On b-colorings and b-continuity of graphs

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Thesis

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Dr. rer. nat.,

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Dipl.-Math. Mais Alkhateeb

born on January 24, 1979 in Tartous, Syria

Referees:

Prof. Dr. rer. nat. Ingo Schiermeyer

Prof. Dr. rer. nat. Margit Voigt

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To my Home Syria

I feel it's my duty to offer my sincere gratitude to each person who has helped me reach my aim here, either directly or indirectly. I am deeply indebted to my supervisor, Professor Ingo Schiermeyer, and Doctor Anja Kohl for all their help and support. They were there for me whenever I needed them, answered my questions, monitored my progress, helped me and corrected me when I got off the subject. Some shared their knowledge, some shared their experience, some inspired me, but all of them shared the passion I had for this project and made this possible for me. I thank also the Institute of Discrete Mathematics and Algebra, the professors, and all the institute members who gave me the direction and walked with me all along this way. Special thanks go to Professor Margit Voigt for all her interesting suggestions and for refereeing this PhD thesis.

I want to remember a few people in my life at this moment, who made this possible for me. I start with my parents who had this vision of providing the best education for their child and created all possible avenues for growth, making numerous personal sacrifices in the process. The belief that they implanted in my growing years has helped me cross all the odds and be here today. No amount of words would be sufficient to express my gratitude to my husband Msc. -Eng. Mosa Alokla, families and friends, I have been very fortunate to have them around me.

I would also like to take this opportunity to thank the gentle people of this beautiful town of Freiberg. This town has given me nice memories; the people have been friendly and supportive.

Finally, I have not to forget the people of my country. I would like to remember and thank all the Syrians all around the world, from all walks of life making a mark in what they do and contributing to the development and peace of Syria and that of the world.

Contents

1 Introduction

Clustering problems occur in many different fields, such as data mining processes, web services classifications, and decompositions of large distributed systems. Let us consider the following problem:

There are n customers $C_1, C_2, \ldots, C_n$ in a bank and for each customer a set of data is given. In the customer relationship management it is useful to divide the customers into groups, such that customers within the same group have almost similar data, but have sufficiently enough dissimilarities to customers in any other group. In order to do this, we have to determine the similarity of any two customers on the basis of their data. These values are stored in a symmetric *similarity matrix* S in which the element $s_{i,j}$ in row i and column j is a natural number that shall express the similarity of the customers C_i and C_j, where a large number means a large similarity. Since no two customers have exactly the same data, we have to define a *threshold* τ such that customers C_i and C_j are considered as similar if and only if $s_{i,j} \geq \tau$. Now we intend to divide the customers into a small as possible number of groups such that any two customers C_i and C_j within the same group satisfy $s_{i,j} \geq \tau$.

This clustering problem can be modeled and solved as a graph coloring problem. In order to do this, we construct a *threshold graph* G, where each customer is represented by a vertex, and any two vertices are joined by an edge if and only if the corresponding customers have a similarity less than τ. A clustering of the customers now corresponds to a partition of the vertex set of G into disjoint independent subsets, i.e. subsets where the vertices are pairwise non-adjacent.

If we assign a color to each of these independent subsets, we obtain a proper coloring of the vertices of G where the independent subsets are the *color classes* of this coloring. Hence, we can determine an optimal clustering of the customers by determining a proper coloring of the threshold graph G with as few as possible colors. Often the problem is too complex to find an optimal solution. So we have to be content with a solution obtained by heuristic methods. One of these heuristics is the clustering algorithm stated below that is based on b-colorings of graphs (see e.g. [Dek08, Dek08-1, Dek08-2, Eff06, Elg07, Yos08] for further information on clustering approaches based on b-colorings).

A *b-coloring* of a graph G is a proper vertex coloring of G such that each color class contains a *color-dominating vertex*, that is, a vertex which is adjacent to at least one vertex in every other color class. A coloring of G that is no b-coloring can be easily improved by choosing a color class without a color-dominating vertex and redistributing all vertices of this color class to suitable other color classes.

Clustering algorithm:

1. Threshold Graph

   Construct the corresponding threshold graph G.

2. Greedy Coloring

   Define an ordering of the vertices of G and color it in this order step-by-step by the smallest possible color each in such a way that no two adjacent vertices receive the same color (the colors are considered as natural numbers).

3. b-Coloring

   Choose a color class that does not contain a color-dominating vertex and redistribute its vertices to suitable other color classes. Repeat this until the obtained coloring is a b-coloring of G.

4. Clustering

   Output the color classes of the b-coloring as clusters.

Example:

Let $C_1, C_2, \ldots, C_{11}$ be 11 customers of a bank and the threshold is set to $\tau = 9$. Moreover, let the following matrix be the corresponding similarity matrix S.

Now we apply to this graph G the clustering algorithm mentioned above.

$$S = \begin{pmatrix} 0 \\ 2 & 0 \\ 1 & 8 & 0 \\ 3 & 7 & 17 & 0 \\ 14 & 3 & 24 & 6 & 0 \\ 21 & 15 & 15 & 8 & 6 & 0 \\ 11 & 11 & 12 & 12 & 5 & 6 & 0 \\ 14 & 11 & 5 & 9 & 9 & 8 & 7 & 0 \\ 9 & 3 & 3 & 12 & 9 & 11 & 9 & 8 & 0 \\ 10 & 14 & 11 & 12 & 9 & 6 & 7 & 8 & 8 & 0 \\ 11 & 1 & 11 & 14 & 5 & 12 & 4 & 18 & 8 & 1 & 0 \end{pmatrix}$$

Step 1: We construct the corresponding threshold graph $G = (V, E)$ with vertex set $V = \{C_1, C_2, \ldots, C_{11}\}$ and edge set $E = \{C_iC_j \mid i, j \in \{1, 2, \ldots, 11\},\ s_{ij} < 9\}$. The result is shown in Figure 1.1.

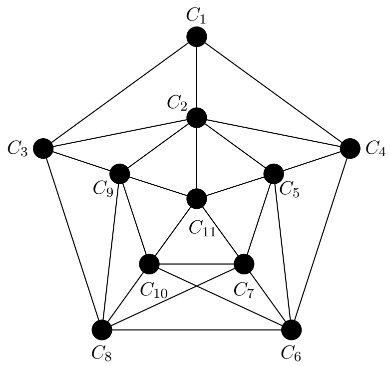


Figure 1.1: Threshold graph G

Step 2: We color the vertices $C_1, C_2, \ldots, C_{11}$ in this order in such a way that the vertex C_i $(i = 1, 2, \ldots, 11)$ receives the smallest possible color that is not assigned to any of C_i's neighbors among $C_1, \ldots, C_{i-1}$. This yields the coloring by 6 colors given in Figure 1.2.

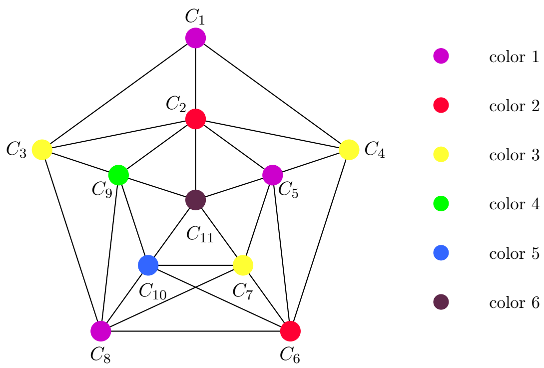


Figure 1.2: Greedy coloring of G

Step 3: We check which of the color classes contains a color-dominating vertex. In Figure

1.3 all color-dominating vertices are marked. So we see that the color classes 4,5, and 6 have a color-dominating vertex each while the color classes 1,2, and 3 have no such vertices.

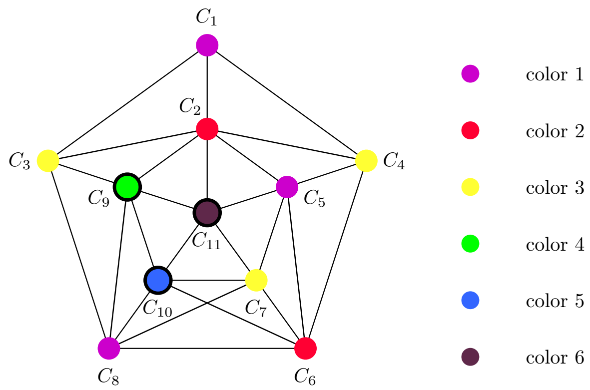


Figure 1.3: All color-dominating vertices

We redistribute the vertices of color class 1 to the other color classes by recoloring C_1 and C_5 by color 4 and C_8 by color 6. This yields the coloring by 5 colors shown in Figure 1.4.

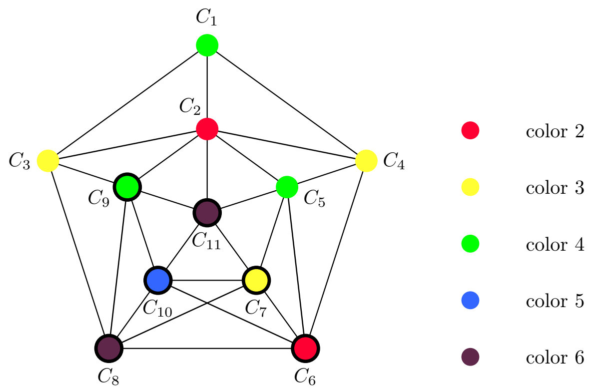


Figure 1.4: b-coloring of G by 5 colors

After this recoloring, all remaining color classes now contain a color-dominating vertex (see the marked vertices in Figure 1.4). So this coloring is a b-coloring of G by 5 colors.

Step 4: We take the 5 color classes as clusters and obtain the following clustering:

$$\begin{aligned}
&\text{Cluster 1} &&= \{C_2, C_6\} &&\text{(color class 2),}\\
&\text{Cluster 2} &&= \{C_3, C_4, C_7\} &&\text{(color class 3),}\\
&\text{Cluster 3} &&= \{C_1, C_5, C_9\} &&\text{(color class 4),}\\
&\text{Cluster 4} &&= \{C_{10}\} &&\text{(color class 5),}\\
&\text{Cluster 5} &&= \{C_8, C_{11}\} &&\text{(color class 6).}
\end{aligned}$$

Note that the obtained coloring is not optimal because we can find a coloring of G by 4 colors (see Figure 1.5). This coloring is even an optimal coloring of G since the vertices

C_6, C_7, C_8, and C_{10} are pairwise adjacent and, therefore, we already need 4 colors. #

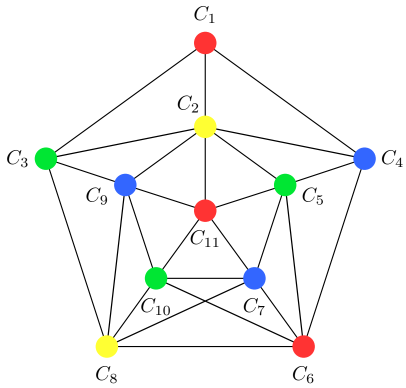


Figure 1.5: b-coloring of G by 4 colors

This was an example how b-colorings can be applied to solve clustering problems. In order to make these colorings applicable for solving other practical problems as well, it makes sense to investigate the b-colorings from the theoretical point of view at first. The present doctoral thesis shall contribute to this investigation.

The outline of the thesis is as follows: We start in Chapter 2 with some basic terminology and a short overview on classical vertex colorings. After this, we introduce a-colorings which are a special type of vertex coloring and in some sense the predecessor of b-colorings. In Chapter 3, we consider b-colorings of graphs. At first, we present in Section 3.1 general bounds and properties of the b-chromatic number. Then we determine in Section 3.2. the exact value of the b-chromatic number for special graphs. In Section 3.3. we investigate the b-coloring problem on bipartite graphs. In doing so, we define the so-called bicomplement and we use it to determine the b-chromatic number of special bipartite graphs, in particular those whose bicomplement has a simple structure. Furthermore, we investigate in Section 3.4. some graphs whose b-chromatic number is close to its t-degree. Chapter 4 deals with graph properties concerning b-colorings, mainly with the b-continuity. At first, we consider in Section 4.1. the b-continuity of graphs whose b-chromatic number was established in Chapter 3. Then, we list in Section 4.2. all b-continuous graph classes that are known so far and we prove the b-continuity of Halin graphs. We finish the chapter with Section 4.3 where we briefly introduce the b-perfectness and b-monotonicity of graphs. The thesis ends with a conclusion in Chapter 5, where we summarize the obtained results and present some open problems for future research.

2 Preliminaries

2.1 Basic terminology

In this section some standard definitions on graph theory are given. Good references for any undefined terms or notations are [Bra99, Gro99, Wes99].

A *graph* $G = (V(G), E(G))$ is defined by the *vertex set* $V(G) = \{v_1, ..., v_n\}$ and the *edge set* $E(G) \subseteq \{e : e = uv \text{ and } u, v \in V(G)\}$. The number of vertices of a graph is its *order*, written as $n(G)$, and the number of edges of a graph is its *size*, written as $m(G)$.

In this thesis, only simple and undirected graphs are considered, unless otherwise stated.

We call u and $v \in V$ *adjacent*, if there exists an edge $e = uv \in E$. A *neighbor* of a vertex v is a vertex which is adjacent to v. Let $N_G(v)$ or $N(v)$ be the *neighborhood* of a vertex $v \in V$, i.e. the set of all neighbors of v, $N_G[v] := N_G(v) \cup \{v\}$ or simply $N[v]$, and $d_G(v) := |N(v)|$ or simply $d(v)$ is the *degree* of vertex v in G. A vertex of degree 0 is an *isolated vertex*. The *maximum degree* is denoted by $\Delta(G)$ and the *minimum degree* is denoted by $\delta(G)$. If $\Delta(G) = \delta(G) = r$, which means that all vertices of G have degree r, then G is called an *r-regular graph*. The *non-neighborhood* of a vertex v is given by $\overline{N}_G(v) := V(G) \setminus N_G[v]$ or simply $\overline{N}(v)$. Two edges are said to be *adjacent* if they have a common vertex.

For $S \subseteq V(G)$, $M \subseteq E(G)$, we denote by $G - S$ and $G - M$ the subgraphs of G obtained by deleting a set of vertices S and a set of edges M, i.e. $G - S := G[V(G) \setminus S]$ and $G - M := (V(G), E(G) \setminus M)$. We simply write $G - v$ or $G - e$ instead of $G - \{e\}$ or $G - \{v\}$.

Furthermore, we use some non-standard notation and terminology in this thesis, which we will define in the corresponding chapter or we include it in the Appendix for easy reference.

2.2 Colorings of graphs

Many concepts in graph theory are induced by investigating the famous Four Color Problem: Can one color the countries of every map by at most four colors, so that neighbor countries receive different colors?
The problem seems first to have been mentioned in 1852 in a written form in a letter from De Morgan to Hamilton, but it was barely known until 1878. Then the Four Color Problem became widely famous through the talk of Cayley who introduced the problem in the London Mathematical Society. In the same year Kempe presented a supposed proof, that turned out to be wrong, which was detected by Heawood in 1890. The proof was settled in 1976 by Appel, Haken, and Koch who were the first that applied computers in solving a graph theory problem. Some flaws in the proof let people question the correctness of the proof. Therefore, Robertson, Sanders, Seymour, and Thomas gave in 1996 a better and shorter proof of the Four-Color-Theorem that uses similar methods as the proof by Appel, Haken, and Koch.

In the meantime the field of graph colorings has grown rapidly. Besides the classical vertex colorings many variants of colorings were introduced. A good overview about coloring problems and other variants of colorings is given in [Jen95]. Two examples of such colorings are a-colorings, which are shortly summarized in Subsection 2.2.2, and b-colorings which are the main topic of this thesis.

2.2.1 Vertex colorings

Definition 2.1. [k-vertex coloring] *A k-*vertex coloring *of a graph G is a mapping $c : V(G) \longrightarrow \{1, ..., k\}$, where every two adjacent vertices u, v receive different colors $c(u) \neq c(v)$.*
The chromatic number *of a graph G, denoted by $\chi(G)$, is the minimum number k of all proper k-colorings of G. A graph G is k-*colorable *if it has a proper k-coloring by k colors.*

The determination of the chromatic number of a graph is $\mathcal{NP}-$hard which means that if $\mathcal{P} \neq \mathcal{NP}$, then there is no efficient algorithm that can solve this problem.
So we are interested in bounds on the chromatic number and results for special graph classes.

General bounds on the chromatic number

Clearly, the vertices of a clique in G have to be colored pairwise differently. The trivial case is to color the vertices of a graph G with $n(G)$ colors. Moreover, a coloring of G by k colors is a partition of $V(G)$ into k independent sets each of them of order at most the independence number $\alpha(G)$. Altogether we obtain:

$$\max\left\{\omega(G), \frac{n(G)}{\alpha(G)}\right\} \leq \chi(G) \leq n(G).$$

Let $V(G) = \{v_1, ..., v_n\}$ and consider a *Greedy algorithm* that colors the vertices $v_1, ..., v_n$ in that order such that v_i receives the smallest available color, i.e. the smallest color which is not already assigned to a vertex in $N(v_i) \cap \{v_1, ..., v_{i-1}\}$. We can prove that this Greedy algorithm uses at most $\Delta(G) + 1$ colors. Therefore:

$$\chi(G) \leq \Delta(G) + 1.$$

This bound is sharp for complete graphs and odd cycles. For the other graphs we know:

Theorem 2.1. (Brooks, [Bro41])
If a connected graph G is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

Szekeres and Wilf (see e.g. [Wes99]) proved that for every graph G, $\chi(G) \leq 1 + \max_{H\subseteq G} \delta(H)$.
Reed [Ree98] conjectured that the Theorem of Brooks can be improved by taking the arithmetic mean of the trivial upper bound $\Delta(G) + 1$ and the trivial lower bound $\omega(G)$:

Conjecture 2.1. (Reed, [Ree98])
For any graph G,

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil$$

The main result in [Ree98] states that if $\Delta(G)$ is sufficiently large and $\omega(G)$ is sufficiently close to $\Delta(G)$, then Conjecture 2.1 holds.
Also, there are many graph classes for which the Conjecture 2.1 is already proved. For example, such graph classes are line graphs of multigraphs [Kin05], quasi-line graphs [Kin08], graphs G with $\alpha(G) \leq 2$ [Kin09, Koh09, Rab08], almost-split graphs [Koh09], $K_{1,3}$-free graphs [Kin09], odd hole free graphs [Ara11], $2K_2, C_4$-free graphs [Ger08], graphs with

disconnected complements [Rab08], graphs G with $\Delta(G) \geq n(G) - 7$ [Koh09], graphs G with $\Delta(G) \geq n(G) - \alpha(G) - 4$ [Koh09], graphs G with $\chi(G) > \lceil \frac{n(G)}{2} \rceil$ [Rab08], graphs G with $\chi(G) > \frac{n(G)-\alpha(G)+3}{2}$ [Rab08], and graphs G with $\chi(G) \leq \omega(G) + 2$ [Ger08].

2.2.2 a-colorings

The a-colorings were introduced by Harary, Hedetniemi, and Prins [Har67] and by Harary and Hedetniemi [Har09].

Definition 2.2. [a-coloring] *An* a-coloring *of a graph G is a proper vertex coloring of G such that, for any pair of colors, there is at least one edge of G whose end vertices are colored with this pair of colors.*

Definition 2.3. [achromatic number] *The* achromatic number *of G, denoted by $\chi_a(G)$, is the largest integer k such that there is an a-coloring of G.*

It is known [Gra86] that the decision problem whether $\chi_a(G) > K$ for a given graph G and an integer K is $\mathcal{NP}-$hard in general. In [Far86], Farber et al. showed that the problem is $\mathcal{NP}-$hard on bipartite graphs. It is even $\mathcal{NP}-$hard for trees [Cai97]. In [Bod80], Bodlaender proved that the problem is $\mathcal{NP}-$hard on cographs and interval graphs.

In [Kor01], they gave the first hardness result for approximating the achromatic number. They showed that for every ϵ there is no $2 - \epsilon$ approximation algorithm, unless $\mathcal{P} = \mathcal{NP}$. More information on approximation algorithms for the problem of determining the achromatic number can be found in [Chau97, Kor01, Kor03, Kor05, Kor07, Kry98, Kry99, Kry06].

General bounds on the achromatic number

We note that a proper coloring that uses $\chi(G)$ colors is an a-coloring of G. Assume that such a coloring is not an a-coloring, then there exist two colors k, l such that there is no edge of G whose end vertices are colored with this pair of colors. It follows that we can recolor all vertices with colors k and l with either the color k or l. But then we obtain a proper coloring of G by $\chi(G) - 1$ colors, a contradiction. Hence:

$$\chi(G) \leq \chi_a(G).$$

Figure 2.1 shows a graph G with chromatic number $\chi(G) = 3$ and achromatic number $\chi_a(G) = 5$.

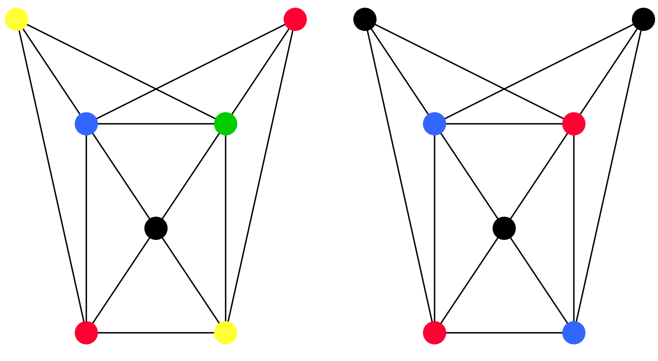


Figure 2.1: Two a-colorings of a graph

Moreover, besides this bound there are the following bounds with respect to the order $n(G)$ and the size $m(G)$, or the matching number $\mu(G)$ of the graph G.

Proposition 2.1. (Harary and Hedetniemi, [Har67])
Every graph G satisfies $\chi_a(G) \leq n(G) - \alpha(G) + 1$.

Since $n(G) - \alpha(G) \leq 2\mu(G)$ it follows:

Corollary 2.1. (Harary and Hedetniemi, [Har67])
For every graph G, $\chi_a(G) \leq 2\mu(G) + 1$.

It is known that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$ for every graph G. Gupta showed the following "Nordhaus-Gaddum"-type result:

Proposition 2.2. (Gupta, [Gup69])
For every graph G, $\chi_a(G) + \chi_a(\overline{G}) \leq \lceil \frac{4}{3} n(G) \rceil$.

Moreover, Harary and Hedetniemi proved the following upper bound:

Proposition 2.3. (Harary and Hedetniemi, [Har67])
For every graph G, $\chi_a(G) + \chi(\overline{G}) \leq n(G) + 1$.

Further "Nordhaus-Gaddum"-bounds on the achromatic number are presented in [Gup69, Aki83, Bha89].

Moreover, in [Aki83] they determined the 41 graphs for which both G and $\overline{G}$ have achromatic number 3. Two examples of such graphs are shown in Figure 2.2. The graph P_4 is also a sharpness example for Proposition 2.2 because it satisfies $\chi_a(P_4) + \chi_a(\overline{P_4}) = \lceil \frac{4}{3} n(P_4) \rceil$. The graph C_5 is also a sharpness example for Proposition 2.3 since it satisfies $\chi_a(C_5) + \chi(\overline{C_5}) = n(C_5) + 1$.

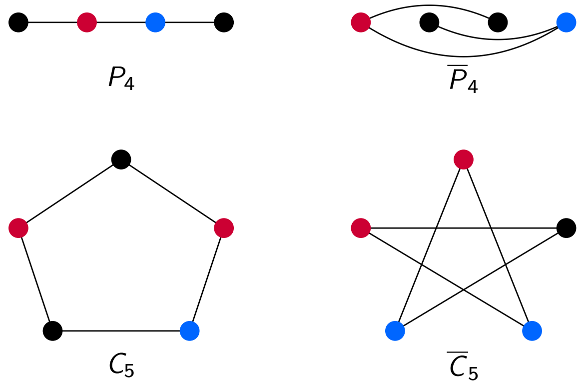


Figure 2.2: a-colorings of P_4 and C_5

Proposition 2.4. (Edwards, [Eda97])
For every graph G, $\chi_a(G) \leq \lfloor \frac{1}{2}(1 + \sqrt{8m(G) + 1}) \rfloor$.

It is known [Eda97] that almost all trees T satisfy $\chi_a(T) = \lfloor \frac{1}{2}(1 + \sqrt{8m(T) + 1}) \rfloor$.

Proposition 2.5. (Cairnie and Edwards, [Cai97])
Let G be a graph with maximum degree $\Delta(G)$ and size $m(G)$. Then for a fixed positive integer $\epsilon > 0$ there exists an integer $N_0 = N_0(\Delta(G), \epsilon)$ such that $(1 - \epsilon)(\lfloor \frac{1}{2}(1 + \sqrt{8m(G) + 1}) \rfloor) \leq \chi_a(G) \leq \lfloor \frac{1}{2}(1 + \sqrt{8m(G) + 1}) \rfloor$.

They also gave a polynomial-time algorithm for determining the achromatic number of a tree with maximum degree at most d, where d is a fixed positive integer. They showed that there is a natural number $N(d)$ such that if T is any tree with $m(T) > N(d)$ edges and maximum degree at most d, then $\chi_a(T)$ is k or $k - 1$, where k is the largest integer such that $\binom{k}{2} < m$.

There are also results on determining the achromatic number of other graph classes such as bounded degree trees [Cai98], maximal outerplanar graphs [Har02], graphs with small achromatic number [Hel76], the cartesian product of two graphs [Hel92], unions of cycles or paths [Lee04, Mac01], random graphs [McD82], permutation graphs [Mil86], extremal regular graphs [Mil82], trees, grids, cubes, and boolean cube graphs [Roi91], hypercubes [Roi00], central graphs and split graphs [Thi09], star graph families [Ver09-1], and double star graph families [Ver09-2].
Further information on a-colorings and the achromatic number can be found in the surveys written by Hughes and MacGillivray [Hug94, Hug97] and Edwards [Eda97], and the theses written by Hara [Har99] and Shanthi [Sha90].

General properties of the achromatic number

Harary et al. [Har67] published the following result about a-colorings of graphs:

Proposition 2.6. (Harary, Hedetniemi, and Prins, [Har67])
If a graph G has a-colorings by j and l colors, then for every integer k, $j \leq k \leq l$, G has an a-coloring by k colors.

This proposition implies that a graph G has an a-coloring by k colors for every integer k satisfying $\chi(G) \leq k \leq \chi_a(G)$. This is an interesting fact in comparison to the behavior of b-colorings.

Geller and Kronk investigated how the deletion of a single vertex or edge effects the achromatic number of a graph. They proved:

Proposition 2.7. (Geller and Kronk, [Gel74])
Let $G = (V, E)$ be a graph with $v \in V$ and $e \in E$. Then
1. $\chi_a(G) - 1 \leq \chi_a(G - v) \leq \chi_a(G)$.
2. $\chi_a(G) - 1 \leq \chi_a(G - e) \leq \chi_a(G) + 1$.

This proposition implies the following two corollaries:

Corollary 2.2. (Geller and Kronk, [Gel74])
For every induced subgraph H of a graph G, $\chi_a(H) \leq \chi_a(G)$.

Corollary 2.3. (Geller and Kronk, [Gel74])

Let G be a graph and $e = uv \in E(G)$. Then:

1. $\chi_a(G - v) = \chi_a(G) - 1$ *if* $\chi_a(G - e) = \chi_a(G)$.

2. $\chi_a(G - v) = \chi_a(G) - 1$ *if* $\chi_a(G - e) = \chi_a(G) - 1$.

For the case that G or $\overline{G}$ is disconnected we know the following:

Let G be a graph with disconnected complement $\overline{G}$ and let $\overline{G}_1, ..., \overline{G}_r$ be the components of $\overline{G}$. Then it is obvious that G is the join of the graphs $G_i = \overline{\overline{G}}_i$ ($i \in \{1, ..., r\}$), i.e. $G = G_1 \oplus ... \oplus G_r$.

It is obvious that $\chi(G) = \sum_{i=1}^{r} \chi(G_i)$. Harary and Hedetniemi proved a similar result for $\chi_a(G)$.

Proposition 2.8. (Harary and Hedetniemi, [Har67])

Let G_1,...,G_r be graphs. Then $\chi_a(G_1 \oplus ... \oplus G_r) = \chi_a(G_1) + ... + \chi_a(G_r)$.

Hell and Miller presented the best possible upper and lower bounds for the achromatic number of the disjoint union $G_1 \cup G_2$ of two graphs G_1 and G_2.

Proposition 2.9. (Hell and Miller, [Hel92])

Let G_1 and G_2 be two graphs. Then

$$\max\{\chi_a(G_1), \chi_a(G_2)\} \leq \chi_a(G_1 \cup G_2) \leq \chi_a(G_1) \cdot \chi_a(G_2).$$

3 *b*-colorings

In 1999, Irving and Manlove [Irv99] introduced the concept of b-colorings and the b-chromatic number.

Definition 3.1. *A* b-coloring *of a graph G by k colors is a proper vertex coloring such that there is a vertex in each color class, which is adjacent to at least one vertex in every other color class. Such a vertex is called a* color-dominating vertex.

Definition 3.2. *The* b-chromatic number *of a graph G, denoted by $\chi_b(G)$, is the largest integer k such that there is a b-coloring of G.*

Let c be a proper vertex coloring of G by k colors and let $\{V_1, \dots, V_k\}$ be the corresponding partition of $V(G)$ into the k color classes. If the union of any two color classes is not independent, then c is an a-coloring, see Definition 2.3. If for every integer $i \in \{1, ..., k\}$ it is impossible to redistribute the vertices of V_i among the other independent sets from $\{V_1, ..., V_k\} \setminus \{V_i\}$ in order to get another proper coloring c' with a fewer number of colors, then each V_i contains a color-dominating vertex and therefore c is a b-coloring of G.

It is known [Irv99] that the decision problem whether $\chi_b(G) > K$ for a given graph G and an integer K is $\mathcal{NP}-$complete in general. It is even $\mathcal{NP}-$hard for bipartite graphs [Kra02] but polynomial for trees [Irv99].
There are also other complexity results with respect to the t-degree of a graph that was introduced by Irving and Manlove [Irv99]:

Definition 3.3. *Let $V = \{v_1, v_2, ..., v_n\}$ be the vertex set of G such that the vertices are ordered so that $d(v_1) \geq d(v_2)... \geq d(v_n)$. Then the* t-degree *$t(G)$ of G is defined by $t(G) := max\{1 \leq i \leq n : d(v_i) \geq i - 1\}$.*

For instance, Kratochvíl et al. proved in [Kra02] that the decision problem whether $\chi_b(G)$ is equal to the t-degree $t(G)$ is $\mathcal{NP}-$complete for connected bipartite graphs G with $t(G) = \Delta(G) + 1$ in general, but polynomial for connected bipartite graphs G satisfying $t(G) \leq 3$ and connected bipartite planar graphs G satisfying $t(G) = 3$.

Furthermore, Corteel et al. [Cor05] investigated the problem of approximating the b-chromatic number of a graph. They proved that the b-chromatic number is hard to approximate in polynomial time within a factor of $120/113-\epsilon$, for any $\epsilon > 0$, unless $\mathcal{P} = \mathcal{NP}$. This was the first hardness result for approximating the b-chromatic number.

Katrenic and Galclík improved the last result. They proved in [Kat10] that for all $\epsilon > 0$, it is $\mathcal{NP}-$hard to approximate the B-COLORING problem for a graph with n vertices within a factor $n^{\frac{1}{4}-\epsilon}$, where the B-COLORING problem is to find a b-coloring of G with a maximum number of colors. The existence of a $n^{1-\epsilon}$-approximation algorithm for the b-chromatic number of general graphs is still an open problem.

Because of the $\mathcal{NP}-$hardness of determining the b-chromatic number in general, we are interested in bounds on the b-chromatic number for general graphs and exact values of the b-chromatic number for special graph classes.

In Section 2.1 we give an overview on known bounds on the b-chromatic number for general graphs. Section 2.2 provides the exact value of the b-chromatic number for graphs with a special structure. For instance, we determine the b-chromatic number of graphs whose maximum degree is at most 2 and graphs whose independence number, clique number, or minimum degree is close to its order. Then we consider the b-chromatic number of powers of paths and cycles. Furthermore, in Section 2.3 we restrict our research to bipartite graphs. Mainly, we study the b-chromatic number of bipartite graphs by using the bicomplement which will be defined in Definition 3.3.2. Moreover, some results on graphs G whose b-chromatic number $\chi_b(G)$ is at least $t(G) - 1$ are presented in Section 2.4. In particular, we consider d-regular graphs and investigate the problem of characterizing those graphs satisfying $\chi_b(G) = d + 1$.

3.1 General bounds on the b-chromatic number

It is obvious that every coloring of a graph G by $\chi(G)$ colors is a b-coloring of G. Moreover, every b-coloring of G is also an a-coloring. Therefore, every graph G satisfies the following bounds:

$$\chi(G) \leq \chi_b(G) \leq \chi_a(G) \tag{3.1}$$

Figure 3.1 shows an example of an a-coloring and a b-coloring of the cycle C_8. This graph satisfies $\chi(C_8) = 2 < \chi_b(C_8) = 3 < \chi_a(C_8) = 4$.

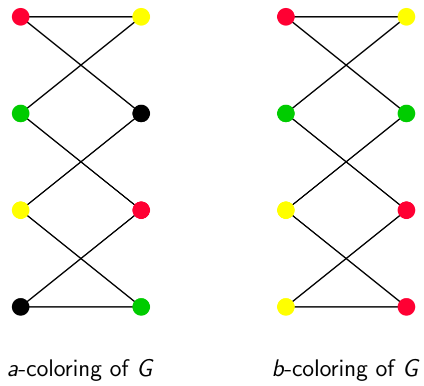


Figure 3.1: a-coloring and b-coloring of the cycle C_8

Kratochvíl et al. [Kra02] showed that the difference $\chi_b(G) - \chi(G)$ may be arbitrarily large. For instance, every graph G that is obtained from a complete bipartite graph $K_{r,r}$, $r \geq 2$, by removing a perfect matching satisfies $\chi_b(G) - \chi(G) = r - 2$.

In a b-coloring of a graph G by $\chi_b(G)$ colors each color-dominating vertex has degree at least $\chi_b(G) - 1$ and all vertices of a clique are colored pairwise distinct. So we conclude that:

$$\omega(G) \leq \chi_b(G) \leq \Delta(G) + 1 \tag{3.2}$$

The upper bound can be improved by taking the t-degree:

Proposition 3.1. (Irving and Manlove, [Irv99])

For every graph G, $\chi_b(G) \leq t(G)$.

This bound is e.g. attained for complete graphs and cycles of length at least 5. On the other hand, Irving and Manlove [Irv99] showed that the difference $t(G) - \chi_b(G)$ can be arbitrarily large by taking complete bipartite graphs $K_{r,r}, r \geq 2$, which satisfy $t(K_{r,r}) - \chi_b(K_{r,r}) = (r+1) - 2 = r - 1$.

Moreover, besides these bounds there are the following bounds with respect to the order $n(G)$ and the size $m(G)$ of the graph G.

Proposition 3.2. (Kouider and Maheó, [Kou07])

Every graph G satisfies $\chi_b(G) \leq n(G) - \alpha(G) + 1$.

Proposition 3.3. (Kohl, [Alk10])

For every graph G with clique number $\omega(G) < n(G)$, $\chi_b(G) \leq \lceil \frac{n(G)+\omega(G)}{2} \rceil - 1$.

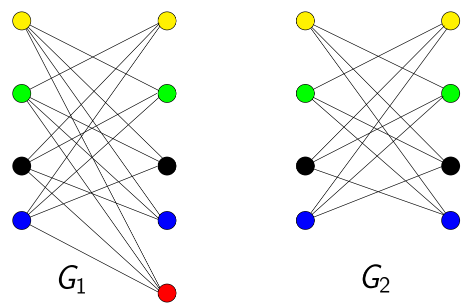


Figure 3.2: b-colorings of two graphs

Figure 3.2 shows b-colorings of two graphs G_1 and G_2, respectively. G_1 is a sharpness example for Proposition 3.2 and G_2 is a sharpness example for Proposition 3.3.

Proposition 3.4. (Kohl, [Alk10])

If G is a graph whose complement $\overline{G}$ has matching number $\nu(\overline{G})$, then $\chi_b(G) \leq n(G) - \lceil \frac{2\nu(\overline{G})}{3} \rceil$.

Hence, if the complement $\overline{G}$ of G has a perfect matching, then $\chi_b(G) \leq \lfloor \frac{2n(G)}{3} \rfloor$.

Proposition 3.5. (Kouider and Maheó, [Kou02])

For every graph G, $\chi_b(G) \leq \frac{1}{2} + \sqrt{2m(G) + \frac{1}{4}}$.

Kouider et al. proved also a "Nordhaus-Gaddum"-type result for the b-chromatic number:

Proposition 3.6. (Kouider and Maheó, [Kou02])

For every graph G, $\chi_b(G) + \chi_b(\overline{G}) \leq n(G) + 1$.

The *clique cover number* $\theta(G)$ is the minimum number of cliques in G needed to cover $V(G)$, i.e. $\theta(G) = \chi(\overline{G})$. Note that the pigeonhole principle yields $n(G) \leq \theta(G)\omega(G)$ for every graph G (since $n(G)$ vertices are distributed among $\theta(G)$ cliques in a minimum clique cover).

Proposition 3.7. (Kouider and Zaker, [Kou06])
Let G be a graph with clique cover number $\theta(G) = t$. Then $\chi_b(G) \leq \frac{t^2\omega(G)}{2t-1}$.

This can slightly be improved to:

Proposition 3.8. *For every graph G with clique cover number $\theta(G) = t$,* $\chi_b(G) \leq \lfloor \frac{t\omega(G)+(t-1)n(G)}{2t-1} \rfloor$.

Proof. Let $n := n(G)$, $\omega := \omega(G)$, $t := \theta(G)$, and $\{Q_1, \dots, Q_t\}$ shall be a minimum clique cover of G. We consider a proper b-coloring of $G = (V, E)$ by k colors. Let $V_1, \dots, V_k$ be the corresponding color classes such that $|V_1| \leq |V_2| \leq \dots \leq |V_k|$. Choose a color-dominating vertex $v_i \in V_i$ for all $i = 1, \dots, k$. By a we denote the number of color classes of cardinality 1. Obviously, $n \geq a + 2(k - a)$, i.e. $a \geq 2k - n$.

Case 1. $a = 0$.

We immediately obtain $k \leq \frac{n}{2}$.

Case 2. $a > 0$.

Since $v_1, \dots, v_a$ are color-dominating vertices and the only vertices with colors $1, \dots, a$ they must be pairwise adjacent. Moreover, because every vertex in V belongs to a clique in the clique cover $\{Q_1, \dots, Q_t\}$, the pigeonhole principle implies that there is an integer $i \in \{1, \dots, t\}$ such that Q_i contains at least $s := \lceil \frac{k-a}{t} \rceil$ vertices from the set $\{v_{a+1}, \dots, v_k\}$. W.l.o.g. assume $v_{a+1}, \dots, v_{a+s} \in Q_i$. Since Q_i is a clique it follows that $v_{a+1}, \dots, v_{a+s}$ are pairwise adjacent. Moreover, since these vertices are color-dominating they must be also pairwise adjacent to all vertices in $\{v_1, \dots, v_a\}$. Hence, $\{v_1, \dots, v_{a+s}\}$ induces a clique of order $a + s$ in G. This yields $\omega \geq a + s = a + \lceil \frac{k-a}{t} \rceil = \lceil \frac{k+a(t-1)}{t} \rceil$ and by $a \geq 2k - n$ we deduce $\omega \geq \lceil \frac{k+(2k-n)(t-1)}{t} \rceil$. This implies $k \leq \lfloor \frac{t\omega+(t-1)n}{2t-1} \rfloor$.

By the inequality $n \leq t\omega$ mentioned above we can show that the upper bound $\frac{n}{2}$ from Case 1 is not larger than the upper bound from Case 2 because $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n(2t-1)}{2(2t-1)} \rfloor = \lfloor \frac{n+2(t-1)n}{2(2t-1)} \rfloor \leq \lfloor \frac{t\omega+2(t-1)n}{2(2t-1)} \rfloor \leq \lfloor \frac{2t\omega+2(t-1)n}{2(2t-1)} \rfloor = \lfloor \frac{t\omega+(t-1)n}{2t-1} \rfloor$.

Altogether, $k \leq \lfloor \frac{t\omega+(t-1)n}{2t-1} \rfloor$ for every b-coloring by k colors and, therefore, $\chi_b(G) \leq \lfloor \frac{t\omega+(t-1)n}{2t-1} \rfloor$. $\square$

Since $n \leq t\omega$, we notice that $\frac{t\omega+(t-1)n}{2t-1} \leq \frac{t^2\omega}{2t-1}$. So the upper bound from Proposition 3.8 is never larger than the bound of Kouider and Zaker, and even improves the bound in case of $n < t\omega$.

According to Proposition 3.8 the complement of a bipartite graph satisfies $\chi_b(G) \leq$

$\lfloor \frac{2\omega(G)+n(G)}{3} \rfloor$.

Since all previous bounds contain $\alpha(G), \omega(G)$, or $\theta(G)$ which are themselves hard to determine it makes sense to mention a bound which can be determined in polynomial time and in spite of this provides good results especially for regular graphs.

Proposition 3.9. (Kohl, [Alk10])
For every graph G, $\chi_b(G) \leq \lfloor \frac{2n(G)-\Delta(G)-\delta(G)-3}{3n(G)-2\Delta(G)-\delta(G)-4} n(G) \rfloor$.

The proposition yields that if G is a $(n(G)-1-r)$-regular graph, then $\chi_b(G) \leq \lfloor \frac{2r-1}{3r-1} n(G) \rfloor$. If G is a $(n(G)-2)$-regular graph, then we obtain $\chi_b(G) \leq \lfloor \frac{n(G)}{2} \rfloor$, which will turn out in Corollary 3.1 to be the exact value on $\chi_b(G)$. Moreover, if G is a $(n(G)-3)$-regular graph we deduce that $\chi_b(G) \leq \lfloor \frac{3n(G)}{5} \rfloor$. This bound is close to the exact value given in Corollary 3.3.

For $(n(G)-4)$-regular graphs G, there exist sharpness examples for the bound $\chi_b(G) \leq \lfloor \frac{5n(G)}{8} \rfloor$. Figure 3.3 shows the complement $\overline{G}$ of a 10-regular graph G of order 14 which has a b-coloring by $8 = \lfloor \frac{5n(G)}{8} \rfloor$ colors.

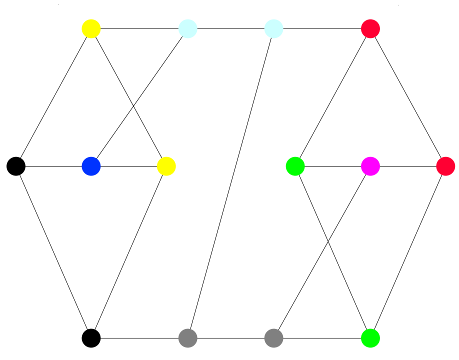


Figure 3.3: Complement of a $(n(G)-4)-$regular graph G with $\chi_b(G) = \lfloor \frac{5n(G)}{8} \rfloor$

For $(n(G)-5)$-regular graphs G, there exist sharpness examples for the bound $\chi_b(G) \leq \lfloor \frac{7n(G)}{11} \rfloor$ as well. Figure 3.4 shows the complement $\overline{G}$ of a $13-$regular graph G of order 18 which has a b-coloring by $11 = \lfloor \frac{7n(G)}{11} \rfloor$ colors. Furthermore, since $\omega(G) = 7$ and $\theta(G) = 3$, it follows that this graph is also a sharpness example for the upper bound in Proposition 3.8.

For the case that G or $\overline{G}$ is disconnected we know the following:

Let G be a graph with disconnected complement $\overline{G}$ and let $\overline{G}_1, ..., \overline{G}_r$ be the components

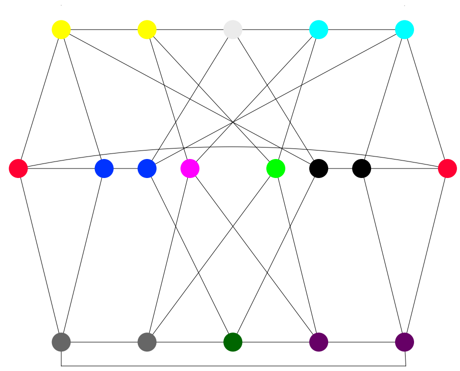


Figure 3.4: Complement of a $(n(G)-5)-$regular graph G with $\chi_b(G) = \lfloor \frac{7n(G)}{11} \rfloor$

of $\overline{G}$. Then it is known that $\chi(G) = \sum_{i=1}^{r} \chi(G_i)$ and $\chi_a(G) = \sum_{i=1}^{r} \chi_a(G_i)$ (see page 13). The following proposition gives a similar result for $\chi_b(G)$.

Proposition 3.10. (Barth et al., [Bar07])

Let $G_1,...,G_r$ be graphs. Then $\chi_b(G_1 \oplus ... \oplus G_r) = \chi_b(G_1) + ... + \chi_b(G_r)$.

For a disconnected graph the b-chromatic number is not always equal to the maximum value of the b-chromatic numbers of its components. For example, Figure 3.5 shows three graphs G_1, G_2, G_3 satisfying $\chi_b(G_1) = \chi_b(G_2) = \chi_b(G_3) = 2$, but $\chi_b(G_1 \cup G_2 \cup G_3) = 3 \neq \max_{1 \leq i \leq 3} \chi_b(G_i)$. Moreover, $\chi_b(G_1 \cup G_2 \cup G_3) = 3 \neq \sum_{i=1}^{3} \chi_b(G_i)$.

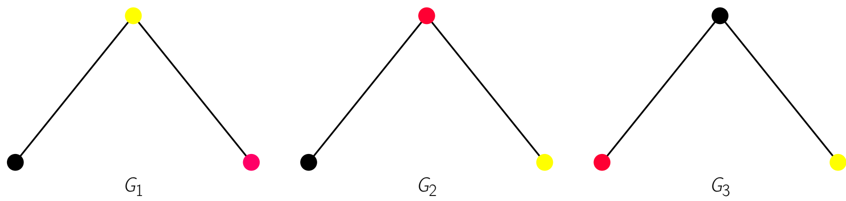


Figure 3.5: b-coloring of a union of graphs

Proposition 3.11. (Kouider and Maheó, [Kou02])

For every disconnected graph G with components $G_1,...,G_r$, $\chi_b(G) \geq \max_{1 \leq i \leq r} \chi_b(G_i)$.

Proposition 3.12. *Let G be a disconnected graph with components $G_1,...,G_r$. Then $\chi_b(G) \leq \sum_{i=1}^{r}(n(G_i) - \chi_b(\overline{G_i})) + 1$.*

Proof.

Since $\chi_b(\overline{G}) = \sum_{i=1}^{r} \chi_b(\overline{G_i})$ by Proposition 3.10 and $\chi_b(G) \leq n+1-\chi_b(\overline{G})$ by Proposition 3.6, we deduce that $\chi_b(G) \leq n+1-\sum_{i=1}^{r} \chi_b(\overline{G_i}) = \sum_{i=1}^{r}(n(G_i) - \chi_b(\overline{G_i})) + 1$. $\square$

Proposition 3.13. (Kohl, [Alk10]) *Let G be a disconnected graph with components $G_1,\ldots,G_r$. Then $\chi_b(G) \le \sum_{i=1}^{r} \chi_b(G_i)$.*

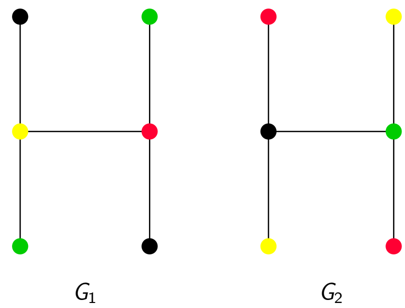


Figure 3.6: *b*-coloring of a disconnected graph

Figure 3.6 shows a *b*-coloring of a union of two graphs G_1 and G_2 satisfying $\chi_b(G_1) = \chi_b(G_2) = 2$ and $\chi_b(G_1 \cup G_2) = \chi_b(G_1) + \chi_b(G_2) = 4$. So $G_1 \cup G_2$ is a sharpness example for the upper bound in Proposition 3.13.

For a coloring c of the graph G let $n_c(G)$ be the number of color classes that contain a color-dominating vertex. We define $d_k(G) := \max_c n_c(G)$ where the maximum is taken over all colorings c of G by exactly k colors.

Theorem 3.1. (Kohl, [Alk10]) *Let G be a disconnected graph with components $G_1, \ldots, G_r$. Then G has a b-coloring by k colors if and only if $k \le \sum_{i=1}^{r} d_k(G_i)$.*

3.2 Exact values of the b-chromatic number for special graphs

G	$\chi_b(G)$
Path P_n	$\begin{cases} 1 & \text{, if } n = 1 \\ 2 & \text{, if } 1 < n < 5 \\ 3 & \text{, if } n \ge 5 \end{cases}$
Cycle C_n	$\begin{cases} 2 & \text{, if } n = 4 \\ 3 & \text{, if } n \ne 4 \end{cases}$
Complete r-partite graph	r
Star $K_{1,n}$	2
Complete graph K_n	n
Wheel $K_1 \oplus C_n$	$\begin{cases} 3 & \text{, if } n = 4 \\ 4 & \text{, if } n \ne 4 \end{cases}$

Table 3.1: Exact values for simple graphs (cf. [Kou02])

3.2.1 Graphs with maximum degree at most 2

If G is a connected graph with maximum degree $\Delta(G) \leq 2$ or an empty graph, then $\chi_b(G)$ is already presented in Table 3.1. If G is a disconnected graph with maximum degree $\Delta(G) = 1$, then G is the union of components of order 2 or 1 and therefore $\chi_b(G) = 2$. So it remains to consider the case that G is disconnected and has maximum degree $\Delta(G) = 2$.

Theorem 3.2. *Let G be a disconnected graph with maximum degree $\Delta(G) = 2$ and let G' be the subgraph of G induced by the components of order at least 3. Then $\chi_b(G) = 2$ if G' is isomorphic to $P_3, P_4, C_4, P_3 \cup P_3, C_4 \cup C_4$, or $C_4 \cup P_3$, respectively. Else, $\chi_b(G) = 3$.*

Proof.
Obviously, G is the union of paths and/or cycles. Since G contains at least two edges, we deduce that $\chi_b(G) \geq \omega(G) \geq 2$. Moreover, $\chi_b(G) \leq \Delta(G) + 1 = 3$.
Case 1: G' is isomorphic to $P_3, P_4, C_4, P_3 \cup P_3, C_4 \cup C_4$, or $C_4 \cup P_3$, respectively.
Assume that G has a b-coloring by 3 colors. At first we notice that components of order at most 2 do not play a role for b-colorings by at least 3 colors because they cannot contain a color-dominating vertex. Hence, G' has a b-coloring by 3 colors. If $G' \simeq P_3, P_4,$ or C_4, then $\chi_b(G') = 2$ according to Table 3.1, a contradiction.

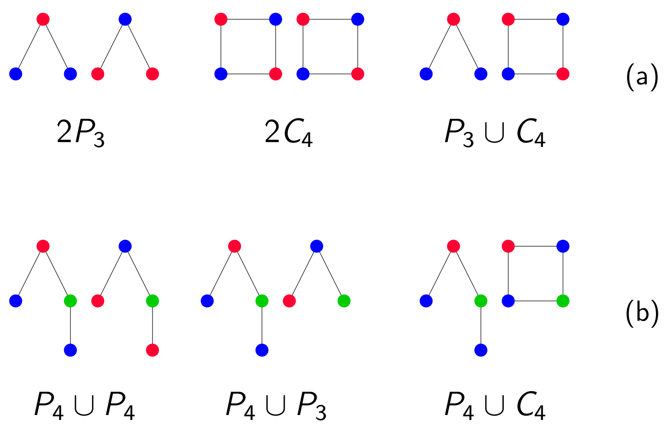


Figure 3.7: b-coloring of graphs with $\Delta = 2$

If $G' \simeq P_3 \cup P_3, C_4 \cup C_4$, or $C_4 \cup P_3$, it follows that at least one of the components of G' contains at least two color-dominating vertices. But neither P_3 nor C_4 can contain more than one color-dominating vertex in a b-coloring by 3 colors, a contradiction. Thus, we deduce that $\chi_b(G) = 2$. Figure 3.7(a) shows b-colorings by 2 colors for the cases where G' is disconnected.

Case 2: G' is not isomorphic to $P_3, P_4, C_4, P_3 \cup P_3, C_4 \cup C_4$, or $C_4 \cup P_3$, respectively.
If G' has at least three components, then we can easily find a b-coloring by 3 colors. Hence, $\chi_b(G) = 3$. If G' has at most two components and one of them is isomorphic to a path of order at least 5 or a cycle different from C_4, then $\chi_b(G) = 3$ according to Table 3.1 and Proposition 3.11. Else, G' is isomorphic to one of the graphs in Figure 3.7(b). Because each of these graphs has a b-coloring by 3 colors (see Figure 3.7(b)), we deduce that $\chi_b(G) \geq 3$. Altogether it follows that $\chi_b(G) = 3$. $\square$

3.2.2 Graphs with independence number close to its order

We note that according to Proposition 3.2 if G is a graph with an independence number $\alpha(G) = n(G) - k$, then $\chi_b(G) \leq k + 1$. Moreover, if $\alpha(G) = n(G) - 1$, then $G \simeq K_{1,i} \cup \underbrace{K_1 \cup ... \cup K_1}_{n-i-1}$ for $i \geq 1$ and thus $\chi_b(G) = 2$.

Theorem 3.3. *Let G be a graph of order n and independence number $\alpha(G) = n-2$. Then $\chi_b(G) = 3$ if $\omega(G) = 3$ or G contains an induced path of order 5. Otherwise, $\chi_b(G) = 2$.*

Proof. Let I be a maximum independent set of order $n - 2$. According to Proposition 3.2 and Inequality (3.2) we obtain $2 \leq \chi_b(G) \leq 3$. So it suffices to prove that there exists a b-coloring by 3 colors if and only if $\omega(G) = 3$ or G contains an induced path of order 5.

[" $\Longrightarrow$ "] Suppose that there is a b-coloring of G by 3 colors.
Let V_1, V_2, V_3 be the corresponding color classes. Moreover, let v_1, v_2, v_3 be color-dominating vertices of colors 1, 2, 3, respectively.
Case 1: $\{v_1, v_2, v_3\}$ induces a clique of order 3. Then $\omega(G) = 3$.
Case 2: $\{v_1, v_2, v_3\}$ does not induce a clique of order 3. W.l.o.g. suppose that $v_1v_2 \in E(\overline{G})$.

– Assume that $\{v_1, v_2\} \subseteq I$. Since v_1, v_2 are color-dominating vertices of colors 1 and 2, respectively, it follows that $V \setminus I$ has to contain at least 3 vertices of colors 1,2 and 3, a contradiction to $|V(G) \setminus I| = 2$.

– Assume that I contains either v_1 or v_2, w.l.o.g. suppose that $v_1 \in I$. Then since $v_1v_2 \in E(\overline{G})$ and v_1 is a color-dominating vertex of color 1 it follows that $V(G) \setminus I$ has to contain two vertices different from v_2 of color 2 and 3, again a contradiction.

Hence we conclude that $V(G) \setminus I = \{v_1, v_2\}$.
Since v_3 is a color-dominating vertex and $v_3 \in I$ we deduce that $v_1v_3 \in E(G)$ and $v_2v_3 \in E(G)$. Because v_1 and v_2 are color-dominating vertices and $v_1v_2 \notin E(G)$, there exist $u_1 \in N(v_1) \cap V_2$ and $u_2 \in N(v_2) \cap V_1$ and hence $(u_1, v_1, v_3, v_2, u_2)$ is a path of order 5 in G which is an induced path because $u_1, u_2, v_3 \in I$, $u_1, v_2 \in V_2$ and $v_1, u_2 \in V_1$ and $v_1v_2 \notin E(G)$.

[" $\Longleftarrow$ "] Suppose that $\omega(G) = 3$ or G contains an induced path of order 5.
If $\omega(G) = 3$, then $3 = \omega(G) \leq \chi_b(G) \leq 3$ and thus $\chi_b(G) = 3$.
Else, $\omega(G) \leq 2$ and G contains an induced path (u_1, v_1, w, v_2, u_2). If at most two independent vertices from $\{u_1, v_1, w, v_2, u_2\}$ belong to I, then $V(G) \setminus I$ has to contain at least 3 vertices, a contradiction to $|I| = n - 2$. Therefore, we deduce that I has to contain at least 3 independent vertices from the set $\{u_1, v_1, w, v_2, u_2\}$ and hence $\{u_1, u_2, w\} \subseteq I$, $\{v_1, v_2\} = V(G) \setminus I$.

Since (u_1, v_1, w, v_2, u_2) is an induced path of order 5 it follows that $u_1 \in N(v_1) \setminus N(v_2)$, $u_2 \in N(v_2) \setminus N(v_1)$, $w \in N(v_1) \cap N(v_2)$. Now we define a coloring c of G by 3 colors as follows: Set $c(x) = 1$ for $x \in \{v_1\} \cup (N(v_2) \setminus N(v_1))$, $c(x) = 2$ for $w \in \{v_2\} \cup (N(v_1) \setminus N(v_2))$ and the remaining uncolored vertices $w \in V(G)$ receive the color $c(x) = 3$.
Since $\{v_1\} \cup (N(v_2) \setminus N(v_1))$, $\{v_2\} \cup (N(v_1) \setminus N(v_2))$, and $N(v_1) \cap N(v_2)$ are independent sets, it follows that the coloring c is proper.
Because of $c(u_1) = 2$, $c(w) = 3$ and $\{u_1, w\} \subseteq N(v_1)$ we deduce that v_1 is a color-dominating vertex of color 1. Analogously, we can prove that v_2 is a color-dominating vertex of color 2. Moreover, since $c(w) = 3$ and w is adjacent to v_1 and to v_2 which received the colors 1 and 2, respectively, it follows that w is a color-dominating vertex of color 3.
Altogether, c is a b-coloring of G. $\square$

Remark 3.1. *Let G be a graph with independence number $\alpha(G) = n(G) - 3$. According to Proposition 3.2 we deduce that $2 \leq \chi_b(G) \leq 4$.*
By Theorem 3.11, we can characterize the graphs G with $\chi_b(G) = \chi(G) = 2$. Moreover, it

is obvious that $2 \leq \omega(G) \leq 4$.

If $\omega(G) \in \{3, 4\}$, *then we can determine* $\chi_b(G)$ *by using the results from Subsection 3.2.4.*

It remains to investigate all graphs G *with* $\omega(G) = 2$ *and* $3 \leq \chi_b(G) \leq 4$.

3.2.3 Graphs with minimum degree close to its order

In this subsection we are interested in graphs with large minimum degree. In particular, we establish the exact value of $\chi_b(G)$ for graphs with minimum degree $\delta(G) \geq n(G) - 3$.

Obviously, if G is a graph with minimum degree $\delta(G) = n(G) - 1$, then $\chi_b(G) = \chi(G) = n(G)$.

Theorem 3.4. *Let* G *be a graph of order* n *and minimum degree* $\delta(G) = n - 2$. *Moreover, let* ζ *be the number of vertices of degree* $n - 2$. *Then* $\chi_b(G) = \chi(G) = n - \frac{\zeta}{2}$.

Proof. Let $s := \frac{\zeta}{2}$. Obviously, the ζ vertices of degree $n - 2$ induce a matching M of cardinality s in $\overline{G}$. Let $M = \{e_1, ..., e_s\}$ such that $e_i = u_i v_i \in E(\overline{G})$ for $i \in \{1, \ldots, s\}$. We notice that $V(G) \setminus \{v_1, ..., v_s\}$ induces a clique Q of order $n - s$ in G. Hence, $\chi_b(G) \geq \chi(G) \geq \omega(G) \geq n - s$.

Suppose that there is a b-coloring c of G by $n - s + a$ colors for $a \geq 1$. W.l.o.g. let $c(V(Q)) = \{1, ..., n - s\}$ such that $c(u_h) = h$ for $h \in \{1, \ldots, s\}$. Since the clique Q can contain at most $n - s$ color-dominating vertices, there exists an integer $i \in \{1, ..., s\}$ such that v_i is a color-dominating vertex with color $c(v_i) > n - s$. Moreover, since $u_i v_i \notin E(G)$, there exists an integer $j \in \{1, ..., s\} \setminus \{i\}$ such that $v_j \in N(v_i)$ and $c(v_j) = i$. Because of $u_i v_j \in E(G)$ and $c(u_i) = c(v_j)$, c is not a proper coloring of G, a contradiction. Thus, $\chi_b(G) \leq n - s$ and altogether, $\chi_b(G) = \chi(G) = n - s = n - \frac{\zeta}{2}$. □

Corollary 3.1. *If* G *is a* $(n(G) - 2)$*-regular graph, then* $\chi_b(G) = \chi(G) = \frac{n(G)}{2}$.

Let G be a connected graph with maximum degree $\Delta(G) = 2$. A *segmentation* $S(G)$ of G shall denote a set of disjoint paths that cover all vertices of G.

If $V(P^1) \cup V(P^2)$ induces a path of order $|V(P^1)| + |V(P^2)|$ in G, then we say that P^1 and P^2 are *consecutive*.

If P^1 and P^2 are non-consecutive, then there exists a so-called *separating set* $\{Q^1, ..., Q^l\} \subseteq S(G)$ of $l \geq 1$ paths such that by setting $Q^0 := P^1$ and $Q^{l+1} := P^2$ the subset $V(Q^i) \cup$

$V(Q^{i+1})$ induces a path of order $|V(Q^i)| + |V(Q^{i+1})|$ in G for $i \in \{0, ..., l\}$. In case that every separating set for P^1 and P^2 contains at least two paths of order 2 we say that P^1 and P^2 are *separated by at least two paths of order 2.* Note that if G is a path, then the separating set is unique and if G is a cycle, then there exist exactly two distinct separating sets.

Lemma 3.1. (Alkhateeb and Kohl) *Let G be a graph of order $n \geq 4$, minimum degree $\delta(G) = n - 3$, and with connected complement $\overline{G}$. Moreover, let c be a vertex coloring of G by k colors where $V_1, ..., V_k$ are the corresponding color classes.*
Then c is a b-coloring of G if and only if $\{\overline{G}[V_1], \ldots, \overline{G}[V_k]\}$ is a segmentation of $\overline{G}$ into paths of order 1 and 2 such that any two paths of order 1 are separated by at least two paths of order 2.

Proof. Since $\overline{G}$ is connected and $\Delta(\overline{G}) = n - 1 - \delta(G) = 2$, $\overline{G}$ is isomorphic to a cycle C_n or a path P_n of order $n \geq 4$. Moreover, because of $\alpha(G) = \omega(\overline{G}) = 2$, it is obvious that c is a proper vertex coloring of G if and only if $|V_i| \in \{1, 2\}$ and $\overline{G}[V_i] \simeq P_{|V_i|}$ for $i \in \{1, ..., k\}$. This implies that c is a proper vertex coloring of G if and only if $\{\overline{G}[V_1], \ldots, \overline{G}[V_k]\}$ is a segmentation of $\overline{G}$ into paths of order 1 and 2.

In the following let $V_h = \{u_h, v_h\}$, $V_i = \{u_i\}$ and $V_j = \{u_j\}$ denote three distinct color classes of cardinality 2 and 1, respectively ($h, i, j \in \{1, \ldots, k\}$).

[" $\Rightarrow$ "] Assume that c is a b-coloring of G.
– Suppose that $\overline{G}[V_i]$ and $\overline{G}[V_j]$ are consecutive. Then $u_iu_j \in E(\overline{G})$ and u_i has no neighbor in color class V_j. Hence, V_i has no color-dominating vertex, a contradiction.
– Suppose that $\overline{G}[V_h]$ is a separating set for $\overline{G}[V_i]$ and $\overline{G}[V_j]$ and w.l.o.g. let
$\overline{G}[V_h \cup V_i \cup V_j] = (u_i, u_h, v_h, u_j)$. Since $u_iu_h \in E(\overline{G})$ and $v_hu_j \in E(\overline{G})$, u_h has no neighbor in color class V_i and v_h has no neighbor in color class V_j. So there is no color-dominating vertex in V_h, a contradiction.
It follows from this that any two paths of order 1 from $\{\overline{G}[V_1], \ldots, \overline{G}[V_k]\}$ are separated by at least two paths of order 2.

[" $\Leftarrow$ "] Assume that $S(\overline{G})$ is a segmentation of $\overline{G}$ into paths of order 1 and 2 such that any two paths of order 1 are separated by at least two paths of order 2.
– Consider the color class V_i of cardinality 1.

Since $\overline{G}[V_i]$ and $\overline{G}[V_j]$ are not consecutive, it follows that $u_i u_j \in E(G)$. Moreover, because of $\alpha(G) = 2$, u_i has at least one neighbor in each color class of cardinality 2. So, u_i is a color-dominating vertex of the color class V_i.

– Consider the color class V_h of cardinality 2.

Because of $\Delta(\overline{G}) = 2$ and $u_h v_h \in E(\overline{G})$, it follows that $|\overline{N}(u_h) \setminus \{v_h\}| \leq 1$ and $|\overline{N}(v_h) \setminus \{u_h\}| \leq 1$. If $|\overline{N}(u_h) \setminus \{v_h\}| = 0$ or ($|\overline{N}(u_h) \setminus \{v_h\}| = 1$ and $w_u \in \overline{N}(u_h) \setminus \{v_h\}$ belongs to a color class of cardinality 2), then u_h has a neighbor in each other color class, i.e. u_h is a color-dominating vertex. The same can be shown for v_h. So it remains to consider the case where $|\overline{N}(u_h)\setminus\{v_h\}| = |\overline{N}(v_h)\setminus\{u_h\}| = 1$ and $w_u \in \overline{N}(u_h)\setminus\{v_h\}$, $w_v \in \overline{N}(v_h)\setminus\{u_h\}$ belong to color classes of cardinality 1. Because of $\omega(\overline{G}) = 2$, we know $w_u \neq w_v$. So, w.l.o.g. let $w_u = u_i$ and $w_v = u_j$. Then $\overline{G}[V_i]$ and $\overline{G}[V_j]$ are separated by $\overline{G}[V_h]$, a contradiction to the properties of the segmentation. Hence, we can deduce that V_h contains a color-dominating vertex.

We conclude that every color class of cardinality 1 and every color class of cardinality 2 has a color-dominating vertex. Thus, c is a b-coloring of G. □

Theorem 3.5. (Alkhateeb and Kohl) *Let G be a graph of order $n \geq 3$ and minimum degree $\delta(G) = n - 3$ such that $\overline{G}$ is connected. Then*

$$\chi_b(G) = \begin{cases} \lfloor \frac{3n}{5} \rfloor & \text{, if } \overline{G} \simeq C_n \;\vee\; \big(\overline{G} \simeq P_n \;\wedge\; 2 \mid (n \bmod 5)\big) \\ \lceil \frac{3n}{5} \rceil & \text{, if } \overline{G} \simeq P_n \;\wedge\; 2 \nmid (n \bmod 5) \end{cases}.$$

Proof. Since $\overline{G}$ is connected and $\Delta(\overline{G}) = 2$, $\overline{G} \simeq C_n$ or $\overline{G} \simeq P_n$.

Let $n = 3$. If $\overline{G} \simeq C_3$, then G is an empty graph and, therefore, $\chi_b(G) = 1 = \lfloor \frac{3n}{5} \rfloor$. If $\overline{G} \simeq P_3$, then $G = K_1 \cup K_2$ yielding $\chi_b(G) = 2 = \lceil \frac{3n}{5} \rceil$.

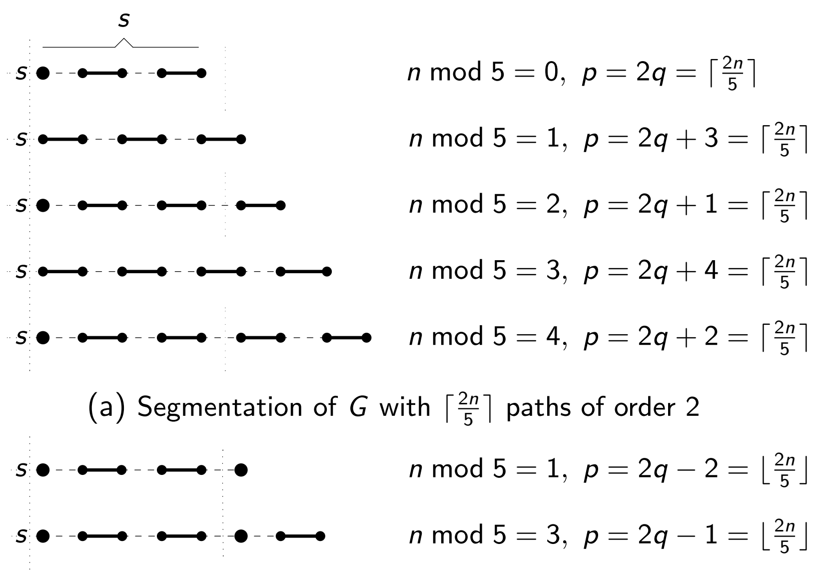
Now consider $n \geq 4$ and let c be a b-coloring of G by k colors where V_1,...,V_k are the corresponding color classes. By Lemma 3.1 we know that $\{\overline{G}[V_1], \ldots, \overline{G}[V_k]\}$ is a segmentation of $\overline{G}$ into paths of order 1 and 2 such that any two paths of order 1 are separated by at least two paths of order 2.

Let p and q denote the number of color classes of cardinality 2 and 1, respectively. Then we obtain $q = n - 2p$ and $k = p + q = p + (n - 2p) = n - p$.

Moreover, since any two paths of order 1 are separated by at least two paths of order

2, we deduce that $p \geq 2q$ if $\overline{G} \simeq C_n$ and $p \geq 2(q-1)$ if $\overline{G} \simeq P_n$. In case of $\overline{G} \simeq P_n$ and $2 \mid (n \bmod 5)$, we can verify that $p \geq 2q$ as follows: $2 \mid (n \bmod 5)$ implies $\exists Q \in \mathbb{Z}$: $n = 5Q+5$. Hence, $n = 5Q+5 = 2p+q$ and therefore $5 \mid (2p-2+q)$. This is not possible for $p = 2(q-1)$ and $p = 2q-1$.

Obviously, the b-chromatic number $\chi_b(G)$ is the largest possible value for k. Since $k = n-p$, we obtain this maximum integer k by minimizing p. So it remains to determine $p_{\min}$ which shall denote the smallest integer p that satisfies the inequality mentioned above.



(a) Segmentation of G with $\lceil \frac{2n}{5} \rceil$ paths of order 2

(b) Segmentation of G with $\lfloor \frac{2n}{5} \rfloor$ paths of order 2

Figure 3.8: Segmentation of $\overline{G}$

– If $\overline{G} \simeq C_n$ or $\big(\overline{G} \simeq P_n$ and $2 \mid (n \bmod 5)\big)$, then $p \geq 2q = 2(n-2p)$ and, therefore, $p \geq \frac{2n}{5}$. Thus, $p_{\min} \geq \lceil \frac{2n}{5} \rceil$. There is a segmentation of $\overline{G}$ with exactly $\lceil \frac{2n}{5} \rceil$ paths of order 2 (see Figure 3.8(a)). So we can deduce that $p_{\min} = \lceil \frac{2n}{5} \rceil$ and, therefore, $\chi_b(G) = n - p_{\min} = \lfloor \frac{3n}{5} \rfloor$.

– If $\overline{G} \simeq P_n$ and $2 \nmid (n \bmod 5)$, then $p \geq 2(q-1) = 2(n-2p-1)$ and, therefore, $p \geq \frac{2(n-1)}{5}$. Hence, $p_{\min} = \lceil \frac{2(n-1)}{5} \rceil$. Moreover, since $\lceil \frac{2(n-1)}{5} \rceil = \lfloor \frac{2n}{5} \rfloor$ for $(n \bmod 5) \in \{1,3\}$ we deduce that $p_{\min} \geq \lfloor \frac{2n}{5} \rfloor$. We can find a segmentation of $\overline{G}$ with exactly $\lfloor \frac{2n}{5} \rfloor$ paths of order 2 (see Figure 3.8(b)). This yields $p_{\min} = \lfloor \frac{2n}{5} \rfloor$ and therefore $\chi_b(G) = n - p_{\min} = \lceil \frac{3n}{5} \rceil$. $\square$

The following Figure 3.9 shows a b-coloring by $\chi_b(G)$ colors for two graphs G with minimum degree $\delta(G) = n(G) - 3$. Note that these colorings correspond to segmentations of $\overline{G}$ into paths of order 2 and 1 (compare Lemma 3.1).

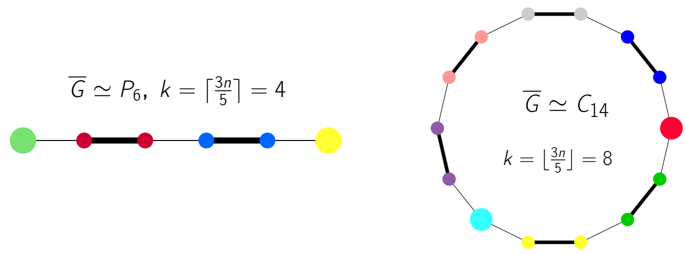


Figure 3.9: Examples of b-colorings of graphs G with $\delta(G) = n(G) - 3$.

Theorem 3.5 immediately implies:

Corollary 3.2. *Let G be a graph of order $n \geq 3$ and minimum degree $\delta(G) = n - 3$ such that $\overline{G}$ is connected. Then*

$$\chi_b(G) = \begin{cases} \lfloor \frac{3n}{5} \rfloor & , \text{ if } \overline{G} \simeq C_n \\ \lfloor \frac{3n+2}{5} \rfloor & , \text{ if } \overline{G} \simeq P_n \end{cases}.$$

If the complement $\overline{G}$ is disconnected and $\overline{G}_1, \ldots, \overline{G}_s$ are the components of $\overline{G}$, then we already know from page 19 that $\chi(G) = \sum_{i=1}^{s} \chi(G_i)$ and $\chi_b(G) = \sum_{i=1}^{s} \chi_b(G_i)$. This allows us to determine $\chi_b(G)$ for all graphs with minimum degree $\delta(G) = n(G) - 3$.

Remark 3.2. *Let G be a graph of order $n \geq 3$ and minimum degree $\delta(G) = n - 3$ such that $\overline{G}$ is disconnected. Moreover let $\overline{G}_1, \ldots, \overline{G}_s$ be the components of $\overline{G}$ and $G_i = \overline{\overline{G}}_i$ $(i = 1 \ldots, s)$. As already mentioned in Section 3.1, $\chi_b(G) = \sum_{i=1}^{s} \chi_b(G_i)$ holds. So we only have to determine $\chi_b(G_i)$ for $i \in \{1, \ldots, s\}$.*

Obviously, $\delta(G_i) \geq n(G_i) - 3$ and $\overline{G}_i$ is connected. If $\delta(G_i) = n(G_i) - 1$ or $\delta(G_i) = n(G_i) - 2$, then $G_i \simeq K_1$ or $G_i \simeq K_1 \cup K_1$, respectively. Hence we can deduce that $\chi_b(G_i) = 1$ in both cases. If $\delta(G_i) = n(G_i) - 3$, then we can apply Theorem 3.5 yielding $\lfloor \frac{3n(G_i)}{5} \rfloor$ or $\lceil \frac{3n(G_i)}{5} \rceil$ depending on $\overline{G}_i$.

If G is a $(n(G) - 3)$-regular graph, then every component of $\overline{G}$ is a cycle. So we deduce:

Corollary 3.3. *If G is a $(n(G) - 3)$-regular graph, then $\chi_b(G) = \sum_{i=1}^{s} \lfloor \frac{3n(G_i)}{5} \rfloor$ where $\overline{G}_1, \ldots, \overline{G}_s$ are the components of $\overline{G}$.*

Since the complements of graphs G with minimum degree $\delta(G) = n(G) - 4$ are graphs with maximum degree 3 which cannot be easily characterized like graphs with maximum degree 2, we believe that a result on the b-chromatic number like Theorem 3.5 cannot be obtained for $\delta(G) = n(G) - 4$.
It may be even possible, that the determination of $\chi_b(G)$ for these graphs is $\mathcal{NP}-$hard.

However, it makes sense to provide some bounds on the b-chromatic number of such graphs. For $\delta(G) = n(G)-4$, it follows that $\Delta(\overline{G}) = 3$ and, therefore, $\theta(G) = \chi(\overline{G}) \leq 4$. Moreover, Brooks' Theorem implies that $\theta(G) \leq 3$ if no component of $\overline{G}$ is a K_4. According to $n(G) \leq \theta(G) \cdot \omega(G)$, $\chi_b(G) \geq \chi(G)$, and Proposition 3.8 we obtain:

Corollary 3.4. *Let G be a graph with $\delta(G) = n(G) - 4$ and $\theta(G) = t$. Then:*
(a) $\lceil \frac{n(G)}{4} \rceil \leq \lceil \frac{n(G)}{t} \rceil \leq \chi_b(G) \leq \frac{t\omega(G)+(t-1)n(G)}{2t-1} \leq \frac{4\omega(G)+3n(G)}{7}$,
(b) $\lceil \frac{n(G)}{2} \rceil \leq \chi_b(G) \leq \lfloor \frac{2\omega(G)+n(G)}{3} \rfloor$ *if $\overline{G}$ is bipartite.*

Moreover, Kohl established other bounds by using Proposition 3.9 for graphs satisfying $\Delta(\overline{G}) = 3$, in particular for graphs whose minimum and maximum degree are close to each other.

Corollary 3.5. (Kohl, [Alk10])
Let G be a graph with $\delta(G) = n(G) - 4$. Then:
(a) $\chi_b(G) \leq \lfloor \frac{2n(G)}{3} \rfloor$ *and* $\lfloor \frac{3n(G)}{4} \rfloor$ *if* $\Delta(G) = \delta(G) + 1$ *and* $\delta(G) + 2$*, respectively,*
(b) $\chi_b(G) \leq \lfloor \frac{5n(G)}{8} \rfloor$ *if G is regular,*
(c) $\lceil \frac{n(G)}{2} \rceil \leq \chi_b(G) \leq \lfloor \frac{5n(G)}{8} \rfloor$ *if G is regular and $\alpha(G) = 2$.*

Note that the complements of the graphs in (c) are triangle-free 3-regular graphs. The gap between lower and upper bound is here at most $\frac{n(G)}{8}$. Both bounds are sharp since the cycle C_6 satisfies $\chi_b(C_6) = 3 = \frac{n(C_6)}{2}$. Furthermore, Figure 3.3 shows a sharpness example for the upper bound $\lfloor \frac{5n(G)}{8} \rfloor$.

3.2.4 Graphs G with $\alpha(G) + \omega(G) \geq n(G)$

Since every graph G satisfies $\chi_b(G) \geq \omega(G)$ and $\chi_b(\overline{G}) \geq \alpha(G)$ we deduce that $\chi_b(G) + \chi_b(\overline{G}) \geq \omega(G) + \alpha(G)$. So by Proposition 3.6 we immediately obtain:

Proposition 3.14. *For every graph G, $\omega(G) + \alpha(G) \leq \chi_b(G) + \chi_b(\overline{G}) \leq n(G) + 1$.*

For example, the lower bound is sharp for every $(n(G)-2)$-regular graph since $\chi_b(G) = \omega(G) = \frac{n(G)}{2}$ and $\chi_b(\overline{G}) = \alpha(G) = 2$. It is also sharp for split graphs because every split graph G satisfies $\chi_b(G) = \omega(G)$ ([Kra02]) and the complement $\overline{G}$ is a split graph as well.

The upper bound is sharp e.g. for all complete graphs and for the graphs C_5 and P_6 because $\chi_b(C_5) = \chi_b(P_6) = 3$, $\chi_b(\overline{C_5}) = \lfloor \frac{3\cdot 5}{5} \rfloor = 3$, $\chi_b(\overline{P_6}) = \lceil \frac{3\cdot 6}{5} \rceil = 4$ (compare Theorem 3.5). If $\alpha(G) + \omega(G) = n(G) + 1$ for a graph G, then Proposition 3.14 yields $\chi_b(G) + \chi_b(\overline{G}) = n(G) + 1$. Moreover, from above we further deduce that $\chi_b(G) = \omega(G)$ and $\chi_b(\overline{G}) = \alpha(G)$.

Also by Proposition 3.14 we deduce that every graph G of order $n(G) = \alpha(G) + \omega(G)$ satisfies $n(G) \leq \chi_b(G) + \chi_b(\overline{G}) \leq n(G) + 1$.
So by Proposition 3.2 we conclude the following:

Corollary 3.6. *For every graph G of order $n(G) = \alpha(G) + \omega(G)$ we obtain:*
(a) $\chi_b(\overline{G}) = \alpha(G)$ if $\chi_b(G) = \omega(G) + 1$, and
(b) $\chi_b(G) = \omega(G)$ if $\chi_b(\overline{G}) = \alpha(G) + 1$.

According to Proposition 3.2 and Inequality (3.2) it follows that $\omega(G) \leq \chi_b(G) \leq \omega(G)+1$ for every graph G of order $n(G) = \alpha(G) + \omega(G)$.

Now we want to characterize the graphs having b-chromatic number $\chi_b(G) = \omega(G) + 1$.

Theorem 3.6. *Let G be a graph of order $n = \alpha(G) + \omega(G)$. Then $\chi_b(G) = \omega(G) + 1$, if there exist maximum independent sets I and $\overline{I}$ in G and $\overline{G}$, respectively, such that there exist two vertices $w \in I \cap \overline{I}$, $x \in V(G) \setminus (I \cup \overline{I})$, and the following properties are satisfied:*

- *(a) $\overline{G}$ contains a matching M of size $q := |\overline{N}(x) \cap (\overline{I} \setminus \{w\})|$ such that for every edge $e \in M$ one end vertex belongs to $\overline{N}(x) \cap (\overline{I} \setminus \{w\})$ and the other end vertex belongs to $N(x) \cap (I \setminus \{w\})$.*
- *(b) If $xw \in E(\overline{G})$, then $q \geq 0$ and there exists a vertex $u \in I \setminus \{w\}$ with $d_G(u) = \omega(G)$. If $xw \in E(G)$, then $q \geq 1$ and G contains a set $D \subseteq \overline{N}(x) \cap I$ which dominates the vertices from $\overline{N}(x) \cap \overline{I}$ in G.*

Otherwise, $\chi_b(G) = \omega(G)$.

Proof.

Since $\omega(G) \leq \chi_b(G) \leq \omega(G)+1$ it suffices to prove that there exists a b-coloring by $w(G)+1$ colors if and only if the conditions mentioned in the Theorem are satisfied.

[" $\Longleftarrow$ "] Suppose that there exist maximum independent sets I and $\overline{I}$ in G and $\overline{G}$, respectively, such that there exist two vertices $w \in I \cap \overline{I}$ and $x \in V(G) \setminus (I \cup \overline{I})$, and the properties (a) and (b) are satisfied.

We denote by $v_1, ..., v_{\omega(G)-1}$ the vertices in $\overline{I} \setminus \{w\}$. If $q > 0$, then w.l.o.g. let $v_1, ..., v_q$ be the vertices in $\overline{N}(x) \cap (\overline{I} \setminus \{w\})$. Moreover, let $u_1, ..., u_q \in N(x) \cap (I \setminus \{w\})$ be the matching partners of $v_1, ..., v_q$. i.e. $M = \cup_{i=1}^{q}\{u_iv_i\}$. Now we consider the following two cases:

Case 1: $xw \in E(\overline{G})$, $\overline{G}$ contains the matching M of size $q \geq 0$ and there exists the vertex $u \in I \setminus \{w\}$ with $d_G(u) = \omega(G)$.

We define a coloring c of G by $\omega(G)+1$ colors as follows:

$c(u_i) = c(v_i) = i$ for $i \in \{1, ..., q\}$, $c(v_i) = i$ for $i \in \{q+1, ..., \omega(G)-1\}$, $c(x) = c(w) = \omega(G)$ and $c(v) = \omega(G)+1$ for every $v \in I \setminus \{u_1, ...u_q, w\}$. Note that $u \in I \setminus \{u_1, ..., u_q, w\}$ and thus $c(u) = \omega(G)+1$.

Since $u_iv_i \in M$ for $i \in \{1, ..., q\}$, $xw \in E(\overline{G})$ and I is an independent set in G we can easily conclude that the color classes are independent sets in G. Therefore, this coloring is proper.

Because $d_G(u) = \omega(G)$, it follows that $N(u) = V(G) \setminus I$ which implies $c(N(u)) = \{1, ..., \omega(G)\}$ and therefore, u is a color-dominating vertex of color $\omega(G)+1$. Analogously, for $i \in \{1, ..., \omega(G)-1\}$, vertex v_i is adjacent to u and to all other vertices in $\overline{I} \setminus \{v_i\}$ and thus $c(N(v_i)) = \{1, ..., \omega(G)+1\} \setminus \{i\}$. Therefore, $v_1, ..., v_{\omega(G)-1}$ are color-dominating vertices of colors $1, ..., \omega(G)-1$, respectively. Furthermore, since $xu \in E(G)$, $xu_i \in E(G)$ for $i \in \{1, ..., q\}$ and $xv_j \in E(G)$ for $j \in \{q+1, ..., \omega(G)-1\}$ we deduce that x has a neighbor in every other color class. It follows that x is a color-dominating vertex of color $\omega(G)$.

Thus, c is a b-coloring of G by $\omega(G)+1$ colors.

Case 2: $xw \in E(G)$, $\overline{G}$ contains the matching M of size $q \geq 1$ and there exists the set $D \subseteq \overline{N}(x) \cap I$ which dominates the vertices from $\overline{N}(x) \cap \overline{I}$ in C.

We define a coloring c of G by $\omega(G)+1$ colors as follows:

$c(u_i) = c(v_i) = i$ for $i \in \{1, ..., q\}$, $c(v_i) = i$ for $i \in \{q+1, ..., \omega(G)-1\}$, $c(v) = \omega(G)$ for every $v \in (N(x) \cap I) \setminus V(M)$ and $c(v) = \omega(G)+1$ for every $v \in \{x\} \cup (\overline{N}(x) \cap I)$.

By little efforts we can prove that the color classes are independent sets in G. Therefore, this coloring is proper.

Moreover, for $i \in \{1, ..., \omega(G)-1\}$, vertex v_i is adjacent to all other vertices in $\overline{I} \setminus \{v_i\}$, to x and if $xv_i \notin E(G)$, then there exists $v \in D$ such that $v_i v \in E(G)$ and $c(v) = \omega(G)+1$. This yields $c(N(v_i)) = \{1, ..., \omega(G)+1\} \setminus \{i\}$ and $v_1, ..., v_{\omega(G)-1}$ are color-dominating vertices of colors $1, ..., \omega(G)-1$, respectively. Since $w \in \overline{I}$ and $xw \in E(G)$ it follows that w has a neighbor in every other color class. Therefore, w is a color-dominating vertex of color $\omega(G)$. Similarly to the last case we can prove that x is a color-dominating vertex of color $\omega(G)+1$.

Altogether, c is a b-coloring of G by $\omega(G)+1$ colors.

[" $\Longrightarrow$ "] Suppose that there is a b-coloring c of G by $\omega(G)+1$ colors and let $V_1, ..., V_{\omega(G)+1}$ be the corresponding color classes.

Let I and $\overline{I}$ be maximum independent sets in G and $\overline{G}$, respectively. Obviously, $|I \cap \overline{I}| \leq 1$ and $\overline{I}$ induces a clique in G. Assume that $|I \cap \overline{I}| = 0$, i.e. $V(G) \setminus \overline{I} = I$. Then since $\overline{I}$ can contain at most $\omega(G)$ color-dominating vertices there has to exist a color-dominating vertex $v \in I$. This implies $N(v) = \overline{I}$ and thus, $\overline{I} \cup \{v\}$ induces a clique of order $\omega(G)+1$, a contradiction. Hence, $|I \cap \overline{I}| = 1$ and we denote the vertex from $I \cap \overline{I}$ by w. Furthermore, this implies that there is exactly one vertex $x \in V(G) \setminus (I \cup \overline{I})$. Now we denote by $v_1, ..., v_{\omega(G)-1}$ the vertices from $\overline{I} \setminus \{w\}$. Since $|\overline{N}(x) \cap \overline{I}| \geq 1$ we deduce $q := |\overline{N}(x) \cap \overline{I} \setminus \{w\}| \geq 0$. Moreover, for $q > 0$ let $v_1, ..., v_q$ be the vertices from $\overline{N}(x) \cap \overline{I} \setminus \{w\}$. Because $v_1, ..., v_{\omega(G)-1}, w \in \overline{I}$ they have to belong to different color classes, we can suppose w.l.o.g. that $w \in V_{\omega(G)}$ and $v_i \in V_i$ for $i \in \{1, ..., \omega(G)-1\}$. Now we distinguish between the following three cases:

Case 1: $\exists\, i \in \{1, ..., \omega(G)-1\}$ such that $x \in V_i$.

Then, because no vertex from $\overline{I} \cup \{x\}$ has color $\omega(G)+1$, we deduce that there exists a color-dominating vertex $u \in I \cap V_{\omega(G)+1}$. However, since $N(u) \subseteq (\overline{I} \setminus \{w\}) \cup \{x\}$ and $\omega(G) \notin c((\overline{I} \setminus \{w\}) \cup \{x\})$ we deduce that u has no neighbor of color $\omega(G)$, a contradiction.

Case 2: $x \in V_{\omega(G)}$.

This implies $xw \in E(\overline{G})$ and $V_{\omega(G)+1} \subseteq I$. Hence, there exists a color-dominating vertex $u \in I \cap V_{\omega(G)+1}$ which is only possible if $N(u) = (\overline{I} \setminus \{w\}) \cup \{x\}$. Therefore, $d_G(u) = \omega(G)$. Moreover, since $V_{\omega(G)+1} \subseteq I$, all color-dominating vertices from the color classes

$V_1, ..., V_{\omega(G)}$ belong to $V(G) \setminus I$. This implies that $v_1, ..., v_{\omega(G)-1}, x$ are the only color-dominating vertices of colors $1, ..., \omega(G)$. Because x is color-dominating and $v_1, ..., v_q \in \overline{N}(x) \cap (\overline{I} \setminus \{w\})$, it follows that there have to exist q vertices $u_1, ..., u_q \in N(x)$ such that $u_i \in V_i \cap I$ for $i \in \{1, ..., q\}$ and $u_i v_i \in E(\overline{G})$. Thus, $M = \cup_{i=1}^{q} \{u_i v_i\}$ is a matching in $\overline{G}$ of size $q > 0$.

Case 3: $x \in V_{\omega(G)+1}$.
Then $V_{\omega(G)} \subseteq I$ and it follows that all other color-dominating vertices from the color classes $V_1, ..., V_{\omega-1}, V_{\omega+1}$ belong to $V(G) \setminus I$. Therefore, $v_1, ..., v_{\omega(G)-1}, x$ are the only color-dominating vertices of colors $1, ..., \omega(G) - 1, \omega(G) + 1$.

- If $wx \in E(G)$, then w is a color-dominating vertex of color class $V_{\omega(G)}$. Moreover, since $v_1, ...v_q$ are the only color-dominating vertices of colors $1, ..., q$ and $v_1, ...v_q \in \overline{N}(x)$, we conclude that there has to exist a non-empty set $D \subseteq \overline{N}(x) \cap I$ such that $D \subseteq V_{\omega(G)+1}$ and every vertex in $\{v_1, ...v_q\}$ is adjacent to at least one vertex in D i.e. D dominates the vertices $v_1, ..., v_q$ in G.
- If $wx \in E(\overline{G})$, then w cannot be a color-dominating vertex of color $\omega(G)$ and, therefore, there exists a color-dominating vertex $u \in I \cap V_{\omega(G)}$ and this vertex has to be adjacent to x and to all vertices in $\overline{I} \setminus \{w\}$. Therefore, $d_G(u) = \omega(G)$.

Finally, because x is a color-dominating vertex and $v_1, ..., v_q \in \overline{N}(x)$, it follows that there have to exist q vertices $u_1, ..., u_q \in N(x)$ such that $u_i \in I \cap V_i$ for $i \in \{1, ..., q\}$ and $u_i v_i \in E(\overline{G})$. Thus, $M = \cup_{i=1}^{q} \{u_i v_i\}$ is a matching M in $\overline{G}$ of size q. Moreover, if $q = 0$, then $xw \in E(G)$ and we set $M = \varnothing$. Else $q > 0$ and $|M| = q \geq 1$. □

Remark 3.3. *If G is a graph with $\alpha(G) + \omega(G) = n(G) - 1$, then $n(G) - 1 \leq \chi_b(G) + \chi_b(\overline{G}) \leq n(G) + 1$ by Proposition 3.14. Moreover, $\omega(G) \leq \chi_b(G) \leq \omega(G) + 2$ according to Proposition 3.2 and Inequality (3.2).*

3.2.5 Further known results for special graphs

Graphs with clique number close to its order

Other interesting simple graphs are graphs with large clique number. Kohl has determined in [Alk10] the b-chromatic number of graphs G which have a clique number at least $n(G)-3$. Obviously, a graph G with clique number $\omega(G) = n(G)$ satisfies $\chi_b(G) = \chi(G) = n(G)$.

If $\omega(G) < n(G)$, then G satisfies $\omega(G) \leq \chi(G) \leq \chi_b(G) < n(G)$ and Proposition 3.3 implies:

Corollary 3.7. *If G is a graph with clique number $\omega(G) \geq n(G) - 2$, then $\chi_b(G) = \omega(G)$. Moreover, $\chi_b(G) = n(G) - 1$ if and only if $\omega(G) = n(G) - 1$.*

Additionally, by Proposition 3.3 and Inequality (3.2) we obtain $n(G) - 3 \leq \chi_b(G) \leq n(G) - 2$ for every graph G with clique number $\omega(G) = n(G) - 3$.

Theorem 3.7. (Kohl, [Alk10]) *Let G be a graph of order n and clique number $\omega(G) = n - 3$. If $\overline{G}$ contains a (not necessarily induced) subgraph H which is*

(a) a path $(w_1, v_1, u_1, u_2, v_2, w_2)$ of length 5 such that $d_G(u_1) = d_G(u_2) = n-3$, $d_G(w_1) = d_G(w_2) = n - 2$, and $d_G(v_1), d_G(v_2) \leq n - 3$, or

(b) a cycle $(w_1, v_1, u_1, u_2, v_2, w_1)$ of length 5 such that $d_G(u_1) = d_G(u_2) = d_G(w_1) = n-3$ and $d_G(v_1), d_G(v_2) \leq n - 3$,

and $V(G) \setminus V(H)$ induces a clique in G, then $\chi_b(G) = n - 2$. Otherwise, $\chi_b(G) = n - 3$.

According to Proposition 3.3 for a graph G with clique number $\omega(G) = n(G) - 4$ we can deduce that $n(G) - 4 \leq \chi_b(G) \leq n(G) - 3$.

Powers of paths and cycles

The rth-power G^r of a graph $G = (V, E)$ is a graph with vertex set $V(G^r) = V$ and edge set $E(G^r) = \{uv : u, v \in V \text{ and } d_G(u, v) \leq r\}$.

Let P_n^r and C_n^r denote the rth power of a path P_n and a cycle C_n, respectively. Effantin et al. established the exact value for the b-chromatic number of powers of paths and the exact value or bounds on the b-chromatic number of powers of cycles.

Theorem 3.8. (Effantin and Kheddouci, [Eff03])

The b-chromatic number of P_n^r for $r \geq 1$ is

$$\chi_b(P_n^r) = \begin{cases} n & \text{, if } n \leq r+1 \\ r + 1 + \lfloor \frac{n-r-1}{3} \rfloor & \text{, if } r+2 \leq n \leq 4r+1 \\ 2r+1 & \text{, if } n \geq 4r+1 \end{cases} .$$

Theorem 3.9. (Effantin and Kheddouci, [Eff03])

The b-chromatic number of C_n^r for $r \geq 1$ is

$$\chi_b(C_n^r) = \begin{cases} n & \text{, if } n \leq 2r+1 \\ r+1 & \text{, if } n = 2r+2 \\ r+1+\lfloor \frac{n-r-1}{3} \rfloor & \text{, if } 3r+1 \leq n \leq 4r \\ 2r+1 & \text{, if } n \geq 4r+1 \end{cases},$$

and if $2r+3 \leq n \leq 3r$ then $\chi_b(C_n^r) \geq \min\{n-r-1, r+1+\lfloor \frac{n-r-1}{3} \rfloor\}$.

Proposition 3.15. *For $r \geq 1$ and $n = 2r+3$, $\chi_b(C_n^r) = r+2+\lfloor \frac{r-1}{5} \rfloor$.*

Proof.

Since $\overline{G}$ is a $(n-1-2r)$-regular graph and $n = 2r+3$ it follows that $\overline{G}$ is a 2-regular graph and n is odd. This implies that $\overline{G} \simeq C_n$. Hence, $\chi_b(G) = \lfloor \frac{3n}{5} \rfloor = \lfloor \frac{6r+9}{5} \rfloor = r+2+\lfloor \frac{r-1}{5} \rfloor$ by Theorem 3.5. □

By Theorem 3.9 and Proposition 3.15 it follows:

Corollary 3.8. *The b-chromatic number of C_n^3 is*

$$\chi_b(C_n^3) = \begin{cases} n & \text{, if } n \leq 7 \\ 4 & \text{, if } n = 8 \\ 5 & \text{, if } n = 9 \\ 6 & \text{, if } 10 \leq n \leq 12 \\ 7 & \text{, if } n \geq 13 \end{cases}.$$

Proposition 3.16. *Let $r \geq 4$ and $2r+4 \leq n \leq 2r+7$. Then*

1. $r+3 \leq \chi_b(C_n^r) \leq r+3+\lfloor \frac{2(r-2)}{8} \rfloor$, if $n = 2r+4$ and $r \geq 4$.
2. $r+4 \leq \chi_b(C_n^r) \leq r+4+\lfloor \frac{3(r-3)}{11} \rfloor$, if $n = 2r+5$ and $r \geq 5$.
3. $\min\{r+5, r+1+\lfloor \frac{r+5}{3} \rfloor\} \leq \chi_b(C_n^r) \leq r+5+\lfloor \frac{4(r-4)}{14} \rfloor$, if $n = 2r+6$ and $r \geq 6$.
4. $\min\{r+6, r+1+\lfloor \frac{r+6}{3} \rfloor\} \leq \chi_b(C_n^r) \leq r+6+\lfloor \frac{5(r-5)}{17} \rfloor$, if $n = 2r+7$ and $r \geq 7$.

Proof.

Since $\overline{G}$ is a $(n-1-2r)$-regular graph it follows that $\chi_b(G) \leq \lfloor \frac{4r-1}{6r-1} n(G) \rfloor$ by Proposition

3.9. Furthermore $m := \min\{n-r+1, r+1+\lfloor\frac{n-r+1}{3}\rfloor\} \leq \chi_b(G)$ by Theorem 3.9. By a little effort we can deduce the bounds of $\chi_b(C_n^r)$ for every $2r+4 \leq n \leq 2r+7$. $\square$

By Theorem 3.9 and Propositions 3.15 and 3.16 we obtain the following two Corollaries:

Corollary 3.9. *The b-chromatic number of C_n^4 is*

$$\chi_b(C_n^4) = \begin{cases} n & , \text{ if } n \leq 9 \\ 5 & , \text{ if } n = 10 \\ 6 & , \text{ if } n = 11 \\ 7 & , \text{ if } 12 \leq n \leq 13 \\ 8 & , \text{ if } 14 \leq n \leq 16 \\ 9 & , \text{ if } n \geq 17 \end{cases}.$$

Corollary 3.10. *The b-chromatic number of C_n^5 is*

$$\chi_b(C_n^5) = \begin{cases} n & , \text{ if } n \leq 11 \\ 6 & , \text{ if } n = 12 \\ 7 & , \text{ if } n = 13 \\ 8 & , \text{ if } n = 14 \\ 9 & , \text{ if } 15 \leq n \leq 17 \\ 10 & , \text{ if } 18 \leq n \leq 20 \\ 11 & , \text{ if } n \geq 21 \end{cases}.$$

Recently, Kohl [Koh11] has determined the exact value of the b-chromatic number of power of cycle C_n^r for $2r+3 \leq n \leq 3r$.

Graphs with Independence Number 2

Since every graph G with independence number 2 satisfies $\chi_b(G) \geq \chi(G) = n(G) - \nu(\overline{G}) \geq \lceil\frac{n(G)}{2}\rceil$, we deduce that by Proposition 3.4:

Proposition 3.17. *For a graph G with independence number $\alpha(G) = 2$, $n(G) - \nu(\overline{G}) \leq \chi_b(G) \leq n(G) - \lceil\frac{2\nu(\overline{G})}{3}\rceil$.*

Moreover, since $\chi(G) \leq \frac{\Delta(G)+\omega(G)+2}{2}$ is satisfied for every graph G with independence number 2 ([Koh09]), Kohl deduced another upper bound on $\chi_b(G)$ with respect to the maximum degree $\Delta(G)$, namely $\chi_b(G) \leq \frac{n(G)+\Delta(G)+\omega(G)+2}{3}$.

By use of Ramsey numbers the Proposition 3.3 can be improved for graphs with independence number 2.

Theorem 3.10. (Kohl, [Alk10]) *If G is a graph with independence number $\alpha(G) = 2$ and clique number $\omega(G) \leq n(G) - 4$, then $\chi_b(G) \leq \lfloor \frac{n(G)+\omega(G)+1-\sqrt{n(G)-\omega(G)+3}}{2} \rfloor$.*

$K_{1,s}$-free Graphs

In [Kou06] some upper bounds for the b-chromatic number of $K_{1,s}$-free graphs with $s \geq 3$ are presented.

Proposition 3.18. (Kouider and Zaker, [Kou06])
Let G be a $K_{1,s}$-free graph with $s \geq 3$. Then $\chi_b(G) \leq (s-1)(\chi(G)-1)+1$.

We note that if $s = 3$, then we get an upper bound for claw-free graphs and, therefore, also for line graphs G namely, $\chi_b(G) \leq 2\chi(G) - 1$.

Other special graphs

The b-chromatic number of Cartesian products and Hamming graphs were studied in [Cha07, Jak11, Kou02, Kou07, Jav08].

Definition 3.4. *Let G_1 and G_2 be two disjoint graphs. The cartesian product $G_1 \times G_2$ is the graph defined by $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices (a, u) and (b, v) are adjacent if $[a = b$ and $u \in N(v)]$ or $[u = v$ and $a \in N(b)]$.*

Proposition 3.19. (Kouider and Maheó, [Kou07])
Let G_1 and G_2 be two disjoint graphs. Then $\chi_b(G_1 \times G_2) \geq \chi_b(G_1) + \chi_b(G_2) - 1$.

Furthermore, there are articles about the b-chromatic number of the cographs, P_4-sparse graphs, Mycielskian of some families of graphs, Kneser graphs, random graphs, central graphs, and star graph families [Bon09, Bal07, Haj09, Jav09, Kra02, Thi10, Ven10]. Incidently, the exact values of the b-chromatic number of powers of complete binary trees, complete caterpillars and the k-ary trees are determined [Eff05, Eff03, Eff051].

3.3 Bipartite graphs

Kratochvíl et al. [Kra02] proved that determining $\chi_b(G)$ is $\mathcal{NP}-$hard even for bipartite graphs.

Theorem 3.11. (Kratochvíl et al., [Kra02])
The decision problem whether $\chi_b(G) > K$ for a given graph and an integer K is $\mathcal{NP}-$complete even for connected bipartite graphs.

However, one can check in polynomial time whether a bipartite graph has b-chromatic number 2:

Theorem 3.12. (Kratochvíl et al., [Kra02])
Let G be a bipartite graph and $G_1, ..., G_r$ its components of order at least 3. Then $\chi_b(G) > 2$ if and only if:

(a) $r = 1$ and $A \subseteq \bigcup_{v\in B} \overline{N}(v)$ or $B \subseteq \bigcup_{v\in A} \overline{N}(v)$, where A and B are the bipartition classes of G_1, or

(b) $r = 2$ and at least one of G_1, G_2 is not complete bipartite, or

(c) $r \geq 3$.

In this subsection we are interested in bounds on the b-chromatic number of bipartite graphs in general and in exact values on $\chi_b(G)$ of some bipartite graph classes.

3.3.1 General bounds on the b-chromatic number for bipartite graphs

Let G be a factor of the complete bipartite graph $K_{|A|,|B|}$. Since $\alpha(G) \geq \max\{|A|, |B|\}$ we conclude the following proposition according to Proposition 3.2:

Proposition 3.20. *If G is a bipartite graph with the bipartition classes A and B, then*

$$\chi_b(G) \leq \min\{|A|, |B|\} + 1 \leq \left\lfloor \frac{n(G)}{2} \right\rfloor + 1.$$

For a bipartite graph G we call a complete bipartite subgraph K_{n_1,n_2} with $n_1, n_2 \geq 1$ a *biclique* of G. Kouider showed an upper bound on the b-chromatic number of bipartite graphs with respect to the biclique cover number.
The *biclique cover number* $\sigma(G)$ is the minimum number of disjoint complete bipartite subgraphs which cover the vertices of G.

Proposition 3.21. (Kouider and Zaker, [Kou06])
Let G be a bipartite graph with order n and biclique cover number $\sigma(G)$. Then $\chi_b(G) \leq \lfloor \frac{n-\sigma(G)+4}{2} \rfloor$.

Moreover, there is the following necessary condition for b-colorings of bipartite graphs:

Lemma 3.2. (Kratochvíl et al., [Kra02])
Let G be a bipartite graph. If c is a b-coloring of G by $k \geq 3$ colors, then one bipartition class contains vertices of all k colors and the other class contains vertices of at least $k-1$ colors.

Obviously, if G is a complete bipartite graph, then $\chi_b(G) = 2$. Moreover, if G is a union of bicliques, then we can determine $\chi_b(G)$ as follows:

Observation 3.1. *Let G be a disconnected bipartite graph with components $G_1, \ldots, G_r$, $r \geq 2$, such that every component of G is a biclique, i.e. $G_i \simeq K_{a_i,b_i}$, with $a_i, b_i \geq 1$ for $i \in \{1, \ldots, r\}$. Moreover, assume that $a_i \geq b_i$ for $i \in \{1, \ldots, r\}$ and $a_1 \geq a_2 \geq \ldots \geq a_r$. Then, $\chi_b(G) = \max\{i : 1 \leq i \leq r \text{ and } a_i \geq i-1\}$.*

Proof. Let $k := \max\{i : 1 \leq i \leq r \text{ and } a_i \geq i-1\}$.
($\leq$) Assume that G has a b-coloring by $k+a$ colors for $a \geq 1$.
Case 1: There exists an integer $i \in \{1, ..., r\}$ such that G_i contains at least two color-dominating vertices with different colors.
W.l.o.g. suppose that u, v are these two color-dominating vertices and $c(u) = 1$ and $c(v) = 2$.
If u, v belong to the same bipartition class, then there have to exist two vertices u', v' in the other bipartition class which have the color 1 and 2, respectively. Thus, the coloring is not proper, a contradiction.
Else, u, v belong to different bipartition classes. Since $k+a \geq k+1 \geq 3$, there has to exist

the color 3 that occurs in both bipartition classes and thus this coloring is not proper, a contradiction.

Case 2: Every G_i contains at most one color-dominating vertex.

Then there exist at least $k+a$ components each of these components having a color-dominating vertex, i.e. a vertex of degree at least $k+a-1$. By pigeonhole principle, at least one of these components, say G_i, has index $i \in \{k+a, ..., r\}$. However, this implies that $a_{k+a} \geq a_i \geq k+a-1$, a contradiction to the definition of k.

($\geq$) We define a partial b-coloring c of G by k colors as follows: For $i \in \{1, ..., k\}$, we color each vertex in $V(G_i) \cap B$ with color i and color the vertices in $V(G_i) \cap A$ with colors from $\{1, ..., k\} \setminus \{i\}$ such that each color in this set occurs at least one time in $V(G_i) \cap A$. This coloring is proper because each color class is an independent set. Moreover, each component G_i contains a color-dominating vertex of color i for $i \in \{1, ..., k\}$. Thus, c is a partial b-coloring of G. Afterwards, since we have $k \geq 2$ colors, it follows that the coloring c can be extended to the whole graph by coloring the non-colored-vertices in $V(G_j) \cap B$ with color 1 and the non-colored-vertices in $V(G_j) \cap A$ with color 2 for $j \in \{k+1, ..., r\}$. Therefore, $\chi_b(G) = k = \max\{i : 1 \leq i \leq r \text{ and } a_i \geq i-1\}$. □

Corollary 3.11. *Let G be a d-regular bipartite graph with the components $G_1, ..., G_r$, $r \geq 2$, such that each of these components is a biclique. Then $\chi_b(G) = \min\{d+1, r\}$.*

3.3.2 The bicomplement

Definition 3.5. *Let $G = (A \cup B, E)$ be a bipartite graph with the* bipartition classes *A and B. The bipartite graph $\widetilde{G} = (A \cup B, \widetilde{E})$ with edge set $E(\widetilde{G}) := \{uv : u \in A, v \in B, uv \notin E\}$ is called the* bicomplement *of G. Moreover, $\widetilde{I}$ denotes the set of isolated vertices in $\widetilde{G}$.*

Note that the bicomplement $\widetilde{G}$ is not unique if G is disconnected and the bipartition classes A and B are not specified. For instance, the graph $H := K_{1,2} \cup K_1$ can be considered as a factor of $K_{2,2}$ or $K_{1,3}$. In the former case we obtain $\widetilde{H} \simeq H$ and in the latter case, $\widetilde{H} \simeq K_{1,1} \cup K_1 \cup K_1$.

In the following, the considered bipartite graphs may have isolated vertices in one bipartition class. So we need to adapt the definition of the biclique cover number to such graphs. We call a subgraph of the form $K_{n_1,0}$ for $n_1 \geq 1$ (which corresponds to an independent

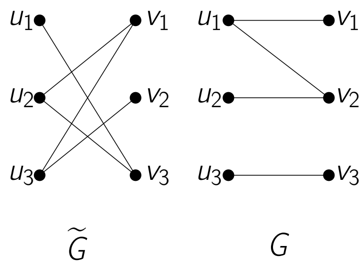


Figure 3.10: Bicomplement of a bipartite graph

set belonging to one of the bipartition classes of G), a *pseudo-biclique* of G. A *biclique cover* of G is now a set of disjoint bicliques which cover all vertices of G, where at most one pseudo-biclique is allowed. The *biclique cover number* $\sigma(G)$ of G is the minimum cardinality of a biclique cover of G.

We have the following relationship between the biclique cover number of G and the connectivity of $\widetilde{G}$ or vice versa:

Observation 3.2. (Kohl, [Alk11]) *Let $G = (A \cup B, E)$ be a non-complete bipartite graph. Then:*

(a) $\sigma(G) = 2$ if and only if $\widetilde{G}$ is disconnected,

(b) $\sigma(\widetilde{G}) = 2$ if and only if G is disconnected.

We note that the upper bound on $\chi_b(G)$ given in Proposition 3.21 is only better than the trivial upper bound $\lfloor \frac{n(G)}{2} \rfloor + 1$ for $\sigma(G) > 2$. So, for $\sigma(G) = 2$, i.e. for the case that $\widetilde{G}$ is disconnected, non-trivial upper bounds are still missing. On page 44 such graphs are considered.

In the following we consider only graphs with components of order at least 3, because the components of order 2 do not play an important role in a b-coloring by at least 3 colors since it cannot contain color-dominating vertices.

The definition of $\widetilde{G}$ and $\widetilde{I}$ allows us to give another formulation of Theorem 3.12 in terms of $\widetilde{G}$ and $\widetilde{I}$:

Proposition 3.22. *Let $G = (A \cup B, E)$ be a bipartite graph and r the number of components (all of order at least 3) of G. Then $\chi_b(G) > 2$ if and only if:*

(a) $r = 1$ and $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$

(b) $r \geq 2$ and $\widetilde{G}$ is connected.

Proof. Let $G_i = (A_i \cup B_i, E_i),\ i = 1, \ldots, r$, be the components of G of order at least 3. Assume that $r = 1$.
According to Theorem 3.12(a), $\chi_b(G) > 2$ if and only if $A \subseteq \bigcup_{v \in B} N_{\overline{G}}(v)$ or $B \subseteq \bigcup_{v \in A} N_{\overline{G}}(v)$. The condition $A \subseteq \bigcup_{v \in B} N_{\overline{G}}(v)$ is equivalent to $A = \bigcup_{v \in B} N_{\widetilde{G}}(v)$ and therefore also to $\widetilde{I} \cap A = \emptyset$. Analogously, $B \subseteq \bigcup_{v \in A} N_{\overline{G}}(v)$ is equivalent to $\widetilde{I} \cap B = \emptyset$. So, we can deduce that $\chi_b(G) > 2$ if and only if $\widetilde{I} \cap A = \emptyset$ or $\widetilde{I} \cap B = \emptyset$, i.e. $\widetilde{I} \subseteq B$ or $\widetilde{I} \subseteq A$.
Assume that $r \geq 2$.
By Theorem 3.12(b),(c) we know that $\chi_b(G) > 2$ if and only if $r \geq 3$ or ($r = 2$ and G_1 or G_2 is not a biclique). This is equivalent to $\sigma(G) \geq 3$, and according to Observation 3.2(a) this is also equivalent to the property that $\widetilde{G}$ is connected. □

The last proposition implies:

Corollary 3.12. *If $\widetilde{I} \nsubseteq A$ and $\widetilde{I} \nsubseteq B$ for a bipartite graph $G = (A \cup B, E)$, then G is connected and $\chi_b(G) = 2$.*

So it suffices to consider in the following only bipartite graphs $G = (A \cup B, E)$ satisfying $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$.

Proposition 3.23. *Let $G = (A \cup B, E)$ be a bipartite graph such that $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$. Then $\chi_b(G) \geq \sigma(\widetilde{G})$.*

Proof. Let c be a b-coloring of G by $k := \chi_b(G)$ colors and let $V_1, \ldots, V_k$ be the corresponding color classes. Since V_i is an independent set for $i \in \{1, \ldots, k\}$, it induces a biclique in $\widetilde{G}$ in case of ($V_i \cap A \neq \emptyset$ and $V_i \cap B \neq \emptyset$) and it induces a pseudo-biclique in $\widetilde{G}$ in case of ($V_i \subseteq A$ or $V_i \subseteq B$).

If $k \geq 3$, then Lemma 3.2 states that one bipartition class contains vertices of all k colors and the other class contains vertices of at least $k - 1$ colors. This implies that there is at most one color class that induces a pseudo-biclique. Hence, $\{\widetilde{G}[V_1], \ldots, \widetilde{G}[V_k]\}$ is a biclique cover of $\widetilde{G}$ and therefore $k \geq \sigma(\widetilde{G})$.

If $k = 2$, then by Proposition 3.22 and the premise $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$ we deduce that G has at least two components, say G_1 and G_2, of order at least 3. For $i = 1, 2$ let $A_i := V(G_i) \cap A$

and $B_i := V(G_i) \cap B$. Since $n(G_i) \geq 3$ and G_i is connected, it follows that $|A_i|, |B_i| \geq 1$. Thus, $A_1 \cup (B \setminus B_1)$ and $B_1 \cup (A \setminus A_1)$ induce two bicliques in $\widetilde{G}$ that cover all vertices in $V(\widetilde{G})$. This yields $\sigma(\widetilde{G}) \leq 2$ and therefore $k \geq \sigma(\widetilde{G})$. $\square$

Bijoins

For $i \in \{1, \ldots, s\}$ let $G_i = (A_i \cup B_i, E_i)$ be a bipartite graph with bipartition classes A_i and B_i.

The *bijoin* $G = (A \cup B, E)$ of the graphs G_i $(i = 1, \ldots, s)$, written as $G = G_1 \diamond \ldots \diamond G_s$, is defined as the bipartite graph with bipartition classes $A = \bigcup_{i=1}^{s} A_i$ and $B = \bigcup_{i=1}^{s} B_i$ and edge set $E = \bigcup_{i=1}^{s} E_i \cup \{\{u_i, v_j\} \mid u_i \in A_i,\ v_j \in B_j,\ i, j \in \{1, \ldots, s\},\ i \neq j\}$.

Kohl gave an upper bound on the b-chromatic number of bijoins of bipartite graphs:

Theorem 3.13. (Kohl, [Alk11])
If $G = (A \cup B, E)$ is the bijoin of the graphs $G_i = (A_i \cup B_i, E_i)$ $(i = 1, \ldots, s)$, then $\chi_b(G) \leq 1 + \sum_{i=1}^{s} \chi_b(G_i)$.

Corollary 3.13. *If G is a bipartite graph whose bicomplement $\widetilde{G}$ is disconnected and $\widetilde{G}_1, \ldots, \widetilde{G}_s$ are the components of $\widetilde{G}$, then $G = G_1 \diamond \ldots \diamond G_s$ where $G_i := \widetilde{\widetilde{G}}_i$ for $i \in \{1, \ldots, s\}$.*

Remark 3.4. (Kohl, [Alk11])
Let $G = (A \cup B, E)$ be a bijoin of the bipartite graphs $G_i = (A_i \cup B_i, E_i)$ $(i = 1, \ldots, s)$. Then:

(a) If $\chi_b(G) = 1 + \sum_{i=1}^{s} \chi_b(G_i)$, then every b-coloring c of G by $\chi_b(G)$ colors has a color that occurs only in one of the bipartition classes A and B.

(b) If for every integer $i \in \{1, \ldots, s\}$ there exists a b-coloring of G_i by $\chi_b(G_i)$ colors such that both bipartition classes A_i and B_i contain all colors, then $\chi_b(G) \geq \sum_{i=1}^{s} \chi_b(G_i)$.

(c) If there exists an integer $j \in \{1, \ldots, s\}$ such that G_j is a biclique, then $\chi_b(G) = 2$.

3.3.3 Bicomplements with simple structure

We intend to determine $\chi_b(G)$ for bipartite graphs G whose bicomplements $\widetilde{G}$ have a simple structure. In particular, we consider the case that every component of $\widetilde{G}$ is complete bipartite or an isolated vertex. Furthermore, we study the case where $\Delta(\widetilde{G}) \leq 2$. Finally, some results on $\chi_b(G)$ of bipartite graphs G with large minimum degree are presented.

Every component of $\widetilde{G} - \widetilde{I}$ is complete bipartite:

Theorem 3.14. *Let $G = (A \cup B, E)$ be a bipartite graph such that ($\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$) and every component of $\widetilde{G} - \widetilde{I}$ is a biclique. Moreover, s denotes the number of non-trivial components of $\widetilde{G}$. Then*

$$\chi_b(G) = \sigma(\widetilde{G}) = \begin{cases} s & , \text{ if } \widetilde{I} = \emptyset \\ s+1 & , \text{ if } \widetilde{I} \neq \emptyset \end{cases} .$$

Proof. Since every component of $\widetilde{G} - \widetilde{I}$ is complete bipartite we can easily deduce that $\sigma(\widetilde{G}) = s$ if $\widetilde{I} = \emptyset$ and $\sigma(\widetilde{G}) = s+1$ if $\widetilde{I} \neq \emptyset$. By Proposition 3.23 we obtain $\chi_b(G) \geq \sigma(\widetilde{G})$. We now prove that also $\chi_b(G) \leq \sigma(\widetilde{G})$ is satisfied.

Case 1. $\widetilde{I} = \emptyset$.

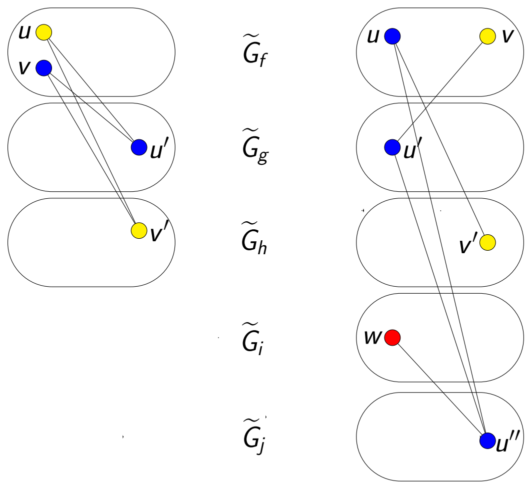
Let $\widetilde{G}_1, ..., \widetilde{G}_s$ be the components of $\widetilde{G}$.

Suppose that there is a b-coloring c of G by $k > s$ colors. By pigeonhole principle, there exists an integer $f \in \{1, ..., s\}$ such that $\widetilde{G}_f$ contains two color-dominating vertices u and v with $c(u) \neq c(v)$. Since $\{u, v\} \notin E(G)$, there have to exist integers $g, h \in \{1, ..., s\} \setminus \{f\}$ such that $u' \in V(\widetilde{G}_g) \cap N(v)$, $c(u) = c(u')$, and $v' \in V(\widetilde{G}_h) \cap N(u)$, $c(v) = c(v')$. Note that if $s = 1$ then g and h do not exist which contradicts our assumption. So it follows $s \geq 2$.

– If u and v belong to the same bipartition class, then $\{u, u'\} \in E(G)$ and $c(u) = c(u')$ (compare Figure 3.11(a)). Hence, c is not a proper coloring of G, a contradiction.

– If u and v do not belong to the same bipartition class, then because of $k + a \geq 3$, there exists a color-dominating vertex w having color $c(w) \notin \{c(u), c(v)\}$. Moreover, there is an integer $i \in \{1, ..., k\}$ such that $w \in V(\widetilde{G}_i)$.

W.l.o.g. assume that w, $u \in A$. This implies also $u' \in A$. It follows that there have to exist an integer $j \in \{1, ..., k\} \setminus \{i\}$ and a vertex $u'' \in B$ such that $u'' \in V(\widetilde{G}_j) \cap N(w)$ and $c(u) = c(u') = c(u'')$. If $j \neq f$, then $\{u, u''\} \in E(G)$ and if $j \neq g$ then $\{u', u''\} \in E(G)$ (compare Figure 3.11(b)). This contradicts the hypothesis that c is a proper coloring of



(a) (b)

Figure 3.11: Example of the case $\widetilde{I} = \varnothing$.

G.

Thus, $\chi_b(G) \leq s$ and altogether $\chi_b(G) = s$.

Case 2. $\widetilde{I} \neq \varnothing$.

Let $\widetilde{G}_1, ..., \widetilde{G}_s$ be the components of $\widetilde{G} - \widetilde{I}$ and $\widetilde{G}_{s+1} := \widetilde{I}$.

Suppose that there is a b-coloring c of G by $k > s+1$ colors. Then we can show by applying similar methods as in Case 1 that the coloring c does not exist. This yields $\chi_b(G) \leq s+1$ and altogether $\chi_b(G) = s+1$. $\square$

$\widetilde{G}$ has maximum degree at most 2

If $G = (A \cup B, E)$ is a bipartite graph satisfying $\Delta(\widetilde{G}) \leq 1$, then every component of $\widetilde{G}$ is isomorphic to K_1 or K_2. So we can apply Theorem 3.14 to determine $\chi_b(G)$.

Note that the number s of components in $\widetilde{G} - \widetilde{I}$ is equal to the number of vertices $v \in B$ satisfying $d_{\widetilde{G}}(v) = 1$. Moreover, $|\widetilde{I} \cap A| = |A| - s$ and $|\widetilde{I} \cap B| = |B| - s$. So we can deduce:

Theorem 3.15. *Let $G = (A \cup B, E)$ be a bipartite graph such that ($\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$) and $\Delta(\widetilde{G}) \leq 1$. Moreover, set $s := |\{v \in B \mid d_{\widetilde{G}}(v) = 1\}|$. Then*

$$\chi_b(G) = \begin{cases} s & , \text{ if } |A| = |B| = s \\ s+1 & , \text{ if } |A| > |B| = s \end{cases}.$$

We now consider bipartite graphs $G = (A \cup B, E)$ satisfying $\Delta(\widetilde{G}) = 2$. If $\widetilde{I} \cap A \neq \emptyset$ and

$\widetilde{I} \cap B \neq \emptyset$, then $\chi_b(G) = 2$. So it remains to consider the case where $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$.

Clearly, in a biclique cover $\mathcal{B}(\widetilde{G})$ of $\widetilde{G}$, every biclique is either a pseudo-biclique or it is isomorphic to P_2, P_3, or C_4.

Let P^1 and P^2 be two distinct paths in $\mathcal{B}(\widetilde{G})$ which belong to the same component C of $\widetilde{G}$.
If $V(P^1) \cup V(P^2)$ induces a path of order $|V(P^1)| + |V(P^2)|$ in $\widetilde{G}$, then we say that P^1 and P^2 are *consecutive*.
If P^1 and P^2 are non-consecutive, then there exists a so-called *separating set* $\mathcal{S} = \{Q^1, \ldots, Q^l\} \subseteq \mathcal{B}(\widetilde{G})$ of $l \geq 1$ paths such that by setting $Q^0 := P^1$ and $Q^{l+1} := P^2$ the subset $V(Q^i) \cup V(Q^{i+1})$ induces a path of order $|V(Q^i)| + |V(Q^{i+1})|$ in $\widetilde{G}$ for $i \in \{0, \ldots, l\}$. We call $\mathcal{S}$ a *separating set for the paths* P^1 *and* P^2.

Note that if the component C is a path, then the separating set for P^1 and P^2 is unique and if the component C is a cycle, then there exist exactly two distinct separating sets for P^1 and P^2. Moreover, if C is a cycle, then it makes sense to define the term 'separating set' for the case $P^1 = P^2$ as well. But here we only mean the non-empty separating set that covers all vertices from $V(G) \setminus (V(P^1) \cup V(P^2))$. $\mathcal{S}$ is then called a *separating set for the paths* P^1 *and* P^1.

Lemma 3.3. (Alkhateeb and Kohl) *Let $G = (A \cup B, E)$ be a bipartite graph such that $\Delta(\widetilde{G}) = 2$. Assume that G has a coloring c by k colors where $V_1, \ldots, V_k$ are the corresponding color classes and*

$$V_i \cap A \neq \emptyset \text{ and } V_i \cap B \neq \emptyset \text{ for } i \in \{1, \ldots, k\}. \quad (*)$$

Then c is a b-coloring of G if and only if $\mathcal{B}(\widetilde{G}) := \{\widetilde{G}[V_1], \ldots, \widetilde{G}[V_k]\}$ is a biclique cover of $\widetilde{G}$ with bicliques of the form P_2, P_3, C_4 such that:

(a) there exist no three consecutive P_2's and

(b) every component of $\widetilde{G}$ that is isomorphic to C_4 is a biclique in $\mathcal{B}(\widetilde{G})$.

Proof. Since $\Delta(\widetilde{G}) = 2$, every component of $\widetilde{G}$ is isomorphic to a path or a cycle. Moreover, by $(*)$ we can easily deduce that $|V_i| \geq 1$ for $i \in \{1, \ldots, k\}$ and no component of $\widetilde{G}$ is isomorphic to P_1. Therefore, $\delta(\widetilde{G}) \geq 1$.

Let $i \in \{1, ..., k\}$. If $|V_i| \geq 5$, then $|V_i \cap A| \geq 3$ or $|V_i \cap B| \geq 3$ which implies $d_{\widetilde{G}}(v) \geq 3 > \Delta(\widetilde{G})$ for $v \in V_i \cap B$ or $v \in V_i \cap A$, a contradiction. If $|V_i| = 4$, then we can show as above that $|V_i \cap A| = |V_i \cap B| = 2$. Thus, $\widetilde{G}[V_i] \simeq K_{2,2} \simeq C_4$. If $|V_i| \in \{2, 3\}$, then it is obvious that $\widetilde{G}[V_i] \simeq P_{|V_i|}$. Hence, $\{\widetilde{G}[V_1], \ldots, \widetilde{G}[V_k]\}$ is a biclique cover of $\widetilde{G}$ with bicliques of the form P_2, P_3, C_4.

In the following let $V_f = \{u_f, v_f, u'_f, v'_f\}$ denote a color class of cardinality 4, $V_g = \{u_g, v_g, u'_g\}$ denote a color class of cardinality 3, and $V_h = \{u_h, v_h\}$, $V_i = \{u_i, v_i\}$, and $V_j = \{u_j, v_j\}$ denote three distinct color classes of cardinality 2, respectively ($f, g, h, i, j \in \{1, \ldots, k\}$). Moreover, w.l.o.g. we assume that $u_f, u'_f, u_g, u'_g, u_h, u_i, u_j \in A$ and $v_f, v'_f, v_g, v_h, v_i, v_j \in B$.

[" $\Rightarrow$ "] Assume that c is a b-coloring of G.
(a) Suppose that $\widetilde{G}[V_h]$, $\widetilde{G}[V_i]$, $\widetilde{G}[V_j]$ are consecutive paths in that order. W.l.o.g. let $\widetilde{G}[V_h \cup V_i \cup V_j] = (u_h, v_h, u_i, v_i, u_j, v_j)$. Then, u_i has no neighbor in V_h and v_i has no neighbor in V_j. Hence, V_i has no color-dominating vertex, a contradiction.
(b) Suppose that there is a component C of $\widetilde{G}$ that is isomorphic to C_4, but is no biclique in $\mathcal{B}(\widetilde{G})$. This implies that the vertices of C are colored by at least two different colors and because of $(*)$ we can deduce that $V(C)$ is the union of exactly two color classes of cardinality 2. W.l.o.g. let $C \simeq \widetilde{G}[V_i \cup V_j] = (u_i, v_i, u_j, v_j, u_i)$. Then, neither u_i nor v_i has a neighbor in V_j and so V_i has no color-dominating vertex, a contradiction.
Altogether, the conditions (a) and (b) are satisfied.

[" $\Leftarrow$ "] Assume that the conditions (a) and (b) are satisfied.
– Consider the color classes V_f and V_g of cardinality 4 and 3, respectively.
W.l.o.g. let $\widetilde{G}[V_f] = (u_f, v_f, u'_f, v'_f, u_f)$ and $\widetilde{G}[V_g] = (u_g, v_g, u'_g)$. Then $N(v_f) \cap A = A \setminus \{u_f, u'_f\}$ and $N(v_g) \cap A = A \setminus \{u_g, u'_g\}$, and since $V_l \cap A \neq \emptyset$ for $l \in \{1, \ldots, k\}$ it follows that v_f and v_g have a neighbor in every other color class, i.e. v_f and v_g are color-dominating vertices of the color classes V_f and V_g.
– Consider the color class V_h of cardinality 2.
Assume that $d_{\widetilde{G}}(u_h) = 1$ or $d_{\widetilde{G}}(v_h) = 1$.
W.l.o.g. let $d_{\widetilde{G}}(v_h) = 1$. Then $N(v_h) \cap A = A \setminus \{u_h\}$ and since $V_l \cap A \neq \emptyset$ for $l \in \{1, \ldots, k\}$ we can deduce as above that v_h is a color-dominating vertex of the color class V_h.
Assume that $d_{\widetilde{G}}(u_h) = d_{\widetilde{G}}(v_h) = 2$.

By the properties of the biclique cover (especially by condition (b)) and by $(*)$, there exist two distinct integers $d, e \in \{1, \ldots, k\} \setminus \{h\}$ such that $\widetilde{G}[V_d]$ and $\widetilde{G}[V_e]$ are separated by $\widetilde{G}[V_h]$ and at least one of the color classes V_d and V_e has cardinality 3. W.l.o.g. let $d = g$ and $\widetilde{G}[V_g \cup V_h] = (u_g, v_g, u'_g, v_h, u_h)$. Thus, $N(v_h) \cap A = A \setminus \{u'_g, u_h\}$ and again by $V_l \cap A \neq \emptyset$ for $l \in \{1, \ldots, k\}$ we can deduce that v_h is a color-dominating vertex of the color class V_h. We conclude that every color class has a color-dominating vertex. Thus, c is a b-coloring of G. $\square$

Lemma 3.4. (Alkhateeb and Kohl) *Let $G = (A \cup B, E)$ be a bipartite graph such that $\Delta(\widetilde{G}) = 2$. Assume that G has a b-coloring c by k colors where $V_1, \ldots, V_k$ are the corresponding color classes and*

$$\begin{aligned} &V_i \cap A \neq \emptyset \text{ and } V_i \cap B \neq \emptyset \text{ for } i \in \{1, \ldots, k-1\}, \\ &V_k \subseteq A. \end{aligned} \qquad (**)$$

Then $\mathcal{B}(\widetilde{G}) := \{\widetilde{G}[V_1], \ldots, \widetilde{G}[V_k]\}$ is a biclique cover of $\widetilde{G}$ with the pseudo-biclique $\widetilde{G}[V_k]$, that is regarded as the union of $|V_k|$ P_1's, and with bicliques of the form P_2, P_3, C_4 such that:

- *(a) there exist no two consecutive P_2's,*
- *(b) there exist no three consecutive paths P^1, P^2, P^3 in that order, where $|V(P^1)| = |V(P^3)| = 2$, $|V(P^2)| = 3$, and $|V(P^2) \cap A| = 1$,*
- *(c) there exist no three consecutive paths P^1, P^2, P^3 in that order, where $|V(P^1)| = |V_k| = 1$, $|V(P^2)| = 3$, and $|V(P^3)| = 2$,*
- *(d) there exist no two consecutive paths P^1, P^2, where $|V(P^2)| = 2$, $|V(P^1)| = 1$, and $x \in V(P^1) \subseteq V_k$ is a color-dominating vertex of the color class V_k, and*
- *(e) a separating set for any two P_1's that does not contain a P_1 contains an odd number of P_3's.*

Proof. Since every component of $\widetilde{G}$ is isomorphic to a cycle or a path, we can easily deduce that there can only exist bicliques isomorphic to P_2, P_3, or C_4 in $\mathcal{B}(\widetilde{G})$ (compare the proof of Lemma 3.3). Moreover, since $V_k \subseteq A$ (by $(**)$) it follows that $\widetilde{G}[V_k]$ is a pseudo-biclique of order $|V_k|$ and we can consider it as a union of $|V_k|$ isolated vertices, i.e. P_1's.

In the following let $V_g = \{u_g, v_g, u'_g\}$ denote a color class of cardinality 3 with $|V_g \cap A| = 2$, $V_h = \{v_h, u_h, v'_h\}$ denote a color class of cardinality 3 with $|V_h \cap A| = 1$, and $V_i = \{u_i, v_i\}$ as well as $V_j = \{u_j, v_j\}$ denote two distinct color classes of cardinality 2, respectively $(g, h, i, j \in \{1, \ldots, k-1\})$. Moreover, w.l.o.g. let $u_g, u'_g, u_h, u_i, u_j \in A$ and $v_g, v_h, v'_h, v_i, v_j \in B$.

(a) Suppose that $\widetilde{G}[V_i]$ and $\widetilde{G}[V_j]$ are consecutive paths and w.l.o.g. let $\widetilde{G}[V_i \cup V_j] = (u_i, v_i, u_j, v_j)$. Then v_i has no neighbor in V_j and, therefore, u_i must be the color-dominating vertex of the color class V_i. However, since $u_i \in A$, this vertex has no neighbor in color class V_k, a contradiction.

(b) Suppose that $\widetilde{G}[V_i]$, $\widetilde{G}[V_h]$, $\widetilde{G}[V_j]$ are consecutive paths in that order and w.l.o.g. let $\widetilde{G}[V_h \cup V_i \cup V_j] = (v_i, u_i, v_h, u_h, v'_h, u_j, v_j)$. Since $u_h \in A$, this vertex has no neighbor in color class V_k. Moreover, v_h has no neighbor in color class V_i and v'_h has no neighbor in color class V_j. Hence, there is no color-dominating vertex in V_h, a contradiction.

(c) Let $V_k = \{x\}$ and suppose that $\widetilde{G}[\{x\}]$, $\widetilde{G}[V_h]$, $\widetilde{G}[V_i]$ are consecutive paths in that order. W.l.o.g. let $\widetilde{G}[\{x\} \cup V_h \cup V_i] = (x, v_h, u_h, v'_h, u_i, v_i)$. Then we can prove analogously to (b) that V_h has no color-dominating vertex, a contradiction.

(d) Let x be a color-dominating vertex of color class V_k and suppose that $\widetilde{G}[\{x\}]$ and $\widetilde{G}[V_i]$ are consecutive paths. Then x has no neighbor in color class V_i, a contradiction.

(e) Let $x, y \in V_k$ and let $\mathcal{S} = \{P^1, \ldots, P^l\}$, $l \geq 1$, be a separating set for the paths $\widetilde{G}[\{x\}]$ and $\widetilde{G}[\{y\}]$ such that $P^i \not\simeq P_1$ (i.e. $P^i \simeq P_2$ or $P^i \simeq P_3$) for $i \in \{1, \ldots, l\}$. Then $x, y \in A$ and, therefore, $\sum_{i=1}^{l} |V(P^i)|$ must be odd. This implies that the number of P_3's in $\mathcal{S}$ is odd. $\Box$

Theorem 3.16. (Alkhateeb and Kohl) *Let G be a bipartite graph of order $n \geq 3$ such that $\widetilde{G}$ is connected and satisfies $\Delta(\widetilde{G}) = 2$. Then*

$$\chi_b(G) = \begin{cases} \lfloor \frac{3n}{7} \rfloor & , \text{ if } \widetilde{G} \simeq C_n \\ \lfloor \frac{3n+2}{7} \rfloor & , \text{ if } \widetilde{G} \simeq P_n \end{cases}.$$

Proof. Since $\widetilde{G}$ is connected and $\Delta(\widetilde{G}) = 2$, it follows that $\widetilde{G} \simeq C_n$ or $\widetilde{G} \simeq P_n$.
Let $n = 3$. Then $\widetilde{G} \simeq P_3$, i.e. G is an empty graph and, therefore, $\chi_b(G) = 1 = \lfloor \frac{3n+2}{7} \rfloor$.
Let $n = 4$. If $\widetilde{G} \simeq C_4$, then G is an empty graph and, therefore, $\chi_b(G) = 1 = \lfloor \frac{3n}{7} \rfloor$. If $\widetilde{G} \simeq P_4$, then $G = K_1 \cup K_1 \cup K_2$ and we obtain $\chi_b(G) = 2 = \lfloor \frac{3n+2}{7} \rfloor$.
Now consider $n \geq 5$ and let c be a b-coloring of G by k colors where $V_1, \dots, V_k$ are the corresponding color classes. Moreover, p and q denote the number of color classes which induce paths of order 3 and 2, respectively.
Case 1. $\forall i \in \{1, \dots, k\} : \; V_i \cap A \neq \emptyset \; \wedge \; V_i \cap B \neq \emptyset$.
We intend to determine $k^1_{\max}$ which shall denote the largest integer k that can be attained in Case 1.
By Lemma 3.3, $|V_i| \in \{2, 3\}$ and $\widetilde{G}[V_i] \simeq P_{|V_i|}$ for $i \in \{1, \dots, k\}$. Hence, $k = p + q$. Moreover, since G has n vertices and every color occurs at least twice and at most three times, the pigeonhole principle yields that there exist $p = n - 2k$ color classes of cardinality 3 and $q = 3k - n$ color classes of cardinality 2. By Lemma 3.3(a) we know that there exist no three consecutive paths of order 2. This implies that $p \geq \frac{q}{2}$ if $\widetilde{G} \simeq C_n$ and $p \geq \frac{q}{2} - 1$ if $\widetilde{G} \simeq P_n$. So by using the above equalities for p and q we obtain $k \leq \frac{3n}{7}$ if $\widetilde{G} \simeq C_n$ and $k \leq \frac{3n+2}{7}$ if $\widetilde{G} \simeq P_n$.
Subcase 1.1.: Let $\widetilde{G} \simeq C_n$. Then

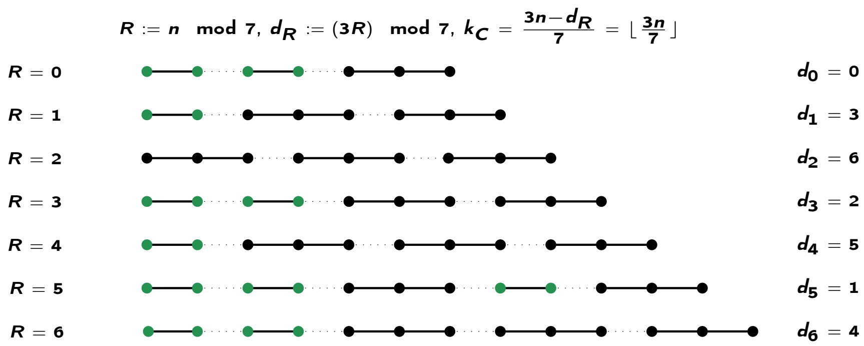
$$\begin{aligned} & q && \leq \; 2p \\ \Leftrightarrow \; & 3k - n && \leq \; 2(n - 2k) \\ \Leftrightarrow \; & 7k && \leq \; 3n \end{aligned}$$

and, therefore, $k^1_{\max} \leq \lfloor \frac{3n}{7} \rfloor =: k_C$. Consider $p = n - 2k_C$ and $q = 3k_C - n$. Then $3p + 2q = n$, $p + q = k_C$, and $q \leq 2p$. So we can find a biclique cover of $\widetilde{G}$ with $p = n - 2k_C$ bicliques isomorphic to P_3 and $q = 3k_C - n$ bicliques isomorphic to P_2, such that there exist no three consecutive P_2's (see Figure 3.12(a)). Due to Lemma 3.3 this biclique cover corresponds to a b-coloring of G by k_C colors. Hence, $k^1_{\max} \geq k_C = \lfloor \frac{3n}{7} \rfloor$, and altogether $k^1_{\max} = \lfloor \frac{3n}{7} \rfloor$.

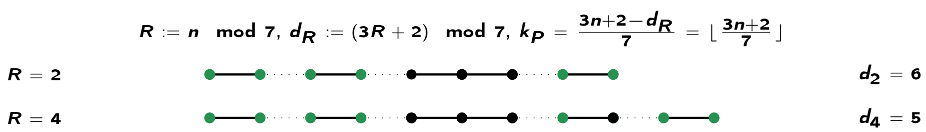
Subcase 1.2.: Let $\widetilde{G} \simeq P_n$. Then:

$$\begin{aligned} & q && \leq && 2p+2 \\ \Leftrightarrow\ & 3k-n && \leq && 2(n-2k)+2 \\ \Leftrightarrow\ & 7k && \leq && 3n+2 \end{aligned}$$

and, therefore, $k^1_{\max} \leq \lfloor \frac{3n+2}{7} \rfloor =: k_P$. Consider $p = n - 2k_P$ and $q = 3k_P - n$. Then $3p + 2q = n$, $p + q = k_P$, and $q \leq 2p + 2$. So we can construct a biclique cover of $\widetilde{G}$ with $p = n - 2k_P$ bicliques isomorphic to P_3 and $q = 3k_P - n$ bicliques isomorphic to P_2 that corresponds to a b-coloring of G by k_P colors (see Figure 3.12(b)). Note that for



(a) Biclique cover of $\widetilde{G}$ with k_C bicliques of the form P_2, P_3



(b) Biclique cover of $\widetilde{G}$ with k_P bicliques of the form P_2, P_3

Figure 3.12: Biclique cover of $\widetilde{G}$ into paths of order 2 and 3

$(n \bmod 7) \notin \{2, 4\}$ it follows that $k_P = k_C$ and we obtain the same biclique cover as in the case $\widetilde{G} \simeq C_n$. So we deduce that $k^1_{\max} \geq k_P = \lfloor \frac{3n+2}{7} \rfloor$, and altogether $k^1_{\max} = \lfloor \frac{3n+2}{7} \rfloor$.

Case 2. $\exists j \in \{1, \dots, k\}:\ V_j \cap A = \emptyset\ \vee\ V_j \cap B = \emptyset$.

We intend to estimate $k^2_{\max}$ which shall denote the largest integer k that can be attained in Case 2. If $k^2_{\max} = 2$, then $k^2_{\max} \leq \lfloor \frac{3n}{7} \rfloor \leq k^1_{\max}$ since $n \geq 5$. Now let $k^2_{\max} \geq 3$ and we consider only b-colorings c with $k \geq 3$ colors.

According to Lemma 3.2, there exists exactly one integer $j \in \{1, \dots, k\}$ such that $V_j \cap A = \emptyset$ or $V_j \cap B = \emptyset$, and $V_i \cap A \neq \emptyset$ and $V_i \cap B \neq \emptyset$ for $i \in \{1, \dots, k\} \setminus \{j\}$. W.l.o.g. let $j = k$ and $V_k \subseteq A$. Moreover, we set $n_k := |V_k|$.

Similarly to Case 1, we now consider p and q, i.e. the number of color classes which induce paths of order 3 and 2, respectively.

At first we notice that $k = p + q + 1$. Moreover, $G[V \setminus V_k]$ has $n - n_k$ vertices of $k - 1$ colors and every color class has cardinality 2 or 3. By pigeonhole principle, there exist $p = (n - n_k) - 2(k - 1)$ color classes of cardinality 3 and $q = k - 1 - p = 3(k - 1) - (n - n_k)$ color classes of cardinality 2.

Let $V_k = \{v_1, ..., v_{n_k}\}$ are ordered in that way such that S_i is a separating set between the two paths $\widetilde{G}[\{v_i\}]$ and $\widetilde{G}[\{v_{i+1}\}]$ so that S_i contains only paths of the form P_2 and P_3 for $i \in \{1, ..., n_k - 1\}$. Moreover, S_0 and S_{n_k} shall be the sets which are only incident to v_0 and v_{n_k}, respectively. For $\widetilde{G} \simeq C_n$ it follows that $S_0 = S_{n_k}$ (see Figure 3.13(a)).

Subcase 2.1.: If $\widetilde{G} \simeq C_n$, then for $i \in \{1, ..., n_k\}$ we deduce that the separating set S_i contains an odd number $p_i \geq 1$ P_3's by Lemma 3.4(e) and at most $q_i \leq \frac{p_i+1}{2}$ P_2's by Lemma 3.4(a) and (b) (see Figure 3.13(b)).

Since C_n contains n_k separating sets S_i, it follows that $p = \sum_{i=1}^{n_k} p_i \geq n_k$ and $q = \sum_{i=1}^{n_k} q_i \leq \frac{p+n_k}{2}$.

This implies that

$$
\begin{aligned}
& q && \leq \frac{n_k + p}{2} && (3.3)\\
\Leftrightarrow\ & q && \leq 2p + \frac{n_k - 3p}{2} \\
\Leftrightarrow\ & 3(k-1) - (n - n_k) && \leq 2[(n - n_k) - 2(k-1)] + \frac{n_k - 3p}{2} \\
\Leftrightarrow\ & 7k && \leq 3n + \frac{14 - (5n_k + 3p)}{2} && (3.4)
\end{aligned}
$$

Recall that $p \geq n_k$. Hence, it follows that $7k \leq 3n$ if $n_k \geq 2$ or ($n_k = 1$ and $p \geq 3$). It remains to investigate the case $n_k = 1$ and $p \leq 2$. Let $x \in V_k$. Clearly, x is a color-dominating vertex. Moreover, since $n \geq 5$ and $\widetilde{G}$ is a cycle, there exist two distinct color classes V_g, V_h such that $\widetilde{G}[V_g], \widetilde{G}[\{x\}], \widetilde{G}[V_h]$ are consecutive paths in that order. From Lemma 3.4(d) we deduce that $|V_g| = |V_h| = 3$ and, therefore, $p \geq 2$. Hence, $p = 2$, but this is a contradiction to Lemma 3.4(e) stating that p must be odd.

Altogether, $k^2_{\max} \leq \lfloor \frac{3n}{7} \rfloor = k^1_{\max}$.

Subcase 2.2.*:* If $\widetilde{G} \simeq P_n$, then we deduce that a separating set S_i contains an odd number $p_i \geq 1$ P_3's by Lemma 3.4(d) and at most $q_i \leq \frac{p_i+1}{2}$ P_2's (see Figure 3.13(b)) for $i \in \{1, ..., n_k - 1\}$. Moreover, since $\widetilde{G}$ is a path, it can contain two sets S_0, S_{n_k} which contain only paths P_2 and P_3 and do not belong to any of the separating sets S_i for $i \in \{1, ..., n_k-1\}$. Thus, if S_i for $i \in \{0, n_k\}$ contains an odd number p_i P_3's, then there are at most $q_i \leq \frac{p_i+1}{2}$ P_2's (see Figure 3.13(d)).Else, p_i is even and there are at most $q_i \leq \frac{p_i+2}{2}$ of P_2's (see Figure 3.13(c)).

Since P_n contains at least n_k-1 separating sets S_i, it follows that $p = p_0+p_{n_k}+\sum_{i=1}^{n_k-1} p_i \geq n_k - 1$ and $q = q_0 + \sum_{i=1}^{n_k-1} q_i + q_{n_k} \leq \frac{p_0+2}{2} + \sum_{i=1}^{n_k-1} \frac{p_i+2}{2} + \frac{p_{n_k}+2}{2} \leq \frac{p+n_k+3}{2}$.

Hence:

$$\begin{aligned} & q && \leq \frac{n_k + p + 3}{2} && (3.5) \\ \Leftrightarrow\ & q && \leq 2p + \frac{n_k - 3p + 3}{2} \\ \Leftrightarrow\ & 3(k-1) - (n - n_k) && \leq 2[(n - n_k) - 2(k-1)] + \frac{n_k - 3p + 3}{2} \\ \Leftrightarrow\ & 7k && \leq 3n + 2 + \frac{13 - (5n_k + 3p)}{2} && (3.6) \end{aligned}$$

and therefore $k^2_{\max} \leq \lfloor \frac{3n+2}{7} + \frac{13-(5n_k+3p)}{14} \rfloor$.

Recall that $p \geq n_k - 1$. So we can deduce that $k^2_{\max} \leq \lfloor \frac{3n+2}{7} \rfloor$ if $n_k \geq 2$ or ($n_k = 1$ and $p \geq 3$). It remains to investigate the case $n_k = 1$ and $p \leq 2$:

Let again x be the color-dominating vertex in V_k.

Assume that $d_{\widetilde{G}}(x) = 2$.

Then we can show as above the existence of two distinct color classes V_g and V_h of cardinality 3 such that $\widetilde{G}[V_g], \widetilde{G}[\{x\}], \widetilde{G}[V_h]$ are consecutive paths in that order. Thus, $p = 2$ and by Lemma 3.4(c) we further obtain $q = 0$. This implies $n = 7$ and from above we deduce that $k^2_{\max} \leq \lfloor \frac{24}{7} \rfloor = \lfloor \frac{3n+2}{7} \rfloor$.

Assume that $d_{\widetilde{G}}(x) = 1$.

Since $n \geq 5$ and by Lemma 3.4(d),(c), there exist two distinct color classes V_g and V_h of cardinality 3 such that $\widetilde{G}[\{x\}], \widetilde{G}[V_g], \widetilde{G}[V_h]$ are consecutive paths in that order. Hence, $p = 2$. Moreover, according to Lemma 3.4(a) there can exist at most one color class V_i of cardinality 2 (in this case $\widetilde{G}[V_h]$ and $\widetilde{G}[V_i]$ are consecutive) and therefore $q \leq 1$. If $q = 0$, then $n = 7$ and as before $k^2_{\max} \leq \lfloor \frac{3n+2}{7} \rfloor$. If $q = 1$, then $n = 9$ and from above we obtain

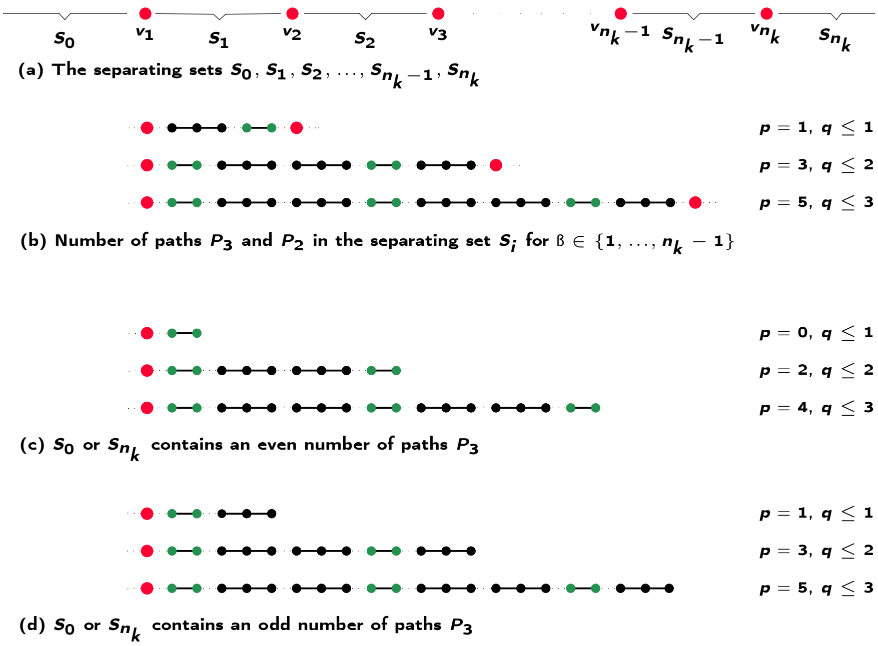


Figure 3.13: Separating sets of $\widetilde{G}$ into paths of order 2 and 3

$k^2_{\max} \leq \lfloor \frac{30}{7} \rfloor = \lfloor \frac{3n+2}{7} \rfloor$.
Altogether, $k^2_{\max} \leq \lfloor \frac{3n+2}{7} \rfloor = k^1_{\max}$.
Finally, since $\chi_b(G) = \max\{k^1_{\max}, k^2_{\max}\} = k^1_{\max}$, we conclude that $\chi_b(G) = \lfloor \frac{3n}{7} \rfloor$ if $\widetilde{G} \simeq C_n$ and $\chi_b(G) = \lfloor \frac{3n+2}{7} \rfloor$ if $\widetilde{G} \simeq P_n$. $\square$

The following Figure 3.14 shows a b-coloring by $\chi_b(G) = 4$ colors for a bipartite graph G whose bicomplement is isomorphic to C_{10}. Note that this coloring corresponds to a biclique cover of $\widetilde{G}$ into paths of order 2 and 3 (compare Lemma 3.3).

Theorem 3.17. *Let G be a bipartite graph such that $\widetilde{G}$ is disconnected, satisfies $\Delta(\widetilde{G}) = 2$ and $\widetilde{I} = \varnothing$. Moreover, let $\widetilde{G}_1, \ldots, \widetilde{G}_s$ be the components of $\widetilde{G}$ and $G_i := \widetilde{\widetilde{G}}_i$ for $i \in \{1, \ldots, s\}$. Then*
$$\sum_{i=1}^{s} \chi_b(G_i) \leq \chi_b(G) \leq \sum_{i=1}^{s} \chi_b(G_i) + 1.$$

Proof. Set $k := \sum_{i=1}^{s} \chi_b(G_i)$. The upper bound $\chi_b(G) \leq 1 + k$ follows immediately from Theorem 3.13 since G is a bijoin of the graphs $G_1, \ldots, G_s$. Moreover, since $\Delta(\widetilde{G}) = 2$ and $\widetilde{I} = \emptyset$, we can easily deduce that for $i \in \{1, \ldots, s\}$ the component $\widetilde{G}_i$ is isomorphic to a

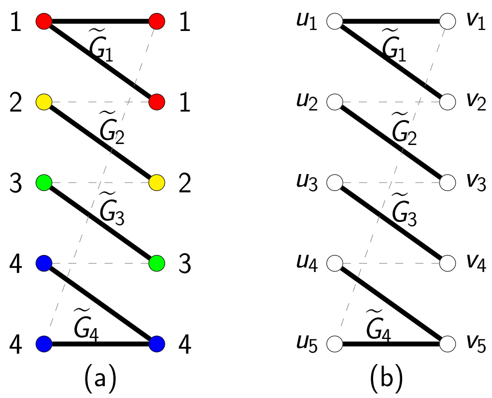


Figure 3.14: b-coloring of a bipartite graph G with $\widetilde{G} \simeq C_{10}$

path or a cycle and $n(G_i) \geq 2$. In the proof of Theorem 3.16 (Case 1) it was shown that for $n(G_i) \geq 3$ the subgraph G_i admits a b-coloring by $\chi_b(G_i)$ colors in such a way that both $V(G_i) \cap A$ and $V(G_i) \cap B$ contain vertices of all $\chi_b(G_i)$ colors. This is even true in case of $n(G_i) = 2$ since then $\chi_b(G_i) = 1$. So we can apply Remark 3.4(b) which yields $\chi_b(G) \geq k$. $\square$

Now we want to characterize graphs for which the lower bound in Theorem 3.17 is attained:

Theorem 3.18. (Alkhateeb and Kohl) *Let G be a bipartite graph such that $\widetilde{G}$ is disconnected, satisfies $\Delta(\widetilde{G}) = 2$, and $\widetilde{I} = \varnothing$. Moreover, let $\widetilde{G}_1, \ldots, \widetilde{G}_s$ be the components of $\widetilde{G}$ and $G_i := \widetilde{\widetilde{G}}_i$ for $i \in \{1, \ldots, s\}$. Then $\chi_b(G) = \sum_{i=1}^{s} \chi_b(G_i)$ if there is no integer $i \in \{1, ..., s\}$ such that $\widetilde{G}_i \simeq P_6$.*

Proof. Let G be a bipartite graph such that $\widetilde{G}$ is disconnected, has maximum degree $\Delta(\widetilde{G}) = 2$, $\widetilde{I} = \varnothing$ and all components of $\widetilde{G}$ are not isomorphic to P_6. Moreover, let $K := \sum_{i=1}^{s} \chi_b(G_i) + 1$. Assume that G has a b-coloring c by K colors where $V_1, \ldots, V_K$ are the corresponding color classes. Then by Remark 3.4(a) there exists a color such that this color occurs in only one bipartition class, w.l.o.g. we suppose that the color K satisfies $V_K \subseteq A$.

For $i \in \{1, ..., s\}$ let c_i be the coloring c restricted to the subgraph G_i, $k_i = |c(V(G_i) \setminus V_K)|$, $A_i = A \cap V(G_i)$, and $B_i = B \cap V(G_i)$.

To complete the proof we have to consider the following two facts:

Fact 3.1. *If $f \in c(A_i) \ \wedge \ f \in c(B_i)$ for an integer $i \in \{1, ..., s\}$, then $f \notin c(A_j \cup B_j)$ for*

$j \in \{1, ..., s\} \setminus \{i\}$.

Proof.

Let $f \in c(A_i) \ \wedge \ f \in c(B_i)$ for an integer $i \in \{1, ..., s\}$. Assume that $f \in c(A_j \cup B_j)$ for $j \in \{1, ..., s\} \setminus \{i\}$. Then there exist $u \in A_i, , v \in B_i$ and $v' \in V(G_j)$ such that $c(u) = c(v) = c(v') = f$. W.l.o.g. let $v \in A_j$. Then $vv' \in E(G)$ by the Definition of a bijoin on page 44. Thus, c is not a proper coloring, a contradiction. #

Fact 3.2. *$k_i = \chi_b(G_i)$ for an integer $i \in \{1, ..., s\}$.*

Proof.

($\leq$) Assume that there exists an integer $i \in \{1, ..., s\}$ such that $k_i > \chi_b(G_i)$.
If there is a color different from K which belongs only to one bipartition class A_i or B_i, then this coloring c cannot be a b-coloring according to Remark 3.4(a), a contradiction. Hence, all color classes from $c(V(G_i) \setminus V_K)$ occur in both bipartition classes A_i and B_i, and therefore, all color classes from $c(V(G_i) \setminus V_K)$ have to be contained in $V(G_i)$ by Fact 3.1. We distinguish between the following cases:
Case 1: Let $K \notin A_i$.
Since $K \notin A_i$ it follows that $V(G_i) \setminus V_K = V(G_i)$. This implies from above that all color classes from $c(V(G_i) \setminus V_K) = cV(G_i))$ occur in both bipartition classes A_i and B_i and thus $c(V(G_i))$ has to be contained in $V(G_i)$ by Fact 3.1. Therefore, all color-dominating vertices in $V(G_i)$ are color-dominating vertices in G_i according to coloring c_i. Thus, c_i is a b-coloring of G_i by k_i colors, a contradiction.
Case 2: Let $K \in c(A_i)$ and there is no color-dominating vertex in $V_K \cap A_i$.
We deduce that for every $u \in V_K \cap A_i$ there is a color $l \in c(V(G_i) \setminus V_K)$ such that $l \notin c(N(u))$. So we recolor every vertex $u \in V_K \cap A_i$ by $c'_i(u) = l$. This yields a new coloring c'_i of G_i by k_i colors. Since all color classes from $c(V(G_i) \setminus V_K)$ have to be contained in $V(G_i)$, it follows that c'_i is a b-coloring of G_i by k_i colors, a contradiction.
Case 3: Let $K \in c(A_i)$ and $V_K \cap A_i$ contains a color-dominating vertex v_k.
Since $K = \sum_{i=1}^{s} \chi_b(G_i) + 1$ and $V_K \cap A_i$ contains a color-dominating vertex it follows that $|c(V(G_i) \setminus V_K)| \geq 1 + \chi_b(G_i)$. So, c_i cannot be a b-coloring of G_i. Thus, we deduce that there is a color $l \in c(V(G_i) \setminus V_K)$ such that every $u \in V_l \cap B_i$ satisfies either $K \notin c(N(u))$ or there is $m \in c(V(G_i) \setminus V_K)$ such that $m \notin c(N(u))$. So we delete the color l and recolor every vertex $u \in V_l$ by $c'_i(u) = K$ if $K \notin c(N(u))$ or $c'_i(u) = m$ if $K \in c(N(u))$.

Since $\{u\} \cup V_K$ is an independent set in G_i if $K \notin c(N(u))$ and $\{u\} \cup V_m$ is an independent set in G_i if $K \in c(N(u))$ this yields a new proper coloring c'_i of G_i by k_i colors.
Since $K \in c(A_i)$ and $v_K \in V_K \cap A_j$ is a color-dominating vertex according to c, it follows that $m \in c_i(N(v_K)) \setminus \{l\} \subseteq c'_i(N(v_K))$ for every $m \in c_i(V(G_j)) \setminus \{l\}$ and thus $v_K \in V_K$ is a color-dominating vertex according to the coloring c'_i in G_i.
Now let $m \in c_i(V(G_i)) \setminus \{l\}$ and choose a color-dominating vertex $v_m \in V_m$ according to the b-coloring c. It follows that there is $N(v_m) \cap V_K \neq \varnothing$ which implies that $v_m \in B_i$. Since v_m is a color-dominating vertex, there has to exist $v_l \in N(v_m) \cap V_l \cap A$. After recoloring we obtain $c'_i(v_l) = k$ and, therefore, $k \in c'_i(N(v_m))$. Moreover, since $m' \in c_i(N(v_m)) \setminus \{l\} \subseteq c'_i(N(v_m))$ for every $m' \in c_i(V(G_j)) \setminus \{l\}$ it follows that v_m is a color-dominating vertex according to the coloring c'_i. Thus, c'_i is a b-coloring of G_i by k_i colors, a contradiction and therefore $\chi_b(G_i) \geq k_i$.

($\geq$) Assume that there exists $i \in \{1, ..., s\}$ such that $k_i < \chi_b(G_i)$. Since $\sum_{i=1}^{s} k_i \geq K - 1 = \sum_{i=1}^{s} \chi_b(G_i)$ according to the previous considerations, it follows that there is an integer $j \in \{1, ..., s\}$ such that $k_j > \chi_b(G_j)$, which is a contradiction to the last Cases.
Altogether we conclude that $\chi_b(G_i) = k_i$. #

Since c is a b-coloring G it follows that there is an integer $h \in \{1, ..., s\}$ such that the component G_h contains a color-dominating vertex from $V_K \cap A_h$.
Let p, q denote the number of color classes which induce paths of order 3 and 2 in $\widetilde{G}_h$. Set $k := k_h = p + q$ and $n_h := n(G_h) = 3p + 2q + n_K$, where $n_k = |V_K \cap A_h|$.
Note that $|c(V(G_h))| \geq 1 + \chi_b(G_h)$. If all color classes which have a color from $c(V(G_h))$ are contained in $V(G_h)$, then this implies that c_h is a b-coloring of G_h by $1+\chi_b(G_h)$ colors, a contradiction. Therefore, there is a color from $c(V(G_h))$ that occurs in at least one other component of $\widetilde{G}$. According to Fact 3.2 this color has to be the color K.
We can deduce that $q \leq \frac{p+n_k}{2}$ and $p \geq n_k$ if $\widetilde{G}_h \simeq C_{n_h}$ as well as $q \leq \frac{p+n_k+3}{2}$ and $p \geq n_k - 1$ if $\widetilde{G}_h \simeq P_{n_h}$ by Lemma 3.4(a),(b),(e) (compare the case 2 in the proof of Theorem 3.16).

(1) Let $\widetilde{G}_h \simeq C_{n_h}$.
Then $\chi_b(G_h) = \chi_b(\widetilde{C}_{n_h}) = \lfloor \frac{3n_h}{7} \rfloor$ by Theorem 3.16. Moreover, this implies that the Inequality (3.4) is satisfied by setting $n := n_h$.
Recall that $p \geq n_k$ according to Lemma 3.4(e). So we can deduce that $k < \chi_b(G_h) = \lfloor \frac{3n_h}{7} \rfloor$ if $n_k \geq 2$ or ($n_k = 1$ and $p \geq 3$), which is a contradiction.

It remains to investigate the case ($n_k = 1$ and $p \leq 2$):

There is no cycle C_{n_h} with ($n_k = 1$ and $p = 2$) by Lemma 3.4(e).

Hence, $n_k = p = 1$ which implies that $q \leq 1$ by Inequality (3.3). Thus, if $q = 0$, then $C_{n_h} \simeq C_4$ and if $q = 1$, then $C_{n_h} \simeq C_6$. Moreover, $v \in V_K$ cannot be a color-dominating vertex because it is not adjacent to a vertex in V_i such that ($|V_i| = 3$, if $C_{n_h} \simeq C_4$) and ($|V_i| = 2$ if $C_{n_h} \simeq C_6$) (compare Lemma 3.4(c) and (d)).

Hence, this is a contradiction to the assumption that $\widetilde{G}_h \not\simeq C_{n_h}$ has a color-dominating vertex of color k.

(2) Let $\widetilde{G}_h \simeq P_{n_h}$.

Then $\chi_b(G_h) = \chi_b(\widetilde{P}_{n_h}) = \lfloor \frac{3n_h+2}{7} \rfloor$ by Theorem 3.16. Moreover, this implies that the Inequality (3.6) is satisfied by setting $n := n_h$.

Recall that $p \geq n_k - 1$. So we can deduce that $k < \chi_b(G_h) = \lfloor \frac{3n_h+2}{7} \rfloor$ if $n_k \geq 3$, ($n_k = 2$ and $p \geq 2$), or ($n_k = 1$ and $p \geq 4$), which is a contradiction.

It remains to investigate the cases ($n_k = 2$ and $p \leq 1$) and ($n_k = 1$ and $p \leq 3$).

- If $n_k = 2$ and $p \leq 1$, then there is at least one P_3 by Lemma 3.4(e). Therefore, $p = 1$. So we deduce that $q \leq 3$ by Inequality (3.5).

Since $0 \leq q \leq 3$ it follows that $k = p+q = 1+q$, but $\chi_b(G_h) = \lfloor \frac{3n_h+2}{7} \rfloor = \lfloor \frac{3(n_k+3p+2q)+2}{7} \rfloor = \lfloor \frac{3(5+2q)+2}{7} \rfloor = \lfloor \frac{17+6q}{7} \rfloor = 1 + q + \lfloor \frac{10-q}{7} \rfloor \geq 2 + q > k$. Thus, $k < \chi_b(G_h)$, a contradiction.

- If $n_k = 1$ and $p \leq 3$. then $q \leq \frac{p}{2} + 2$ by Inequality (3.5).

For $p = 0, q \leq 2$ it follows that there is no suitable P_{n_h} by Lemma 3.4(c) and because of $n_h \geq 2$.

For $p = 1, q \leq 2$ it follows that if $q = 0$, then $\widetilde{G}_h \simeq P_4$ which implies that $1 = k < \chi_b(\widetilde{P}_4) = 2$, a contradiction.

Moreover, if $q = 1$, then $\widetilde{G}_h \simeq P_6$, a contradiction to the premise $G_h \not\simeq P_6$,

and if $q = 2$, then there is no suitable P_{n_h} by Lemma 3.4(a) and (d).

For $p \in \{2, 3\}, q \leq 3$ it follows that $k = p + q$, but $\chi_b(G_h) = \lfloor \frac{3n_h+2}{7} \rfloor = \lfloor \frac{3(n_k+3p+2q)+2}{7} \rfloor = p + q + \lfloor \frac{9-q}{7} \rfloor > k$ for $p, q \leq 2$. Thus, $k < \chi_b(G_h)$, a contradiction.

For $p \in \{2, 3\}, q = 3$ we conclude that there is no component $\widetilde{G}_h$ by Lemma 3.4(a),(d) and (c).

This implies that $\widetilde{G}_h \not\simeq P_{n_h}$.

Thus, there is no component G_h such that $K \in c(A_h)$ and $V_K \cap A_h$ contains a color-dominating vertex. This implies that there is no b-coloring c of G by K colors and, therefore, $\chi_b(G) = \sum_{i=1}^{s} \chi_b(G_i)$. $\square$

A Nordhaus-Gaddum-type result for $\widetilde{G}$

It is known that every graph G satisfies $\chi_b(G) + \chi_b(\overline{G}) \leq n(G) + 1$ by Proposition 3.6. Kohl proved a similar result for every bipartite graph G and their bicomplement $\widetilde{G}$. For small values of $n(G)$ we easily obtain:

Observation 3.3. *Let G be a bipartite graph.*

- *If $n(G) = 2$, then $\{G, \widetilde{G}\} = \{K_2, K_1 \cup K_1\}$ and, therefore, $\chi_b(G) + \chi_b(\widetilde{G}) = 3 = n(G) + 1$.*
- *If $n(G) = 3$, then $\{G, \widetilde{G}\} = \{K_{1,2}, K_1 \cup K_1 \cup K_1\}$ or $\{K_2 \cup K_1, K_2 \cup K_1\}$. In the first case, $\chi_b(G)+\chi_b(\widetilde{G}) = 3 = n(G)$ and in the latter case, $\chi_b(G)+\chi_b(\widetilde{G}) = 4 = n(G)+1$.*

Theorem 3.19. (Kohl, [Alk11]) *For every bipartite G of order $n \geq 4$, $\chi_b(G) + \chi_b(\widetilde{G}) \leq \frac{6n+8}{7}$.*

The last theorem implies that $\chi_b(G) + \chi_b(\widetilde{G}) \leq n(G) + 1$ for every bipartite graph G. Moreover, we deduce that $\chi_b(G) + \chi_b(\widetilde{G}) = n(G) + 1$ is only possible if $n(G) \leq 3$ and $\chi_b(G) + \chi_b(\widetilde{G}) = n(G)$ is only possible if $n(G) \leq 8$. Moreover, all pairs of graphs $\{G, \widetilde{G}\}$ for which $\chi_b(G) + \chi_b(\widetilde{G}) = n(G)$ are characterized in [Alk11].

3.4 Graphs with b-chromatic number close to its t-degree

In this section we consider graphs G with $\chi_b(G) = \Delta(G) + 1$, in particular we derive some results on d-regular graphs with $\chi_b(G) = d + 1$ or graphs whose b-chromatic number is close to the t-degree.
Clearly, $\chi_b(G) \leq \Delta(G) + 1$. We now want to consider graphs where this bound is attained. Kratochvíl et al. showed the following sufficient condition:

Proposition 3.24. (Kratochvíl et al., [Kra02])
Let G be a graph containing vertices $v_1, ..., v_{\Delta(G)+1}$ such that $d(v_i) = \Delta(G)$ for all i and $d(v_i, v_j) \geq 4$ for all $i \neq j$. Then $\chi_b(G) = \Delta(G) + 1$.

Moreover, planar graphs G with t vertices of degree at least $t - 1$ satisfy $\chi_b(G) = t$ under certain sufficient conditions. [Kra02].

Proposition 3.25. (Kratochvíl et al., [Kra02])
Let G be a planar graph of girth at least 5, and $t \geq 4$ an integer. If G contains t vertices $v_1, ..., v_t$ such that $d(v_i) \geq t - 1$ for all i and $d(v_i, v_j) \geq 4$ for all $i \neq j$. Then $\chi_b(G) \geq t$.

3.4.1 Regular graphs

If only regular graphs are considered, then we know the following:

Proposition 3.26. (Kratochvíl et al., [Kra02])
For every d-regular graph G with at least d^4 vertices, $\chi_b(G) = d + 1$.

Recently, this result was improved as follows:

Proposition 3.27. (Cabello and Jakovac, [Cab10])
For every d-regular graph G with at least $2d^3$ vertices, $\chi_b(G) = d + 1$.

Moreover, Kohl achieved the following two results:

Proposition 3.28. (Kohl, [Alk11])
If G is a non-complete d-regular graph with $d \geq \frac{2n(G)}{3} - 1$, then $\chi_b(G) < d + 1$.

Proposition 3.29. (Kohl, [Alk11])
If G is a d-regular graph with disconnected complement and $d \leq n-2$, then $\chi_b(G) < d+1$.

Moreover, there are the following results with respect to the diameter $diam(G)$, vertex connectivity $\kappa(G)$, and the girth $g(G)$ of the graph G (see Definition on page 102).

Proposition 3.30. (Cabello and Jakovac, [Cab10])
Let G be a d-regular graph with no cycle of length 4 and $diam(G) \leq d$, then $\chi_b(G) = d+1$.

Proposition 3.31. (Shaebani, [Sha11])
Let G be a d-regular graph that contains no cycle of length 4. Then $\chi_b(G) \geq \lfloor \frac{d+3}{2} \rfloor$. Besides, if G has a triangle, then $\chi_b(G) \geq \lfloor \frac{d+4}{2} \rfloor$.

These lower bounds are sharp for the Petersen graph.

Proposition 3.32. (Shaebani, [Sha11])
Let G be a d-regular graph that contains no cycle of length 4. If $diam(G) \geq 6$, then $\chi_b(G) = d+1$.

Proposition 3.33. (Shaebani, [Sha11])
Let G be a d-regular graph that contains no cycle of length 4. If $\kappa(G) \leq \frac{d+1}{2}$, $\chi_b(G) = d+1$.

This upper bound is sharp for the Petersen graph as well.

Kouider showed in [Kou05] that for every graph G with girth at least 6, $\chi_b(G) \geq \Delta(G)$. Moreover, for d-regular graphs the following theorem was proved independently by El Sahili et al. [ElS06], Kouider et al. [Kou05], and Blidia et al. [Bli09].

Theorem 3.20. *Every d-regular graph G with girth $g \geq 6$ satisfies $\chi_b(G) = d+1$.*

El Sahili and Kouider [ElS06] showed that this result is not extendable to all regular graphs. They also gave partial results of d-regular graphs G with girth $g = 5$ satisfying $\chi_b(G) = d+1$ e.g.:

Proposition 3.34. (El Sahili and Kouider, [ElS06])
Let G be a d-regular graph with girth $g = 5$ and containing no cycles of length 6. Then $\chi_b(G) = d+1$.

Furthermore, Blidia and Maffray conjectured in [Bli09] that:

Conjecture 3.1. *Every d-regular graph G with girth $g \geq 5$ at least different from the Petersen graph satisfies $\chi_b(G) = d + 1$.*

This conjecture is proved for d-regular graphs with $d \leq 6$.

Proposition 3.35. (Cabello and Jakovac, [Cab10])
Let G be a d-regular graph with girth $g = 5$. Then $\chi_b(G) \geq \lfloor \frac{d+1}{2} \rfloor$.

In [Jak09-1] the b-chromatic number of connected cubic graphs is studied. It is shown that all but four connected cubic graphs have the b-chromatic number equal to 4.

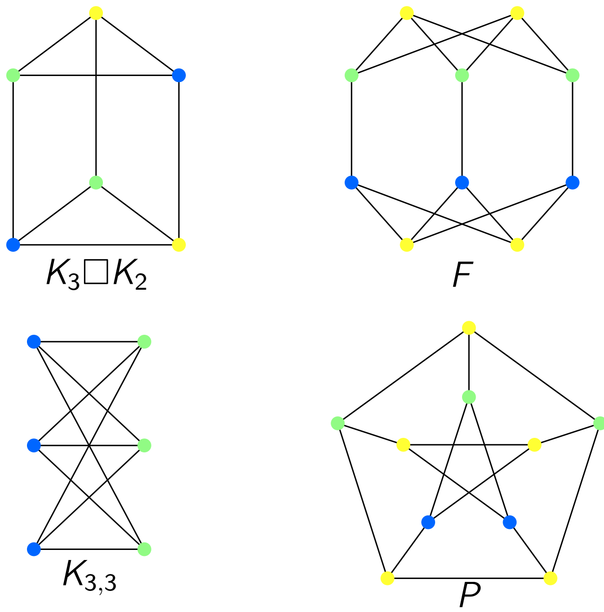


Figure 3.15: The four exception graphs.

Proposition 3.36. (Jakovac and Klavzar, [Jak09-1])
Let G be a connected 3-regular graph. Then $\chi_b(G) = 4$ unless G is the Petersen graph P, a Prism over K_3 graph $K_3 \square K_2$, $K_{3,3}$, or F (see Figure 3.15). In these cases $\chi_b(P) = \chi_b(K_3 \square K_2) = \chi_b(F) = 3$ and $\chi_b(K_{3,3}) = 2$.

We now consider d-regular bipartite graphs. Obviously, all d-regular bipartite graphs have an even girth. So by Theorem 3.20 we conclude:

Proposition 3.37. *For every d-regular bipartite graph G with girth $g \neq 4$, $\chi_b(G) = d+1$.*

If $d = 1$, then the edges of G induce a matching and clearly $\chi_b(G) = d + 1 = 2$.
It remains to investigate the b-chromatic number of d-regular bipartite graphs with girth $g = 4$ and $2 \le d \le r - 2$.

In the following let G be a d-regular bipartite graph with girth 4. Since the bipartition classes of G must have the same cardinality, we may suppose that G is a d-regular factor of $K_{r,r}$ for some $r \ge d$.

Observation 3.4. *Let G be a d-regular bipartite graph with girth $g = 4$. If there exists a vertex $u \in V(G)$ which does not belong to a cycle of length 4, then $\chi_b(G) = d + 1$.*

Proof.
Since $\chi_b(G) \le d+1$, it suffices to prove that G has a b-coloring by $d+1$ colors. The main idea of the proof is to find a partial b-coloring of G with $d+1$ colors. Afterwards, since we have $d+1$ colors it follows that the coloring can be extended to the whole graph by a Greedy coloring algorithm. Let u be a vertex from $V(G)$ which does not belong to a cycle of length 4 and let $v_1, \dots, v_d$ denote the neighbors of u. Moreover, for $i = 1, \dots, d$ we set $N_i := N(v_i) \setminus \{u\}$. We define a partial coloring c of G by $d+1$ as follows: Set $c(u) = d+1$ and $c(v_i) = i$ for $i \in \{1, ..., d\}$ and color the vertices from N_i with pairwise distinct colors from $\{1, \dots, d\} \setminus \{i\}$.
Note that $\{u\} \cup \bigcup_{i=1}^{d} N_i$ is an independent set, because the whole set belongs to the same bipartition class of G. Since there exists no C_4 that does contain the vertex u, we deduce that $N_i \cap N_j = \emptyset$ for $i, j \in \{1, \dots, d\},\ i \ne j$. So the partial coloring c always exists and is proper. Moreover, since $c(N(v_i)) = \{1, ..., d+1\} \setminus \{i\}$ and $c(u) = d+1$, it follows that $v_1, ..., v_d$ are color-dominating vertices of colors $1, ..., d$, respectively. Moreover, because of $c(N(u)) = \{1, ..., d\}$ we conclude that u is a color-dominating vertex of color $d+1$. Thus, c is a partial b-coloring of G. $\square$

Lemma 3.5. *Let G be a d-regular factor of $K_{r,r}$. Then*
(a) If $r > 2d$, then $\widetilde{G}$ is connected.
(b) If $r = 2d$, then $\widetilde{G}$ is disconnected $\Longleftrightarrow$ G is disconnected $\Longleftrightarrow$ $\widetilde{G} \simeq G \simeq K_{d,d} \cup K_{d,d}$.
(c) If $r < 2d$, then G is connected.

Proof.
Let $\widetilde{G}_1, ..., \widetilde{G}_\zeta$ be the components of $\widetilde{G}$. Since G is d-regular it follows that $\widetilde{G}$ is an

$(r-d)$-regular graph and thus each component $\widetilde{G}_i$ is $(r-d)$-regular. This implies that the bipartition classes of $\widetilde{G}_i$ must have the same cardinality. So we may suppose that for $i \in \{1, ..., \zeta\}$, G_i is a factor of K_{r_i,r_i} with $r_i \geq r-d$.

(a) Assume that $\widetilde{G}$ is disconnected. Then $\zeta \geq 2$. If $r \geq 2d+1$, then we deduce that $r-d \geq \frac{r+1}{2}$ and thus $r_i \geq r-d \geq \frac{r+1}{2}$.

Then, $r = \sum_{i=1}^{\zeta} r_i \geq \sum_{i=1}^{\zeta} \frac{r+1}{2} = \zeta \frac{r+1}{2} \geq r+1$, a contradiction.

(b) If $r = 2d$ and $\widetilde{G}$ is disconnected, then $\zeta \geq 2$ and $r_i \geq r-d \geq r-\frac{r}{2} = \frac{r}{2}$ and this implies that $r = \sum_{i=1}^{\zeta} r_i \geq \sum_{i=1}^{\zeta} \frac{r}{2} \geq \zeta \frac{r}{2}$. Thus, $\zeta = 2$ and $\widetilde{G}_i \simeq K_{\frac{r}{2},\frac{r}{2}}$ for $i \in \{1,2\}$. So we conclude that $\widetilde{G} \simeq G \simeq K_{d,d} \cup K_{d,d}$.

If $r = 2d$, $G = \widetilde{\widetilde{G}}$ is disconnected, and $\widetilde{\zeta}$ is the number of components in G, then $\widetilde{\zeta} \geq 2$ and by applying the similar methods we deduce that $\widetilde{\widetilde{G}} \simeq \widetilde{G} \simeq K_{d,d} \cup K_{d,d}$.

(c) Let $\widetilde{d} = r-d$. Then since $\widetilde{G}$ is a $\widetilde{d}$-regular factor of $K_{r,r}$ and $r < 2d$, i.e. $r > 2\widetilde{d}$, we can apply (a) which yields that $\widetilde{\widetilde{G}} = G$ is connected. $\square$

Proposition 3.38. *Let G be a 2-regular factor of $K_{r,r}$. Then*

$$\chi_b(G) = \begin{cases} 2 & , \text{ if } r \neq 3 \text{ and } \widetilde{G} \text{ is disconnected} \\ 3 & , \text{ if } r = 3 \text{ or } \widetilde{G} \text{ is connected} \end{cases}.$$

Proof.

Since G is 2-regular it follows that $r \geq 2$.

If $r = 2$, then $\widetilde{G}$ is an empty graph of order 4 implying that $\widetilde{G}$ is disconnected. Moreover, $G \simeq C_4$ and, therefore, $\chi_b(G) = 2$ by Table 3.1.

If $r = 3$, then $\widetilde{G}$ is a union of three edges implying that $\widetilde{G}$ is disconnected. Moreover, $G \simeq C_6$ and $\chi_b(G) = 3$ by Table 3.1.

If $r = 4$ and $\widetilde{G}$ is connected, then G is also connected by Lemma 3.5(b). Thus, $G \simeq \widetilde{G} \simeq C_8$ and $\chi_b(G) = 3$ by Table 3.1. Else, $r = 4$ and $\widetilde{G}$ is disconnected and then $G \simeq \widetilde{G} = C_4 \cup C_4$ by Lemma 3.5(b). Therefore, $\chi_b(G) = 2$ by Theorem 3.2.

If $r > 4$, then $\widetilde{G}$ is connected according to Lemma 3.5(a) and clearly $G \not\simeq C_4 \cup C_4$. Therefore, $\chi_b(G) = 3$ by Theorem 3.2. $\square$

Proposition 3.39. *Let G be a 3-regular factor of $K_{r,r}$. Then*

$$\chi_b(G) \begin{cases} < 4 & , \text{ if } (r \neq 4 \text{ and } \widetilde{G} \text{ is disconnected) or } G \simeq K_{3,3} \cup K_{3,3} \cup K_{3,3} \\ = 4 & , \text{ if } (r = 4 \text{ or } \widetilde{G} \text{ is connected) and } G \not\simeq K_{3,3} \cup K_{3,3} \cup K_{3,3} \end{cases}.$$

Proof.

Since G is 3-regular it follows that $r \geq 3$.

If $r = 3$, then $\widetilde{G}$ is an empty graph of order 6 implying that $\widetilde{G}$ is disconnected. Moreover, $G \simeq K_{3,3}$ and, therefore, $\chi_b(G) = 2$ by Table 3.1.

If $r = 4$, then $\widetilde{G}$ is a union of four edges implying that $\widetilde{G}$ is disconnected. Moreover, G is an $(r-1)$-regular factor of $K_{r,r}$ and, therefore, $\chi_b(G) = 4$ by Theorem 3.15.

If $r = 5$ and $\widetilde{G}$ is disconnected, then $\widetilde{G}$ is a 2-regular graph and has at least two components. This is only possible if $\widetilde{G}$ is the union of two cycles of length 4 and 6 , respectively. So we deduce $\widetilde{G} \simeq C_4 \cup C_6$ (see Figure 3.16(a)). Therefore, $\chi_b(G) = \chi_b(\widetilde{C}_4) + \chi_b(\widetilde{C}_6) = 1 + 2 = 3$ by Theorems 3.16 and 3.18.

If $r = 5$ and $\widetilde{G}$ is connected, then G is an $(r-2)$-regular factor of $K_{r,r}$. Thus, $\chi_b(G) = 4$ by Theorem 3.16.

If $r = 6$ and $\widetilde{G}$ is disconnected, then $G \simeq \widetilde{G} = K_{3,3} \cup K_{3,3}$ by Lemma 3.5(b) and, therefore, $\chi_b(G) = 2$ by Observation 3.1 (see Figure 3.16(b)).

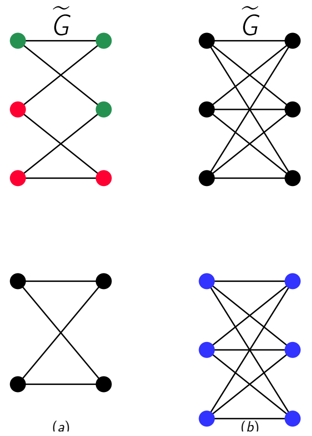


Figure 3.16: 3-regular bipartite graph with disconnected bicomplement

Else, $r = 6$ and $\widetilde{G}$ is connected. Then G is connected by Lemma 3.5(b). As G and $\widetilde{G}$ are connected and have order 12 it follows that G cannot be isomorphic to $K_{3,3}$ or to F in Figure 3.15. Hence, $\chi_b(G) = 4$ by Proposition 3.36.

If $r > 6$, then $\widetilde{G}$ is connected by Lemma 3.5(a). Moreover, if G is connected, then we can easily deduce that $\chi_b(G) = 4$ by Proposition 3.36. Else, G is disconnected and has $\zeta \geq 2$ components $G_1, ...G_\zeta$, and we consider the following subcases: If there is a component G_i with $\chi_b(G_i) = 4$, then $4 \leq \max_{1 \leq i \leq \zeta} \chi(G_i) \leq \chi_b(G) \leq \Delta(G) + 1 \leq 4$ by Proposition 3.11 and thus $\chi_b(G) = 4$.

If all components G_i have $\chi_b(G_i) \leq 3$, then we deduce that each component G_i is isomorphic to F or to $K_{3,3}$.

– If G contains at least one component which is isomorphic to F, w.l.o.g. $G_1 \simeq F$, then G has a coloring c by 4 colors such that G_1 contains three color-dominating vertices of colors 1,2,3 and G_i contains one color-dominating vertex of color 4 for $i \in \{2, ..., \zeta\}$. Therefore, c is a b-coloring of G by 4 colors and thus $\chi_b(G) = 4$.

– If $G_i \not\simeq F$ for every $i \in \{1, ..., \zeta\}$, then since $r > 6$ and $d = 3$ we deduce that G contains at least three components isomorphic to $K_{3,3}$.

If G has exactly three components, i.e. $G \simeq K_{3,3} \cup K_{3,3} \cup K_{3,3}$, then $\chi_b(G_i) = 3$ by Observation 3.1.

If G contains at least 4 components, then $\chi_b(G) = 4$ by Observation 3.1. □

Proposition 3.40. *Let G be a 4-regular factor of $K_{r,r}$. Then $\chi_b(G) < 5$* if $r \neq 5$ and $\widetilde{G}$ is disconnected. *Incidently , $\chi_b(G) = 5$ for $r = 5$.*

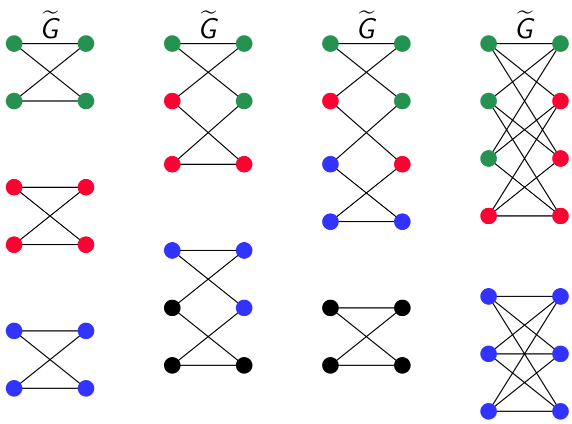
Proof.

Let G be a 4-regular factor of $K_{r,r}$ with disconnected bicomplement. Then $4 \leq r \leq 8$.

If $r = 4$, then $G \simeq K_{4,4}$ and therefore $\chi_b(G) = 2$ by Table 3.1.

If $r = 5$, then $\widetilde{G}$ is an $(r-1)$-regular factor of $K_{r,r}$ and $\chi_b(G) = 5$ by Theorem 3.15.

If $r = 6$, then $\widetilde{G}$ is a 2-regular graph and disconnected. This yields $\widetilde{G} \simeq C_4 \cup C_4 \cup C_4$, $\widetilde{G} \simeq C_6 \cup C_6$, or $\widetilde{G} \simeq C_8 \cup C_4$ (see (a),(b) and (c) in Figure 3.17).



(a) (b) (c) (d)

Figure 3.17: 4-regular bipartite graphs with disconnected bicomplement

This implies that $\chi_b(G) = \chi_b(\widetilde{C}_4) + \chi_b(\widetilde{C}_8) = 1 + 3 = 4$, $\chi_b(G) = \chi_b(\widetilde{C}_6) + \chi_b(\widetilde{C}_6) =$

$2+2=4$, or $\chi_b(G)=\chi_b(\widetilde{C}_4)+\chi_b(\widetilde{C}_4)+\chi_b(\widetilde{C}_4)=3$ by Theorems 3.16 and 3.18.
If $r=7$, then $\widetilde{G}$ is a 3-regular graph and disconnected. This implies that the components are factors of $K_{3,3}$ and $K_{4,4}$, respectively.
Hence, $\widetilde{G} \simeq K_{3,3} \cup Q_3$, (see (d) in Figure 3.17). Moreover, $\chi_b(G) = \chi_b(\widetilde{K}_{3,3} \diamond \widetilde{Q}_3) \leq 1+\chi_b(\widetilde{K}_{3,3})+\chi_b(\widetilde{Q}_3) \leq 1+1+2=4$ by Theorem 3.13.
If $r=8$, then $G \simeq \widetilde{G} \simeq K_{4,4} \cup K_{4,4}$ by Lemma 3.5(b). Therefore, $\chi_b(G)=2$ by Observation 3.1. □

Recently, the last propositions about d-regular bipartite graphs with disconnected bicomplement were generalized:

Theorem 3.21. (Kohl, [Alk11])
Let G be a d-regular factor of $K_{r,r}$ with disconnected bicomplement $\widetilde{G}$ and $2 \leq d \leq r-2$. Then $\chi_b(G) < d+1$.

So it remains to answer:

Question 3.1. *Which d-regular bipartite graphs G with connected bicomplement satisfy $\chi_b(G)=d+1$?*

From Propositions 3.38 and 3.39 we deduce that $\chi_b(G)=d+1$ for connected d-regular bipartite graphs G with connected bicomplement and $d \leq 3$. So we ask:

Question 3.2. *Let $d \in \{4,5\}$ and $d \leq r-2$. Does every connected d-regular factor G of $K_{r,r}$ with connected bicomplement $\widetilde{G}$ satisfy $\chi_b(G)=d+1$?*

This question cannot be answered in the affirmative for all integers d, since for $d=r-2$ there exists an $(r-2)$-regular factor G of $K_{r,r}$ with $\chi_b(G)<d+1$. For instance, if $r=8$, then every 6-regular factor G of $K_{8,8}$ has the b-chromatic number $\chi_b(G)=\lfloor \frac{6 \cdot 8}{7} \rfloor = 6 < 7 = d+1$ by Theorem 3.16.

3.4.2 Trees and Cacti

As we already know, every graph G satisfies $\chi_b(G) \leq t(G)$. For example, this bound is attained for the complete graph K_n. But for the complete bipartite graph $K_{r,r}$, the difference $t(K_{r,r})-\chi_b(K_{r,r})=r+1-2=r-1$ can be large.

Irving and Manlove determined the b-chromatic number of trees with respect to the t-degree. They proved:

Proposition 3.41. (Irving and Manlove, [Irv99])
For a tree T with t-degree $t(T)$, $t(T) - 1 \leq \chi_b(T) \leq t(T)$.

We call a vertex $v \in V(G)$ a *dense vertex* if v has degree $d(v) \geq t(G) - 1$. Irving and Manlove also introduced the notion of a pivoted tree:

Definition 3.6. *Let T be a tree with t-degree $t(T)$. We say that T is* pivoted *if it has exactly $t(T)$ dense vertices and at least one non-dense vertex $v \in V(T)$ such that:*
(a) Each dense vertex is adjacent either to v or to a dense vertex adjacent to v.
(b) Any dense vertex adjacent to v and to another dense vertex has degree $t(T) - 1$.
Also we call v the pivot *of the graph T.*

Note that Irving and Manlove proved in [Irv99] that the pivot of a pivoted tree T is unique. Moreover, they characterized the trees with $\chi_b(T) = t(T) - 1$ and with $\chi_b(T) = t(T)$:

Proposition 3.42. (Irving and Manlove, [Irv99])
Let T be a tree with t-degree $t(T)$. Then:
(1) $\chi_b(T) = t(T) - 1$ if and only if T is a pivoted tree.
(2) $\chi_b(T) = t(T)$ if and only if T is a non-pivoted tree.

Figure 3.18 shows a non-pivoted tree T with $\chi_b(T) = t(T) = 5$.

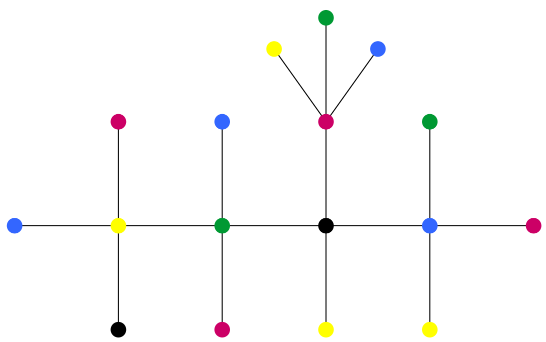


Figure 3.18: b-coloring of a tree

Campos et al. [Cam09] adapted Irving and Manlove's notion of 'pivoted' to cacti and they could prove a statement like Proposition 3.42 for cacti with t-degree at least 7. This implies:

Proposition 3.43. (Campos et al., [Cam09])
If G is a cactus and $t(G) \geq 7$, then $t(G) - 1 \leq \chi_b(G) \leq t(G)$.

3.4.3 Halin graphs

Definition 3.7. *A Halin graph $H = T \cup C$ is a plane graph, where T is a plane tree with no vertex of degree 2 and C is a cycle connecting the leaves of T such that C crosses no edge of T.*

Figure 3.19 shows a Halin graph H with $\chi_b(H) = t(H) = 5$.

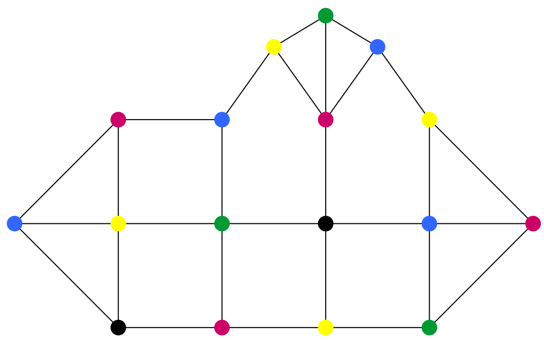


Figure 3.19: b-coloring of a Halin graph

Let $H = T \cup C$ be a Halin graph. Then, since there is no vertex in T of degree 2 by the Definition 3.7 it follows that all vertices in H have degree at least 3 which implies that $\delta(H) = 3$.

Halin studied these graphs H which are minimal 3-connected graphs with $\delta(H) = 3$. If $T \cong K_{1,l}$, $l \geq 3$, then the Halin graph H is a wheel (see Table 3.1 for the b-chromatic number of wheels).

In the following, we consider only Halin graphs $H = T \cup C$ which are not wheels.

Let $v \in V(C)$ be a *leaf vertex* of H and a vertex $v \in V(H \setminus C)$ be a *non-leaf vertex* of H.

Since H is not a wheel, we can easily deduce that $\omega(H) \leq 3$ by the Definition 3.7. Moreover, the tree $H \setminus C$ has order at least 2, so there are at least two vertices $u, v \in V(H \setminus C)$ with $d_{H\setminus C}(u) = d_{H\setminus C}(v) = 1$ and $d(u), d(v) \geq \delta(H) = 3$. So there exist two vertices $x, y \in N_C(u)$ such that $H[\{u, x, y\}]$ is a triangle in H. Thus, $\omega(H) = 3$. From this it follows that $\chi_b(H) \geq \chi(H) \geq \omega(H) = 3$.
Moreover, $N[u] \cup N[v]$ is a set of at least 4 vertices each of them of degree at least 3, which implies that $t(H) \geq 4$.

Observation 3.5. *Let $H = T \cup C$ be a Halin graph with $|V(H \setminus C)| = 2$. Then*
1. $\chi_b(H) = t(H) - 1 = 3$ if $\Delta(H) = 3$.
2. $\chi_b(H) = t(H) = 4$ if $\Delta(H) > 3$.

Proof. We know that $t(H) \geq 4$ as we already mentioned. Since there exist at most two vertices of degree larger than 3 it immediately follows that $t(H) = 4$.

Let $V(H \setminus C) = \{u, v\}$ and w.l.o.g. let $d(u) \geq d(v)$.
If $\Delta(H) = 3$, then $H \simeq K3\Box K2$ (see Figure 3.15) and, therefore, $\chi_b(H) = t(H) - 1 = 3$ by Proposition 3.36.
If $\Delta(H) > 3$, then $d_C(u) \geq 3$. Since $d_{H\setminus C}(u) = d_{H\setminus C}(v) = 1$ and H is plane, we can deduce that $N_C(u)$ and $N_C(v)$ induce two paths $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_l)$, respectively, such that $u_k v_1, u_1 v_l \in E(H)$. From above we know that $k \geq 3,\ l \geq 2$, and $k \geq l$.

So we can define a partial coloring c of H by 4 colors as follows:
- If k is odd, then we set $c(u) = 1,\ c(v) = c(u_2) = c(u_{k-1}) = 2,\ c(u_1) = c(v_1) = 3$, and $c(u_k) = c(v_l) = 4$.
- If k is even, then we set $c(u) = 1,\ c(v) = c(u_1) = c(u_4) = c(u_k) = 2,\ c(u_2) = c(v_l) = 3$, and $c(u_3) = c(v_1) = 4$.

One can easily check that this partial coloring is proper (see Figure 3.20). Moreover, if k is odd, then u, v, u_1, u_k are color-dominating vertices of colors $1, 2, 3, 4$, respectively and if k is even, then u, v, u_2, u_3 are color-dominating vertices of colors $1, 2, 3, 4$, respectively. Thus, c is a partial b-coloring of H. The remaining uncolored vertices all belong to C and, thus, all have degree 3. So we can extend the partial b-coloring c to a b-coloring of H by 4 colors by coloring the remaining vertices using a Greedy algorithm. This yields a b-coloring of H by $t(H) = 4$ colors and, therefore, $\chi_b(H) = t(H) = 4$. $\square$

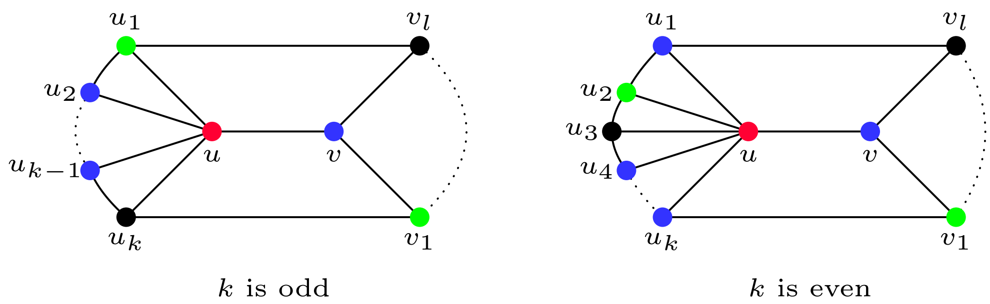


Figure 3.20: b-coloring of a Halin graph by 4 colors

Definition 3.8. *A Halin graph $H = T \cup C$ is called* pivoted *if T is pivoted. Moreover, the pivot of the tree T shall be the* pivot *of the Halin graph H.*

Since the pivot of T is unique, it follows that the pivot of the Halin graph H is unique as well.

Theorem 3.22. *Let $H = T \cup C$ be a pivoted Halin graph. Then $\chi_b(H) = t(H) - 1$.*

Proof.

Let $H = T \cup C$ be a pivoted Halin graph with $t := t(H)$ and v be the pivot of H. We note that v is a non-leaf vertex because v is pivot of the tree T which does not belong to the leaves of T. Moreover, let $v_1, ..., v_t$ be the t dense vertices in H. Suppose w.l.o.g. that $v_1, ..., v_s$ are the dense vertices which are adjacent to v, where $v_1, ..., v_r$ are the dense vertices which are adjacent to v and to at least one other dense vertex from $\{v_{s+1}, ..., v_t\}$. Each vertex of $v_1, ..., v_r$ has exactly $t-1$ neighbors and one of them is v by Definition 3.6(b). If we assume that $s = 1$, then v_1 has to be adjacent to v and to all other $t-1$ dense vertices, a contradiction. So it follows that $s \geq 2$. Moreover, since v is non-dense it follows that $s < t$ and $r > 0$ (compare Definition 3.6(a) and (b)). Hence, $2 \leq s \leq t-2$ and $r \geq 1$.

From $3 = \delta(H) \leq d(v) \leq t-2$ we obtain $t \geq 5$. Therefore, since $d(u) = 3$ for each vertex $u \in V(C)$, we deduce that u cannot be a dense vertex. So it follows by Definition 3.8 that H contains exactly $t > 4$ non-leaf dense vertices.

By Proposition 3.1 we know that $\chi_b(H) \leq t$. Similar to the proof of Irving and Manlove [Irv99], we now prove that $\chi_b(G) < t$ by showing that there exists no b-coloring of H by t colors.

Assume that H has a b-coloring c_t by t colors. Then the dense vertices $v_1, ..., v_t$ have to be the color-dominating vertices for the coloring c_t. Suppose w.l.o.g. that $c_t(v_i) = i$ for $i \in \{1, ..., t\}$. Since each of $v_1, ..., v_s$ is adjacent to v it follows that v has to receive a color j such that $j \in \{s+1, ..., t\}$. This color j exists because of $t \geq s+2$. Moreover, there has to exist an integer $i \in \{1, ..., r\}$ such that v_i is adjacent to v_j and v by Definition 3.6(a). Since $c_t(v_j) = c_t(v) = j$ and by Definition 3.6(b) it follows that $|c_t(N(v_i))| \leq t-2$ and thus v_i is not a color-dominating vertex, a contradiction.

Now we want to prove that H has a b-coloring c by $t-1$ colors where we use the colors $2, \ldots, t$.

We color v and the dense vertices $v_1, \ldots, v_t$ as follows: set $c(v_i) = i$ for $i \in \{2, ..., t\}$, $c(v) = t$, and $c(v_1) = 2$.

This yields a proper partial coloring because $v_1 v_2 \notin E(H)$ (otherwise $\{v, v_1, v_2\}$ would induce a triangle in the tree $H \setminus C$) and $vv_t \notin E(H)$ (since $s \leq t-2$).

For $i \in \{2, ..., t\}$ let

– $U_i := \{u : u \in N(v_i) \wedge u \text{ is uncolored}\}$ (the uncolored neighbors of v_i),

– $E_i := \{c(u) : u \in N(v_i) \wedge u \text{ is colored}\}$ (the colors that are assigned to neighbors of v_i),

– $R_i := \{2, \ldots, t\} \setminus (\{i\} \cup E_i)$ (the required colors in order to make v_i color-dominating).

For $i \in \{2, ..., t\}$, we now intend to make v_i color-dominating.

Let $I = \{i : \ i \in \{2, \ldots, t\} \ \wedge \ U_i \cap V(C) = \emptyset\}$ and $\overline{I} := \{2, \ldots, t\} \setminus I$.

Analogously to the proof of Irving and Manlove [Irv99] for trees, we can show that the vertices of $\bigcup_{i \in I} U_i$ can be colored in such a way that $R_i \subseteq c(U_i)$ for $i \in I$. This implies that v_i for $i \in I$ is color-dominating. Moreover, no vertex from $\bigcup_{i \in I} U_i$ is adjacent to a vertex from $\bigcup_{i \in \overline{I}} U_i$ (because of $\bigcup_{i \in I} U_i \cap V(C) = \emptyset$). So the coloring of the vertices from $\bigcup_{i \in I} U_i$ causes no further forbidden colors for the vertices in $\bigcup_{i \in \overline{I}} U_i$.

Now let $i \in \overline{I}$. Since $U_i \cap V(C) \neq \varnothing$ for $i \in \overline{I}$, every vertex v_i has at least one neighbor on the cycle C. So, the subgraph of H induced by all vertices v_i, $i \in \overline{I}$ and their neighbors has a structure similar to the one shown in Figure 3.21.

Moreover, because of $d_H(v_i) \geq t-1$ and $d_H(v_r) = t-1$ it can be verified that $|U_i| > |R_i|$ for $i \in \overline{I} \setminus \{r\}$ and $|U_i| = |R_i|$ for $i \in \overline{I} \cap \{r\}$. Therefore, we delete for each integer $i \in \overline{I} \setminus \{r\}$ $|U_i| - |R_i|$ vertices from the set U_i and obtain a new set U_i' satisfying $|U_i'| = |R_i|$ (for $i \in \overline{I} \cap \{r\}$ we set $U_i' := U_i$).

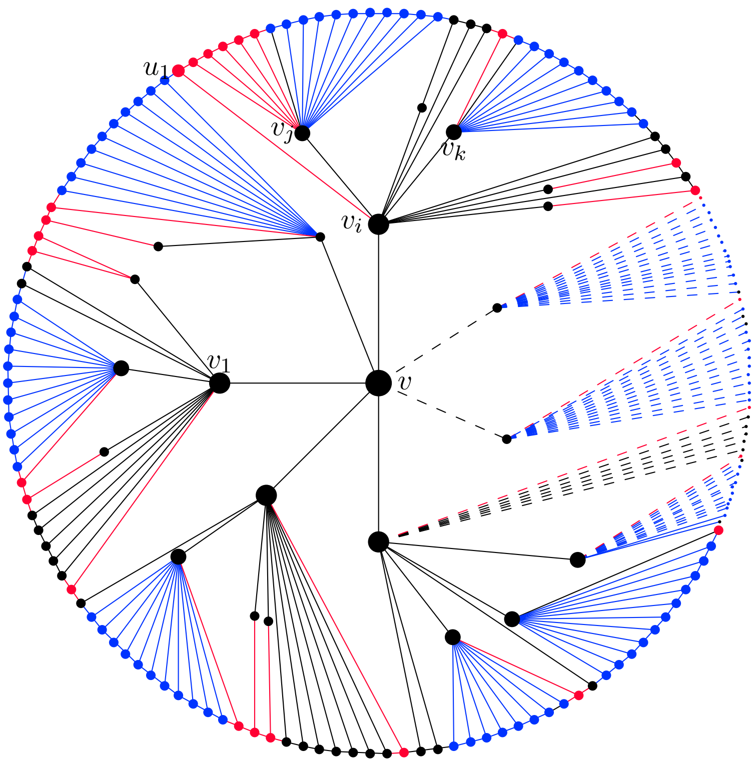


Figure 3.21: A pivoted Halin graph

Hereby, the deletion process is executed in such a way, that we start by deleting vertices from $U_i \cap V(C)$ which are the first that occur when we walk along the cycle C in clockwise order (starting with u_1). This will be done until we reach the number $|U_i| - |R_i|$ or until there is no vertex from U_i' on the cycle anymore (compare the red vertices in Figure 3.21). The the remaining vertices to delete can be chosen arbitrarily.

Let $H' = (V', E')$ be the graph with $V' := \bigcup_{i \in \overline{I}} U_i'$ and $E' := E(H[V']) \cup \{u_i^1 u_i^2 \ : \ u_i^1, u_i^2 \in U_i',\ i \in \overline{I}\}$ and let $\mathcal{L} := \{L(x) : x \in V'\}$ be a list assignment for H' where $L(x) := R_i$ for every vertex $x \in U_i'$.

Fact 3.3. *H' has a list-coloring c' for the given list assignment $\mathcal{L}$.*

Now we want to give a sketch of proof that the list coloring c' of H' for the given list assignment $\mathcal{L}$ always exists.

Sketch of proof.

Let $n_i := |E_i \setminus \{c(v)\}|$ for $i \in \{2, ..., r-1\}$, $n_r = |E_r|$, and suppose w.l.o.g. that $n_1 \geq n_2 \geq ... \geq n_r$.

- For $r = 2$, we know that $t = s + n_1 + n_2$ which implies that $n_2 = t - s - n_1 \leq t - s - n_2$, and thus, $n_2 \leq \lfloor \frac{1}{2}(t-s) \rfloor$. This yields $|L(x)| = |R_2| = t - 2 - n_2 \geq t - 2 - \lfloor \frac{1}{2}(t-s) \rfloor = \lceil \frac{1}{2}(t+s) - 2 \rceil \geq 2$ because of $t \geq 5$ and $s \geq 2$.
- For $r \geq 3$, we know that $t = s + n_1 + ... + n_r$ and we obtain:

$$|R_i| = \begin{cases} t-3 & \text{if } i \in \{r+1, \ldots, t\}, \\ t - n_i - 3 & \text{if } i \in \{2, \ldots, r-1\}. \\ t - n_r - 2 & \text{if } i = r \end{cases} \quad . \tag{3.7}$$

Since $t \geq 5$, $r \geq 3$, and $t - n_i = s + n_1 + ... + n_r - n_i \geq r + (n_1 - n_i) + n_2 + ... + n_r \geq 3 + 0 + r - 1 \geq r + 2 \geq 5$ this yields that for every $i \in \{2 \ldots, t\}$ and a vertex $x \in U_i'$ there are $|L(x)| = |R_i| \geq 2$ allowed colors.

Let H'' be the graph obtained from H' by deleting the sets U_i' for $i \in \overline{I} \cap \{s+1, ..., t\}$ (compare the blue vertices in Figure 3.21). After this deletion, the graph H'' can be properly list-colored by walking along the cycle C in anti-clockwise order and coloring each vertex by a suitable color from its list (if $r \in \overline{I}$, then we start with U_r').

Then we insert the sets U_i' for $i \in \overline{I} \cap \{s+1, ..., t\}$ again, and color these sets step by step in anti-clockwise order on the cycle C.

It can be shown because of $|R_i| \geq 2$ for $i \in \overline{I}$, that this coloring algorithm is always possible. #

Since there exists a proper list coloring c' of H' for the list assignment $\mathcal{L}$ we can set $c(x) := c'(x)$ for every vertex $x \in V' \subseteq V(H)$. This yields $R_i = c(U_i')$ (since U_i' induces a clique of order $|R_i|$ in H') and, thus, every vertex v_i for $i \in \overline{I}$ is now color-dominating.

Thus, the vertices in $\{v_2, ..., v_t\}$ are color-dominating vertices of colors $2, 3, .., t$, respectively. Furthermore, the uncolored vertices in $V(H)$ are non-dense vertices, i.e. each of them has at most $t-2$ neighbors in H. Therefore, we can color these vertices by colors $2, ..., t$ using a Greedy algorithm.
Thus, H has a b-coloring by $t-1$ colors, and since $\chi_b(H) < t$ we deduce that $\chi_b(H) = t-1$.
□

4 b-continuity

It is known that every graph G has an a-coloring by k colors for every integer k satisfying $\chi(G) \leq k \leq \chi_a(G)$ (see page 12). However, this property does not always hold for a b-coloring. Irving and Manlove [Irv99] already noticed that there are graphs which have b-colorings by k colors and by $k+2$ colors, but not by $k+1$ colors. For example, Figure 4.1 shows two bipartite graphs which both have a b-chromatic number 4 but do not admit b-colorings by 3 colors.

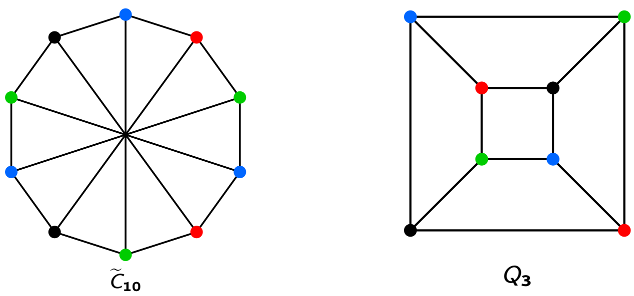


Figure 4.1: Two bipartite graphs with b-chromatic number 4

In 2004, Faik et al. [Fai03] introduced the concept of b-continuity:

Definition 4.1. *A graph G is* b-continuous *if and only if G has a b-coloring by k colors for every integer k satisfying $\chi(G) \leq k \leq \chi_b(G)$.*

Figure 4.2 shows a bipartite graph G having 7 vertices. G has b-chromatic number 4 but does not allow a b-coloring by 3 colors. By case analysis we can verify that the graph G is the smallest non-b-continuous graph and the only one with 7 vertices.

The B-CONTINUITY problem is the problem of deciding whether a given graph G is b-continuous or not. Barth et al. [Bar07] showed that the b-continuity problem is $\mathcal{NP}-$complete even if b-colorings by $\chi(G)$ and $\chi_b(G)$ colors are part of the input. It is still $\mathcal{NP}-$complete for bipartite graphs [Fai05].

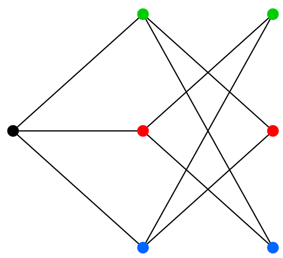


Figure 4.2: Smallest non-b-continuous graph [Alk10]

Definition 4.2. *The* b-spectrum $S_b(G)$ *of a graph* G *is the set of integers* k *such that* G *has a* b*-coloring by* k *colors.*

For instance, a $(r-1)$-factor of the complete bipartite graph $K_{r,r}$ has b-spectrum $\{2, r\}$ [Kra02], see Figure 4.3.

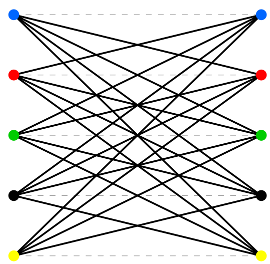


Figure 4.3: b-coloring of a 4-factor of $K_{5,5}$

For a given graph G and an integer k the problem of deciding whether $k \in S_b(G)$ is $\mathcal{NP}-$complete [Irv99]. Moreover, Barth, Cohn, and Faik [Bar07] proved that for every finite set of integers I, there exists a graph G with $S_b(G) = I$.

In this chapter we intend to investigate the b-continuity. The outline of this chapter is as follows: In Section 4.1 we are interested in the b-spectrum of graphs whose b-chromatic numbers were already determined in Chapter 3. In Section 4.2 we summarize the b-continuous graph classes that are known so far and we prove that Halin graphs are b-continuous. We finish this chapter with Section 4.3 where we give a short overview about other interesting properties concerning b-colorings, namely the b-monotonicity and b-perfectness.

4.1 b-spectrum of special graphs

Graphs with independence number or clique number close to its order

Theorems 3.3, 3.7, and 3.6 imply that every graph G with clique number at least $n(G)-4$, independence number at least $n(G)-2$, or order at most $\alpha(G)+\omega(G)$ satisfies $\chi_b(G) \leq \chi(G)+1$. So we immediately obtain:

Proposition 4.1. *If G is a graph of clique number $\omega(G) \geq n(G)-4$, independence number $\alpha(G) \geq n(G)-2$, or order $n(G) \leq \alpha(G)+\omega(G)$, then G is b-continuous.*

This proposition is best possible in the following sense:

Proposition 4.2. (Kohl, [Alk10])
For every integer $n \geq 7$ there exists a non-b-continuous graph G of order n and clique number $\omega(G) = n-5$, and there exists a non-b-continuous graph G of order n and independence number $\alpha(G) = n-3$.

Moreover, there exist graphs G of order $n(G) = \alpha(G)+\omega(G)+1$ which are non-b-continuous. For instance, Figure 4.2 shows a graph G of order $n(G) = 7$ which has $\alpha(G) = 4$ and $\omega(G) = 2$.

Graphs with minimum degree close to its order

Lemma 4.1. (Kohl, [Alk10])
Let G be a graph with minimum degree $\delta(G) = n(G)-3$. Moreover, let $\overline{G}_1, ..., \overline{G}_t$ be the components of $\overline{G}$ which are not isomorphic to C_3 and d denotes the number of components of $\overline{G}$ which are isomorphic to C_3. Then $\chi_b(G) = \sum_{i=1}^{t} \chi_b(G_i)+d$ and $\chi(G) = \sum_{i=1}^{t} \chi(G_i)+d = \sum_{i=1}^{t} \lceil \frac{n(Gi)}{2} \rceil + d$.

Proposition 4.3. (Alkhateeb and Kohl, [Alk10])
If G is a graph with minimum degree $\delta(G) = n(G)-3$, then G is b-continuous.

Proof. Let $\overline{G}_1, ..., \overline{G}_t$ be the components of $\overline{G}$ which are not isomorphic to C_3 and d denotes the number of components of $\overline{G}$ which are isomorphic to C_3. Additionally, let $G' := G_1 \oplus ... \oplus G_t$. By Lemma 4.1 we can deduce that $\chi_b(G) = \chi_b(G')+d$ and $\chi(G) = \chi(G')+d$.

Moreover, since $\Delta(\overline{G'}) \leq 2$ and $\overline{G'}$ has no component which is isomorphic to C_3 it follows that $\alpha(G') \leq 2$. If $\alpha(G') = 1$, then G' is complete and therefore b-continuous. If $\alpha(G') = 2$, then G' is b-continuous as well [Alk10]. It follows from this that G' has a b-coloring c' by k' colors for $\chi(G') \leq k' \leq \chi_b(G')$. Let $c'(V(G')) = \{1, ..., k'\}$. We can extend c' to a coloring c of G by $k' + d$ colors by coloring the independent sets of cardinality 3 by d pairwise different colors from $\{k' + 1, ..., k' + d\}$. Due to the properties of a join it is easy to check, that c is a b-coloring by $k := k' + d$ colors for $\chi(G) = \chi(G') + d \leq k \leq \chi_b(G') + d = \chi_b(G)$. Hence G is b-continuous. □

There exist graphs G with minimum degree $\delta(G) = n(G) - 5$ which are non-b-continuous. For instance, the cube graph Q_3 (see Figure 4.1) is a bipartite graph with 8 vertices and minimum degree 3. So we ask:

Question 4.1. *Is every graph G with minimum degree $\delta(G) = n(G) - 4$ b-continuous?*

Bipartite graphs with special bicomplement

Recall that $\widetilde{G}$ is the bicomplement of the bipartite graph G, a *biclique cover* of $\widetilde{G}$ is a set of disjoint bicliques of $\widetilde{G}$ which cover all vertices of the graph where at most one pseudo-biclique is allowed, $\sigma(\widetilde{G})$ is the biclique cover number of $\widetilde{G}$, and $\widetilde{I}$ is the set of isolated vertices in $\widetilde{G}$ (see Subsection 3.3.2 on page 41).

Proposition 4.4. *Let G be a bipartite graph such that $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$. Then $S_b(G) \subseteq \{2, \sigma(\widetilde{G}), ..., \chi_b(G)\}$.*

Proof.

Assume that G has a b-coloring by k colors for an integer k satisfying $3 \leq k \leq \sigma(\widetilde{G}) - 1$. Moreover, let $V_1, V_2, ..., V_k$ be the corresponding color classes. Since each of $V_1, V_2, ..., V_k$ is an independent set in G, we deduce that each of $\widetilde{G}[V_1], \widetilde{G}[V_2], ..., \widetilde{G}[V_k]$ is a biclique or a pseudo-biclique in $\widetilde{G}$. Because of $k \geq 3$, Lemma 3.2 implies that $\{\widetilde{G}[V_1], \widetilde{G}[V_2], ..., \widetilde{G}[V_k]\}$ contains at most one pseudo-biclique. Hence, $\{\widetilde{G}[V_1], \widetilde{G}[V_2], ..., \widetilde{G}[V_k]\}$ is a biclique cover of $\widetilde{G}$ with $k \leq \sigma(\widetilde{G}) - 1$ (pseudo-)bicliques, a contradiction to the definition of $\sigma(\widetilde{G})$. □

Corollary 4.1. *Let G be a bipartite graph such that $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$. Then G is non-b-continuous for $\sigma(\widetilde{G}) \geq 4$.*

In Theorem 3.14, we already determined the b-chromatic number $\chi_b(G)$ of bipartite graphs G where each component of $\widetilde{G} - \widetilde{I}$ is a biclique. We proved that

$$\chi_b(G) = \sigma(\widetilde{G}) \ = \begin{cases} s & , \text{ if } \widetilde{I} = \varnothing \\ s+1 & , \text{ if } \widetilde{I} \neq \varnothing \end{cases} ,$$

where s is the number of non-trivial components of $\widetilde{G}$. By Proposition 4.4 we deduce:

Corollary 4.2. *Let G be a bipartite graph such that $\widetilde{I} \subseteq A$ or $\widetilde{I} \subseteq B$ and every component of $\widetilde{G} - \widetilde{I}$ is a biclique. Then $S_b(G) = \{2, \sigma(\widetilde{G})\}$. Hence, G is non-b-continuous if and only if $\sigma(\widetilde{G}) \geq 4$.*

This corollary generalizes the result about $(r-1)$-factors G of $K_{r,r}$ whose bicomplement $\widetilde{G}$ consists of bicliques of order 2 (compare page 77).

We now consider bipartite graphs G whose bicomplement $\widetilde{G}$ is connected and has maximum degree 2. According to Theorem 3.16, the b-chromatic number of these graphs is

$$\chi_b(G) = \begin{cases} \lfloor \frac{3n}{7} \rfloor & , \text{ if } \widetilde{G} \simeq C_n \\ \lfloor \frac{3n+2}{7} \rfloor & , \text{ if } \widetilde{G} \simeq P_n \end{cases} .$$

For the biclique cover number of $\widetilde{G}$ of these graphs we obtain:

Proposition 4.5. *Let G be a bipartite graph of order $n \geq 5$ such that $\widetilde{G}$ is connected and satisfies $\Delta(\widetilde{G}) = 2$. Then $\sigma(\widetilde{G}) = 2$ if $n = 6$, and otherwise:*

$$\sigma(\widetilde{G}) = \begin{cases} \lceil \frac{n+4}{4} \rceil & , \text{ if } \widetilde{G} \simeq C_n, \ n \neq 6 \\ \lceil \frac{n+3}{4} \rceil & , \text{ if } \widetilde{G} \simeq P_n, \ n \neq 6 \end{cases} .$$

Proof.

Let $\{\widetilde{G}_1, ..., \widetilde{G}_k\}$ be a biclique cover of $\widetilde{G}$ with at most one pseudo-biclique and let p and q be the number of bicliques of order 3 and 2, respectively, and n_k be the order of the pseudo-biclique.

If $n_k = 0$, then $n = 3p + 2q$ and $k = p + q = \frac{n-2q}{3} + q = \frac{n+q}{3} \geq \frac{n}{3}$. Hence, $k \geq k_B := \lceil \frac{n}{3} \rceil$. If $n_k > 0$, then we obtain $n = 3p + 2q + n_k$ and $k = p + q + 1$. Moreover, since $n - n_k = 3p + 2q \leq 3(p+q)$ it follows that $k = p + q + 1 \geq \lceil \frac{n-n_k}{3} \rceil + 1$.

Analogously to the proof of Theorem 3.16 and Lemma 3.4(e) (a separating set for any two P_1's that does not contain a P_1 contains an odd number of P_3's) we deduce that $n_k \leq p$ if $\widetilde{G} \simeq C_n$ and $n_k \leq p+1$ if $\widetilde{G} \simeq P_n$.

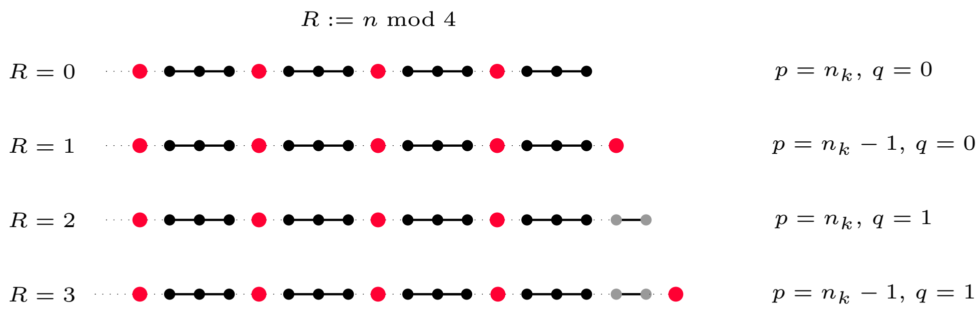


Figure 4.4: Biclique covers of $\widetilde{G}$

Case 1: $\widetilde{G} \simeq C_n$.

Then $n = 3p+2q+n_k \geq n_k+3p \geq 4n_k$, which implies that $n_k \leq \frac{n}{4}$. Thus, $k \geq \lceil\frac{n-n_k}{3}\rceil+1 \geq \lceil\frac{n-\frac{n}{4}}{3}\rceil + 1 = \lceil\frac{n+4}{4}\rceil =: k_C$.

Considering both cases $n_k = 0$ and $n_k > 0$, we obtain $k \geq \min\{k_C, k_B\}$ and therefore also $\sigma(\widetilde{G}) \geq \min\{k_C, k_B\}$.

For $n = 6$ this yields $\sigma(\widetilde{G}) \geq k_B = 2$. Since we can easily find two bicliques which cover $\widetilde{G}$ it follows that $\sigma(\widetilde{G}) = k_B = 2$.

For the case $n \neq 6$ we obtain $k \geq k_C$. Moreover, Figure 4.4 shows a biclique cover of $\widetilde{G}$ with k_C (pseudo-)bicliques (note that only $R = 0$ or $R = 2$ is possible). Hence, $\sigma(\widetilde{G}) \leq k_C$, and altogether, $\sigma(\widetilde{G}) = k_C$.

Case 2: $\widetilde{G} \simeq P_n$.

Then $n = 3p + 2q + n_k \geq n_k + 3p \geq 4n_k - 3$, which implies that $n_k \leq \frac{n+3}{4}$. Therefore, $k \geq \lceil\frac{n-n_k}{3}\rceil + 1 \geq \lceil\frac{n-\frac{n+3}{4}}{3}\rceil + 1 = \lceil\frac{n+3}{4}\rceil =: k_P$.

Considering both cases $n_k = 0$ and $n_k > 0$, we deduce that $k \geq \min\{k_P, k_B\}$ implying that $\sigma(\widetilde{G}) \geq \min\{k_P, k_B\}$.

For $n = 6$ this yields $\sigma(\widetilde{G}) \geq k_B = 2$. Again we can find a biclique cover of $\widetilde{G}$ with two bicliques and thus, $\sigma(\widetilde{G}) = k_B = 2$.

For the case $n \neq 6$ we obtain $k \geq k_P$. Figure 4.4 shows a biclique cover of $\widetilde{G}$ with k_P (pseudo-)bicliques. Hence, $\sigma(\widetilde{G}) \leq k_P$, and altogether, $\sigma(\widetilde{G}) = k_P$. $\square$

By Propositions 4.4, 4.5 and Theorem 3.16 we deduce that:

Corollary 4.3. *Let G be a bipartite graph of order $n \geq 5$ such that $\widetilde{G}$ is connected and satisfies $\Delta(\widetilde{G}) = 2$. Then $S_b(G) = \{2\}$ if $n = 6$, and otherwise:*

$$S_b(G) \subseteq \{2, \sigma(\widetilde{G}), ..., \chi_b(G)\} = \begin{cases} \{2, \lceil\frac{n+4}{4}\rceil, ..., \lfloor\frac{3n}{7}\rfloor\} & , \text{ if } \widetilde{G} \simeq C_n, \ n \neq 6 \\ \{2, \lceil\frac{n+3}{4}\rceil, ..., \lfloor\frac{3n+2}{7}\rfloor\} & , \text{ if } \widetilde{G} \simeq P_n, \ n \neq 6 \end{cases}.$$

Observation 4.1. *Let G be a bipartite graph of order $n \geq 5$ such that $\widetilde{G}$ is connected and satisfies $\Delta(\widetilde{G}) = 2$. Then $S_b(G) \supseteq \{2, \lceil\frac{n}{3}\rceil, ..., \chi_b(G)\}$.*

Proof.

Since $\chi(G) = 2$ it is clear that $2 \in S_b(G)$. Set $k := \chi_b(G) = \lfloor\frac{3n}{7}\rfloor$ if $\widetilde{G} \simeq C_n$ and $k := \chi_b(G) = \lfloor\frac{3n+2}{7}\rfloor$ if $\widetilde{G} \simeq P_n$. Moreover, set $p := n - 2k$ and $q := 3k - n$.

Let $l \in \mathbb{N}, \ l \leq k - \lceil\frac{n}{3}\rceil$. Moreover, set $p' := p + 2l$ and $q' := q - 3l$.
Then, $p', q' \in \mathbb{N}$, $p' + q' = p + q - l = k - l$, and $3p' + 2q' = 3p + 2q = n$.

In Case 1 of the proof of Theorem 3.16 we showed that there exists a biclique cover of $\widetilde{G}$ with p bicliques isomorphic to P_3 and q bicliques isomorphic to P_2, such that there exist no three consecutive P_2's (see Figure 3.12). Therefore, since $p' \geq p$ and $q' \leq q$, we can also construct a biclique cover of $\widetilde{G}$ with p' bicliques isomorphic to P_3 and q' bicliques isomorphic to P_2, such that there exist no three consecutive P_2's. According to Lemma 3.3, this biclique cover corresponds to a b-coloring of G by $k - l$ colors.

This implies, that G admits a b-coloring by $k - l$ colors for each $l \in \{0, \ldots, k - \lceil\frac{n}{3}\rceil\}$. Hence, $\{\lceil\frac{n}{3}\rceil\}, \ldots, \chi_b(G)\} \subseteq S_b(G)$. $\square$

From Corollary 4.3 and Observation 4.1 we conclude:

Corollary 4.4. *Let G be a bipartite graph of order $n \geq 5$ such that $\widetilde{G}$ is connected and satisfies $\Delta(\widetilde{G}) = 2$. Then G is b-continuous if and only if $n \leq 9$.*

4.2 b-continuous graph classes

4.2.1 Known b-continuous graph classes

The b-continuity is already proved for the following graph classes:

- Chordal graphs (independently proved by Faik [Fai04], and Kará, Kratochvíl, and Voigt [Kar04]),
- Hypercubes Q_n with $n \neq 3$ (Faik and Scale, [Fai03]),
- 3-regular graphs except for two outliers which are mentioned in Figure 4.1 (Faik and Scale, [Fai03]),
- Graphs with independence number 2 (Kohl, [Alk10]),
- K_4-minor-free graphs (Kohl, [Koh07]),
- Cographs (Bonomo et al., [Bon09]),
- P_4-sparse graphs (Bonomo et al., [Bon09]),
- P_4-tidy graphs (Velasquez, Bonomo, and Koch, [Vel10]).

Moreover, there exist other b-continuous subclasses of graphs such as some Kneser graphs (Javadi and Omoomi, [Jav09]) and some planar graphs under certain conditions (Kará, Kratochvíl, and Voigt, [Kar04]):

Proposition 4.6. (Kara et al., [Kar04]) *Let G be a connected planar graph of girth at least 5 and $t = t(G)$. If G contains t vertices $v_1, v_2, ..., v_t$ such that $d(v_i) \geq t-1$ for all $i \in \{1, ..., t\}$ and distance $d_G(v_i, v_j) \geq 5$ for all $i \neq j$ then $\chi_b(G) = t(G)$ and G is b-continuous.*

Since all non-b-continuous graphs that are known so far contain a claw as an induced subgraph we ask:

Question 4.2. *Does there exist a claw-free graph that is non-b-continuous?*

We conjecture that there is no such graph. This is reason to pose the following conjecture:

Conjecture 4.1. *Line graphs are b-continuous.*

Kohl [K11] recently proved that line graphs of 2-degenerate graphs are b-continuous.

Observation 4.2. *If G is a claw-free graph of maximum degree at most 3, then G is b-continuous.*

Proof. The case $\Delta(G) = 0$ is trivial. If $1 \leq \Delta(G) \leq 2$, then G is a union of paths and/or cycles. Therefore, $2 \leq \chi(G) \leq \chi_b(G) \leq \Delta(G) + 1 = 3$. Moreover, if $\Delta(G) = 3$, then G cannot be a bipartite graph because G has no claw. This implies that $3 \leq \chi(G) \leq \chi_b(G) \leq \Delta(G) + 1 = 4$. So in both cases, $\chi_b(G) \leq \chi(G) + 1$ and thus, G is b-continuous. $\square$

4.2.2 Halin graphs

Recall that a Halin graph $H = T \cup C$ (see page 70) is a plane graph where T is a tree which has no vertex of degree 2 and C is a cycle connecting the leaves of T such that C crosses no edge of T. Figure 4.5 shows a Halin graph where the blue edges are the edges of the tree T and the red edges are the edges of C.

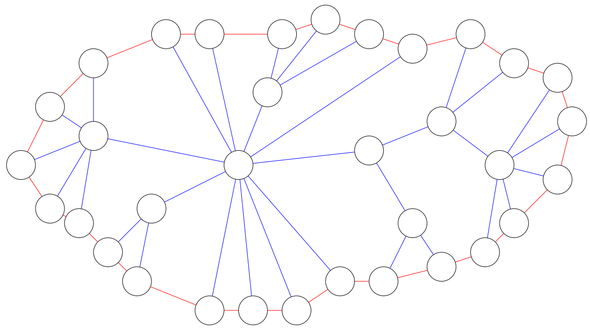


Figure 4.5: A Halin graph

Theorem 4.1. *Every Halin graph $H = T \cup C$ is b-continuous.*

Proof.

Suppose that H is a wheel. Then $\chi_b(H) \leq \chi(H) + 1$ (compare Table 3.1) and, therefore, H is b-continuous.

Suppose that H is not a wheel. Then $\omega(H) = 3$ (see page 70).

If $\chi(H) \leq \chi_b(H) \leq \chi(H) + 1$, then H is obviously b-continuous.

Now consider $\chi_b(H) > \chi(H) + 1$. If we can reduce each b-coloring of H by k colors to a

b-coloring of H by $k-1$ colors, for each k satisfying $\chi(H)+2 \leq k \leq \chi_b(H)$, then it follows that there exists a b-coloring of H by k colors, for each k satisfying $\chi(H) \leq k \leq \chi_b(H)$. Thus, H is b-continuous.

Let c be a b-coloring of H by $k \geq \chi(H) + 2$ colors and set $T^1 := T$. Now we define a vertex ordering $v_1, ..., v_{n-1}$ and a family of induced subtrees $T^2, ..., T^n$ of T such that $T^{i+1} := T^i - v_i$ for $i \in \{1, ..., n-1\}$ and $d_{T^i}(v_i) = 1$. This vertex ordering always exists because T^i is a tree of order at least two and, therefore, it has at least two leaves for $i \in \{1, ..., n-1\}$.

Moreover, let c_i be the coloring c restricted to the subtree T^i and j be the smallest integer such that the coloring c_{j+1} is not a proper b-coloring of T^{j+1} by k colors. Hence, the last removed vertex v_j is either the only color-dominating vertex of color $c(v_j)$ in T^j or it is the only neighbor of color $c(v_j)$ of the only color-dominating vertex w of color $c(w)$ in T^j. The former is not possible since $d_{T^j}(v_j) = 1 < k-1$ and therefore, the latter holds. Suppose w.l.o.g. that $c(v_j) = 1$ and $c(w) = k$.

Now we construct a proper b-coloring c'_{j+1} of $H^{j+1} := H[V(T^{j+1})]$ by $k-1$ colors as follows: We color the vertex w with color 1. Since w was the unique color-dominating vertex of color k we deduce that the neighbors of each other vertex v of color k are colored by at most $k-2$ different colors. So, we can color v with a suitable color from $\{1, .., k-1\}$. All other vertices in $V(T^{j+1})$ which have colors from $\{1, .., k-1\}$ keep their colors. This proper coloring is a b-coloring of H^{j+1} because there exist $k-1$ color-dominating vertices of pairwise different colors from $\{1, ..., k-1\}$ in H^{j+1}.

We extend this b-coloring c'_{j+1} to a proper b-coloring of H by coloring the removed vertices $v_j, ..., v_1$ in that order by suitable colors. This is always possible since $k-1 \geq \chi_b(H)+1 \geq \omega(H)+1 \geq 4$ and for $i \in \{1, ..., j\}$, v_i has degree at most 3 in $H[V(T^i)]$, i.e. when v_i is next to be colored it has at most three colored neighbors and so there is at least one color for v_i available. This yields a b-coloring of H by $k-1$ colors and this completes the proof. $\square$

4.3 Further graph properties concerning b-colorings

4.3.1 b-monotonicity

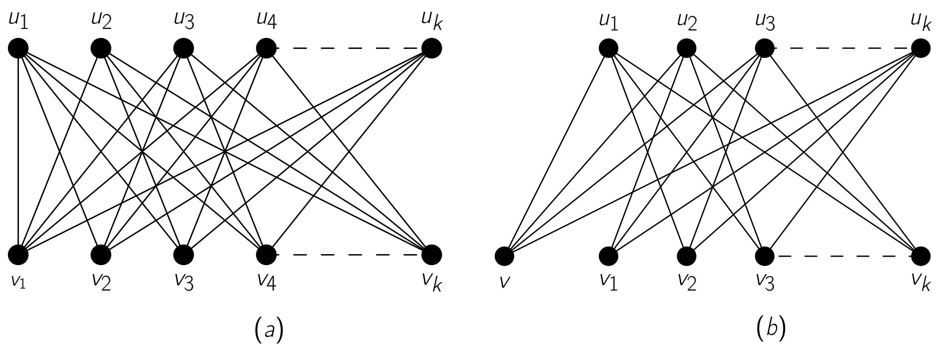


Figure 4.6: Two bipartite graphs, [Bal07].

It is known that $\chi(H) \leq \chi(G)$ for any induced subgraph H of a graph G. Moreover, Geller and Kronk [Gel74] proved this property also for the achromatic number, see Definition 2.3. However, for the b-chromatic number this property does not always hold. Balakrishnan et al. presented in [Bal07] two examples which show that the deletion of a single vertex of the graph can cause an increase or a decrease of the b-chromatic number. The graph G in Figure 4.6(a) satisfies $\chi_b(G) = 2$ and $\chi_b(G - v_1) = k$ while the graph G in Figure 4.6(b) satisfies $\chi_b(G) = k + 1$ and $\chi_b(G - v_1) = 2$. So the gap between $\chi_b(G - v)$ and $\chi_b(G)$ for a vertex $v \in V(G)$ can be arbitrarily large. Therefore, it is also interesting to characterize the graphs G with $\chi_b(G) = \chi_b(G - v)$ or $\chi_b(G) = \chi_b(G - v) + 1$.

Question 4.3. (Balakrishnan et al., [Bal07]) *Which connected graphs G satisfy $\chi_b(G) - 1 \leq \chi_b(G - v) \leq \chi_b(G)$ for any vertex $v \in V(G)$?*

Moreover, Balakrishnan et al. gave a lower and an upper bound for $\chi_b(G - v)$ in terms of $\chi_b(G)$ and proved that both bounds are attained.

Proposition 4.7. (Balakrishnan et al., [Bal07])
For every connected graph G with $n(G) \geq 5$ and $v \in V(G)$,

$$\chi_b(G) - \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) \leq \chi_b(G - v) \leq \chi_b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

Bonomo et al. introduced in [Bon09] the concept of b-monotonic graphs.

Definition 4.3. (Bonomo et al., [Bon09]) *The graph G is b-*monotonic *if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced subgraph H_1 of G, and every induced subgraph H_2 of H_1.*

In [Bon09, Kle] it is proved that cographs and P_4-sparse graphs (see definition in Appendix **A**) are b-monotonic. They also describe a polynomial time algorithm to compute the b-chromatic number for such classes of graphs.
Velasquez et al. generalized in [Vel10] the last results to P_4-tidy graphs (see definition in Appendix **A**) and proved that these graphs are b-monotonic as well. They also designed a dynamic programming algorithm to compute the b-chromatic number in polynomial time within this graph class. Moreover, Bonomo et al. [Bon09] posed the problem, whether it is possible to characterize b-monotonic graphs by forbidden induced subgraphs and to find some other b-monotonic graph classes, as for example, the class of distance-hereditary graphs.

4.3.2 b-perfectness

In the study of graph colorings and the determination of the chromatic number, there is an important class of graphs, called perfect graphs. The perfect graphs are those for which $\chi(H) = \omega(H)$ is satisfied for every induced subgraph H of G.
For the b-colorings it is known that $\chi_b(G) \geq \chi(G) \geq \omega(G)$. In 2005, Hoáng and Kouider introduced the concept of b-perfect graphs.

Definition 4.4. (Hoáng and Kouider, [Hoa05]) *A graph G is b-*perfect *if each induced subgraph H of G satisfies $\chi_b(H) = \chi(H)$.*

Figure 4.7 shows four graphs which are b-perfect.
All b-perfect bipartite graphs and all b-perfect P_4-sparse graphs are characterized by minimal forbidden induced subgraphs in [Hoa05]. Also, all $2K_2$-free and $\overline{P_5}$-free graphs are b-perfect [Hoa05]. Figure 4.8 presents a set of 22 forbidden induced graphs $\mathcal{F} = \{F_1, .., F_{22}\}$. Hoáng and Kouider [Hoa05] could characterize the b-perfect bipartite graphs and cographs in terms of these forbidden induced subgraphs. They proved that:

Theorem 4.2. (Hoáng and Kouider, [Hoa05]) *A bipartite graph is b-perfect if it contains no F_1, F_2, and F_3 as induced subgraphs.*

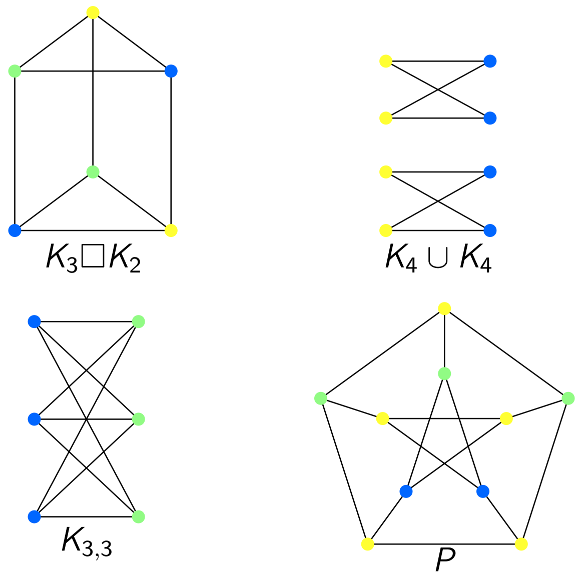


Figure 4.7: b-perfect graphs.

Theorem 4.3. (Hoáng and Kouider, [Hoa05]) *A cograph is b-perfect if it contains no F_3 and F_6 as induced subgraphs.*

In 2007, Hoáng, Sales, and Maffray [Hoa09] conjectured that a graph is b-perfect if and only if it is $\mathcal{F}$-free and proved this conjecture for diamond-free graphs and for graphs with a chromatic number at most 3.
Maffray and Mechebbek [Maf08] could verify this conjecture for chordal graphs and then also for C_4-free graphs.
Recently, a major progress was made by Hoáng, Maffray, and Mechebbek [Hoa10], who proved:

Theorem 4.4. (Hoáng, Maffray, and Mechebbek, [Hoa10]) *A graph is b-perfect if and only if it is $\mathcal{F}$-free.*

Also Hoáng et al. [Hoa09] introduced the class of minimally b-imperfect graphs:

Definition 4.5. (Hoáng et al., [Hoa09, Hoa05])
A graph is minimally b-imperfect *if it is not b-perfect and each of its proper induced subgraphs is b-perfect.*

Conjecture 4.2. (Hoáng et al., [Hoa09])
A minimally b-imperfect graph G that is not triangle-free has $\chi_b(G) = 4$ and $\omega(G) = 3$.

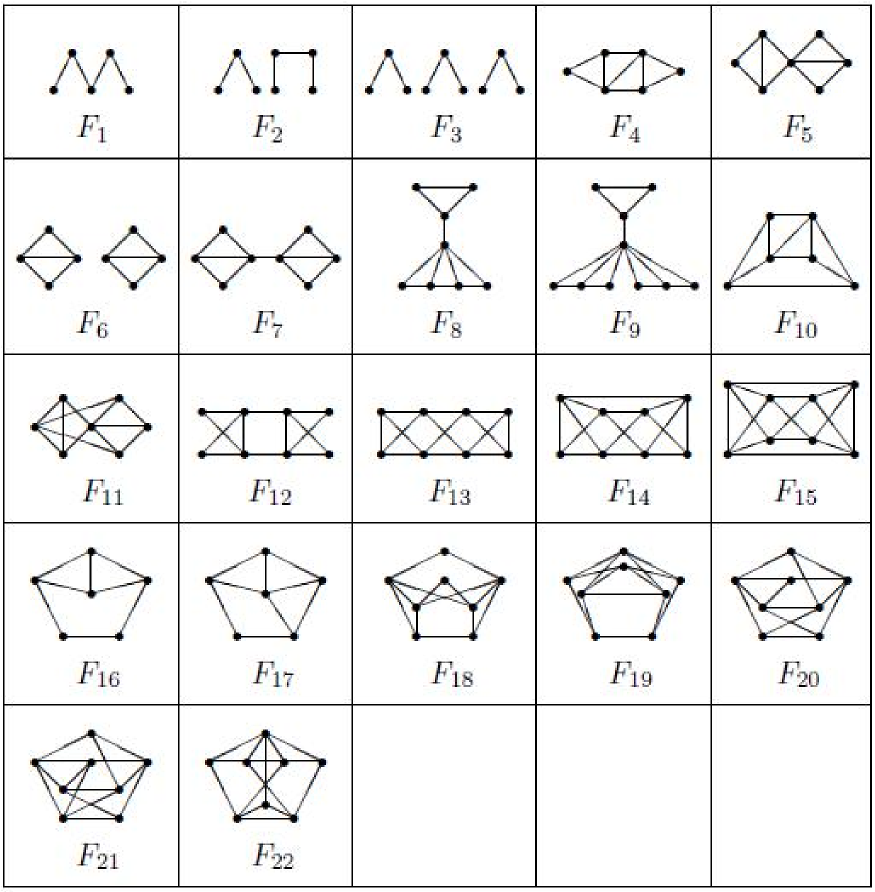


Figure 4.8: Class $\mathcal{F} = \{F_1, ..., F_{22}\}$, [Maf08, Hoa09].

At last, we want to mention that also another variant of b-perfectness is studied by Klein and Kouider in [Kle04]:

Definition 4.6. (Klein and Kouider, [Kle04])
A graph G is $b\omega$-perfect *if each induced subgraph H of G satisfies $\chi_b(H) = \omega(H)$.*

They proved in [Kle04] that a cograph is $b\omega$-perfect if and only if it is F_3-free and F_6-free.

5 Conclusion

When b-colorings were introduced, they were only interesting from a theoretical point of view as a reasonable successor of ordinary vertex colorings and a-colorings (see Chapter 2). In the meantime, also applications for b-colorings were considered. One possible application for solving clustering problems was shown in a simplified version in Chapter 1.
The focus, however, is still on the theoretical investigation of this type of coloring. There are already more than 50 published papers on b-colorings, mainly on bounds for the b-chromatic number of general graphs or special graph classes. The present thesis gives an overview of the current state of knowledge on b-colorings and supplements the research in this field with a range of new results.

The main results were given in Chapter 3. At first, we summarized in Section 3.1 general properties and known bounds on the b-chromatic number and established a new upper bound on χ_b with respect to the clique cover number (see Proposition 3.8). In Section 3.2 we restricted our research to special graphs. Mainly, we obtained results for the following graphs:

Graphs G with:	Exact Value or Bounds for $\chi_b(G)$ in:
$\Delta(G) \leq 2$	Subsection 3.2.1 (Theorem 3.2)
$\alpha(G) \geq n(G) - 2$	Subsection 3.2.2 (Theorem 3.3)
$\delta(G) \geq n(G) - 3$	Subsection 3.2.3 (Theorem 3.4, 3.5, Remark 3.2)
$\alpha(G) + \omega(G) \geq n(G)$	Subsection 3.2.4 (Theorem 3.6)

Since we determined $\chi_b(G)$ for all graphs G with independence number $\alpha(G) \geq n(G) - 2$ or minimum degree $\delta(G) \geq n(G) - 3$, it is natural to try to go one step further, i.e.:

Problem 5.1. *Determine the b-chromatic number for all graphs G with independence number $\alpha(G) = n(G) - 3$ or minimum degree $\delta(G) = n(G) - 4$.*

For graphs G with independence number $\alpha(G) = n(G) - 3$ only the case $\omega(G) = 2$, $\chi_b(G) \in \{3, 4\}$ is open (compare Remark 3.1). With some effort, it may be possible to characterize these graphs with b-chromatic number 3 or 4, respectively.
For graphs G with minimum degree $\delta(G) = n(G) - 4$ we already gave bounds on $\chi_b(G)$ in Corollaries 3.4 and 3.5. Since the complements of these graphs are graphs with maximum degree 3 which cannot be easily characterized like graphs with maximum degree 2, we believe that a simple closed formula for $\chi_b(G)$ like in Theorem 3.5 cannot be achieved. The determination of $\chi_b(G)$ for these graphs might even be $\mathcal{NP}-$hard.

In Section 3.3 we considered bipartite graphs. At first, we presented known results on the b-chromatic number of bipartite graphs. Then we introduced the bicomplement $\widetilde{G}$ of a bipartite graph G and used it to prove a lower bound on $\chi_b(G)$ (see Proposition 3.23). This bound is attained when each component of $\widetilde{G}$ is complete bipartite or an isolated vertex (compare Theorem 3.14). Therefore, the following question arises:

Question 5.1. *For which bipartite graphs G is the lower bound in Proposition 3.23 tight, i.e. $\chi_b(G) = \sigma(\widetilde{G})$?*

We already know that the determination of the b-chromatic number of bipartite graphs is $\mathcal{NP}-$hard. Therefore, one should consider bipartite graphs with a simple structure at first. We did this by investigating $\chi_b(G)$ for bipartite graphs whose bicomplements have maximum degree 2 (Theorems 3.15–3.18). This determination already took about 15 pages, although we did not even consider all possible cases because the effort for determining $\chi_b(G)$ for these remaining cases is too large compared to the use. Therefore, considering bipartite graphs whose bicomplements have maximum degree 3 does not make much sense.

It is known that $\chi_b(G) \leq t(G)$ for every graph G (Proposition 3.1). In Section 3.4 we dealt with the question which graphs have a b-chromatic number close to its t-degree. Since $t(G) \leq \Delta(G) + 1$, one should start to characterize graphs G whose b-chromatic number is equal to $\Delta(G) + 1$. When we restrict ourselves to regular graphs, we can formulate the following problem:

Problem 5.2. *Characterize the d-regular graphs G that satisfy $\chi_b(G) < d + 1$ and those that satisfy $\chi_b(G) = d + 1$.*

For d-regular bipartite graphs G with small values of d, namely, $d \in \{2, 3, 4\}$, we could solve Problem 5.2 (compare Observation 3.4 and Propositions 3.38 – 3.40). For d-regular bipartite graphs G with $d \geq 5$, this problem is still open, but only in case that $\widetilde{G}$ is connected (compare Theorem 3.21 as well as Questions 3.1 and 3.2).

If G is a tree or a cactus with t-degree at least 7, then G satisfies $t(G) - 1 \leq \chi_b(G) \leq t(G)$ (see Propositions 3.41 and 3.43). Since Halin graphs have a tree-like structure, it is very likely that every Halin graph H satisfies $t(H) - 1 \leq \chi_b(H) \leq t(H)$. We already could prove that $\chi_b(H) = t(H) - 1$ if H is a pivoted Halin graph (Theorem 3.22). It remains to investigate non-pivoted Halin graphs:

Question 5.2. *Does every non-pivoted Halin graph H satisfies $t(H) - 1 \leq \chi_b(H) \leq t(H)$?*

Chapter 4 mainly deals with the b-continuity and the b-spectrum of graphs. Since the determination of the b-spectrum and even the decision whether a graph is b-continuous are both $\mathcal{NP}-$hard, one should investigate the b-continuity of graphs with simple structure at first. We did this in Section 4.1 by verifying the b-continuity of graphs whose b-chromatic number was determined in Section 3.2 and 3.3. After this, in Section 4.2 we listed all b-continuous graph classes that are known so far and we proved the b-continuity of Halin graphs (Theorem 4.1). Moreover, we posed a question and a conjecture on the b-continuity of claw-free graphs and line graphs, respectively, that we want to raise here again:

Question 5.3. *Does there exist a claw-free graph that is non-b-continuous?*

Conjecture 5.1. *Line graphs are b-continuous.*

Since the last conjecture deals with line graphs, it makes sense to investigate b-edge colorings as well.

Definition 5.1. *A* b-edge coloring *of a graph G by k colors is a proper edge coloring of G such that there is an edge in each color class, which is adjacent to at least one edge in every other color class. The* b-chromatic index *of a graph G, denoted by $\chi'_b(G)$, is the largest integer k such that there is a b-edge coloring of G by k colors.*

Surprisingly, nobody ever considered b-edge colorings before, although there are many papers concerning a-edge colorings of graphs. So, the edge version of b-colorings is surely an interesting new field and a fruitful topic for future research.

Bibliography

[Aki83] J. Akiyama, F. Harary, P. Ostrand, *A graph and its complement with specified properties VI: Chromatic and achromatic numbers*, Pacific Journal of Mathematics, 104 (1983), 15-27.

[Alo92] N. Alon, M. Tarsi, *Colorings and orientations of graphs*, Combinatorica 12(2) (1992), 125-134.

[Alk10] M. Alkhateeb, A. Kohl, *Upper bounds on the b-chromatic number and results for restricted graph classes*, Discussiones Mathematicae Graph Theory, 31(4) (2011), 709-735.

[Alk11] M. Alkhateeb, A. Kohl, *Investigating the b-chromatic number of bipartite graphs by using the bicomplement*, submitted to Discrete Applied Mathematics, manuscript (2011).

[Ara11] N.R. Aravind, T. Karthick, C.R. Subramanian, *Bounding χ in terms of ω and Δ for some classes of graphs*, Discrete Mathematics, 311(12) (2011), 911-920.

[Bal07] R. Balakrishnan, S. Francis Raj, *Bounds for the b-chromatic number of the Mycielskian of some families of graphs*, manuscript, (2009).

[Bal07] R. Balakrishnan, S. Francis Raj, *Bounds for the b-chromatic number of vertex-deleted subgraphs and the extremal graphs*, Discrete Mathematics, 34 (2009), 353-358.

[Bar07] D. Barth, J. Cohn, T. Faik, *On the b-continuity property of graphs*, Discrete Applied Mathematics, 155 (2007), 1761-1768.

[Bha89] V. N. Bhat-Nayak, M. Shanti, *Achromatic numbers of a graph and its complement*, Bulletin of the Bombay Mathematical Colloquium, 6 (1989), 9-14.

[Bli09] M. Blidia, F. Maffray, Z. Zemir, *On b-coloring in regular graphs*, Discrete Applied Mathematics, 157 (2009), 1787-1793.

[Bod80] H.L. Bodlander, *Achromatic number is $\mathcal{NP}-$complete for cographs and interval graphs*, Inform. process. Lett., 31 (1989), 135-138.

[Bon09] F. Bonomo, G. Durán, F. Maffray, J. Marenco, M. Valencia-Pabon, *On the b-coloring of cographs and P_4-sparse graphs*, Graphs and Combinatorics, 25 (2009), 153-167.

[Bra99] A. Brandstädt, V. B. Le, J. Sprinrad, *Graph classes: A survey*, SIAM, (1999).

[Bro41] R.L. Brooks, *On coloring the nodes of a network*, Proc. Cambridge Phil. Soc., 37 (1941), 194-197.

[Cab10] S. Cabello, M. Jakovac, *On the b-chromatic number of regular graphs*, Discrete Applied Mathematics, 159(13) (2011), 1303-1310.

[Cai97] N. Cairnie, K. Edwards, *Some results on the achromatic number*, Journal of Graph Theory, 26(3) (1997), 129-136.

[Cai98] N. Cairnie, K. Edwards, *The achromatic number of bounded degree trees*, Discrete Mathematics, 188 (1998), 87-97.

[Cam09] V. Campos, C. Sales, F. Maffray, A. Silva, *b-chromatic number of cacti*, Electronic Notes in Discrete Mathematics, 35 (2009), 281-286.

[Cha07] F. Chaouche, A. Berrachedi, *Some bounds for the b-chromatic number of generalized Hamming graphs*, Far East Journal of Applied Mathematics, 26 (2007), 375-391.

[Chau97] A. Chaudhary, S. Vishwanathan, *Approximation algorithms on the achromatic number*, Journal of Algorithms, 41 (2001), 404-416.

[Cor05] S. Corteel, M. Valencia-Pabon, J-C. Vera, *On approximating the b-chromatic number*, Discrete Applied Mathematics, 146 (2005), 106-110.

[Dek08] L. Dekar, H. Kheddouci, *A graph b-coloring based scheme for composition-oriented web services abstraction: COWSA*, ICSOC PhD Symposium, (2008).

[Dek08-1] L. Dekar, H. Kheddouci, *A graph b-coloring based method for composition-oriented web services classification*, ISMIS, 3 (2008), 599-604.

[Dek08-2] L. Dekar, H. Kheddouci, *Distance-2 self-stabilizing algorithm for a b-coloring of graphs*, SSS, (2008), 19-31.

[Eda97] K. J. Edwards, *The harmonious chromatic number and the achromatic number, In: surveys in combinatorics* (Invited papers for 16th British Combinatorial Conference)(Ed., R. A. Bailey), Cambridge University Press, Cambridge, (1997), 13-47.

[Eff03] B. Effantin, H. Kheddouci, *The b-chromatic number of some power graphs*, Discrete Mathematics and Theoretical Computer Science, 6 (2003), 45-54.

[Eff05] B. Effantin, *The b-chromatic number of power graphs of complete caterpillars*, Journal of Discrete Mathematical Sciences & Cryptography, 8 (2005), 483-502.

[Eff051] B. Effantin, H. Kheddouci, *Exact values for the b-chromatic number of a power complete k-ary tree*, Journal of Discrete Mathematical Sciences & Cryptography, (2005), 117-129.

[Eff06] B. Effantin, H. Kheddouci, *A distributed algorithm for a b-coloring of a graph*, ISPA 2006, Lecture Notes in Computer Science, 4330 (2006), 430-438, .

[Elg07] H. Elghazel, K. Benabdeslem, A. Dussauchoy, *Constrained graph b-coloring based clustering approach*, DaWaK 2007, Lecture Notes in Computer Science, 4654 (2007), 262-271.

[ElS06] A. EL Sahili, M. Kouider, *About b-colourings of regular graphs*, Technical Report N 1432, CNRS-Université de Paris Sud-LRI, (2006).

[Fai03] T. Faik, J.-F. Saclé, *Some b-continuous classes of graphs*, Technical Report N1350, LRI, Université de Paris Sud, (2003).

[Fai04] T. Faik, *About the b-continuity of graphs*, Electronic Notes in Discrete Mathematics, 17 (2004), 151-156.

[Fai05] T. Faik, *La b-continuité des b-colorations: complexité, propriétés structurelles et algorithmes*, Ph.D. Thesis no. 7880, Université Paris XI Orsay, 2005 (in French).

[Far86] M. Farber, G. Hahn, P. Hell, D. Miller, *Concerning the achromatic number of graphs*, Journal of Combinatorial Theory B, 40 (1986), 21-39.

[Gel74] D.P. Geller, H.V. Kronk, *Further results on the achromatic number*, Fundamenta Mathematicae, 85 (1986), 285-290.

[Ger08] D. Gernert, L. Rabern, *A computerized system for graph theory, illustrated by partial proofs for graph-coloring problems*, Graph Theory Notes of New York LV, (2008), 14-24.

[Gra86] F. Gavril, M. Yannakakis, *Edge dominating sets in graphs*, SIAM Journal of Applied Mathematics, 38(3)(1980), 364-372.

[Gro99] J. Gross, J. Yellen, *Graph theory and its applications*, CRC Press, (1999).

[Gup69] R. P. Gupta, *Bounds on the chromatic and achromatic numbers of complementary graphs*, in Recent Progress in Combinatorics (Proceedings of Third Waterloo Conference on Combinatorics, Waterloo, 1968) (ed. W. T. Tutte), Academic Press, New York, (1969), 229-235.

[Haj09] H. Hajiabohlhassan, *On the b-chromatic number of Kneser graphs*, Discrete Applied Mathematics, 158 (2010), 232-244.

[Har99] S. Hara, *Complete coloring and the achromatic number of graphs*, Thesis, Yokohama National University, Japan, (1999).

[Har02] S. Hara, A. Nakamoto, *Achromatic numbers of maximal outerplanar graphs*, Yokohama Mathematical Journal, 49 (2002), 181-186.

[Har09] F. Harary, S. Hedetniemi, *The achromatic number of a graph*, Journal of Combinatorial Theory, 8(2) (1970), 154-161.

[Har67] F. Harary, S. T. Hedetniemi, G. Prins, *An interpolation theorem for graphical homomorphisms*, Portugal Mathematics, 26 (1967), 453-462.

[Hel76] H. Hell, D.J. Miller, *Graph with given achromatic number*, Discrete Mathematics, 16 (1976), 195-207.

[Hel92] H. Hell, D.J. Miller, *Achromatic numbers and graph operations*, Discrete Mathematics, 108 (1992), 297-305.

[Hoa05] C.T. Hoáng, M. Kouider, *On the b-dominating coloring of graphs*, Discrete Applied Mathematics, 152 (2005), 176-186.

[Hoa09] C.T. Hoáng, F. Maffray, C.L. Sales, *On minimally b-imperfect graphs*, Discrete Applied Mathematics, 157(17) (2009), 3519-3530.

[Hoa10] C.T. Hoáng, F. Maffray, M. Mechebbek, *A characterization of b-perfect graphs*, submitted to Discrete Mathematics, (2010).

[Hug94] F. Hughes, *On the achromatic number of graphs*, M.Sc. Thesis, University of Victoria, B.C., (1994).

[Hug97] F. Hughes, G. MacGillivray, *The achromatic number of graphs: a survey and some new results*, Bull. Institute of Combinatorics and Its Applications, 19 (1997), 27-56.

[Irv99] R.W. Irving, D.F. Manlove, *The b-chromatic number of a graph*, Discrete Applied Mathematics, 91 (1999), 127-141.

[Jak09-1] M. Jakovac, S. Klavzar, *The b-chromatic number of cubic graphs*, Graphs and Combinatorics, 26 (2009), 107-118.

[Jak11] M. Jakovac, I. Peterin, *On the b-chromatic number of some graph products*, University of Maribor, manuscript, (2011).

[Jav08] R. Javadi, B. Omoomi, *On b-coloring of the Cartesian product of graphs*, manuscript, (2008).

[Jav09] R. Javadi, B. Omoomi, *On b-coloring of the Kneser graphs*, Discrete Mathematics, 309 (2009), 4399-4408.

[Jen95] T.R. Jensen, B. Toft, *Graph coloring problems*, Wiley New York, (1995).

[Kar04] J. Kará, J. Kratochvíl, M. Voigt, *b-continuity*, Technical University Ilmenau, Faculty for Mathematics and Natural sciences, preprint No.M14/04, (2004).

[Kat10] J. Katrenic, F. Galclík, *A note on approximating the b-chromatic number*, submitted to Discrete Applied Mathematics, (2010).

[Kin05] A.D. King, B.A. Reed, A. Vetta, *An upper bound for the chromatic number of line graphs*, European Journal of Combinatorics, 28 (2007), 2182-2187.

[Kin08] A.D. King, B. Reed, *Bounding χ in terms of ω and Δ for quasi-line graphs*, Journal of Graph Theory, 59 (2008), 215-228.

[Kin09] A.D. King, *Claw-free graphs and two conjectures on ω, Δ, and χ*, Ph.D. Thesis, McGill University, (2009).

[Kle] S. Klein, M. Kouider, *b-coloration and P_4-free graphs*, manuscript.

[Kle04] S. Klein, M. Kouider, *On b-perfect graphs*, In Annals of the XII Latin-Ibero-American Congress on Operations Research, Havanna, Cuba, October (2004).

[Koh07] A. Kohl, *On two b-continuous graph classes and upper bounds for the b-chromatic number*, manuscript, (2008).

[Koh09] A. Kohl, I. Schiermeyer, *Some results on Reed's conjecture about ω, Δ, and χ with respect to α*, Discrete Mathematics, 310 (2009), 1429-1438.

[Koh11] A. Kohl, *The b-chromatic number of powers of cycles*, submitted to Discrete Mathematics and Theoretical Computer Science, manuscript (2011).

[K11] *private communication with A. Kohl*, 2011.

[Kou05] M. Kouider, *b-chromatic number of a graph, subgraphs and degrees*, Technical Report N1392, LRI, Université de Paris Sud, (2005).

[Kou02] M. Kouider, M. Maheó, *Some bounds for the b-chromatic number of a graph*, Discrete Mathematics, 256 (2002), 267-277.

[Kou06] M. Kouider, M. Zaker, *Bounds for the b-chromatic number of some families of graphs*, Discrete Mathematics, 306 (2006), 617-623.

[Kou07] M. Kouider, M. Maheó, *The b-chromatic number of the Cartesian product of two graphs*, Studia Sci. Math. Hungar., 44 (2007), 49-55.

[Kou11] M. Kouider, M. Valencia-Pabon, *On lower bounds for the b-chromatic number of connected bipartite graphs*, Electronic Notes in Discrete Mathematics, 37 (2011), 399-404.

[Kor01] G. Kortsarz, R. Krauthgamer, *On approximating the achromatic number*, SIAM Journal on Discrete Mathematics, 14 (2011), 408-422.

[Kor05] G. Kortsarz, J. Radhakrishnan, S. Sivasubramanian, *Complete partitions of graphs*, in Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms (Vancouver) ACM, New York , (2005), 860-869.

[Kor03] G. Kortsarz, S. Shende, *Approximating the achromatic number problem on bipartite graphs*, in Proceedings of 11th European Symposium on Algorithms, Lecture Notes in Computer Science, Springer, Berlin (2003), 385-396.

[Kor07] G. Kortsarz, S. Shende, *An improved approximation of the achromatic number on bipartite graphs*, , SIAM Journal on Discrete Mathematics, 21 (2007), 361-373.

[Kra02] J. Kratochvíl, Zs. Tuza, M. Voigt, *On the b-chromatic number of graphs*, Lecture Notes In Computer Science, 2573 (2002), 310-320.

[Kry98] P. Krysta, *Approximation algorithms for combinatorial optimization problems in graph coloring and network design*, Ph.D. Thesis, University of Saarland, Germany, (2001).

[Kry99] P. Krysta, K. Lorys', *New approximation algorithms for the achromatic number*, Research report No. 98-1-016, Max-Planck-Institut für Informatik, Saarbrücken, Germany, (1998).

[Kry06] P. Krysta, K. Lorys', *Efficient approximation algorithms for the achromatic number*, in Proceedings of 7th European Symposium on Algorithms, Lecture Notes in Computer Science, 1643, Springer, Berlin (1999), 402-413.

[Lee04] J. Lee, Y-h. Shin, *The achromatic number of the union of cycles*, Discrete Applied Mathematics, 143 (2004), 330-335.

[Mac01] G. MacGillivray, A. Rodriguez, *The achromatic number of the union of paths*, Discrete Mathematics, 231 (2001), 331-335.

[Maf08] F. Maffray, M. Mechebbek, *On the b-perfect chordal graphs*, Graphs and Combinatorics, 25 (2009), 365-375.

[McD82] C. J. H. McDiarmid, *Achromatic numbers of random graphs*, Mathematical Proceedings of the Cambridge Philosophical Society, 92 (1982), 21-28.

[Mil86] F. Milazzo, V. Vacirca, *On the achromatic number of permutation graphs*, in Proceedings of First Catania International Combinatorial Conference on Graphs, Steiner Systems and their Applications, (2) 24B (1987), 71-76.

[Mil82] Z. Miller, *Extremal regular graphs for the achromatic number*, Discrete Mathematics, 40 (1982), 235-253.

[Rab08] L. Rabern, *A note on Reed's conjecture*, SIAM Journal on Discrete Mathematics, 22(2) (2008), 820-827.

[Ree98] B.A. Reed, *ω, Δ and χ*, Journal of Graph Theory, 27(4) (1998), 177-212.

[Roi91] Y. Roichman, *The achromatic number of trees, grids and cubes*, M.Sc. Thesis, Hebrew University of Jerusalem, (1991).

[Roi98] Y. Roichman, *On the achromatic number of boolean cubes*, Bar-Ilan University, manuscript, (1998).

[Roi00] Y. Roichman, *On the achromatic number of hypercubes*, Journal of Combinatorial Theory B, 79 (2000), 177-182.

[Sal09] C.L. Sales, L. Sampaio, *b-coloring of m-tight graphs*, Electronic Notes in Discrete Mathematics, 35 (2009) 209-214.

[Sha11] S. Shaebani, *On the b-chromatic number of regular graphs without 4-cycle*, Discrete Applied Mathematics, 160 (2012), 1610-1614.

[Sha90] M. Shanthi, *Achromatic numbers*, Ph.D. Thesis, University of Bombay, (1990).

[Thi09] K. Thilagavathi, N. Roopesh, *Achromatic colouring of central graphs and split graphs*, Far East Journal of Applied Mathematics, 30 (2008), 359-369.

[Thi09-1] K. Thilagavathi, N. Roopesh, *Generalisation of achromatic colouring of central graphs*, Electronic Notes in Discrete Mathematics, 33 (2009), 147-152.

[Thi09-2] K. Thilagavathi, K. P. Thilagavathi, N. Roopesh, *The achromatic colouring of graphs*, Electronic Notes in Discrete Mathematics, 33 (2009), 153-156.

[Thi10] K. Thilagavathi, D.Vijayalakshmi, N.Roopesh, *b-colouring of central graphs*, International Journal of Computer Applications, 11(3) (2010), 0975-8887.

[Vel10] C.I.B. Velasquez, F. Bonomo, I. Koch, *On the b-coloring of P_4- tidy graphs*, Discrete Applied Mathematics, 159 (2010), 60-68.

[Ven10] J. Vernold Vivin, M. Venkatachalam, *The b-chromatic number on star graph families*, Le Matematiche, LXV Fasc. I, 65(1) (2000), 119-125.

[Ver09-1] J. Vernold Vivin, M. Venkatachalam, M. M. Akbar Ali, *A note on achromatic coloring of star graph families*, Filomat, 23 (2000), 251-255.

[Ver09-2] J. Vernold Vivin, M. Venkatachalam, M. M. Akbar Ali, *Achromatic coloring on double star graph families*, International Journal of Mathematical Combinatorics, 3 (2009), 71-81.

[Wes99] D.B. West, *Introduction to graph theory*, Prentic Hall Upper sadle River, NJ07458,(2001).

[Yos08] T. Yoshida, H. Elghazel, V. Deslandres, M-S. Hacid, A. Dussauchoy, *Toward improving b-coloring based clustering using a Greedy re-coloring algorithm*, Advances in Greedy Algorithms, (2007), 553-568.

Definitions

1. A ***cactus*** is a connected graph in which any two cycles have at most one vertex in common.

2. A ***clique*** is a set of pairwise adjacent vertices in G. The *clique number* of G, denoted by $\omega(G)$, is the maximum order of a clique in G.

3. A graph G is ***cograph*** if G does not contain P_4 as an induced subgraph.

4. The graph $\overline{G} = (V, \overline{E})$ is the ***complement*** of $G = (V, E)$ if and only if every $uv \in \overline{E}$ satisfies $uv \notin E(G)$.

5. The ***connectivity*** of G, denoted by $\kappa(G)$, is the minimum size of the vertex set S, so that $G - S$ is disconnected or has only one vertex.

6. Let $G = (V, E)$ and $u, v \in V$. Then the ***distance*** of u and v, denoted by $d(u, v)$ is the number of edges in a shortest path that connects u and v. The ***diameter***, denoted by $diam(G)$, is maximum value of $d(u, v)$ such that $u, v \in V(G)$.

7. A graph H is a ***factor*** of G if H is a subgraph of G with $V(H) = V(G)$. A graph H is an r***-regular factor*** of G if H is a factor of G and r-regular (i.e $\delta(H) = \Delta(H) = r$).

8. The ***girth*** $g := g(G)$ is the length of a shortest cycle in G.

9. An ***independent set*** is a set of pairwise non-adjacent vertices in G. The independence number of G, denoted by $\alpha(G)$, is the maximum order of an independent set in G.

10. Let $G = (V, E)$ be a graph and $W \subseteq V$. Then we call the graph (W, F) where $F = \{uv : u, v \in W \wedge uv \in E\}$ the subgraph ***induced*** by W and denote it by $G[W]$. We call H ***an induced subgraph*** of G if $H \subseteq G$ and $H = G[V(H)]$. Let G and H be two graphs. We say that G is H***-free*** if G does not contain H as an induced subgraph.

11. Let G_1,...,G_r be graphs. The ***join*** $G_1 \oplus ... \oplus G_r$ is the graph defined by
    – $V(G_1 \oplus ... \oplus G_r) = V(G_1) \cup ... \cup V(G_r)$.

– $E(G_1 \oplus ... \oplus G_r) = E(G_1) \cup ... \cup E(G_r) \cup \{uv : u \in V(G_i) \wedge v \in V(G_j),$ for $i, j \in \{1, ..., r\}$ and $i \neq j\}$.

12. A ***matching*** is a set of pairwise non-adjacent edges in G. The matching number of G, denoted by $\nu(G)$, is the maximum size of a matching in G.

13. A graph is d-***regular*** if every vertex of G has degree equal to d.

14. A graph is P_4-***sparse*** if every set of five vertices contains at most one induced P_4.

15. A graph G is called a split graph if there exists a partition $V(G) = I \cup K$ such that the subgraphs of G induced by I and K are empty and complete graphs, respectively.

16. A graph $G' = (V', E')$ is called a ***subgraph*** of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$. In this case we write $G' \subseteq G$.

17. A *partner* A of a P_4 in G is a vertex v in $V(G - A)$ such that $A + v$ induces at least two P_4's. A graph G is P_4-***tidy*** if any P_4 has at most one partner.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts habe ich Unterstützungsleistungen von folgenden Personen erhalten:

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Weitere Personen waren an der Abfassung der vorliegenden Arbeit nicht beteiligt. Die Hilfe eines Promotionsberaters habe ich nicht in Anspruch genommen. Weitere Personen haben von mir keine geldwerten Leistungen für Arbeiten erhalten, die nicht als solche kenntlich gemacht worden sind.

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Dipl.-Math. Mais Alkhateeb

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