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# Relaxed Strong Colorings of Hypergraphs

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# Relaxed Strong Colorings of Hypergraphs

Harrison Piehowski

Advisor: Dr. Peh Ng

University of Minnesota, Morris

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## Outline of Discussion

I - Hypergraphs

II - Relaxed Strong Colorings

III - Complexity

IV - The Case of Two Colors

V - Minimal Edges

VI - Families

VII - Special Classes of Hypergraphs

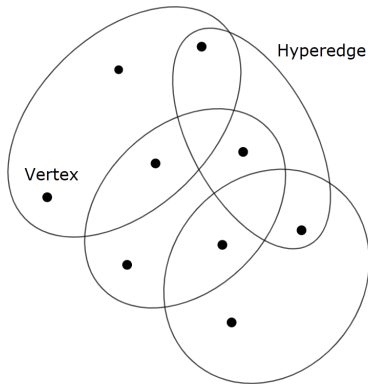
VIII - Future Work

Acknowledgements

References

# I - Hypergraphs

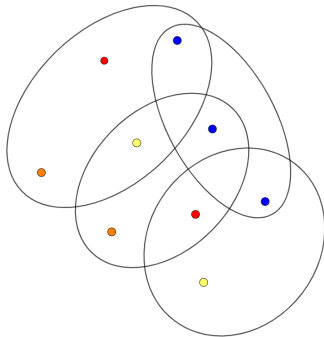
A *hypergraph*  $H = (V, E)$  is a pair of sets  $V = \{1, 2, \dots, n\}$ ,  $n \in \mathbb{Z}^+$ ,  $E$  such that  $E \subseteq \mathcal{P}(V)$  and  $\bigcup E = V$ . A *simple* hypergraph is a hypergraph such that if  $e_i, e_j \in E$  and  $e_i \subseteq e_j$ , then  $e_i = e_j$ .



**Figure:** This is a simple hypergraph with 9 vertices and 4 hyperedges.

## II - Relaxed Strong Colorings

Given a simple hypergraph  $H$ , we wish to assign colors to the vertices such that each hyperedge contains only different colors or only the same color, using at least two colors. In the former case, the hyperedge is called *multi-colored*. In the latter case, the hyperedge is called *mono-colored*. Such a coloring is called a *Relaxed Strong Coloring of  $H$* . Given the hypergraph  $H$ , we wish to find the least number of colors used in any relaxed strong coloring.

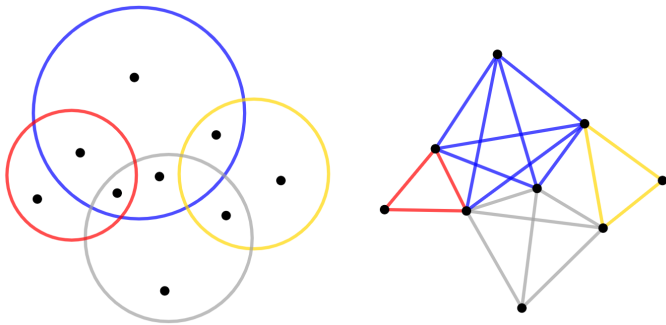


**Figure:** This hypergraph is relaxed-strongly colored using four colors, the fewest necessary.

### III - Complexity

Claude Berge [1987]:

Let  $H = (V, E)$  be a hypergraph. The 2-section of  $H$  is the simple graph  $G = (V, E')$  such that if  $e \in E$  and  $\{v_1, v_2\} \in e$ , then  $\{v_1, v_2\} \in E'$ .



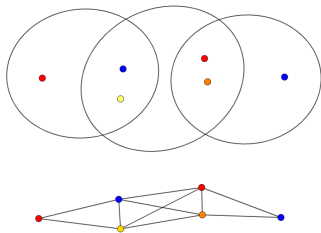
**Figure:** To obtain a 2-section, we turn the hyperedges into graph edges, i.e. each hyperedge induces a clique in the 2-section.

# III - Complexity

## Proposition

*If in no relaxed strong coloring of a hypergraph  $H$  there is a mono-colored hyperedge, then the relaxed strong coloring number of  $H$  is the chromatic number of the 2-section of  $H$ .*

This problem is a complication of graph coloring, making it an *NP-Complete* problem, meaning that there is no known time-efficient way to solve this problem, and there probably will not be one.



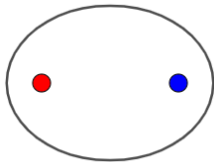
**Figure:** No hyperedge may be colored with all one color, so the question reduces to graph coloring.

## IV - The Case of Two Colors

### Proposition

*For any hypergraph  $H = (V, E)$  with every hyperedge of size at least 3,  $H$  is disconnected if and only if  $N^*(H) = 2$ .*

*Sketch of proof:* Using two colors implies the hypergraph is disconnected or there is an edge of size two. A disconnected hypergraph requires only two colors to be relaxed-strong colored optimally.



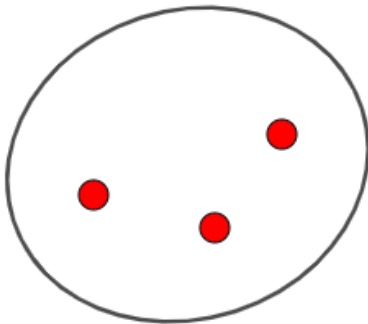
**Figure:** A hyperedge of size two can be colored with only two colors but still be connected.



## IV - The Case of Two Colors

### Proposition

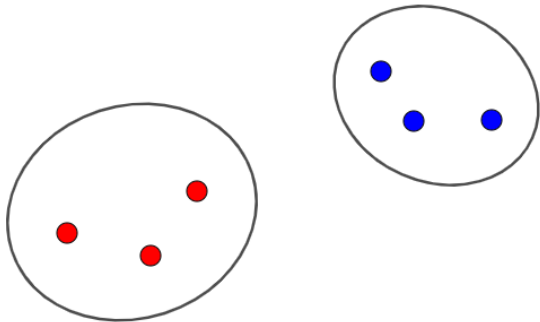
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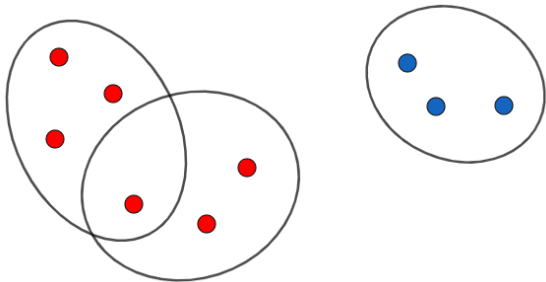
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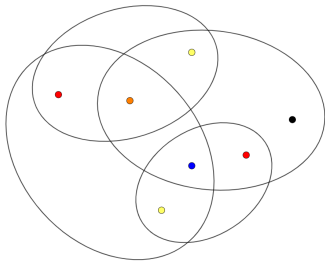


# V - Minimal Edges

## Proposition

*If a hypergraph  $H$  is connected and each hyperedge has size at least  $k$ , then the number of colors used in any proper relaxed strong coloring of  $H$  is at least  $k$ .*

*Sketch of Proof: Coloring a hyperedge of size  $k$  with  $k - 1$  colors forces that edge to be mono-colored. A set of mono-colored hyperedges (with at least two colors) composes a disconnected hypergraph.*



**Figure:** This hypergraph is connected and has hyperedges of size at least three, so the fewest colors necessary to relaxed-strong color it is at least three.

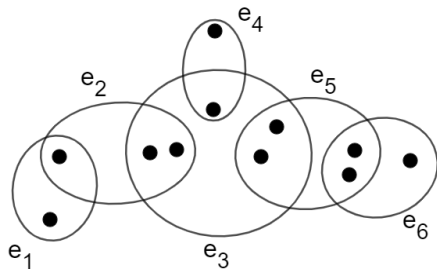
## VI - Families

### Definition

Let  $H = (V, E)$  be a hypergraph with each hyperedge having size at least 2. Let  $R \subseteq E^2$  be a relation such that if  $e_1, e_2 \in E$ , then  $(e_1, e_2) \in R$  iff  $|e_1 \cap e_2| \geq 2$ . Let  $F$  be the transitive closure of  $R$ . We call  $F$  the *Family relation* of  $H$ .

Each edge shares two vertices with itself, so  $F$  is reflexive. The intersection of sets is symmetric, so  $F$  is symmetric.  $F$  is the transitive closure of  $R$ , so it is transitive. Hence  $F$  is an equivalence relation.

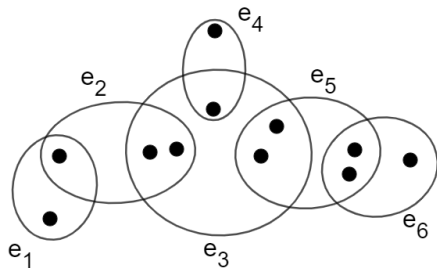
## VI - Families



**Figure:** The Family Relation on the set of hyperedges of this hypergraph gives the equivalence classes  $\{\{e_1\}, \{e_2, e_3, e_5, e_6\}, \{e_4\}\}$ . The hyperedges  $e_2$  and  $e_3$  share two vertices and  $e_3$  and  $e_5$  share two vertices, so  $e_2$  and  $e_5$  are also related.

## VI - Families

An equivalence class of the family relation, or *family*, indicates which hyperedges ought to be colored with one or all distinct colors. If two hyperedges  $e_1, e_2 \in E$  are in the same family, then in any coloring of  $H$ ,  $e_1$  is multi-colored iff  $e_2$  is.



**Figure:** The hyperedge  $e_2$  is multi-colored iff  $e_3$  is multi-colored. The hyperedge  $e_5$  is multi-colored iff  $e_6$  is multi-colored. The hyperedge  $e_1$  may be mono-colored independently of any other hyperedge.

## VII - Special Classes of Hypergraphs

### Definition

A hypersegment  $H = (V, E)$  is a hypergraph such that if  $E = \{e_1, e_2, \dots, e_k\}$ , then for  $i = 1, 2, \dots, k - 1$ ,  $|e_i \cap e_{i+1}| \geq 1$  and if  $|i - j| \geq 2$ , then  $e_i \cap e_j = \emptyset$ .

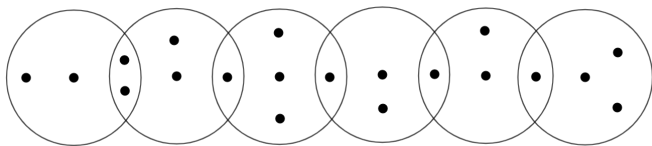
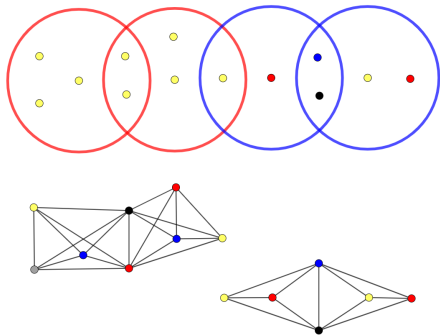


Figure: This hypergraph is a hypersegment.



## VII - Special Classes of Hypergraphs

In a hypersegment, we may make entire families mono- or multi-colored without affecting other families. Since we cannot have every family contain only mono-colored hyperedges (this would indicate a hypergraph colored with only one color), we must have at least one family containing only multi-colored edges, and we may mono-color the rest of the families.



**Figure:** This hypersegment can be colored with four colors by using a Greedy algorithm starting with the smallest hyperedge.

## VIII - Future Work

In the future, this research may be furthered by exploring the relaxed strong coloring numbers of basic classes of hypergraphs. We may also increase our understanding of hypergraph coloring by applying concepts from graph coloring. We seek certain transformations on hypergraphs that do not change the relaxed strong coloring number, of which simplification is one.

# Acknowledgements

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