrow \min_x \\ \text{s.t. } (\tilde{c}, x) &\in \mathbf{P} \end{aligned} \quad (8.1)$$

where x is an n -dimensional vector of decision variables, \tilde{c} is a fixed fuzzy vector, $p_1 \in \mathbb{R}^n$ and $p_2 \in \mathbb{R}^m$ are also fixed.

To make our reasoning clear, let us make the following

Assumption 8.1. *Let us assume that*

- *The feasible region defined by \mathbf{P} is bounded and each basis is nondegenerate.*
- *The polytope \mathbf{P} has a finite number of vertices.*

Definition 8.1. *A set of optimal solutions of fuzzy optimization problem (8.1) is*

$$\Psi(\tilde{c}) = \arg \min_x \{p_1^\top \tilde{c} + p_2^\top x : (\tilde{c}, x) \in \mathbf{P}\}. \quad (8.2)$$

$\Psi(\tilde{c})$ is also called the follower's rational set.

In turn, having complete information on the possible reactions of the follower, the leader selects the parameters to optimize his / her own objective function. Thus, the linear fuzzy bilevel programming problem can be stated as

$$\begin{aligned} F(\tilde{c}, x) = d_1^\top \tilde{c} + d_2^\top x &\rightarrow \min_{\tilde{c} \in \mathfrak{F}} \\ \text{s.t. } x &\in \Psi(\tilde{c}). \end{aligned} \quad (8.3)$$

8.2 Solution approach

In this Section we describe a k -th best algorithm for fuzzy linear bilevel optimization problem (8.3).

Proposition 8.1. *An optimal solution to fuzzy bilevel optimization problem (8.3) (if one exist) is a subset of the extreme point of the decision space \mathbf{P} .*

Proof. In Chapter 4 it is shown that only an extreme points of the crisp polytope P can be optimal solutions of the lower level optimization problem.

As soon as the upper level optimization problem is linear, Theorem 6.2 can be applied. Thus, an optimal solution of problem (8.3) can only be located in the extreme point of the decision space \mathbf{P} . □

Solution algorithm for fuzzy bilevel programming problem (8.3) is as follows.

Let us consider fuzzy single-level optimization problem

$$\begin{aligned} F(\tilde{c}, x) &\rightarrow \min_{(\tilde{c}, x)} \\ x &\in P \\ \tilde{c} &\in \mathfrak{F}. \end{aligned} \quad (8.4)$$

Note, that problem (8.4) is a fuzzy linear optimization problem with fuzzy objective and fuzzy constraints. It can be solved with the technique described in Chapter 6 if only we consider (\tilde{c}, x) as a fuzzy variable and \mathbf{P} as a fuzzy polytope. Thus, its level-cut can

be decomposed into two crisp polytopes $\mathbf{P}_L(\alpha)(= \mathfrak{P}_L(\alpha) \times P)$ and $\mathbf{P}_R(\alpha)(= \mathfrak{P}_R(\alpha) \times P)$ for $\alpha \in [0, 1]$.

As shown earlier, a fuzzy solution of problem (8.4) is a union of the convex hulls of solutions of problems

$$\begin{aligned} F(c_L(\alpha), x) &\rightarrow \min_{(c_L(\alpha), x)} \\ (c_L(\alpha), x) &\in \mathbf{P}_L(\alpha) \end{aligned} \quad (8.5)$$

and

$$\begin{aligned} F(c_R(\alpha), x) &\rightarrow \min_{(c_R(\alpha), x)} \\ (c_R(\alpha), x) &\in \mathbf{P}_R(\alpha) \end{aligned} \quad (8.6)$$

for all $\alpha \in [0, 1]$. It is clear, that for calculations we have to choose particular α -cuts.

Following Remark 6.6, we have to compute a maximum convex hull of problems (8.5) and (8.6). Since the maximal convex hull is obtained for $\alpha = 0$, this α -cut has to be necessarily considered.

Note that, the strongest solution (see Definition 4.5) can only be obtained, if we consider the level-cut for $\alpha = 1$.

For a set of level-cuts $A = \{\alpha_1, \dots, \alpha_T\}$ ($0 = \alpha_1 < \dots < \alpha_T = 1$) we have to compute the vertices of feasible sets of these problems: $\mathbf{P}_L(\alpha)$ and $\mathbf{P}_R(\alpha)$. All further discussions are made for a fixed $\alpha := \alpha_t \in A$.

We sort all vertices in ascending order with respect to the value of the fuzzy objective function. The ordered set of all vertices of $\mathbf{P}_L(\alpha)$ and $\mathbf{P}_R(\alpha)$ is called S^α . Let us denote the first vertex in the set S^α as (c_1^α, x_1^α) . Obviously, this is an optimal solution of either of the problem (8.5) or problem (8.6).

Consider the lower level optimization problem for the fixed parameter c_1^α :

$$\begin{aligned} f(c_1^\alpha, x) &\rightarrow \min_x \\ x &\in P. \end{aligned} \quad (8.7)$$

This problem is a crisp linear optimization problem, which can be easily solved.

If its optimal solution is x_1^α , STOP with optimal solution (c_1^α, x_1^α) . Else go to the next vertex (c_k^α, x_k^α) (here $k = 2$) in the ordered set S^α and solve the lower level problem again for the new parameter c_k^α :

$$\begin{aligned} f(c_k^\alpha, x) &\rightarrow \min_x \\ x &\in P. \end{aligned} \quad (8.8)$$

Repeat the algorithm, until

$$\arg \min_x \{f(c_k^\alpha, x)\} = x_k^\alpha, \quad (8.9)$$

i.e. the optimal solution (c^α, x^α) is found.

Repeat the algorithm for all chosen α -cuts and compute a fuzzy optimal solution according to formula (6.10) as a convex hull of all solutions found for all chosen α -cuts

$$(\tilde{c}^*, x^*) = \bigcup_{\alpha \in [0, 1]} \text{conv}\{(c^\alpha, x^\alpha)\}. \quad (8.10)$$

As soon as we deal with a *fuzzy* bilevel optimization problem, we have to endow the fuzzy solution with its membership function.

As it is described in Section 4.4, in a vast majority of cases we do not need the membership function itself, but only membership function values in certain points. They can be calculated with Definition 6.12. Membership function values provide enough information to rank every point $(c_*, x_*) \in (\tilde{c}^*, x^*)$. According to Definition 4.4, we call the point with a maximal membership function value is the best optimal solution.

8.3 Algorithm

Set $t := 1$.

STEP 1 Choose a set of α -cuts $A = \{\alpha_1, \dots, \alpha_T\}$. Initialize $\alpha := \alpha_t \in A$. Consider fuzzy linear (single-level) optimization problem (8.4).

STEP 2 For a fixed α split problem (8.4) into two crisp problems (8.5) and (8.6). Compute the vertices of feasible sets of these problems.

STEP 3 Sort all the vertices of polytopes $\mathbf{P}_L(\alpha)$ and $\mathbf{P}_R(\alpha)$ in an ascending order with respect to the value of the fuzzy objective function in an ordered set $S^\alpha = \cup_k \{(c_k^\alpha, x_k^\alpha)\}$.

STEP 4 Set $k := 1$.

STEP 5 Solve the lower level fuzzy linear optimization problem (8.8) for c_k^α .

STEP 6 If its solution $x^\alpha \neq x_k^\alpha$, go to STEP 4 for $k := k + 1$, i.e. repeat STEP 4 for the next vertex in the set S^α .

STEP 7 Save the pair $(c^\alpha, x^\alpha) := (c_k^\alpha, x_k^\alpha)$. Set $\alpha := \alpha_{t+1}$ and go to STEP 2.

STEP 8 The optimal solution of the initial fuzzy bilevel optimization problem (8.3) is computed from formula (8.10) (\tilde{c}^*, x^*) .

- If the aim is to provide the leader with some crisp solution c_* :

STEP 8a Find the membership function values of all crucial points of fuzzy solution \tilde{c}^* . Choose the best solution c_* .

- If the task consist in computing the best optimal solution x_* for the follower:

STEP 8b For the optimal \tilde{c}^* solve the lower level problem

$$f(\tilde{c}^*, x) \rightarrow \min_x \quad (8.11)$$

$$x \in P.$$

and use the method described in Chapter 4.

Theorem 8.1. *The algorithm finds an optimal solution.*

Proof. This follows immediately from Proposition 8.1. □

Theorem 8.2. *The algorithm is convergent.*

Proof. Convergence is established by noting Assumption 8.1. □

8.4 Example

Let us consider a following simple linear fuzzy bilevel optimization problem

$$\begin{aligned} F(\tilde{c}, x) &= \tilde{c} + 5x \rightarrow \min_{\tilde{c}} \\ 0 &\preceq \tilde{c} \preceq \tilde{4} \end{aligned} \quad (8.12)$$

where x solves

$$\begin{aligned} f(\tilde{c}, x) &= x + 2 \rightarrow \max_x \\ \text{s.t. } &-x + 2\tilde{c} \preceq 4 \\ &2x + \tilde{c} \preceq 16 \\ &x - \tilde{c} \preceq 6 \\ &x \geq 0, \end{aligned} \quad (8.13)$$

where $\tilde{4} = (3, 4, 5)$. The aim is to find the best optimal solution c^* for the leader. Let us assume that the fuzzy variable \tilde{c} is presented as a normalized fuzzy number.

Using the Algorithm we obtain the following:

STEP 1 (Iteration 1.1). We initialize the set $A := \{0, 1\}$. We consider the fuzzy linear (single-level) optimization problem

$$\begin{aligned} F(\tilde{c}, x) &= \tilde{c} + 5x \rightarrow \min \\ \text{s.t. } &0 \preceq \tilde{c} \preceq \tilde{4} \\ &-x + 2\tilde{c} \preceq 4 \\ &2x + \tilde{c} \preceq 16 \\ &x - \tilde{c} \preceq 6 \\ &x \geq 0. \end{aligned} \quad (8.14)$$

STEP 2 (Iteration 1.1). Solving problem (8.14) for $\alpha = 0$ we obtain the following two optimization problems:

$$\begin{aligned} F(c_L, x) &= c_L + 5x \rightarrow \min \\ \text{s.t. } &0 \leq c_L \leq 3 \\ &-x + 2c_L \leq 4 \\ &2x + c_L \leq 16 \\ &x - c_L \leq 6 \\ &x \geq 0 \end{aligned} \quad (8.15)$$

and

$$\begin{aligned} F(c_R, x) &= c_R + 5x \rightarrow \min \\ \text{s.t. } &0 \leq c_R \leq 5 \\ &-x + 2c_R \leq 4 \\ &2x + c_R \leq 16 \\ &x - c_R \leq 6 \\ &x \geq 0. \end{aligned} \quad (8.16)$$

STEP 3 (Iteration 1.1). Sorting all solutions in ascending order with respect to the value of the fuzzy objective functions we obtain the ordered set

$$S^0 = \{(0, 0), (2, 0), (3, 2), (0, 6), (4.8, 5.6), (3, 6.5), (1.3, 7.3)\}.$$

The first vertex in the set S^0 is $(c_1^0, x_1^0) = (0, 0)$.

STEP 4 (Iteration 1.1). We have to solve now the lower level fuzzy linear optimization problem (8.13) for $c_1^0 = 0$:

$$f(0, x) = x + 2 \rightarrow \max_{x \in [0, 6]} . \quad (8.17)$$

The optimal solution of (8.17) is $x^0 = 6$. As soon as $x^0 \neq x_1^0$, we have to go to the neighbouring vertex in the set S^0 . Namely, to $(c_2^0, x_2^0) = (2, 0)$.

STEP 4 (Iteration 1.2). We have to solve lower level optimization problem (8.13) again for $c_2^0 = 2$:

$$f(2, x) = x + 2 \rightarrow \max_{x \in [0, 7]} . \quad (8.18)$$

The optimal solution of problem (8.18) is $x^0 = 7$. Clear, that $x^0 \neq x_2^0$ and we go to an successive vertex in S^0 . Namely to $(c_3^0, x_3^0) = (3, 2)$.

STEP 4 (Iteration 1.3). For $c_3^0 = 3$ we solve following optimization problem:

$$f(3, x) = x + 2 \rightarrow \max_{x \in [2, 6.5]} . \quad (8.19)$$

The optimal solution of (8.19) is $x^0 = 6.5$. As soon as $x^0 \neq x_3^0$, we have to consider next vertex $(c_4^0, x_4^0) = (0, 6)$.

STEP 4 (Iteration 1.4). Now we have to solve problem (8.17) again. Its optimal solution, as expected, is $x^0 = x_4^0$.

STEP 5 (Iteration 1.4). Optimal solution $(c^0, x^0) = (0, 6)$. We go to STEP 2 with $\alpha = 1$.

STEP 2 (Iteration 2.1). Problem (8.14) for $\alpha = 1$ is formulated as

$$\begin{aligned} F(c, x) = c + 5x &\rightarrow \min \\ \text{s.t. } 0 &\leq c \leq 4 \\ -x + 2c &\leq 4 \\ 2x + c &\leq 16 \\ x - c &\leq 6 \\ x &\geq 0. \end{aligned} \quad (8.20)$$

STEP 3 (Iteration 2.1). Solving problem (8.20) and sorting its solutions in asserting order we obtain the set

$$S^1 = \{(0, 0), (2, 0), (4, 4), (0, 6), (4, 6), (1.3, 7.3)\}.$$

STEP 4 (Iteration 2.1). For the first vertex $(c_1^1, x_1^1) = (0, 0)$ we consider lower level fuzzy linear optimization problem (8.17). As soon as its optimal solution $x^1 \neq x_1^1$, we go to the next vertex $(c_2^1, x_2^1) = (2, 0)$.

STEP 4 (Iteration 2.2). For $c_2^1 = 2$ lower level problem states as (8.18). Its optimal solution $x^1 \neq x_2^1$ and we consider the next vertex $(c_3^1, x_3^1) = (4, 4)$.

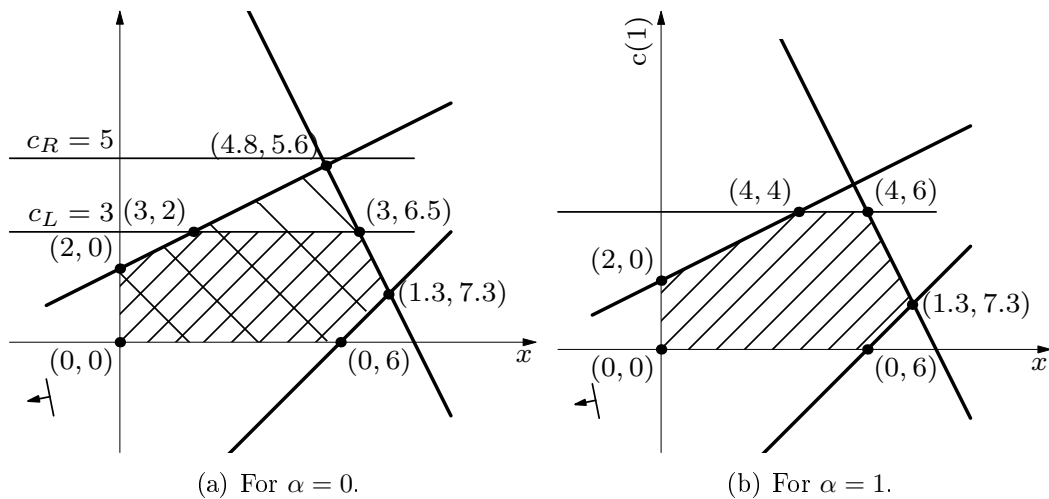


Fig. 8.1: Different feasible sets are obtained for different α -cuts.

STEP 4 (Iteration 2.3). For $c_3^1 = 4$ lower level optimization problem is

$$f(4, x) = x + 2 \rightarrow \max_{x \in [4, 6]} . \quad (8.21)$$

An optimal solution is $c^1 = 6$. It is easy to see, that $c^1 \neq c_3^1$. We consider now the next in the set S^1 vertex $(c_4^1, x_4^1) = (0, 6)$.

STEP 4 (Iteration 2.4). For $c_4^1 = 0$ lower level problem states as (8.17). Its optimal solution $x^1 = x_4^1$.

STEP 5 (Iteration 2.4). An optimal solution is $(c^1, x^1) = (0, 6)$.

Now it is clear, that solution $(c^*, x^*) = (0, 6)$ is valid for all α -cuts (for all $\alpha \in [0, 1]$) and, thus, is a best optimal solution.

9 Conclusions

In the dissertation the solution approaches for different fuzzy optimization problems are presented.

The single-level optimization problem with fuzzy objective function is solved by its reformulation into a related biobjective optimization problem. This problem, in turn, is solved by methods of the multiobjective optimization problem's scalarization technique. Elements of the Pareto set of the corresponding biobjective optimization problem are interpreted as optimal solutions of the initial optimization problem with fuzzy objective. It is also discussed, that the set of the optimal solutions depends on scalarization parameters as soon as each different single objective optimization problem can determine a different solution set.

A special attention is given to the computation of the membership function of the fuzzy solution of the fuzzy optimization problem in the linear case. Knowledge of the membership function values of the elements of the set of fuzzy optimal solutions enables the decision-maker to make an *educated* choice between these solutions. Moreover, using our approach, a decision-maker can see a correlation among solutions and quantitatively measure the advantage of his / her choice over other solutions.

The membership function value of such a solution equals to the geometric measure of all α such that this solution is Pareto optimal for the corresponding biobjective optimization problem. Explicit formulas for computing this membership function value are given. The theory for continuous triangular fuzzy numbers could be extended to the general LR-numbers. For this, formulas for computing left- and right-hand side functions $c_L(\alpha)$, $c_R(\alpha)$ need to be used (see Chanas (1989); Chanas and Kuchta (1994)). But in general case these functions are no longer linear with respect to α . Therefore, the computation of $z_i^+(\lambda)$ and $z_i^-(\lambda)$ is cumbersome. Moreover, under convexity assumptions the discussions could be extended to nonlinear optimization problems with fuzzy objective function. This leads to a more complicated formula for the membership function.

Further, it is also discussed that, necessary and sufficient conditions for the optimal solution of the the convex nonlinear optimization problem with fuzzy objective function can be explained through the necessary and sufficient conditions for the Pareto optimal solution of the corresponding biobjective optimization problem (and for solutions of its scalarized problem). Optimality conditions for differentiable fuzzy optimization problems have a form of Karush-Kuhn-Tucker optimality conditions.

Moreover, using the Hadamard upper and lower directional α -derivatives, necessary and sufficient conditions for local / global optimality of the nondifferentiable nonconvex optimization problem with fuzzy objective function are derived.

A fuzzy optimization problem (with both fuzzy objectives and constraints) is also investigated in the thesis in the linear case. A solution approach is based on the notion of the fuzzy polytope. The approach is also based on taking level-cuts. Thus, for each α -cut the initial fuzzy linear optimization problem is splitted into two crisp linear optimization

problems. Then, considering all α -cuts a fuzzy optimal solution is found as a union of the convex hulls of corresponding optimal solutions.

To simplify a crisp choice of the decision-maker, for all crucial points of the fuzzy solution, corresponding membership function values can be found. This makes a choice of the decision-maker well-grounded.

This approach is quite innovative one and we are trying to extend it to nonlinear fuzzy optimization problems in our coming work.

Our solution approach can be applied to a class of more complicated problems, namely fuzzy bilevel optimization problems. In the present work two main cases of fuzzy bilevel optimization problem are discussed:

1. bilinear optimization problems with fuzzy objective functions;
2. linear optimization problems with fuzzy objective functions and fuzzy constraints.

In the case of bilevel optimization problem with fuzzy objective functions (see point 1.), two algorithms are presented and compared using an illustrative example, that represents a real-world problem.

The first algorithm we call Yager index approach. Its main ideas are stated in the following. At the lower level the fuzzy optimization problem is solved by the index ranking technique. This solution is then taken to find an optimal solution on the upper level.

It is clear that this algorithm can be extended to convex continuous problems without any difficulty. The problem is stated as bilinear to make the comparison with the next algorithm more transparent.

The second algorithm is based on the membership function approach. The lower level fuzzy optimization problem is solved by methods of the scalarization technique. Elements of the Pareto set of each biobjective optimization problem are interpreted as potential optimal solutions of the lower level fuzzy optimization problem on certain level-cuts. The optimal solution is selected due to the highest membership function value. Then, this solution is used on the upper level such that, with response to its region of stability, the optimal solution of the leader is found. Comparing all optimal solutions with respect to the upper level function value, the optimal solution of the fuzzy bilevel optimization problem is found. Moreover, this solution is shown to be a global optimal solution.

Both methods are illustrated using the example of the traffic problem with given fuzzy costs. In this particular example as the leader we can see a government agency that can use tolls to motivate the users of the network (the drivers) to avoid certain regions, such as UNESCO world heritage sites on the path ($2 \rightarrow 3$). On the other hand, using the toll policy it is also possible to force the users to choose the certain path (e.g. $1 \rightarrow 2 \rightarrow 4, 1 \rightarrow 3 \rightarrow 4$).

For the case of fuzzy linear bilevel optimization problem with both fuzzy objectives and constraints (see point 2.) we adopt the k -th best algorithm. This algorithm is based on the solution approach for the fuzzy linear optimization problem with fuzzy objective and fuzzy constraints. The adopted k -th best algorithm provides us with an optimal solution. Membership function values can be calculated for the solutions and for solutions on both levels.

Following this line of thought, some suggestions can be made for future research:

1. The extension of the procedure for calculating the membership function value to nonlinear fuzzy optimization problems (including the case of fuzzy constraints).
2. The extension of the solution approach for problems with fuzzy objective and fuzzy constraints for the nonlinear case.
3. The development of necessary and sufficient optimality conditions for the problem in point 2. (in differentiable and nondifferentiable cases).
4. The development of a solution algorithm for fuzzy bilevel optimization problem in nonlinear case, that provides the decision-maker with the membership function values of the possible solutions.

Bibliography

- Ammar, E. I., 2000. Interactive stability of multiobjective NLP problems with fuzzy parameters in the objective functions and constraints. *Fuzzy Sets and Systems* 109, 83–90.
- Arora, S., Gupta, R., 2009. Interactive fuzzy goal programming approach for bilevel programming problem. *European Journal of Operational Research* 194, 368 – 376.
- Audet, C., Savard, G., Zghal, W., 2008. Multiobjective optimization through a series of single-objective formulations. *SIAM Journal on Optimization* 19, 188–210.
- Avis, D., Fukuda, K., 1992. A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra. Tech. Rep. B-237, Tokyo Institute of Technology.
- Banks, H., Jacobs, M., 1970. A differential calculus for multifunctions. *Journal of Mathematical Analysis and Application* 29, 246–272.
- Bard, J., 1982. A grid search algorithm for the linear bilevel programming problem. In: *Proceedings of the 14th Annual Meeting of the American Institute for Decision Science*. pp. 256–258.
- Bard, J., 1983. An efficient point algorithm for a linear two-stage optimization problem. *Operational Research* 31, 670–684.
- Bard, J., Moore, J., 1990. A branch and bound algorithm for the bilevel programming problem. *SIAM Journal on Scientific and Statistical Computing* 11, 281–292.
- Bard, J., Plummer, J., Sourie, J., 1998. Determining tax credits for converting nonfood crops to biofuels: an application of bilevel programming. In: Migdalas, A., Pardalos, P., Värbrand, P. (Eds.), *Multilevel Optimization: Algorithms and Applications*. Kluwer Academic Publishers, Dordrecht, pp. 23–50.
- Bellman, R., Zadeh, L., 1970. Decision making in a fuzzy environment. *Management Science* 17, 141–164.
- Ben-Ayed, O., 1988. Bilevel linear programming: analysis and application to the network design problem. Ph.D. thesis, University of Illinois at Urbana-Champaign.
- Ben-Ayed, O., Blair, C., 1990. Computational difficulties of bilevel linear programming. *Operations Research* 38, 556–560.
- Ben-Ayed, O., Blair, C., Boyce, D., LeBlanc, L., 1992. Construction of a real-world bilevel linear programming model of the highway design problem. *Annals of Operations Research* 34, 219–254.

- Bertsimas, D., Tsitsiklis, J., 1997. *Introduction to Linear Optimization*. Athena Scientific, Massachusetts.
- Bialas, W., Karwan, M., 1978. Multilevel linear programming. Tech. Rep. 78-1, State University of New York, Buffalo.
- Bialas, W., Karwan, M., 1984. Two-level linear programming. *Management Science* 30, 1004–1020.
- Bialas, W., Karwan, M., Shaw, J., 1980. A parametric complementary pivot approach for two-level linear programming. Tech. Rep. 80-2, State University of New York, Buffalo.
- Bjørndal, M., Jørnsten, K., 2005. The deregulated electricity market viewed as a bilevel programming problem. *Journal of Global Optimization* 33, 465–475.
- Blair, C., 1992. The computational complexity of multi-level linear programs. *Annals of Operations Research* 34, 13–19.
- Bomze, I., Demyanov, V., Fletcher, R., Terlaky, T., 2010. *Nonlinear Optimization (Lecture notes in mathematics Vol. 1889)*. Springer Berlin / Heidelberg.
- Buckley, J., 1995. Joint solution to fuzzy programming problems. *Fuzzy Sets and Systems* 72, 215–220.
- Buckley, J., Feuring, T., 2000. Fuzzy differential equations. *Fuzzy Sets and Systems* 110, 43–54.
- Cadenas, J., Verdegay, J., 2009. Towards a new strategy for solving fuzzy optimization problems. *Fuzzy Optimization and Decision Making* 8, 231–244.
- Calvete, H., Galé, C., Dempe, S., Lohse, S., 2011. Bilevel problems over polyhedra with extreme point optimal solutions. *Journal of Global Optimization* 236, 1–14.
- Camacho, F., 2006. Two examples of a bilevel toll setting problem. In: *Proceedings of the International Business and Economics Research Conference, Las Vegas*.
- Candler, W., 1988. A linear bilevel programming algorithm: a comment. *Computers and Operations Research* 15, 297–298.
- Candler, W., Fortuny-Amat, J., McCarl, B., 1981. The potential role of multilevel programming in agricultural economics. *American Journal of Agricultural Economics* 63, 521–531.
- Cassidy, R., Kirby, M., Raike, W., 1971. Efficient distribution of resources through three levels of government. *Management Science* 17, 462–473.
- Chanas, S., 1983. The use of parametric programming in fuzzy linear programming. *Fuzzy Sets and Systems* 11, 243–251.
- Chanas, S., 1989. Fuzzy programming in multiobjective linear programming – a parametric approach. *Fuzzy Sets and Systems* 29, 303–313.

- Chanas, S., Kuchta, D., 1994. Linear programming problem with fuzzy coefficients in the objective function. In: Delgado, M., Kacprzyk, J., Verdegay, J. L., Vila, M. (Eds.), *Fuzzy Optimization*. Physica-Verlag, Wurzburg, pp. 148–157.
- Chanas, S., Kuchta, D., 1996a. A concept of the optimal solution of the transportation problem with fuzzy cost coefficients. *Fuzzy Sets and Systems* 82, 299–305.
- Chanas, S., Kuchta, D., 1996b. Multiobjective programming in optimization of interval objective functions - a generalized approach. *European Journal of Operational Research* 94, 594–598.
- Clarke, F., 1983. *Optimization and Nonsmooth Analysis*. John Wiley & Sons, New York.
- Delgado, M., Verdegay, J., Vila, M., 1989. A general model for fuzzy linear programming. *Fuzzy Sets and Systems* 29, 21–29.
- Dempe, S., 1987. A simple algorithm for the linear bilevel programming problem. *Optimization* 18, 373–385.
- Dempe, S., 2000. A bundle algorithm applied to bilevel programming problems with non-unique lower level solutions. *Computational Optimization and Applications* 15, 145–166.
- Dempe, S., 2002. *Foundations of Bilevel Programming*. Kluwer Academic Publishers, Dordrecht.
- Dempe, S., Dutta, J., Lohse, S., 2006. Optimality conditions for bilevel programming problems. *Optimization* 55, 505–524.
- Dempe, S., Fanghänel, D., Starostina, T., 2009. Optimal toll charges: fuzzy optimization approach. In: Heyde, F., Löhne, A., Tammer, C. (Eds.), *Methods of Multicriteria Decision – Theory and Applications*. Shaker Verlag, Aachen, pp. 29–45.
- Dempe, S., Ruziyeva, A., 2011. The Karush-Kuhn-Tucker optimality conditions in fuzzy optimization. *The Annals of Fuzzy Sets, Fuzzy Logic and Fuzzy Systems* 1, 201–213.
- Dempe, S., Ruziyeva, A., 2012. On the calculation of a membership function for the solution of a fuzzy linear optimization problem. *Fuzzy Sets and Systems* 188, 58–67.
- Dempe, S., Schreier, H., 2006. *Operations Research: Deterministische Modelle und Methoden*. Teubner Verlag, Wiesbaden.
- Dempe, S., Starostina, T., 2006. Optimal toll charges in a fuzzy flow problem. In: Reusch, B. (Ed.), *Computational Intelligence, Theory and Applications*. Springer Verlag, Berlin, pp. 405–413.
- Dempe, S., Starostina, T., 2007. On the solution of fuzzy bilevel programming problems. *Optimization Online*.
- Dubois, D., Prade, H., 1978. Operations on fuzzy numbers. *International Journal of Systems Science* 9, 613–626.

- Dyer, M., Proll, L., 1977. An algorithm for determining all extreme points of a convex polytope. *Mathematical Programming* 12, 81–96.
- Ehrgott, M., 2005. *Multicriteria Optimization*. Springer Verlag, Berlin.
- Ekel, P., Pedrycz, W., Schinzinger, R., 1998. A general approach to solve a wide class of fuzzy optimization problems. *Fuzzy Sets and Systems* 97, 49–66.
- Fadel, G. M., Li, Y., 2002. Approximating the pareto curve to help solve biobjective design problems. *Structural and Multidisciplinary Optimization* 23, 280–296.
- Fliege, J., 2004. Gap-free computation of Pareto-points by quadratic scalarizations. *Mathematical Methods of Operations Research* 59, 69–89.
- Fortuny-Amat, J., McCarl, B., 1981. A representation and economic interpretation of a two-level programming problem. *Journal of the Operational Research Society* 32, 783–792.
- Haurie, A., Savard, G., White, D., 1990. A note on: an efficient point algorithm for a linear two-stage optimization problem. *Operations Research* 38, 553–555.
- Hobbs, B., Nelson, S., 1992. A nonlinear bilevel model for analysis of electric utility demand-side planning issues. *Annals of Operations Research* 34, 255–274.
- Ishibuchi, H., Tanaka, H., 1990. Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research* 48, 219 – 225.
- Ishizuka, Y., Aiyoshi, E., 1992. Double penalty method for bilevel optimization problems. *Annals of Operations Research* 34, 73–88.
- Jiménez, F., Cadenas, J., Sánchez, G., Gómez-Skarmeta, A., Verdegay, J., 2006. Multi-objective evolutionary computation and fuzzy optimization. *International Journal of Approximate Reasoning* 43, 59–75.
- Kasperski, A., Zieliński, P., 2006. On combinatorial optimization problems on matroids with uncertain weights. *European Journal of Operational Research* 177, 851–864.
- Kim, I., Weck, O., 2005. Adaptive weighted sum method for bi-objective optimization: Pareto front generation. *Structural and Multidisciplinary Optimization* 29, 149– 158.
- Kim, T., Suh, S., 1988. Toward developing a national transportation planning model: a bilevel programming approach for Korea. *Annals of Regional Science* 22, 65–80.
- Labbe, M., Marcotte, P., Savard, G., 1998. A bilevel model of taxation and its application to optimal highway pricing. *Management Science* 44, 1608–1622.
- Löhne, A., 2011. *Vector Optimization with Infimum and Supremum*. Springer-Verlag, Berlin.
- Li, Y., Fadel, G., Wiecek, M., Blouin, V., 2003. Minimum effort approximation of the Pareto space of convex bi-criteria problems. *Optimization and Engineering* 4, 231–261.

- Libor, V., Duda, J., Vesely, L., Zajicek, L., 2003. On d.c. functions and mappings. *Seminario Matematico e Fisico, Unniversita di Modena* 1, 111–138.
- Liu, S.-T., Kao, C., 2004. Network flow problems with fuzzy arc lengths. *Transactions on Systems, Man, and Cybernetics* 34, 765–769.
- Lohse, S., 2011. Eine spezielle Klasse von Zwei-Ebenen-Optimierungsaufgaben. Ph.D. thesis, Technical University Bergakademie Freiberg.
- Marcotte, P., Savard, G., 1991. A note on the Pareto optimality of solutions to the linear bilevel programming problem. *Computers and Operations Research* 18, 355 – 359.
- Marcotte, P., Savard, G., 2001. Bilevel programming: Applications. In: *Encyclopedia of Optimization*. Kluwer Academic Publishers, Dordrecht.
- Migdalas, A., 1995. Bilevel programming in traffic planning: models, methods and challenge. *Journal of Global Optimization* 7, 381–405.
- Nemhauser, G., Wolsey, L., 1988. *Integer and Combinatorial Optimization*. John Wiley & Sons, New York.
- Önal, H., 1993. A modified simplex approach for solving bilevel linear programming problems. *European Journal of Operational Research* 67, 126–135.
- Orlovski, S., 1985. Mathematical programming problems with fuzzy parameters. In: Kacprzyk, J., Yager, R. R. (Eds.), *Management Decision Support Systems using Fuzzy Sets and Possibility Theory*. Verlag TÜV, Köln, pp. 136–145.
- Panigrahi, M., Panda, G., Nanda, S., 2008. Convex fuzzy mapping with differentiability and its application to fuzzy optimization. *European Journal of Operational Research* 185, 47–62.
- Parraga, F., 1981. Hierarchical programming and applications to economic policy. Ph.D. thesis, Systems and Industrial Engineering Department, University of Arizona.
- Peidro, D., Mula, J., Jiménez, M., Botella, M., 2010. A fuzzy linear programming based approach for tactical supply chain planning in an uncertainty environment. *European Journal of Operational Research* 205, 65–80.
- Pham, B., 2001. Representation of fuzzy shapes. In: Arcelli, C., Cordella, L., di Baja, G. (Eds.), *IWVF-4, Proceedings (Lecture notes in computer science Vol. 2059)*. Springer Berlin / Heidelberg, pp. 239–248.
- Rockafellar, R., 1970. *Convex analysis*. Princeton University Press, Princeton.
- Rommelfanger, H., Hanuscheck, R., Wolf, J., 1989. Linear programming with fuzzy objectives. *Fuzzy Sets and Systems* 29, 31–48.
- Rosenfeld, A., 1984. The fuzzy geometry of image subsets. *Pattern Recognition Letters* 2, 311–317.

- Ruszczynski, A., 2006. *Nonlinear Optimization*. Princeton University Press, Princeton.
- Ruziyeva, A., Dempe, S., 2012. Yager ranking index in fuzzy bilevel optimization. *Artificial Intelligence Research*, (accepted for publication).
- Sakawa, M., Yano, H., 1989. An interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters. *Fuzzy Sets and Systems* 30, 221 – 238.
- Sengupta, A., Pal, T., 2000. On comparing interval numbers. *European Journal of Operational Research* 127, 28–43.
- Sengupta, A., Pal, T., 2006. Solving the shortest path problem with interval arcs. *Fuzzy Optimization and Decision Making* 5, 71–89.
- Shi, C., Zhou, H., Lu, J., Zhang, G., Zhang, Z., 2007. The k th-best approach for linear bilevel multifollower programming with partial shared variables among followers. *Applied Mathematics and Computation* 188, 1686–1698.
- Shih, H.-S., Lai, Y.-J., Lee, E., 1996. Fuzzy approach for multi-level programming problems. *Computers and Operations Research* 23, 73–91.
- Shih, H.-S., Lee, E., 1999. Fuzzy multi-level minimum cost flow problems. *Computers and Operations Research* 107, 159–176.
- Sinha, S., 2003. Fuzzy mathematical programming applied to multi-level programming problems. *Computers and Operations Research* 30, 1259–1268.
- Stackelberg, H., 1934. *Marktform und Gleichgewicht*. Verlag von Julius Springer, Berlin.
- Tanaka, H., Ichihashi, H., Asai, K., 1984. A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers. *Control and Cybernetics* 13, 185–194.
- Tuy, H., Ghannadan, S., 1998. A new branch and bound method for bilevel linear programs. In: Migdalas, A., Pardalos, P., Värbrand, P. (Eds.), *Multilevel Optimization: Algorithms and Applications*. Kluwer Academic Publishers, Dordrecht, pp. 231–249.
- Unger, T., Dempe, S., 2010. *Lineare Optimierung: Modell, Lösung, Anwendung*. Springer Verlag, Wiesbaden.
- Verdegay, J., 1982. Fuzzy mathematical programming. *Fuzzy Information and Decision Processes*, 231–237.
- Weber, R., Werners, B., Zimmerman, H.-J., 1990. Planning models for research and development. *European Journal of Operational Research* 48, 175–188.
- Wen, U., Huang, A., 1996. A simple Tabu Search method to solve the mixed-integer linear bilevel programming problem. *European Journal of Operational Research* 88, 563–571.
- Wen, U.-P., Hsu, S.-T., 1989. A note on a linear programming algorithm based on bicriteria programming. *Computers and Operations Research* 16, 79–83.

- Werners, B., 1987. An interactive fuzzy programming system. *Fuzzy Sets and Systems* 23, 131 – 147.
- White, D., Anandalingam, G., 1993. A penalty function approach for solving bi-level linear programs. *Journal of Global Optimization* 3, 397–419.
- Wu, D., Yang, Z., Vela, S., Liang, L., 1997. Sensitivity analysis for the elastic-demand network equilibrium problem with applications. *Transportation Research Part B* 31, 55–70.
- Wu, H.-C., 2004. An (α, β) -optimal solution concept in fuzzy optimization problems. *Optimization* 53, 203–221.
- Wu, H.-C., 2007. The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function. *European Journal of Operational Research* 176, 46–59.
- Wu, H.-C., 2008. The optimality conditions for optimization problems with fuzzy-valued objective functions. *Optimization* 57, 473–489.
- Wu, H.-C., 2009. The optimality conditions for optimization problems with convex constraints and multiple fuzzy-valued objective functions. *Fuzzy Optimization and Decision Making* 8, 295–321.
- Wu, S., Chen, Y., Marcotte, P., 1998. A cutting plane method for linear bilevel programming. *Systems Science and Mathematical Science* 11, 125–133.
- Wu, Z., Xu, J., 2008. Nonconvex fuzzy mapping and the fuzzy pre-variational inequality. *Fuzzy Sets and Systems* 159, 2090–2103.
- Yager, R., 1981. A procedure for ordering fuzzy subsets of the unit interval. *Information Sciences* 24, 143–161.
- Zadeh, L., 1963. Optimality and non-scalar-valued performance criteria. *Automatic Control, IEEE Transactions on* 8, 59 – 60.
- Zadeh, L., 1965. Fuzzy sets. *Information and Control* 8, 338–353.
- Zadeh, L., 1975a. The concept of a linguistic variable and its application to approximate reasoning-I. *Information Sciences* 8, 199–249.
- Zadeh, L., 1975b. The concept of a linguistic variable and its application to approximate reasoning-II. *Information Sciences* 8, 301–357.
- Zadeh, L., 1975c. The concept of a linguistic variable and its application to approximate reasoning-III. *Information Sciences* 9, 43–80.
- Zhang, G., Lu, J., 2010. Fuzzy bilevel programming with multiple objectives and cooperative multiple followers. *Journal of Global Optimization* 47, 403–419.

- Zhang, G., Lu, J., Dillon, T., 2006. An approximation branch-and-bound algorithm for fuzzy bilevel decision making problems. In: Proceedings of the International Multiconference on Computer Science and Information Technology. pp. 223–231.
- Zhang, G., Lu, J., Dillon, T., 2008. An approximation Kuhn–Tucker approach for fuzzy linear bilevel decision making. In: Phillips-Wren, G., Ichalkaranje, N., Jain, L. (Eds.), Intelligent Decision Making: An AI-Based Approach. Vol. 97. Springer Berlin / Heidelberg, pp. 157–171.
- Zhang, X., Huang, G.-H., Chan, C., Liu, Z., Lin, Q., 2010. A fuzzy-robust stochastic multiobjective programming approach for petroleum waste management planing. Applied Mathematical Modelling 34, 2778–2788.
- Zimmermann, H.-J., 1976. Description and optimization in fuzzy systems. International Journal of General Systems 2, 209–215.
- Zimmermann, H.-J., 1978. Fuzzy programming and linear programming with several objective functions. Fuzzy Sets and Systems 1, 45–55.
- Zimmermann, H.-J., 1991. Fuzzy Set Theory and its Applications. Kluwer Academic Publishers, Dordrecht.