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Contents

Chapter 1

Introduction

Within the past decades, wavelets and associated wavelet transforms have been intensively investigated in both applied and pure mathematics. They and the related multi-scale analysis provide essential tools to describe, analyse and modify signals, images or, in rather abstract concepts, functions, function spaces and associated operators. A comprehensive exposition about the theory is given for instance by Daubechies [Dau93] and Coifman-Mayer [MC97].
One of the reasons for the great interest in this subject is that one of their applications is the field of signal processing and compression, which is becoming more and more important in our technological world. Since the early 1990s, wavelet transform has been propagated as a mile stone in image and audio compression, and the methods currently used are based on wavelets. An introductory book on wavelet theory is provided in [LMR94].
A mathematically important milestone was the development of the Fourier transform, which was introduced in the famous "Théorie analytique de la chaleur" by Jean Baptiste Joseph Fourier in 1822. There it was described the decomposition of a signal into frequencies and amplitudes. The wavelet transform is an improvement on this theory, which is motivated by the necessity of a more flexible tool.
The most influential constructions of wavelets in $\mathbb{R}^n$ can be found in the works of Haar [Haa11], Grossmann Morlet [GM84] and Daubechies [Dau88].
Theoretical investigations in that direction belong to the field of harmonic analysis. From the modern point of view, harmonic analysis is the theory of locally compact groups. By having a look at this abstract approach, the algebraic structure behind wavelet transforms and related questions is revealed.
The constructions of wavelet transforms can be entirely based on an abstract group-theoretical and representation-theoretical approach. An abstract exposition of this topic can be found in Kisil [Kis99b], [Kis99a]. For the particular situation of the Lorentz group $SO(n+1,1)$ acting on the sphere S^n, an associated wavelet construction was carried out by Antoine and Vandergheynst [AV99], [ADJV02]; see also [Fer09], [Fer08]. Their approach is extended to further non-Euclidean manifolds such as the hyperboloid, by Bogdanova [Bog05].

We aim to investigate functions on Lie groups and homogeneous spaces. Thereby our desire is to develop wavelets on these manifolds. Therefore we have to discuss the harmonic analysis in a very general way such that its algebraic and group theoretical nature can be understood. It is also important to look at the wavelet transform from the group theoretical point of view in order to formulate (admissibility) conditions for wavelets.

An alternative approach to wavelets was followed by Coifman-Maggioni [CM06] and on the sphere By .

Classical wavelet theory on $\mathbb{R}^n$ is based on the group which is generated by translations and dilations. It is evident that translations are rotations on a sphere (seen as homogeneous space of the rotation group), but there is no canonical choice for dilations. Some alternative constructions on the sphere are given by Freeden [FGS98] or for graphs there are constructions by Coifman-Maggioni [CM06]. The key idea of diffusive wavelets is to generate dilations from a diffusive semigroup, e.g. from time evolution of fundamental solution to the heat equation on the homogeneous space. The advantage of using compact groups is the availability of powerful tools like the Peter-Weyl theorem and the connected classifications of irreducible representations. A related concept which is based on spectral calculus of the Laplace operator on closed manifolds was proposed by Geller [GM09].

Discrete wavelet transforms in such a setting were discussed in [CM06] and [BCMS06], where heat evolution is combined with an orthogonalization procedure to model a multi-resolution analysis within $L^2(S^3)$.

Due to the generality of this concept, we can formulate a Fourier analysis on compact Lie Groups, homogeneous spaces, and also some noncompact manifolds.

The Fourier analysis on these manifolds helps us to solve partial differential equations such as the heat equation, as our most important application. The analytical approach for wavelets uses semigroups of operators. The fundamental solution of the heat equation is the basis for the semigroup of convolution operators, which we use to formulate diffusive wavelets.

The structure of the present thesis is as follows. At first, we give an introduction about the harmonic analysis on compact Lie groups by using the group theoretical approach. Therefore, important points and relations between famous theorems of representation theory are worked out and discussed in an appropriate way for our use. Since we aim to discuss the harmonic analysis on manifolds, especially on Lie groups, the necessary theory of Lie groups and Lie algebras is presented emphasizing its relation to geometrical and analytical aspects.

In the third chapter, we describe the basic idea of diffusive wavelets and we formulate the theory for compact Lie groups and their homogeneous spaces.

We discuss special cases of diffusive wavelets which possess additional symmetries, in the sense that they are invariants under the action of some group. The construction of wavelets possessing those symmetries is also investigated in [BBCK10].

In the fourth chapter, we discuss a row of important examples for which the explicit realization of wavelets is given in terms of their Fourier series. The torus is the most natural manifold to discuss wavelets, since the periodizations of functions in $\mathbb{R}^n$ can be regarded as functions on the torus of appropriate dimension. The Fourier analysis of those functions simplifies to the usual exponential Fourier series. Hence the form of the wavelets on the torus is easy to understand and can even be visualized.

Another object of great interest is the sphere, since it appears in many applications. For instance in geoscience the case of the two-dimensional sphere is interesting [FGS98, Fer09, AV07]. Other fields, e.g. texture analysis, ask for wavelets on the rotation group $SO(3)$ and its double covering manifold which is the three-dimensional sphere ([BS05, Hie07, BE10] and others). In [Ebe08] constructions of wavelets on S^3 are discussed. We investigate the n-dimensional sphere as homogeneous space $SO(n+1)/SO(n)$. Thus, we also discuss the n-dimensional rotation group as the first non-commutative example of a compact Lie group. Here, we formulate the general construction and we also consider special cases, such as zonal wavelets which are common for spherical constructions. Especially for applications, the discretization of wavelets is an important task. Nevertheless we do not aim here to discretize our wavelet. We only give some hints in Section 3.4.1 about scale-discretized wavelets.

A crucial point for the construction on Lie groups is the existence of a Plancharel measure, which is ensured in the compact case. In the noncompact case, our constructions will work if the Plancharel measure exists.

As noncompact example we consider the Heisenberg group. The Heisenberg group is one of the most important Lie groups in time-frequency analysis and the Plancharel measure is explicitly known [Str91], [Tha98]. Furthermore the construction on compact groups uses the Peter-Weyl theorem which requires the compactness of the group. For the Heisenberg group all the irreducible representations are characterized by the Stone von-Neumann theorem. As appropriate equation for our diffusion process on the Heisenberg group, we use the heat equation with respect to the sub-Laplacian, since its natural structure is the sub-Riemannian one.

Another interesting example is the spin group $\mathrm{S}pin(m)$. In [Som96], we find an outline for properties about functions on $\mathrm{S}pin(m)$. We consider and develop it to introduce diffusive wavelets on $\mathrm{S}pin(m)$. In [CFKS07], [CFK06] we can also find some investigations of wavelets related to Clifford analysis. For our investigations, we have to discuss the representations of $\mathrm{S}pin(m)$. We know half of the representations of $\mathrm{S}pin(m)$ from the rotation group $SO(m)$, since $\mathrm{S}pin(m)$ is a double covering of it, but this is not enough. For a comprehensive discussion we have to introduce weights and routs of all irreducible representations; in that way all irreducible representations can be characterized and explicitly realized, see [VLSC01].

We introduce $\mathrm{S}pin(m)$ as a group in the Clifford algebra and we aim to construct diffusive wavelets on $\mathrm{S}pin(m)$ also for Clifford valued functions. In order to give a clear exposition we will present the necessary calculations extensively, at least in the Appendix. As a homogeneous space of $\mathrm{S}pin(m+1)$ we will consider the sphere S^m.

In the closing chapter we consider the Radon transform as a further object of interest, where the theory of diffusive wavelets can be applied successfully. The amount of publications and results in this research field is huge. We consider the Radon transform on compact Lie groups and we rewrite it in our language for compact Lie groups. The resulting transform differs from that one investigated by Helgason [Hel99, Hel11]. The Radon transform of our type is motivated by some applications in texture analysis during the investigation of crystals with respect to their structure.

The Radon transform can be inverted with the help of diffusive wavelets, for the special case of $SO(3)$, which comes from the application in texture analysis. The fact that the Radon transform of wavelets on $SO(3)$ gives wavelets on S^2 is described in detail. The related concept of Gabor frames enables an inversion too, see [CFKT11].

For applications, it is not possible to measure all the data which give the continuous Radon transform. Consequently, we can only consider a finite set of measurements. The inversion of this incomplete[1] Radon transform is discussed in Section 5.2. A work of Peasenson [Pes04] fits very well to our situation.

Eventually, we are able to formulate a Shannon-sampling-theorem for compact Lie groups which assumes a very convenient form in terms of representation theory. Beside the theoretical value of the Shannon-sampling theorem, it can also be used to discuss many questions for the applications, such as the optimal choice of points of measuring to obtain a stable inversion.

[1] in the sense that there are only finitely many measurements

Chapter 2

General theory

2.1 Preliminaries on representation theory

The list of literature about Lie groups is enormous long and even if we restrict to the very important contributions, we can not list an appropriate collection here. A collection of important theorems and proofs is given by Fegan in [Feg91]. More detailed investigations can be found in [Bum04] and a comprehensive overview of the theory and explicit examples are given by the three books of Vilenkin and Klimyk [VK93, VK91, VK92].
It is often seen, that authors consider Lie groups as matrix groups ([Bir37] for instance), at least in the finite dimensional case. This identification is possible because there exists a faithful representations of finite dimensional Lie groups and in $\mathbb{R}^n$ (for n large enough). We start by introducing the notion of representations. Afterwards we will also introduce the concept of Fourier transform on compact Lie groups, which is closely connected to representation theory. The Fourier expressed in terms of representation theory will turn out to be one of the fundamental concepts for our study.

Definition 2.1.1 (Representation)**.** Let $\mathcal{G}$ be a Lie group and $\mathcal{H}$ a d-dimensional Hilbert space with inner product $\langle\ ,\ \rangle_{\mathcal{H}}$. Further $\mathrm{GL}(\mathcal{H})$ denotes the group of linear, invertible and bounded operators on $\mathcal{H}$. A *representation of* $\mathcal{G}$ *in* $\mathcal{H}$ is a continuous group homomorphism π from $\mathcal{G}$ to $\mathrm{GL}(\mathcal{H})$, i.e. $\pi : \mathcal{G} \to \mathrm{GL}(\mathcal{H})$, with

$$\begin{aligned} \pi(g_1 g_2) &= \pi(g_1)\pi(g_2) && \forall g_1, g_2 \in \mathcal{G}, \\ \pi(e) &= \mathrm{Id}_{\mathcal{H}}, \end{aligned}$$

where e denotes the unit element in $\mathcal{G}$ and $\mathrm{Id}_{\mathcal{H}}$ the identity mapping on $\mathcal{H}$. The dimension of the representation is denoted by d_π and equals the dimension of $\mathcal{H}$.

A representation π is *unitary* if $\pi(g)$ is an unitary operator for all $g \in \mathcal{G}$.

A representation is *faithful*, if π is injective or equivalently $\pi(g) \neq \mathrm{Id}_{\mathcal{H}}$ for all $g \neq e$.

Let π_j be a representation in the Hilbert space $\mathcal{H}_j$ $(j = 1, 2)$. One says, that π_1 is *equivalent* to π_2 (writing $\pi_1 \sim \pi_2$) if there exists a bounded, linear operator

$$A : \mathcal{H}_1 \to \mathcal{H}_2$$

so that

$$A\pi_1(g) = \pi_2(g)A \qquad \forall g \in \mathcal{G}.$$

A is called *intertwining operator* between π_1 and π_2.

This defines an equivalence relation on the set of irreducible representations and enables us to investigate equivalence classes of representations.

Remark 2.1.2 (Integration on Lie groups). Integration on manifolds or Lie groups can be given by the usual concept of partition of unity. On locally compact Lie groups we have an invariant measure, the so called Haar measure, i.e.

$$\int_{\mathcal{G}} f(g)\,\mathrm{d}g = \int_{\mathcal{G}} f(sg)\,\mathrm{d}g \qquad \text{left Haar measure}$$
$$\int_{\mathcal{G}} f(g)\,\mathrm{d}g = \int_{\mathcal{G}} f(gs)\,\mathrm{d}g \qquad \text{right Haar measure.}$$

In case there exist a measure which is left- and right-invariant, $\mathcal{G}$ is called to be *unimodular*. Every compact group is unimodular and the invariant measure is unique up to equivalence.

Furthermore, to every representation π in $\mathcal{H}$ can be associated an equivalent unitary representation in the following way. Let $\langle\ ,\ \rangle$ be the scalar product on $\mathcal{H}$, then

$$(u, v) := \int_G \langle \pi(g)(u), \pi(g)(v) \rangle\,\mathrm{d}\mu(g).$$

defines another scalar product in $\mathcal{H}$. The integration is taken with respect to the (right-invariant) Haar measure $\mathrm{d}\mu$. Obviously π is unitary with respect to the scalar product $(\cdot, \cdot)$ on $\mathcal{H}$, which is defined in (2.1.1):

$$\begin{aligned}(\pi(g)u, \pi(g)v) &= \int_G \langle \pi(g')\pi(g)(u), \pi(g')\pi(g)(v) \rangle\,\mathrm{d}\mu(g') \\ &= \int_G \langle \pi(g'g)(u), \pi(g'g)(v) \rangle\,\mathrm{d}\mu(g') = (u, v).\end{aligned}$$

So it is enough to look at unitary representations.

Definition 2.1.3 (irreducibility)**.** Let π be a representation of $\mathcal{G}$ in $\mathcal{H}$. A subspace $U \subset \mathcal{H}$ is invariant under π if

$$\{\pi(g)u,\ u \in U\} \subset U \qquad \forall g \in \mathcal{G}.$$

If the only invariant subspaces of π are the trivial ones (i.e. $\{0\}$ and $\mathcal{H}$), π is called to be *irreducible*.

Equivalently one can say that for any linear operator $A : \mathcal{H} \to \mathcal{H}$, $\pi(g)A = A\pi(g)$ implies that $A = c\,Id_{\mathcal{H}}$ for some constant $c \in \mathbb{C}$. Later on a generalization of this fact will give Schur's Lemma 2.1.6.
For two representations π_j in $\mathcal{H}_j$ $(j = 1, 2)$, the direct sum $\pi_1 \oplus \pi_2$ of π_1 and π_2 in $\mathcal{H}_1 \oplus \mathcal{H}_2$ is given by

$$\pi_1 \oplus \pi_2(g)(x, y) = \pi(g)(x, y) = (\pi_1(g)x, \pi_2(g)y), \quad x \in \mathcal{H}_1,\ y \in \mathcal{H}_2, \tag{2.1.1}$$

where in the above formula $(\pi_1(g)x, \pi_2(g)y)$ should be understood as a pair and not as an inner product.
In this way the *direct orthogonal sum* $\sum_n^{\oplus} T_n$ is defined and can be extended to the *direct integral* $\int_\Lambda^{\oplus} T_\lambda \,\mathrm{d}\mu(\lambda)$ of representations, where $\mathrm{d}\mu$ denotes the Plancherel measure.
If π_1 and π_2 are irreducible representations of dimension d_1 and d_2, the direct sum $\pi_1 \oplus \pi_2$ is a reducible representation of dimension $d_1 + d_2$, which posses exactly two nontrivial invariant subspaces, namely $(0, \mathcal{H}_2)$ and $(\mathcal{H}_1, 0)$. The restriction of $\pi_1 \oplus \pi_2$ to $(\mathcal{H}_1, 0)$ is equivalent to π_1, while the restriction to $(0, \mathcal{H}_2)$ is equivalent to π_2.
A precise form is given in

Lemma 2.1.4. *Let U be an invariant subspace with respect to representation π of $\mathcal{G}$ in $\mathcal{H}$. Then the orthogonal complement $U^\perp$ is also invariant under π. Hence $\pi_1 \oplus \pi_2$ decomposes into two irreducible components.*

Proof: For $u \in U$ and $v \in U^\perp$ it holds

$$0 = \langle \pi(g^{-1})u, v\rangle = \langle u, \pi(g)v\rangle \qquad \forall g \in \mathcal{G}.$$

□

Obviously, we have $\pi(g^{-1}) = \pi^*(g)$.

Corollary 2.1.5. More important than the lemma itself is the conclusion, that every finite dimensional unitary representation is complete reducible, i.e. can be written as the direct sum of irreducible representations.

One of the fundamental theorems of representation theory is the following lemma.

Lemma 2.1.6 (Schur)**.** *Let A be the intertwining operator between irreducible representations π_1 in $\mathcal{H}_1$ and π_2 in $\mathcal{H}_2$, then A is either the null operator or invertible.*

A proof can be found in [Fol95, Chapter 3].
A consequence is the uniqueness (up to a constant) of intertwining operators of equivalent representations. To see this fact, one chooses another intertwining operator B, then for all scalars λ: $B - \lambda A$ is an other intertwining operator between π_1 and π_2. Choosing $\lambda = \lambda_0$, so that $\det(B - \lambda_0 A) = 0$ and hence $B - \lambda_0 A$ is not invertible. By Schur's Lemma (Lemma2.1.6) $B - \lambda_0 A$ is the null operator, consequently $B = \lambda_0 A$.

2.1.1 Matrix coefficients and characters

Every finite dimensional representation can be identified with a matrix subgroup, this comes from the fact that for a fixed basis in the representation Hilbert space $\mathcal{H}$ of the representation π of $\mathcal{G}$ one can fix a basis $\mathfrak{B} = \{u_i, i = 1, ..., d_\pi\}$ so that one can identify the linear mapping $\pi(g)$ with the corresponding matrix with respect to $\mathfrak{B}$.

Definition 2.1.7. The entries of the matrix corresponding to the representation π are of the form

$$\pi_{ij}(g) := \langle \pi(g)u_i, u_j \rangle_{\mathcal{H}} \qquad i, j = 1, ..., d_\pi. \tag{2.1.2}$$

π_{ij} are the *matrix coefficients* of π.

We will make use of both notations, if we have a certain basis in mind we will write π_{ij} as we defined in (2.1.2). In general we also write $\pi_{xy}(g) = \langle \pi(g)x, y \rangle$ for $x, y \in \mathcal{H}$.
Let $A \in GL(d_\pi)$ be a change of the basis, than the matrix changes to a similar matrix $A^{-1}(\pi_{ij}(g))_{i,j=1}^{d_\pi} A$. Independent of the choice of the basis is the notion of the character.

Definition 2.1.8. The *character* of a representation π is given by

$$\chi_\pi(g) = \text{ trace}(\pi(g)) = \sum_{i=1}^{n} \pi_{ii}(g).$$

The characters χ_π posses the following invariance property of being a class function.

Definition 2.1.9. If a function on a Lie group is constant over conjugate classes, i.e.

$$f(g) = f(h^{-1}gh) \qquad \forall h \in \mathcal{G}$$

than f is called *class function*.

2.1.2 Regular representation and Peter Weyl theorem

Definition 2.1.10 (Regular representations)**.** The (right- and left-) regular representation is a representation in the Hilbert space $L^2(\mathcal{G})$, given by

$$L_g : f(x) \mapsto f(g^{-1}x) \qquad \text{left-regular representation}$$
$$R_g : f(x) \mapsto f(xg) \qquad \text{right-regular representation,}$$

for $f \in L^2(\mathcal{G})$. Indeed L_g is a representation. Setting $L_g f = f_g$:

$$L_{g_1}L_{g_2}f(x) = L_{g_1}f_{g_2}(x) = f_{g_2}(g_1^{-1}x) = f(g_2^{-1}g_1^{-1}x) = L_{g_1g_2}f(x).$$

The reason, why one can use the presented tools of representation theory to establish a harmonic analysis on Lie Groups is given in the following theorem. An improvement of it will give the Peter-Weyl theorem later which asserts the complete reducibility of the regular representations.

Theorem 2.1.11. *Every irreducible representation π of $\mathcal{G}$ in $\mathcal{H}$ is equivalent to the right-regular representation in a certain vector space of scalar valued functions on $\mathcal{G}$.*

The certain vector space is spanned by the matrix coefficients $\pi_{xy}(g)$. We make use of the notation

$$\pi_{x\mathcal{H}} := \text{span}\{\pi_{xy}|\ y \in \mathcal{H}\}$$
$$\pi_{\mathcal{H}x} := \text{span}\{\pi_{yx}|\ y \in \mathcal{H}\}.$$

Proof: Let π be a irreducible representation of $\mathcal{G}$ in $\mathcal{H}$ and $\mathcal{H} \ni a \neq 0$. The linear mapping $A : \mathcal{H} \to \pi_{\mathcal{H}a}$ shall be defined by

$$Ax = \pi_{xa}(g) = \langle \pi(g)x, a\rangle_{\mathcal{H}} \in \pi_{\mathcal{H}a}.$$

Setting $y = \pi(g_0)x$ one finds

$$(A\pi(g_0)x)(g) = Ay = \langle \pi(g)y, a\rangle_{\mathcal{H}} = \langle \pi(gg_0)x, a\rangle_{\mathcal{H}} = Ax(gg_0) = R_{g_0}Ax(g),$$

where R_g denotes the restriction of the right-regular representation to $A\mathcal{H} = \{Ax|\ x \in \mathcal{H}\} = \pi_{\mathcal{H}a}$. Obviously A is the intertwining operator between π and R.
To ensure the equivalence of π and R one has to show the invertibility of A.
For $x \in Ker(A)$ it follows that $A\pi(g)x = R_gAx = 0$, hence $Ker(A)$ is invariant under π. Irreducibility of π and $A\pi(e)a = \langle a, a\rangle_{\mathcal{H}} \neq 0$ implies $Ker(A) = \{0\}$. □

By the continuity of the representation for compact groups $\mathcal{G}$ it follows $\pi_{ax}(g) \in L^2(\mathcal{G})$ and $\pi_{\mathcal{H}x}$, $\pi_{x\mathcal{H}} \subset L^2(\mathcal{G})$.
The assertion of the above theorem is also valid for the left-regular representation $L_{g_0}f(g) = f(g_0^{-1}g)$. Replacing A in the proof by $Ax : x \mapsto \pi_{ax}(g)$ results in

$$(L_{g_0}A)y(g) = \langle \pi(g_0^{-1})\pi(g)a, y\rangle = \langle \pi(g)a, \pi(g_0)y\rangle = (A\pi(g_0))y(g).$$

Hence A is the intertwining operator between π in $\mathcal{H}$ and L in $\pi_{a\mathcal{H}}$.
From the transitivity of the equivalence of representations follows now $R \sim L$.
The space $\pi(\mathcal{H}) = \{\pi_{xy}(g) = \langle \pi(g)x, y\rangle_{\mathcal{H}}|\ x, y \in \mathcal{H}\}$, spanned by matrix coefficients, is right- and left-invariant. This can be easily seen. Let be $z = \pi(g_0)x$, $v = \pi^*(g_0^{-1})y$, then one has

$$\pi_{xy}(gg_0) = \langle \pi(g)\pi(g_0)x, y\rangle_{\mathcal{H}} = \langle \pi(g)z, y\rangle_{\mathcal{H}} = \pi_{zy}(g) \qquad \Rightarrow \pi_{\mathcal{H}y} \text{ is right- invariant}$$
$$\pi_{xy}(g_0^{-1}g) = \langle \pi(g_0^{-1})\pi(g)x, y\rangle_{\mathcal{H}} = \langle \pi(g)x, \pi^*(g_0^{-1})y\rangle_{L^2(\mathcal{H})} = \pi_{xv}(g) \qquad \Rightarrow \pi_{x\mathcal{H}} \text{ is left-invariant.}$$

Lemma 2.1.12. *The spaces $\pi(\mathcal{H}_\pi)$ and $\xi(\mathcal{H}_\xi)$ are equal if π and ξ are equivalent representations in $\mathcal{H}_\pi$ and $\mathcal{H}_\xi$, respectively.*

Proof: Let A be the intertwining operator between π and ξ: $\pi(g) = A^{-1}\xi(g)A$, $z = Ax$, $v = (A^*)^{-1}y$, then

$$\xi_{zv}(g) = \langle\xi(g)z, v\rangle_{\mathcal{H}_\xi} = \langle\xi(g)Ax, (A^*)^{-1}y\rangle_{\mathcal{H}_\xi} = \langle A^{-1}\xi(g)Ax, y\rangle_{\mathcal{H}_\pi} = \langle\pi(g)x, y\rangle_{\mathcal{H}_\pi} = \pi_{xy}(g)$$

□

The invariance of $\pi(\mathcal{H})$ under right- and left-translations should not be missunderstood as invariance of $\pi_{\mathcal{H}y}$. $\pi_{\mathcal{H}y}$ is only invariant under right-translations. The space $\pi_{y\mathcal{H}}$ is left-invariant, i.e. $\pi(g_0^{-1})\pi_{xy}(g) = \pi_{x\pi(g_0^{-1})y}(g)$.

Theorem 2.1.13 (Burnside)**.** *For an irreducible representation π of a compact group $\mathcal{G}$ in the Hilbert space $\mathcal{H}$ with a basis $\{u_i, i = 1, ..., d_\pi\}$, the matrix coefficients $\pi_{u_iu_j}$ are linearly independent and span $\pi(\mathcal{H})$, $\dim\pi(\mathcal{H}) = d_\pi^2$.*

Proof: By π_i we denote the function space $\pi_{\mathcal{H}u_i}$. The functions π_{xu_i} and π_{xu_j} are linearly independent for $i \neq j$. To show this fact one uses a contraposition.
Form the contrary assumption$\pi_{xu_i}(g) = \sum_{i\neq j}\lambda_j\pi_{xu_j}$ it follows that

$$\langle\pi(g)x, u_i\rangle = \sum_{j\neq i}\lambda_j\langle\pi(g)x, u_j\rangle = \langle\pi(g)x, \sum_{j\neq i}\lambda_j u_j\rangle \Rightarrow u_i = \sum_{j\neq i}\lambda_j u_j.$$

But this contradicts to the assumption of linear independence of $\{u_i,\, i = 1, ..., d_\pi\}$.
In order to obtain, that the whole spaces π_i and π_j are orthogonal to each other we show that $K := \pi_i \cap \pi_j = \{0\}$ for $i \neq j$.
Because π_i and π_j are right-invariant, also $K = \pi_i \cap \pi_j$ is right-invariant. By irreducibility of π and the equivalence of π to the (right-regular) representation in π_i, π is either $K = \{\pi_i\}$ or $\{0\}$. If $K = \{\pi_i\}$, it follows that $\pi_i = \pi_j$ and hence $i = j$, which contradicts to $\pi_i \neq \pi_j$. Consequently, we have $K = \{0\}$.
So all $\pi_{xy}(g) \in \pi(\mathcal{H})$ can be uniquely decomposed into $\pi_{xy}(g) = \sum_{i=1}^{n}\alpha_i\pi_{xe_i}(g)$, where $y = \sum_{i=1}^{n}\alpha_i u_i$ and

$$\pi(\mathcal{H}) = \bigoplus_{i=1}^{n}\pi_i.$$

One obtains as well the decomposition of $\pi(\mathcal{H})$ into left-invariant subspaces $\pi_i^l := \pi_{u_i\mathcal{H}}$

$$\pi(\mathcal{H}) = \bigoplus_{i=1}^{n}\pi_i^l.$$

□

One of the most important theorems of harmonic analysis was proven by Hermann Weyl and his student Peter in [PW27]. The theorem can also be found in [Feg91, Tay86, Bum04, Fol95] and many others.

Theorem 2.1.14 (Peter-Weyl; 1927). *Let $\widehat{\mathcal{G}} := \{\pi_\alpha,\ \alpha \in I\}$ be the set of all equivalence classes of irreducible representations of the compact Lie group $\mathcal{G}$, then the following orthogonal decomposition of $L^2(\mathcal{G})$ into translation invariant subspaces hold true*

$$L^2(\mathcal{G}) = \bigoplus_{\pi_\alpha,\ \alpha\in I} \pi_\alpha(\mathcal{G}).$$

Because of the compactness of $\mathcal{G}$ the parameter set of irreducible representation I is discrete, just like the spectrum of the Laplace operator on $\mathcal{G}$. Furthermore, the translation invariant subspaces π_α are exactly the π_i from above, spanned by the corresponding matrix coefficients. We will discuss later the difficulties in the case of the Heisenberg group, which arise when $\mathcal{G}$ is not compact. (see Chapter 4.4.3)
There is a one-to-one correspondence between irreducible representations π and their characters $\chi_\pi(g) = \operatorname{trace}(\pi(g))$ (see Definition 2.1.8) and we have the following corollary.

Corollary 2.1.15. Let π_1, π_2 are two irreducible representations of $\mathcal{G}$ then it is

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle_{L^2(\mathcal{G})} = \begin{cases} 1, & \pi_1 \sim \pi_2 \\ 0, & else \end{cases}.$$

We know, that χ_{π_α} and χ_{π_β} are living in the translation invariant subspaces $\pi_\alpha(\mathcal{H})$ and $\pi_\beta(\mathcal{H})$ of $L^2(\mathcal{G})$. $\pi_\alpha(\mathcal{G})$ and $\pi_\beta(\mathcal{G})$ are orthogonal to each other. It is left to show that $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle_{L^2(\mathcal{G})} = 1$ for $\alpha = \beta$, but this follows from (2.1.5).

2.1.3 Fourier transform on compact Lie groups

We choose a unitary representation from the equivalence class of irreducible representations $[\pi_\alpha]$. Because the corresponding matrix of matrix coefficients $(\pi^\alpha_{ij})^n_{i,j=1}$ is unitary, it follows $\pi^\alpha_{ij}(g) = \overline{\pi^\alpha_{ji}(g)}$ and for all $i = 1, ..., d_\pi$

$$\sum_{j=1}^n |\pi^\alpha_{ij}(g)|^2 = 1. \tag{2.1.3}$$

Note that by compactness of $\mathcal{G}$, it is also unimodular. Integration over $\mathcal{G}$ yields

$$\sum_{j=1}^n \int_{\mathcal{G}} |\pi^\alpha_{ij}(g)|^2 \,\mathrm{d}g = 1,$$

where $\mathrm{d}g$ denotes the normalized Haar measure. Furthermore, $\int\limits_{\mathcal{G}} |\pi^\alpha_{ij}(g)|^2 \,\mathrm{d}g = \int\limits_{\mathcal{G}} |\pi^\alpha_{ji}(g)|^2 \,\mathrm{d}g$. By irreducibility of π_α there is a $g_0 \in \mathcal{G}$, so that $\pi_\alpha(g_0)e_j = u_k$ and hence:

$$\begin{aligned}\int_{\mathcal{G}} |\pi^\alpha_{ij}(g)|^2 \,\mathrm{d}g &= \int_{\mathcal{G}} |\langle \pi_\alpha(g_0^{-1}g)u_i, u_j\rangle|^2 \,\mathrm{d}g \\ &= \int_{\mathcal{G}} |\langle \pi_\alpha(g)u_i, \pi_\alpha(g_0)u_j\rangle|^2 \,\mathrm{d}g = \int_{\mathcal{G}} |\pi^\alpha_{ik}(g)|^2 \,\mathrm{d}g,\end{aligned}$$

where for integration we make use of the Haar measure. Consequently, for $1 \le i, j, l, m \le n$ we have

$$\int_{\mathcal{G}} |\pi^\alpha_{ij}(g)|^2 \,\mathrm{d}g = \int_{\mathcal{G}} |\pi^\alpha_{lm}(g)|^2 \,\mathrm{d}g \tag{2.1.4}$$

With (2.1.3) and (2.1.4) we can choose an orthogonal basis in Theorem2.1.13. By unitarity of π then follows that

$$\langle \pi^\alpha_{ij},\, \pi^\alpha_{kl}\rangle_{L^2(\mathcal{G})} = \delta_{ik}\delta_{jl}\frac{1}{n}, \tag{2.1.5}$$

where $n = d_{\pi_\alpha}$ is the dimension of the representation π_α and δ_{ij} denotes the Kronecker delta. Therewith we obtain the orthonormal system $\{\sqrt{d_{\pi_\alpha}}\pi^\alpha_{ij}, 1 \le i, j \le d_{\pi_\alpha}\}$.

Definition 2.1.16. Let $\mathcal{G}$ be a compact Lie group. With decomposition

$$L^2(\mathcal{G}) = \bigoplus_{\pi_\alpha,\, \alpha \in I} \pi_\alpha(\mathcal{G}),$$

then the expansion of $f \in L^2(\mathcal{G})$ with respect to the basis $\{\sqrt{d_{\pi_\alpha}}\pi^\alpha_{ij}\}$ which is given by

$$f(g) = \sum_{\alpha \in I} \sum_{i,j=1}^{d_{\pi_\alpha}} c^\alpha_{ij}\pi^\alpha_{ij}(g); \tag{2.1.6}$$

$$c^\alpha_{ij} = d_{\pi_\alpha} \int_{\mathcal{G}} f(g)\overline{\pi^\alpha_{ij}(g)} \,\mathrm{d}g \tag{2.1.7}$$

is the *Fourier transform* on $\mathcal{G}$.

This shows that the Fourier coefficients $(c^\alpha_{ij})_{i,j=1}^{d_{\pi_\alpha}}$ for functions on non-commutative, compact Lie groups are matrix-valued.
Coming from (2.1.6) one can represent the above Fourier expansion in terms of characters, which gives the spectral decomposition. Therefore

$$f_\alpha(g) := \sum_{i,j=1}^{d_{\pi_\alpha}} c^\alpha_{ij}\pi^\alpha_{ij}(g) \in \pi_\alpha(\mathcal{G}). \tag{2.1.8}$$

Therewith (2.1.6) assumes the form

$$f(g) = \sum_{\alpha \in I} f_\alpha(g), \tag{2.1.9}$$

where I parameterizes $\hat{\mathcal{G}}$, the set of all equivalence classes of irreducible representations.

Definition 2.1.17. The convolution product of $f, h \in \mathcal{G}$ is defined by

$$(f * h)(g) := \int_{\mathcal{G}} f(a)h(a^{-1}g) \, \mathrm{d}a. \tag{2.1.10}$$

The projection of f onto $\pi_\alpha(\mathcal{G})$ is denoted by $f_\alpha(g)$ and we will show, that $f_\alpha(g)$ can by given by

$$f_\alpha(g) = d_{\pi_\alpha}(f * \chi_{\pi_\alpha})(g). \tag{2.1.11}$$

Because π^α is unitary $\overline{\pi^\alpha_{ij}(h)} = \pi^\alpha_{ji}(h^{-1})$ holds true. To verify (2.1.11) we pluck (2.1.7) into (2.1.8). This results in

$$\begin{aligned} f_\alpha(g) &= \sum_{i,j=1}^{d_{\pi_\alpha}} d_{\pi_\alpha} \int_{\mathcal{G}} f(h)\pi^\alpha_{ji}(h^{-1})\pi^\alpha_{ij}(g) \, \mathrm{d}h = d_{\pi_\alpha} \operatorname{trace}\Big(\underbrace{\int_{\mathcal{G}} f(h)\pi_\alpha(h^{-1}) \, \mathrm{d}h}_{=:\widehat{f}(\pi_\alpha)} \pi_\alpha(g) \Big) \\ &= d_{\pi_\alpha} \int_{\mathcal{G}} f(h)\chi_{\pi_\alpha}(h^{-1}g) \, \mathrm{d}h = d_{\pi_\alpha}(f * \chi_{\pi_\alpha})(g). \end{aligned}$$

The Fourier coefficients of $f \in L^2(\mathcal{G})$ are given by the operator-valued integral

$$\widehat{f}(\pi_\alpha) = \int_{\mathcal{G}} f(h)\pi^*_\alpha(h) \, \mathrm{d}h.$$

Theorem 2.1.18 (Convolution theorem)**.** *It holds*

$$\widehat{\phi * \psi} = \widehat{\phi}\,\widehat{\psi}, \qquad \forall \phi, \psi \in L^2(\mathcal{G}).$$

Proof:

$$\begin{aligned} \int_{\mathcal{G}} \int_{\mathcal{G}} \phi(h)\psi(h^{-1}g) \, \mathrm{d}h \, \pi^*_\alpha(g) \, \mathrm{d}g &= \int_{\mathcal{G}} \int_{\mathcal{G}} \psi(g)\pi_\alpha(g^{-1}h^{-1}) \, \mathrm{d}g \, \phi(h) \, \mathrm{d}h \\ = \int_{\mathcal{G}} \psi(g)\pi_\alpha(g^{-1}) \, \mathrm{d}g \int_{\mathcal{G}} \phi(h)\pi_\alpha(h^{-1}) \, \mathrm{d}h &= \int_{\mathcal{G}} \psi(g)\pi^*_\alpha(g) \, \mathrm{d}g \int_{\mathcal{G}} \phi(h)\pi^*_\alpha(h) \, \mathrm{d}h = \widehat{\phi}\,\widehat{\psi}. \end{aligned}$$

□

We finish this section by introducing an involution, which will make use of later.

Definition 2.1.19. Let f be a function on $\mathcal{G}$, we define

$$\check{f}(g) := \overline{f(g^{-1})}. \tag{2.1.12}$$

With

$$\check{f}(g) = \sum_{\pi \in \widehat{\mathcal{G}}} d_\pi \operatorname{trace}(\overline{\widehat{f}(\pi)\pi^*(g)}) = \sum_{\pi \in \widehat{\mathcal{G}}} d_\pi \operatorname{trace}(\widehat{f}^*(\pi)\pi(g)),$$

where $\widehat{\mathcal{G}}$ denotes again the set of all equivalence classes of irreducible representations of $\mathcal{G}$, we have

$$\widehat{\check{f}}(\pi) = \widehat{f}^*(\pi). \tag{2.1.13}$$

Definition 2.1.20 (Hilbert Schmidt operator)**.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $\{u_i\}$ a basis of $\mathcal{H}_1$. A *Hilbert Schmidt operator* from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a continuous linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ with

$$\|A\|_{HS} := \sum_{i=1}^{\infty} \|Au_i\|_{\mathcal{H}_2} < \infty. \tag{2.1.14}$$

For Hilbert Schmidt operators from $\mathcal{H}_1$ onto itself one has

$$\|A\|_{HS} = \sum_i \|Ae_i\|_{\mathcal{H}_1} = \sum_i \langle A^* Ae_i, e_i\rangle_{\mathcal{H}_1} = \operatorname{trace}(A^* A). \tag{2.1.15}$$

$\|\cdot\|_{HS}$ denotes the *Hilbert Schmidt norm.*

Theorem 2.1.21 (Parseval identity)**.**

$$\|f\|^2_{L^2(\mathcal{G})} = \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \|\widehat{f}(\pi)\|^2_{HS} \qquad \forall f \in L^2(\mathcal{G}). \tag{2.1.16}$$

Proof: We expand f in a Fourier series and use the index notation for the trace. With $|f|^2 = \overline{f}f$ an easy calculation yields

$$\begin{aligned}\int_{\mathcal{G}} |f(g)|^2 \,\mathrm{d}g &= \sum_{\xi,\pi\in\widehat{\mathcal{G}}} d_\xi \mathrm{d}_\pi \int_{\mathcal{G}} \overline{\sum_{i,j=1}^{d_\pi} \widehat{f}_{ij}(\pi)\pi_{ji}(g)} \sum_{l,m=1}^{d_\xi} f_{ml}(\xi)\xi_{lm}(g)\,\mathrm{d}g \\ &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \sum_{i,j=1}^{d_\pi} \overline{f_{ij}(\pi)} f_{ij}(\pi) = \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \operatorname{trace}(f^*(\pi) f(\pi)),\end{aligned}$$

under consideration of (2.1.5). □

2.2 Quasi regular representations and functions on homogeneous spaces

A homogeneous space of a Lie group is a manifold X with a given (left) action[1] A of the group $\mathcal{G}$, $A : \mathcal{G} \times X \to X$ so that the action is transitive[2], i.e. $\forall x, y \in X, \exists g \in \mathcal{G} : g \cdot x = y$, where we use the notation $A(g, x) = g \cdot x$. The fundamental difference between a Lie group and a homogeneous space is, that there is a distinguished element in $\mathcal{G}$, namely the neutral element e but no distinguished point exists in the homogeneous space. This is given in a non-canonical way to X. So we choose an arbitrary point $x_0 \in X$. Let $\mathscr{H}$ be the stabilizer of the point x_0: $\mathscr{H} = \{h \in \mathcal{G} |\ h \cdot x_0 = x_0\}$. Clearly $\mathscr{H}$ is a subgroup since $e \in \mathscr{H}$ and by $(g_1 g_2) \cdot x_0 = g_1 \cdot (g_2 \cdot x_0)$ it is closed under group multiplication.
The stabilizer of another point $y \in X$ is like follows. By transitivity of the group action there is a $g \in \mathcal{G}$ with $g \cdot x_0 = y$. Hence the stabilizer of $y \in X$ is $g\mathscr{H}g^{-1}$. Here one sees in which way the construction is independent of the choice of the base point. The change of the base point on X corresponds to an conjugate action on $\mathcal{G}$.
Hence, every point $x \in X$ can be identified with a fiber of the form $g\mathscr{H} = \{gh |\ h \in \mathscr{H}\}$, the set of $g_y \in \mathcal{G}$ for which $g_y \cdot x_0 = y$.[3]
Of course we have to distinguish between left- and right-factorization, since $g\mathscr{H} = \{gh |\ h \in \mathscr{H}\} \neq \mathscr{H}g = \{hg |\ h \in \mathscr{H}\}$ and such that

$$\mathcal{G}/\mathscr{H} \neq \mathscr{H}\backslash\mathcal{G}.$$

2.2.1 Functions on homogeneous spaces

In this section we want to investigate properties of functions on homogeneous spaces. An important point will be to extend the definition of the Fourier transform to functions on homogeneous spaces. This will reveal how the restriction of functions to the homogeneous space looks like in Fourier domain.
We introduce the following isomorphism between function spaces on $\mathcal{G}$ and corresponding function spaces on $X \simeq \mathcal{G}/\mathscr{H}$.
Let f be a function on X with base point x_0 then it is clear that $f(g \cdot x_0)$ can be viewed as function on $\mathcal{G}$ with variable g. To make this precise we look at the canonical projection $P : \mathcal{G} \to \mathcal{G}/\mathscr{H}$ $(g \mapsto g\mathscr{H})$. The pullback applied to functions on X then gives a corresponding function on $\mathcal{G}$:

$$\tilde{f}(g) = f(P(g)) \tag{2.2.1}$$

Obviously $\tilde{f}(g)$ is constant over fibers of the form $g\mathscr{H}$. For functions on $\mathscr{H}\backslash\mathcal{G}$ a similar construction yields functions, which are constant over fibers of the form $\mathscr{H}g$.

[1]By definition a group action is associative and $e \cdot x = x$
[2]Such X are also refereed as $\mathcal{G}$-*space*.
[3]Defining a right action as group action on X the same construction leads to the homogeneous space $\mathscr{H}\backslash\mathcal{G}$.

In the other direction we introduce a push forward method to project functions from $\mathcal{G}$ to X:

$$\mathbb{P}f(x) = \int_{P^{-1}(x)} f(g)\, \mathrm{d}_{\mathscr{H}} g, \tag{2.2.2}$$

where $\mathrm{d}_{\mathscr{H}}$ denotes the normalized Haar measure on $\mathscr{H}$. The measure on X can be chosen so that

$$\int_{\mathcal{G}} f(g)\, \mathrm{d}g = \int_X \mathbb{P}f(y)\, \mathrm{d}_X y, \tag{2.2.3}$$

with a quasi-invariant[1] measure d_X, where the quasi-invariant measure in opposite to the invariant one is not unique. A comprehensive discussion about appropriate measures can be found in [Füh05]. In the present study, no difficulties arise since $\mathcal{G}$ is compact and

$$\mathbb{P}(\tilde{f}) = f. \tag{2.2.4}$$

In what follows we identify functions on X with those which are constant over the appropriate fibers $g\mathscr{H}$. This allow us to write (2.2.2) as

$$\mathbb{P}f(g) = \int_{\mathscr{H}} f(gh)\, \mathrm{d}_{\mathscr{H}} h, \tag{2.2.5}$$

Where now $x = [g]$ is the equivalence class of g with respect to the equivalence relation $g_1 \sim g_2 \Leftrightarrow \exists h \in \mathscr{H} :\ g_1 h = g_2$. In a similar way one has $x = g \cdot x_0$.

Definition 2.2.1. A function on $X \simeq \mathcal{G}/\mathscr{H}$ is called *zonal* if it is invariant under the action of the stabilizer of the base point of X, i.e.

$$f(x) = f(h \cdot x) \qquad \forall h \in \mathscr{H}.$$

Class one – and quasi regular representations

One important point in what follows will be to understand the Fourier transform of functions on homogeneous spaces of the group $\mathcal{G}$ and the corresponding symbol action of projection and lifting method on the Fourier domain.

Definition 2.2.2. Let $\mathcal{G}$ be a Lie group and $\mathscr{H}$ be a subgroup of $\mathcal{G}$, which fact we denote by $\mathscr{H} < \mathcal{G}$. A representation π of $\mathcal{G}$ is called to be of *class one with respect to $\mathscr{H}$* if the corresponding matrix coefficients are invariant under $\mathscr{H}$, i.e.

$$\pi_{ij}(g) = \pi_{ij}(gh) \qquad (\text{ or } \qquad \pi_{ij}(g) = \pi_{ij}(hg)) \qquad \forall h \in \mathscr{H}.$$

Later we will use matrix coefficients of class one representations to span the space of functions on the homogeneous space $\mathcal{G}/\mathscr{H}$(or $\mathscr{H}\backslash\mathcal{G}$).

[1] Sets of measure zero are preserved under translation.

Definition 2.2.3. Let $\mathscr{H} < \mathcal{G}$. If for any representation π (of $\mathcal{G}$ in $\mathcal{H}$) the set of $\mathscr{H}$ invariant vectors in $\mathcal{H}$ is at most of dimension one, $\mathscr{H}$ is called a *massive subgroup of* $\mathcal{G}$.

Applying the projection method to the regular representation we obtain a quasi regular representation in $L^2(X)$.

Definition 2.2.4. [1] Let $\mathcal{G}$ be a compact Lie group and $\mathscr{H} < \mathcal{G}$. The *quasi regular representation* of $\mathcal{G}$ is a representation in $L^2(\mathcal{G}/\mathscr{H})$ given by

$$\pi_{qreg}(g) : f(x) \mapsto f(g^{-1} \cdot x) \qquad f \in L^2(\mathcal{G}/\mathscr{H}). \tag{2.2.6}$$

As an example we want to look at the equivalence irreducible components of quasi regular representation to irreducible components of the regular representation.
Let again $X \sim \mathcal{G}/\mathscr{H}$ be a homogeneous space of a compact Lie group $\mathcal{G}$. Let π_k be the (left) regular representation of $\mathcal{G}$ in $L^2(X)$, restricted to an minimal invariant subspace (so that it is irreducible) of dimension d_k. The left-regular representation L_k of $\mathcal{G}$, restricted to span $\{\pi^k_{ij},\ 1 \leq i,j \leq d_k\}$ and π_k posses the same character and hence are equivalent by irreducibility.

$$L^k_{(i,j)(l,m)}(g) = \int_{\mathcal{G}} \pi^k_{ij}(gh)\overline{\pi^k_{lm}(h)}\,\mathrm{d}h = \sum_{p=1}^{d_k} \int_{\mathcal{G}} \pi^k_{ip}(g)\pi^k_{pj}(h)\overline{T_{lm}k(h)}\,\mathrm{d}h$$
$$= \frac{1}{d_k}\sum_{p=1}^{d_k} \delta_{pl}\delta_{jm}\pi^k_{ip} = \frac{1}{d_k}T^k_{il}(g)\delta_{jm}$$

And hence

$$\chi_{L_k} = \sum_{i,j=1}^{d_k} L^k_{(i,j)(i,j)}(g) = \sum_{i,j=1}^{d_k} \frac{1}{d_k}\pi^k_{ii}(g)\delta_{jj} = \chi_{\pi_k}$$

Hence every quasi regular representation is equivalent to a regular representation as we have asserted before. Here we have seen the concrete construction.
The converse is in general not true i.e. not every irreducible representation is equivalent to a quasi regular representation.
As we will see later in Chapter 4.3 in the case of $SO(3)$ we are in the comfortable situation that also the converse is true.

Fourier transform of functions on $X \simeq \mathcal{G}/\mathscr{H}$

Let f be a function on X. (2.2.4) is written as $f(g) = \int_{\mathscr{H}} f(gh)\,\mathrm{d}_{\mathscr{H}}h$, this uses the identification of functions which are constant over $g\mathscr{H}$ and functions on X, i.e. holds true if there is a function g defined on X with $\tilde{g} = f$.

[1] This definition can be adapted to homogeneous spaces $\mathscr{H}\backslash\mathcal{G}$ and the corresponding right action (here denoted in the same way) of $\mathcal{G}$ on $\mathscr{H}\backslash\mathcal{G}$. While that $\pi_{qreg}(f(x)) = f(g \cdot x)$

Remark 2.2.5.

$$\widehat{\mathbb{P}f}(\pi) = \pi_{\mathscr{H}} \widehat{f}(\pi), \qquad \text{with} \qquad \pi_{\mathscr{H}} = \int_{\mathscr{H}} \pi(h)\, \mathrm{d}_{\mathscr{H}} h. \tag{2.2.7}$$

Using the Fourier series expansion of f we find:

$$\begin{aligned} f(g) &= \int_{\mathscr{H}} \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \operatorname{trace}\left(\widehat{f}(\pi)\pi(gh)\right)\, \mathrm{d}_{\mathscr{H}} h \\ &= \int_{\mathscr{H}} \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \operatorname{trace}\left(\widehat{f}(\pi)\pi(gh)\right)\, \mathrm{d}_{\mathscr{H}} h \\ &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \operatorname{trace}\left(\pi_{\mathscr{H}}\widehat{f}(\pi)\pi(g)\right), \end{aligned} \tag{2.2.8}$$

where

$$\pi_{\mathscr{H}} = \int_{\mathscr{H}} \pi(h)\, \mathrm{d}_{\mathscr{H}} h, \tag{2.2.9}$$

and we remark, that we are taking the trace and hence can make a cyclic permutation of matrices.

Lemma 2.2.6. *$\pi_{\mathscr{H}}$ is a projection matrix onto the subspace of $\mathscr{H}$ invariant vectors in $\mathcal{H}$.*

Parts of the idea of the proof can also be found in [VK91]. Regarding the case of $\mathscr{H}\backslash\mathcal{G}$, equation (2.2.8) changes to $f(g) = \sum d_\pi \operatorname{trace}(\widehat{f}(\pi)\pi_{\mathscr{H}}\pi(g))$, with $\pi_{\mathscr{H}}$ as in (2.2.9).

Proof: We have to show two things.
At first $\pi_{\mathscr{H}}\pi_{\mathscr{H}} = \pi_{\mathscr{H}}$: This can be easily seen by

$$\pi_{\mathscr{H}}^2 = \left(\int_{\mathscr{H}} \pi(h)\, \mathrm{d}h\right)^2 = \int_{\mathscr{H}}\int_{\mathscr{H}} \pi(h_1h_2)\, \mathrm{d}h_1\, \mathrm{d}h_2 = \int_{\mathscr{H}}\int_{\mathscr{H}} \pi(h)dh\, \mathrm{d}h_2 = \int_{\mathscr{H}} \pi(h)\, \mathrm{d}h = \pi_{\mathscr{H}}.$$

This implies that $\pi_{\mathscr{H}}$ is the projection onto the space of Fourier coefficients in $\mathbb{C}^{d_\pi\times d_\pi}$ of functions which are invariant on fibers of the form $g\mathscr{H}$. In other words which are Fourier coefficients of functions on X.
The second point is to show $\pi_{\mathscr{H}} v = v \in \mathcal{H}$, if and only if $\pi(h)v = v\ \forall h \in \mathscr{H}$. Equivalently, $\pi_{\mathscr{H}}$ is the null projection if $\mathcal{H}$ contains no $\mathscr{H}$ invariant vectors.
Let $\mathcal{H}$ be the representation Hilbert space of π and $\mathcal{H}_{\mathscr{H}} := \{v \in \mathcal{H} |\, \pi(h)(v) = v \quad \forall h \in \mathscr{H}\}$.
If $\mathcal{H}_{\mathscr{H}} = \emptyset$, the restriction of π to $\mathscr{H}$ gives an irreducible representation of $\mathscr{H}$. Due to Peter-Weyl theorem the matrix coefficients of this representation are orthogonal to the character of the trivial representation of $\mathscr{H}$, which is the identity, hence

$$\int_{\mathscr{H}} \pi(h)\, \mathrm{d}h = \langle Id, \pi\rangle_{L^2(\mathscr{H})} = 0 \Rightarrow \widehat{f}(\pi) = 0 \qquad \forall \pi \text{ with } \mathcal{H}_{\mathscr{H}} = \emptyset. \tag{2.2.10}$$

□

We sort the basis $\{u_i,\, i = 1, ..., d_\pi\}$ of $\mathcal{H}$ in that way, that $\{u_i, i = 1, ..., k\}$ spans $\mathcal{H}_{\mathscr{H}}$ (the k-dimensional subspace of $\mathscr{H}$ invariant vectors). Consequently,

$$\pi_{\mathscr{H}} = \begin{pmatrix} I_k & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{2.2.11}$$

where I_k denotes the k-dimensional identity matrix. $\mathbf{O}$ are zero-matrices of appropriate dimension.
Therefrom we see:

Corollary 2.2.7. Fourier coefficients of functions on X are of the form

$$\widehat{f}(\pi) = \begin{pmatrix} A \\ \mathbf{O} \end{pmatrix}, \tag{2.2.12}$$

where A is a matrix of dimension $k \times d_\pi$.

This is equivalent to say that functions on $\mathcal{G}$ which are invariant on fibers $g\mathcal{H}$, can be expanded in a series of matrix coefficients π_{ij} with $j \leq k$.

Remark 2.2.8. For the case of functions on $\mathscr{H}\backslash\mathcal{G}$ the assertion of Corollary 2.2.7 assumes the form

$$\widehat{f}(\pi) = \begin{pmatrix} A & \mathbf{O} \end{pmatrix}, \tag{2.2.13}$$

where A is a matrix of dimension $d_\pi \times k$.

The property of a function f to be zonal can be also be expressed in the special form of its Fourier coefficients. For a zonal function f on X the function $\tilde{f}$ is invariant under right- and left-shifts with $h \in \mathscr{H}$, i.e. $\tilde{f}(g) = \tilde{f}(hg) = \tilde{f}(gh)\ \forall h \in \mathscr{H}$. Hence $\tilde{f}$ is a function on $\mathcal{G}/\mathscr{H}$ as well as it is a function on $\mathscr{H}\backslash\mathcal{G}$. Corollary 2.2.7 and Remark 2.2.8 implies the Fourier coefficients of zonal functions are of the following form:

$$\widehat{f}(\pi) = \begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{2.2.14}$$

with $A \in \mathbb{C}^{k\times k}$ (k is again the number of $\mathscr{H}$ invariant vectors in the representation Hilbert space $\mathcal{H}$ of π) and $\mathbf{O}$ of appropriate dimension.

Also for class functions we want to deduce the special shape of their Fourier coefficients. Writing down the class function property $f(a) = f(g^{-1}ag)$ for the Fourier series expansion of such a function one finds

$$\sum_\pi d_\pi \operatorname{trace}(\widehat{f}(\pi)\pi(a)) = \sum_\pi d_\pi \operatorname{trace}(\widehat{f}(\pi)\pi(g^{-1}ag)). \tag{2.2.15}$$

By Peter-Weyl theorem this is equivalent to

$$\operatorname{trace}\left(\left(\widehat{f}(\pi) - \pi(g)\widehat{f}(\pi)\pi(g^{-1})\right)\pi(a)\right) = 0. \tag{2.2.16}$$

$\mathbb{C}^{d_\pi \times d_\pi}$ shall denote the space of matrices of dimension $d_\pi \times d_\pi$. In a first step we verify

$$\text{span}\{\pi(g), g \in \mathcal{G}\} = \mathbb{C}^{d_\pi \times d_\pi}. \tag{2.2.17}$$

$\mathbb{C}^{d_\pi \times d_\pi}$ becomes a Hilbert space equipped with the Hilbert-Schmidt inner product

$$\langle A, B \rangle_{\mathbb{C}^{d_\pi \times d_\pi}} = \text{trace}(B^* A).$$

For every $A \in \mathbb{C}^{d_\pi \times d_\pi}$ there is a function having A as a Fourier coefficient of it. At least there is $\phi(g) = d_\pi \, \text{trace}(A\pi(g))$, where only A appears in its Fourier series.
By Peter-Weyl Theorem, character functions (matrix coefficients) are linearly independent. Hence from (2.2.16) one deduces, that $(\widehat{f}(\pi) - \pi(g^{-1})\widehat{f}(\pi)\pi(g))$ is the orthogonal complement of $\text{span}\{\pi(g), g \in \mathcal{G}\}$ with respect to the Hilbert-Schmidt inner product $\text{trace}(B^* A)$ in $\mathbb{C}^{d_\pi \times d_\pi}$, so that

$$\widehat{f}(\pi) - \pi(g^{-1})\widehat{f}(\pi)\pi(g) = 0 \qquad \Leftrightarrow \qquad f(\pi)\pi(g) = \pi(g)f(\pi).$$

Matrices, which are commuting with every matrix in $\mathbb{C}^{d_\pi \times d_\pi}$ are a multiples of the identity matrix. This results in the following Corollary.

Corollary 2.2.9. Fourier coefficients of class functions are multiplies of the identity. Hence every class function posseses an expansion in character functions.

Remark 2.2.10. The projection of a class function to X results in a zonal function.

2.3 General remarks on wavelets

For construction of wavelets on a manifold M the general idea is to form a frame in $L^2(M)$ by dilating and translating a *mother wavelet* Ψ. Starting with a mother wavelet $\Psi \in L^2(M)$ one has to introduce a dilation and translation operator D_ρ and T_x, where the set of dilations is parameterized by $\rho \in R$ and that of translations by $x \in L$. In general one needs not to restrict to dilations and translations but can also call in further operators. For example on $\mathbb{R}^n$ beside the canonical dilation and translation operator in $L^2(\mathbb{R}^n)$ it is possible to add the rotation operator.[1] On the sphere dilations and translations are part of the Möbius group [Cno94]; all possible dilations and translations are worked out in [Fer09, Fer08], this involves comprehensive discussions of geometrical aspects from [Cno93, Kna02, Por81] and others. The mother wavelet has to be chosen in a way, that

$$\{T_x D_\rho \Psi,\ (x,\rho) \in L \times R\}$$

forms a frame in $L^2(M)$. This requirement or equivalent conditions leads to admissibility conditions for $\Psi \in L^2(M)$ to be a mother wavelet.

2.3.1 Group theoretical formulation of wavelet theory

From the group theoretical point of view dilation and translation operators are provided by an irreducible representation π of a group $\mathcal{G}$ in the Hilbert space $L^2(M)$.[1] The general formulation for Banach spaces, where the crucial notions can be formulated with a convenient measure of generality, can be found in [Kis99b, Füh05].

The condition, that $0 \not\equiv \Psi \in L^2(M)$ is admissible if $\{\pi(g)\Psi,\ g \in \mathcal{G}\}$ forms a frame in $L^2(M)$ is equivalent to say, there are constants $c_1, c_2 > 0$, so that

$$c_1 \|f\|_{L^2(M)} \leq \int_{\mathcal{G}} \left|\langle \pi(g)\Psi, f\rangle_{L^2(M)}\right|^2 \, \mathrm{d}g \leq c_2 \|f\|_{L^2(M)} \qquad \forall f \in L^2(M). \tag{2.3.1}$$

Definition 2.3.1. Let $\Psi \in L^2(M)$ be admissible, than the wavelet transform $WT : L^2(M) \to L^2(\mathcal{G})$ is defined by

$$WT : f \mapsto \langle \pi(g)\Psi, f\rangle_{L^2(M)}.$$

In (2.3.1) we have two conditions. The estimate to below and hence the invertibility of WT is ensured by irreducibility of π. This can be seen by a contraposition. We assume a $0 \not\equiv f \in L^2(M)$ with $WT(f) = \langle \pi(g)\Psi, f\rangle \equiv 0$. This is $f \perp \operatorname{span}\{\pi(g)\Psi,\ g \in \mathcal{G}\} = L^2(M)$

[1] Here some difficulties arise, since there is no square-integrable non zero $\Psi \in L^2(\mathbb{R}^2)$. Therefore the parameter set of rotations can not be independent of that of dilations and translations. One has to choose an admissible section in the sense of (2.3.2).

[1] Here and in the rest of this section $L^2(M)$ can be replaced by any other Hilbert space $\mathcal{H}$. In consequence one defines the wavelet transform in $\mathcal{H}$ corresponding to π.

by irreducibility of π and $\Psi \not\equiv 0$. But this implies $f \equiv 0$, which is a contradiction. So $\mathrm{Ker}(WT) = \{0\}$.
The upper estimate gives a proper admissibility condition of Ψ to be square-integrable

$$\int_{\mathcal{G}} \|\pi(g)\Psi\|^2_{L^2(M)} \, \mathrm{d}g < \infty$$

and guaranties that WT is a bounded operator from $L^2(M)$ into $L^2(\mathcal{G})$.
Since $L^2(M)$ is infinite dimensional there is no compact $\mathcal{G}$ for which an irreducible representation π exists so that a wavelet transform is provided in the way we have sketched above. But the crucial tools of harmonic analysis which we have introduced in the first chapter, such as the Peter-Weyl theorem requires the condition of a compact Lie group $\mathcal{G}$.
In most cases there is no irreducible representation which is also square-integrable. This case appears for example discussing the sphere [Fer09, ADJV02, AV07, AV99, BE10]. There the sphere is observed as homogeneous space of the Lorentz group $SO(1, n+1)/SO(1, n) \simeq S^n$ so that there is a canonical action of $SO(1, n+1)$ on S^n. Nevertheless all irreducible representations of $SO(1, n+1)$ in $L^2(S^n)$ are not square-integrable.
The concept can be weaken in the following way. Let π be a irreducible representation of $\mathcal{G}$ in $L^2(M)$. Instead of asking for the square- integrability of the whole group one restricts to a homogeneous space $X \simeq \mathcal{G}/\mathscr{H}$ of $\mathcal{G}$. Let $\sigma : \mathcal{G}/\mathscr{H} \to \mathcal{G}$ be a section, satisfying

$$c_1 \|f\|_{L^2(M)} \leq \int_X |\langle \pi(\sigma(x))\psi, \, f \rangle_{L^2(M)}|^2 \leq c_2 \|f\|_{L^2(M)}, \tag{2.3.2}$$

then σ is called an *admissible section*. That means, the set of dilations and translation is parameterized now by X. The set of dilated and translated wavelets $\{\sigma(x)\psi, \, x \in X\}$ forms a frame in $L^2(M)$. Different admissible section leads to different looking dilations[1].
An even more general formulation of admissibility condition is assumed by Dahlke, Steidel and Teschke in [DST07], considering that the transformation, which in our case gives the identity, gives a bounded, invertible operator A_σ, namely

$$A_\sigma f = \int_X \langle f, U(\sigma(x)\Psi) \rangle U(\sigma(x))\Psi \, \mathrm{d}x.$$

2.3.2 The idea of diffusive wavelets

To motivate the subjects of the following chapter we introduce here the general idea of diffusive wavelets.
With the concept of diffusive wavelets we are able to use the powerful tools of harmonic analysis to construct wavelets on compact[2] Lie groups and homogeneous spaces.

[1] Also different looking translations are possible, but here one chose usually some natural action of the Group.
[2] We will sketch how one can overcome the critical points of non compactness and apply the method the Heisenberg group.

In the concept of diffusive wavelets dilation and the translation operator are separated from each other. Translation will be given as left shift operator, hence as left-regular representation. Of course the left-regular representation is not irreducible in $L^2(\mathcal{G})$ but will decompose into irreducible components, where each of it can be viewed as a scale space. In order to find an admissible mother wavelet one has to add a dilation operator which changes between different scale spaces. In the concept of diffusive wavelets this is achieved by an evolution process comparable to the heat evolution of the heat kernel. Hence dilations will be parameterized by $\mathbb{R}_+$ and translations by the compact group $\mathcal{G}$.
The reconstruction property, which we need if we want to invert the wavelet transform comes from the action of a certain semigroup, defined by an evolution process. The following two definitions are usual and can be found for instance in [AR05] and elsewhere. Later we will adjust the motions to our special purposes which will give an almost similar notion.

Definition 2.3.2. Let $\{D_\rho,\, \rho > 0\}$ be a continuous family of operators on $L^2(\mathcal{G})$. This family is called an *admissible semigroup* if the following conditions are satisfied:

- D_ρ is a bounded operator, independent of ρ
- $\lim\limits_{\rho \to 0} D_\rho = Id$, s.t. D_ρ approximates the identity operator
- D_ρ is positive for all ρ
- $D_{\rho_1} D_{\rho_2} = D_{\rho_1+\rho_2}$, such that $\{D_\rho, \rho > 0\}$ forms a semigroup.

For understanding the construction as usual dilation one would need only the first and the second condition. For a convenient formulation one requires the positivity and the semigroup property. This is not a big restriction of generality and most of the imaginable and all of the appearing examples here satisfy these conditions.
Many important examples of approximate identities come from a diffusion process. As solution of the corresponding partial differential equation those process is often given by convolution with the fundamental solution.

Definition 2.3.3. If $\{D_\rho,\, \rho > 0\}$ is an admissible semigroup and D_ρ can be written as convolution operator, i.e. there is a family of kernels $\{K_\rho,\, \rho > 0\} \subset L^1(\mathcal{G})$ so that $D_\rho(f) = f * K_\rho$, $\{D_\rho,\, \rho > 0\}$ is called an *approximate identity with kernel K_ρ.*

Remark 2.3.4. From $K_\rho \in L^1(\mathcal{G})$ it follows that the corresponding convolution operator $K_\rho * : f \mapsto K_\rho * f$ is bounded from $L^p(\mathcal{G})$ to $L^p(\mathcal{G})$.

The aim is now to find families of convolution kernels $\{\psi_\rho,\, \rho > 0\}$ and $\{\Psi_\rho,\, \rho > 0\}$, so that

$$K_R = \int_R^\infty \check{\psi}_\rho * \Psi_\rho \, \alpha(\rho) \, \mathrm{d}\rho, \tag{2.3.3}$$

forms a family of kernels of an approximate identity. We use again of the notation $\check{\psi}(g) = \overline{\psi(g^{-1})}$. Both families, $\{\psi_\rho\}$ and $\{\Psi_\rho\}$ shall be in $L^1(\mathcal{G})$, so that the convolution is a mapping $L^p \to L^p$. For a function f we can then define the transformation

$$WT : f \mapsto (f * \check{\psi}_\rho)(g) = \int_{\mathcal{G}} f(h)\check{\psi}(h^{-1}g)\,\mathrm{d}h = \langle f, T_g\psi_\rho\rangle_{L^2(\mathcal{G})}.$$

Hereby T_g is the translation operator and the dilations are parameterized by $\rho \in \mathbb{R}_+$. By Assumption (2.3.3) this transform can be inverted via

$$\begin{aligned} f &= \lim_{R\to 0} \int_R^\infty WT(f)(\rho,\cdot) * \Psi_\rho\,\alpha(\rho)\,\mathrm{d}g \\ &= \lim_{R\to 0} f * \int_R^\infty \check{\psi}_\rho * \Psi_\rho\,\alpha(\rho)\,\mathrm{d}\rho. \end{aligned}$$

The dilation operator in that approach is given as choice of the parameter ρ of ψ_ρ

$$D_{\rho_2}\psi_{\rho_1} \mapsto \psi_{\rho_1+\rho_2}$$

This approach works for arbitrary approximate convolution identities K_ρ and we will see, that also classical wavelets can be described in that way.
In particular we are interest in those approximate identities for which the operator $*\partial_\rho K_\rho$: $f \mapsto f * \partial_\rho K_\rho$ is positive. Then the corresponding Fourier coefficients of the kernel functions $\partial_\rho K_\rho$ are positive matrices and the choice $\psi_\rho = \Psi_\rho$ seems reasonable. We will later implement this general philosophy in the particular situation where K_ρ is the heat kernel and where both families coincide.
Four our purpose we translate Definition 2.3.3 into the Fourier domain.

Corollary 2.3.5. Let $\widehat{\mathcal{G}}_+ \subset \widehat{\mathcal{G}}$ be co-finite. If $\{K_\rho,\, \rho > 0\}$ is the kernel of an approximate identity if and only if it is a subfamily of $L^1(\mathcal{G})$ which satisfies

- $\|\widehat{K}_\rho(\pi)\|_{HS} \leq C$ independent of $\pi \in \widehat{\mathcal{G}}$ and $t \in \mathbb{R}_+$
- $\lim_{\rho\to 0} \widehat{K}_\rho = Id$ for all $\pi \in \widehat{\mathcal{G}}$
- $-\partial_\rho \widehat{K}_\rho$ is a positive matrix for all $\pi \in \widehat{\mathcal{G}}_+$ and $t \in \mathbb{R}_+$
- $\widehat{K}_{\rho_1}\widehat{K}_{\rho_2} = \widehat{K}_{\rho_1+\rho_2}$.

We would like to remark, that our point of view on the construction of wavelets is not contrary to the classical wavelet theory. That means, that classical wavelets can also be obtained from our construction, i.e. the dilation of a usual wavelet construction can always be observed as coming from the action of an operator family in the sense of Definition 2.3.3 as it will be the case for diffusive wavelets. For diffusive wavelets the family of operators will be given as a diffusion process.

As a representative example for showing, that the concept is valid for classical wavelets, we chose the Mexican hat wavelet on $\mathbb{R}$, which is given by

$$\psi(x) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{-x^2/2} = (1 - x^2) e^{-x^2/2}. \qquad (2.3.4)$$

In Fourier domain these wavelets are of the following form

$$\hat{\psi}(x) := \frac{1}{\sqrt{2\pi}} \omega^2 e^{-\omega^2/2}.$$

For a description of the theory behind the Mexican hat wavelet and other classical wavelets like Haar and Daubchies wavelets we recommend [Dau92, LMR94, Grö01]. As expected, no difficulties rise from the translation operator. This is given as left-regular representation of $(\mathbb{R}, +)$ on $L^2(\mathbb{R})$. The dilation operator in $L^2(\mathbb{R})$ is given by the following action of the affine-linear group in $L^2(\mathbb{R})$

$$D_\rho : \; \psi(x) \mapsto \frac{1}{\sqrt{\rho}} \psi\left(\frac{x}{\rho}\right). \qquad (2.3.5)$$

The Haar measure of the affine-linear group, also called $ax + b$-group. This group is the set $\mathbb{R}_+ \times \mathbb{R}$ with the multiplication law $(a, b)(c, d) = (ac, ad + b)$ for $(a, b), (c, b) \in \mathbb{R}_+ \times \mathbb{R}$ and the Haar-measure $\frac{\mathrm{d}a}{|a|^2}\,\mathrm{d}b$. The dilation D_ρ in (2.3.5) comes from the representation of the sub-group $(\rho, 0)$ of the $(ax + b)$-group in $L^2(\mathbb{R})$.

To find the corresponding approximate identity, so that the dilation in the case of the Mexican hat wavelet can be given as a dilation from the diffusive wavelet approach we verify that the kernel of the convolution approximate identity

$$K_t(x) = \int_t^\infty (\check{\psi}_\rho * \psi\rho)(x) \frac{\mathrm{d}\rho}{|\rho|^2}$$

satisfies conditions of Corollary 2.3.5, such that it is an approximate identity in the sense of Definition 2.3.3.

Therefore we note, that the dilation operator D_ρ for classical wavelets which is given in (2.3.5) on Fourier domain corresponds to the dilation operator $D_{a^{-1}}$

$$\widehat{D_a f} = D_{a^{-1}} \hat{f}.$$

Since the Mexican hat wavelets are real and even functions we have $\check{\psi}_\rho = \psi_\rho$ and hence $K_t(x) = \int_t^\infty (\psi_\rho * \psi_\rho)(x) \frac{\mathrm{d}\rho}{|\rho|^2}$. Consequently, for the Mexican hat wavelets the corresponding approximate identity K_t has Fourier coefficients of the following form:

$$\begin{aligned}
\hat{K}_t(\omega) &= \int_t^\infty D_{\rho^{-1}} \hat{\psi}_\rho^2(\omega) \frac{\mathrm{d}\rho}{|\rho|^2} \\
&= \int_t^\infty \rho(\rho\omega)^4 e^{-|\omega\rho|^2} \frac{\mathrm{d}\rho}{|\rho|^2} \\
&= \left[-\frac{1}{2} e^{-\omega^2\rho^2} (\omega^2\rho^2 + 1) \right]_t^\infty \\
&= \frac{1}{2} e^{-\omega^2 t^2} (\omega^2 t^2 + 1).
\end{aligned}$$

Since $\mathbb{R}$ is non-compact and hence the set $\hat{\mathbb{R}} = \mathbb{R}_+$ of irreducible representations is continuous as well as the spectrum of the Laplacian. Nevertheless one sees immediately, that K_t is the kernel of an approximate identity.
For the construction of diffusive wavelets we shall use the notion of

Definition 2.3.6 (diffusive approximate identity)**.** Let $\widehat{\mathcal{G}}_+ \subset \widehat{\mathcal{G}}$ be co-finite. A continuous differentiable[1] family of functions $\{p_t,\, t > 0\} \subset L^1(\mathcal{G})$ forms a *diffusive approximate identity* if

$$\|\widehat{p}_t(\pi)\|_{HS} < C \text{ independent of } \pi \in \widehat{\mathcal{G}} \text{ and } t \in \mathbb{R}_+ \tag{2.3.6}$$

$$\lim_{t\to 0} \widehat{p}_t = Id \text{ for all } \pi \in \widehat{\mathcal{G}} \tag{2.3.7}$$

$$\lim_{t\to \infty} \widehat{p}_t = 0 \text{ for all } \pi \in \widehat{\mathcal{G}}_+ \tag{2.3.8}$$

$$\tag{2.3.9}$$

To $\widehat{\mathcal{G}}_+$ we associate the subspace of $L^2(\mathcal{G})$, which is spanned by the matrix coefficients of the corresponding representations. Later we will make use of the notation

$$L^2_0(\mathcal{G}) = \bigoplus_{\pi_\alpha \in \widehat{\mathcal{G}}} \pi_\alpha(\mathcal{G}) \tag{2.3.10}$$

For an approximate identity as well as for a diffusive approximate identity holds

$$\lim_{t\to 0} p_t * f \to f, \qquad f \in L^p(\mathcal{G}),$$

where the convergence is in the L^p-sense. This follows from condition (2.3.6) and (2.3.7). Since by (2.3.6) it is $\|p_t * f\|_{L^p} \le \|f\|_{L^p}$, hence one can investigate $p_t * f$ in Fourier domain and by convolution theorem (Theorem 2.1.18) $\widehat{p_t * f}(\pi) = \widehat{f}(\pi)\widehat{p}_t(\pi) \to \widehat{f}(\pi)$ as $t \to 0$.
From (2.3.6), (2.3.7), (2.3.8) and the fact that $p_t \in C^1(\mathbb{R}_+, L^1(\mathcal{G}))$ on deduces that

$$p_t|_{\widehat{\mathcal{G}}_+} = -\int_t^\infty \partial_t p_t \,\mathrm{d}t.$$

The most important example of an diffusive approximate identity is the heat kernel. It satisfies in addition the semigroup property

$$p_{t_1} * p_{t_2} = p_{t_1+t_2}.$$

Another important example, especially for the case of the sphere are diffusive wavelets corresponding to the Abel-Poisson kernel [Ebe08, FGS98]. The Abel-Poisson kernel arises as the integral kernel to solve the Dirichlet problem of the Laplace equation $\Delta u = 0$ on the unit ball. One can also ask for this construction for arbitrary manifolds, that are surfaces of higher-dimensional manifolds. The convolution operator with the corresponding Abel-Poisson kernel

[1] i.e. the mapping $t \mapsto p_l$ is $C^1(\mathbb{R}_+, L^1(\mathcal{G}))$.

will always give a diffusive approximate identity, where the dilation parameter can be given as the distance to the boundary. This can be seen in the example of the sphere. We have to observe the radius variable as evolution (dilation) parameter. Since the radius is a quantity $0 < r < 1$, the substitution $t = -\ln(r)$ gives the right diffusive evolution parameter for our definition, where the parameter varies over $\mathbb{R}_+$.
The difference of the approximate identity coming from the Abel-Poisson kernel and the approximate identity coming from the heat kernel are the eigenvalues of the corresponding kernel with respect to the Laplacian. We will list both examples for the case of the sphere.

2.3.3 Universal enveloping algebra

In our construction of diffusive wavelets we have to investigate the fundamental solution of the heat equation which is closely related to the Laplace-Beltrami operator of the corresponding manifold.
In order to understand the Laplace operator[1] we have to investigate its geometrical rise. In general, first order differential operators can be identified with tangential fields. Especially the left invariant fields forming the Lie algebra are interesting and give left invariant Differential operators. Since the Laplace operator is not only left but also right invariant and of order two it is not enough to look at the Lie algebra and its representation regarding it as differential operators. We introduce the universal enveloping algebra in order to represent also higher order differential operators with the help of appropriate representations. The invariance will come from the property of the Casimir element to be in the center of the universal enveloping algebra. The Laplace operator appears as the result of the appropriate representation of the Casimir element.
The constructions we give here rises from a collection of contributions from [Bum04, Str91, Feg91, VK93].
The notion of representations can be transferred to algebras. So the representation ζ of a Lie Algebra $\mathfrak{g}$ is a Lie Algebra homomorphism into the the Lie Algebra of Linear operators on a Hilbert space $\mathcal{H}$.

$$\zeta : \mathfrak{g} \to End(\mathcal{H})^1 \qquad (2.3.11)$$

$$\zeta([h_1, h_2]) = \zeta(h_1)\zeta(h_2) - \zeta(h_2)\zeta(h_1) = [\zeta(h_1), \zeta(h_2)]. \qquad (2.3.12)$$

Remark 2.3.7. There is a one to one correspondence between representations of simply connected Lie groups and Lie algebras. The differential of the representation of a Lie group gives a representation of its Lie algebra.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\zeta} & End(\mathcal{H}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ \mathcal{G} & \xrightarrow[\pi]{} & GL(\mathcal{H}) \end{array}$$

With $d\pi = \zeta$.

To investigate general properties of mathematical objects one often uses some isomorphic object and investigate it instead of the original one. In this way one can translate questions into different languages like from analysis to algebra or representation theory. But of course one has to care that the properties of interest are invariant under the mapping. Some properties

[1] Since we are on a manifold rather than in $\mathbb{R}^n$ one often says Laplace-Beltrami operator instead of Laplace operator.

are even not invariant under an isomorphism. A stronger connection exists between the so called universal enveloping algebras and a corresponding object. The crucial property is the universal property of it.

Definition 2.3.8. Beside the usual notion of a Lie algebra homomorphism between Lie algebras, we define for a associative, unital[1] algebra A a *Lie algebra homomorphism* h_A between a Lie algebra $\mathfrak{g}$ and A as a linear mapping $h_A : \mathfrak{g} \to A$ with $h_A([X,Y]) = h_A(X)h_A(Y) - h_A(Y)h_A(X)$.

Every unital algebra A becomes a Lie algebra $Lie(A)$ equipping A with the Lie bracket $[a,b] = ab - ba$.
The map $j : A \to Lie(A)$ from the set of associative, unital algebras to the set of Lie algebras is not surjective, i.e. not for every Lie algebra $\mathfrak{g}$ there is an associative, unital algebra A in with $Lie(A) = \mathfrak{g}$. But one can always find a algebra A so that $\mathfrak{g}$ is embedded in $Lie(A)$. In this way A arises as the universal enveloping algebra of $\mathfrak{g}$.

Definition 2.3.9 (Universal enveloping algebra)**.** Let $\mathfrak{g}$ be a Lie algebra. The *universal enveloping algebra of* $\mathfrak{g}$ is the associative, unital algebra $U_\mathfrak{g}$ which posses the *universal property.* The universal property is defined as follows:
Let $h_{U_\mathfrak{g}}$ be a Lie algebra homomorphism $h_{U_\mathfrak{g}} : \mathfrak{g} \to U_\mathfrak{g}$, so that for any unital algebra A with Lie algebra homomorphism

$$h_A : \mathfrak{g} \to A$$

exists a algebra homomorphism

$$h : U_\mathfrak{g} \to A,$$

with

$$h_A = h\, h_{U_\mathfrak{g}}.$$

The uniqueness of the universal enveloping algebra holds in the sense of equivalence class with respect to the equivalence relation of algebras being homomorphic to each other. A construction of the universal enveloping of the Lie algebra $\mathfrak{g}$ uses the tensor algebra of $\mathfrak{g}$. The construction is rather formal. Let $\otimes\mathfrak{g}$ be the tensor algebra of $\mathfrak{g}$, i.e. $\otimes\mathfrak{g} = \bigoplus_{k=0}^{\infty} \otimes^k \mathfrak{g}$, where $\otimes^k\mathfrak{g}$ denotes the module of tensors of order k over the field $\mathbb{C}$ or $\mathbb{R}$ respectively.
Now let I be the ideal in $\otimes\mathfrak{g}$ which is generated by elements of the form $[X,Y]+X\otimes Y-Y\otimes X$. Constructing the Quotient $\otimes\mathfrak{g}/I$ identifies elements a, b in $\otimes\mathfrak{g}$ for which there is an $i \in I$, so that $i \cdot a = b$, where $\cdot$ denotes the multiplication in $\otimes\mathfrak{g}$.
Every (Lie) algebra $\mathfrak{g}$ is naturally embedded in $\otimes\mathfrak{g}$ via the subspace $\otimes^1 V$. This embedding shall be denoted by $j : \mathfrak{g} \to \otimes\mathfrak{g}$.

[1] unital means there is a unit element in A

Now one has to verify, that the universal property is satisfied by $\otimes\mathfrak{g}/I$. Let $\phi : \mathfrak{g} \to Lie(A)$ be a Lie algebra homomorphism.
Let now h_A be any Lie algebra homomorphism $\mathfrak{g} \to A$, then h_A can be extended to $\otimes\mathfrak{g}$ by setting $\otimes h_A(X_1 \otimes ... \otimes X_k) = h_A(X_1)...h_A(X_k)$. It is left to show that the kernel of $\otimes h_A$ is I. This is strait forward

$$\otimes h_A([X,y] - X \otimes Y + Y \otimes X) = h_A([X,Y]) - h_A(X)h_A(Y) + h_A(Y)h_A(X) = 0.$$

From the fundamental *Poincaré-Birkhoff-Witt theorem*, which can be found in [Bir37] it follows, that $j : \mathfrak{g} \to U_{\mathfrak{g}}$ is injective.

2.3.4 Killing form and adjoint representation

A Lie group can always be regarded as Riemannian manifold. This is done by equipping the tangential space with the naturally given Killing Form.
For a comprehensive understanding of the Laplace operator on Lie groups we want to discuss the geometrical rise of it. Therefore one uses the natural, geometrical induced notion of the killing form. We have to have a look at the adjoint representation of Lie groups and its Lie algebra.
The conjugate mapping ${}_g(h) = ghg^{-1}$ induces an action of $\mathcal{G}$ on itself. The differential of ${}_g$ at the neutral element e gives an invertible, linear mapping in $\mathfrak{g}$.

$$\mathrm{d}_g \in GL(\mathfrak{g}). \tag{2.3.13}$$

Definition 2.3.10. The *adjoint representation of a Lie Group G* is defined by

$$Ad : \mathcal{G} \to GL(\mathfrak{g}) \tag{2.3.14}$$
$$g \mapsto \mathrm{d}_g \tag{2.3.15}$$

Corresponding to Remark 2.3.7 the differential of Ad at e will give a representation ad of the Lie algebra of $\mathcal{G}$[1]- the *adjoint representation of* $\mathfrak{g}$:

$$ad = \mathrm{d}(Ad)_e. \tag{2.3.16}$$

Using the notion of integral curves one finds the comfortable relation: $ad(x) = [x, \cdot]$. Let $V, W \in \mathfrak{g}$, left invariant vector fields on $\mathcal{G}$ and let ϕ_t^V be the integral curve passing through e for $t = 0$. Than we can write

$$(ad(V))(W) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{d}\phi_{-t}^V)_{\phi_t^V(e)} W(\phi_t^V(e)) \right|_{t=0} = [V, W],$$

that gives $End(\mathfrak{g}) \ni ad(V) = [V, \cdot]$. A proof can bee found in [Bum04].

[1]Here we understand the Lie algebra as the tangential space of $\mathcal{G}$ at e.

Therewith one can obtain easily the Lie homomorphism property of *ad*. By the Jacobian identity $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ and antisymmetry of the Lie brackets it is

$$ad([X,Y]) = [[X,Y],\cdot] = [X,[Y,\cdot]] - [Y,[X,\cdot]] = ad(X)ad(Y) - ad(Y)ad(X). \qquad (2.3.17)$$

Since the Laplace operator can be defined for Riemannian manifolds we will demonstrate how a Riemannian structure arises naturally on a Lie group. The Lie algebra of a Lie group can be equipped in a natural way with a bilinear form, from which then we can deduce the corresponding Laplace operator.

Definition 2.3.11 (Killing form)**.** The *killing form* $B(,) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is a symmetric bilinear form:

$$B(X,Y) := \text{ trace}(ad(X)ad(Y)) = \text{ trace}([X,[Y,\cdot]]), \qquad X,Y \in \mathfrak{g} \qquad (2.3.18)$$

Let $\{X_i,\, i=1,...,n\}$ be a basis of $\mathfrak{g}$, and $[X,[Y,X_i]] = \sum_{j=1}^n \xi^{ij}X_j$ then

$$\text{trace}([X,[Y,\cdot]]) = \sum_{i=1}^n \xi^{ii}. \qquad (2.3.19)$$

Definition 2.3.12. A Lie algebra $\mathfrak{g}$ is *semi simple*, if its killing form is non-degenerated (positive definite). A Lie group $\mathcal{G}$ is *semi simple*, if its Lie algebra is semi simple.

2.3.5 Casimir element and Casimir operator

The Laplace operator can be identified with the Casimir element of the universal enveloping algebra, i.e. the tensor of order two, which is in the center of $U_{\mathfrak{g}}$.

Definition 2.3.13. Let $\mathfrak{g}$ and $\mathcal{G}$ be semi simple. Let B be the killing form and $\{X_i\}$ a orthogonal[1] basis of $\mathfrak{g}$. Further let X^i be the corresponding dual basis of the dual space of $\mathfrak{g}$. Then the *Casimir element* is defined by

$$\Omega = \sum_{i=1}^n X_i \otimes X^i \qquad (2.3.20)$$

By Riesz representation theorem X^i can be identified with a basis X_i in $\mathfrak{g}$.

$$\Omega = \sum_{i=1}^n X_i B(X_i,\cdot) \in U_{\mathfrak{g}} \qquad (2.3.21)$$

is in the centre of $U_{\mathfrak{g}}$ and independent of the choice of X_i.

[1] Orthogonality with respect to B.

For a representation ζ of $\mathfrak{g}$ in

$$\Delta_{\mathcal{G}} = \zeta(\Omega) = \sum_{i=1}^{n} \zeta(X_i)\zeta(X^i). \tag{2.3.22}$$

A natural representation in the vector space[2] C^∞ and the one we want to use here is defined by:

$$X_i \mapsto \frac{\partial}{\partial x_i}, \tag{2.3.23}$$

where $\frac{\partial}{\partial x_i}$ denotes the derivation in C^∞, mapping every f to its Lie derivative. The Lie derivative gives the first derivative in the direction of the tangential vector given by X_i at every point on $\mathcal{G}$. Precisely said, let $\{(U_i, \varphi_i)\}$ be an atlas on $\mathcal{G}$, then

$$\frac{\partial}{\partial x_i} f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi^{-1}(\varphi(g) + tv))|_{t=0}, \qquad \varphi^{-1}(tv)|_{t=0} = X_i(g) \in T_g\mathcal{G} \ (v \in \mathbb{R}^n), \tag{2.3.24}$$

where n denotes the dimension of $\mathcal{G}$ and $T_g\mathcal{G}$ the tangential space of $\mathcal{G}$ at g. The extension of ζ to $U_{\mathfrak{g}}$ works in the usual way, so that

$$\Delta_{\mathcal{G}} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 \tag{2.3.25}$$

To show that $\Delta_{\mathcal{G}}$ is translation invariant one has to verify that Ω is in the centre of $U_{\mathfrak{g}}$.

Definition 2.3.14. Let π be a representation of $\mathcal{G}$ in $\mathcal{H}$ and let B be a bilinear form in $\mathcal{H}$. B is invariant with respect to π, if

$$B(\pi(g)v, \pi(g)u) = B(v, u). \tag{2.3.26}$$

In that case B is also *invariant* for the corresponding representation $\mathrm{d}\pi$ of the Lie Algebra $\mathfrak{g}$. The defining equation one obtains by deriving (2.3.26)

$$\frac{\mathrm{d}}{\mathrm{d}t} B(\pi(\exp(tX))v, \pi(\exp(tX))u)|_{t=0} = B(\pi_*(X)v, u) + B(v, \pi_*(X)u) = 0. \tag{2.3.27}$$

For $z \in \mathfrak{g}$ there are constant coefficients a_{ij}, with $[z, x_i] = \sum_{j=1}^{n} a_{ij} x_j$ and by the invariance:

$$0 = B([z, x_i], x_j) + B(x_i, [z, x_j]) = \alpha_{ij} + \alpha_{ji}. \tag{2.3.28}$$

Further

$$z\Omega = z \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} ([z, x_i]x_i + x_i z x_i) = \sum_{i,j=1}^{n} a_{ij} x_j x_i + \sum_{i=1}^{n} x_i z x_i \tag{2.3.29}$$

[2] The notion of representations is similar defined for an topological vector space in place of a Hilbert space. In the case of $\mathcal{G}$ being compact we have $C^\infty \subset L^2(\mathcal{G})$ is dense.

and on the other hand

$$\Omega z = \sum_{i=1}^{n} x_i x_i z = \sum_{i=1}^{n} (-x_i[z, x_i] + x_i z x_i) = -\sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} x_i z x_i, \tag{2.3.30}$$

by (2.3.28) we have $a_{ij} = -a_{ji}$, hence $z\Omega = \Omega z$. Since $\mathfrak{g}$ generates $U_{\mathfrak{g}}$, consequently Ω commutes with all $u \in U_{\mathfrak{g}}$.

Remark 2.3.15. The killing form B is ad invariant. That is the adjoint representation of $\mathcal{G}$ in unitary and that of $\mathfrak{g}$ is has the killing form as invariant bilinear form.

$$B([x, y], z) = \text{trace}(ad(x)ad(y)ad(z) - ad(y)ad(x)ad(z)) \tag{2.3.31}$$

$$B(y, [x, z]) = \text{trace}(ad(y)ad(x)ad(z) - ad(y)ad(z)ad(x)) \tag{2.3.32}$$

Since the trace is invariant under change of the sequence in cyclic order, i.e. $\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(BCA)$ it is $B([x, y], z) = B(y, [x, z])$.

2.4 Eigenfunctions of differential operators on $\mathcal{G}$

The concept of identification of the Lie algebra of $\mathcal{G}$ with the set of left invariant operators acting on smooth functions on $\mathcal{G}$ is well known. In the same way left invariant operators of higher order can be represented with the help of the universal enveloping algebra of Lie group $\mathcal{G}$ (see also Chapter 2.3.3).

Let D be a left invariant differential operator. The corresponding element of the Lie algebra of $\mathcal{G}$ is denoted in the same way. For the representation π of $\mathcal{G}$ in the Hilbert space $\mathcal{H}$ one can consider the operator D in $\mathcal{H}$ by

$$\pi_*(D)f := \frac{\mathrm{d}}{\mathrm{d}t}\pi(\exp(tX))u\bigg|_{t=0}. \tag{2.4.1}$$

The straightforward extension to the universal enveloping algebra gives all left-invariant differential operators to $\mathcal{H}$.

Since we are looking for the eigenfunctions of the Laplacian in particular, the following assertion is very interesting

$$D\langle \pi(g)u_i, u_j\rangle_{\mathcal{H}} = \langle \pi(g)\pi_*(D)u_i, u_j\rangle_{\mathcal{H}}. \tag{2.4.2}$$

This follows from the direct calculation. Let $D \in Lie(\mathcal{G})$, then $Af(e) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tA))|_{t=0}$ and by left invariance we have $L_g D f(e) = Af(g)$. Hence

$$\begin{aligned} D\langle \pi(g)u_i, u_j\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\,\langle \pi(g\exp tD)u_i, u_j\rangle|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle \pi(g)\pi(\exp tD)u_i, u_j\rangle|_{t=0} \\ &= \langle \pi(g)\pi_*(D)u_i, u_j\rangle \end{aligned}$$

The crucial assertion is that $\langle \pi(g)u_i, u_j\rangle_{\mathcal{H}}$ is an eigenfunction of D if u_i is an eigenvector of $\pi_*(D)$.

2.5 Heat kernel and heat equation on compact Lie groups

The Heat kernel on a Group $\mathcal{G}$ is the fundamental solution $e^h : \mathcal{G} \times \mathbb{R}_+ \to \mathbb{C}$ of the heat equation, $(\Delta_\mathcal{G} - \partial_t)u = 0$. So that the initial value problem

$$\Delta_\mathcal{G} u(g,t) - \partial_t u(g,t) = 0 \tag{2.5.1}$$

$$u(g,0) = f(g) \tag{2.5.2}$$

has the solution

$$u(g,t) = (e_t^{heat} * f)(g). \tag{2.5.3}$$

The Laplace operator on $\mathcal{G}$ can also be characterized as the second order differentiable operator $\Delta_\mathcal{G}$ which is translation invariant under right and left shifts.

$$\Delta L_g = L_g \Delta \qquad \text{and} \qquad \Delta R_g = R_g \Delta. \tag{2.5.4}$$

The correct way of its description is connected to the Casimir element of the universal enveloping algebra and the extension of the adjoint representation of the Lie algebra of $\mathcal{G}$ to the universal enveloping algebra as saw in Section 2.3.3. The image of the Casimir element under the extended representation will give the Laplace operator on $\mathcal{G}$.

2.5.1 Heat kernel on $\mathcal{G}$

Eigenfunctions of the Laplacian are given by matrix coefficients of irreducible representations. From (2.5.4) we see that

$$R_{g_1}\Delta_\mathcal{G}\langle \pi(g_2)v, u\rangle_\mathcal{H} = \langle \pi(g_2)\pi_*(\Delta_\mathcal{G})\pi(g_1)v, u\rangle_\mathcal{H} \tag{2.5.5}$$

$$= \langle \pi(g_2)\pi(g_1)\pi_*(\Delta_\mathcal{G})v, u\rangle_\mathcal{H} = \Delta_\mathcal{G} R_{g_1}\langle \pi(g_2)v, u\rangle_\mathcal{H}, \tag{2.5.6}$$

which means

$$\pi_*(\Delta_\mathcal{G})\pi(g_2) = \pi(g_2)\pi_*(\Delta_\mathcal{G}) \qquad \forall g_2 \in \mathcal{G}. \tag{2.5.7}$$

Consequently the linear operator $\pi_*(\Delta_\mathcal{G}) = -\lambda_\pi^2 Id$ is a multiple of the identity operator and depends on π. Consequently, the projection of any function $f \in L^2(\mathcal{G})$ to a translation invariant subspace $\pi_\alpha(\mathcal{G})$ (given in (2.1.11)) is an eigenfunction of $\Delta_\mathcal{G}$:

$$\Delta_\mathcal{G}(f * \chi) = -\lambda_\alpha^2 (f * \chi). \tag{2.5.8}$$

We have $\Delta_\mathcal{G}(f * \chi_{\pi_\alpha}) = (f * \Delta_{\chi_\alpha}\chi_{\pi_\alpha})$. We will use the characters as a system of eigenfunctions of $\Delta_\mathcal{G}$ to express the heat kernel on $\mathcal{G}$ by

$$e_t^{heat}(g) = \sum_{\pi_\alpha} d_{\pi_\alpha} e^{-\lambda_\alpha^2 t}\chi_\alpha(g). \tag{2.5.9}$$

It is obvious, that $e_t^{heat}(g)$ satisfies the heat equation (2.5.1). A short calculation shows, that for $u(g,0) = f(g)$ the initial value problem of the heat equation is solved by $u(g,t) = (f * e_t^{heat})(g)$:

$$\lim_{t \to 0} f * e_t^{heat} = \lim_{t \to 0} \sum_{\pi_\alpha} e^{\lambda_\alpha^2 t}(f * \chi_\alpha) = \sum_{\pi_\alpha} f_\alpha = f. \tag{2.5.10}$$

Chapter 3

Diffusive wavelets on Lie groups and homogeneous spaces

3.1 Diffusive wavelets on compact Lie groups

In this chapter we give the general construction of diffusive wavelets for compact Lie groups $\mathcal{G}$ and homogeneous spaces. We will follow the idea we described in Chapter 2.3.2. For concrete constructions we restrict us here to the heat kernel on $\mathcal{G}$ to construct the corresponding diffusive wavelets. Let $L^2_0(\mathcal{G})$ the subspace of $L^2(\mathcal{G})$ as defined in (2.3.10), corresponding to the approximate diffusive identity $\{p_t,\ t > 0\}$ arising from the heat kernel. For $f \in L^2(\mathcal{G})$ the projection onto $L^2_0(\mathcal{G})$ is denoted by

$$f|_{\widehat{\mathcal{G}}_+} = \sum_{\pi_\alpha \in \widehat{G}_+} f * \chi_{\pi_\alpha}. \tag{3.1.1}$$

The Fourier transform and its inversion will be defined for functions in $L^2_0(\mathcal{G})$.

Definition 3.1.1. Let p_t be the kernel of an diffusive approximate identity and $\alpha(\rho) > 0$ a weight function. A family $\{\psi_\rho,\ \rho > 0\} \subset L^2_0(\mathcal{G})$ is called *diffusive wavelet family*, if it satisfies the admissibility condition

$$p_t|_{\widehat{\mathcal{G}}_+} = \int_t^\infty \check{\psi}_\rho * \psi_\rho \, \alpha(\rho)\, \mathrm{d}\rho. \tag{3.1.2}$$

where again $\check{\psi}_\rho(g) = \overline{\psi_\rho(g^{-1})}$.
Thanks to the convolution theorem the admissibility condition (3.1.2) can be studied in Fourier domain. An application of Fourier transform to both sides yields:

$$\widehat{p}_t(\pi_\alpha) = \int_t^\infty \widehat{\psi}_\rho(\pi)\widehat{\psi}^*_\rho(\pi)\, \alpha(\rho)\, \mathrm{d}\rho, \qquad \forall \pi \in \widehat{\mathcal{G}}_+. \tag{3.1.3}$$

Differentiation with respect to t results in

$$-\partial_t \widehat{p}_t(\pi) = \widehat{\psi}_\rho(\pi)\widehat{\psi}^*_\rho(\pi)\, \alpha(\rho), \qquad \forall \pi \in \widehat{\mathcal{G}}_+. \tag{3.1.4}$$

Let ψ_ρ be a wavelet with Fourier coefficients $\widehat{\psi}_\rho(\pi)$. We would like to mention a certain freedom in the choice of the Fourier coefficients of the wavelets. If $\widehat{\psi}_\rho(\pi)$ are Fourier coefficients of wavelets, then a multiplication with a unitary matrix $\eta_\rho(\pi)$ from the right still leads to Fourier coefficients of a wavelet ψ'_ρ. Later on we will take a closer look at the choice of $\eta_\rho(\pi)$ and the weight function $\alpha(\rho)$. First we consider the special case of the diffusive wavelets based on the heat kernel.

Let p_t be the heat kernel e_t^{heat}, given in (2.5.9). First we have to determine the appropriate spectrum $\widehat{\mathcal{G}}_+$. From the definition of diffusive approximate identity (Definition 2.3.6) we know that

$$\lim_{t\to\infty} \widehat{e}_t^{heat}(\pi) = 0 \tag{3.1.5}$$

for all $\pi \in \widehat{\mathcal{G}}_+$. This is the case for all nontrivial representations π_0 of $\mathcal{G}$. Trivial representation means $\pi_0(g) \equiv Id_{\mathcal{H}}$. Since the character of the trivial representation is $\chi_{\pi_0} \equiv 1$ the corresponding translation invariant subspace in $L^2(\mathcal{G})$ is the space of constant functions. Consequently, the corresponding eigenvalue of $\Delta_{\mathcal{G}}$ vanishes $\lambda_0 = 0$. Hence (3.1.5) is not satisfied by $\widehat{e}_t^{heat}(\pi_0) = Id$. The Fourier coefficients of all other irreducible representations, $\widehat{e}_t^{heat}(\pi)$, $\pi \neq \pi_0$ satisfies this condition and we find for the heat kernel

$$\widehat{\mathcal{G}}_+ = \widehat{\mathcal{G}} \backslash \{\pi_0\}. \tag{3.1.6}$$

The admissibility condition (3.1.2) is formulated in Fourier domain by (3.1.4). For diffusive wavelets corresponding to the heat kernel from (3.1.4) follows

$$\begin{aligned}
\widehat{\psi}_\rho(\pi)\widehat{\psi}^*_\rho(\pi) &= -\frac{1}{\alpha(\rho)}\partial_\rho \widehat{e}_\rho^{heat}(\pi) &(3.1.7)\\
&= -\frac{1}{\alpha(\rho)}\partial_\rho e^{-\lambda_\pi^2\rho} Id &(3.1.8)\\
&= \frac{1}{\alpha(\rho)}\lambda_\pi^2 e^{-\rho\lambda_\pi^2} Id, &(3.1.9)
\end{aligned}$$

such that

$$\widehat{\psi}_\rho(\pi) = \frac{1}{\alpha(\rho)}\lambda_\pi e^{-\lambda_\pi^2\rho/2} Id. \tag{3.1.10}$$

If we multiply $\widehat{\psi}_\rho(\pi)$ with any unitary matrix $\eta_\rho(\pi)$, the result still satisfies the admissibility (3.1.4).

Now, the expansion of the wavelet has the form

$$\psi_\rho(g) = \frac{1}{\sqrt{\alpha(\rho)}} \sum_{\pi\in\widehat{\mathcal{G}}_+} d_\pi \lambda_\pi e^{-\rho\lambda_\pi^2/2}\,\mathrm{trace}(\eta_\rho(\pi)\pi(g)). \tag{3.1.11}$$

We remark, that the freedom of the choice of $\eta_\rho(\pi)$ corresponds to the freedom of choosing an admissible section σ for the construction which was sketched in section 2.3.1 (formula (2.3.2)).

Hence here one can adapt the special form of dilations but also the focus of localization of the wavelets is fixed by the choice of $\eta_\rho(\pi)$. Since translating the wavelet corresponds to multiplying the Fourier coefficients with the unitary matrix $\pi(g)$ from the right, this corresponds to the choice $\pi(g) = \eta_\rho(\pi)$.
A natural choice seems to be $\eta_\rho(\pi) = Id_{d_\pi \times d_\pi}$. In this case the wavelet family localizes at $e \in \mathcal{G}$ for $\rho \to 0$.
The weight function $\alpha(\rho)$ shall be used to normalize the wavelet family in $L^2(\mathcal{G})$. By Parsevals Identity we have

$$\int_{\mathcal{G}} |f(g)|^2 \,\mathrm{d}g = \sum_{\widehat{\mathcal{G}}} d_\pi \int_{\mathcal{G}} |\operatorname{trace}(\widehat{f}(\pi)\pi(g))|^2 \,\mathrm{d}g. \tag{3.1.12}$$

For the wavelet Ψ_ρ we have the expansion in terms of character functions, hence

$$\|\Psi_\rho\|_{L^2(\mathcal{G})} = \frac{1}{\alpha(\rho)} \sum_{\pi \in \widehat{\mathcal{G}}_+} d_\pi^2 \lambda_\pi e^{-\rho\lambda_\pi^2} \|\chi_\pi(g)\|_{L^2(\mathcal{G})} = \frac{1}{\alpha(\rho)} \sum_{\pi \in \widehat{\mathcal{G}}_+} d_\pi^2 \lambda_\pi e^{-\rho\lambda_\pi^2}. \tag{3.1.13}$$

For normalized wavelet family $\{\psi_\rho,\ \rho > 0\}$ we choose

$$\alpha(\rho) = \sum_{\pi \in \widehat{\mathcal{G}}_+} d_\pi^2 \lambda_\pi e^{-\rho\lambda_\pi^2}. \tag{3.1.14}$$

Looking at the expansion of the heat kernel (2.5.9), one sees that this choice of $\alpha(\rho)$ means

$$\alpha(\rho) = -\partial_\rho e_\rho^{heat}(e) = -\Delta_{\mathcal{G}} e_\rho^{heat}(e). \tag{3.1.15}$$

Theorem 3.1.2 (Parsevals Identity)**.** *The wavelet transform, defined in the usual way by*

$$WTf(\rho, g) := (f * \check{\psi}_\rho)(g) = \langle f, T_g \psi_\rho \rangle_{L^2(\mathcal{G})}, \tag{3.1.16}$$

is an unitary operator $WT : L_0^2(\mathcal{G}) \to L^2(\mathbb{R}^+ \times \mathcal{G}, \alpha(\rho)\,\mathrm{d}\rho,\ \mathrm{d}g)$.

Proof:

$$\begin{aligned}
&\langle WTf(\rho, g), WTh(\rho, g)\rangle_{L^2(\mathbb{R}^+ \times \mathcal{G}, \alpha(\rho)\,\mathrm{d}\rho,\ \mathrm{d}g)} \\
&= \int_{\mathbb{R}^+} \int_{\mathcal{G}} WTf(\rho, g)\overline{WTh(\rho, g)} \,\mathrm{d}g\, \alpha(\rho)\,\mathrm{d}\rho \\
&= \int_{\mathbb{R}^+} \int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\mathcal{G}} f(x)\check{\psi}_\rho(x^{-1}g)\, \overline{h(y)\check{\psi}_\rho(y^{-1}g)} \,\mathrm{d}x\ \mathrm{d}y\ \ \mathrm{d}g\, \alpha(\rho)\,\mathrm{d}\rho \\
&= \lim_{t \to 0} \int_{\mathcal{G}} \int_{\mathcal{G}} f(x)\overline{h(y)} \int_t^\infty \int_{\mathcal{G}} \check{\psi}_\rho(x^{-1}g)\psi_\rho(g^{-1}y)\ \mathrm{d}g\, \alpha(\rho)\,\mathrm{d}\rho\ \mathrm{d}x\ \mathrm{d}y \\
&= \lim_{t \to 0} \int_{\mathcal{G}} \int_{\mathcal{G}} f(x)\overline{h(y)} e_t^{heat}|_{\widehat{\mathcal{G}}_+}(yx^{-1})\ \mathrm{d}x\ \mathrm{d}y \\
&= \langle f, h\rangle_{L_0^2(\mathcal{G})}.
\end{aligned}$$

□

By Theorem 3.1.2 the inverse of the wavelet transform is given by the adjoint.

Theorem 3.1.3. *The wavelet transform is invertible on its range by*

$$\int_{\mathbb{R}^+}\int_{\mathcal{G}} WTf(\rho,x)\psi_\rho(x^{-1}g)\,\mathrm{d}x\,\alpha(\rho)\,\mathrm{d}\rho = f(g) \qquad \forall f\in L^2_0(\mathcal{G}). \tag{3.1.17}$$

While the invertibility follows from the previous theorem, we would like to give a direct proof here, based on the property of approximate identity.

Proof: This is straitforward via

$$\int_{\mathbb{R}^+}\int_{\mathcal{G}} WTf(\rho,x)\psi_\rho(x^{-1}g)\,\mathrm{d}x\,\alpha(\rho)\,\mathrm{d}\rho = \int_{\mathbb{R}^+}\int_{\mathcal{G}} (f*\check{\psi}_\rho)(x)\psi_\rho(x^{-1}g)\,\mathrm{d}x\,\alpha(\rho)\,\mathrm{d}\rho$$
$$= \lim_{t\to 0}\int_t^\infty f*\check{\psi}_\rho*\psi_\rho\alpha(\rho)\,\mathrm{d}\rho = \lim_{t\to 0}(f*e_t^{heat})(g) = f(g).$$

□

3.2 Diffusive wavelets on homogeneous spaces

Let again $X\simeq\mathcal{G}/\mathscr{H}$, where $\mathscr{H}$ is a subgroup of $\mathcal{G}$, be a homogeneous space with base point x_0. In this chapter we extend the construction of wavelets from the previous chapter, done for the case of compact Lie Groups $\mathcal{G}$ to their homogeneous spaces $X\simeq\mathcal{G}/\mathscr{H}$. In Chapter 2.2.1 we have already discussed the question of functions on homogeneous spaces. There are two basic approaches. On one hand following the idea of projection - and lifting method we can lift functions from X to $\mathcal{G}$ and work on $\mathcal{G}$. On the other hand we can discuss the transfer of wavelets from $\mathcal{G}$ to X. In the latter case it is more complicate to understand the corresponding wavelet transform and inversion formula. This will become clear when we look at the Fourier transform of these transformations.
We will discuss both approaches in this chapter.

3.2.1 Diffusive wavelets of class type

We start by taking a function from X and lift it to $\mathcal{G}$ by the lifting method (2.2.1). Then we can apply the wavelet transform on the lifted function $\tilde{f}$, make inversion on $\mathcal{G}$ and project the result back to X.
The wavelet transform on X, denoted by WT_X assumes the following form: Let $f\in L^2_0(X) = L^2_0(\mathcal{G})\cap L^2(X)$, then

$$WT_Xf(\rho,g) = WT\tilde{f}(\rho,g) = \int_{\mathcal{G}} f(x\cdot x_0)\check{\Psi}_\rho(x^{-1}g)\,\mathrm{d}x \tag{3.2.1}$$
$$= \int_{\mathcal{G}} f(x\cdot x_0)\overline{\Psi_\rho(g^{-1}x)}\,\mathrm{d}x \tag{3.2.2}$$
$$= \int_X f(y)\overline{\mathbb{P}_x\Psi_\rho(g^{-1}\cdot y)}\,\mathrm{d}y = \langle f, T_g\Psi_\rho\rangle_{L^2(X)}. \tag{3.2.3}$$

Hence, in this case the wavelet transform can be written as an integral over the homogeneous space and assumes the well-known form, where the translation is given by the canonical action of $\mathcal{G}$ on X. In this case the wavelet transform of any function assumes values in $L^2(\mathbb{R}^+ \times \mathcal{G}, \alpha(\rho)\,\mathrm{d}\rho,\,\mathrm{d}g)$ and is also unitary according to Theorem 3.1.2.
Let us make some observations on this construction.
Remembering Definition 2.1.9, a class function is a function f on a Lie group $\mathcal{G}$, which is constant over conjugate classes, i.e. $f(g) = f(h^{-1}gh) \qquad \forall h \in \mathcal{G}$.
For the first step let us assume wavelets of class type functions. In particular this corresponds to the choice of $\eta_\rho(\pi) = Id_{d_\pi \times d_\pi}$ for the diffusive wavelet which we obtained in (3.1.11).
Here we can continue the above formulation of the wavelet transform and get

$$WT_X f(\rho, g) = \int_{\mathcal{G}} f(x \cdot x_0)\overline{\Psi_\rho(g^{-1}x)}\,\mathrm{d}x \tag{3.2.4}$$
$$= \int_{\mathcal{G}} f(xg \cdot x_0)\overline{\Psi_\rho(x)}\,\mathrm{d}x \tag{3.2.5}$$
$$= \int_{\mathcal{G}} f(x \cdot y)\overline{\Psi_\rho(x)}\,\mathrm{d}x, \qquad \text{with } y = g \cdot x_0, \tag{3.2.6}$$

since $\Psi_\rho(g^{-1}x) = \Psi_\rho(xg^{-1})$ for class type functions and the invariance of $\mathrm{d}x$.

Corollary 3.2.1. Let ψ_ρ be a diffusive wavelet family and assume that ψ_ρ are class type functions on $\mathcal{G}$. Then the associated zonal wavelet transform $WT_X : L^2_0(X) \to L^2(\mathbb{R}_+ \times X, \alpha(\rho)\,\mathrm{d}\rho \otimes \,\mathrm{d}x)$ is unitary.

Proof: The calculation is reduced to the one on $\mathcal{G}$. Indeed,

$$\begin{aligned}\langle WT_X\phi_1, WT_X\phi_2\rangle &= \int_0^\infty \int_X WT_X\phi_1(\rho, x)\overline{WT_X\phi_2(\rho, x)}\,\mathrm{d}x\alpha(\rho)\,\mathrm{d}\rho \\ &= \langle WT\tilde{\phi}_1, WT\tilde{\phi}_2\rangle = \langle\tilde{\phi}_1, \tilde{\phi}_2\rangle = \langle\phi_1, \phi_2\rangle\end{aligned}$$

follows from Theorem 3.1.2 and (2.2.3). □

3.2.2 Zonal wavelets

We recall Definition 2.2.1 of a zonal function, that is a function on X which is invariant under the action of the stabilizer of the base point x_0. Also in Chapter 2.2.1 we have seen, that the Fourier coefficients of zonal functions are matrices with the special form

$$\widehat{f}(\pi) = \begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}. \tag{3.2.7}$$

Let us remark that the Projection of a class function $\mathbb{P}_X$ is always zonal, but the lifted zonal function is not necessarily constant over conjugate classes.
Nevertheless the special form of the wavelet transform (3.2.6) can be kept for zonal wavelets.

For any diffusive approximate identity p_t, the zonal average

$$p_t^X(g \cdot x_0) = \int_{\mathscr{H}} \int_{\mathscr{H}} p_t(h_1 g h_2) \, \mathrm{d}\mu_{\mathscr{H}}(h_1) \, \mathrm{d}\mu_{\mathscr{H}}(h_2) \tag{3.2.8}$$

gives a zonal approximate identity p_t^X on X.

Definition 3.2.2. Let p_t^X be a zonal diffusive approximate identity on X and let $\alpha(\rho) > 0$ be a given weight function. A family $\psi_\rho \in L^2(X)$ is called *zonal diffusive wavelet family* if

1. ψ_ρ is zonal with respect to x_0,
2. the admissibility condition

$$p_t^X(x)\big|_{\widehat{\mathcal{G}}_+} = \int_t^\infty \check{\psi}_\rho * \psi_\rho(x) \, \alpha(\rho) \, \mathrm{d}\rho \tag{3.2.9}$$

is satisfied.

The reason why it is comfortable to formulate the wavelet transform for zonal wavelets can be seen from the definition $WTf = f * \check{\Psi}_\rho$. Here appears the $\vee$-involution that maps functions which are invariant over right fibers $g\mathscr{H}$ into those which are invariant over left fibers $\mathscr{H}g$. Hence, if Ψ_ρ is a function on $\mathcal{G}/\mathscr{H}$, then $\check{\Psi}_\rho$ is a function on $\mathscr{H}\backslash\mathcal{G}$ and cannot be defined on $\mathcal{G}/\mathscr{H}$. On the Fourier side the $\vee$-involution acts on the Fourier coefficients by taking them to their adjoint. This means, that $\vee$-involution does not preserve the special form of the Fourier coefficients of functions on X (c.f. Corollary 2.2.7). Furthermore, the Fourier coefficients of a zonal function are of the form (3.2.7) and the adjoint gives a matrix of the same form. Consequently the $\vee$-involution maps zonal function to zonal functions, such that in time-domain by zonality, i.e. $f(h \cdot x) = f(x)$ for all $h \in \mathscr{H}$ we have that $\check{f}$ lives on $\mathcal{G}/\mathscr{H}$:

$$\check{f}(h \cdot x) = \check{\tilde{f}}(hg) = \overline{\tilde{f}(g^{-1}h^{-1})} \qquad \text{with } g \cdot x_0 = x \tag{3.2.10}$$

Because of zonality this equals

$$\check{f}(x) = \check{\tilde{f}}(g) = \overline{\tilde{f}(g^{-1})} \qquad \forall h \in \mathscr{H}, \tag{3.2.11}$$

hence $\check{\tilde{f}}$ is a function, which is constant over right fibers $g\mathscr{H}$. We have

$$\check{f}(g \cdot x_0) = \overline{f(g^{-1} \cdot x_0)}. \tag{3.2.12}$$

Theorem 3.2.3. *The zonal wavelet transform* $WT_X : L_0^2(X) \to L^2(\mathbb{R}_+ \times X, \alpha(\rho) \, \mathrm{d}\rho \otimes \, \mathrm{d}x)$ *is unitary and invertible by*

$$\phi = \int_0^\infty \int_{\mathcal{G}} WT_X \phi(\rho, g) \psi_\rho(g^{-1} \cdot x) \, \alpha(\rho) \, \mathrm{d}\rho. \tag{3.2.13}$$

Proof: By admissibility condition 3.2.9 and its formulation in Fourier-domain (c.f. Remark 2.2.5) we have $\int_0^\infty \hat{\psi}_\rho(\pi)\hat{\psi}_\rho^*(\pi)\alpha(\rho)\,\mathrm{d}\rho = \pi_{\mathscr{H}}$. The explicit form of $\pi_{\mathscr{H}}$ is given in (2.2.11).

$$\begin{aligned}\langle\phi_1,\phi_2\rangle &= \sum_{\pi\in\widehat{\mathcal{G}}_+} d_\pi\ \mathrm{trace}(\hat{\phi}_2^*(\pi)\hat{\phi}_1(\pi)) = \sum_{\pi\in\widehat{\mathcal{G}}_+} d_\pi\ \mathrm{trace}(\hat{\phi}_2^*(\pi)\pi_{\mathscr{H}}\hat{\phi}_1(\pi))\\ &= \int_0^\infty \sum_{\pi\in\widehat{\mathcal{G}}_+} d_\pi\ \mathrm{trace}(\hat{\phi}_2^*(\pi)\hat{\psi}_\rho(\pi)\hat{\psi}_\rho^*(\pi)\hat{\phi}_1(\pi))\alpha(\rho)\,\mathrm{d}\rho\\ &= \langle WT_X\phi_1, WT_X\phi_2\rangle.\end{aligned}$$

The inversion formula is similar to the inversion formula on $\mathcal{G}$, which is given in (3.1.17). □

3.2.3 General case

We now would like to consider the general case of nonzonal wavelets. As stated in the previous section, there major problem arises due to the admissibility condition, where we make use of the $\vee$-involution. To this end we will outline our previous approach.
To formulate the wavelet transform and the inversion formula for wavelets on X we consider the transformation in Fourier domain. There the wavelet transform is given by

$$\widehat{WTf(\rho,g)} = \widehat{\Psi}_\rho^*\widehat{f}. \tag{3.2.14}$$

Hence, this transform corresponds to a multiplication with the adjoint Fourier coefficient from the left.
The reconstruction formula (3.1.17) is a usual convolution, combined with an integration over all scales

$$\int_{\mathbb{R}^+} WT\widehat{f(\rho,\cdot)} * \Psi_\rho\ \alpha(\rho)\,\mathrm{d}\rho = \int_{\mathbb{R}^+} \widehat{\Psi}_\rho\widehat{\Psi}_\rho^*\widehat{f}\ \alpha(\rho)\,\mathrm{d}\rho. \tag{3.2.15}$$

The admissibility condition (3.1.2) has the form (3.1.4) in Fourier domain, where appears the multiplication with the adjoint from the right.

$$\widehat{p_t}(\pi_\alpha) = \int_t^\infty \widehat{\psi}_\rho(\pi)\widehat{\psi}_\rho^*(\pi)\,\alpha(\rho)\,\mathrm{d}\rho, \qquad \forall\pi\in\widehat{\mathcal{G}}_+. \tag{3.2.16}$$

Now we are going to formulate these three steps for the case of the homogeneous spaces. Let us start by introducing a few notations.

Definition 3.2.4. Let $\phi,\psi\in L^1(X)$. Then we define

1. the *group convolution*

$$\phi*\psi(x) = \int_{\mathcal{G}} \phi(g\cdot x_0)\psi(g^{-1}\cdot x)\,\mathrm{d}g \in L^1(X); \tag{3.2.17}$$

2. the $\bullet$-*product*

$$\phi \bullet \psi(g) = \int_X \phi(x)\overline{\psi(g^{-1} \cdot x)}\, \mathrm{d}x = \langle \phi, T_g\psi \rangle \in L^1(\mathcal{G}); \tag{3.2.18}$$

3. the *zonal product*

$$\phi \,\hat{\bullet}\, \psi(x) = \int_{\mathcal{G}} \overline{\phi(g \cdot x_0)}\psi(g \cdot x)\, \mathrm{d}g \in L^1(X). \tag{3.2.19}$$

For these products we have the following properties.

Proposition 3.2.5. Let $\phi, \psi \in L^1(X)$.

1. $\widetilde{\phi * \psi} = \tilde{\phi} * \tilde{\psi}$ and thus $\widehat{\phi * \psi}(\pi) = \hat{\psi}(\pi)\hat{\phi}(\pi)$.
2. $\phi \bullet \psi = \tilde{\phi} * \check{\tilde{\psi}}$ and thus $\widehat{\phi \bullet \psi}(\pi) = \widehat{\psi}^*(\pi)\widehat{\phi}(\pi)$.
3. If ψ is zonal with respect to x_0 then $\phi \bullet \psi$ is constant on cosets $g\mathscr{H}$ and thus defines a function on X.
4. $\widetilde{\phi \,\hat{\bullet}\, \psi} = \check{\tilde{\phi}} * \tilde{\psi}$ and $\widehat{\phi \,\hat{\bullet}\, \psi}(\pi) = \widehat{\psi}(\pi)\widehat{\phi}^*(\pi)$.
5. $\phi \,\hat{\bullet}\, \psi$ is zonal with respect to x_0.

Proof: (1) obvious, from $\widehat{\phi * \psi}(\pi) = \hat{\psi}(\pi)\hat{\phi}(\pi)$ one sees that the form of Fourier coefficients of functions on X is preserved.
(2) Calling in (2.2.3) and (2.2.4) one finds

$$\begin{aligned}(\phi \bullet \psi)(g) &= \int_X \phi(x)\overline{\psi(g^{-1} \cdot x)}\, \mathrm{d}x \\ &= \int_{\mathcal{G}} \tilde{\phi}(h)\overline{\tilde{\psi}(g^{-1}h)}\, \mathrm{d}h = \int_{\mathcal{G}} \tilde{\phi}(h)\check{\tilde{\psi}}(h^{-1}g)\, \mathrm{d}h = (\tilde{\phi} * \check{\tilde{\psi}})(g)\end{aligned}$$

(3) A calculation based on the zonality of ψ yields

$$\phi \bullet \psi(gh) = \int_X \phi(x)\overline{\psi(h^{-1}g^{-1} \cdot x)}\, \mathrm{d}x = \int_X \phi(x)\overline{\psi(g^{-1} \cdot x)}\, \mathrm{d}x = \phi \bullet \psi(g).$$

On the Fourier side one sees directly from (2), that for functions acting on a zonal function ψ the Fourier coefficient $\widehat{\psi}^*(\pi)\widehat{\phi}(\pi)$ has the form of a function on X.
(4)

$$\begin{aligned}\widetilde{\phi \,\hat{\bullet}\, \psi} &= \int_{\mathcal{G}} \overline{\phi(g \cdot x_0)}\psi(g \cdot x)\, \mathrm{d}g = \int_{\mathcal{G}} \overline{\tilde{\phi}(g)}\tilde{\psi}(g \cdot h)\, \mathrm{d}g \qquad \text{with } h \cdot x_0 = x \\ &= \int_{\mathcal{G}} \check{\tilde{\phi}}(g^{-1})\tilde{\psi}(g \cdot h)\, \mathrm{d}g = (\check{\tilde{\phi}} * \tilde{\psi})(h)\end{aligned}$$

The second relation follows immediately.

(5)For $h \in \mathcal{H}$ a direct calculation shows

$$\phi \,\hat{\bullet}\, \psi(h \cdot x) = \int_{\mathcal{G}} \overline{\phi(g \cdot x_0)}\psi(gh \cdot x)\,\mathrm{d}g = \int_{\mathcal{G}} \overline{\phi(gh^{-1} \cdot x_0)}\psi(g \cdot x)\,\mathrm{d}g = \phi \,\hat{\bullet}\, \psi(x)$$

by the right invariance of d. On the Fourier side one sees immediately that for functions ϕ and ψ on X we have that from $\widehat{\phi \,\hat{\bullet}\, \psi}(\pi) = \widehat{\psi}(\pi)\widehat{\phi}^*(\pi)$ it follows that $\widehat{\psi}(\pi)\widehat{\phi}^*(\pi)$ has form (3.2.7) and hence is the Fourier coefficient of a zonal function. □

Remark 3.2.6. We have introduced now all necessary transformations on X. The problem is how can we define a wavelet directly on X? Our idea is to use the projection of the heat kernel on $\mathcal{G}$ to X in order to obtain the heat kernel on X. Therefore we have to ensure that the projection of an $\mathcal{G}$-invariant operator on $\mathcal{G}$ gives an $\mathcal{G}$-invariant differential operator on X i.e. the following diagram commutes:

$$\begin{array}{ccc} C^\infty(\mathcal{G}) & \xrightarrow{D} & C^\infty(\mathcal{G}) \\ \downarrow{\scriptstyle \mathbb{P}_{\mathscr{H}}} & & \downarrow{\scriptstyle \mathbb{P}_{\mathscr{H}}} \\ C^\infty(X) & \xrightarrow[\mu(D)]{} & C^\infty(X) \end{array}$$

where μ is a surjective homomorphism from $\mathcal{G}$-invariant operators on $\mathcal{G}$ to those on X. This question is investigated by Helgason [Hel11, Hel01], Wolf [Wol84] and others. The commutation of the diagram and the existence of μ is true if the homogeneous space X is reductive, i.e. when $\mathscr{H}$ contains no normal subgroup of $\mathcal{G}$ or equivalently, there is an $Ad(\mathscr{H})$-invariant subspace in $\mathfrak{g}$ which is complementary to the Lie algebra of $\mathscr{H}$ in $\mathfrak{g}$. In this case, μ is given by $\mu(D)\mathbb{P}_{\mathscr{H}} f = D(f \circ \pi)$.

Consequently, in all what follows X is a reductive homogeneous space[1].

$$\mathbb{P}_X e_t^{heat} = e_t^{heat,X}, \tag{3.2.20}$$

where $e_t^{heat,X}$ denotes the heat kernel on X.

We already know, that the wavelet transform shall be of the form

$$WT\phi(\rho, g) = \phi \bullet \psi_\rho(g) = \langle \phi, T_g \psi_\rho \rangle. \tag{3.2.21}$$

Let us remark, that the transform lives on $\mathcal{G}$ rather than on X for nonzonal wavelets. We aim for an inversion formula of the kind

$$\phi(x) = \mathbb{P}_{\mathscr{H}} \int_0^\infty WT\phi(\rho, \cdot) * \tilde{\Psi}_\rho(g)\,\alpha(\rho)\,\mathrm{d}\rho \tag{3.2.22}$$

with a second family $\Psi_\rho \in L^2(X)$. By a short computation we see that $(\phi \bullet \psi) * \tilde{\chi} = \phi \bullet (\chi \,\hat{\bullet}\, \psi)$ for $\phi, \psi, \chi \in L^1(X)$. Since our reconstruction formula is of the form $(\phi \bullet \psi) * \tilde{\Psi}$, this motivates us to give the following definition.

[1]In our construction the homogeneous space is reductive if and only if it is isotropy irreducible[WZ91]

Definition 3.2.7. Let p_t be a diffusive approximate identity and $\alpha(\rho) \geq 0$ be a given weight function. A family $\psi_\rho \in L^2(X)$ is called (non-zonal) *diffusive wavelet family* if the admissibility condition

$$p_t^X(x)\big|_{\widehat{\mathcal{G}}_+} = \int_t^\infty \psi_\rho \,\hat{\bullet}\, \psi_\rho(x)\, \alpha(\rho)\, \mathrm{d}\rho \tag{3.2.23}$$

is satisfied.

We have seen, that under the projection $\mathbb{P}_{\mathscr{H}}$ a family of diffusive wavelets on $\mathcal{G}$ $\{\Psi_\rho,\, \rho > 0\}$ becomes a family of wavelets on $\mathcal{G}/\mathscr{H}$. On $\mathcal{G}$ we have the freedom to multiply the Fourier coefficients of a wavelet by unitary matrix $\eta_\rho(\pi)$ from the right in order to obtain another wavelet. Under the projection to the homogeneous space such a multiplication means a deformation of the wavelet in the sense that a zonal wavelet becomes a nonzonal wavelet. For a nonzonal wavelet $\eta_\rho(\pi)$ can be chosen in such a way, that we obtain a zonal wavelet.
Let $\{\Psi_\rho,\, \rho_0\}$ be a zonal diffusive wavelet on $\mathcal{G}/\mathscr{H}$. Then a family of L^2-functions $\{\Psi'_\rho,\, \rho > 0\}$ with $\widehat{\Psi}'_\rho(\pi) = \widehat{\Psi}_\rho(\pi)\eta_\rho(\pi)$ where

$$\eta_\rho(\pi)^*\eta_\rho(\pi) = \pi_{\mathscr{H}} \tag{3.2.24}$$

forms a (possibly nonzonal) wavelet on $\mathcal{G}/\mathscr{H}$.
In that way all diffusive wavelets, corresponding to a fixed diffusive approximate identity can be obtained from zonal wavelets, unique up to $\alpha(\rho)$, which corresponds to the diffusive approximate identity.

Remark 3.2.8. The wavelets Ψ_ρ and $T_g\Psi_{\rho'}$ are not orthogonal in general. A calculation yields, for heat wavelet families, the identity

$$\langle \Psi_\rho, T_g\Psi_{\rho'}\rangle_{L^2(\mathcal{G})} = \frac{1}{\sqrt{\alpha(\rho)\alpha(\rho')}} \sum_{\pi\in\hat{\mathcal{G}}} d_\pi \lambda_\pi^2 e^{-\lambda_\pi^2(\rho+\rho')/2}\chi_\pi \tag{3.2.25}$$

$$= -\frac{1}{\sqrt{\alpha(\rho)\alpha(\rho')}}\Delta_{\mathcal{G}} p^{heat}_{(\rho+\rho')/2} \tag{3.2.26}$$

Orthogonal wavelets are obtained by diffusion wavelets by Coifman and Maggioni [CM06], there a discrete diffusion method is combined with a orthogonalization method.

3.3 Further symmetries

One can ask for wavelets on manifolds which satisfy additional symmetries. This question is investigated in [BBCK10]. There the symmetry is given by the invariance of the wavelets under action of a finite reflection group, which involves the theory of Coxeter groups.
The property, that a wavelet Ψ_ρ satisfies further symmetries on a manifold means, that Ψ_ρ is invariant under the action of a certain subgroup $\mathcal{J}$ of $\mathcal{G}$,

$$\Psi_\rho(j\cdot x) = \Psi_\rho(x)\,\forall j\in\mathcal{J}.$$

i.e. $\tilde{\Psi}_\rho$ is invariant over right cosets $g\mathscr{H}$ as well as over $\mathcal{J}g$.
In Fourier domain this corresponds to

$$\hat{\Psi}_\rho = \pi_{\mathscr{H}} \hat{\Psi}_\rho \pi_{\mathcal{J}},$$

where J is the projection onto the $\mathcal{J}$ invariant subspace in representation space $\mathfrak{L}^{\pi_\alpha}$.

If we ask wavelets Ψ_ρ to satisfy this additional symmetry, also the translates of Ψ_ρ should have this property. This means that

$$\widehat{T_g \Psi_\rho} = \hat{\Psi}_\rho \pi(g^{-1}),$$

but $T_g \Psi_\rho$ is only invariant under the action of $\mathcal{J}$ if

$$\hat{\Psi}_\rho \pi_{\mathcal{J}} \pi(g^{-1}) = \hat{\Psi}_\rho \pi(g^{-1}) \pi_{\mathcal{J}} \qquad \Leftrightarrow \qquad \pi_{\mathcal{J}} \pi(g^{-1}) = \pi(g^{-1}) \pi_{\mathcal{J}}.$$

Consequently, $\pi_{\mathcal{J}}$ is multiple of the unitary matrix, since $\pi(g^{-1})$ is not. Furthermore $\pi_{\mathcal{J}}$ comes from an unitary representation, such that the only possibility is $\pi_{\mathcal{J}} = \pm Id$.
In [BBCK10] the construction is expressed by an Intertwining operator between the usual action of dilation and translation on $\mathcal{G}/\mathscr{H}$ and the action on which is given via the projection $\mathcal{J}\backslash\mathcal{G}/\mathscr{H}$. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{G}/\mathscr{H} & \xrightarrow{D_a} & \mathcal{G}/\mathscr{H} \\ \Big\downarrow \mathbb{P} & & \Big\downarrow \mathbb{P} \\ \mathcal{J}\backslash\mathcal{G}/\mathscr{H} & \xrightarrow[\mathbb{P}(D_a)]{} & \mathcal{J}\backslash\mathcal{G}/\mathscr{H} \end{array}$$

where $\mathbb{P}(D_a) = \mathbb{P} \circ D_a \circ \mathbb{P}^{-1}$. As we see from our investigations in Fourier domain a explicit calculation of $\mathbb{P}(D_a)$ is not possible, such that one has to lift the wavelet to $\mathcal{G}/\mathscr{H}$, apply the dilation operator D_a and project it back to $\mathcal{J}\backslash\mathcal{G}/\mathscr{H}$ in order to obtain the dilation operator $\mathbb{P}(D_a)$ an $\mathcal{J}\backslash\mathcal{G}/\mathscr{H}$.

3.4 The non-compact case

At least we have to mention the critical points for non- compact groups, which is the reason to restrict the general investigations to the compact case. In special cases we will also investigate the construction for non-compact groups. In this thesis we look at the Heisenberg group for that purpose (section 4.4.3).
The spectrum of the Laplacian of non-compact groups becomes continuous. Consequently the expansion in Eigenfunctions of the Laplacian becomes a direct integral

$$f(g) = \int_{\mathbb{R}}^{\oplus} \hat{f}(\lambda) \pi_\lambda(g) \, \mathrm{d}\mu(\lambda). \tag{3.4.1}$$

Furthermore the expansion in matrix coefficients of irreducible representations in the compact case is weighted with the dimension of the representation, which is always finite but can become infinite in the non-compact case. The critical question hence is, if there is a measure on $\hat{\mathcal{G}}$, so that the integral

$$\int_{\hat{\mathcal{G}}} \hat{f}(\lambda)\pi_\lambda \,\mathrm{d}\mu(\lambda), \text{ with } \hat{f}(\lambda) := \int_{\mathcal{G}} \pi_\lambda^*(g) f(g) \,\mathrm{d}g \tag{3.4.2}$$

is well defined for some function space on $\mathcal{G}$. The measure $\mathrm{d}\mu(\lambda)$ is the so called Plancherel-measure. If the Plancherel measure exists, the construction of diffusive wavelets works similar to the compact case.

The existence of the Plancherel-measure, and hence the construction of diffusive wavelets can not be guarantied for general locally compact groups. But since the Plancherel-measure exists for nilpotent Lie groups [CG90], one can extend the investigations of our work to nilpotent Lie groups.

3.4.1 Scale discretized diffusive wavelets

A naturally rising task in wavelets theory is the discretization of continuous wavelets. The full discretization is not our aim, nevertheless we want to make the step into the direction of application and give the discretization of the scaling parameter.

Definition 3.4.1. Let $\{\rho_j,\, j \in \mathbb{Z}\}$ be a strictly decreasing sequence of real numbers, satisfying

$$\lim_{j\to\infty} \rho_j = 0, \qquad \lim_{j\to-\infty} \rho_j = \infty. \tag{3.4.3}$$

Further let $\{\Psi_\rho,\, \rho > 0\}$ be a family of diffusive wavelets.
The family of scale discretized wavelets (a wavelet packet) is defined by

$$\hat{\Psi}_j^P(\pi) = \left(\int_{\rho_{j+1}}^{\rho_j} (\hat{\Psi}_\rho)^2 \alpha(\rho) \,\mathrm{d}\rho \right)^{\frac{1}{2}}, \tag{3.4.4}$$

which is in space domain

$$\Psi_j^P = \sum_{\pi\in\hat{\mathcal{G}}} d_\pi \lambda_\pi \left(\int_{\rho_{j+1}}^{\rho_j} e^{-\rho\lambda_\pi^2} \,\mathrm{d}\rho \right)^{\frac{1}{2}} \operatorname{trace}(\eta(\pi)\pi(g)) \tag{3.4.5}$$

The admissibility condition for scale discretized wavelets the reads now as

$$p_{\rho_m}^{heat}(g) = \sum_{j=-\infty}^{m} (\check{\Psi}_\rho^P * \Psi_\rho^P)(g). \tag{3.4.6}$$

It is easily seen that by our assumptions the admissibility condition (3.4.6) is satisfied:

$$\sum_{j=-\infty}^{m} (\check{\Psi}_\rho * \Psi_\rho)(g) = \sum_{j=-\infty}^{m} \sum_{\pi\in\hat{\mathcal{G}}} d_\pi \lambda_\pi \left(\int_{\rho_{j+1}}^{\rho_j} e^{-\rho\lambda_\pi^2} \,\mathrm{d}\rho \right) \operatorname{trace}(\eta^*(\pi)\eta(\pi)\pi(g)) \tag{3.4.7}$$

$$= \int_{\rho_m}^{\infty} \sum_{\pi\in\hat{\mathcal{G}}} d_\pi \lambda_\pi e^{-\rho\lambda_\pi^2} \operatorname{trace}(\pi(g)) \,\mathrm{d}\rho = \int_{\rho_m}^{\infty} (\check{\Psi}_\rho * \Psi_\rho)(g)\alpha(\rho) \,\mathrm{d}\rho = p_{\rho_m}^{heat}(g). \tag{3.4.8}$$

The wavelet transform is now given naturally by

$$WT^P f(j,g) := \langle f, T_g \Psi_j^P \rangle_{L^2(\mathcal{G})} = (f * \check{\Psi}_j^P)(g). \tag{3.4.9}$$

Theorem 3.4.2. *The wavelet transform WT^P is an isometry between $L^2(\mathcal{G})$ and $L^2(\mathbb{Z}\times\mathcal{G})$*[2]

Proof: A direct calculation yields

$$\|WTf(j,g)\|^2_{L^2(\mathbb{Z}\times\mathcal{G})} = \sum_{j\in\mathbb{Z}} \int_{\mathcal{G}} WTf(j,g)\overline{WTf(j,g)} \,\mathrm{d}g \tag{3.4.10}$$

$$= \sum_{j\in\mathbb{Z}} \int_{\mathcal{G}} (f * \check{\Psi}_{\rho_j}^P)(g)\, \overline{(f * \check{\Psi}_{\rho_j}^P)(g)} \,\mathrm{d}g \tag{3.4.11}$$

$$= \sum_{j\in\mathbb{Z}} \int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\mathcal{G}} f(a) \check{\Psi}_j^P(g^{-1}a)\overline{f(b)} \Psi_{\rho_j}^P(b^{-1}g) \,\mathrm{d}g \,\mathrm{d}a \,\mathrm{d}b \tag{3.4.12}$$

$$= \lim_{t\to 0} \int_{\mathcal{G}} \int_{\mathcal{G}} f(a)\overline{f(b)} (\check{\Psi}_\rho * \Psi_\rho)(ba^{-1})\alpha(\rho) \,\mathrm{d}\rho \,\mathrm{d}a \,\mathrm{d}b \tag{3.4.13}$$

$$= \|f\|^2_{L^2(\mathcal{G})}. \tag{3.4.14}$$

□

Theorem 3.4.3. *The scale discretized Wavelet transform is invertible on its range by the following inversion formula*

$$f(g) = \sum_{j\in\mathbb{Z}} (WT^P f(j,\cdot) * \Psi_j^P(\cdot))(g). \tag{3.4.15}$$

Proof: We just have to use the definition of WT^P and see

$$\lim_{m\to\infty} \sum_{j=-\infty}^{m-1} (WT^P f(j,\cdot) * \Psi_j^P(\cdot))(g) \tag{3.4.16}$$

$$= \lim_{j\to\infty} \int_{\rho_m}^{\infty} (WTf(\rho,\cdot) * \Psi_\rho(\cdot))(g)\alpha(\rho) \,\mathrm{d}\rho, \tag{3.4.17}$$

which coincides with our usual reconstruction formula. □

[2] Where the measure is the tensor product of that of $l^2(\mathbb{Z})$ and that of $L^2(\mathcal{G})$.

A common strategy is to build up a multiresolution analysis corresponding to Ψ^P. For a detailed discussion of multi resolution analysis we refer to [Dau92] or [LMR94]. Here we mention that this can be done also for our scale discretized wavelets. Since we do not aim the complete discretization give only a short description.

Definition 3.4.4. The scaling function, corresponding to Ψ_j^P is defined via its Fourier coefficients by

$$\hat{\Phi}_j^P(\pi) := \begin{cases} I_{d_\pi} & \pi \notin \widehat{\mathcal{G}}_+ \\ \left(\int_{\rho_j}^{\infty} (\hat{\Psi}_\rho(\pi))^2 \alpha(\rho)\, \mathrm{d}\rho\right)^2 & \pi \in \widehat{\mathcal{G}}_+ \end{cases}.$$

For the filtering properties we define further

$$P_\rho(f) := \Phi_\rho^P * \Phi_\rho^P * f \tag{3.4.18}$$

$$S_\rho(f) := \Psi_\rho^P * \Psi^P *_\rho f, \tag{3.4.19}$$

for $f \in L^2(\mathcal{G})$.

By construction we have that P_ρ is an approximation of the identity operator. Defining

$$V_R(\mathcal{G}) = P_R(L^2(\mathcal{G})) = \left\{P_R(f), f \in L^2(\mathcal{G})\right\} \qquad R \in \mathbb{R}_+, \tag{3.4.20}$$

$$W_R(\mathcal{G}) = S_R(L^2(\mathcal{G})) = \left\{S_R(f), f \in L^2(\mathcal{G})\right\} \qquad R \in \mathbb{R}_+, \tag{3.4.21}$$

it is clear that $V_R \subset V_{R'}$ for $R \geq R'$.
From the definition of scaling function and the above property we conclude:

- $L^2(\mathcal{G})\backslash L_0^2(\mathcal{G}) \subset V_{R'}(\mathcal{G}) \subset V_R(\mathcal{G}) \subset L^2(\mathcal{G}),\ 0 < R < R' < \infty$
- $\{\lim\limits_{\rho\to\infty} \Phi_\rho^{(2)} * f |\ f \in L^2(\mathcal{G})\} = L^2(\mathcal{G})\backslash L_0^2(\mathcal{G})$
- $\overline{\{f \in V_R |\ R \in (0,\infty)\}}^{\|\cdot\|_{L^2}} = L^2(\mathcal{G}).$

By definition of Φ_j^P and Ψ_j^P and under consideration of (3.4.18) and (3.4.19) we have

$$V_{\rho_j} = V_{\rho_{j-1}} \oplus W_{\rho_j}. \tag{3.4.22}$$

Chapter 4

Explicit realizations for important groups and manifolds

4.1 The torus

In the case of the torus we are treating an abelian structure, hence all representations are one-dimensional. Let $\mathbb{T}_k$ denote the k-dimensional torus which can be identified with[1]

$$\mathbb{T}_k = \mathbb{R}^k/(2\pi\mathbb{Z})^k. \tag{4.1.1}$$

Hence functions on $\mathbb{T}_k$ are regarded as k-fold periodic functions on $\mathbb{R}^k$. We identify elements on $\mathbb{T}_k$ with equivalence classes of elements on $\mathbb{R}^k$ via projection (4.1.1): $x \sim x \mod \mathbb{Z}^k$, where the modulus is taken componentwise. The character functions corresponding to the one-dimensional representations π_α are given by the standard Laplace operator

$$\chi_\alpha(x) = \frac{1}{(2\pi)^k} e^{i\sum_{l=1}^k \alpha_l x_l}, \tag{4.1.2}$$

where α is a k-dimensional multi index $\alpha \in \mathbb{Z}^k$ and we write

$$|\alpha|_p = \left(\sum_{j=1}^k |\alpha|^p \right)^{\frac{1}{p}} \qquad \text{for } p \in \mathbb{N}.$$

The Laplace operator on $\mathbb{T}^k$ is given by $\Delta_{\mathbb{T}^k} = \sum_{l=1}^k \partial^2_{x_l}$. Hence the corresponding eigenvalues are $-\sum_{l=1}^k \alpha_l^2 = -|\alpha|_2^2$. Consequently, corresponding to (2.5.9) the heat kernel on $\mathbb{T}^k$ has the series expansion

$$e_t^{heat,\mathbb{T}^k} = \frac{1}{(2\pi)^k} \sum_{\alpha\in\mathbb{Z}^k} e^{-|\alpha|_2^2 t} e^{i\alpha\cdot x}, \tag{4.1.3}$$

where we make use of the notation $\alpha \cdot x = \sum_{l=1}^k \alpha_l x_l$.
Also on $\mathbb{T}^k$ we fix the function space $L_0^2(\mathbb{T})$, which we wish to investigate by diffusive wavelets, as the span of all eigenfunctions of $\Delta_{\mathbb{T}^k}$ with non vanishing eigenvalue. We set $\widehat{\mathbb{T}}^k_+$ to be $\widehat{\mathbb{T}}^k \backslash \{\pi_\alpha(\mathbb{T}^k),\ \Delta_{\mathbb{T}^k}\chi_\alpha = 0\}$, i.e. we choose $L_0^2(\mathbb{T}^k)$ again to be the space of L^2-functions with vanishing mean value, i.e. the standard L^2-space without the constant functions. Hence, in what follows we exclude the vanishing multi index $\alpha_l^0 := 0$ for $l = 1, ..., k$.
For the definition of diffusive wavelets on $\mathbb{T}^k$ corresponding to the heat kernel on $\mathbb{T}^k$ we follow (3.1.11) and find the family $\{\Psi_\rho,\ \rho > 0\}$ of diffusive wavelets, defined by

$$\Psi_\rho(x) = \frac{1}{\sqrt{(2\pi)^k \alpha(\rho)}} \sum_{\alpha\in\mathbb{Z}^k\backslash\{0\}} |\alpha|_1 e^{-|\alpha|_2^2\rho/2} e^{i\alpha\cdot x} \tag{4.1.4}$$

where $\alpha(\rho)$ is an appropriate weight function.

[1] It would be enough to assume k linearly independent vectors and factorize $\mathbb{T}^k = \mathbb{R}^k/\Omega^k$, where the lattice Ω^k is given by $\mathbb{Z}\omega_1 + ... + \mathbb{Z}\omega_k$. Our stronger formulation represents no loss of generality.

To determine more explicit formulae we restrict to the case $k = 1$. Since $\mathbb{T}^k = S^1 \times ... \times S^1$ and hence $L^2(\mathbb{T}^k) = \otimes_{l=1}^k L^2(S^1)$, one can construct wavelets on $\mathbb{T}^k$ by tensor products of those on S^1. The heat kernel becomes

$$e_t^{heat,\mathbb{T}^1}(x) = \frac{1}{2\pi} \sum_{\alpha=-\infty}^{\infty} e^{-\alpha^2 t}(e^{ix})^\alpha = 1 + 2\sum_{k=1}^{\infty} e^{-\alpha^2 t}\cos(\alpha x) = \frac{1}{2\pi}\vartheta_3(x/2, e^{-t}) \tag{4.1.5}$$

in terms of Jacobi's ϑ_3-function, cf. [WW96, Chapter XXI].
Therewith the heat kernel on $\mathbb{T}^k$, given by (4.1.3) can be written as

$$e_t^{heat,\mathbb{T}^k} = \frac{1}{(2\pi)^k} \prod_{l=1}^{k} \vartheta_3(x_l/2, e^{-t}). \tag{4.1.6}$$

The corresponding wavelet on T^1 can be written as

$$\Psi_\rho(x) = \frac{1}{\sqrt{2\pi\alpha(\rho)}} \sum_{\alpha=-\infty}^{\infty} |\alpha| e^{-\alpha^2\rho/2} e^{ix\alpha} \tag{4.1.7}$$

$$= \frac{1}{\sqrt{2\pi\alpha(\rho)}} \sum_{\alpha=-\infty}^{\infty} \eta_\rho(\alpha)|\alpha| e^{-\alpha^2\rho/2} e^{ix\alpha} \tag{4.1.8}$$

Here we use the choice $\eta_\rho(\alpha) = -i \text{ sign}\alpha$ and find

$$\Psi'_\rho(x) = \frac{1}{\sqrt{2\pi}} \partial_x \vartheta_3(x/2, e^{-\rho/2}) \tag{4.1.9}$$

The wavelet (4.1.3) now reads as

$$\Psi_\rho(x) = \frac{1}{\sqrt{(2\pi)^k\alpha(\rho)}} \prod_{l=1}^{k} \partial_{x_l}\vartheta_3(x_l/2, e^{-\rho/2}). \tag{4.1.10}$$

The corresponding wavelet transform of a function $f \in L^2[0, 2\pi] \simeq L^2(\mathbb{T})$ with normalisation $\alpha(\rho) = 1$ is

$$\begin{aligned} WT\phi(\rho, \theta) &= \int_0^{2\pi} f(\tau)\partial_\tau\vartheta_3\left(\frac{1}{2}(\tau - \theta), e^{-\rho/2}\right) \, \mathrm{d}\tau \\ &= \int_0^{2\pi} f'(\theta - \tau)\vartheta_3(\frac{1}{2}\tau, e^{-\rho/2}) \, \mathrm{d}\tau \end{aligned} \tag{4.1.11}$$

with inversion formula

$$\phi(\theta) = \int \phi(\tau) \, \mathrm{d}\tau - \int_0^\infty \int_0^{2\pi} WT\phi(\rho, \theta - \tau) \, \partial_\tau\vartheta_3(\frac{1}{2}\tau, e^{-\rho/2}) \, \mathrm{d}\tau \, \mathrm{d}\rho. \tag{4.1.12}$$

The wavelet transform $WT\phi(\rho, \theta)$ describes for *small* ρ the 'high-frequency part' of ϕ localized near the point θ.
We conclude this example with some pictures of the family ψ_ρ on $\mathbb{T}^1$ for different ρ depicted in Figure 5.1.

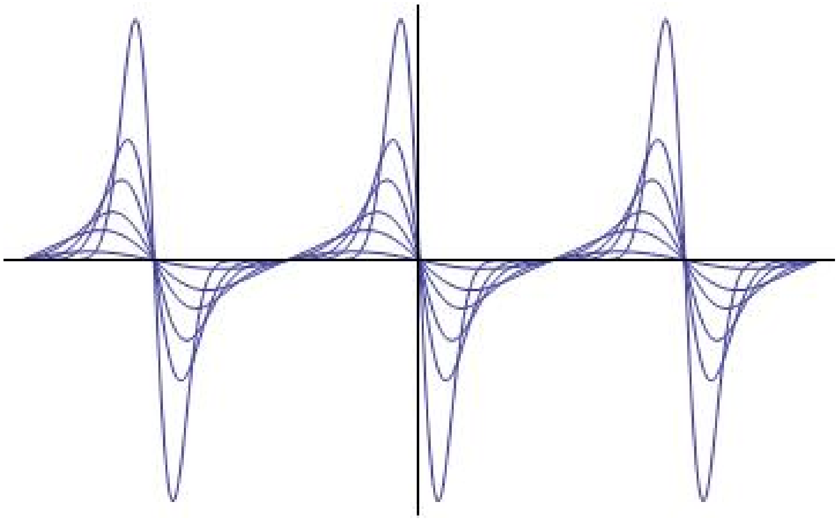

Figure 4.1: The toroidal family $\vartheta_3'(\theta/2, e^{-\rho/2})$ for $-3\pi \leq \theta \leq 3\pi$ and scale parameters $\rho \in \{0.005, 0.01, 0.015, 0.025, 0.04, 0.1\}$.

4.1.1 Second possibility for the torus

In the previous section we regarded the torus as a compact abelian group. A second possibility is to define a projection of functions on $\mathbb{R}^k$ to those on $\mathbb{T}^k$, which can be done by periodization. Basically one identifies k-fold periodic functions on $\mathbb{R}^k$ with those on $\mathbb{T}^k$.

While that we consider the group $(\mathbb{R}^n, +)$ which is commutative. The Fourier theory for $\mathbb{R}^n$ is well known and inversion formulae as well as convolution theorem, which is necessary for our construction of diffusive wavelets, are available in that setting. Consequently in this case we do not need to discuss the non-compactness and formulation of diffusive wavelets on $\mathbb{R}^n$ is straightforward and give no rise of difficulties. We will consider the subgroup $(\mathbb{Z}^n, +)$ and look at the projection from which we already mentioned in (4.1.1). The corresponding projection of functions on $\mathbb{R}^n$ onto $\mathbb{T}^k$ will be called *periodization* of the function and is defined by

$$\mathbb{P}f(x) = f_p(x) := \sum_{\omega \in 2\pi\mathbb{Z}^k} f(x + \omega). \tag{4.1.13}$$

We will see, that the periodization of the heat kernel on $\mathbb{R}^k$ will give that one on $\mathbb{T}^k$. The construction leads to the same wavelets which we obtained in the previous section.

The function f_p is k-fold periodic, but it is not clear if the sum (4.1.13) is well defined.

Lemma 4.1.1. *For $f \in L^p(\mathbb{R}^k)$ with $1 \leq p < \infty$ the projection of f_p belongs to $L^p(\mathbb{T}^k)$.*

Proof: Let $\mathbb{Q}_f$ be the fundamental domain of $\mathbb{T}^k$, i.e. $[-\pi,\pi]^k$,

$$\begin{aligned}\int_{\mathbb{T}^k} |f_p(x)|^p \,\mathrm{d}x &= \int_{Q_f} \left| \sum_{\omega\in 2\pi\mathbb{Z}^k} f(x+\omega) \right|^p \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x.\end{aligned}$$

□

The heat kernel on $\mathbb{R}^n$ is given in the usual way. The eigenfunctions of the Laplacian Δ in $\mathbb{R}^n$ are $e^{i\lambda\cdot x}$ with respect to the eigenvalues $-\lambda^2$ with $\lambda\in\mathbb{R}^n$. The corresponding heat kernel can by given be given by

$$e_t^{heat,\mathbb{R}^n}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^n} e^{-\lambda^2 t} e^{i\lambda\cdot x}\,\mathrm{d}\lambda \tag{4.1.14}$$

$$= \frac{1}{2\pi}\int_0^\infty e^{-\lambda^2 t}\int_{S^{n-1}} e^{i|\lambda|\xi\cdot x}\,\mathrm{d}\xi\,\mathrm{d}|\lambda|, \tag{4.1.15}$$

which is the expansion in a direct integral i.e. the Fourier integral of the heat kernel

$$e_t^{heat,\mathbb{R}^n}(x) = \frac{1}{2(\pi t)^n} e^{-\frac{\|x\|^2}{4t}}. \tag{4.1.16}$$

Obviously, $e_t^{heat,\mathbb{R}^n}$ belongs to $L^p(\mathbb{R}^n)$ and hence the periodization

$$\mathbb{P}_n e_t^{heat,\mathbb{R}^n} \tag{4.1.17}$$

exists, is n-fold periodic and satisfies the heat equation in every point. Thus it represents the heat kernel on the n-dimensional torus.

Since $\mathbb{T}^n$ is a compact manifold, the spectrum of the Laplacian and the expansion in a Fourier series is discrete (see (4.1.3)).

For simplicity we write $\mathbf{m} = (m_1,\dots,m_n)^T \in \mathbb{Z}^n$. For an $f\in L^2(\mathbb{T}^n)$ we have

$$f(\mathbf{x}) = \sum_{m\in\mathbb{Z}^n} \hat{f}(m) e^{i\sum_{j=1}^n m_j x_j} \quad \hat{f}(m) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\mathbf{x}) e^{-i\sum_{j=1}^n m_j x_j}\,\mathrm{d}x_1\ldots\,\mathrm{d}x_n, \tag{4.1.18}$$

where $x = (x_1,\dots,x_n)^T$.

Remark 4.1.2. Let f, $g\in L^2(\mathbb{T}^n)$ with Fourier coefficients f_m, g_m respectively. The convolution theorem can be can be written in the form

$$\begin{aligned}(f*g)(\mathbf{x}) &= \frac{1}{(2\pi)^n}\sum_{m\in\mathbb{Z}^n} \hat{f}(m)\,\hat{g}(m)\int_0^{2\pi}\cdots\int_0^{2\pi} e^{-i\sum_{j=1}^n m_j y_j} e^{-i\sum_{j=1}^n m_j(x_j-y_j)}\,\mathrm{d}y_1\ldots\,\mathrm{d}y_n \\ &= \sum_{m=\in\mathbb{Z}^n} \hat{f}(m)\,\hat{g}(m) e^{i\sum_{j=1}^n m_j x_j}.\end{aligned}$$

Definition 4.1.3. Let $\{h_t,\, t > 0\}$ be a diffusive approximate identity, and let $\hat{h}_t(m)$ be the Fourier coefficients of the kernel functions h_t. Then the corresponding diffusive wavelet is defined by

$$\psi_\rho(\mathbf{x}) := \sum_{m \in \mathbb{Z}^n} \left(-\frac{\mathrm{d}}{\mathrm{d}t} \hat{h}_t(m) \right)^{\frac{1}{2}} e^{i \sum_{j=1}^n m_j x_j}. \tag{4.1.19}$$

Utilizing the convolution theorem (Remark 4.1.2), by our construction we find

$$\int_t^\infty \int_{\mathbb{T}^k} \psi_\rho(\mathbf{y}) \psi_\rho(\mathbf{x} - \mathbf{y}) \,\mathrm{d}\mathbf{y}\,\mathrm{d}\rho = \sum_{m \in \mathbb{Z}^n} \hat{h}_t(m) e^{i \sum_{j=1}^n m_j x_j} = h_t(\mathbf{x}).$$

Since the approximate identity h_t is uniformly bounded in $L^1(\mathbb{T}^n)$, we get

$$\int_t^\infty \int_{T^n} |(\psi_\rho * \psi_\rho)(\mathbf{x})| \,\mathrm{d}\mathbf{x}\,\mathrm{d}\rho = \int_{\mathbb{T}^n} \left| \sum_{m \in \mathbb{Z}^n}^{\infty} \hat{h}_t(m) e^{i \sum_{j=1}^n m_j x_j} \right| \mathrm{d}\mathbf{x} < \infty,$$

independently of t.
As a concrete example of a diffusive wavelet on $\mathbb{T}^n$ we will present diffusive wavelets which corresponds to the heat kernel. The construction is already given in Definition 4.1.3, we just need to calculate.
The Fourier coefficients for of the expansion (4.1.3) of the heat kernel $e_t^{heat,\mathbb{T}^n}$ can be given explicitly

$$\begin{aligned}
\hat{e}_t^{heat,\mathbb{T}^n}(m) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{\omega \in Q} e_t^{heat,\mathbb{R}^n}(x + \omega) e^{-i \sum_{j=1}^n m_j x_j} \,\mathrm{d}x_1 \ldots \mathrm{d}x_n \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{2(\pi t)^{n/2}} e^{-\frac{\|x\|^2}{4t}} e^{-i \sum_{j=1}^n m_j x_j} \,\mathrm{d}x \\
&= \frac{1}{2\pi^n} e^{-\sum_{j=1}^n m_j^2 t}
\end{aligned}$$

Definition 4.1.4. Let $\{\psi_\rho\}$ be a subfamily of $L^2(\mathbb{T}^n)$ with Fourier series expansion (4.1.18). The wavelet we are looking for has the Fourier series expansion:

$$\psi_\rho(x) = \sum_{m \in \mathbb{Z}^n} \frac{1}{\sqrt{2\pi^n}} \sum_{j=1}^n m_j^2 e^{-\sum_{j=1}^n m_j^2 \rho} e^{i \sum_{j=1}^n m_j x_j}$$

In the case of the two dimensional torus $\mathbb{T}^2$ the explicit form of the diffusion wavelet corresponding to the heat kernel is

$$\psi_\rho(x) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} (m^2 + n^2) e^{-m^2 \rho} e^{-n^2 \rho} e^{imx_1} e^{inx_2}.$$

A visualization for different dilation parameters $\rho = 0.3,\, 0.5,\, 0.7,\, 0.9$ and similar translation parameter is given in figure 5.1-4.5. The figures illustrates the localization property of the wavelets for dilation parameter tending to zero.

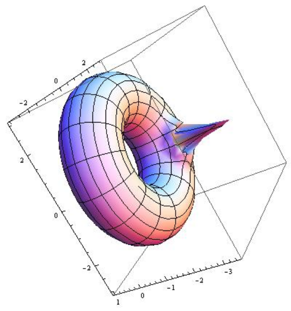

Figure 4.2: Wavelet on T_2, $\rho = 0.3$

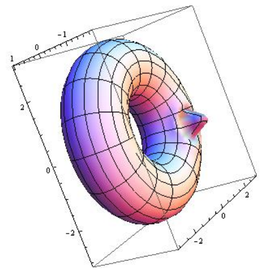

Figure 4.3: Wavelet on T_2, $\rho = 0.5$

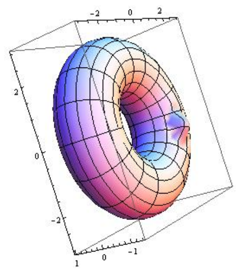

Figure 4.4: Wavelet on T_2, $\rho = 0.7$

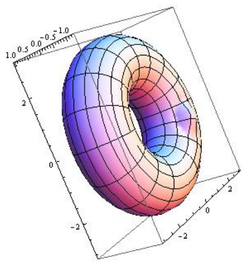

Figure 4.5: Wavelet on T_2, $\rho = 0.9$

4.2 Spherical diffusive wavelets

There is a big interest in wavelets on the sphere. For geosciences this rises from the outer form of the earth and investigations by Freeden for the two-dimensional sphere can be found in [FGS98] where the approach is chosen via special functions. Behind the construction of Freeden one can find the action of a semigroup given by convolution integrals which can be seen as special type the same type of convolution kernel which we use. A group theoretical approach is investigated by Antoine and Vandergheynst in [AV99]. Here the dilation and the translation are given as representation of a group and the sphere is viewed as homogeneous space of the Lorentz group $SO(n,1)$. In this approach the difficulty is to overcome the problem, that there is no irreducible representation of $SO(n,1)$, which is square-integrable.
In this section we investigate the group $SO(n)$ of rotations in $\mathbb{R}^n$. As homogeneous space we are particularly interested in $SO(n+1)/SO(n) \sim S^n$. As we mentioned in Remark 3.2.6, we need that the subgroup $SO(n)$ contains no normal subgroup of $SO(n+1)$. This is obvious, since the conjugate classes of $SO(n)$ in $SO(n+1)$ are stabilizer of different points on S^n.
From Definition 2.2.2 and 2.2.3 we deduce, that a irreducible representation π of $\mathcal{G}$ is of class one with respect to $\mathscr{H}$ if and only if $\mathrm{rank}\pi_{\mathscr{H}} \geq 1$ and $\mathscr{H}$ is a massive subgroup of $\mathcal{G}$ if and only if $\mathrm{rank}\pi_{\mathscr{H}} \leq 1$ for all $\pi \in \widehat{\mathcal{G}}$.

Lemma 4.2.1. *$SO(n)$ is a massive subgroup of $SO(n+1)$ for all $n \in \mathbb{N}$.*

A proof can be found in Vilenkin [VK93, Chapter IX.2.6].
In the case of $SO(3)$ we have a comfortable situation since all irreducible representations are of class one with respect to $SO(2)$. Before we take a closer look at $SO(3)$ we pursue the aim of wavelets on S^n.
An orthonormal system in $L^2(S^n)$ is provided by *spherical harmonics* $\{\mathcal{Y}_k^i,\, k \in \mathbb{N}_0,\, i = 1, ..., d_k(n)\}$, where

$$d_k(n) = (2k+n-1)\frac{(k+n-2)!}{k!(n-1)!} \tag{4.2.1}$$

denotes the dimension of the subspace spanned by spherical harmonics of degree k. These subspaces

$$\mathcal{H}_k := \text{span}\{\mathcal{Y}_k^i,\, i = 1, ..., d_k(n)\} \tag{4.2.2}$$

are the rotation/translation invariant subspaces, hence are the invariant subspaces of the quasi regular representation T. In that way the quasi-regular representation

$$T(g) : f(\xi) \mapsto f(g^{-1} \cdot \xi) \quad f \in L^2(S^n) \tag{4.2.3}$$

decomposes into $d_k(n)$-dimensional irreducible representations $T^k(g)$ in $\mathcal{H}_k$. The corresponding matrix coefficients are the Wigner-polynomials

$$T_{ij}^k(g) = \langle T^k(g)\mathcal{Y}_k^i,\, \mathcal{Y}_k^j\rangle_{L^2(S^n)}. \tag{4.2.4}$$

Consequently, we have

$$\mathcal{Y}_k^i(g^{-1} \cdot \xi) = \sum_{j=1}^{d_k(n)} T_{ij}^k(g)\mathcal{Y}_k^j(\xi). \tag{4.2.5}$$

By Lemma 4.2.1 it follows, that the subspace of zonal functions in $\mathcal{H}_k$ is one-dimensional. It is spanned by Gegenbauer polynomials of order $\lambda = \frac{n-1}{2}$ denoted by $C_k^{(n-1)/2}(\xi_0 \cdot \xi)$ where ξ_0 denotes the base point on $SO(n+1)/SO(n) \sim S^n$. Usually ξ_0 is chosen to be the north pole.
Remark 4.2.2. There is a natural identification of zonal functions on S^n and functions on $[0, \pi]$, since zonal functions depends only on the angle between their argument and the point to which they are zonal. For any function f on $[-1, 1]$ (so that $f(\cos(\cdot))$ is defined on $[0, \pi]$) the function $f(\xi_0 \cdot \eta)$ is zonal with respect to $\xi_0 \in S^n$ as a function of $\eta \in S^n$. It is clear that

$$\int_{S^n} f(\xi_0 \cdot \eta)\, \mathrm{d}\eta = \Omega_{n-1} \int_0^\pi f(\cos\theta) \sin(\theta)^{2\lambda}\, \mathrm{d}\theta \qquad \lambda = \frac{n-1}{2}. \tag{4.2.6}$$

Here and later on we denote the surface measure of S^n by $\Omega_n = |S^n|$. So for zonal wavelets we will make use of the notation $f(\eta) = f(\xi_0 \cdot \eta)$. There is no danger of confusion since the domain S^n or $[0, \pi]$ of f makes clear in which way we look at it.
The Gegenbauer polynomials $C_k^\lambda(\cos(\cdot))$ form an orthogonal system on $L^2([0, \pi])$ with respect to the measure $\sin(\theta)^{2\lambda}\, \mathrm{d}\theta$.

There exists a long list of interesting formulas described for example in [BBP69]. Since the theory of special functions can be described in a natural way by representation theory we discuss some of them from our point of view.

Theorem 4.2.3 (Addition theorem). *For all* $\xi, \eta \in \mathbb{S}^n$ *and* $k \in \mathbb{N}_0$ *we have*

$$\frac{\mathcal{C}_k^{(n-1)/2}(\xi \cdot \eta)}{\mathcal{C}_k^{(n-1)/2}(1)} = \frac{\Omega_n}{d_k(n)} \sum_{i=1}^{d_k(n)} \mathcal{Y}_k^i(\xi)\overline{\mathcal{Y}_k^i(\eta)}. \tag{4.2.7}$$

Proof: It suffices to check that the right hand side is zonal with respect to ξ, which follows from (4.2.5) with g being in the stabilizer of ξ and exchanging orders of summation yields

$$\sum_{i=1}^{d_k(n)} \mathcal{Y}_k^i(\xi)\overline{\mathcal{Y}_k^i(g^{-1} \cdot \eta)} = \sum_{i=1}^{d_k(n)} \mathcal{Y}_k^i(\xi) \overline{\sum_{j=1}^{d_k(n)} T_{ij}^k(g)\mathcal{Y}_k^j(\eta)} \tag{4.2.8}$$

$$= \sum_{j=1}^{d_k(n)} \sum_{i=1}^{d_k(n)} T_{ji}^k(g^{-1})\mathcal{Y}_k^i(\xi)\overline{\mathcal{Y}_k^j(\eta)} = \sum_{j=1}^{d_k(n)} \mathcal{Y}_k^j(g \cdot \xi)\overline{\mathcal{Y}_k^j(\eta)} \tag{4.2.9}$$

Then, in order to find the constants it suffices to choose $\xi = \eta$ and integrate both sides over $\mathbb{S}^n$. □

Since we are interested here in wavelets on S^n, which we obtain by projection from $SO(n+1)$, we have to consider all irreducible representations of $SO(n+1)$ which do not have vanishing matrix coefficients under the projection $\mathbb{P}_{SO(n)}$. These are the representations of class one with respect to $SO(n)$ and we realize them by the usual quasi-regular representations in $L^2(S^n)$. To express the heat kernel on S^n we have to calculate the projection of matrix coefficients $\mathbb{P}_{SO(n)}T_{ij}^k$. Therefore the following lemmas are useful.

Lemma 4.2.4. *Let* $\xi, \xi_0 \in S^n$*, where* ξ_0 *is the base point,* $k \in \mathbb{N}_0$ *and* $i = 1, ..., d_k(n)$*, then*

$$\int_{SO(n)} \mathcal{Y}_k^i(h \cdot \xi) \, \mathrm{d}h = \frac{\mathcal{Y}_k^i(\xi_0)}{C_k^\lambda(1)} C_k^\lambda(\xi_0 \cdot \xi) \qquad \lambda = \frac{n-1}{2}. \tag{4.2.10}$$

With our *zonal averaging method*, (see subsection 3.2.2) every function on S^n can be averaged over orbits of $SO(n)$ on S^n to become a zonal function with respect to the point with stabilizer $SO(n)$.

Proof: Since the result is obviously a zonal (with respect to ξ_0) it it a multiple of C_k^λ. To determine the right constant we only have to choose $\xi = \xi_0$ that gives $\int_{S^n} \mathcal{Y}_k^i(h \cdot \xi_0) \, \mathrm{d}h = \mathcal{Y}_k^i(\xi_0)$. □

Theorem 4.2.5 (Funk-Hecke). *Let* f *be a zonal* L^1*-function. Then for* $i = 1, ..., d_k(n)$ *it is*

$$\int_{S^n} f(\xi \cdot \eta)\mathcal{Y}_k^i(\eta) \, \mathrm{d}\eta = \mathcal{Y}_k^i(\xi) \frac{\Omega_{n-1}}{C_k^\lambda(1)} \int_0^\pi f(\cos(\theta))C_k^\lambda(\cos(\theta)) \sin(\theta)^{2\lambda} \, \mathrm{d}\theta. \tag{4.2.11}$$

Proof: We decompose the integral over S^n into one over $[0,\pi] \times SO(n)$. $SO(n)$ shall be the stabilizer of ξ. Further let $\gamma(t)$ be a geodesic from ξ to $-\xi$ that we parameterize by the angle between ξ and $\gamma(\theta)$, namely $\xi \cdot \gamma(\theta) = \theta \in [0,\pi]$. Since f is zonal with respect to ξ there is constant on $SO(n) \cdot \gamma(\theta)$ such that

$$\begin{aligned}
&\int_0^\pi \int_{SO(n)} f(h \cdot \gamma(\theta)) \mathcal{Y}_k^i(h \cdot \gamma(\theta))\, \mathrm{d}h\, \mathrm{d}\theta \\
&= \Omega_{n-1} \int_0^\pi f(\cos(\theta)) \int_{SO(n)} \mathcal{Y}_k^i(h \cdot \gamma(\theta))\, \mathrm{d}h \sin(\theta)^{2\lambda}\, \mathrm{d}\theta \\
&= \mathcal{Y}_k^i(\xi) \frac{\Omega_{n-1}}{C_k^\lambda(1)} \int_0^\pi f(\cos(\theta)) C_k^\lambda(\cos(\theta)) \sin(\theta)^{2\lambda}\, \mathrm{d}\theta
\end{aligned}$$

□

Therefrom we can deduce the projection of matrix coefficients T_{ij}^k of representations of $SO(n+1)$, which are class one with respect to $SO(n)$. Using the zonal averaging formula (4.2.10) and Funk-Hecke Theorem 4.2.5 we find

$$\begin{aligned}
\mathbb{P}_{SO(n)} T_{ij}^k(g) &= \int_{SO(n)} \langle \mathcal{Y}_k^i(g^{-1}\cdot), \mathcal{Y}_k^j(h\cdot) \rangle_{L^2(S^n)}\, \mathrm{d}h \\
&= \frac{\overline{\mathcal{Y}_k^j(\xi_0)}}{C_k^\lambda(1)} \langle \mathcal{Y}_k^i(g^{-1}\cdot), C_k^\lambda(\xi_0 \cdot) \rangle_{L^2(S^n)} \\
&= \mathcal{Y}_k^i(g \cdot \xi_0) \overline{\mathcal{Y}_k^j(\xi_0)} \frac{\Omega_{n-1}}{(C_k^\lambda(1))^2} \int_0^\pi C_k^\lambda(\cos(\theta)) C_k^\lambda(\cos(\theta)) \sin(\theta)^{2\lambda}\, \mathrm{d}\theta \\
&= \mathcal{Y}_k^i(g \cdot \xi_0) \overline{\mathcal{Y}_k^j(\xi_0)} \frac{\Omega_n}{d_k(n)}.
\end{aligned} \tag{4.2.12}$$

Hereby the normalization relation of Gegenbauer polynomials

$$\int_0^\pi C_l^\lambda(\cos\theta) C_k^\lambda(\cos\theta) (\sin\theta)^{2\lambda}\, \mathrm{d}\theta = (C_k^\lambda(1))^2 \frac{\Omega_n}{d_k(n)\Omega_{n-1}}. \tag{4.2.13}$$

The eigenvalues of the Laplacian on S^n and hence that of the Laplacian on $SO(n+1)$ are $-\lambda_k^2 = -k(k+n-2)$ with respect to the eigenfunctions Y_k^i and T_{ij}^k, respectively.
Now we can formulate the heat kernel on S^n, that is

$$e_t^{heat,S^n}(\xi) = \sum_{k=0}^\infty d_k(n) e^{-\lambda_k^2 t} \frac{C_k^\lambda(\xi_0 \cdot \xi)}{C_k^\lambda(1)} \tag{4.2.14}$$

$$= \sum_{k=0}^\infty \frac{2k+n-1}{n-1} e^{-k(k+n-2)t} C_k^\lambda(\xi_0 \cdot \xi), \tag{4.2.15}$$

whereby $d_k(n) = \binom{n+k}{n} - \binom{n+k-2}{n}$ and $C_k^\lambda(1) = \binom{n+k-2}{k}$.
Consequently, we have shown

Theorem 4.2.6 (zonal diffusive wavelets on S^n)**.** *Let $\alpha(\rho)$ be a weight function on S^n. Then zonal diffusive wavelets on S^n are given by*

$$\Psi_\rho(\xi) = \frac{1}{\sqrt{\alpha(\rho)}} \sum_{k=0}^{\infty} \frac{(2k+n-1)\lambda_k}{n-1} e^{-\lambda_k^2\rho/2} C_k^{\lambda}(\xi_0 \cdot \xi), \tag{4.2.16}$$

where $\lambda_k = \sqrt{k(k+n-2)}$.

Also here it is interesting to discuss diffusive wavelets corresponding to any other diffusive approximate identity. In fact λ_k can be replaced by any other monoton sequence $\lambda_k \to \infty$. This leads to replacing the Laplacian by any other left and right translation invariant operator, having the same eigenspaces span$\{T_{ij}^k,\ i,j = 1,...,d_k\}$.
A second important approximate identity on the sphere comes from the Abel-Poisson kernel. This kernel has eigenvalues $-k$ with respect to the Laplacian. So the corresponding choice $\lambda = \sqrt{k}$ gives the diffusive approximate identity, corresponding to the Abel-Poisson kernel.
The kernel itself is a zonal function, and hence depends only on an angle $\theta \in [-\pi, \pi]$. In Figure 5.1 we find a visualization of the Abel-Poisson kernel. It localizes much faster than the Weierstrass kernel, which is visualized in Figure 4.2

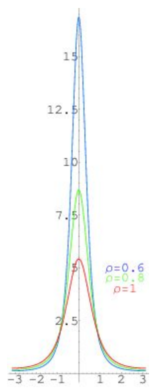

Figure 4.6: Kernel of the Abel-Poisson kernel on the two-dimensional sphere

Whenever the construction of diffusive wavelets is done on a manifold $\mathcal{M}$ that is the surface of another Riemannian manifold $\mathcal{N}$ with metric d, one can use use the Abel-Poisson kernel as fundamental solution of the Laplace equation on $\mathcal{N}$ as approximate diffusive identity on $\mathcal{M}$. The dilation/ diffusive parameter can be chosen as $-\ln(r)$, where r shall be the distance of a point in $\mathcal{N}$ to the boundary that is $\mathcal{M}$.

4.2.1 Nonzonal wavelets

In this section we want apply the construction of nonzonal diffusive wavelets and show how they are obtained on the sphere S^n.

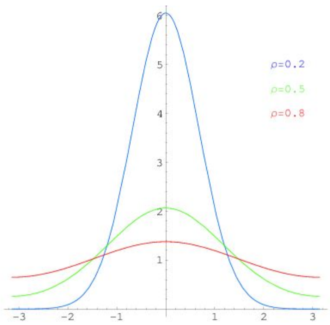

Figure 4.7: Heat kernel on the two-dimensional sphere

From Lemma 4.2.1 it follows, that the bases in $L^2(S^n)$ can be chosen, so that

$$\pi_{SO(n)} = \begin{pmatrix} 1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}. \tag{4.2.17}$$

Because the projection in Fourier domain corresponds to left multiplication of the Fourier coefficients by $\pi_{SO(n)}$ with (4.2.12) and $\mathcal{Y}_k^i(\xi_0) = \sqrt{\frac{d_k(n)}{\Omega_n}}$ for $i = 1$. This gives

$$\mathbb{P}_{SO(n)} T_{i1}^k(g) = \sqrt{\frac{\Omega_n}{d_k(n)}} Y_k^i(g \cdot \xi_0) \tag{4.2.18}$$

Now, nonzonal wavelets can be obtained by multiplying Fourier coefficients from the right by $\eta_\rho(k)$, which is determined by (3.2.24). So let $\omega(k) = (\omega_i(k))_{i=1}^{d_k(n)} \in \mathbb{C}^{d_k(n)}$ be the unit length vector of entries of the first (and the only non-zero) line of $\eta_\rho(k)$.
The one-dimensional subspace of zonal functions in $\mathcal{H}_k$ is spanned by $T_{11}^k(g)$ hence $T_{11}^k(g) = c\, C_k^\lambda(g \cdot \xi_0)$. The constant c can be determined from (4.2.12) and gives $c = \frac{1}{C_k^\lambda(1)}$.
As we have seen in the previous subsection Fourier coefficients of zonal wavelets are of the from

$$\widehat{\psi}_\rho(k) = \lambda_k e^{-\lambda_k^2 \rho/2} \begin{pmatrix} 1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \tag{4.2.19}$$

Consequently a nonzonal wavelet on S^n has the from

$$\psi_\rho(g) = \sum_{k=0}^{\infty} d_k(n) \lambda_k e^{-\lambda_k^2 \rho/2} \operatorname{trace}(\eta_\rho(k) T^k(g)) \tag{4.2.20}$$

$$= \sum_{k=0}^{\infty} d_k(n) \lambda_k e^{-\lambda_k^2 \rho/2} \sqrt{\frac{\Omega_n}{d_k(n)}} \sum_{i=1}^{d_k(n)} \omega_i(k) \mathcal{Y}_k^i(g \cdot \xi_0), \tag{4.2.21}$$

where $T^k(g) := (T_{ij}^k(g))_{i,j}$.
Also in [BCEK09] we calculated nonzonal wavelets with formulae of special functions, which we used here. The context in which we presented the result here posses a completeness in the sense that it follows, that all diffusive spherical wavelets are of the form (4.2.20).

4.3 The case of $SO(3)$ and S^2

Now we are going to apply the diffusive wavelet method to the special case of the two dimensional sphere. All results we need are given in Section 4.2.1. Later we will make use of the results of this section, in order to discuss the behaviour of our diffusive wavelets under Radon transform on $SO(3)$, when we discuss the Radon transform on compact Lie groups.
In the same way as in $L^2(S^n)$, in $L^2(S^2)$ the translation invariant subspaces $\mathcal{H}_k$ are spanned by the spherical harmonics of same degree of homogeneity $\{\mathcal{Y}_k^i,\ i = 1, ..., 2k+1\}$ ([Mül66]). Now we find $d_k(2)$ =dim$\mathcal{H}_k = 2k+1$ and the eigenvalue of the Laplacian corresponding to the subspace $\mathcal{H}_k$ is $-\lambda_k^2 = -k(k+1)$. The corresponding matrix coefficients are known as *Wigner polynomials* ([AW82])

$$T_{ij}^k(g) = \langle T^k(g)\mathcal{Y}_k^i,\ \mathcal{Y}_k^j\rangle_{L^2(S^2)}. \tag{4.3.1}$$

Using the results of the previous section, for the two-dimensional sphere we find the general form of diffusive wavelets corresponding to the heat kernel to be

$$\Psi_\rho(\xi) = \sum_{k=0}^{\infty}(2k+1)\sqrt{k(k+1)}e^{-k(k+1)\rho/2}\sqrt{\frac{4\pi}{2k+1}}\sum_{i=1}^{2k+1}\omega_i(k)\mathcal{Y}_k^i(\xi). \tag{4.3.2}$$

We remark that these wavelets where already constructed in [BCEK09].
Zonal wavelets on S^2 are of the form

$$\Psi_\rho(\xi) = \frac{1}{\sqrt{\alpha(\rho)}}\sum_{k=0}^{\infty}(2k+1)\sqrt{k(k+1)}e^{-k(k+1)\rho/2}C_k^{\frac{1}{2}}(\xi_0\cdot\xi). \tag{4.3.3}$$

These wavelets, which we obtain here as diffusive wavelets where also constructed in [FGS98] using the appropriate formulae of special functions. We would like to point out, that our construction, based on representation theory, is more general.

4.3.1 Diffusive wavelets on $SO(3)$

As we have mentioned in the previous section, for $SO(3)$ all irreducible representations are unitary equivalent to one of the irreducible components of the quasi-regular representation in $L^2(S^2)$, i.e. all irreducible representations are of class one with respect to $SO(2)$. In [BE10] the double covering property of S^3 and $SO(3)$ is used to project wavelets from S^3 to $SO(3)$, which results in deleting the odd Fourier coefficients from a wavelets on S^3, which is discussed in [Ebe08]. The manifold S^3 has the advantage, that it can be equipped with a group structure by identifying it with the set of unit quaternions $\mathbb{H}_u$. The group structure is given by the usual multiplication of quaternions. There are two ways to embed $\mathbb{R}^3$ into the quaternions. One can identify $\mathbb{R}^3$ with the vectorial part of quaternion $\mathbb{H}$ and arbitrary scalar part, or one considers quaternions with vanishing scalar part. A rotation in $\mathbb{R}^3$, hence an action of $SO(3)$ in $\mathbb{R}^3$ is realized by the map $s \mapsto q^{-1}sq$ with $q \in \mathbb{H}_u$. Consequently q and $-q$ give the same

rotation. Even functions, i.e. functions f on S^3 with $f(-x) = f(x)$, can be identified with functions on $SO(3)$. Since S^3 in that manner is a group we can also calculate wavelets on S^3 as group and project it to $S^3/\{\pm 1\}$ in order to construct wavelets on a manifold which is diffeomorphic to $SO(3)$ as a manifold.
Here we want to go the direct way and calculate the characters of $SO(3)$.
The eigenvalues of the Laplacian on $SO(3)$ corresponding to the eigenfunctions in

$$\mathcal{H}_k = \operatorname{span}\{T^k_{ij},\, i,j = 1,...,2k+1\}$$

are the same as the eigenvalues of the eigenfunctions of the corresponding subspace on S^2:

$$\Delta_{\mathcal{G}} T^k_{ij} = (2k+1)T^k_{ij}. \tag{4.3.4}$$

The characters are given by

$$\begin{aligned}
\chi_k(g) = \ &\operatorname{trace}(T^k(g)) = \sum_{k=1}^{d_k} \langle \mathcal{Y}^i_k(g^{-1}(\cdot)), \mathcal{Y}^i_k \rangle \\
&= \sum_{k=1}^{d_k} \int_{\mathbb{S}^2} \mathcal{Y}^i_k(g^{-1}(\xi)\mathcal{Y}^i_k(\xi)\, \mathrm{d}\mu(\xi) \\
&= \frac{(2k+1)}{4\pi} \int_{\mathbb{S}^2} C^{\frac{1}{2}}_k(g^{-1}(\xi)\cdot\xi)\, \mathrm{d}\mu(\xi).
\end{aligned}$$

To calculate $\chi(g)$ for $SO(3)$ we use polar coordinates, which identify $\eta \in \mathbb{S}^2$ with the values of its Euler angles $\eta = (\theta_1, \theta_2) \in [0, 2\pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. Rotations $g \in SO(3)$ are parameterized by a rotational axis and a rotational angle. Since the characters on $SO(3)$ are independent of the rotational axis, one can choose the axis, which contains the north pole. Hence $g^{-1} \cdot \eta = (\theta_1 + \gamma, \theta_2)$. With $\cos(g^{-1}\cdot\eta, \eta) = \sin^2\theta_2 + \cos\gamma\cos^2\theta_2$ and (see [GR65]):

$$C^\lambda_k(x) = \frac{\Gamma(2\lambda + k)}{\Gamma(k+1)\Gamma(2\lambda)}\ {}_2F_1\left(2\lambda + k, -k; \lambda + \frac{1}{2}; \frac{1-x}{2}\right),$$

hence for $\lambda = \frac{1}{2}$ we have

$$C^{\frac{1}{2}}_k(x) = {}_2F_1\left(1 + k, -k; 1; \frac{1-x}{2}\right),$$

and therewith

$$\begin{aligned}
\chi_k(g) &= \frac{(2k+1)}{2\pi} \int_0^{\frac{\pi}{2}} C^{\frac{1}{2}}_k(\sin^2\theta_2 + \cos\gamma\cos^2\theta_2)\cos\theta_2\, \mathrm{d}\theta_2 \\
&= \frac{(2k+1)}{2\pi} \int_0^1 C^{\frac{1}{2}}_k(x^2 + \cos\gamma(1 - x^2))\, \mathrm{d}x, \qquad \text{with } x = \sin\theta_2 \\
&= \frac{(2k+1)}{2\pi} \int_0^1 {}_2F_1(k+1, -k; 1; \frac{1}{2}(1-\cos\gamma)\underbrace{(1-x^2)}_{=y})\, \mathrm{d}x \\
&= \frac{(2k+1)}{2\pi} \int_0^1 {}_2F_1(k+1, -k; 1; \underbrace{\frac{1}{2}(1-\cos\gamma)}_{=\sin^2(\frac{\gamma}{2})} y)\frac{1}{2}(1-y)^{-\frac{1}{2}}\, \mathrm{d}y.
\end{aligned}$$

Since

$$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}\ {}_2F_1(a_1,a_2;\nu;ax)\,\mathrm{d}x = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\ {}_2F_1(a_1,a_2;\mu+\nu;a)$$

with $\nu = 1$ and $\mu = \frac{1}{2}$ we get

$$\chi_k(g) = \frac{(2k+1)}{4\pi}\frac{\Gamma(\frac{3}{2})\Gamma(1)}{\Gamma(\frac{5}{2})}\ {}_2F_1\left(k+1,-k;\frac{3}{2};\sin^2\left(\frac{\gamma}{2}\right)\right)$$

and further (also from [GR65]) by

$$C_{2k}^{\lambda}(t) = \frac{(-1)^k}{(\lambda+k)}\binom{k+1}{\lambda}^{-1}\ {}_2F_1(k+\lambda,-k;\frac{1}{2};t^2)|$$

we obtain

$$\chi_k(g) = \frac{(2k+1)}{4\pi}\binom{2k+1}{2k}^{-1}C_{2k}^1\left(\sin\left(\frac{\gamma}{2}\right)\right) = \frac{1}{4\pi}C_{2k}^1\left(\sin\left(\frac{\gamma}{2}\right)\right).$$

Hence the heat kernel on $SO(3)$ is given by

$$p^{SO(3)}(t,g) = \frac{1}{4\pi}\sum_{k=0}^{\infty}(2k+1)e^{-k(k+1)t}C_{2k}^1\left(\sin\left(\frac{\gamma(g)}{2}\right)\right),$$

where $\gamma(g)$ denotes the angle of g, which is parameterized by a rotational axis and a rotational angle. It holds

$$\gamma(g) = \arccos\left(\frac{\mathrm{trace}(g)-1}{2}\right).$$

For more details about the discussion of such parameterizations can be found in [Hie07].
By our construction a family of wavelets on $SO(3)$ corresponding to the heat kernel is given by

$$\Psi_\rho(g) = \frac{1}{\sqrt{\alpha(\rho)}}\frac{1}{4\pi}\sum_{k=0}^{\infty}(2k+1)\sqrt{k(k+1)}e^{-\frac{k(k+1)}{2}\rho}C_{2k}^1\left(\sin\left(\frac{\gamma(g)}{2}\right)\right). \tag{4.3.5}$$

We will come back to this in the discussion of the Radon transform of wavelets on $SO(3)$.
Before we generalize the wavelets to Clifford-valued wavelets on the spin group and on the sphere which involves further constructions on representation theory, we investigate the construction of diffusive wavelets on a non-compact Lie group, the Heisenberg group. With the knowledge of the previous chapters this can be done by a little supplement which is devoted to the harmonic analysis on the Heisenberg group which is connected to the continuous spectrum of the Laplacian and the Sub-Riemanian structure.

4.4 Introduction to Heisenberg group H_n

There are many different branches in mathematics where the Heisenberg group plays an important role. Consequently it is one of most investigated groups. It is also a good example for a noncompact group. That is why after a short introduction we want to show that our notion of diffusive wavelets can be extended to the Heisenberg group.
Central points of the facts about the Heisenberg group are collected from [Grö01, Str91, Tha98, BFI11].
In Quantum mechanics the Heisenberg group is generated by position and momentum operators. The Fourier transform $\mathcal{F}f = \hat{f}$ in $\mathbb{R}^n$ is defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-i\,x\cdot\xi} f(x)\,\mathrm{d}x.$$

The position operator is given as a translation operator in space domain

$$T(y)f(x) = f(x+y),$$

while the translation in frequency domain, which corresponds to the momentum operator, is defined by

$$e(\eta)f(x) = \mathcal{F}^{-1}((\mathcal{F}f)(\xi+\eta)) = \mathcal{F}^{-1}T(\eta)\mathcal{F}f(x).$$

It is easy to see that is a modulation in space domain

$$e(\eta)f(x) = e^{ix\cdot\eta}f(x).$$

By Borel functional calculus (see [KR97] for an introduction and Appendix A.3 for a brief definition) the corresponding generating operators are $Q_j = x_j$, with $Q\cdot x = \sum_{j=1}^n x_j Q_j$ and $D_j = -i\frac{\partial}{\partial x_j}$ with $D\cdot y = \sum_{j=1}^n y_j D_j$ with , i.e.

$$e(x) = e^{iQ\cdot x} \qquad\qquad T(y) = e^{iD\cdot y}.$$

The Heisenberg uncertainty principle tells us, that momentum and position operators of the same index do not commute. The physical meaning of this fact is that the location in space and the momentum of some particle cannot be determined simultaneously. From the mathematical point of view we have

$$[Q_j, D_j] = i\,Id, \qquad\qquad j = 1,...,n. \tag{4.4.1}$$

where Id denotes the identity operator.
Since the Q_j and D_j span the Heisenberg algebra, the Lie group we get in the usual way by exponentiation. The commutator relation 4.4.1 implies that H_n is a nilpotent Lie group. Every nilpotent Lie group is completely determined by its commutation relation of its Lie

algebra (For a brief description see Appendix A.2). A discussion of nilpotent Lie groups and their geometry can be found, e.g. in [BFI11].
From the relations (4.4.1) the group law of H_n follows as

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(yx' - xy')). \tag{4.4.2}$$

As usual there is a corresponding matrix group, which is obtained by a faithful representation m, which is here given by the subgroup of matrices of the form

$$m(x, y, t) = \begin{pmatrix} 1 & x^T & y^T & t/2 \\ 0, & Id_{2n} \times 2n & & -y \\ & & & x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case the Lie Algebra can be easily represented by the set of matrices of the form

$$m(x, y, t) = \begin{pmatrix} 0 & x^T & y^T & t/2 \\ 0, & \mathbf{O}_{2n} \times 2n & & -y \\ & & & x \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the exponential mapping is the usual matrix exponential.
We want to go back to the abstract view, identifying H_n with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, equipped with the group law (4.4.2).

4.4.1 Lie algebra of H_n

We are interested in the analysis of the Heisenberg group. Especially the heat equation plays an important role in our purpose to investigate diffusive wavelets. Therefore we need a better understanding of the Lie Algebra as the set of left invariant differential operators rather than looking at it as the matrix Lie algebra of the homomorphic matrix subgroup.
Later we will emphasize the special form of the Lie algebra, which admits a sub-Riemannian structure on H_n.
A basis in the linear space of first order differential operators is given by

$$X_j = \frac{\partial}{\partial x_j}, \qquad Y_j = \frac{\partial}{\partial y_j}, \qquad T = \frac{\partial}{\partial t},$$

such that a vector field/ first order differential operator V can be written in coordinate form

$$V = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} \right) + c \frac{\partial}{\partial t}.$$

V is a left-invariant vector field, if it commutes with the left translation L_g,i.e. $L_g V = V L_g$, this means that the following diagram commutes:

$$\begin{array}{ccc} T_a\mathcal{G} & \xrightarrow{dL_g} & T_{ga}\mathcal{G} \\ {\scriptstyle V(a)}\uparrow & & \uparrow{\scriptstyle V(ga)} \\ \mathcal{G} & \xrightarrow{L_g} & \mathcal{G} \end{array}$$

A left-invariant vector field is uniquely determined if it is known at one point a on the group. One can choose the unit element e to be this point. So for every tangential vector at the unit there is a corresponding left-invariant vector field and vice versa. $V(g) = \mathrm{d}L_g X(e)$, where $\mathrm{d}L_g$ is the differential of L_g.

The left-invariant vector fields V_{X_k} shall have the tangential vector X_k at the unit element, which has coordinates $a_j = \delta_{jk}$, $b_j = 0$ for $j = 1, ..., n$ and $c = 0$. Y_k has only the coordinate $b_k = 1$ to be non-zero at $(0, 0, 0)$ and for T it is $c = 1$.

Let f be a smooth function on H_n and let $\gamma = (x_\gamma(t), y_\gamma(t), s_\gamma(t))$ be a curve in H_n with $(\gamma_{X_k}(0)) = (0, 0, 0)$ and $\dot\gamma_{X_k} = X_k$ such that by left-invariance of X_k we know

$$(V_{X_k} f)(h) = \frac{\mathrm{d}}{\mathrm{d}t} f(h \cdot \gamma(s))\bigg|_{s=0}.$$

Let $h = (x, y, t)$, such that

$$(h \cdot \gamma(s)) = (x + x_\gamma(s), y + y_\gamma(s), t + t_\gamma(s) + 2(yx_\gamma(s) - xy_\gamma(s))),$$

hence

$$V_{X_k} = \frac{\partial}{\partial x_k} + 2y_k \frac{\partial}{\partial t}.$$

Analogously one obtains

$$V_{Y_k} = \frac{\partial}{\partial y_k} - 2x_k \frac{\partial}{\partial t} \qquad \text{and} \qquad V_T = \frac{\partial}{\partial t}.$$

The commutation relations are now

$$[V_{X_k}, V_{Y_k}] = 4T,$$

whereby all other commutators vanish. Therefore, $\{V_{X_k}, V_{Y_k}, T, k = 1, ..., n\}$ form a basis of the Lie algebra of H_n.

4.4.2 Sub-Riemannian structure on H_n

A manifold posses a sub-Riemannian structure, if its tangent bundle contains a subbundle H, such that all linear combinations of H are in H and a finite application of the Lie Bracket to elements from H generates the whole tangent space.

The Heisenberg group possesses a sub-Riemannian structure and it is convenient to look at the geometry of the group by considering this natural structure.

Therefore, also the Laplacian shall be considered as the sub-Laplacian coming from the sub-Riemannian structure.

The sub-bundle $B := \{V_{X_k}, Y_{Y_k}, k = 1, ..., n\}$ of the tangent bundle is bracket generating, i.e. if we add the vector fields, obtained by application of the Lie-bracket we get the whole tangent bundle. Therefore B is called non-holonomic and defines a sub-Riemannian structure of step two on H_n. Hereby the step two means, that one has to add Vector fields, which are obtained by applying the Lie bracket one time.
Therefore, the sub-Laplacian (sub-Riemannian Laplacian) results in

$$\Delta_{sub} := \sum_{j=1}^{2n} V_{X_k}^2 + V_{Y_k}^2.$$

We do not discuss here the physical meaning of sub-Rimannian structures. Further discussions about sub-Riemannian structure can be found in [BFI11]. Here we will only mention the following important theorem

Theorem 4.4.1 (Chow's Theorem). *Any two points on a sub-Riemannian Manifold can be joined by a piece-wise smooth horizontal curve.*

A curve $\gamma : \mathbb{R} \supset I \ni t \mapsto \gamma(t) \in H_n$ is horizontal if $\dot{\gamma} \in B$ for all $t \in I$.

4.4.3 Harmonic analysis on H_n

Since H_n is a noncompact group it is not clear that constructions we have for compact groups due to Peter-Weyl theorem can be obtained on H_n. Fortunately, there are similar tools like Stone-von-Neumann theorem which creates hope to achieve some results for H_n. Thanks to the existence of a Plancherel measure the Fourier transform can be developed in a similar way, where of course now the sum over irreducible representation becomes an integral, since the spectrum of Laplacian now is continuous.
Where in the compact case every irreducible component is multiplied by the dimension of the corresponding representation it is not clear what happens in the case of infinite dimensional representations for noncompact groups. But this is precisely the question for a Plancherel measure, which ensures that the integral over all irreducible representations exists.

Schrödinger representation and Fourier transform

Since for the Heisenberg group we need a replacement of the Peter-Weyl theorem, we need to take a look at all irreducible representations. A classification of all irreducible representations is given by

Theorem 4.4.2 (Stone-von-Neumann). *Every irreducible representation of H_n is unitary equivalent to one and only one of the representations*

- $\pi_\lambda(x, y, t)\varphi(\xi) = e^{i\lambda t/4} e^{i\lambda(x\xi + \frac{1}{2}xy)} \varphi(\xi + y)$, $\lambda \in \mathbb{R}\backslash\{0\}$ *in* $L^2(\mathbb{R})$
- $\chi_{(\xi,\eta)}(x, y, t) = e^{i(x\xi + y\eta)}$ *in* $\mathbb{C}$.

Since the representations χ are one-dimensional and of Plancherel measure zero, for the Fourier transform we have to consider only the infinite dimensional representations π_λ. For our purposes it is enough that we treat here only the Schrödinger representations π_λ, but at least we have to mention that there is a further possibility to look at representations of H_n, the so-called Bargmann-Fock representations [Fol95].
The left- and right translation invariant measure on H_n, with respect to the underlying manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is given by usual Lebesgue-measure $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t$.

Definition 4.4.3. The Fourier transform of a function $f \in L^1(H_n)$ is given by the operator valued (Bochner) integral

$$\hat{f}(\lambda) = \int_{H_n} f(x,y,t)\pi_\lambda^*(x,y,t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t.$$

Like in the compact case, the convolution theorem holds:

$$\widehat{(f * g)}(\lambda) = \hat{g}(\lambda)\hat{f}(\lambda).$$

Analogously to the well-known Fourier transform in the Euclidean setting one finds for the Heisenberg group the following results (for proofs we refer to [Str91] and [Tha98]).
Because all π_λ are unitary it follows, that

$$|\langle \hat{f}(\lambda)\phi, \psi\rangle_{L^2(\mathbb{R}^n)} \leq \|\phi\|_2 \|\psi\|_2 \|f\|_1,$$

hence the Fourier transform $\hat{f}(\lambda)$ of $f \in L^1(H_n)$ gives for every $\lambda \in \mathbb{R}\backslash\{0\}$ a bounded operator on $L^2(\mathbb{R}^n)$.

Definition 4.4.4. A linear operator A on a separable Hilbert space H is a *p-schatten operator*, if its p-schatten norm

$$\|A\|_p := (\ \mathrm{trace}|A|^p)^{\frac{1}{p}} \tag{4.4.3}$$

is finite. Hereby, we have $|A| := \sqrt{A^*A}$ in the sense of functional calculus. For the special case $p = 1$ the operator A is a *trace class operator*. For the case $p = 2$, i.e. if the Hilbert-Schmidt norm $\|A\|_{HS} :=\ \mathrm{trace}(A^*A)$ is finite, A is a *Hilbert-Schmidt operator*. The Hilbert space $HS(L^2(\mathbb{R}))$ contains all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$, the inner product of $A, B \in HS(L^2(\mathbb{R}^n))$ is given by $\mathrm{trace}(A^*B) = \sum_\alpha \langle Ae_\alpha, Be_\alpha\rangle_{L^2(\mathbb{R}^n)}$, where $\{e_\alpha\}$ is a basis in $L^2(\mathbb{R}^n)$.

In the following $L^q(\mathbb{R}\backslash\{0\}, HS(L^2(\mathbb{R})),\ \mathrm{d}\mu(\lambda))$ will denote the space of mappings $\mathbb{R}\backslash\{0\} \to HS(L^2(\mathbb{R}))$, so that $\|f\|^p := \int_{\mathbb{R}\backslash\{0\}} \|m(\lambda)\|_{HS(L^2(\mathbb{R}))}\ \mathrm{d}\mu(\lambda) < \infty$ where the Plancherel measure is given by $\mathrm{d}\mu(\lambda) = (2\pi)^{-(n+1)}|\lambda|^n\, \mathrm{d}\lambda$.

The necessity of the existence of the Plancherel measure:

$$\langle f, g\rangle_{L^2(H_n)} = \int_{\mathbb{R}\backslash\{0\}} \operatorname{trace}(\hat{f}(\lambda)\hat{g}^*(\lambda))\, \mathrm{d}\mu(\lambda). \tag{4.4.4}$$

appears in the following theorem.

Theorem 4.4.5. *Let $\mathfrak{B}(L^2(\mathbb{R})$ denote the set of bounded operators on $L^2(\mathbb{R})$. Fourier transform is continuous from $L^1(H_n)$ into $L^\infty(\mathbb{R}\backslash\{0\}, \mathfrak{B}(L^2(\mathbb{R})), \mathrm{d}\mu(\lambda))$. If $f \in L^2(H_n)$, then $\hat{f}(\lambda)$ gives a Hilbert-Schmidt operator and*

$$\|\hat{f}\|_{L^2(\mathbb{R}\backslash\{0\}, HS(L^2(\mathbb{R})),\, \mathrm{d}\mu(\lambda))} = \|f\|_{L^2(H_n)}.$$

By some interpolation argument one can obtain:

Theorem 4.4.6. *Let $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the Fourier transform maps $L^p(H_n)$ continuously into $L^q(\mathbb{R}\backslash\{0\}, p-\text{Schatten}(L^2(\mathbb{R})), \mathrm{d}\mu(\lambda))$, where $p-\text{Schatten}(L^2(\mathbb{R}))$ denotes all linear operators on mapping $L^2(\mathbb{R})$ onto itself and having finite $p-$schatten norm.*

Theorem 4.4.7. *The Fourier transform of $f \in L^2(H_n)$ is invertible by*

$$f(x, y, t) = \int_{-\infty}^{\infty} \operatorname{trace}(\hat{f}(\lambda)\pi_\lambda(x, y, t))\, \mathrm{d}\mu(\lambda) \tag{4.4.5}$$

Here some difficulties arise. For instance it is not clear whether $\hat{f}(\lambda)\pi_\lambda(x, y, t)$ is of trace class. To verify (4.4.5) one uses that $L^1(\mathbb{R}, \text{trace class})$ is dense in $L^2(\mathbb{R}, HS)$. This will be shown by the continuity of the Fourier transform and the well-known fact that the space of test functions $S(H_n)$ is dense in $L^2(H_n)$.
For a test function f, the Fourier transform $\widehat{f}(\lambda)$ for $\lambda \neq 0$ is a Hilbert-Schmidt operator.

$$\widehat{f}(\lambda)\varphi(\xi) = \int_{H_n} f(x, y, t)\pi_\lambda^*(x, y, t)\varphi(\xi)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \tag{4.4.6}$$

$$= \int_{H_n} f(x, y, t)e^{i\lambda t/4}e^{i\lambda(x\xi - \frac{1}{2}xy)}\varphi(\xi - y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t, \tag{4.4.7}$$

substituting now $s = \xi - y$ (consequently $y = \xi - s$)

$$= \int_{H_n} f(x, \xi - s, t)e^{i\lambda t/4}e^{i\lambda(\frac{1}{2}x(\xi - s))}\varphi(s)\, \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t \tag{4.4.8}$$

$$= (\varphi *_{\mathbb{R}^n} K_\lambda)(\xi), \tag{4.4.9}$$

where

$$K_\lambda(y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}} f(x, y, t)e^{i\lambda t/4}e^{i\lambda(\frac{1}{2}xy)}\, \mathrm{d}x\, \mathrm{d}t. \tag{4.4.10}$$

Consequently, the operator which corresponds to the Fourier coefficient $\hat{f}(\lambda)$ is nothing but the convolution operator with kernel K_λ.

To calculate the norm $\|K_\lambda\|_{HS}$ we use the usual basis of trigonometric polynomials $\{e^{i\omega\cdot x},\ \omega\in\mathbb{R}^n\}$ in $L^2(\mathbb{R}^n)$ and the Plancherel theorem in $\mathbb{R}^n$.

$$\begin{aligned}
\|K_\lambda\|_{HS}^2 &= \int_{\mathbb{R}^n}\langle K_\lambda^* K_\lambda e^{i\omega}, e^{i\omega}\rangle_{L^2(\mathbb{R}^n)}\ \mathrm{d}\omega = \int_{\mathbb{R}^n}\langle K_\lambda e^{i\omega}, K_\lambda e^{i\omega}\rangle_{L^2(\mathbb{R}^n)}\ \mathrm{d}\omega \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_\lambda(\xi-s)e^{i\omega s}\ \mathrm{d}s\ \overline{\int_{\mathbb{R}^n} K_\lambda(\xi-z)e^{i\omega z}\ \mathrm{d}z}\ \mathrm{d}\xi\ \mathrm{d}\omega \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\widehat{T_s K_\lambda}\omega(\omega)|^2\ \mathrm{d}s\ \mathrm{d}\omega \\
&= \int_{\mathbb{R}^n} |T_s K_\lambda(\omega)|^2_{L^2(\mathbb{R}^n)}\ \mathrm{d}s \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K_\lambda(\xi-s)|^2\ \mathrm{d}\xi\ \mathrm{d}s
\end{aligned}$$

In our special case, this is

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}} f(x,\xi-s,t)e^{i\lambda t/4}e^{i\lambda(\frac{1}{2}x(\xi-s))}\ \mathrm{d}x\ \mathrm{d}t\right|^2\ \mathrm{d}\xi\ \mathrm{d}s \tag{4.4.11}$$

For the detailed calculation we use the following notation for partial Fourier transform. $\mathcal{F}_{x\to\xi}(f(...,\cdot,x,\cdot,...))(\xi)$ stands for the partial Fourier transform of a function f of many variables, where the Fourier transform is taken with respect to x only.
Substituting $2o=\xi-s$ in (4.4.11) yields

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}} f(x,2o,t)e^{i\lambda t/4}e^{i\lambda(x(\xi-o))}\ \mathrm{d}x\ \mathrm{d}t\right|^2\ \mathrm{d}\xi(-2)\ \mathrm{d}o \tag{4.4.12}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\mathcal{F}_{x\to\lambda(\xi-o)}\mathcal{F}_{t\to\lambda}(f((x,2o,t)))(\lambda(o-\xi),-\frac{\lambda}{4})\right|^2\ \mathrm{d}\xi(-2)\ \mathrm{d}o \tag{4.4.13}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(\lambda(\xi-o),2o,-\frac{\lambda}{4})\right|^2\ (-2)\ \mathrm{d}\xi\ \mathrm{d}o, \tag{4.4.14}$$

by using again Plancherel theorem in $\mathbb{R}^n$. Now substituting $(o-\xi)=z$ leads to

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(\lambda z,2(z+\xi),-\frac{\lambda}{4})\right|^2\ 2\ \mathrm{d}\xi\ \mathrm{d}z \tag{4.4.15}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(\lambda z,\xi,-\frac{\lambda}{4})\right|^2\ \mathrm{d}\xi\ \mathrm{d}z \tag{4.4.16}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f(z,\xi,-\frac{\lambda}{4})\right|^2\ |\lambda|^{-n}\ \mathrm{d}\xi\ \mathrm{d}z. \tag{4.4.17}$$

Integration with respect to λ now yields

$$\int_{H_n} |f(x,y,t)|^2\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}t = \frac{1}{4}\int_{\mathbb{R}} \|\widehat{f}(\lambda)\|_{HS}|\lambda|^n\ \mathrm{d}\lambda. \tag{4.4.18}$$

This gives the Plancherel measure.

4.4.4 Spectral decomposition and heat kernel on H_n

The spectral decomposition of a function with respect to the irreducible components is given by the convolution with the characters. The component, corresponding to the representation π is obtained by convolution with the character of π. We aim to obtain the spectral decomposition of the heat kernel (of the heat equation, which involves the sub-Laplacian) for our purpose to develop diffusive wavelets on H_n.
While the Laplacian coming from the Casimir element involves a complete basis of the Lie Algebra (c.f. (2.3.25)), the sub-Laplacian involves only those operators which corresponds to vector fields belonging to the sub-Riemanian structure.
The eigen-subspaces of the Laplacian decomposes into smaller eigen-spaces of the sub-Laplacian, since the sub-Laplacian is only left-invariant but not right-invariant.
For our purpose we calculate $(\pi_\lambda)_*(\Delta_{sub})$. We start with the calculation of

Lemma 4.4.8. *For the vector fields spanning the sub-Riemannian structure we have*

$$(\pi_\lambda)_*(V_{X_k}) = -i\lambda x_k$$
$$(\pi_\lambda)_*(V_{Y_k}) = \frac{\partial}{\partial x_k}.$$

This reminds us of the beginning of this chapter where the construction of the Heisenberg group is motivated by these operators. Here we have another example, where a mathematical object of a special group can be observed first only via its representations in other applications.

Proof: Let $\{e_k,\, k = 1, ..., n\}$ be the canonical basis in $\mathbb{R}^n$. We have

$$(\pi_\lambda)_*(V_{X_k}) = \frac{\mathrm{d}}{\mathrm{d}t}\, \pi_\lambda \left(\exp\left(t\frac{\partial}{\partial x_k} + 2ty_k \frac{\partial}{\partial s} \right) \right) \varphi(\xi) \Big|_{t=0}.$$

With the Baker-Campbell-Hausdorff formula and $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x_k}\right] = 0$ we obtain

$$\begin{aligned}(\pi_\lambda)_*(V_{X_k}) &= \frac{\mathrm{d}}{\mathrm{d}t}\, \pi_\lambda \left(te_k, 0, 2ty_k s\right) \varphi(\xi)|_{t=0}\,, \qquad \text{while } y_k = 0\\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{i\lambda t \xi_k} \varphi(\xi)\right)\Big|_{t=0}\\ &= i\lambda \xi_k \varphi(\xi).\end{aligned}$$

Analogously, for V_{Y_k} we get

$$\begin{aligned}(\pi_\lambda)_*(V_{Y_k}) &= \frac{\mathrm{d}}{\mathrm{d}t}\, \pi_\lambda \left(exp\left(t\frac{\partial}{\partial y_k} - 2tx_k \frac{\partial}{\partial s} \right) \right) \varphi(\xi) \Big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\, \pi_\lambda \left(0, te_k, -2tx_k s\right) \varphi(\xi)|_{t=0}\,, \qquad \text{while } x_k = 0\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\, \varphi(\xi + te_k)|_{t=0}\\ &= \frac{\partial}{\partial \xi_k} \varphi(\xi).\end{aligned}$$

□

Now it is easily seen, that the sub-Laplacian Δ_{sub} on H_n under transfer by the representation π_λ gives the harmonic oscillator/ Hermite operator on $L^2(\mathbb{R}^n)$.

Corollary 4.4.9. For the sub-Laplacian we have

$$\begin{aligned}(\pi_\lambda)_*(\Delta_{sub}) = (\pi_\lambda)_*\left(\sum_{j=1}^n V_{X_j}^2 + V_{Y_j}^2\right) &= \sum_{j=1}^n \left(\frac{\partial}{\partial \xi_j}\right)^2 - \lambda^2|\xi_j|^2 \\ &= \Delta - \lambda^2|x|^2. \end{aligned} \tag{4.4.19}$$

Since the Hermite functions

$$\phi_\alpha^\lambda = |\lambda|^{\frac{n}{4}} \Phi_\alpha(|\lambda|^{\frac{1}{2}} x)$$

are eigenfunctions of (4.4.19) with respect to the eigenvalues

$$(2|\alpha| + n)|\lambda|. \tag{4.4.20}$$

and $\{\Phi_\alpha,\ \alpha \in \mathbb{Z}^k,\ k \in \mathbb{N}\}$ form a basis in $L^2(\mathbb{R}^n)$, the eigenfunctions of Δ_{sub} on H_n are of the form

$$\langle \pi_\lambda(x, y, t)\phi_\alpha^\lambda, \phi_\beta^\lambda\rangle_{L^2(\mathbb{R}^n)}.$$

A calculation shows that this equals

$$\langle \pi_\lambda(x, y, t)\phi_\alpha^\lambda, \phi_\beta^\lambda\rangle_{L^2(\mathbb{R}^n)} = \begin{cases} (2\pi)^{\frac{n}{2}} e^{i\lambda t} \Phi_{\alpha,\beta}(\sqrt{|\lambda|}(x + iy)) & \lambda > 0 \\ (2\pi)^{\frac{n}{2}} e^{i\lambda t} \overline{\Phi}_{\alpha,\beta}(\sqrt{|\lambda|}(x + iy)) & \lambda < 0, \end{cases} \tag{4.4.21}$$

where $\Phi_{\alpha,\beta}$ are the special Hermite functions which form an orthonormal system in $L^2(\mathbb{C}^n)$. (See also Appendix A.1).

Simultaneously (4.4.21) are eigenfunctions of T with respect to the eigenvalue $i\lambda$. Hence (4.4.21) provides eigenfunctions of the Laplacian $\Delta = \Delta_{sub} + T^2$ on H_n with respect to the eigenvalues $(2|\alpha| + n)|\lambda| - |\lambda|^2$. This enables us to construct diffusive wavelets also for the heat equation which involves the whole Laplacian. Nevertheless we continue to investigate the diffusive wavelets for Δ_{sub}, which is more appropriate with respect to the geometry of H_n.

The radial-symmetric eigenfunctions of Δ_{sub} are given by

$$\phi_k^\lambda(x, y, t) = e^{i\lambda t} \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(\sqrt{|\lambda|}(x + iy)), \tag{4.4.22}$$

Hence the characters are

$$\chi_\lambda(x, y, t) = (2\pi)^{\frac{n}{2}} \sum_{k=0}^\infty \phi_k^\lambda(x, y, t).$$

And hence $f^\lambda(x, y, t) = (f * \chi_\lambda)(x, y, t)$ are the spectral components of $f \in L^2(H_n)$ and furthermore

$$f(x, y, t) = \int_{-\infty}^\infty f^\lambda(x, y, t)\ \mathrm{d}\mu(\lambda),$$

holds true. One can also obtain this spectral decomposition for $f \in L^p(H_n)$ with $1 < p < \infty$, see [Str91].

4.4.5 Diffusive wavelets on H_n

Now we are at the point where we can construct our diffusive wavelets on H_n.
Since the eigenvalues of Δ_{sub} (4.4.20) depend on $|\alpha| = k$ the expansion with respect to characters is not suitable for the heat kernel. The expansion into radial-symmetric eigenfunctions of Δ_{sub}, given in (4.4.22), is a more appropriate one.
We can now write down the fundamental solution of the heat equation involving the sub-Laplacian

$$(\Delta_{sub} - \partial_r)u((x, y, t), r) = 0,$$

the following expression

$$p_r(x, y, t) = \int (2\pi)^{\frac{n}{2}} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} e^{-((2|\alpha|+n)|\lambda|)r} \phi_k^\lambda(x, y, t) \, \mathrm{d}\mu(\lambda).$$

Since the number of multi-indexes with $|\alpha| = k$ is k^n

$$\hat{p}_r(\lambda) = (2\pi)^{\frac{n}{2}} \bigoplus_{k=0}^{\infty} e^{-(2k+n)|\lambda|r} Id_{k^n \times k^n}.$$

and hence we find for the Fourier transform of a diffusive wavelet $\{\psi_r, \, r > 0\}$ on H_n the condition

$$\hat{\psi}_r(\lambda) = \left((2\pi)^{\frac{n}{4}} \bigoplus_{k=0}^{\infty} e^{-(2k+n)|\lambda|r/2} Id_{k^n \times k^n} \right) U,$$

where U is an unitary operator on $L^2(\mathbb{R}^n)$ expressed as a matrix with respect to the basis of Hermite functions $\{\Phi_\alpha, \, \alpha \in \mathbb{N}_0^n\}$. As in the compact case the freedom of choosing U represents the choice of the point on H_n where the wavelet ψ_r localizes for r tending to 0. Since we choose $e = (0, 0, 0)$ to be that point U is uniquely determined to be the identity operator.
We want to calculate explicitly the diffusive wavelets for the special example of the three dimensional Heisenberg group H^1.

$$\Psi_\rho(x, y, t) = \int_{\mathbb{R}} \operatorname{trace} \left(\pi_\lambda(x, y, t) \bigoplus_{k=0}^{\infty} e^{-(2k+1)|\lambda|\frac{\rho}{2}} Id_{k^n \times k^n} \right) \mathrm{d}\mu(\lambda) \tag{4.4.23}$$

$$= \sqrt{2\pi} \int_{\mathbb{R}} \sum_{k=0}^{\infty} e^{-(2k+1)|\lambda|\frac{\rho}{2}} e^{i\lambda t} \sum_{|\alpha|=k} \phi_{\alpha,\alpha}(\sqrt{|\lambda|}|x + iy|) \, \mathrm{d}\mu(\lambda) \tag{4.4.24}$$

$$= \sqrt{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \sum_{k=0}^{\infty} e^{-(2k+1)|\lambda|\frac{\rho}{2}} \phi_{k,k}(\sqrt{|\lambda|}|x + iy|) \, \mathrm{d}\mu(\lambda) \tag{4.4.25}$$

with (A.1.2) this equals to

$$= \int_{\mathbb{R}} e^{i\lambda t} \sum_{k=0}^{\infty} e^{-(2k+1)|\lambda|\frac{\rho}{2}} L_k \left(\frac{1}{2} |\lambda| |x + iy|^2 \right) e^{-\frac{1}{4}|x+iy|^2} \, \mathrm{d}\mu(\lambda) \tag{4.4.26}$$

To solve this we have to calculate the integral

$$\int_{\mathbb{R}} e^{i\lambda t}\, e^{-(2k+1)|\lambda|\frac{\rho}{2}}\ L_k\left(\frac{1}{2}|\lambda||x+iy|^2\right)\ \mathrm{d}\mu(\lambda). \tag{4.4.27}$$

By (A.1.1), this equals to

$$\int_{\mathbb{R}} e^{i\lambda t}\, e^{-(2k+1)|\lambda|\frac{\rho}{2}}\ \frac{e^{\left(\frac{1}{2}|\lambda||x+iy|^2\right)}}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}\left(\frac{1}{2}|\lambda||x+iy|^2\right)}\right)^k (e^{-\left(\frac{1}{2}|\lambda||x+iy|^2\right)} \left(\frac{1}{2}|\lambda||x+iy|^2\right)^k)\ \mathrm{d}\mu(\lambda)$$

$$= \int_{\mathbb{R}} e^{i\lambda t}\, e^{-(2k+1)|\lambda|\frac{\rho}{2}}\ \frac{e^{\left(\frac{1}{2}|\lambda||x+iy|^2\right)}}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}|\lambda|}\right)^k \frac{1}{(\frac{1}{2}|x+iy|^2)^k} (e^{-\left(\frac{1}{2}|\lambda||x+iy|^2\right)} \left(\frac{1}{2}|\lambda||x+iy|^2\right)^k)\ \mathrm{d}\mu(\lambda)$$

$$= \int_0^\infty e^{i\lambda t}\, e^{-(2k+1)\lambda\frac{\rho}{2}}\ \frac{e^{\left(\frac{1}{2}\lambda|x+iy|^2\right)}}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k (e^{-\left(\frac{1}{2}\lambda|x+iy|^2\right)}\lambda^k)\ \mathrm{d}\mu(\lambda)$$

$$+ (-1)^k \int_0^\infty e^{-i\lambda t}\, e^{-(2k+1)\lambda\frac{\rho}{2}}\ \frac{e^{\left(\frac{1}{2}\lambda|x+iy|^2\right)}}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k (e^{-\left(\frac{1}{2}\lambda|x+iy|^2\right)}\lambda^k)\ \mathrm{d}\mu(\lambda)$$

For simplicity we only calculate the first integral (the second integral can be calculated analogously)

$$\int_0^\infty e^{i\lambda t}\, e^{-(2k+1)\lambda\frac{\rho}{2}}\ \frac{e^{\left(\frac{1}{2}\lambda|x+iy|^2\right)}}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k e^{-\left(\frac{1}{2}\lambda|x+iy|^2\right)}\lambda^k\ \mathrm{d}\mu(\lambda)$$

$$= \frac{(-1)^k}{k!} \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k \left(e^{i\lambda t-(2k+1)\lambda\frac{\rho}{2}+\frac{1}{2}\lambda|x+iy|^2}\right) e^{-\left(\frac{1}{2}\lambda|x+iy|^2\right)}\lambda^k\ \mathrm{d}\mu(\lambda)$$

$$= \frac{(-1)^k}{k!} \left(it-(2k+1)\frac{\rho}{2}+\frac{1}{2}|x+iy|^2\right)^k \int_0^\infty \left(e^{i\lambda t-(2k+1)\lambda\frac{\rho}{2}+\frac{1}{2}\lambda|x+iy|^2}\right) e^{-\left(\frac{1}{2}\lambda|x+iy|^2\right)}\lambda^k\ \mathrm{d}\mu(\lambda)$$

$$= \frac{(-1)^k}{k!}(-1)^k \left(it-(2k+1)\frac{\rho}{2}+\frac{1}{2}|x+iy|^2\right)^k \int_0^\infty e^{i\lambda t-(2k+1)\lambda\frac{\rho}{2}} \frac{1}{\left(it-(2k+1)\frac{\rho}{2}\right)^k}\ \mathrm{d}\mu(\lambda)$$

$$= -\frac{1}{k!}\frac{1}{it-(2k+1)\frac{\rho}{2}} \left(1+\frac{\frac{1}{2}|x+iy|^2}{it-(2k+1)\frac{\rho}{2}}\right)^k$$

Consequently, the wavelet assumes the form:

$$\Psi_\rho(x,y,t) = -\sum_{k=0}^{\infty}\left(\frac{1}{k!}\frac{1}{it-(2k+1)\frac{\rho}{2}}\left(1+\frac{\frac{1}{2}|x+iy|^2}{it-(2k+1)\frac{\rho}{2}}\right)^k + \frac{(-1)^k}{k!}\frac{1}{-it-(2k+1)\frac{\rho}{2}}\left(1+\frac{\frac{1}{2}|x+iy|^2}{-it-(2k+1)\frac{\rho}{2}}\right)^k\right) e^{-\frac{1}{4}|x+iy|^2}.$$

For ρ tending to 0 we observe, that the wavelet tends to

$$\Psi_{\rho\to 0}(x,y,t) = \frac{1}{it}\left(e^{\left(1-\frac{i\frac{1}{2}|x+iy|^2}{t}\right)} + e^{-\left(1+\frac{i\frac{1}{2}|x+iy|^2}{t}\right)}\right) e^{-\frac{1}{4}|x+iy|^2} \tag{4.4.28}$$

$$= \frac{1}{it}2\cosh(1)e^{-\frac{i\frac{1}{2}|x+iy|^2}{t}} e^{-\frac{1}{4}|x+iy|^2} \tag{4.4.29}$$

Conspicuous is the singularity, that we have for $t \to 0$ when $x + iy \neq 0$. This is a characteristic phenomena for the Heisenberg group, which is caused by it special geometry - the sub-Riemannian structure. The rest of the form equals the behavior of the heat kernel on H_1, just as we expected.

Remark 4.4.10. Note that the subgroup $\mathcal{T}\{(0,0,t)\}$ of H_n is always normal, such that according to Remark 3.2.6 the construction of diffusive wavelets on the homogeneous space $H_n/\mathcal{T}$ makes no sense. This is no gap of the theory but shows that there is a difficult singularity in $H_n/\mathcal{T}$. Nevertheless it shall be possible to look at Heisenberg manifolds, for which one factorizes a discrete subgroup from H_n.

4.5 The Spin group $Spin(m)$

A further non-trivial but important example of a compact Lie group is the Spin group $Spin(m)$. The main difficulty will be to determine all irreducible representations of $Spin(m)$. Therefore we introduce the notion of roots and weights of representations. This concepts can be used to label all representations. Since we will use regular non-regular representations on Clifford-valued functions on $Spin(m)$ we have to spend some effort for determining the invariant subspaces.

4.5.1 Roots and weights

In this section we collect the assertions about weights of representations, that are necessary for the construction of the weights of $Spin(m)$ that are usually used to label all irreducible representations of $Spin(m)$. A more comprehensive discussion about the theoretical bases can be found in [Bum04], [Feg91], [VK95], [VK92] and elsewhere.
We already mentioned that a representation π is uniquely determined by the values that it character assumes on $\mathbb{T}$. We now restrict π itself to $\mathbb{T}$. What we obtain is the representation $\mathbb{T}$ that decomposes into one-dimensional irreducible components, since $\mathbb{T}$ is commutative.
Since $\mathbb{T}$ is compact, all irreducible representations π are of the form

$$\begin{aligned} \pi : \mathbb{T} &\to \{e^{ix} |\, x \in \mathbb{R}\} \\ t &\mapsto e^{i\theta(t)}. \end{aligned} \tag{4.5.1}$$

Note that $\theta : \mathbb{T} \to \mathbb{R}/(2\pi\mathbb{Z})$ is a homomorphism and hence a representation of $\mathbb{T}$. Consequently, the derivative $d\theta : \mathfrak{t} \to \mathbb{R}$ is a representation of $\mathfrak{t}$, the Lie algebra of $\mathbb{T}$. This defines the weights of π:

Definition 4.5.1. Let π be a representation of $\mathcal{G}$ with $\dim(\mathfrak{t}) = r$. Let $\pi_j(t) = e^{i\theta_j(t)}$, $j = 1, ..., r$ be the one-dimensional representations in which π decomposes when restricted to $\mathbb{T}$.
We denote the restriction of π to $\mathbb{T}$ by $\pi_{\mathbb{T}}$.

The set of weights of π is given by $\{\pm \mathrm{d}\theta_j\} \subset \mathfrak{t}^*$. The *weights* of the adjoint representation are called *roots*.

If one regards π as a matrix with respect to a fixed basis of the representation space its restriction to $\mathbb{T}$ contains 2×2 block matrices (up to change of rows and lines), which correspond to a rotation in the respective plane.

$$\pi|_{\mathbb{T}} = \begin{pmatrix} \Theta_1 & & & & & & \\ & \Theta_2 & & & & & \\ & & \ddots & & & & \\ & & & \Theta_r & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \text{ with } \Theta_j = \begin{pmatrix} \cos(\theta_j(t)) & \sin(\theta_j(t)) \\ -\sin(\theta_j(t)) & \cos(\theta_j(t)) \end{pmatrix}.$$

Further, note that the eigenvalues of the derivative of $\pi|_{\mathbb{T}}$ for $X \in \mathfrak{t}$ are always purely imaginary:

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{i\theta_j(\exp(sX))}\Big|_{s=0} = i \ \mathrm{d}\theta_j(X). \tag{4.5.2}$$

Multiplying it with imaginary unit i determines the weights of π.
The weights $\mathrm{d}\theta_j$ are homomorphisms and are uniquely determined by the values in $\mathfrak{t}$, which are mapped to $0 \mod 2\pi$. Under $\mathrm{d}\theta_j$, the so-called *integer lattice* I is determined by the following property

$$2\pi I = \exp^{-1}(1) \subset \mathfrak{t}.$$

Roughly speaking, the specific form of the weights of the representation π corresponds to the density of $\mathrm{d}\theta_j(2\mathrm{I})$ in $\mathbb{Z}$. This is meant like follows: let $t_j \in \mathfrak{t}$ be so that for every $s \in \mathbb{R}$ (or $\mathbb{C}$) $\mathrm{d}\theta_j(st_j)$ is zero for all j but exactly one $j \in \{1, ..., r\}$. This gives us a direction on $\mathbb{T}$ which we associate to θ_j and we denote it by $t_j \in \mathfrak{t}$. There is a smallest $s_j \in \mathbb{R}$ (in $\mathbb{C}$ one with smallest absolute value) so that $\exp(s_j t_j) = 1$ and hence

$$\mathrm{d}\theta_j(s_j t_j) = m_j \in \mathbb{Z}. \tag{4.5.3}$$

Any integer multiple of $(s_j t_j)$ will be mapped to the corresponding integer multiple of m_j in $\mathbb{Z}$. This is what we mean by the density of $\mathrm{d}\theta_j(I)$ in $\mathbb{Z}$. The correspondence between $\mathrm{d}\theta_j$ and m_j is one to one, so we will also call m_j weight of π.
Let $t_1, ..., t_r \in \mathfrak{t}$ be a normalized (with respect to the killing form) basis of $\mathfrak{t}$ and $m_1, ..., m_r$ be the weights of π, then the mapping

$$\beta : \mathbb{T} \to \mathbb{R}^n/(2\pi m_1\mathbb{Z} \times ... \times 2\pi m_r\mathbb{Z}) \tag{4.5.4}$$

$$\exp\left(\sum_{k=1}^{r} a_k t_k\right) = \Pi_{k=1}^{r} \exp(a_k t_k) \mapsto (a_1, ..., a_r)/(2\pi m_1\mathbb{Z} \times ... \times 2\pi m_r\mathbb{Z}), \tag{4.5.5}$$

gives an embedding of $\mathbb{T}$ in $\mathbb{R}^n$.
In fact for every $(m_1, ..., m_r) \in \mathbb{Z}_+^r$ there is a representation π with weights $m_1, ..., m_r$. In this way we have labeled the representations by its weights $m_1, ..., m_r \in \mathbb{Z}$ and it is necessary to mention the connection (4.5.5) between m_j and $\mathrm{d}\theta_j$. One can also choose the lattice so that $(m_1, ..., m_r) \in (l\mathbb{Z})_+^r$ for any $l \in \mathbb{Q}$ as we will see in the case of $\mathrm{S}pin(m)$, where the appropriate choice of l will be $\frac{1}{2}$.
In $\mathfrak{t}$ we obtain a lattice corresponding to weights that is given by $\{\sum_{j=1}^r k_j m_j t_j,\ k_j \in \mathbb{Z}\}$. The symmetry of this lattice is of importance and can be expressed by the Weyl group of the corresponding representation.
If there are at least two points in $\mathbb{T}$ that belong to the same conjugate class then the information about the representation is the same at all these points. Hence we can factor out these symmetry:

Definition 4.5.2. The Weyl group is defined by

$$W = N(T)/T, \tag{4.5.6}$$

where $N(T)$ is the normalizer of T in $\mathcal{G}$, i.e. $gTg^{-1} = T\, \forall g \in G\mathcal{G}$

W acts on $\mathbb{T}$ by conjugation, and hence on $\mathfrak{t}$ by the adjoint representation $\mathrm{ad}(w)$ for $w \in W$. The weights of the adjoint representation are called roots of the representation. We can look at the hyperplanes in $\mathfrak{t}$ that are the kernel of the roots α_i: $L_{\alpha_i} = \{\alpha_i(t) = 0\}$. The complement of the union of all hyperplanes consists of open connected components; the closure of every of this components is called a Weyl chamber.
The Weyl group permutes the Weyl chambers transitively and hence also the weights that we can identify with elements in $\mathfrak{t}$ by Riesz theorem and that are symmetric to each other in the above sense.
The reflections at the plains L_{α_i} generate W.
One can distinguish an arbitrary Weyl chamber and call it positive. All weights are positive, that are in the dual of the positive Weyl chamber.
A weight $\mathrm{d}\theta$ is a highest weight if it is positive and if $\mathrm{d}\theta - \mathrm{d}\lambda$ is not positive for all other weights $\mathrm{d}\lambda$ of the same representation.
Note that in the construction above (4.5.3) where we obtained $\mathrm{d}\theta_j(rs_jt_j) = rm_j$, the vector $(m_1, ..., m_n)$ corresponds to the highest weight of the representation.
There is a famous theorem of Weyl which says that the correspondence between irreducible representations and highest weights is one to one.

4.6 Clifford algebra setting

Clifford algebras arise in many fields. As algebra of operators they play an enormous role in physics. A realization of it as linear operators on the Grassmann algebra can be found in [GM91], here the realization of the spinor space comes out as the Grassmanian itself.

A comprehensive set of results for Clifford analysis is given by [DSS92]. There the realization of the Clifford algebra is given for instance as a full matrix algebra of appropriate dimension. Further descriptions can be found in [GHS08, GHS06]. Since the spinor spaces are minimal left ideals of the algebra, they can be given very conveniently in this realization of the Clifford algebra.

To every vector space one can associate a corresponding complex-valued Clifford algebra. Here it is sufficient to define the basic properties of the Clifford algebra as starting point.

Let $\{e_i,\, i = 1, ..., m\}$ be a basis of $\mathbb{C}^m$; the corresponding complex-valued Clifford algebra $\mathbb{C}_m$ is determined by the anti commutative relation $-2\delta_{ij} = e_i e_j + e_j e_i$[1]. Therefore the algebra is given by

$$\mathbb{C}_m = \left\{ \sum_{A \subset \{1,...,m\}} a_A e_A,\, a_A \in \mathbb{C} \right\}, \tag{4.6.1}$$

where the set $A = \{\alpha_1, ..., \alpha_k\}$ is sorted, i.e. $\alpha_1 < ... < \alpha_k$, $k \leq m$, and $e_A = e_{\alpha_1} ... e_{\alpha_k}$. The dimension of $\mathbb{C}_m$ is 2^m. The scalars are contained in $\mathbb{C}_m$ as 0-vectors, hence the unit element of $\mathbb{C}_m$ is 1.

We will make use of the main anti-involution (also called conjugation):

$$\overline{a} = \sum_{A \subset \{1,...,m\}} a_A \overline{e_A}, \qquad \overline{e_i e_j} = \overline{e_j}\,\overline{e_i}, \qquad \overline{e_i} = -e_i. \tag{4.6.2}$$

The subspace of $\mathbb{C}_m$ of k-vectors is given by $\text{span}\{e_A,\, |A| = k\}$[2]. The k-vector part of an $a \in \mathbb{C}_m$ is given by $[a]_k = \sum_{|A|=k} a_A e_A$ with $|A| = k$. The subspace of k-vectors in $\mathbb{C}_m$ is denoted by $\mathbb{C}_{m,k}$.

Also of importance is the Clifford inner product

$$\langle a, b\rangle_{\mathbb{C}_m} = [\overline{a}b]_0 = \sum_{|A|=0}^{m} (-1)^{|A|} \overline{a_A} b_A. \tag{4.6.3}$$

This makes $\mathbb{C}_m$ being a Hilbert space with orthonormal basis $\{e_A,\, A \subset \{1, ..., m\}\}$. The outer product in $\mathbb{C}_m$ is defined by

$$a \wedge b = \frac{1}{2}(ab - ba). \tag{4.6.4}$$

4.7 Spin group

There are several important subgroups in $\mathbb{C}_m$. The Clifford group is defined as set of invertible elements. The Pin group is given as the set of products of unit vectors. Hereby a vector a is a unit vector if it is a vector with $\sum_{|A|=1} |a_A|^2 = 1$ and $a_A = 0$ for $|A| \neq 1$.

[1] δ_{ij} denotes the usual Kroneker symbol

[2] $|A|$ denotes the cardinality of A

The Spin group, in which we are interested, is a subgroup of the Pin group and is defined as the set of even products of unit vectors

$$\mathrm{S}pin(m) = \{\prod_{j=1}^{2k} s_j,\ s_j \in S^m\}. \tag{4.7.1}$$

In each of these cases, the group multiplication is given by the usual Clifford multiplication.

4.7.1 Lie algebra of $\mathrm{S}pin(m)$

The Lie algebra $\mathfrak{spin}(m)$ of $\mathrm{S}pin(m)$ is the space of bi-vectors in $\mathbb{C}_m$: $\mathfrak{spin}(m) = \mathbb{C}_{m,2}$. This can be seen as follows: Since we are in the comfortable situation to expand the exponential mapping $\exp : \mathfrak{spin}(m) \to \mathrm{S}pin(m)$ in a series, for $X_{ij} = e_{ij} \in \mathbb{C}_{m,2}$ we find:

$$\begin{aligned} \exp(tX_{jk}) &= \sum_{l=1}^{\infty} \frac{1}{l!}(\frac{1}{2}e_{jk})^l = e_{jk}\sum_{l=1}^{\infty} \frac{1}{(2l-1)!}t^{2l-1} + \sum_{l=1}^{\infty} \frac{1}{(2l)!}t^{2l} \\ &= \cos(t) + e_{jk}\sin(t) = e_j(e_k \sin(t) - e_j \cos(t)), \end{aligned} \tag{4.7.2}$$

obviously $e_j, (e_k \sin(t) - e_j \cos(t)) \in S^m$, hence the exponential of an element from $\mathbb{C}_{m,2}$ gives always an element, that can be written as a sum of an even number of unit vectors.
Since $\mathrm{S}pin(m)$ is a double covering of $SO(m)$ we have $\dim \mathrm{S}pin(m) = \dim SO(m) = \frac{1}{2}n(n+1)$, but this is also the dimension of $\mathbb{C}_{m,2}$ which hence is the complete Lie algebra of $\mathrm{S}pin(m)$.
In order to follow the general concept of determining all irreducible representations we need to look at the maximal torus of $\mathrm{S}pin(m)$. Let us study the weights of $\mathrm{S}pin(m)$.

4.8 Weights of $\mathrm{S}pin(m)$

In order to get the weights we look at the torus of $\mathrm{S}pin(m)$ and its Lie algebra $\mathfrak{t}$. The Lie algebra can be given as the span of a maximal[1] system of commuting vector fields, i.e. $\mathfrak{t} = \mathrm{span}\{Y_i,\ i = 1, ..., r\} \subset \mathfrak{spin}(m)$ with $[Y_i, Y_j] = 0$ for all $Y_i, Y_j \in \mathfrak{t}$. Such a system is obviously given by

$$\{Y_j = X_{2j-1,2j} = e_{2j-1}e_{2j},\ j = 1, ..., \left[\frac{m}{2}\right]\} \tag{4.8.1}$$

and hence $\mathbb{T} = \left\{ \prod_{j=1}^{[\frac{m}{2}]} \exp(t_j Y_j),\ t \in [0, 2\pi) \right\}$. According to (4.5.2), the weights can be given now as the derivative of

$$\theta_j : \mathbb{T} \to \mathbb{R}/2\pi, \tag{4.8.2}$$

[1] maximal in the sense, that there is no further vector field which commutes with all vector field of the system.

where θ_j is given in (4.5.1). This results in

$$\prod_{j=1}^{[\frac{m}{2}]} \exp(t_j Y_j) \mapsto m_j t_j 2\pi \mod 2\pi, \tag{4.8.3}$$

where the derivative has to be taken with respect to all t_j, so that $(m_1, ..., m_{[\frac{m}{2}]})$ stands for the weights. We have to verify, which $(m_1, ..., m_{[\frac{m}{2}]})$ are admissible weights.
From (4.7.2) we see that the natural representation of every element $t = \exp(t_j e_{j,n-j+1}) \in \mathbb{T}$ of the torus is a rotation in the plane $E_f = \text{span}\{e_j, e_{n-j+1}\} \subset \mathbb{C}^m$ by the angle $m_j t_j 2\pi$.
Hence, for any representation π of $Spin(m)$ we obtain its restriction to $\mathbb{T}$ as the direct sum of rotations

$$\pi\left(\prod_{j=1}^{[\frac{m}{2}]} \exp(t_j Y_j)\right) = \pi(t_1, ..., t_{[\frac{m}{2}]})v = e^{i(m_1 t_1 + ... + m_{[\frac{m}{2}]} t_{[\frac{m}{2}]})}, \tag{4.8.4}$$

for some $(m_1, ..., m_{[\frac{m}{2}]})$.
Since the weights corresponds to the dual of the integer lattice in $\mathbb{T}$ we pick out those $(m_1, ..., m_{[\frac{m}{2}]})$, such that $(t_1, ..., t_{[\frac{m}{2}]}) \in \ker(\exp) \Rightarrow (m_1 t_1, ..., m_{[\frac{m}{2}]} t_{[\frac{m}{2}]}) \in \ker(\exp)$.
For eigenvalues of rotations we remark, that the rotation must be by an angle of 0 or π.
From (4.8.4) we see, that for the integer lattice $(m_1 t_1, ..., m_{[\frac{m}{2}]} t_{[\frac{m}{2}]}) \in \ker(\exp)$ we have $e^{i(m_1 t_1 + ... + m_{[\frac{m}{2}]} t_{[\frac{m}{2}]})} = 1$. Consequently $(m_1 t_1, ..., m_{[\frac{m}{2}]} t_{[\frac{m}{2}]}) \in \ker(\exp)$ implies

$$m_j t_j = 0 \text{ or } m_j t_j = \pi \text{ and } m_1 t_1 + ... + m_{[\frac{m}{2}]} t_{[\frac{m}{2}]} = 0 \mod 2\pi, \tag{4.8.5}$$

such that m_j has to be an integer for all j.
If $t_j = 0 \mod 2\pi$ one can always remove this component from $\prod_{j=1}^{[\frac{m}{2}]} \exp(t_j Y_j)$, i.e. setting $t_j = 0$, without loosing the property of being an element of the integer lattice. If $t_j = \pi \mod 2\pi$ one has to remove additionally another component with the same property in order to stay in the integer lattice.
Hence for any choice of $\varepsilon_j = 1$ or 0 ($j = 1, ..., [\frac{m}{2}]$) and $\varepsilon_1 + ... + \varepsilon_{[\frac{m}{2}]}$ is an even integer, $(t_j m_j = \pi \varepsilon_j)$ satisfies (4.8.5).
We assume now, that $(t_1, ..., t_{[\frac{m}{2}]})$ belongs to the integer lattice. Then also $(m_1 t_1, ..., m_{[\frac{m}{2}]} t_{[\frac{m}{2}]})$ shall belong to this lattice. But since if $(\varepsilon_1 t_1, ..., \varepsilon_{[\frac{m}{2}]} t_{[\frac{m}{2}]})$ belongs to it, also $(m_1 \varepsilon_1 t_1, ..., m_{[\frac{m}{2}]} \varepsilon_{[\frac{m}{2}]} t_{[\frac{m}{2}]})$ does, we have that either all m_j are even, or all m_j are odd.
This can also be seen in an easy counterexample, where we assume $t_l, t_k = \pi$ and $\varepsilon_j = 0$ except $j = k$ and $j = l$. Then $(\varepsilon_1 t_1, ..., \varepsilon_{[\frac{m}{2}]} t_{[\frac{m}{2}]})$ belongs to the integer lattice but $(m_1 \varepsilon_1 t_1, ..., m_{[\frac{m}{2}]} \varepsilon_{[\frac{m}{2}]} t_{[\frac{m}{2}]})$ does only for m_l and m_k both even or both odd.
A discussion about the admissible weights can also be found in [GM91], where the connection between m_j and $\mathrm{d}\theta_j$ is another one than the one we have given by (4.5.5), so that the corresponding weights are from $(\frac{1}{2}\mathbb{Z})^{[\frac{m}{2}]}$.

We have to look now at the action of the Weyl group to select the highest weight for every representation.
The Weyl group acts on $\mathbb{T}$ and hence on $\mathfrak{t}$ and $\mathfrak{t}^*$. Its action on the weights is closed[1] and corresponds to a permutation of the m_j; also a change of sign of m_j is possible. In the case where m is odd, an arbitrary number of sign changes is allowed; while in the case of an even m, only an even number of sign changes is possible.
The positive Weyl chamber shall be the chamber where

$$m_1 \geq ... \geq m_{[\frac{m}{2}]-1} \geq |m_{[\frac{m}{2}]}|. \tag{4.8.6}$$

In the case of an odd m, all m_j of positive weights are positive. When m is even, $m_{[\frac{m}{2}]}$ can be negative.
We can also compare weights of different representations by the so called lexicographic order, i.e. $(m_1, ..., m_k) < (l_1, ..., l_k)$, if the difference $l_j - m_j$ in the first component where the weights are different is positive.
For the construction of all irreducible representations of $Spin(m)$ we make use of the so-called Cartan product.
The Cartan product is a procedure to build up an irreducible representation from two known irreducible representations. Let π_1 and π_2 be irreducible representations in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, let $(m_1, ..., m_k)$ and $(l_1, ..., l_k)$ be the highest weights of π_1 and π_2. The canonically given representation $\pi_1 \otimes \pi_2$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is highly reducible. The irreducible component of the maximal weight[1] occurring in $\pi_1 \otimes \pi_2$ has the highest weights $(l_1 + m_1, ..., l_k + m_k)$.
A minimal set of irreducible representations from which we can build up every irreducible representation is called fundamental.

4.9 Representations of $Spin(m)$ and Clifford-valued wavelets

From the previous section we already know, that a fundamental system of irreducible representations of $Spin(m)$ in the case of an odd m is contained in the set of representations with weights of the form $(1, 0, ..., 0), ..., (1, ..., 1)$ and $(\frac{1}{2}, 0, ..., 0), ..., (\frac{1}{2}, ..., \frac{1}{2})$ and in the case of m even in the set of representations of weights $(1, 0, ..., 0), ..., (1, ..., 1)$, $(\frac{1}{2}, 0, ..., 0), ..., (\frac{1}{2}, ..., \frac{1}{2})$ and $(1, ..., 1, -1)$, $(\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2})$.
For convenience we consider the above system instead of the (minimal) fundamental system, for which we would not need to consider $(1, ..., 1)$ or $(1, ..., \pm 1)$.
From this starting point the corresponding irreducible representations are obtained in [LSC01] as representations in some Clifford-valued function spaces of spherical monogenics and harmonic functions.

[1] The Weyl group maps weight to weights
[1] with respect to the lexicographically order

We remark, that for $Spin(m)$ the usual way of harmonic analysis using matrix coefficients as eigenfunctions of the Laplacian leads to problematic calculations of integrals which would give the matrix coefficients (see (4.11.1), (4.11.2)).
Therefore, here we prefer another way of thinking: we directly use representations of the group (here $Spin(m)$) in the function spaces in which we are interested. In that way, we can investigate operators in our function space as derivatives of representations.
In the end we can formulate Clifford-valued diffusive wavelets corresponding to a modified diffusion equation, where the corresponding operator is a realization of the Casimir element, just as in the classical case.
There are two types of fundamental representations of the spin group in the Clifford algebra $\mathbb{C}_m$ given by

$$h(s)a = sas^{-1} \tag{4.9.1}$$

$$l(s)a = sa. \tag{4.9.2}$$

The invariant subspaces, where h is irreducible, are the k-vector spaces. The invariant subspaces of l are the so-called spinor spaces. Obviously they are minimal left ideals in $\mathbb{C}_m$. Spinor spaces can be determined explicit by primitive idempotents ([DSS92],[LSC01]). This goes as follows: Set

$$I_j = \frac{1}{2}(1 + ie_j e_{j+m}), \tag{4.9.3}$$

then one easily sees $I_j^2 = \frac{1}{4}(1+2ie_je_{j+m}+(ie_je_{j+m})^2) = \frac{1}{4}(1+2ie_je_{j+m}-e_je_{j+m}e_je_{j+m}) = I_j$. Furthermore $e_jI_j = \frac{1}{2}(-ie_{j+m} + e_j) = -ie_{j+m}I_j$ and similarly $e_{j+m}I_j = -ie_jI_j$. A minimal left ideal is generated by $I = I_1...I_m$, namely $\mathbb{C}_{2m}I$. Clearly $I^2 = I$.
We introduce also

$$T_j = \frac{1}{2}(e_{2j-1} - ie_{2j}), \tag{4.9.4}$$

and note that $I_j = T_j\overline{T}_j$.
There are many possibilities to realize representations of $Spin(m)$ in $L^2(\mathbb{C}_m)$. For instance one can just take the regular representations h_r and l_r of h and l respectively:

$$h_r(s) : f(a) \mapsto f(sas^{-1}) \tag{4.9.5}$$

$$l_r(s) : f(a) \mapsto f(sa). \tag{4.9.6}$$

h_r is a representation, which does not distinguish between $h_r(s)$ and $h_r(-s)$ and acts exactly like the usual regular representation of $SO(m)$. Here the double covering nature of $Spin(m)$ with respect to $SO(m)$ is revealed.
The L^2-space of Clifford-valued functions involves the choice of an appropriate inner product. This is discussed in chapters 0 and 1 in [DSS92]. Applying the regular representations to

$L^2(S^m \to \mathbb{C}_m)$ separates it into Clifford-valued functions over rotation invariant domains. So it is enough to look at $L^2(S^m, \mathbb{C}_m)$. We can build up all irreducible representations by Cartan product, from the irreducible pieces of these fundamental representations. The inner product in our case shall be given by

$$\langle f, g\rangle_{L^2(S^m)} = \int_{S^m} \langle \overline{f(\xi)} g(\xi)\rangle_{\mathbb{C}_m} \, \mathrm{d}\xi.$$

The tensor product representations $h_r \otimes h$ and $h_r \otimes l$ in $L^2(S^m) \otimes \mathbb{C}_m \simeq L^2(\mathbb{C}_m, \mathbb{C}_m)$ are given by

$$H(s) : f(a) \mapsto s f(s^{-1} a s) s^{-1} \tag{4.9.7}$$

$$L(s) : f(a) \mapsto s f(s^{-1} a s). \tag{4.9.8}$$

Remark 4.9.1. One important observation is, that the representations are unitary:

$$\langle H_s f(a), H_s g(a)\rangle_{L^2(S^m \to \mathbb{C}_m)} = \int_{S^m} \langle s^{-1} f(sas^{-1}) s, s^{-1} g(sas^{-1}) s\rangle_{\mathbb{C}_m} \, \mathrm{d}a \tag{4.9.9}$$

$$= \int_{S^m} \langle f(sas^{-1}) g(sas^{-1})\rangle_{\mathbb{C}_m} \, \mathrm{d}a. \tag{4.9.10}$$

A similar line shows that also L_s is unitary.
By unitary of H and L, the invariant subspaces in the representation Hilbert space $L^2(\mathbb{C}_m \to \mathbb{C}_m)$ are orthogonal.
We should assure us, that we are dealing with bounded operators. This follows from compactness of $Spin(m)$: By smoothness of representations, from compactness follows the finite dimensionality of all irreducible representation spaces and hence the compactness of all derivatives of the representation.

The most interesting question is now to find the invariant subspaces. This is comprehensively investigated in [LSC01]. The desired invariant subspaces are spanned by eigenfunctions of the operators, that one obtains by mapping the Casimir element via the corresponding representation into the representation space.
So we shall look at $H_*(\Omega)$ and $L_*(\Omega)$ according to Definition (2.4.1). We mentioned already that the space of bivectors $\mathbb{C}_{m,2}$ can be identified with the Lie algebra $\mathfrak{spin}(m)$. We equip it with the natural given killing form $B(\cdot,\cdot)$. A calculation (Appendix A.4) yields

$$B(x, y) = -\frac{1}{4} \sum_{i \neq j} \sum_{\substack{k \neq j \\ k \neq i}} (x_{jk} - x_{kj})(y_{kj} - y_{jk}) + (x_{ki} - x_{ik})(y_{ik} - y_{ki}). \tag{4.9.11}$$

So that $\|\frac{1}{2} e_{ij}\|_B = 1$. Consequently we use a basis on $\mathfrak{spin}(m)$, which is orthonormal with respect to B

$$\{\frac{1}{2} e_{ij},\ 1 \leq i < j \leq m\}. \tag{4.9.12}$$

Moreover, as stated in Section 2.3.5 the Casimir element, mapped by π is given by

$$\pi_*(\Omega) = \sum_{\substack{i,j=1,\ldots,m \\ i<j}} \pi_* \left(\frac{1}{2}e_{ij}\right)^2 \left(= \frac{1}{4} \sum_{\substack{i,j=1,\ldots,m \\ i<j}} \pi_* \left(e_{ij}\right)^2 \right). \tag{4.9.13}$$

In [Som96], [DSS92], [VLSC01] and in many other places we find the calculation of the image obtained from mapping Ω by H_* and L_*:

$$H_* \left(\frac{1}{2}e_{ij}\right) = 2(x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}) =: L_{ij} \tag{4.9.14}$$

Having in mind, that our representation Hilbert space is a function space, the operator L_{ij} can be interpreted as a differential operator along the surface of the sphere, also called tangential derivative. The precise direction is given by the section of the plane, spanned by x_i and x_j and the sphere. In consequence we have

$$H_*(\Omega) = \sum_{\substack{i,j=1,\ldots,m \\ i<j}} H_* \left(\frac{1}{2}e_{ij}\right)^2 = \sum_{\substack{i,j=1,\ldots,m \\ i<j}} L_{ij}^2, \tag{4.9.15}$$

and further

$$L_* \left(\frac{1}{2}e_{ij}\right) = H_* \left(\frac{1}{2}e_{ij}\right) + \frac{1}{2}e_{ij}\mathbf{1}, \tag{4.9.16}$$

where $\mathbf{1}$ denotes the identity operator. Hence we have

$$\begin{aligned} L_*(\Omega) &= H_*(\Omega) + \sum_{\substack{i,j=1,\ldots,m \\ i<j}} \frac{1}{2}e_{ij} H_*(\frac{1}{2}e_{ij}) + \sum_{\substack{i,j=1,\ldots,m \\ i<j}} \left(\frac{1}{2}e_{ij}\right)^2 \\ &= H_*(\Omega) + \Gamma - \frac{1}{4}\binom{m+1}{2}\mathbf{1}, \end{aligned} \tag{4.9.17}$$

with

$$\Gamma = \sum_{\substack{i,j=1,\ldots,m \\ i<j}} e_{ij} L_{ij}. \tag{4.9.18}$$

We now briefly introduce a special type of functions, which will be the type of eigenfunctions of $H_*(\Omega)$ and $L_*(\Omega)$ and which give us the possibility to have a new look at functions on $Spin(m)$.

Functions of simplicial variables

In this section we show that the function spaces, consisting of functions which depend on simplicial variables, are invariant under H_s and L_s.

Let $u_1, ..., u_m \in \mathbb{C}^m$ be a orthonormal basis in $\mathbb{C}^m$. The corresponding simplicial variable in $\mathbb{C}_m$ is given by

$$a(u_1, ..., u_m) = u_1 + u_1 \wedge u_2 + u_1 \wedge u_2 \wedge u_3 + ... + u_1 \wedge ... \wedge u_m. \tag{4.9.19}$$

One the other hand one can match a unique right-handed orthonormal basis to a simplicial variable taking u_1 as normalized vector. In a second step one takes a linearly independent vector from the plane that is represented by $u_1 \wedge u_2$ and that is spanned by u_1 and u_2. Now one applies the Gram-Schmidt procedure to obtain a righthanded orthonormal basis after m steps.
In what follows we restrict the function of simplicial type to $a(u_1, ..., u_m)$, while $u_1, ..., u_m$ are assumed to be unit vectors. This gives a one to one correspondence to functions on $SO(m)$, resulting in the following lemma.

Lemma 4.9.2. *Functions that depend on simplicial variables can be identified with functions on $SO(m)$*

Furthermore, by definition of the outer product $x \wedge y = \frac{1}{2}(xy - yx)$ we have

$$\overline{s}a(u_1, ..., u_m)s = \overline{s}u_1 s + \overline{s}u_1 \wedge u_2 s + \overline{s}u_1 \wedge u_2 \wedge u_3 s + ... + \overline{s}u_1 \wedge ... \wedge u_m s = a(\overline{s}u_1 s, ..., \overline{s}u_m s).$$

Consequently, $H(s)f(a) = sf(\overline{s}as)\overline{s}$ is a function of a simplicial variable, if and only if $f(a)$ is such a function. Hence we have:

Corollary 4.9.3. Functions of simplicial type are invariant under H.

Later we will make use of the following lemma.

Lemma 4.9.4. *A function on the spin group can be represented as a pair of functions of a simplicial variable.*

Proof: A function $f(s)$ on $Spin(m)$ can be decomposed in an odd and an even part: $f(s) = \alpha(s) + \gamma(s)$, with $\alpha(s) = \alpha(-s)$ and $\gamma(s) = -\gamma(-s)$. For the odd part $\gamma(s)$, there is an even function $\beta(s)$ so that $s\beta(s) = \gamma(s)$. Hence the pair (α, β) can be identified with f. Since $Spin(m)$ is a double covering of $SO(m)$, even functions on $Spin(m)$ can be identified with functions on $SO(m)$. Furthermore, all right-handed orthonormal bases of $\mathbb{C}^m$ can be obtained by the action of exactly one rotation on one of these bases. This identification gives a faithful and irreducible representation (an identification) of $SO(m)$. We have already discussed, that the set of right-handed orthonormal bases of $\mathbb{C}^m$ are represented by simplicial variables. □

4.9.1 Eigenfunctions of $H_*(\Omega)$ and $L_*(\Omega)$

For a comprehensive discussion of the eigenfunction we refer to [VLSC01, DSS92]. Here we want to recall the results of the discussion in order to use them for further constructions in

the next section, where we are more interested in their restriction to the sphere in order to obtain Clifford-valued wavelets on the sphere.
Simplicial functions are functions can be viewed as functions on many Clifford-variables $x_1, ..., x_k$. Where every variable x_i has components x_{ij}. By operators

$$\Delta_{x_i} f(x_1, ..., x_k), \tag{4.9.20}$$

$$\partial_{x_i} f(x_1, ..., x_k) \tag{4.9.21}$$

we denote the Laplacian and the Dirac operator, acting on the Clifford-variable x_i of f.
For vector variable functions, a rotation -and hence a H- invariant differential operator is the Laplacian. The harmonic polynomials satisfy

$$\begin{aligned} \Delta_{x_i} P(x_1, ..., x_k) &= 0 \qquad \text{for } i = 1, ..., k \\ \partial_{x_i} \partial_{x_j} P(x_1, ..., x_k) &= 0 \qquad \text{for } i \neq j. \end{aligned} \tag{4.9.22}$$

A monogenic function is given, if

$$\partial_{x_i} P(x_1, ..., x_k) = 0 \qquad \text{for } i = 1, ..., k. \tag{4.9.23}$$

Simplicial functions are special kind of functions of vector variables. Its symmetry can be expressed by the characteristic differential equation

$$\langle x_i \partial_{x_{i+1}} \rangle P(x_1, ..., x_k) = 0 \qquad \text{for } i = 1, ..., k-1, \tag{4.9.24}$$

where the definition

$$\langle x_i \partial_{x_{i+1}} f(x_1, ..., x_n) \rangle := -[x_i \partial_{x_{i+1}} f(x_1, ..., x_n)]_0 \tag{4.9.25}$$

is used.
Consequently, the simplicial harmonic system $\mathcal{H}$ consists of polynomials satisfying (4.9.22) and (4.9.24); the simplicial monogenics are polynomials which satisfy (4.9.23) and (4.9.24).
It can be proven, that the simplicial harmonics span the irreducible subspaces spaces for H and the simplicial monogenics span those of L.
This is calculated in [LSC01] and the highest weight vectors for the weight $(\underbrace{2, ..., 2}_{k \text{ times}}, 0, ..., 0)$ is of the form

$$\langle x_1 \wedge ... \wedge x_k, T_1 \wedge ... \wedge T_k \rangle_{\mathbb{C}_m},$$

c.f. (4.9.4).
The tensor products, which we use to represent higher even integer weight representations $(2s_1, ..., 2s_k)$, correspond to the weight vector

$$\langle x_1 T_1 \rangle_{\mathbb{C}_m}^{2s_1} \langle x_1 \wedge x_2, T_1 \wedge T_2 \rangle_{\mathbb{C}_m}^{2s_2} ... \langle x_1 \wedge ... \wedge x_k, T_1 \wedge ... \wedge T_k \rangle_{\mathbb{C}_m}^{2s_k}. \tag{4.9.26}$$

In the case of an odd m, for odd integer weights one just has to multiply the weight vectors given above from the right by the primitive idempotents $I_1, ..., I_k$ (c.f. (4.9.3)) in order to obtain the weight of the even integer weight "$+\frac{1}{2}$" in every component.
For the case of an even m there the concept is nearly the same, except for the weights of type $(2n_1+1, ..., \pm(2n_k+1))$. For the ones with the plus sign one has to multiply the function in (4.9.26) with I_m from the right and for those with a minus sign one has to multiply with $I' = \overline{T}_m T_m$ (notation from (4.9.4)) in place of I_m.
For example in [VLSC01] we find that the eigenvalue of $H_*(\Omega)$ for the simplicial harmonic

$$\mathcal{K}_{\mathfrak{m}} := \langle x_1 T_1\rangle_{\mathbb{C}_m}^{m_1} \langle x_1 \wedge x_2, T_1 \wedge T_2\rangle_{\mathbb{C}_m}^{m_2} ... \langle x_1 \wedge ... \wedge x_k, T_1 \wedge ... \wedge T_k\rangle_{\mathbb{C}_m}^{m_k} \qquad (4.9.27)$$

of weight $m = (m_1, ..., m_k)$ is given by

$$-\sum_{j=1}^{k} k_j(m_j + m - 2j), \qquad (4.9.28)$$

while the eigenvalue of $L_*(\Omega)$ for simplicial monogenic

$$\mathcal{L}_{\mathfrak{m}} \langle x_1 T_1\rangle_{\mathbb{C}_m}^{m_1} \langle x_1 \wedge x_2, T_1 \wedge T_2\rangle_{\mathbb{C}_m}^{m_2} ... \langle x_1 \wedge ... \wedge x_k, T_1 \wedge ... \wedge T_k\rangle_{\mathbb{C}_m}^{m_k} I_1 ... I_k \qquad (4.9.29)$$

is given by

$$-\sum_{j=1}^{k} m_j(m_j + m - 2j + 1) - \frac{m(m-1)}{8}. \qquad (4.9.30)$$

Before we construct diffusive wavelets directly on $Spin(m)$, we look for diffusive wavelets on the sphere which is a homogeneous space of $Spin(m)$. If the reader is only interested in the construction of diffusive wavelets on $Spin(m)$, it is also convenient to continue with Section 4.11.

4.10 Diffusive wavelets on the sphere and Clifford analysis

We have already seen in Section 4.2.1 how we can construct wavelets on the sphere as a homogeneous space of $SO(n+1)$, of course the sphere is also a homogeneous space of the spin group. Since the representations H and L act on the argument of the function by a rotation, the invariant functions will be defined on rotation invariant subspaces. We utilize this fact to consider only functions on the sphere $S^m = \{u \in \mathbb{C}_{m+1}, \sum_A u_A e_A = \sum_{j=1}^{m+1} u_j e_j, \langle u, u\rangle_{C_{m+1}} = 1\} \subset \mathbb{C}_{m+1}$.
Since functions on the sphere depend only on one vector, one sees no longer their simplicial character. In case of simplicial monogenic functions of degree k, after this restriction we end up with the space of spherical monogenics of degree k. Following [DSS92] this space shall be denoted by $\mathcal{M}(k, V)$ or $\mathcal{M}(m, k, V)$ if we wish to emphasize the dimension of the sphere.

Values of spherical monogenics are in V which is chosen to be a spinor space or the whole Clifford algebra.
The spherical monogenics decompose further into two disjoined subspaces, namely

- The so-called inner spherical monogenics, i.e. homogeneous monogenic polynomials of degree k (harmonics of order k): $\mathcal{M}^+(m,k,V)$
- The so-called outer spherical monogenics, i.e. homogeneous monogenic functions of degree $-(k+m)$ (harmonics of order $k+1$): $\mathcal{M}^-(m,k,V)$

$\mathcal{M}^+(m,k,V)$ and $\mathcal{M}^-(m,k,V)$ are eigenspaces of the Gamma operator (c.f. (4.9.18)):

$$\begin{aligned} \Gamma_\xi P_k(\xi) &= (-k)P_k(\xi), \qquad \forall P_k \in \mathcal{M}^+(m,k,V) \\ \Gamma_\xi Q_k(\xi) &= (k+m+1)Q_k(\xi), \qquad \forall Q_k \in \mathcal{M}^-(m,k,V). \end{aligned} \tag{4.10.1}$$

and of the spherical Laplace-Beltami operator Δ_ξ:

$$\Delta_\xi P_k(\xi) = H_*(\Omega)P_k = (-k)(k+m)P_k(\xi), \qquad \forall P_k \in \mathcal{M}^+(m,k,V) \tag{4.10.2}$$

$$\Delta_\xi Q_k(\xi) = H_*(\Omega)Q_k = -(k+1)(k+m+1)Q_k(\xi), \qquad \forall Q_k \in \mathcal{M}^-(m,k,V) \tag{4.10.3}$$

The theory of these function systems is well described in [DSS92] and elsewhere. There one finds the decomposition

$$L^2(S^m,\mathbb{C}_{m+1}) = \bigoplus_{k=0}^{\infty}(\mathcal{M}(k,\mathbb{C}_{m+1})) = \bigoplus_{k=0}^{\infty}(\mathcal{M}^+(k,\mathbb{C}_{m+1}) \oplus \mathcal{M}^-(k,\mathbb{C}_{m+1})) \tag{4.10.4}$$

and P_k and Q_k form an orthogonal basis with respect to the L^2 scalar product

$$\langle f,g\rangle_{L^2} = \int_{S^m} \langle \overline{f(\xi)}g(\xi)\rangle_{\mathbb{C}_{m+1}}\, \mathrm{d}\xi.$$

The space of harmonic functions clearly contains the monogenic functions. The space of k-homogeneous functions $\mathcal{H}(m,k,\mathbb{C}_{m+1})$ can be decomposed into

$$\mathcal{H}(m,k,\mathbb{C}_{m+1}) = \mathcal{M}^+(m,k,\mathbb{C}_{m+1}) \oplus \mathcal{M}^-(m,k-1,\mathbb{C}_{m+1}). \tag{4.10.5}$$

Consequently, considering (4.9.17) and (4.10.1) we have

- The space of spherical monogenics $\mathcal{M}(k,\mathbb{C}_{m+1}) = \mathcal{M}^+(k,\mathbb{C}_{m+1} \oplus \mathcal{M}^-(k,\mathbb{C}_{m+1})$ forms the eigenspace of $L_*(\Omega)$ with respect to the eigenvalue $(-k)(k+m+1) - \binom{m+2}{2}$, i.e.

$$L_*(\Omega)P_k = (-k(k+m+1) - \binom{m+2}{2})P_k \tag{4.10.6}$$

$$L_*(\Omega)Q_k = (-k(k+m+1) - \binom{m+2}{2})Q_k. \tag{4.10.7}$$

From (4.10.3) and (4.9.15) one sees

- The space of harmonic functions $\mathcal{H}(k;\mathbb{C}_{m+1}) = \mathcal{M}^+(k,\mathbb{C}_{m+1}) \oplus \mathcal{M}^-(k-1,\mathbb{C}_{m+1})$ forms the eigenspace of $H_*(\Omega)$ with respect to the eigenvalue $(-k)(k+m)$, i.e.

$$H_*(\Omega)P_k = -k(k+m)P_k(\xi) \tag{4.10.8}$$

$$H_*(\Omega)Q_{k-1} = -k(k+m)Q_{k-1}(\xi). \tag{4.10.9}$$

For concrete calculations one has to construct the functions P_k and Q_k. Let $\alpha = (\alpha_1, ..., \alpha_{m+1}) \in \mathbb{N}^{m+1}$ denote a multi-index, with the usual notations

$$x^\alpha = x_1^{\alpha_1}...x_{m+1}^{\alpha_{m+1}} \quad \text{for } x \in \mathbb{C}^{m+1} \tag{4.10.10}$$

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}...\partial_{x_{m+1}}^{\alpha_{m+1}} \tag{4.10.11}$$

$$\alpha! = \alpha_1!...\alpha_{m+1}! \tag{4.10.12}$$

$$|\alpha| = \sum_{j=1}^{m+1} \alpha_j \tag{4.10.13}$$

Starting from a natural system of polynomials, namely $\{\frac{1}{\alpha}\xi^\alpha\}$, a system of monogenic functions can be given as Cauchy-Kovalevskaya extension of these polynomials

$$V_\alpha(\xi) = CK\left(\frac{1}{\alpha!}\xi^\alpha\right) = \sum_{j=0}^{|\alpha|} \frac{(-1)^j \xi_0^j}{j!}[(\overline{e_0}\partial_\xi)^j \xi^\alpha]. \tag{4.10.14}$$

For details we refer to [DSS92]. A basis of $\mathcal{M}^+(k,\mathbb{C}_m)$ is given by the set:

$$\{V_\alpha,\ |\alpha| = k\}. \tag{4.10.15}$$

Defining further

$$W_\alpha(\xi) = (-1)^{|\alpha|}\partial^\alpha \frac{\overline{\xi}}{A_m}, \tag{4.10.16}$$

where A_m denotes the area of S^m, a basis of $\mathcal{M}^-(k,\mathbb{C}_{m+1})$ can be given by

$$\{W_\alpha,\ |\alpha| = k\}. \tag{4.10.17}$$

Further expansions can be found in [DSS92].
With these function systems we are now in the condition to apply our method of constructing diffusive wavelets in the same way we did it for scalar-valued functions on the sphere ([BE10] and 4.2.1).

4.10.1 Heat kernel of $L_*(\Omega) - \partial_t$

We have mentioned in many places, that the Laplacian can be replaced by other operators. Using any representations U which is different from the left-regular representation but is also

in $L^2(\mathcal{G})$ such that the irreducible components give a orthogonal decomposition of the L^2-space, we can replace the Laplacian by $U_*(\Omega)$. This is exactly the situation which we have for $Spin(m+1)$ and the representation L and we would like to construct wavelets of the diffusive process which involves $L_*(\Omega)$.
Let us start by construction the heat kernel for the heat operator coming from $L_*(\Omega)$ on the sphere.
Since $H_*(\Omega)$ is the usual spherical Laplace Beltrami operator, $H_*(\Omega) - \partial_t$ represents the canonical heat operator. Its fundamental solution is given by

$$P_H(t,\xi) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \sum_{|\beta|=k-1} \exp\left(-k(k+m)t\right)\left(V_\alpha(\xi) + W_\beta(\xi)\right). \tag{4.10.18}$$

This fundamental solution allows us to obtain the series expansion of of the fundamental solution of $L_*(\Omega) - \partial_t$. It has the form

$$P_L(t,\xi) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \exp\left(\left(-k(k+m+1) - \binom{m+2}{2}\right) t\right)\left(V_\alpha(\xi) + W_\alpha(\xi)\right). \tag{4.10.19}$$

As we already mentioned we can expand $f \in L^2(S^m, \mathbb{C}_{m+1})$ into spherical monogenics by

$$f(\xi) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} (\hat{f}_V(\alpha) V_\alpha(\xi) + \hat{f}_V(\alpha) W_\alpha(\xi)), \tag{4.10.20}$$

where $\hat{f}_V(\alpha)$ and $\hat{f}_V(\alpha)$ are the Fourier coefficients.
For the construction of diffusive wavelets we can go the usual way which we developed in Chapter 3. All notations of the following section are taken from there.
There are many ways to consider the sphere as a homogeneous space. Here we look at it as $S^m \simeq SO(m+1)/SO(m)$. Let $f, h \in L^2(S^m, \mathbb{C}_m)$, then the following convolution

$$(f * h)(\xi, \omega) = \int_{SO(m+1)} \overline{f(g(\xi))} h(g(\omega))\, \mathrm{d}g, \tag{4.10.21}$$

where $\mathrm{d}g$ is taken as the Haar measure and $g(\xi)$ stands for the element obtained by the rotation g applied to ξ, gives a function on $SO(m+1)$, which is constant over co-sets $gSO(m)$ and hence defines a function on S^m [EW11].
We shall look at the invariance property of this convolution. There exist an $\eta \in SO(m)$ such that

$$(f * h)(\xi, \eta(\xi)) = (f * h)(g(\xi), g(\eta(\xi))) =: (f * h)(\eta)\ \forall g \in SO(m+1). \tag{4.10.22}$$

Since η is not unique but can be chosen as $\eta\zeta$, with ζ coming from the stabilizer of η we find $(f * h)(\eta SO(m)) = (f * h)(\eta)$ for the subgroup $SO(m)$ in $SO(m+1)$. By factoring this subgroup $(f * h)$ becomes a function on S^m.

On the other hand this function is invariant under the left action of the stabilizer in $SO(m+1)$ of ξ. Functions with this property are called to be zonal (c.f. Definition 2.2.1).
One can formulate the convolution theorem, which assumes the following form.

Theorem 4.10.1. *For $f, h \in L^2(S^m, \mathbb{C}_{m+1})$ we have*

$$f * g = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} (\hat{f}_V(\alpha)\hat{h}_V(\alpha)V_\alpha(\xi) + \hat{f}_W(\alpha)\hat{h}_W(\alpha)W_\alpha(\xi)) \tag{4.10.23}$$

Sketch of the proof: One considers the expansion of the functions into spherical monogenics. Subsequently one changes order of integration and summation, which is possible by Fubini's theorem and uses the orthonormality property of V_α and W_α. □

Definition 4.10.2. The family of functions

$$\left\{ \psi_\rho(\xi) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \exp\left(\left(-k(k+m+1) - \binom{m+2}{2} \right) \frac{t}{2} \right) (V_\alpha(\xi) + \alpha) W_\alpha(\xi) \right\}, \tag{4.10.24}$$

defines diffusive wavelets corresponding to the modified Laplace operator $L_*(\Omega) = \Delta_\xi + \Gamma_\xi - \binom{m+2}{2} \mathbf{1}$.
The corresponding wavelet transform is given by

$$WT\, f(\rho, g) := \langle f(\cdot), \psi_\rho(g^{-1}(\cdot)) \rangle_{L^2(S^m, \mathbb{C}_{m+1})}. \tag{4.10.25}$$

For the wavelet transform we have the following theorem.

Theorem 4.10.3. *The wavelet transform is invertible on its range by*

$$f(\xi) = \int_0^\infty WT\, f(\rho, g) * \psi_\varrho(g(\xi))\, \mathrm{d}\rho\ \forall f \in L^2(S^m, \mathbb{C}_{m+1}). \tag{4.10.26}$$

Proof:

$$\int_0^\infty WT\, f(\rho, g) * \psi_\rho(g(\xi))\, \mathrm{d}\rho = \int_0^\infty \int_{SO(m+1)} \left(\int_{S^m} \overline{f(\zeta)} \psi_\rho(g^{-1}(\zeta))\, \mathrm{d}\zeta \right) \Psi_\rho(g^{-1}(\xi))\, \mathrm{d}g\ \ \mathrm{d}\rho \tag{4.10.27}$$

By construction we are dealing with an diffusive approximate identity, hence the change of order of integration is valid.

$$= \int_{t\to 0}^\infty \int_{S^m} \overline{f(\zeta)} \left(\int_{SO(m+1)} \Psi_\varrho(g^{-1}(\zeta) \Psi_\varrho(g^{-1}(\xi)) \right) \mathrm{d}g)\, \mathrm{d}\zeta\ \ \mathrm{d}\varrho \tag{4.10.28}$$

$$= \int_{S^m} \overline{f(\zeta)} \left(\int_{t\to 0}^\infty (\Psi_\varrho * \Psi_\varrho)(\zeta, \xi) \right)\ \mathrm{d}\varrho\ \ \mathrm{d}\zeta \tag{4.10.29}$$

$$= \lim_{t\to 0} f * P_t(\xi) = f(\xi) \tag{4.10.30}$$

□

4.10.2 Some modifications of the operator $L_*(\Omega)$

In our approach one can easily consider the operator $\Delta - \Gamma$ instead of $L_*(\Omega)$, we just replace the eigenvalues in the series expansion of the fundamental solution by the eigenvalues of $\Delta - \Gamma$, which is obviously $-k(k+m+1)$, since these operators differ from each other only by a multiple of the identity operator

$$\sum_{k=0}^{\infty} \sum_{|\alpha|=k} \exp\left(-k(k+m+1)t\right)\left(V_\alpha(\xi) + W_\alpha(\xi)\right) \tag{4.10.31}$$

The corresponding wavelets are now of the form

$$\left\{ \psi_\rho(\xi) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \exp\left(-k(k+m+1)\frac{t}{2}\right)(V_\alpha(\xi))W_\alpha(\xi) \right\}, \tag{4.10.32}$$

Now we can easily write the diffusive wavelets with respect to further diffusive approximate identities. The importance of the magnetic Laplacian $\Delta_{mag} := \Delta + (1 - \Gamma)\Gamma$ can be motivated by physical meaning. Again from (4.10.1) and (4.10.3) the eigenvalues are of $V_\alpha \in \mathcal{M}^+(k, \mathbb{C}_m)$ with respect to the magnetic Laplacian Δ_{mag} are $-k(2k+m+1)$ and that of $W_\alpha \in \mathcal{M}^-(k, \mathbb{C}_{m+1})$ are $-(2k^2 + 3k(m+1) + (m+1)^2)$. Consequently, the corresponding diffusive wavelets are of the form

$$\psi_\rho(\xi) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \left(\exp\left(-k(2k+m+1)\frac{t}{2}\right) V_\alpha(\xi) \right.$$
$$\left. + \exp\left(-(2k^2 + 3k(m+1) + (m+1)^2)\frac{t}{2}\right) W_\alpha(\xi) \right).$$

4.11 Eigenfunction of $\Delta_{\mathrm{S}pin}$ and the heat kernel on $\mathrm{S}pin(m)$

Let us now take a look at the case of the Spin group. Eigenfunctions of $\Delta_{\mathrm{S}pin}$ on $\mathrm{S}pin(m)$ can be given as matrix coefficients of eigenvectors of $\pi_*(\Omega)$, for any irreducible representation π. All irreducible representations are of the form H or L in the subspace of simplicial harmonics or monogenics, respectively. For the moment we denote the eigenfunction with respect to the weight $(l_1, ..., l_{[\frac{m}{2}]})$ by $v_{(l_1,...,l_{[\frac{m}{2}]})}$. Consequently, all functions of the form

$$h(s) = \int_{\mathbb{C}_m} \overline{H(s)v_{(l_1,...,l_{[\frac{m}{2}]})}(a)}\, v_{(s_1,...,s_{[\frac{m}{2}]})}(a)\, \mathrm{d}a, \qquad (l_1, ..., l_{[\frac{m}{2}]}), (s_1, ..., s_{[\frac{m}{2}]}) \in (2\mathbb{Z})^{[\frac{m}{2}]} \tag{4.11.1}$$

$$l(s) = \int_{\mathbb{C}_m} \overline{L(s)v_{(l_1,...,l_{[\frac{m}{2}]})}(a)}\, v_{(s_1,...,s_{[\frac{m}{2}]})}(a)\, \mathrm{d}a, \qquad (l_1, ..., l_{[\frac{m}{2}]}), (s_1, ..., s_{[\frac{m}{2}]}) \in ((2\mathbb{Z}+1)^{[\frac{m}{2}]} \tag{4.11.2}$$

represents harmonics and harmonic functions are linear combinations of them.

We can also chose the following way:
Since we already know the eigenfunctions of $H_*(\Omega)$ and $L_*(\Omega)$ if we can express Δ_{Spin} in terms of $H_*(\Omega)$ and $L_*(\Omega)$, then we easily obtain the eigenfunctions of Δ_{Spin}.
This can be easily done for the Dirac operator on $Spin(m)$, which we denote by ∂_s. From Lemma 4.9.4 we know that a function on $Spin(m)$ can be regarded as a pair of functions $\alpha(g)$ and $\beta(g)$ on $g \in SO(m)$ or as a pair of a simplicial variable, respectively. Consequently, for a function f on $Spin(m)$ we have

$$f(s) = H(s)\alpha(a(u_1, ..., u_m)) + L(s)\beta(a(u_1, ..., u_m)), \tag{4.11.3}$$

where the simplicial variable $a(u_1, ..., u_m)$ is fixed, in order to have the dependance on s only. For the action of the Dirac operator $\partial_s = \sum_{\mathbb{C}_{m,2}} e_{ij}(H_*(e_{ij}) + L_*(e_{ij}))$ on f, by (4.9.14) and (4.9.16) we have

$$\partial_s f(s) = \sum_{i<j} e_{ij}(H_*(e_{ij})\alpha + L_*(e_{ij})\beta) \tag{4.11.4}$$

$$= \Gamma H(s)\alpha + (\Gamma - \binom{m}{2})L(s)\beta) \tag{4.11.5}$$

Hence, we can immediately deduce the eigensystem of ∂_s. The same construction we would like to have for Δ_s. Therefore we look at the action of Δ_s on $H(s)\alpha$ and $L(s)\beta$ separately:

$$\Delta_{Spin} H(s)\alpha(a(u_1, ..., u_m)) = (\sum_{j=1}^{m} \Delta_{u_j} + \sum_{k<l} \Delta_{u_k u_l})H(s)\alpha(a(u_1, ..., u_m)) \tag{4.11.6}$$

$$\Delta_{Spin} L(s)\beta(a(u_1, ..., u_m)) = (\sum_{j=1}^{m} \Delta_{u_j} + \sum_{k<l} \Delta_{u_k u_l} + \sum_{j=1}^{m} \Gamma_{u_j} - \binom{m}{2})L(s)\beta(a(u_1, ..., u_m)). \tag{4.11.7}$$

Since for the Laplacian in the components u_m we have

$$\Delta_u = \sum_{i<j} L^2_{u,e_{ij}} = \Gamma_u(m - 2 - \Gamma_u), \text{ with } \Gamma_u = u \wedge \partial_u, \tag{4.11.8}$$

the only critical point is the study of the part of the mixed Laplacian

$$\Delta_{uv} = \sum_{i<j} L_{u,e_{ij}} L_{v,e_{ij}}. \tag{4.11.9}$$

To this end we can express the action of Δ_{uv} on monogenics in terms of u, v, ∂_u and ∂_v, as we did for Δ_u. A rather technical calculation, which can be found in Appendix A.5 gives

$$\Delta_{uv} f(u, v) = - < v, \dot{\partial}_u >< u, \partial_v > \dot{f}(u, v). \tag{4.11.10}$$

where the dot means, that the derivative ∂_u is applied directly to $f(u, v)$, but not to $\langle u, \partial_v \rangle$ (Hestenes overdot notation).

Consequently, we have

$$\Delta_{\mathrm{S}pin} H(s)\alpha(a(u_1,...,u_m)) = \left(\sum_{j=1}^{m}\Gamma_{u_j}(m-2-\Gamma_{u_j}) - \sum_{k<l}\langle u_k, \dot{\partial_{u_l}}\rangle\langle u_l, \partial_{u_k}\rangle\right) H(s)\dot{\alpha}(a(u_1,...,u_m))$$

$$\Delta_{\mathrm{S}pin} L(s)\beta(a(u_1,...,u_m)) = \left(\sum_{j=1}^{m}\Gamma_{u_j}(m-2-\Gamma_{u_j}) - \sum_{k<l}\langle u_k, \dot{\partial_{u_l}}\rangle\langle u_l, \partial_{u_k}\rangle + \sum_{j=1}^{m}\Gamma_{u_j} - \binom{m}{2}\right) L(s)\dot{\beta}(a(u_1,...,u_m))$$

A closer look to the operator $\langle u_k, \dot{\partial_{u_l}}\rangle$ shows that

$$\langle u, \partial_v\rangle = \sum_{i=1}^{m} u_i \partial_{v_i}, \tag{4.11.11}$$

which can be viewed as a mixed Euler operator c.f. Appendix A.5. In fact, from the characteristic system of simplicial monogenics (4.9.24) we know that simplicial functions vanish under the mixed Euler operator.

We discussed already simplicial monogenics in Section 4.9. Let $k_1,...,k_m$ $(l_1,...,l_m)$ denote the degree of homogeneity of α (or β) in the variable $u_1,...,u_m$, respectively. Therefore, $\Gamma_{u_i}(H(s)\alpha + L(s)\beta) = (k_i\alpha + l_i\beta)$. Hence for functions $f(s) = H(s)\alpha + L(s)\beta$ on $\mathrm{S}pin(m)$ we have

$$\Delta_{\mathrm{S}pin} H(s)\alpha(a(u_1,...,u_m)) \tag{4.11.12}$$

$$= \left(\sum_{j=1}^{m}\Gamma_{u_j}(m-2-\Gamma_{u_j})\right) H(s)\dot{\alpha}(a(u_1,...,u_m)) \tag{4.11.13}$$

$$= \left(\sum_{j=1}^{m} k_j(m-2-k_j)\right) H(s)\alpha(a(u_1,...,u_m)) \tag{4.11.14}$$

$$\tag{4.11.15}$$

and

$$\Delta_{\mathrm{S}pin} L(s)\beta(a(u_1,...,u_m)) \tag{4.11.16}$$

$$= \left(\sum_{j=1}^{m}\Gamma_{u_j}(m-2-\Gamma_{u_j}) + \sum_{j=1}^{m}\Gamma_{u_j} - \binom{m}{2}\right) L(s)\dot{\beta}(a(u_1,...,u_m)) \tag{4.11.17}$$

$$= \left(\sum_{j=1}^{m} l_j(m-2-l_j)\right) L(s)\beta(a(u_1,...,u_m)). \tag{4.11.18}$$

Such that according to our construct wavelets, Clifford-valued diffusive wavelets on $\mathrm{S}pin(m)$ assume the form

$$\psi_\rho(s) = \sum_{k=1}^{\infty} \sum_{\mathfrak{m}\in\mathbb{Z}^k} \exp\left(\left(\sum_{j=1}^{m} k_j(m-2-k_j)\right)\frac{t}{2}\right) H(s)\mathcal{K}_{\mathfrak{m}} \tag{4.11.19}$$

$$+\exp\left(\left(\sum_{j=1}^{m} k_j(m-2-k_j)\right)\frac{t}{2}\right) L(s)\mathcal{L}_{\mathfrak{m}} \tag{4.11.20}$$

where $\mathcal{K}_{\mathfrak{m}}$ and $\mathcal{L}_{\mathfrak{m}}$ form a complete system of simplicial functions. (see (4.9.27) and (4.9.29)).

Chapter 5

Diffusive wavelets and Radon transform on $SO(3)$

5.1 Radon transform on compact Lie Groups

A comprehensive discussion of Radon transforms on $\mathbb{R}^n$ and also on homogeneous spaces can be found in the book of Helgason [Hel99, Hel11]. In 1917 J. Radon showed that a differentiable function on $\mathbb{R}^2$ and $\mathbb{R}^3$ can be reconstructed from their values of integrals over hyperplanes. What are the submanifolds of integration for a Radon transform on another manifold? Having an application of our wavelets in mind we answer this question for compact Lie groups $\mathcal{G}$. In the example of the spherical Radon transform on S^2 the integrals are taken over great circles, that are orbits of the action of $SO(2)$. The great circles can be parameterized by the points, which are invariant under rotations which has the corresponding great circle as orbit. Introducing $\theta \in S^2$ as the parameter of the great circle $\{\xi \in S^2, \theta \cdot \xi = 0\} \subset S^2$ this transformation is not invertible, since θ and $-\theta$ represents the same great circles. But it clearly becomes invertible if we restrict it to even functions on S^2.

In sketched situation we look at functions on S^2 or equivalently on those on $SO(3)$, which are constant on right co-set of the form $gSO(2)$. Applying further the Radon transform leads to a further averaging over left co-sets $SO'(2)g$, where $SO'(2)$ means that the left co-set can be taken based on another subgroup than the right co-set. Right as well as left co-sets can be parameterized by points on S^2 which are invariant under its action.

This leads to the definition of the Radon transform on $\mathcal{G}$ which shall be defined as an integral over right– and left-translated subgroups $\mathscr{H}$ of $\mathcal{G}$.

Further we will show that the Radon transform of wavelets on $SO(3)$ gives wavelets on S^2. This can also be found in [BE10], while we start the discussion here in a more general manner.

Definition 5.1.1. Let $\mathscr{H}$ be a subgroup of the compact Lie group $\mathcal{G}$. The Radon transform of a integrable function f on $\mathcal{G}$ is defined by

$$\mathscr{R}f(x,y) = \int_{\mathscr{H}} f(xhy^{-1})\,\mathrm{d}h \qquad x,y \in \mathcal{G}, \tag{5.1.1}$$

where d here is the normalized Haar measure on $\mathscr{H}$.

Next we discuss the range of the Radon transform $\mathscr{R}$. Since x, y in (5.1.1) are in $\mathcal{G}$ a first look gives the impression that the Radon transform is defined over $\mathcal{G} \times \mathcal{G}$. But by deeper investigation we see that $\mathscr{R}f(x,y)$ is invariant under right shifts of x and y, hence $\mathscr{R}$ is defined over $\mathcal{G}/\mathscr{H} \times \mathcal{G}/\mathscr{H}$.
To prove this fact we look at $\mathscr{R}$ in Fourier domain. There we find that $\mathscr{R}$ acts in the following way: Let first $y \in \mathcal{G}$ be fixed and regard $\mathscr{R}f(\cdot,y)$ as a function on $\mathcal{G}$ in the first argument, then

$$\widehat{\mathscr{R}f(\cdot,y)}(\pi) = \pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi) \qquad \pi \in \widehat{\mathcal{G}}. \tag{5.1.2}$$

Hence the function $\mathscr{R}f(\cdot,y)$ is invariant under the projection $\mathbb{P}_{\mathscr{H}}$, since the Fourier coefficients are invariant under the left multiplication by $\pi_{\mathscr{H}}$: $\pi_{\mathscr{H}}\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi) = \pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)$. Consequently, we have

$$\mathscr{R}f(x \cdot h, y) = \mathscr{R}f(x,y) \qquad \forall h \in \mathscr{H}. \tag{5.1.3}$$

Now a look at the Radon transform as function in the second argument y, while the first argument x is fixed, we find

$$\begin{aligned}
\mathbb{P}_{\mathscr{H}}\mathscr{R}f(x,y) &= \int_{\mathscr{H}} \mathscr{R}f(x,yh)\,\mathrm{d}h \\
&= \int_{\mathscr{H}} \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \operatorname{trace}(\widehat{f}(\pi)\pi(x))\pi_{\mathscr{H}}\pi(h^{-1}y^{-1})\,\mathrm{d}h \\
&= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \operatorname{trace}(\widehat{f}(\pi)\pi(x))\pi_{\mathscr{H}}\pi^*(y) = \mathscr{R}f(x,y).
\end{aligned} \tag{5.1.4}$$

Hence, $\mathscr{R}f(x,y)$ is constant over fibers of the form $y\mathscr{H}$ and

$$\widehat{\mathscr{R}f(x,\cdot)}(\pi) = \pi_{\mathscr{H}}\overline{\pi^*(x)\widehat{f}(\pi)^*}. \tag{5.1.5}$$

Consequently, $\mathscr{R}$ maps functions over $\mathcal{G}$ to functions over $\mathcal{G}/\mathscr{H} \times \mathcal{G}/\mathscr{H}$. Now an interesting question is to determine the concrete spaces for the domain and range of $\mathscr{R}$. We will restrict us here to consider the Radon transform over the space $L^2(\mathcal{G})$.

Theorem 5.1.2. *Let $\mathscr{H}$ be the subgroup of $\mathcal{G}$ which determines the Radon transform on $\mathcal{G}$ and let $\widehat{\mathcal{G}}_1 \subset \widehat{\mathcal{G}}$ be the set of irreducible representations with respect to $\mathscr{H}$. Then for $f \in C^\infty(\mathcal{G})$ we have*

$$\|\mathscr{R}f\|^2_{L^2(\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H})} = \sum_{\widehat{\mathcal{G}}_1} rank(\pi_{\mathscr{H}})\|\widehat{f}\|^2_{HS}. \tag{5.1.6}$$

Proof: For the proof we expand $\mathscr{R}f(x,y)$ for a fixed y as function in x over $\mathcal{G}$ (or better $\mathcal{G}/\mathscr{H}$) and apply Parseval's identity (2.1.16). With (5.1.2) we have

$$\begin{aligned}
\|\mathscr{R}f\|_{L^2(\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H})} &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \int_{\mathcal{G}} \|\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)\|_{HS}^2 \,\mathrm{d}y \\
&= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \int_{\mathcal{G}} \operatorname{trace}\left(\widehat{f}^*(\pi)\pi(y)\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)\right) \,\mathrm{d}y \\
&= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \operatorname{trace}\left(\widehat{f}\int_{\mathcal{G}} f^*(\pi)\pi(y)\pi_{\mathscr{H}}\pi^*(y)\,\mathrm{d}y\right) \\
&= \sum_{\pi\in\widehat{\mathcal{G}}_1} \operatorname{rank}(\pi_{\mathscr{H}}) \operatorname{trace}(\widehat{f}^*\widehat{f})
\end{aligned}$$

Here we made use of the fact

$$\begin{aligned}
\int_{\mathcal{G}} \widehat{f}^*(\pi)\pi(y)\pi_{\mathscr{H}}\pi^*(y)\,\mathrm{d}y &= \left(\sum_{k=1}^{\operatorname{rank}\pi_{\mathscr{H}}} \int_{\mathcal{G}} \pi_{ik}(y)\overline{\pi_{kj}(y)}\,\mathrm{d}y\right)_{i,j=1}^{d_\pi} \\
&= \frac{\operatorname{rank}(\pi_{\mathscr{H}})}{d_\pi} Id.
\end{aligned}$$

□

The Theorem 5.1.2 give us the important result, that the Radon transform is an isometry between $L^2(\mathcal{G})$ and the some Sobolev space on $\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H}$. The Sobolev space is determined by the class one representations of $\mathcal{G}$ with respect to $\mathscr{H}$ or more precisely by the dimension of the $\mathscr{H}$ invariant vectors for all representations of $\mathcal{G}$. Consequently, the inversion formula can be given as the adjoined operator of the Radon transform.
In the next section we will have a detailed look at this situation for the Radon transform on $SO(3)$.

5.1.1 Radon transform on $SO(3)$

The Radon transform on $SO(3)$ is intensively investigated, examples are [BS05], [Hie07], [Hel99], [Hel11], [BE10]. One of the reasons is, that the subgroup over which the integration is taken is $\mathscr{H} = SO(2)$, which has practical applications in crystallography, a field of texture analysis and geophysics.
We will look at it from our point of view which we build up in the previous chapters.
The practical problem can be described as follows[1]. The desire is to determine the structure of a specimen of crystals. Because of the structure of the crystal one can equip it with an inner orthogonal coordinate system $\{e_1, e_2, e_3\}$. Additionally one distinguish an outer orthogonal coordinate system $\{u_1, u_2, u_3\}$ related to the specimen. The orientation of a crystal in the

[1] For simplicity we neglect here spherical symmetries.

specimen is defined by the unique rotation $\gamma \in SO(3)$ which maps the inner coordinate system to the outer one, i.e. $ge_i = u_i$ for $i = 1, 2, 3$.
Now, the function of interest is the *orientation density fiction (ODF)* $f \in L^2(SO(3))$ that is a probability measure on $SO(3)$. The function value $f(g)$ gives the amount of crystals in the specimen with orientation g.
The practical measurement sends a electron beam through the specimen coming from the direction $h \in S^2$ and measures the intensity of electrons, emitted from the specimen in the direction $r \in S^2$. One can interpret the result as the integral over all orientations $g \in SO(3)$ with $g \cdot h = r$, the set of those orientations are called great circle $C_{hr} = \{g \in SO(3),\, g \cdot h = r\}$ in $SO(3)$. The situation is sketched in Figur5.1.

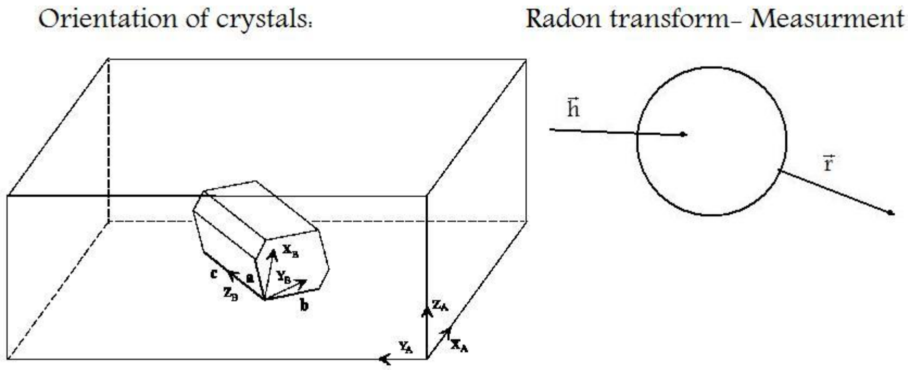

Figure 5.1: Orientation of a crystal in a specimen, Radon measurements

It is clear that the great circle is given by

$$C_{h,r} = h'\, SO(2)\, r'^{-1} := \{h'gr',\, h \in SO(2)\} \qquad h', r' \in SO(3), \tag{5.1.7}$$

where $h', r' \in SO(3)$ satisfy $h' \cdot \xi_0 = h$ and $r' \cdot \xi_0 = r$ with $SO(2)$ being the stabilizer of $\xi_0 \in S^2$, hence ξ_0 is the north pole.
For the Radon transform we have.

Definition 5.1.3. The Radon transform on $SO(3)$ is defined by

$$\mathscr{R} f(x, y) = \int_{C_{x,y}} f(g)\, \mathrm{d}g \qquad f \in L^2(SO(3)). \tag{5.1.8}$$

Definition 5.1.4. The *Sobolev space* $H_t(\mathcal{G})$ on a compact Lie group is defined as the domain of the operator $(Id - \Delta)^t$ in $L^2(\mathcal{G})$:

$$H_t(\mathcal{G}) := \{f \in L^2(\mathcal{G}),\ \|f\|_t^2 = \|(Id - \Delta)^{\frac{t}{2}} f\|_{L^2(\mathcal{G})}^2 < \infty\} \tag{5.1.9}$$

Theorem 5.1.5. *The Radon transform on $SO(3)$ is an invertible mapping*

$$\mathscr{R} : L^2(SO(3)) \to H_{\frac{1}{2}}(S^2 \times S^2) \tag{5.1.10}$$

Proof: With $d_k = 2k+1$ is the dimension of the irreducible representations and $-\lambda_k^2 = -k(k+1)$ are the eigenvalues of the Laplacian Δ we have $d_k = \sqrt{1+4\lambda_k^2}$. Furthermore, for $SO(3)$ we have $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}_1$. Now the assertion follows from (5.1.6)

$$\|\mathscr{R}f\|_{L^2(S^2\times S^2)} = \|(1-4\Delta)^{-\frac{1}{4}} f\|_{L^2(\mathcal{G})}. \tag{5.1.11}$$

□

Remark 5.1.6. From Theorem 5.1.5 we deduce the reconstruction formula for the Radon transform on $SO(3)$ so let

$$f(x,y) = \sum_{k=0}^{\infty}\sum_{i,j=1}^{2k+1} \widehat{f}(k,i,j)\mathcal{Y}_k^i(x)\overline{\mathcal{Y}_k^j(y)} \in H_{\frac{1}{2}}(S^2\times S^2) \tag{5.1.12}$$

be the result of a Radon transform. Then the pre-image $g \in L^2(SO(3))$ is given by

$$g = \sum_{k=0}^{\infty}\sum_{i,j=1}^{2k+1} \frac{(2k+1)}{4\pi}\widehat{f}(k,i,j)T_{ij}^k = \sum_{k=0}^{\infty}(2k+1)\operatorname{trace}(\widehat{g}(k)T^k) \tag{5.1.13}$$

$$\tag{5.1.14}$$

From which follows for the Fourier coefficients

$$\widehat{g}(k)_{ij} = \frac{1}{4\pi}\widehat{f}(k,j,i). \tag{5.1.15}$$

5.1.2 Radon transform of wavelets on $SO(3)$

Let us now take a look at our wavelets on $SO(3)$, which we constructed in Section 4.3, c.f. (4.3.5). For these wavelets we have the following result.

Lemma 5.1.7. *Let $\{\Psi_\rho,\, \rho > 0\}$ be a family of class type*[1] *wavelets on $SO(3)$, then the family of functions $\{\mathscr{R}\Psi_\rho(x,\cdot), \rho > 0, \xi \in S^2$ fixed$\}$ defines a family of zonal wavelets on S^2.*

The lemma can be seen in the following way.
The general formula for the Fourier expansion of the Radon transform (5.1.4) reads in our case as

$$\mathscr{R}f(x,y) = \sum_{k=0}^{\infty}(2k+1)\operatorname{trace}(\widehat{f}(k)T^k(x)\pi_{SO(2)}T^*(y)) \tag{5.1.16}$$

$$= \sum_{k=0}^{\infty}(2k+1)\sum_{i,j=1}^{2k+1}\widehat{f}(k)_{ij}T_{j1}^k(x)\overline{T_{i1}^k(y)} \tag{5.1.17}$$

$$= 4\pi\sum_{k=0}^{\infty}\sum_{i,j=1}^{2k+1}\widehat{f}(k)_{ij}\mathcal{Y}_k^i(x)\overline{\mathcal{Y}_k^j(y)}, \tag{5.1.18}$$

[1] Every wavelet is a class type function in case of $\eta_\rho(\pi) = Id$, which we used in (4.3.5).

where we also considered (4.2.18). This formula can also be found in [BS05].
Recall the form of our diffusive wavelets on $SO(3)$ (4.3.5) we find for the Fourier coefficients

$$\widehat{\psi}_\rho(k) = \frac{1}{\sqrt{\alpha(\rho)}}\frac{1}{4\pi}\sqrt{k(k+1)}e^{-\frac{k(k+1)}{2}\rho}Id. \tag{5.1.19}$$

Hence the Radon transform of ψ_ρ yields

$$\mathscr{R}\psi_\rho(x,y) = \frac{1}{\sqrt{\alpha(\rho)}}\sum_{k=0}^{\infty}\sum_{i,j=1}^{2k+1}\sqrt{k(k+1)}e^{-\frac{k(k+1)}{2}\rho}\delta_{ij}\mathcal{Y}_k^i(x)\overline{\mathcal{Y}_k^j(y)} \tag{5.1.20}$$

$$= \frac{1}{\sqrt{\alpha(\rho)}}\sum_{k=0}^{\infty}(2k+1)\sqrt{k(k+1)}e^{-\frac{k(k+1)}{2}\rho}C_k^{1/2}(x\cdot y). \tag{5.1.21}$$

This can be easily seen by Theorem 4.2.3 and $C_k^{1/2}(1) = 1$. Hence, the image of the wavelets under the Radon transform are exactly the wavelets we constructed for S^2 (c.f. (4.3.3)). The choice of $x \in S^2$ corresponds to the choice of the point to which the wavelets are zonal and by application of the translation operator all wavelets can be mapped onto the zonal wavelet family, given by the choice x being the north pole.

5.1.3 Radon transform of non-class type functions

We chose now wavelets on $SO(3)$, where we make a non-trivial choice of $\eta_\rho(k)$, hence we chose non-zonal wavelets. Furthermore, we assume that $\eta_\rho(\pi)$ is independent of ρ without loss of generality. We will show, that the Radon transform will result in non-zonal wavelets on S^2. The general form of wavelets on $SO(3)$ (c.f. (4.3.5)) is given by the Fourier coefficients

$$\widehat{\psi}_\rho(k) = \frac{1}{\sqrt{\alpha(\rho)}}\frac{1}{4\pi}\sqrt{k(k+1)}e^{-\frac{k(k+1)}{2}\rho}\eta_\rho(k) \qquad \eta_\rho(k) \in U(2k+1). \tag{5.1.22}$$

Now, the Radon transform yields

$$\mathscr{R}\Psi_\rho(x,y) = \sum_{k=0}^{\infty}\sqrt{k(k+1)}e^{-\frac{k(k+1)}{2}\rho}\sum_{i,j=1}^{2k+1}(\eta_\rho(k))_{ij}\mathcal{Y}_k^i(x)\overline{\mathcal{Y}_k^j(y)} \tag{5.1.23}$$

since the vector $(\mathcal{Y}_k^i(x))_{i=1}^{2k+1}$ has Euclidean norm $\sqrt{\frac{2k+1}{4\pi}}$ (by Theorem 4.2.3). Since $\eta_\rho(k)$ is unitary the vector

$$\omega_j(k) := \sqrt{\frac{4\pi}{2k+1}}\eta_\rho(k)(\mathcal{Y}_k^i(x))_{i=j} \tag{5.1.24}$$

has also Euclidean norm 1. Consequently we obtain exactly the form (4.2.20) of a non-zonal spherical diffusive wavelet for S^2:

$$\mathscr{R}\Psi_\rho(x,y) = \sum_{k=0}^{\infty}\frac{(2k+1)}{4\pi}\sqrt{k(k+1)}e^{-\frac{k(k+1)}{2}\rho}\sqrt{\frac{4\pi}{2k+1}}\sum_{j=1}^{2k+1}\omega_j(k)\overline{\mathcal{Y}_k^j(y)} \tag{5.1.25}$$

5.2 Variational interpolation problem

For the application in texture analysis, the number of measurements is finite. Consequently, the invertibility will be lost. Since the Radon transform of a function on a Lie group $\mathcal{G}$ is given as an integral over submanifolds of $\mathcal{G}$ we will look at the situation in the following way. The Radon transform is the collection of functionals $F_{x,y} := \mathscr{R}(x,y)$ which maps a function f to the integral over the submanifold $x\mathscr{H}y$ for some sub-group $\mathscr{H}$ of $\mathcal{G}$ and $x, y \in \mathcal{G}$. In the application in texture analysis we have a finite set of functionals F_{x_ν,y_ν} for $\nu = 1, ..., N$ and we have to find a good approximation of f from $F_{x_\nu,y_\nu}(f)$.
This task is formulated in [Pes04] as a variational spline problem. There a set of functionals F_ν is given as integrals over d_ν-dimensional submanifolds $\mathcal{M}_\nu$ of a d-dimensional Riemannian manifold $\mathcal{M}$ $(0 \le d_\nu \le d)$. We assume a finite number N of manifolds $\mathcal{M}_\nu$,

$$F_\nu(f) = \int_{\mathcal{M}_\nu} f(x)\,\mathrm{d}x = v_\nu \qquad \nu = 1, ..., N. \tag{5.2.1}$$

The variational spline problem fits in some sense optimal to the practical question of determining the ODF f from measurements of the Radon transformed f. On the one hand we are interested in regions where the values of f are large but if the curvature of f is small in those regions it would be more useful to increase the measurements around the maximum of f and those points where the curvature is large. Hence the right criteria is the value of $(1 + Delta)f$ and on should increase the density of measurements around points where $(1 + \Delta)f$ is large. The density of measurements should be high at those points where the interpolation is highly nonlinear.

Definition 5.2.1. The Sobolev space $H_t(\mathcal{M})$ is defined by

$$H_t(\mathcal{M}) = \{\|f\|_t := \|(1 + \Delta_\mathcal{M})^{t/2} f\|_{L^2(\mathcal{M})} < \infty\}, \tag{5.2.2}$$

where $\Delta_\mathcal{M}$ denotes the Laplace-Beltrami operator on $\mathcal{M}$.

Now the question of interest is to find for given v_ν, $\nu = 1, ..., N$, a function $s_t(f) \in H_t(\mathcal{M})$ with

$$\left.\begin{array}{ll} F_\nu(s_t(f)) & = v_\nu \\ \|s_t(v)\|_t & \to \min\,. \end{array}\right\} \tag{5.2.3}$$

Definition 5.2.2. A set of functionals F_ν is called to be independent, if there are test functions $\varphi_\mu \in C_0^\infty(\mathcal{M})$ $(\mu, \nu = 1, ..., N)$ such that:

$$F_\nu(\varphi_\mu) = \delta_{\nu\mu}, \tag{5.2.4}$$

where $\delta_{\nu\mu}$ denotes the Kronecker delta.

The essential result in [Pes04], which we utilize here is the following.

Theorem 5.2.3. *Let F_ν $(\nu = 1, ..., N)$ be a set of linear functionals, independent in the sense of Definition 5.2.2 and belonging to H_{-t_0}[1]. Then for $t > t_0 + d/2$ and a given vector*

[1] In distributional sense, i.e. $F_\nu : H_{t_0}(\mathcal{M}) \to L^2(\mathcal{M})$ and $(H_{t_0})' = H_{-t_0}$

$v = (v_\nu)_{\nu=1}^N \in \mathbb{R}^n$ *the solution of* (5.2.3) *is given by*

$$s_t(v) = \sum_{j=0}^{\infty} c_j(s_t(v))\phi_j, \tag{5.2.5}$$

where ϕ_j *are the eigenfunctions of* $\Delta_{\mathcal{M}}$ *with respect to the eigenvalues* $-\lambda_j^2$*. The Fourier coefficients* $c_j(s_t(v))$ *are given by*

$$c_j(s_t(v)) = (1-\lambda_j^2)^{-t} \sum_{\nu=1}^{N} \alpha_\nu(s_t(v))F_\nu(\varphi_j), \tag{5.2.6}$$

where $\alpha(s_t(v)) = (\alpha_\nu(s_t(v)))_{\nu=1}^N \in \mathbb{R}^N$ *solves*

$$\beta\alpha(s_t(v)) = v, \tag{5.2.7}$$

with $\beta \in \mathbb{R}^{N\times N}$ *is given by*

$$\beta_{\nu\mu} = \sum_{j=1}^{\infty}(1-\lambda_j^2)^{-t}\overline{F_\nu(\varphi_j)}F_\mu(\varphi_j). \tag{5.2.8}$$

By the independence assumption of F_ν the system (5.2.7) is solvable for all $v \in \mathbb{R}^N$.
The assumption $t > t_0 + d/2$ ensures that (5.2.8) converges (see [Pes04] for description).
We continue by applying this theorem to our case of a manifold being a compact Lie group.
We used frequently that the eigenfunctions of $\Delta_{\mathcal{G}}$ for compact Lie groups $\mathcal{G}$ are given by matrix coefficients π_{ij} of all irreducible representations π. Again $\widehat{\mathcal{G}}$ shall denote the set of all irreducible representations and the characters of π are again given by $\chi_\pi = \operatorname{trace}(\pi)$.
The task to find a function f on $\mathcal{G}$ so that for a given finite set $\{\mathscr{R}f(x_\nu, y_\nu),\ \nu = 1, ..., N\}$ the function f solves Problem (5.2.3) for the special case of the Radon transform on $\mathcal{G}$. We substitute the relevant notions in order to obtain the formulation for the case of the Radon transform. Hence $d_\nu = dim\mathscr{H}\ \forall\nu = 1, ..., N$ and $\mathcal{M}_\nu = x_\nu\mathscr{H}y_\nu^{-1}$.
This means we have to solve the linear system (5.2.7) for the special case where β, given in (5.2.8) assumes the form

$$\beta_{\mu\nu} = \sum_{\pi\in\widehat{\mathcal{G}}}(1-\lambda_\pi^2)^{-t} \sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu, y_\nu))}\mathscr{R}(\pi_{ij}(x_\mu, y_\mu)). \tag{5.2.9}$$

In order to determine the entries of the matrix β we have to calculate $\mathscr{R}\pi(g_\nu)$ and $\sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu, y_\nu))}\mathscr{R}(\pi_{ij}(x_\mu, y_\mu))$.
Here we have

$$\sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu,y_\nu))}\mathscr{R}(\pi_{ij}(x_\mu,y_\mu)) = \sum_{i,j=1}^{d_\pi} \int_{\mathscr{H}} \overline{\pi_{ij}(x_\nu h y_\nu^{-1})\,\mathrm{d}h} \int_{\mathscr{H}} \pi_{ij}(x_\mu h y_\mu^{-1})\,\mathrm{d}h \tag{5.2.10}$$

$$= \sum_{i,j=1}^{d_\pi} \int_{\mathscr{H}} \int_{\mathscr{H}} \pi_{ji}(y_\nu h x_\nu^{-1})\,\mathrm{d}h \pi_{ij}(y_\mu h x_\mu^{-1})\,\mathrm{d}h \tag{5.2.11}$$

$$= \operatorname{trace}(\pi_{\mathscr{H}}\pi(y_\nu)\pi_{\mathscr{H}}\pi(x_\nu^{-1})\pi_{\mathscr{H}}\pi(x_\mu)\pi_{\mathscr{H}}\pi(y_\mu^{-1})). \tag{5.2.12}$$

Hence we obtain the function that is zonal in every component. A special case is given by the addition theorem 4.2.3 of spherical harmonics. Since we have the application of the Radon transform on $SO(3)$ in mind and $SO(2)$ is a massive subgroup in $SO(3)$ we would like to study weather there is any simplification for this case. Indeed, when $\mathscr{H}$ is a massive subgroup[1] of $\mathcal{G}$ we find

$$\sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu,y_\nu))}\mathscr{R}(\pi_{ij}(x_\mu,y_\mu)) = \pi_{11}(y_\nu)\pi_{11}(x_\nu^{-1})\pi_{11}(x_\mu)\pi_{11}(y_\mu^{-1}) \tag{5.2.13}$$

$$= \pi_{11}(y_\nu x_\nu^{-1} x_\mu y_\mu^{-1}), \tag{5.2.14}$$

and hence we obtain for the matrix coefficients

$$\beta_{\mu\nu} = \sum_{\pi\in\widehat{\mathcal{G}}} (1-\lambda_\pi^2)^{-t}\pi_{11}(y_\nu x_\nu^{-1} x_\mu y_\mu^{-1}) \tag{5.2.15}$$

where $\widehat{G}_1$ denotes the set of irreducible representations with $\operatorname{rank}\pi_{\mathscr{H}} = 1$.

5.2.1 Variational interpolation problem for the Radon transform on $SO(3)$

Special functions of rotation group of arbitrary dimension and related theorems are discussed in Section 4.2.1. Special functions and relations between them for the special case of $SO(3)$ and S^2 are given in Section 4.3. Here we briefly recall some facts in order to remind the notation and to have all relations at hand for investigations of the Radon transform on $SO(3)$ All irreducible representations are equivalent to an irreducible component of the left regular representation

$$T(g) : f(\xi) \mapsto f(g^{-1}\cdot x), \tag{5.2.16}$$

where $\cdot$ denotes the canonical action of $SO(3)$ on S^2. The T invariant subspaces of $L^2(S^2)$ are $\mathcal{H}_k = \{\mathcal{Y}_k^i, i = 1,...,2k+1\}$- spanned by spherical harmonics of degree k. T^k shall denote the irreducible representation, obtained by restriction of T to $\mathcal{H}_k$.

[1] i.e. $\operatorname{rank}\pi_{\mathscr{H}} \le 1$

The matrix coefficients of T^k are the Wigner polynomials T^k_{ij} of degree k:

$$\mathcal{Y}^j_k(g^{-1}\cdot\xi) = \sum_{i=1}^{2k+1} T^k_{ij}(g) Y^i_k(\xi) \qquad T^k_{ij} = \langle \mathcal{Y}^j_k(g^{-1}\cdot), Y^i_k(\cdot)\rangle_{L^2(S^2)}. \tag{5.2.17}$$

Since matrix coefficients always have the norm $\frac{1}{d_\pi}$, where $d_\pi = 2k+1$, we have

$$T^k_{i1}(g) = \sqrt{\frac{4\pi}{2k+1}}\mathcal{Y}^i_k(g\cdot\xi_0), \tag{5.2.18}$$

where $\xi_0 \in S^2$ is the base point of $SO(3)/SO(2) \sim S^2$, often chosen as north pole. The eigenvalues of Laplacian on $SO(3)$ and on S^2 corresponding to polynomials of degree k is $-k(k+1)$, i.e. $\Delta_{SO(3)}T^k_{ij} = -k(k+1)T^k_{ij}$ and $\Delta_{S^2}\mathcal{Y}^i_k = -k(k+1)\mathcal{Y}^i_k$.
Furthermore the dimension of zonal functions in $\mathcal{H}_k$ is one and is spanned by Gegenbauer polynomial of order $C^{\frac{1}{2}}_k(\xi_0\cdot\xi)$. Consequently, zonal functions depend only on the angle between the argument ξ and the base point (north pole).
The Addition Theorem 4.2.3 for S^2 assumes the following form.

Theorem 5.2.4 (Addition theorem)**.** *For all $\xi,\eta \in \mathbb{S}^2$ and $k \in \mathbb{N}_0$*

$$\mathcal{C}^{\frac{1}{2}}_k(\xi\cdot\eta) = \frac{4\pi}{2k+1}\sum_{i=1}^{2k+1}\mathcal{Y}^i_k(\xi)\overline{\mathcal{Y}^i_k(\eta)}. \tag{5.2.19}$$

Let us now take a look at the concrete case of $\mathscr{R}$ on $SO(3)$.
In order to determine β for our problem at hand, we have to calculate
$\mathscr{R}(T^k_{ij})$. Since $\mathscr{R}(T^k)(x,y) = T^k(x)\pi_{SO(2)}(T^k(y))^*$ we have

$$\mathscr{R}T^k_{ij}(\xi,\eta) = T^k_{i1}(\xi)\overline{T^k_{j1}(\eta)} = \frac{4\pi}{2k+1}\mathcal{Y}^i_k(\xi)\overline{\mathcal{Y}^j_k(\eta)}. \tag{5.2.20}$$

Consequently, for the variational spline problem with $t>1$ we have

$$\beta_{\nu\mu} = \sum_{k=0}^{\infty}(1-k(k+1))^{-t}\left(\frac{4\pi}{2k+1}\right)^2\sum_{i,j=1}^{2k+1}\overline{\mathcal{Y}^i_k(\xi_\nu)}\mathcal{Y}^j_k(\eta_\nu)\mathcal{Y}^i_k(\xi_\mu)\overline{\mathcal{Y}^j_k(\eta_\mu)} \tag{5.2.21}$$

$$= \sum_{k=0}^{\infty}(1-k(k+1))^{-t}C^{\frac{1}{2}}_k(\xi_\nu\cdot\eta_\nu)C^{\frac{1}{2}}_k(\xi_\mu\cdot\eta_\mu), \tag{5.2.22}$$

where we made use of Addition Theorem 5.2.4.
Summarizing the results of this section we have the following theorem. Given the problem of the Radon transform on $SO(3)$

$$\left.\begin{array}{ll}\mathscr{R}(s_t(f))(x_\nu,y_\nu) & = v_\nu \\ \|s_t(v)\|_t & \to \min .\end{array}\right\} \tag{5.2.23}$$

Theorem 5.2.5. *Let $\{(x_1, y_1), ..., (x_N, y_N)\}$ be a set of pairs of points from $SO(3)$, such that there are test functions $\phi_1, ..., \phi_N$ with*

$$\mathscr{R}\phi_\mu(x_\nu, y_\nu) = \delta_{\nu\mu}.$$

Then for $t > \frac{3}{2}$ and a vector (of measurements) $v = (v_\nu)_{\nu=1}^N \in \mathbb{R}^N$ the solution of (5.2.23) is given by

$$s_t(v) = \sum_{k=0}^{\infty} \sum_{i,j=1}^{2k+1} c_k^{ji}(s_t(v)) T_{ij}^k = \sum_{k=0}^{\infty} \operatorname{trace}(c_k(s_t(v))T^k), \tag{5.2.24}$$

where T_{ij}^k are the Wiegner polynomials. The Fourier coefficients $c_k(s_t(v))$ of the solution are given by their matrix entries

$$c_k^{ji}(s_t(v)) = (1 - k(k+1))^{-t} \sum_{\nu=1}^{N} \alpha_\nu(s_t(v)) \mathscr{R}(T_{ij}^k)(x_\nu, y_\nu), \tag{5.2.25}$$

whereby $\alpha(s_t(v)) = (\alpha_\nu(s_t(v)))_{\nu=1}^N \in \mathbb{R}^N$ is the solution of

$$\beta\alpha(s_t(v)) = v, \tag{5.2.26}$$

with $\beta \in \mathbb{R}^{N \times N}$ given by

$$\beta_{\nu\mu} = \sum_{k=0}^{\infty} (1 - k(k+1))^{-t} C_k^{\frac{1}{2}}(\xi_\nu \cdot \eta_\nu) C_k^{\frac{1}{2}}(\xi_\mu \cdot \eta_\mu). \tag{5.2.27}$$

For applications, one just has to apply standard methods to solve (5.2.26) and one will get the solution (5.2.24). A discussion of the stability of the solution will involve the condition number of the matrix β. But the concrete discussion depends on the choice of the solution method. Since here we do not lead the discussion of the numerics we restrict to have a look at the Shannon sampling theorem on a rather abstract level.
We assume a signal $f \in \mathcal{G}$ that is bandlimited i.e. $\hat{f}(\pi) \neq 0$ only for finite many $\pi \in \widehat{\mathcal{G}}$. We denote $\widehat{\mathcal{G}}_f = \{\pi, \ \hat{f}(\pi) \neq 0\}$ and let $d_{max} := \max_{\pi \in \widehat{\mathcal{G}}_f} d_\pi$ the corresponding representation of dimension d_{max} shall be denoted by π_{max}. For a set of points $X = \{g_{a,b} \in \mathcal{G}, \ a, b = 1, ..., d_{max}\}$ with

$$\det \left(\pi_{ij}(g_{(a,b)})\right)_{(a,b),(i,j)=(1,1)}^{(d_{max},d_{max})} \neq 0 \tag{5.2.28}$$

By the expression $\left(\pi(g_{(a,b)})_{ij}\right)_{(a,b),(i,j)=(1,1)}^{(d_{max},d_{max})}$ we denote the matrix where the matrix coefficients of π vary along the rows evaluated at a point $g_{(a,b)}$. The point $g_{(a,b)}$ vary along the lines over X.

Existence of points with (5.2.28)

Let $\{u_i\}_{i=1}^{d_\pi}$ be a basis in the representation Hilbert space $\mathcal{H}$. By irreducibility of π we can chose points $g_{ab} = g_a g_b$ in $\mathcal{G}$ so that $\pi(g_b)u_i = u_{i(b)}$ and $\pi(g_a^{-1}) = u_{j(a)}$. Hence,

$$\pi_{ij}(g_{ab}) = \langle \pi(e)\pi(g_b)u_i,\, \pi^*(g_a)u_j\rangle = \pi_{i(b)j(a)}(e) = \delta_{i(b)j(a)}, \tag{5.2.29}$$

for arbitrary permutations $i(b)$ and $j(a)$.

Theorem 5.2.6 (Shannon)**.**

$$f(g) = \sum_{a=1}^{d_{max}} \sum_{b=1}^{d_{max}} f(g_{(a,b)}) L_{(a,b)}(g), \tag{5.2.30}$$

while

$$L_{(a,b)}(g) := \sum_{\pi \in \widehat{\mathcal{G}}_f} d_\pi \sum_{i,j=1}^{d_\pi} c^\pi_{(a,b)}(i,j)\pi_{ij}(g) \tag{5.2.31}$$

and

$$\sum_{a,b=1}^{d_\pi} c^\pi_{(a,b)}(i,j)\pi_{nm}(g_{(a,b)}) = \delta_{m,j}\delta_{n,i} \qquad \forall \pi \in \widehat{\mathcal{G}}_f \tag{5.2.32}$$

Here $\delta_{k,l}$ denotes the Kroneker Delta.

The solvability of (5.2.32) is ensured by the existence of general distributed points (i.e. those which satisfy (5.2.28)). The matrix $(c^\pi_{(a,b)}(i,j))_{(i,j),(a,b)=(1,1)}^{(d_\pi,d_\pi)}$ is the inverse matrix of $(\pi_{nm}(g_{(a,b)}))_{(n,m),(a,b)=(1,1)}^{(d_\pi,d_\pi)}$.

Proof: Inserting (5.2.31) in (5.2.30) yields

$$\sum_{a,b=1}^{d_{max}} F(g_{ab})L_{(a,b)}(g) = \sum_{\pi \in \widehat{\mathcal{G}}_f} d_\pi \sum_{i,j=1}^{d_{max}} \sum_{a,b=1}^{d_{max}} c^\pi_{(a,b)}(i.j)\pi_{ij}(g)F(g_{ab}). \tag{5.2.33}$$

Using the Fourier series expansion of F and changing the order of summation we obtain

$$= \sum_{\pi \in \widehat{\mathcal{G}}_f} d_\pi \sum_{i,j=1}^{d_{max}} \sum_{a,b=1}^{d_{max}} c^\pi_{(a,b)}(i,j)\pi_{ij}(g) \sum_{n,m=1}^{d_{max}} \hat{F}_{nm}(\pi)\pi_{mn}(g_{ab}) \tag{5.2.34}$$

$$= \sum_{\pi \in \widehat{\mathcal{G}}_f} d_\pi \sum_{i,j=1}^{d_{max}} \sum_{n,m=1}^{d_{max}} \sum_{a,b=1}^{d_{max}} c^\pi_{(a,b)}(i,j)\pi_{mn}(g_{ab})\; \hat{F}_{nm}(\pi)\pi_{ij}(g), \tag{5.2.35}$$

Finely by (5.2.32) this equals the Fourier series expansion of F:

$$= \sum_{\pi \in \widehat{\mathcal{G}}_f} d_\pi \sum_{i,j=1}^{d_{max}} \hat{F}_{ji}(\pi)\,\pi_{ij}(g) = F(g). \tag{5.2.36}$$

□

This generalization of the classical Shannon sampling theorem makes use of the group structure and can be utilized as starting point for further discussion of the choice of points $x, y \in \mathcal{G}$ such that measure points of the Radon transform and the connected discretization.
Beside the theoretical value of the Shannon sampling theorem this can also be used to discuss many questions for the applications, such as the optimal choice of points of measuring in order to obtain a stable inversion.

Appendix A

Appendix

A.1 Hermite polynomials

In order to evaluate the Schrödinger representations of H_n, which are in the Hilbert space $L^2(\mathbb{R}^n)$ we introduce here the basic facts about the orthonormal system of Hermite polynomials and Hermite functions.
The Hermite Polynomials are defined by

$$H_k(t) := (-1)^k e^{t^2} (\frac{\mathrm{d}^k}{\mathrm{d}t^k} e^{-t^2}) \qquad\qquad k \in \mathbb{N}_0,\ t \in \mathbb{R}.$$

The Hermite polynomials form an orthonormal system with respect to the measure e^{-t^2}, in that case their norm is $\|H_k\|_{L^2(\mathbb{R}, e^{-t^2}\ \mathrm{d}t)} = \sqrt{2^k \sqrt{\pi} k!}$. Hence an orthonormal system in $L^2(\mathbb{R})$ is given by the Hermite functions, which are defined as

$$h_k(t) := \left(2^k \sqrt{\pi} k!\right)^{-\frac{1}{2}} H_k e^{-\frac{1}{2}t^2}.$$

The tensor product of Hermite functions gives an orthonormal system in $L^2(\mathbb{R}^n)$

$$\Phi_\alpha(x) = \Pi_{j=1}^n h_{\alpha_j}(x_j),$$

in the usual multi index notation $\alpha = (\alpha_j)_{j=1}^n$ and $|\alpha| = \sum_{j=1}^n \alpha_j$.
The heat equation (corresponding to the sub-Laplacian) has the symbol of the harmonic oscillator $-\Delta + |x^2|$, also called Hermite operator for which the eigenfunctions are the the Hermite functions Φ_α.

$$(-\Delta + |x|^2)\Phi_\alpha(x) = (2|\alpha| + n)\Phi_\alpha(x)$$

Simultaneously $\Phi_\alpha(x)$ is an eigenfunction of the Fourier transform $\mathcal{F}\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha$.
For $\alpha. \beta \in \mathbb{N}_0$, the special Hermite functions are defined by

$$\Phi_{\alpha,\beta}(z) = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Phi_\alpha(\xi + \frac{y}{2}) \Phi_\alpha(\xi - \frac{y}{2}) e^{ix\xi}\ \mathrm{d}\xi \qquad\qquad z = x + iy \in \mathbb{C}^n$$

The special Hermite function form an orthonormal system in $L^2(\mathbb{C}^n)$.
The special hermite functions can be expressed in terms of Laguerre functions.

$$L_k^\delta(t) = \frac{e^t t^{-\delta}}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k (e^{-t}t^{k+\delta}); \qquad L_\alpha^\beta = \Pi_{j=1}^n L_{\alpha_j}^{\beta_j} \tag{A.1.1}$$

The following formulae hold

$$\Phi_{\alpha,\alpha} = (2\pi)^{-\frac{n}{2}}\Pi_{j=1}^n L_{\alpha_j}(\frac{1}{2}|z_j|^2)e^{-\frac{1}{4}|z_j|^2} \tag{A.1.2}$$

$$\Phi_{\alpha+\beta,\alpha} = (2\pi)^{-\frac{n}{2}}\left(\frac{\alpha!}{(\alpha+\beta)!}\right)^{\frac{1}{2}}\left(\frac{i}{\sqrt{2}}\right)^{|\beta|}\overline{z}^\beta L_\alpha^\beta(z)e^{-\frac{1}{4}|z|^2}, \tag{A.1.3}$$

$$\Phi_{\alpha,\alpha+\beta} = (2\pi)^{-\frac{n}{2}}\left(\frac{\alpha!}{(\alpha+\beta)!}\right)^{\frac{1}{2}}\left(-\frac{i}{\sqrt{2}}\right)^{|\beta|} z^\beta L_\alpha^\beta(z)e^{-\frac{1}{4}|z|^2}. \tag{A.1.4}$$

A.2 Nilpotent Lie groups

The property of a Lie group $\mathcal{G}$ possessing an abelian structure is very strong and brings many simplifications for the general theory. The property of a Lie group to be nilpotent is somehow a measure of the non-commutativity of the group. If a group is non-commutative, equivalently the commutators $[X, Y]$ of vector fields X and Y do not vanish for some $X, Y \in \mathfrak{g}$. But if $[X, Y]$ does not vanish, may further applications of the commutator vanishes and we can deduce therefrom structural simplifications.

Definition A.2.1. A Lie group $\mathcal{G}$ is *nilpotent* if for all elements X, Y of the Lie algebra $\mathfrak{g}$ the expression

$$[X, [..., [X, [X, Y]]]] \tag{A.2.1}$$

vanishes after finitely many applications of the commutator. A nilpotent Lie group is of step $n \in \mathbb{N}$, if n is the smallest number for which the expression after n applications of $[\,,\,]$ (A.2.1) vanishes for all $X, Y \in \mathfrak{g}$.

By Baker-Campbell-Hausdorff formula

$$e^X e^Y = e^Z, \qquad Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [[X, Y], Y]) + \cdots \tag{A.2.2}$$

the complete structure is determined if $\mathcal{G}$ is nilpotent, since in that case the sum for Z in (A.2.2) is finite.

A.3 Borel functional calculus

We frequently make use of application of functions to operators. For example the square root of the Laplacian $\Delta_\mathcal{G}$ gives the dilation operator, which we use to vary wavelets in their scale. The Borel functional calculus formulates the general concept.

For a linear, selfadjoint operator A on a Hilbert space $\mathcal{H}$, there is a basis $\{e_j,\ j \in J$ with $|J| = \dim\mathcal{H}\}$ of eigenvectors of $\mathcal{H}$, so that $Ae_j = \lambda_j e_j$ for all $j \in J$. Hence the operator A acts on $v = \sum_{j\in J} c_j e_j \in \mathcal{H}$ by

$$Av = \sum_{j\in J} v_j \lambda_j e_j. \tag{A.3.1}$$

Let now $f : \mathbb{R} \to \mathbb{R}$ be a functions, then the operator $f(A)$ is defined by its action via

$$f(A)v = \sum_{j\in J} v_j f(\lambda_j) e_j \tag{A.3.2}$$

A.4 Killing form of $\mathrm{S}pin(m)$

The Lie Bracket of $\mathrm{S}pin(m)$ is given by

$$[x, y] := xy - yx, \tag{A.4.1}$$

as the usual commutator with respect to the clifford multiplication. For the bases $\{e_{ij},\ i < j\}$ it is

$$[e_{ij}, e_{kl}] = 2(\delta_{ik}e_{jl} + \delta_{jl}e_{ik} - \delta_{il}e_{jk} - \delta_{jk}e_{il}). \tag{A.4.2}$$

The killing form is given by $\operatorname{trace}(\operatorname{ad}(x)\operatorname{ad}(y)) = \operatorname{trace}[x, [y, \cdot]]$. For an general element $y \in \mathfrak{spin}(m) = \mathbb{C}_{m,2}$ we make the convention $y = \sum\limits_{i<j} y_{ij}e_{ij} = \frac{1}{2}\sum\limits_{i,j=1}^{n} y_{ij}e_{ij}$, that the matrix (y_{ij}) is skew symmetric and has zero entries on the diagonal.

$$[y, e_{ij}] = \frac{1}{2}\sum_{kl} y_{kl}[e_{kl}, e_{ij}] \tag{A.4.3}$$

$$= \sum_{kl} y_{kl}(\delta_{ki}e_{lj} + \delta_{lj}e_{ki} - \delta_{li}e_{kj} - \delta_{kj}e_{li}) \tag{A.4.4}$$

$$= \sum_{l}(y_{il}e_{lj} - y_{jl}e_{li}) + \sum_{k}(y_{kj}e_{ki} - y_{ki}e_{kj}) \tag{A.4.5}$$

$$= \sum_{k}(y_{kj} - y_{jk})e_{ki} + (y_{ik} - y_{ki})e_{kj} \tag{A.4.6}$$

And further

$$
\begin{aligned}
&[x,[y,e_{ij}]] \\
&= \frac{1}{2}\sum_{mn}\sum_{k} x_{mn}(y_{kj}-y_{jk})[e_{mn},e_{ki}] + x_{mn}(y_{ik}-y_{ki})[e_{mn},e_{kj}] \\
&= \sum_{mn}\sum_{k} x_{mn}(y_{kj}-y_{jk})(\delta_{mk}e_{ni}+\delta_{ni}e_{mk}-\delta_{mi}e_{nk}-\delta_{nk}e_{mi}) \\
&\quad + x_{mn}(y_{ik}-y_{ki})(\delta_{mk}e_{nj}+\delta_{nj}e_{mk}-\delta_{mj}e_{nk}-\delta_{nk}e_{mj}) \\
&= \sum_{n}\sum_{k} x_{kn}(y_{kj}-y_{jk})e_{ni} - x_{in}(y_{kj}-y_{jk})e_{nk} + x_{kn}(y_{ik}-y_{ki})e_{nj} - x_{jn}(y_{ik}-y_{ki})e_{nk} \\
&\quad + \sum_{m}\sum_{k} x_{mi}(y_{kj}-y_{jk})e_{mk} - x_{mk}(y_{kj}-y_{jk})e_{mi}) + x_{mj}(y_{ik}-y_{ki})e_{mk} - x_{mk}(y_{ik}-y_{ki})e_{mj}) \\
&= \sum_{n}\sum_{k} x_{kn}(y_{kj}-y_{jk})e_{ni} - x_{in}(y_{kj}-y_{jk})e_{nk} + x_{kn}(y_{ik}-y_{ki})e_{nj} - x_{jn}(y_{ik}-y_{ki})e_{nk} \\
&\quad + \sum_{n}\sum_{k} x_{ni}(y_{kj}-y_{jk})e_{nk} - x_{nk}(y_{kj}-y_{jk})e_{ni}) + x_{nj}(y_{ik}-y_{ki})e_{nk} - x_{nk}(y_{ik}-y_{ki})e_{nj}) \\
&= \sum_{n}\sum_{k} (x_{kn}-x_{nk})(y_{kj}-y_{jk})e_{ni} + (x_{ni}-x_{in})(y_{kj}-y_{jk})e_{nk} \\
&\quad + (x_{kn}-x_{nk})(y_{ik}-y_{ki})e_{nj} + (x_{nj}-x_{jn})(y_{ik}-y_{ki})e_{nk}
\end{aligned}
$$

We have to calculate the e_{ij} part of this expression.
The part is $(x_{kn}-x_{nk})(y_{kj}-y_{jk})e_{ni}$ contributes $-(x_{kj}-x_{jk})(y_{kj}-y_{jk}) = (x_{jk}-x_{kj})(y_{kj}-y_{jk})$ for $n=j$.
From $(x_{ni}-x_{in})(y_{kj}-y_{jk})e_{nk}$ we get $(x_{ii}-x_{ii})(y_{jj}-y_{jj}) = 0$ for $n=i$ and $k=j$, but for $n=j$ and $k=i$ this gives a e_{ij} part: $-(x_{ji}-x_{ij})(y_{ij}-y_{ji}) = (x_{ij}-x_{ji})(y_{ij}-y_{ji})$.
The part $(x_{kn}-x_{nk})(y_{ik}-y_{ki})e_{nj}$ gives $(x_{ki}-x_{ik})(y_{ik}-y_{ki})$ for $n=i$.
And the term $(x_{nj}-x_{jn})(y_{ik}-y_{ki})e_{nk}$ brings $(x_{ij}-x_{ji})(y_{ij}-y_{ji})$ for $n=i$ and $k=j$.
Consequently the e_{ij}-part is

$$
\begin{aligned}
&2(x_{ij}-x_{ji})(y_{ij}-y_{ji}) + \sum_{k}(x_{jk}-x_{kj})(y_{kj}-y_{jk}) + (x_{ki}-x_{ik})(y_{ik}-y_{ki}) && \text{(A.4.7)} \\
&= \sum_{\substack{k\neq j\\ k\neq i}}(x_{jk}-x_{kj})(y_{kj}-y_{jk}) + (x_{ki}-x_{ik})(y_{ik}-y_{ki}). && \text{(A.4.8)}
\end{aligned}
$$

In order to obtain the trace, we have to take the sum over all $i<j$ or half the sum over all i,j. This gives:

$$
\begin{aligned}
\operatorname{trace}([x,[y,e_{ij}]]) &= \sum_{i<j}\sum_{\substack{k\neq j\\ k\neq i}}(x_{jk}-x_{kj})(y_{kj}-y_{jk}) + (x_{ki}-x_{ik})(y_{ik}-y_{ki}) && \text{(A.4.9)} \\
&= \frac{1}{4}\sum_{i\neq j}\sum_{\substack{k\neq j\\ k\neq i}}(x_{jk}-x_{kj})(y_{kj}-y_{jk}) + (x_{ki}-x_{ik})(y_{ik}-y_{ki}) && \text{(A.4.10)}
\end{aligned}
$$

We shall look at the norm of a basis vector $e_{mn} = \frac{1}{2}\sum_{i\neq j}(\delta_{im}\delta_{nj} + \delta_{in}\delta_{mj})e_{ij}$ with respect to the killing.

$$\|e_{mn}\|^2 = \frac{1}{4}\sum_{i\neq j}\sum_{\substack{k\neq j\\ k\neq i}}(x_{jk} - x_{kj})(y_{kj} - y_{jk}) + (x_{ki} - x_{ik})(y_{ik} - y_{ki}) \quad (A.4.11)$$

$$= \frac{1}{4}(-4 - 4 - 4 - 4) = -4 \quad (A.4.12)$$

Consequently

$$\{\frac{1}{2}e_{ij}, 1 \leq i < j \leq m\} \quad (A.4.13)$$

is the orthonormal (with respect to the killing form) Bases of $\mathfrak{spin}(m)$

A.5 The mixed Laplacian Δ_{uv}

We make use of the following fundamental Equalities:

$$-x \wedge \partial_x = -\frac{1}{2}\left(\sum_{i,j=1}^{m} x_i e_i \partial_{x_j} e_j - \sum_{i,j=1}^{m} \partial_{x_i} e_i x_j e_j\right) \quad (A.5.1)$$

$$= -\frac{1}{2}(x_i\partial_{x_j} - x_j\partial_{x_i})e_{ij} = \sum_{i<j} L_{ij}e_{ij} \quad (A.5.2)$$

$$= \Gamma_x \quad (A.5.3)$$

$$E_u = \sum_{i=1}^{m} u_i\partial_{u_i} \quad (A.5.4)$$

Consequently:

$$\Gamma_u + E_u = -u\partial_u \quad (A.5.5)$$

The Mixed Laplacian is given by

$$\Delta_{uv} = \sum_{i<j} L_{u,ij}L_{v,ij} = \sum_{i<j}(u_j\partial_{u_i} - u_i\partial_{v_j})(v_j\partial_{v_i} - v_i\partial_{v_j})$$
$$= \sum_{i<j} u_iv_i\partial_{u_j}\partial_{v_j} + u_jv_j\partial_{u_i}\partial_{v_i} - u_iv_j\partial_{v_j}\partial_{v_i} - u_jv_i\partial_{u_i}\partial_{v_j}.$$

Lemma A.5.1.

$$\{\Gamma_u, \Gamma_v\} + \frac{1}{2}[\{(u \wedge \partial_v), (v \wedge \partial_u)\} + \{(u \wedge v), (\partial_u \wedge \partial_v)\}] = -3\Delta_{uv} + (m-2)(\Gamma_u + \Gamma_v) - \binom{m}{2}.$$

We will see, that the scalar part of $\frac{1}{2}(\Gamma_u\Gamma_v + \Gamma_v\Gamma_u)$ is already Δ_{uv}. The rest of the calculation will be devoted to cancelation of the appearing four-vector part and eventually to calculate the corresponding bi-vector part of the expression, which we find for vanishing four-vector part.

Because of $[\Gamma_u\Gamma_v]_2 = -[\Gamma_v\Gamma_u]_2$ it is

$$\frac{1}{2}\{\Gamma_u, \Gamma_v\} := \frac{1}{2}(\Gamma_u\Gamma_v + \Gamma_v\Gamma_u) \tag{A.5.6}$$

$$= [\Gamma_u\Gamma_v]_0 + [\Gamma_u\Gamma_v]_4 \tag{A.5.7}$$

By

$$\Gamma_u\Gamma_v = (-u \wedge \partial_u)(-v \wedge \partial_v) \tag{A.5.8}$$

$$= \frac{1}{4}(-u\partial_u + \partial_u u)(-v\partial_v + \partial_v v) \tag{A.5.9}$$

$$= \sum_{i<j}\sum_{k<l}(u_j\partial_{u_i} - u_i\partial_{u_j})(v_l\partial_{v_k} - v_k\partial_{v_l})e_{ijkl} \tag{A.5.10}$$

we see that

$$[\Gamma_u\Gamma_v]_0 = \sum_{i=k<j=l}(-1)(u_j\partial_{u_i} - u_i\partial_{u_j})(v_l\partial_{v_k} - v_k\partial_{v_l}) = -\sum_{i<j} L_{u,ij}L_{v,ij} = -\Delta_{uv} \tag{A.5.11}$$

$$[\Gamma_u\Gamma_v]_4 = \frac{1}{4}\sum_{i,j,k,l \text{ different}} e_{ijkl}L_{u,ij}L_{v,kl} \tag{A.5.12}$$

The task is now, to cancel the four-vector part!

Therefor we look at

$$-\frac{1}{2}\{(u \wedge \partial_v), (v \wedge \partial_u)\} - (u \wedge v)(\partial_u \wedge \partial_v)$$

and we will find, that the scalar part also contains Δ_{uv} and the four-vector part is proportional to that of $\Gamma_u\Gamma_v$.

$$\begin{aligned}
&-\frac{1}{2}[(u \wedge \partial_v)(v \wedge \partial_u) + (v \wedge \partial_u)(u \wedge \partial_v)]_0 \\
&= \frac{1}{4}\sum_{i\neq j}(u_i\partial_{v_j} - u_j\partial_{v_i})(v_i\partial_{u_j} - v_j\partial_{u_i}) + \sum_{i\neq j}(v_i\partial_{u_j} - v_j\partial_{u_i})(u_i\partial_{v_j} - u_j\partial_{v_i}) \\
&= \frac{1}{2}\sum_{i\neq j} u_iv_i\partial_{u_j}\partial_{v_j} + u_jv_j\partial_{u_i}\partial_{v_i} - u_iv_j\partial_{v_j}\partial_{u_i} - u_jv_i\partial_{u_j}\partial_{v_i} \\
&\qquad + \frac{1}{4}\sum_{i\neq j} -u_i\partial_{u_i} - u_j\partial_{u_j} - v_i\partial_{v_i} - v_j\partial_{v_j} \\
&= \frac{1}{2}\sum_{i\neq j} u_iv_i\partial_{u_j}\partial_{v_j} + u_jv_j\partial_{u_i}\partial_{v_i}\underline{-u_iv_j\partial_{v_j}\partial_{u_i} - u_jv_i\partial_{u_j}\partial_{v_i}} - \frac{(m-1)}{2}(E_u + E_v),
\end{aligned}$$

where the last term comes from the action of ∂_u on u and of ∂_v on v in the term, which we have to consider by the product rule. Furthermore

$$-[(u \wedge v)(\partial_u \wedge \partial_v)]_0 = \frac{1}{2} \sum_{i \neq j} (u_i v_j - u_j v_i)(\partial_{u_i} \partial_{v_j} - \partial_{u_j} \partial_{v_i})$$
$$= \frac{1}{2} \sum_{i \neq j} (\underline{u_i v_j \partial_{u_i} \partial_{v_j} + u_j v_i \partial_{u_j} \partial_{v_i}} - u_i v_j \partial_{u_j} \partial_{v_i} - u_j v_i \partial_{u_i} \partial_{v_j}).$$

We take the sum of the two calculations above and obtain:

$$-[\frac{1}{2}\{(u \wedge \partial_v), (v \wedge \partial_u)\} + (u \wedge v)(\partial_u \wedge \partial_v)]_0$$
$$= \frac{1}{2} \left(\sum_{i \neq j} u_i v_i \partial_{u_j} \partial_{v_j} + u_j v_j \partial_{u_i} \partial_{v_i} - u_i v_j \partial_{u_j} \partial_{v_i} - u_j v_i \partial_{u_i} \partial_{v_j} - (m-1)(E_u + E_v) \right)$$
$$= \frac{1}{2} \left(\sum_{i \neq j} u_i \partial_{u_j} (v_i \partial_{v_j} - v_j \partial_{v_i}) + u_j \partial_{u_i} (v_j \partial_{v_i} - v_i \partial_{v_j}) - (m-1)(E_u + E_v) \right)$$
$$= -\frac{1}{2} \left(\sum_{i \neq j} (u_i \partial_{u_j} - u_j \partial_{u_i})(v_j \partial_{v_i} - v_i \partial_{v_j}) - (m-1)(E_u + E_v) \right) \tag{A.5.13}$$
$$= \Delta_{uv} - \frac{(m-1)}{2}(E_u + E_v). \tag{A.5.14}$$

We have to evaluate also the four-vector part:

$$[(u \wedge \partial_v)(v \wedge \partial_u) + (v \wedge \partial_u)(u \wedge \partial_v)]_4 \tag{A.5.15}$$
$$= \frac{1}{4} \sum_{i,j,k,l \text{ different}} (u_i \partial_{v_j} - u_j \partial_{v_i})(v_k \partial_{u_l} - v_l \partial_{u_k}) e_{ijkl} \tag{A.5.16}$$
$$= \frac{1}{4} \sum_{i,j,k,l \text{ different}} (u_i v_k \partial_{u_l} \partial_{v_j} + u_j v_l \partial_{v_i} \partial_{u_k} - u_j v_k \partial_{u_l} \partial_{v_i} - u_i v_l \partial_{v_j} \partial_{u_k}) e_{ijkl} \tag{A.5.17}$$

The same we obtain for

$$[(u \wedge v)(\partial_u \wedge \partial_v)]_4 = \frac{1}{4} \sum_{i,j,k,l \text{ different}} (u_i v_j - u_j v_i)(\partial_{v_k} \partial_{u_l} - \partial_{v_l} \partial_{u_k}) e_{ijkl}$$
$$= \frac{1}{4} \sum_{i,j,k,l \text{ different}} (u_i v_k \partial_{u_l} \partial_{v_j} + u_j v_l \partial_{v_i} \partial_{u_k} - u_j v_k \partial_{u_l} \partial_{v_i} - u_i v_l \partial_{v_j} \partial_{u_k}) e_{ijkl}$$

And also for

$$[\Gamma_u \Gamma_v]_4 = [(u \wedge \partial_u)(v \wedge \partial_v)]_4$$
$$= \frac{1}{4} \sum_{i,j,k,l \text{ different}} e_{ijkl} (u_i \partial_{u_j} - u_j \partial u_i)(v_k \partial_{v_l} - v_l \partial_{v_k})$$
$$= \frac{1}{4} \sum_{i,j,k,l \text{ different}} e_{ijkl} (u_i v_k \partial_{u_j} \partial_{v_l} + u_j v_l \partial_{u_i} \partial_{v_k} - u_i v_l \partial_{u_j} \partial_{v_k} - u_j v_k \partial_{u_i} \partial_{v_l})$$

So that the Sum

$$\Gamma_u\Gamma_v + \Gamma_v\Gamma_u + \frac{1}{2}\{(u \wedge \partial_v), (v \wedge \partial_u)\} + (u \wedge v)(\partial_u \wedge \partial_v) \tag{A.5.18}$$

has a vanishing four vector part:

$$\frac{1}{2} \sum_{i,j,k,l \text{ different}} (u_i v_k \partial_{u_j} \partial_{v_l} + u_j v_l \partial_{u_i} \partial_{v_k} - u_j v_k \partial_{u_i} \partial_{v_l} - u_i v_l \partial_{u_j} \partial_{v_k}) \tag{A.5.19}$$

$$+ \frac{1}{2} \sum_{i,j,k,l \text{ different}} (u_i v_k \partial_{u_l} \partial_{v_j} + u_j v_l \partial_{v_i} \partial_{u_k} - u_j v_k \partial_{u_l} \partial_{v_i} - u_i v_l \partial_{v_j} \partial_{u_k}) e_{ijkl} \tag{A.5.20}$$

$$= 0. \tag{A.5.21}$$

The Scalar part of (A.5.18) is, according to (A.5.13) and (A.5.11):

$$[\Gamma_u\Gamma_v + \Gamma_v\Gamma_u + \frac{1}{2}\{(u \wedge \partial_v), (v \wedge \partial_u)\} + (u \wedge v)(\partial_u \wedge \partial_v)]_0 = -3\Delta_{uv} + \frac{(m-1)}{2}(E_u + E_v) \tag{A.5.22}$$

Note that the scalar – and the four-vector part of $\{a, b\}$ is the same as that of ab for all bi-vectors. Hence we have formally

$$[\Gamma_u\Gamma_v + \Gamma_v\Gamma_u + (u \wedge \partial_v)(v \wedge \partial_u) + (u \wedge v)(\partial_u \wedge \partial_v)]_0 \tag{A.5.23}$$

$$= [2\{\Gamma_u\Gamma_v\} + \frac{1}{2}(\{(u \wedge \partial_v), (v \wedge \partial_u)\} + \{(u \wedge v), (\partial_u \wedge \partial_v)\})]_0 \tag{A.5.24}$$

Since we are in the situation with operator calculus, we have to be careful with the above equation. There is an action of ∂_u (∂_v) on the appearing $u(v)$. It is left to look at the bi-vector part and to consider the action, which we mentioned just now.
We can look at the bi-vector part as it is build up by two contributions: The FIRST contribution comes from the action of $[\Gamma_u\Gamma_v + \Gamma_v\Gamma_u + \frac{1}{2}((u \wedge \partial_v)(v \wedge \partial_u)) + (u \wedge v)(\partial_u \wedge \partial_v)]_2$ on a function f and a second one by the action of all ∂_u, ∂_v in $[\Gamma_u\Gamma_v + \Gamma_v\Gamma_u + \frac{1}{2}((u \wedge \partial_v)(v \wedge \partial_u)) + (u \wedge v)(\partial_u \wedge \partial_v)]_2$ on all appearing u, v in $[\Gamma_u\Gamma_v + \Gamma_v\Gamma_u + \frac{1}{2}((u \wedge \partial_v)(v \wedge \partial_u)) + (u \wedge v)(\partial_u \wedge \partial_v)]_2$, respectively. Since the bi-vector part $[\{a \wedge b, c \wedge d\}]_2$ vanishes for arbitrary vectors a, b, c, d, also FIRST part of the contribution to the bi-vector part vanishes.
The bi-vector part, that doesn't vanish comes from the action of ∂_u and ∂_v on u and v in the operator itself. Also the part $(m-1)(E_u + E_v)$ in the scalar part (A.5.13) comes from that action. We will calculate the action this kind in a whole and write

$$\Gamma_u\Gamma_v + \Gamma_v\Gamma_u + \frac{1}{2}\{(u \wedge \partial_v), (v \wedge \partial_u)\} + \frac{1}{2}\{(u \wedge v), (\partial_u \wedge \partial_v)\} = -3\Delta_{uv} + A \tag{A.5.25}$$

The part

$$A = C + D \tag{A.5.26}$$

decomposes in the action of ∂_u and ∂_v on u and v in $\{(u \wedge \partial_v), (v \wedge \partial_u)\}$, which we denote by C, and the same action in $\{(u \wedge v), (\partial_u \wedge \partial_v)\}$, which we denote by D.
So we have to calculate:

$$C = \frac{1}{2}((u \wedge \dot{\partial}_v)(\dot{v} \wedge u) + (v \wedge \dot{\partial}_u)(\dot{u} \wedge v)) \tag{A.5.27}$$
$$= \frac{1}{8}([u, \dot{\partial}_v][\dot{v}, \partial_u] + [v, \dot{\partial}_u][\dot{u}, \partial_v]) \tag{A.5.28}$$

For further evaluation of this expression we have a look at

$$[u, \dot{\partial}_v][\dot{v}, \partial_u] = (u\dot{\partial}_v - \dot{\partial}_v u)(\dot{v}\partial_u - \partial_u \dot{v}) \tag{A.5.29}$$
$$= \sum_{i,j,k,l=1}^{m} u_i e_i \dot{\partial}_{v_j} e_j \dot{v}_k e_k \partial_{u_l} e_l + \dot{\partial}_{v_i} e_i u_j e_j \partial_{u_k} e_k \dot{v}_l e_l \tag{A.5.30}$$
$$- u_i e_i \dot{\partial}_{v_j} e_j \partial_{u_k} e_k \dot{v}_l e_l - \dot{\partial}_{v_i} e_i u_j e_j \dot{v}_k e_k \partial_{u_l} e_l \tag{A.5.31}$$
$$= \sum_{i,j,k,l=1}^{m} -u_i e_i \partial_{u_l} e_l + e_i u_j e_j \partial_{u_k} e_k e_i \tag{A.5.32}$$
$$- u_i e_i e_j \partial_{u_k} e_k e_j - e_i u_j e_j e_i \partial_{u_l} e_l \tag{A.5.33}$$
$$= -mu\partial_u + \sum_{j,k,l=1}^{m} e_j u_l e_l \partial_{u_k} e_k e_j - u_l e_l e_j \partial_{u_k} e_k e_j - e_j u_l e_l e_j \partial_{u_k} e_k \tag{A.5.34}$$
$$\left(= -mu\partial_u + \sum_{j=1}^{m} e_j u \partial_u e_j - u e_j \partial_u e_j - e_j u e_j \partial_u \right) \tag{A.5.35}$$
$$=: -mu\partial_u + \sum_{j=1}^{m} \alpha_j - \beta_j - \gamma_j \tag{A.5.36}$$

Because of complexity of the calculation, we look separated at α–, β– and γ-part. For each, we partition the sum over j into four parts: $k \neq j,\ j \notin \{k,l\}$, $k \neq j,\ j \in \{k,l\}$, $k = j,\ j \notin \{k,l\}$ and $k = j,\ j \in \{k,l\}$

$\sum \alpha_j :$	$k \neq j,\ j \notin \{k,l\}$	$-(m-2)\sum_{k\neq l} e_l u_l e_k \partial_{u_k}$
	$k \neq j,\ j \in \{k,l\}$	$2\sum_{k\neq l} u_l e_l \partial_{u_k} e_k = 2(u \wedge \partial_u) = -2\Gamma_u$
	$k = j,\ j \notin \{k,l\}$	$-(m-1)\sum_{k=1}^{m} u_k e_k \partial_{u_k} e_k$
	$k = j = l$	$\sum_{k=1}^{m} u_k \partial u_k$
	$\sum_{k,j,l=1}^{m}$	$-(m-2)u\partial_u + 2E_u + 2(u \wedge \partial_u) = -(m-2)u\partial_u + 2E_u - 2\Gamma_u$
$\sum \beta_j :$	$k \neq j,\ j \notin \{k,l\}$	$(m-2)\sum_{k\neq l} u_k e_k \partial_{u_l} e_l$
	$k \neq j,\ j \in \{k,l\}$	0, da $u_k e_k e_j \partial_{u_l} e_l e_j = -u_l e_l e_j \partial_{u_k} e_k e_j$
	$k = j,\ j \notin \{k,l\}$	$(m-1)\sum_{k=1}^{m} u_k e_k \partial_{u_k} e_k$
	$k = j = l$	$\sum_{k=1}^{m} u_k \partial u_k$
	$\sum_{k,j,l=1}^{m}$	$(m-2)u\partial_u$

	$k \neq j,\ j \notin \{k,l\}$	$(m-2)\sum_{k\neq l} u_k e_k \partial_{u_l} e_l$
	$k \neq j,\ j \in \{k,l\}$	0, da $u_k e_k e_j \partial_{u_l} e_l e_j = -u_l e_l e_j \partial_{u_k} e_k e_j$
$\sum \gamma_j$:	$k = j,\ j \notin \{k,l\}$	$(m-1)\sum_{k=1}^m u_k e_k \partial_{u_k} e_k$
	$k = j = l$	$\sum_{k=1}^m u_k \partial u_k$
	$\sum_{k,j,l=1}^m$	$(m-2)u\partial_u$

such that, with the use of (A.5.5) we have:

$$\text{(A.5.36)} = -mu\partial_u + 2E_u - 2\Gamma_u - (m-2)u\partial_u - (m-2)u\partial_u \tag{A.5.37}$$

$$= (4m-4)E_u + (4m-8)\Gamma_u \tag{A.5.38}$$

Consequently the part C in (A.5.26) is

$$C = \frac{1}{2}(m-1)(E_u + E_v) + \frac{1}{2}(m-2)(\Gamma_u + \Gamma_v) \tag{A.5.39}$$

For the part D we have

$$D = \frac{1}{2}((\dot{\partial}_u \wedge \dot{\partial}_v)(\dot{u} \wedge \dot{v}) + (\dot{\partial}_u \wedge \partial_v)(\dot{u} \wedge v) + (\dot{\partial}_v \wedge \partial_u)(\dot{v} \wedge u)) \tag{A.5.40}$$

We have listed in detail, how the calculations work. A short calculation of the the seen type shows

$$\frac{1}{2}(\dot{\partial}_u \wedge \dot{\partial}_v)(\dot{v} \wedge \dot{u}) = -\binom{m}{2} \tag{A.5.41}$$

$$\frac{1}{2}(\dot{\partial}_v \wedge \partial_u)(\dot{v} \wedge u) = \frac{1}{8}[\dot{\partial}_v, \partial_u][\dot{v}, u] = -\frac{1}{8}[\partial_u, \dot{\partial}_v][\dot{v}, u], \tag{A.5.42}$$

We have already calculated $[u, \dot{\partial}_v][\dot{v}, \partial_u]$, and in the same way we see that (note that $-\partial_u u = E_u - \Gamma_u$, since $E_u - \Gamma_u = \sum_{k=1}^m u_k \partial_{u_k} + \frac{1}{2}(u \wedge \partial_u) = \sum_{k=1}^m -u_k e_k \partial_{u_k} e_k + \frac{1}{2}\sum_{j\neq i} u_i e_i \partial_{u_j} e_j - \frac{1}{2}\sum_{k\neq l} \partial_l e_l u_k e_k = -\sum_{k=1}^m u_k e_k \partial_{u_k} e_k - \sum_{j\neq i} \partial_{u_i} e_i u_j e_j = -\partial_u u$)

$$[\dot{\partial}_v, \partial_u][\dot{v}, u] = -(\partial_u \dot{\partial}_v - \dot{\partial}_v \partial_u)(\dot{v}u - u\dot{v}) \tag{A.5.43}$$

$$= \sum_{j=1}^m -\partial_u e_j e_j u - e_j \partial_u u e_j + \partial_u e_j u e_j + e_j \partial_u e_j u \tag{A.5.44}$$

$$= m\partial_u u + (m-2)\partial_u u - 2E_u + 2\Gamma_u + 2(m-2)\partial_u u \tag{A.5.45}$$

$$= -m(E_u - \Gamma_u) - (m-2)E_u - \Gamma_u - 2E_u - 2\Gamma_u - 2(m-2)(E_u - \Gamma_u) \tag{A.5.46}$$

$$= -(4m-4)E_u + (4m-8)\Gamma_u \tag{A.5.47}$$

consequently:

$$D = -\frac{1}{2}(m-1)(E_u + E_v) + \frac{1}{2}(m-2)(\Gamma_u + \Gamma_v) - \binom{m}{2} \tag{A.5.48}$$

Such that

$$\begin{aligned}A &= C + D \\ &= \frac{1}{2}(m-1)(E_u + E_v) + \frac{1}{2}(m-2)(\Gamma_u + \Gamma_v) - \frac{1}{2}(m-1)(E_u + E_v) \\ &\qquad + \frac{1}{2}(m-2)(\Gamma_u + \Gamma_v) - \binom{m}{2} \\ &= (m-2)(\Gamma_u + \Gamma_v) - \binom{m}{2}\end{aligned}$$

All together gives the result of Lemma A.5.1:

$$\{\Gamma_u, \Gamma_v\} + \frac{1}{2}[\{(u \wedge \partial_v), (v \wedge \partial_u)\} + \{(u \wedge v), (\partial_u \wedge \partial_v)\}] = -3\Delta_{uv} + (m-2)(\Gamma_u + \Gamma_v) - \binom{m}{2}$$

As special case we are interested in the action of the mixed Laplacian on functions $f(u, v)$, which are spherical monogenic in both variables u and v, i.e. $f(u, v)$ is homogeneous of degree (k, l) in (u, v) and $\partial_u f(u, v) = \partial_v f(u, v) = 0$. Consequently,

$$\Gamma_u f(u, v) = -kf(u, v), \qquad \Gamma_v f(u, v) = -lf(u, v), \tag{A.5.49}$$

and

$$\{\Gamma_u, \Gamma_v\} f(u, v) = 2klf(u, v) \quad (m-2)(\Gamma_u + \Gamma_v) f(u, v) = -(m-2)(k+l) f(u, v). \tag{A.5.50}$$

Further, in order to determine the action of the mixed Laplacian, we have to calculate the action of $(u \wedge \partial_v)(v \wedge \partial_u)$ on f. Since f is monogenic, we have

$$(v \wedge \partial_u) f(u, v) = \frac{1}{2}[v, \partial_u] f(u, v) = -\frac{1}{2}\partial_u v f(u, v) = < v, \partial_u > f(u, v), \tag{A.5.51}$$

where the last equality can be seen in the following way.

$$\begin{aligned}-\frac{1}{2}\partial_u v f(u, v) &= -\frac{1}{2}\sum_{i,j=1}^{m} \partial_{u_i} v_j e_{ij} f(u, v) = (\frac{1}{2}\sum_{i,j=1}^{m} v_j \partial_{u_i} e_{ji} + \sum_{i=1}^{m} v_i \partial_{u_i}) f(u, v) \\ &= \frac{1}{2} v \partial_u f(u, v) + < v, \partial_u > f(u, v) = < v, \partial_u > f(u, v).\end{aligned}$$

The action of $(u \wedge \partial_v)(v \wedge \partial_u)$ separates as usual into

$$(u \wedge \partial_v)(v \wedge \partial_u) f(u, v) = (u \wedge \dot{\partial}_v)(\dot{v} \wedge \partial_u) f(u, v) + (u \wedge \dot{\partial}_v)(v \wedge \partial_u)\dot{f}(u, v) \tag{A.5.52}$$

$$= (u \wedge \dot{\partial}_v)(\dot{v} \wedge \partial_u) f(u, v) + < u, \dot{\partial}_v >< v, \partial_u > \dot{f}(u, v) \tag{A.5.53}$$

further,

$$(u \wedge \dot{\partial}_v)(\dot{v} \wedge \partial_u) f(u,v) = -\frac{1}{4}[u, \dot{\partial}_v]\partial_u \dot{v} f(u,v) = -\frac{1}{4}(u\dot{\partial}_v - \dot{\partial}_v u)\partial_u \dot{v} f(u,v) \quad (A.5.54)$$

$$= -\frac{1}{4}(u\dot{\partial}_v \partial_u \dot{v} - \dot{\partial}_v u \partial_u \dot{v}) f(u,v) \quad (A.5.55)$$

$$= -\frac{1}{4}(\sum_{j=1}^{m} u e_j \partial_u e_j - e_j u \partial_u e_j) f(u,v) \quad (A.5.56)$$

$$= -\frac{1}{4}((m-2)u\partial_u + (m-2)u\partial_u - 2E_u - 2(u \wedge \partial_u)) f(u,v) \quad (A.5.57)$$

$$= \frac{1}{2}(E_u - \Gamma_u) f(u,v), \quad (A.5.58)$$

since $f(u,v)$ is monogenic.
We noted already, that $-\partial_u u = E_u - \Gamma_u$, further we have for $f(u,v)$:

$$\Gamma_u f(u,v) = -(u \wedge \partial_u) f(u,v) = \frac{1}{2}\partial_u u,$$

hence:

$$\frac{1}{2}(E_u - \Gamma_u) f(u,v) = -\frac{1}{2}(\partial_u u) f(u,v) = -\Gamma_u f(u,v), \quad (A.5.59)$$

Consequently:

$$E_u f(u,v) = -\Gamma f(u,v) \quad (A.5.60)$$

Together this gives

$$(u \wedge \partial_v)(v \wedge \partial_u) f(u,v) = \frac{1}{2}(E_u - \Gamma_u) f(u,v) + < u, \dot{\partial}_v >< v, \partial_u > \dot{f}(u,v) \quad (A.5.61)$$

$$= E_u + < u, \dot{\partial}_v >< v, \partial_u > \dot{f}(u,v). \quad (A.5.62)$$

The result for $(v \wedge \partial_u)(u \wedge \partial_v)$ is obtained by replacing u by v and vis versa in the above lines. A short calculation shows, that $< u, \dot{\partial}_v >< v, \partial_u > \dot{f}(u,v) =< v, \dot{\partial}_u >< u, \partial_v > \dot{f}(u,v)$, so that

$$\frac{1}{2}\{u \wedge \partial_v, v \wedge \partial_u\} f(u,v) \quad (A.5.63)$$

$$= \frac{1}{2}(E_u + E_v) f(u,v) + \frac{1}{2}(< u, \dot{\partial}_v >< v, \partial_u > \dot{f}(u,v) + < v, \dot{\partial}_u >< u, \partial_v > \dot{f}(u,v)) \quad (A.5.64)$$

$$= \frac{1}{2}(E_u + E_v) f(u,v) + < u, \dot{\partial}_v >< v, \partial_u > \dot{f}(u,v) \quad (A.5.65)$$

$$= \frac{1}{2}(E_u + E_v) f(u,v) + kaart f(u,v). \quad (A.5.66)$$

Finally we have to calculate $\frac{1}{2}\{u \wedge v, \partial_u \wedge \partial_v\}f(u,v)$.

$$\begin{aligned}
\frac{1}{2}\{u \wedge v, \partial_u \wedge \partial_v\}f(u,v) &= \frac{1}{2}\{u \wedge v, \dot{\partial}_u \wedge \dot{\partial}_v\}\dot{f}(u,v) + Df(u,v) \\
&= ([(u \wedge v)(\partial_u \wedge \partial_v)]_0 + [(u \wedge v)(\partial_u \wedge \partial_v)]_4 + [(u \wedge v)(\partial_u \wedge \partial_v)]_2 \\
&\qquad - [(u \wedge v)(\partial_u \wedge \partial_v)]_2 + D)f(u,v) \\
&= (u \wedge v)(\partial_u \wedge \partial_v)f(u,v) - [(u \wedge v)(\partial_u \wedge \partial_v)]_2 f(u,v) + Df(u,v) \\
&= Df(u,v) - [(u \wedge v)(\partial_u \wedge \partial_v)]_2 f(u,v).
\end{aligned}$$

So we calculate the two-vector part

$$\begin{aligned}
[u \wedge v, \partial_u \wedge \partial_v]_2 \dot{f}(u,v) &= \left[\sum_{i \neq j} u_i v_j e_{ij} \sum_{k \neq l} \partial_{u_k} \partial_{v_l} e_{kl}\right]_2 \dot{f}(u,v) \\
&= \left(\sum_{i=k;j\neq l} u_i \partial_{u_i} v_j e_j \partial_{v_l} e_l - \sum_{i=l;j\neq k} u_i v_j e_j \partial_{u_k} e_k \partial_{v_i} \right. \\
&\qquad \left. - \sum_{j=k;i\neq l} u_i e_i v_j \partial_{u_j} \partial_{v_l} e_l + \sum_{j=l;i\neq k} u_i e_i v_j \partial_{u_k} e_k \partial_{v_j}\right) \dot{f}(u,v) \\
&= (E_u(v \wedge \partial_v) - \langle u, \dot{\partial}_v\rangle(v \wedge \partial_u) - \langle v, \dot{\partial}_u\rangle(u \wedge \partial_v) + (u \wedge \partial_u)E_v)\dot{f}(u,v) \\
&= (E_u\Gamma_v - \langle u, \dot{\partial}_v\rangle\langle v, \partial_u\rangle - \langle v, \dot{\partial}_u\rangle\langle u \wedge \partial_v\rangle + \Gamma_u E_v)\dot{f}(u,v) \\
&= -E_u\Gamma_v f(u,v) - \Gamma_u E_v f(u,v) - 2kaart f(u,v)
\end{aligned}$$

Eventually we have:

$$\begin{aligned}
&(\{\Gamma_u, \Gamma_v\} + \frac{1}{2}[\{(u \wedge \partial_v), (v \wedge \partial_u)\} + \{(u \wedge v), (\partial_u \wedge \partial_v)\}])f(u,v) \\
&= \left(\{\Gamma_u, \Gamma_v\} + \frac{1}{2}(E_u + E_v)f(u,v) + kaart + D + 2kaart + (E_u\Gamma_v + E_v\Gamma_u)\right) f(u,v) \\
&= (2kl + \frac{1}{2}(k+l) - \frac{1}{2}(m-1)(E_u + E_v) + \frac{1}{2}(m-2)(\Gamma_u + \Gamma_v) - \binom{m}{2} - 2kl + 3kaart)f(u,v) \\
&= (3\,kaart - (m-2)(k+l) - \binom{m}{2})f(u,v).
\end{aligned}$$

From Lemma A.5.1 we know, that this is equal to

$$\begin{aligned}
&= -3\Delta_{u,v} + (m-2)(\Gamma_u + \Gamma_v) - \binom{m}{2} f(u,v) \\
&= (-3\Delta_{u,v} - (m-2)(k+l) - \binom{m}{2})f(u,v),
\end{aligned}$$

such that

$$\Delta_{uv} f(u,v) = - < v, \dot{\partial}_u >< u, \partial_v > \dot{f}(u,v).$$

Appendix A

List of symbols

Bibliography

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