View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by University of Minnesota, Morris (UMM): Digital Well

University of Minnesota Morris Digital Well
University of Minnesota Morris Digital Well

Mathematics Publications

Faculty and Staff Scholarship

2004

Nonic 3-adic Fields

John W. Jones
Arizona State University

David P. Roberts
University of Minnesota - Morris, roberts@morris.umn.edu

Follow this and additional works at: https://digitalcommons.morris.umn.edu/mathematics

Part of the Number Theory Commons

Recommended Citation
John W. Jones and David P. Roberts. Nonic 3-adic Fields. In ANTS VI: Algorithmic Number Theory. Berlin: Springer (2004), 293-308.

This Conference Proceeding is brought to you for free and open access by the Faculty and Staff Scholarship at University of Minnesota Morris Digital Well. It has been accepted for inclusion in Mathematics Publications by an authorized administrator of University of Minnesota Morris Digital Well. For more information, please contact skulann@morris.umn.edu.

Nonic 3-adic Fields*

John W. Jones[1] and David P. Roberts[2]

[1] Department of Mathematics and Statistics, Arizona State University, Box 871804, Tempe, AZ 85287
jj@asu.edu

[2] Division of Science and Mathematics, University of Minnesota–Morris, Morris, MN 56267
roberts@mrs.umn.edu

Abstract. We compute all nonic extensions of $\mathbf{Q}_3$ and find that there are 795 of them up to isomorphism. We describe how to compute the associated Galois group of such a field, and also the slopes measuring wild ramification. We present summarizing tables and a sample application to number fields.

1 Introduction

This paper is one of three accompanying our online database of low degree p-adic fields, located at `http://math.la.asu.edu/~jj/localfields/`. The first paper, [10], describes the database in general. There are two cases enormously more complicated than all the others in the range considered, octic 2-adic fields and nonic 3-adic fields. The paper [9] describes the 1823 octic 2-adic fields and this paper describes the 795 nonic 3-adic fields.

Our online database has an interactive feature which allows one to enter an irreducible polynomial $f(x) \in \mathbf{Z}[x]$ and obtain a thorough analysis of the ramification in the corresponding number field $K = \mathbf{Q}[x]/f(x)$. The inclusion of octic 2-adic fields and nonic 3-adic fields in the database greatly extends the number fields K that can be analyzed mechanically by our programs. Certainly, the degree of K can be very much larger than 9.

◊ Section 2 discusses standard resolvent constructions and then three more specialized resolvent constructions for nonic fields with a cubic subfield. Section 3 centers on the Galois theory of nonic fields over general ground fields, describing the 34 possibilities for the Galois group associated to a nonic field. Also this section gives further information useful for our particular ground field $\mathbf{Q}_3$. For example, 11 of the 34 possible Galois groups can be immediately ruled out over $\mathbf{Q}_3$, because they don't have appropriate filtration subgroups. Somewhat coincidentally, the 23 groups that remain are exactly those with a normal Sylow 3-subgroup.

* In the published version, the T_{15}-ϵ slot of Table 3.3 is mistakenly filled with a $+$ rather than a $-$. This error caused several other parts of the published version to be in error. In this posted version, we have fixed this error and its consequences, flagging altered paragraphs and table entries with the marginal symbol ◊.

◊ The nonic 3-adic field section of our database would run to some twenty printed pages, so here we give only summarizing tables. Section 4 centers on Table 4.1 which sorts the 795 fields we find according to discriminant and Galois group. All of the 23 eligible groups appear. Section 5 describes ramification in nonic 3-adic fields in terms of slopes, with Tables 5.1 and 5.2 summarizing our results. Finally, Section 6 gives an application to number fields.

2 Resolvent Polynomials

Resolvents play a major role in the computation of Galois groups over $\mathbf{Q}$. Some resolvents can be computed quickly using exact arithmetic with resultants. However more often one computes resolvents via approximations to complex roots, knowing *a priori* that the resolvents in question have integer coefficients.

The fields studied here are represented by monic polynomials $f \in \mathbf{Z}[x]$. The computation of absolute resolvents can then follow standard methods. However, most applications of resolvents to computing Galois groups in high degree utilize relative resolvents. In the relative case, one somehow gives structure to the roots of f to reflect the fact that $\mathrm{Gal}(f)$ is known to lie in some proper subgroup G^u of S_n. One speaks of resolvents relative to the upper bound group G^u.

A complication for us is that for a given nonic polynomial $\tilde{f} \in \mathbf{Z}[x]$, we may have $\mathrm{Gal}_{\mathbf{Q}_p}(\tilde{f}) \leq G^u$ but $\mathrm{Gal}_{\mathbf{Q}}(\tilde{f}) \not\leq G^u$. In this case, the resolvent will have coefficients which are p-adic integers, but generally not rational integers, and so the method of computing via complex approximations does not work directly. The method [6] of using p-adic approximations rather than complex approximations does not help here. It involves choosing a prime p unramified for the given extension of $\mathbf{Q}$, whereas we are starting with p-adic extensions which are highly ramified.

For our three relative resolvents, the upper bound group G^u is the wreath product $S_3 \wr S_3$, which is the generic Galois group of nonic fields with a cubic subfield. Given $\tilde{f} \in \mathbf{Z}[x]$ which defines a nonic extension of $\mathbf{Q}_3$ with a unique cubic subfield, we work around the problem described in the preceding paragraph by computing $f \in \mathbf{Z}[x]$ which defines the same nonic extension of $\mathbf{Q}_3$, but where f has a corresponding cubic subfield over $\mathbf{Q}$. Then we use complex roots in the resolvent construction applied to f. The computation of f from $\tilde{f}$ is described further in the last paragraph of this section.

◊ We now describe the five resolvent constructions which we will use systematically in the sequel. The first is a standard absolute resolvent. One starts with a degree n polynomial $f(x)$ with complex roots $\alpha_1, \ldots, \alpha_n$. The resolvent corresponds to the subgroup $S_2 \times S_{n-2}$ of S_n and is given by

$$f_{\mathrm{disc}}(x) = \prod_{i<j} (x - (\alpha_i - \alpha_j)^2) \in \mathbf{Z}[x]. \tag{1}$$

It can be computed quickly without approximations to roots via the formula

$$f_{\mathrm{disc}}(x^2) = \mathrm{Resultant}_y(f(y), f(x+y))/x^n. \tag{2}$$

In one case, we will also make use of the variant $f_{\mathrm{Disc}}(x) = f_{\mathrm{disc}}(x^2)$, which is itself an absolute resolvent for $S_1 \times S_1 \times S_{n-2} < S_n$. In general, we will systematically denote polynomial resolvent constructions by $f \mapsto f_*$ for some symbol $*$. We will use similar notation for the same resolvent constructions on the level of fields starting in the next section.

$\diamondsuit$ In one case we use an absolute resolvent of degree 72 associated to $(S_2 \times S_7) \cap A_9$, denoted here by f_{72} (in [7] it is denoted R_6). Its roots correspond to the orbit of

$$(\alpha_1 - \alpha_2) \prod_{3 \le i < j \le 9} (\alpha_i - \alpha_j).$$

Factoring this resolvent over $\mathbf{Q}_3$ is the most difficult resolvent computation considered here.

For the remaining three resolvents, we start with an irreducible monic degree nine polynomial $f \in \mathbf{Z}[x]$ such that $\mathbf{Q}[x]/f(x)$ has a unique cubic subfield. We choose $g(y) \in \mathbf{Z}[y]$ so that $\mathbf{Q}[y]/g(y)$ is isomorphic to this subfield. Let

$$h(y,x) = x^3 + \sum_{k=0}^{2} \sum_{\ell=0}^{2} c_{k\ell} y^k x^\ell \tag{3}$$

be a cubic factor of $f(x)$ over $\mathbf{Q}[y]/g(y)$. Let $\beta_1, \beta_2, \beta_3$ be the complex roots of $g(y)$. For $i = 1, 2, 3$, let $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ be the complex roots of $h(\beta_i, x)$. Then we can recover

$$f(x) = \prod_{i=1}^{3} \prod_{j=1}^{3} (x - \alpha_{i,j}) \,. \tag{4}$$

A formula bypassing the $\alpha_{i,j}$ is

$$f(x) = \mathrm{Resultant}_y(g(y), h(y,x)). \tag{5}$$

The three resolvents are

$$f_{18}(x) = \prod_{i=1}^{3} \prod_{\sigma \in S_3} \left(x - \sum_{j=1}^{3} \alpha_{i,j} \alpha_{i+1,\sigma(j)} \right), \tag{6}$$

$$f_{27}(x) = \prod_{i=1}^{3} \prod_{j=1}^{3} \prod_{k=1}^{3} \left(x - (\alpha_{1,i} + \alpha_{2,j} + \alpha_{3,k}) \right), \tag{7}$$

$$f_{36}(x) = \prod_{\sigma \in S_3} \prod_{\tau \in S_3} \left(x - \sum_{i=1}^{3} \alpha_{1,i} \alpha_{2,\sigma(i)} \alpha_{3,\tau(i)} \right). \tag{8}$$

Now we return to describing how we adjust a given polynomial $\tilde{f}$ to a better one f. We first find a global model $\mathbf{Q}[y]/g(y)$ for the cubic subfield of the extension $\mathbf{Q}_3[x]/\tilde{f}(x)$ of $\mathbf{Q}_3$; this is easy using the database [10]. Then, we use Algorithm 3.6.4 from [2] to reduce the factorization of $\tilde{f}(x)$ over $\mathbf{Q}_3[y]/g(y)$ to a factorization problem over $\mathbf{Q}_3$. We loosely approximate a cubic factor

$\tilde{h}(y,x) \in \mathbf{Z}_p[x]$ that we obtain by a polynomial $h(y,x) \in \mathbf{Z}[y,x]$, and use Equation (5) to compute a candidate for f. Finally, we test if our candidate f defines the same nonic field over $\mathbf{Q}_3$ as $\tilde{f}$ by using Panayi's p-adic root finding algorithm [13]. If it doesn't, we repeat with a better approximation to $\tilde{h}(y,x)$. Since we ultimately use complex roots in computing resolvents f_{18}, f_{27}, and f_{36}, we aim throughout to keep the coefficients of f relatively small.

3 Galois Theory of Nonics

We will set things up over a general base field F, and specialize when necessary to our case of interest, $F = \mathbf{Q}_3$. We work in the context of abstract separable fields of finite degree over F. We say a nonic field K is multiply imprimitive, uniquely imprimitive, or primitive, iff it has ≥ 2, 1, or 0 cubic subfields respectively.

To bring in Galois theory, we imagine that a separable closure $\overline{F}$ of F is given. The number of subfields of $\overline{F}$ isomorphic to a separable degree n extension K/F is $n/|\operatorname{Aut}(K)|$. We associate to K is the n-element set X of homomorphisms $\sigma : K \to \overline{F}$. Let K^{gal} be the subfield of $\overline{F}$ generated by the images of these n homomorphisms. Then we call $G = \operatorname{Gal}(K^{\mathrm{gal}}/F)$ the Galois group of K with respect to the fixed separable closure of $\overline{F}$. So G is a transitive subgroup of the symmetric group S_X and a quotient group of the absolute Galois group $\operatorname{Gal}(\overline{F}/F)$. Occasionally we will use this notation when K is a separable algebra which is only a product of fields. Then G is no longer a transitive subgroup of S_X as indeed its minimal orbits correspond to the factor fields of K.

In the case $F = \mathbf{Q}$, one can take $\overline{\mathbf{Q}} \subset \mathbf{C}$ as a separable closure. However in other cases, like our case $F = \mathbf{Q}_3$, there is no simple choice of $\overline{F}$ and in practice one must work with objects which are independent of the choice of $\overline{F}$. We will therefore consider the Galois group of K to be a subgroup of S_n which is only defined up to conjugation.

There are 34 transitive subgroups of S_9 up to conjugation, 4 corresponding to multiply primitive fields, 19 to uniquely imprimitive fields, and 11 to primitive fields. So, given a nonic field K, one wants first to identify its Galois group among the 34 possibilities. The literature contains several accounts of computing Galois groups, with [8] being a recent survey. Some of these accounts pay particular attention to nonics [7, 4]. The approach presented here is tailored to 3-adic fields, where it is easier to compute subfields and automorphism groups than it is to work with many different relative resolvents and/or large degree absolute resolvents.

Twenty-three of the thirty-four groups have just one Sylow 3-subgroup, while the remaining seven solvable groups have four Sylow 3-subgroups. The twenty-three groups will be particularly important for us and a partial inclusion diagram for just these groups is given in Figure 1. We use the T-notation of [1] to indicate the possible Galois groups, with T standing for transitive. This T-notation will be our main notation in the sequel as well. However in Tables 3.1, 3.2, and 3.3 we will also present a more descriptive notation based on [5].

In Figure 1, a line from T_i down to T_j means there are subgroups $G_i, G_j \subset S_n$ of type T_i, T_j respectively, with $G_i \subset G_j$ and $|G_i| = |G_j|/2$. The text in Figure 1 briefly indicates how some of the phenomena presented in Tables 3.1, 3.2, and 3.3 relate to index two inclusions.

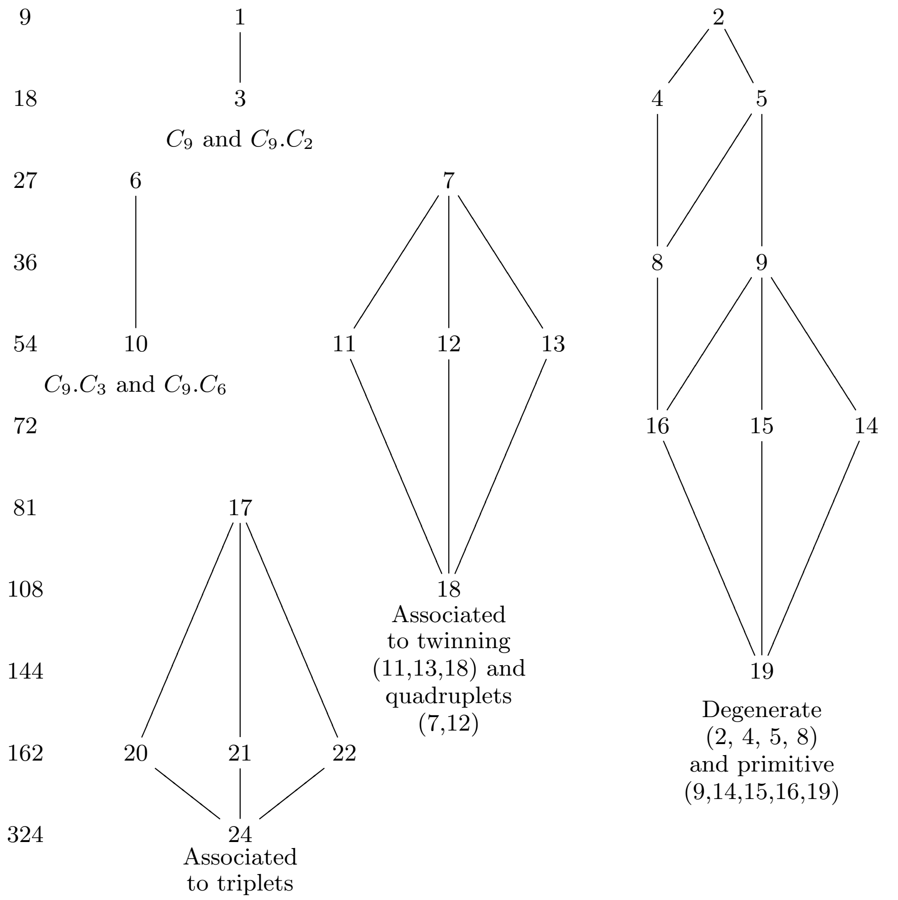

Fig. 1. Nonic groups having a normal Sylow 3-subgroup and their index two inclusions

Over $F = \mathbf{Q}_3$, some of the groups can be easily ruled out. In general, let F be a p-adic field, meaning a finite extension of either $\mathbf{Q}_p$ of $\mathbf{F}_p((t))$. Then a Galois extension K^{gal}/F has a filtration by Galois subfields,

$$F \subseteq K^{\mathrm{gal},u} \subseteq K^{\mathrm{gal},t} \subseteq K^{\mathrm{gal}}, \tag{9}$$

with $K^{\mathrm{gal},u}$ being the maximal unramified subextension and $K^{\mathrm{gal},t}$ being the maximal tamely ramified subextension. The group $\mathrm{Gal}(K^{\mathrm{gal},u}/F)$ is necessarily cyclic. Similarly, $\mathrm{Gal}(K^{\mathrm{gal},t}/K^{\mathrm{gal},u})$ is cyclic; moreover, this subquotient has order prime to p. Finally $\mathrm{Gal}(K^{\mathrm{gal}}/K^{\mathrm{gal},t})$ is a p-group. Eleven of the thirty-four candidate Galois groups fail to have a corresponding chain of normal subgroups. Five of these excluded groups are uniquely imprimitive and six are primitive. They are presented with a dash in the last column in Table 3.2 and Table 3.3 respectively. The twenty-three groups which do have a corresponding chain are exactly those in Figure 1.

There are different approaches for identifying $\mathrm{Gal}(K^{\mathrm{gal}}/F)$. We begin by computing three quantities directly associated to K, the cubic subfields of K, the automorphism group of K, and the parity of K. There can be one, two, or four cubic subfields, and the possible Galois groups for a cubic subfield are C_3 and S_3. The automorphism group $\mathrm{Aut}(K)$ has nine elements exactly when K is itself Galois, in which case $\mathrm{Aut}(K) \cong \mathrm{Gal}(K^{\mathrm{gal}}/F)$. Otherwise $|\mathrm{Aut}(K)|$ is 1 or 3 while $|\mathrm{Gal}(K^{\mathrm{gal}}/F)| > 9$. In our case of $F = \mathbf{Q}_3$, we used Panayi's root finding algorithm [13] for the computation of both subfields and automorphism groups. Parities are easier. By definition $K = \mathbf{Q}[x]/f(x)$ has parity $\epsilon = +$ if the polynomial discriminant of f is a square in F and $\epsilon = -$ otherwise. In the former case $\mathrm{Gal}(K^{\mathrm{gal}}/F)$ is in A_9 and in the latter case it is not.

Tables 3.1, 3.2, and 3.3 have columns corresponding to the objects just discussed. A blank for $|\mathrm{Aut}(K)|$ signifies that K has only the identity automorphism. Tables 3.2 and 3.3 also present information related to resolvents. A resolvent construction $f \mapsto f_*$ on the level of polynomials induces a well-defined resolvent construction $K \mapsto K_*$ on the level of algebras, where $K = F[x]/f(x)$ and $f(x)$ is chosen so that $f_*(x)$ is separable, in which case $K_* = F[x]/f_*(x)$. Finally, the column headed by # previews the next section by giving the number of nonic 3-adic fields with the given Galois group. If the Galois group is ruled out by the lack of an unramified/tame/wild filtration, we print a dash.

3.1 Multiply Imprimitive Fields

Suppose K is a nonic field with more than one cubic subfield. Then if K_1 and K_2 are any two distinct cubic subfields, $K = K_1 \otimes K_2$. Table 3.1 gives the four possible Galois groups. In two cases, there are two cubic subfields, and in two cases there are four cubic subfields, as indicated. In this category, no resolvents are necessary for distinguishing the Galois groups.

Table 3.1. Nonic groups corresponding to nonic fields with more than one cubic subfield

G	**Name**	$\lvert G\rvert$	$\lvert A\rvert$	**Subs**	ϵ	#
2	$E(9)$	3^2	9	$CCCC$	$+$	1
4	$S_3 \times 3$	$2^1 3^2$	3	SC	$-$	24
5	$3^2 : 2$	$2^1 3^2$		$SSSS$	$+$	1
8	$S_3 \times S_3$	$2^2 3^2$		SS	$-$	9

3.2 Uniquely Imprimitive Fields

Table 3.2. Nonic groups corresponding to nonic fields with exactly one cubic subfield. The first fourteen groups have one Sylow 3-subgroup and the last five groups have four Sylow 3-subgroups.

G	**Name**	$\lvert G\rvert$	$\lvert A\rvert$	**Sub**	ϵ	K_{9b}	K_{9x}	K_{27}	K_{27b}	K_{9y}	m	#
1	$C(9)$	3^2	9	C	+	T_1	C_3^3	T_1^3	T_1^3	C_3^3		12
3	$D(9)$	$2^1 3^2$		S	+	T_3	S_3^3	T_3^3	T_3^3	S_3^3		5
6	$\frac{1}{3}[3^3]3$	3^3	3	C	+	T_6	T_2	27_+	27_+	T_2		8
7	$[3^2]3$	3^3	3	C	+	T_7	T_2	$T_7' T_7'' T_7'''$	27_+	$C_3' C_3'' C_3'''$	4	4
10	$[3^2]S_{3;6}$	$2^1 3^3$		S	+	T_{10}	T_4	27_+	27_+	T_4		49
11	$E(9):6$	$2^1 3^3$		S	+	T_{11}	T_4	$T_{13} 18_-$	27_+	$G_{6,18-} C_3$	2^*	20
12	$[3^2]S_3$	$2^1 3^3$	3	S	−	T_{12}	T_5	$T_{12}' T_{12}'' T_{12}'''$	27_+	$S_3' S_3'' S_3'''$	4	36
13	$E(9):D_6$	$2^1 3^3$		C	−	T_{13}	T_4	$T_{11} 18_-$	27_+	$G_{6,18-} C_3$	2^*	20
17	$3 \wr 3$	3^4	3	C	+	T_{17}'	T_7	27_+	27_+	T_7	3	36
18	$E(9):D_{12}$	$2^2 3^3$		S	−	T_{18}	T_8	$T_{18}' 18_-$	27_+	$G_{6,36-} S_3$	2	48
20	$3 \wr S_3$	$2^1 3^4$	3	S	−	T_{20}'	T_{11}	27_-	27_+	T_{13}	3	180
21	$\frac{1}{2}[3^2:2]S_3$	$2^1 3^4$		S	+	T_{21}'	T_{12}	27_+	27_+	T_{12}	3	108
22	$[3^3:2]3$	$2^1 3^4$		C	−	T_{22}'	T_{13}	27_-	27_+	T_{11}	3	60
24	$[3^3:2]S_3$	$2^2 3^4$		S	−	T_{24}'	T_{18}	27_-	27_+	T_{18}	3	144

G	**Name**	$\lvert G\rvert$	$\lvert A\rvert$	**Sub**	ϵ	K_{18}	K_{27}	K_{36}	m	#
25	$[\frac{1}{2}S_3^3]3$	$2^2 3^4$		C	+	18_+	27_+	36_+		−
28	$S_3 \wr 3$	$2^3 3^4$		C	−	18_+	27_-	36_+		−
29	$[\frac{1}{2}S_3^3]S_3$	$2^3 3^4$		S	−	18_-	27_-	36_-		−
30	$\frac{1}{2}[S_3^3]S_3$	$2^3 3^4$		S	+	18_-	27_+	36_-		−
31	$S_3 \wr S_3$	$2^4 3^4$		S	−	18_-	27_-	36_-		−

It is here that we will use the three specialized resolvent constructions of the previous section. They let us canonically construct algebras K_{18}, K_{27}, and K_{36} of the indicated degree from a nonic field K with a unique cubic subfield. Table 3.2 gives the nineteen possibilities for the Galois group of a nonic field with a unique cubic subfield.

The resolvents K_{18}, K_{27} and K_{36} are sometimes irreducible. In this case, the corresponding slots on Table 3.2 contain 18_+, 18_-, 27_+, 27_-, 36_+ or 36_-. In general, we indicate a field of degree > 9 by giving its degree and its parity. We indicate a field of degree ≤ 9 by giving its Galois group.

When G has just one Sylow 3-subgroup there are unique factorizations

$$K_{18} = K_{9b} \times K_{9x}, \tag{10}$$

$$K_{36} = K_{27b} \times K_{9y}, \tag{11}$$

with factors having the indicated degree and the properties

$$\begin{aligned} K_{9b}^{\mathrm{gal}} &= K_{27b}^{\mathrm{gal}} = K^{\mathrm{gal}}, \\ K_{9x}^{\mathrm{gal}} &= K_{9y}^{\mathrm{gal}} \overset{3}{\subset} K^{\mathrm{gal}}. \end{aligned} \tag{12}$$

Here the superscript 3 means that $[K^{\mathrm{gal}} : K_{9y}^{\mathrm{gal}}] = 3$. Table 3.2 indicates the structure of K_{9b}, K_{9x}, K_{27b}, and K_{9y}. In the K_{9y} column, $G_{6,18-}$ and $G_{6,36-}$ are sextic groups of order 18 and 36 respectively which are not contained in A_6.

The m heading a column in Table 3.2 stands for multiplicity. In general, we say that a k-tuplet of degree n is a complete list of non-isomorphic degree n fields $K_1, \ldots, K_k$ with K_j^{gal} all the same. One speaks of singletons, twins, triplets, and quadruplets, for $k = 1, 2, 3,$ and 4 respectively. The T_i row has a k in the m column iff a nonic field K with $\mathrm{Gal}(K^{\mathrm{gal}}/F) = T_i$ belongs to a k-tuple. Here, 1 is indicated by a blank. In most cases, fields in a tuplet share the same T_i. The one exception, indicated by a $*$ in Table 3.2, is twins consisting of a T_{11} nonic field and a T_{13} nonic field. In fact, the thirty-four T_i fall into thirty-three isomorphism classes as abstract groups, the one coincidence being $T_{11} \cong T_{13}$.

Our priming convention in the resolvent columns of Table 3.2 is to distinguish different fields with the same Galois group. A T_i in the i row means a nonic field isomorphic to the original one. Note that in the instances of twinning, one can always pass from a field to its twin via K_{27}. In the five instances of triplets, it is K_{18} which lets one pass from a nonic field $K = K_a$ to a second triplet K_{9b}. Applying this degree eighteen resolvent construction to K_{9b} gives the remaining triplet K_{9c}. Finally, applying it to K_{9c} returns K_{9a}. Thus any collection of nonic triplets comes with a natural cyclic order. Finally, in the two instances of quadruplets, one can pass from a given field to the three others via K_{27}.

Table 3.2 makes clear how one can compute Galois groups of uniquely imprimitive nonic fields. The cases of one versus four Sylow 3-subgroups are distinguished by the reducible versus reducibility of K_{18} or equally well K_{36}. Within the one Sylow 3-subgroup case, $|A|$, **Sub**, ϵ, and K_{9y} suffice to distinguish groups. Within the four Sylow 3-subgroup case, $|A|$, **Sub**, and ϵ alone distinguish groups, except for T_{29} versus T_{31} for which the printed resolvent information doesn't help. In our setting of $F = \mathbf{Q}_3$, neither T_{29} or T_{31} arises, but one way to distinguish them is by the discriminant resolvent. Recall $f_{\mathrm{Disc}}(x) = f_{\mathrm{disc}}(x^2)$. In both cases, K_{Disc} factors as a degree 18 field times a degree 54 field. The degree 18 field has even parity in the case T_{29} and odd parity in the case T_{31}.

3.3 Primitive Fields

Neither subfields nor automorphisms can help with determining the Galois group of a primitive nonic field. The parity ϵ remains helpful, as shown in Table 3.3.

Table 3.3 also gives the degrees and parities of the field factors of the degree 36 resolvent K_{disc} and the resolvent K_{72}. The information presented in Table 3.3 clearly suffices to identify Galois groups associated to nonic 3-adic fields, with

Table 3.3. Nonic groups corresponding to primitive nonic fields

	G	**Name**	$\|G\|$	ϵ	K_{disc}	K_{72}	#
	9	$E(9):4$	$2^2 3^2$	$+$	18_-^2	18_-^4	2
	14	$E(9):Q_8$	$2^3 3^2$	$+$	36_+	36_+^2	4
$\diamondsuit$	15	$E(9):8$	$2^3 3^2$	$-$	36_-	36_-^2	4
	16	$E(9):D_8$	$2^3 3^2$	$-$	18_- $18'_-$	36_+^2	4
	19	$E(9):2D_8$	$2^4 3^2$	$-$	36_-	72_+	16
	23	$E(9):2A_4$	$2^3 3^3$	$+$	36_+	36_+^2	$-$
	26	$E(9):2S_4$	$2^4 3^3$	$-$	36_-	72_+	$-$
	27	$PSL(2,8)$	$2^3 3^2 7$	$+$	36_+	36_+^2	$-$
	32	$P\Sigma L(2,8)$	$2^3 3^3 7$	$+$	36_+	36_+^2	$-$
	33	A_9	$2^6 3^5 5^1 7$	$+$	36_+	36_+^2	$-$
	34	S_9	$2^7 3^5 5^1 7$	$-$	36_-	72_+	$-$

the computationally expensive K_{72} needed only for distinguishing T_{15} from T_{19}. For general bases, more resolvents of higher degrees would be required ([4, 7]).

4 Nonic 3-adic Fields by Discriminant and Galois Group

In general, let $\mathcal{K}(p,n)$ be the set of isomorphism classes of degree n extension fields of $\mathbf{Q}_p$. The paper [14] describes how one goes about finding polynomials $f_i(x) \in \mathbf{Z}[x]$ such that $\mathbf{Q}_p[x]/f_i(x)$ runs over $\mathcal{K}(p,n)$. One key ingredient is the root finding algorithm of [13], mentioned already in both of the previous sections. Here, it is used to ensure that $\mathbf{Q}_p[x]/f_i(x)$ and $\mathbf{Q}_p[x]/f_j(x)$ are not isomorphic for $i \neq j$. The other key ingredient is the mass formula of [11, 15], as refined in [14]. One knows that one has found enough polynomials to cover all of $\mathcal{K}(p,n)$ by the use of this formula. In this section, we present Table 4.1 summarizing the result of the calculation in the case $(p,n) = (3,9)$, and comment on several features of the table.

Given a nonic 3-adic field K, let K^u be its maximal subfield unramified over $\mathbf{Q}_3$. Let $f = [K^u : \mathbf{Q}_3]$ and $e = 9/f$. The original mass formula of [11, 15] makes it natural to divide nonic fields K into three classes according to e. The unique field with $e = 1$ corresponds to the boldface entry with $c = 0$ in the row for group $T_1 = C_9$. The 41 fields with $e = 3$ correspond to italicized entries, and the remaining 753 are given in ordinary type.

The refined mass formula of [14] makes it natural to further divide nonic fields K according to their discriminant exponent c. This exponent indexes columns in Table 4.1. Mass formulae can be used to count the number of subfields of $\overline{F}$ of a given type. Recall from Section 3 that if a field K has automorphism group A, then this count will reflect the $9/|A|$ isomorphic copies of K in $\overline{F}$. Collecting these contributions to the mass formula and dividing by the degree 9 gives a count of isomorphism classes of fields, each weighted by their mass $1/|A|$. The sums of these masses for a given (e,c) are given by the refined mass formula in

Table 4.1. Discriminants and Galois groups of nonic extensions of $\mathbf{Q}_3$. Fields with $e = 1$, 3, 9 are respectively indicated by bold, italic, and regular type.

G	$\lvert A\rvert$	0	9	10	12	13	15	16	18	19	20	21	22	23	24	25	26	#
2	9				*1*													1
4	3		*2*		*1*		6, *3*	3		9								24
5									1									1
8					1		2		3	3								9
1	9	**1**			*2*								9					12
3									1				1				3	5
6	3				*2*								6					8
7	3				*1*			3										4
10								6	11				8				24	49
11					2			1	8				9					20
12	3									9		27						36
13			*2*		*1*		*2*		3	3		9						20
17	3				*9*				9				18					36
18						2	4		3	12		18	9					48
20	3					6	12		9	45			27			81		180
21											27				27		54	108
22			*6*		*3*		*6*					9	9	27				60
24								6	12	9	27	9		27	27	27		144
9					1			1										2
14				1					3									4
◇ 15					2			2										4
16				1					3									4
◇ 19			2			2	6			6								16
$\#_1$		**1**																**1**
M_1		$\mathbf{0.\overline{1}}$																
$\#_3$			*10*		*20*		*11*											*41*
M_3			$\mathit{8.\overline{6}}$		$\mathit{8.\overline{6}}$		*9*											
$\#_9$			2	2	6	10	30	22	66	96	54	72	96	54	54	108	81	753
M_9			2	2	6	6	18	18	54	54	54	54	54	54	54	54	81	
$M_{9,3}$					4	4	12	12	36	36	54							
$M_{9,4}$							4	4	12	12		36	36	54				
$M_{9,5}$									6	6		18	18		54	54	81	

[14], and are presented as $M_e(c)$ towards the bottom of Table 4.1. Of course, one has $\#_e(c) \geq M_e(c)$ with $\#_e(c)$ the total number of fields for a given (e, c).

The lines corresponding to the four multiply imprimitive groups can be constructed directly by tensoring pairs of cubic fields. For example, as K_c runs over the four C_3 fields and K_s runs over the six S_3 fields, $K_c \otimes K_s$ runs over the twenty-four $T_4 = C_3 \times S_3$ fields. Similarly, the primitive groups T_9 and T_{16} are isomorphic to the sextic transitive groups $C_3^2.C_4$ and $C_3^2.D_4$ respectively. In each of these two cases, one nonic field K comes from two sextic fields K_{6a}, K_{6b} with $K^{\mathrm{gal}} = K_{6a}^{\mathrm{gal}} = K_{6b}^{\mathrm{gal}}$. Otherwise the lines of Table 4.1 cannot be constructed from the lower degree tables of [10].

Twins, triplets, and quadruplets are visible in varying degrees on Table 4.1. In general, for a nonic 3-adic field K with discriminant 3^c, one has $c = 2s_u + 6s_v$ with the slopes $s_u \leq s_v$ discussed further in the next section. In a tuplet, s_v is always constant. For twins, s_u typically varies within a twin pair; however one can at least see that the total number of fields for T_{11} and T_{13} is the same and the total number for T_{18} is even. In a triplet, the cubic subfield is constant; if $3^{c_{\mathrm{sub}}}$

is its discriminant then $s_u = c_{\mathrm{sub}}/2$, so c_u and hence c is constant. This explains conceptually why all entries on rows 17, 20, 21, 22, and 24 are multiples of 3. For quadruplets, the possible c_{sub}'s are $(0,4,4,4)$ in the case T_7 and $(3,5,5,5)$ in the case T_{12}, explaining the structure of these rows.

The mass formulas mentioned already come from specializing mass formulas for general p-adic base fields F to the case F is an unramified extension of $\mathbf{Q}_3$ of degree 9, 3, or 1. One can also specialize these formulas to the case that F is a ramified cubic extension of $\mathbf{Q}_3$, and one gets the last three lines of Table 4.1. The number in the $M_{9,c_{\mathrm{sub}}}$ row and the c column is the total mass of isomorphism classes of nonic fields of discriminant 3^c with a specified cubic subfield of discriminant $3^{c_{\mathrm{sub}}}$, where a pair $K_{\mathrm{sub}} \subset K$ is counted with mass $1/|A_0|$ where A_0 is the group of automorphisms of K stabilizing K_{sub}. This is simplest in the cases $c \geq 20$ where all fields are uniquely primitive and thus $|A| = |A_0|$.

5 Slopes in Nonic 3-adic Fields

The paper [10] provides background on slopes in general. Here we will keep the discussion focused on nonic 3-adic fields K and their associated Galois fields K^{gal}. The results of the calculations are summarized in Tables 5.1 and 5.2.

Table 5.1. Slopes in nonic 3-adic fields with discriminant exponent $c \leq 18$. Visible slopes are in boldface and hidden slopes in ordinary type.

c	G	Slopes	#
0	1	$\mathbf{0}, \mathbf{0}$	1
9	4	$\mathbf{0}, \mathbf{1.5}$	2
9	13	$\mathbf{0}, \mathbf{1.5}, 1.5$	2
9	22	$\mathbf{0}, \mathbf{1.5}, 1.5, 1.5$	6
9	19	$\mathbf{1.125}, \mathbf{1.125}$	2
10	14	$\mathbf{1.25}, \mathbf{1.25}$	1
10	16	$\mathbf{1.25}, \mathbf{1.25}$	1
12	2	$\mathbf{0}, \mathbf{2}$	1
12	4	$\mathbf{0}, \mathbf{2}$	1
12	8	$\mathbf{1.5}, \mathbf{1.5}$	1
12	1	$\mathbf{0}, \mathbf{2}$	2
12	6	$\mathbf{0}, \mathbf{2}, 2$	2
12	7	$\mathbf{0}, \mathbf{2}, 2$	1
12	11	$0, \mathbf{1.5}, \mathbf{1.5}$	2
12	13	$\mathbf{0}, \mathbf{2}, 2$	1
12	17	$\mathbf{0}, \mathbf{2}, 2, 2$	9
12	22	$\mathbf{0}, \mathbf{2}, 2, 2$	3
12	9	$\mathbf{1.5}, \mathbf{1.5}$	1
12	15	$\mathbf{1.5}, \mathbf{1.5}$	2
13	18	$\mathbf{1.5}, 1.5, \mathbf{1.667}$	2
13	20	$0, \mathbf{1.5}, 1.5, \mathbf{1.667}$	6
13	19	$\mathbf{1.625}, \mathbf{1.625}$	2
15	4	$\mathbf{1.5}, \mathbf{2}$	6
15	4	$\mathbf{0}, \mathbf{2.5}$	3
15	8	$\mathbf{1.5}, \mathbf{2}$	2
15	13	$\mathbf{0}, 1.5, \mathbf{2.5}$	2
15	18	$\mathbf{1.5}, 1.5, \mathbf{2}$	4
15	20	$0, \mathbf{1.5}, 1.5, \mathbf{2}$	12
15	22	$\mathbf{0}, 1.5, 1.5, \mathbf{2.5}$	6
15	19	$\mathbf{1.875}, \mathbf{1.875}$	6
16	4	$\mathbf{2}, \mathbf{2}$	3
16	7	$0, \mathbf{2}, \mathbf{2}$	3
16	10	$\mathbf{1.5}, 2, \mathbf{2.167}$	6
16	11	$0, \mathbf{2}, \mathbf{2}$	1
16	24	$\mathbf{1.5}, 1.5, 2, \mathbf{2.167}$	6
16	9	$\mathbf{2}, \mathbf{2}$	1
16	15	$\mathbf{2}, \mathbf{2}$	2
18	5	$\mathbf{1.5}, \mathbf{2.5}$	1
18	8	$\mathbf{1.5}, \mathbf{2.5}$	3
18	3	$\mathbf{1.5}, \mathbf{2.5}$	1
18	10	$0, \mathbf{1.5}, \mathbf{2.5}$	2
18	10	$\mathbf{1.5}, 2, \mathbf{2.5}$	9
18	11	$0, \mathbf{1.5}, \mathbf{2.5}$	2
18	11	$\mathbf{2}, 2, \mathbf{2.333}$	3
18	11	$\mathbf{1.5}, 2, \mathbf{2.5}$	3
18	13	$\mathbf{2}, 2, \mathbf{2.333}$	3
18	17	$0, \mathbf{2}, 2, \mathbf{2.333}$	9
18	18	$\mathbf{1.5}, 2, \mathbf{2.5}$	3
18	20	$0, \mathbf{2}, 2, \mathbf{2.333}$	9
18	24	$\mathbf{1.5}, 1.5, 1.667, \mathbf{2.5}$	3
18	24	$\mathbf{1.5}, 1.5, 2, \mathbf{2.5}$	9
18	14	$\mathbf{2.25}, \mathbf{2.25}$	3
18	16	$\mathbf{2.25}, \mathbf{2.25}$	3

Table 5.2. Visible and hidden slopes in nonic 3-adic fields with discriminant exponent $c \geq 19$.

c	G	Slopes	#
19	4	$\mathbf{2}, \mathbf{2.5}$	9
19	8	$\mathbf{2}, \mathbf{2.5}$	3
19	12	$\mathbf{1.5}, 2.5, \mathbf{2.667}$	9
19	13	$1.5, \mathbf{2}, \mathbf{2.5}$	3
19	18	$1.5, \mathbf{2}, \mathbf{2.5}$	3
19	18	$\mathbf{1.5}, 2.5, \mathbf{2.667}$	9
19	20	$0, \mathbf{1.5}, 2.5, \mathbf{2.667}$	18
19	20	$\mathbf{1.5}, 2, 2.5, \mathbf{2.667}$	27
19	24	$\mathbf{1.5}, 2, 2.5, \mathbf{2.667}$	9
19	19	$\mathbf{2.375}, \mathbf{2.375}$	6
20	21	$\mathbf{1.5}, 2.5, 2.667, \mathbf{2.833}$	27
20	24	$\mathbf{1.5}, 2.5, 2.667, \mathbf{2.833}$	27
21	12	$1.5, \mathbf{2.5}, \mathbf{2.667}$	27
21	13	$\mathbf{2}, 2.5, \mathbf{2.833}$	9
21	18	$1.5, \mathbf{2.5}, \mathbf{2.667}$	9
21	18	$\mathbf{2}, 2.5, \mathbf{2.833}$	9
21	22	$1.5, \mathbf{2}, 2.5, \mathbf{2.833}$	9
21	24	$1.5, \mathbf{2}, 2.5, \mathbf{2.833}$	9

c	G	Slopes	#
22	1	$\mathbf{2}, \mathbf{3}$	9
22	3	$\mathbf{2}, \mathbf{3}$	1
22	6	$0, \mathbf{2}, \mathbf{3}$	6
22	10	$0, \mathbf{2}, \mathbf{3}$	2
22	10	$\mathbf{2}, 2, \mathbf{3}$	6
22	11	$2, \mathbf{2.5}, \mathbf{2.833}$	9
22	17	$0, \mathbf{2}, 2, \mathbf{3}$	18
22	18	$2, \mathbf{2.5}, \mathbf{2.833}$	9
22	20	$\mathbf{2}, 2, 2.333, \mathbf{3}$	27
22	22	$\mathbf{2}, 2, 2.333, \mathbf{3}$	9
23	22	$\mathbf{2}, 2.5, 2.833, \mathbf{3.167}$	27
23	24	$\mathbf{2}, 2.5, 2.833, \mathbf{3.167}$	27
24	21	$1.5, \mathbf{2.5}, 2.667, \mathbf{3.167}$	27
24	24	$1.5, \mathbf{2.5}, 2.667, \mathbf{3.167}$	27
25	20	$2, \mathbf{2.5}, 2.833, \mathbf{3.333}$	81
25	24	$2, \mathbf{2.5}, 2.833, \mathbf{3.333}$	27
26	3	$\mathbf{2.5}, \mathbf{3.5}$	3
26	10	$0, \mathbf{2.5}, \mathbf{3.5}$	6
26	10	$2, \mathbf{2.5}, \mathbf{3.5}$	18
26	21	$1.5, \mathbf{2.5}, 2.667, \mathbf{3.5}$	54

Suppose $\mathrm{Gal}(K^{\mathrm{gal}}/\mathbf{Q}_3)$ has order $2^a 3^b$, so that $b \in \{2,3,4\}$. Then there are b slopes to compute, which we always index in weakly increasing order, $s_1 \leq s_2 \ (\leq \cdots)$. Consider a chain of subfields

$$\mathbf{Q}_3 \overset{2^a}{\subseteq} K^{\mathrm{gal},0} \overset{3}{\subset} K^{\mathrm{gal},1} \overset{3}{\subset} K^{\mathrm{gal},2} \left(\overset{3}{\subset} \cdots\right), \tag{13}$$

with, as indicated, $K^{\mathrm{gal},j}$ having degree $2^a 3^j$ over $\mathbf{Q}_3$. There is only one choice for $K^{\mathrm{gal},0}$ as, in all 23 cases not excluded by (9), the group T_i has only one Sylow 3-subgroup. There may be several choices for some of the intermediate $K^{\mathrm{gal},j}$, and then of course $K^{\mathrm{gal},b} = K^{\mathrm{gal}}$. We require that the intermediate fields be chosen such that the discriminant exponents c_j are the minimum possible for Galois subfields of degree $2^a 3^j$; the c_j are then uniquely defined. The j^{th} slope is then given by the formula

$$s_j = \frac{c_j - c_{j-1}}{2^a 3^j - 2^a 3^{j-1}}, \tag{14}$$

an instance of Proposition 3.4 of [10]. Note that we are allowing 0 as a slope here. Otherwise the slopes are > 1 and correspond to wild ramification.

Two of the b slopes just discussed are visible in K itself in the following sense. Let 3^c be the discriminant of K. If K has a cubic subfield, let

K_{sub} be a cubic subfield of minimum discriminant $3^{c_{\text{sub}}}$. Then from the tower $\mathbf{Q}_3 \subset K_{\text{sub}} \subset K$, one has slopes $s_u = c_{\text{sub}}/2$ and $s_v = (c - c_{\text{sub}})/6$, another instance of Proposition 3.4 of [10]. In this setting, $s_u \leq s_v$. If K is primitive, then the visible slopes are $s_u = s_v = c/8$. An important point is that the highest slope is always visible, i.e. $s_v = s_b$.

We call the remaining $b - 2$ slopes hidden. *A priori*, one might have expected that to calculate them, one would have to compute field discriminants of resolvents of relatively high degree, perhaps 27 say, and apply Proposition 3.4 of [10] yet again. However, as we explain next, in fact one needs to compute discriminants only of associated nonics.

First, if K has ≥ 2 cubic subfields or 0 cubic subfields, then $b = 2$ and so there are no hidden slopes to compute. Now consider the 14 possible Galois groups of a uniquely imprimitive 3-adic nonic. From Table 3.2, T_1 and T_3 have $b = 2$, while T_6, T_7, T_{10}, T_{11}, T_{12}, T_{13} and T_{18} have $b = 3$, and finally T_{17}, T_{20}, T_{21}, T_{22}, and T_{24} have $b = 4$.

In the two $b = 2$ cases, we are again done. For the seven $b = 3$ cases, we can consider the nonic resolvent K_{9x}. One has an exact sequence

$$C_3 \hookrightarrow \text{Gal}(K^{\text{gal}}/\mathbf{Q}_3) \twoheadrightarrow \text{Gal}(K_{9x}^{\text{gal}}/\mathbf{Q}_3). \tag{15}$$

Now the two nonic groups of order 27, namely T_6 and T_7, both have only one normal subgroup of order 3. So the remaining groups presently under consideration, T_{10}, T_{11}, T_{12}, T_{13} and T_{18}, also have only one normal subgroup of order 3. So it is necessarily the highest slope s_3 of K which disappears upon passage from K to K_{9x}. Only one of the remaining slopes s_1 and s_2 is visible in K, but both are visible in K_{9x}, allowing us to identify all three of s_1, s_2, and s_3. This computation would work equally well replacing K_{9x} by K_{9y}, as indeed $K_{9x}^{\text{gal}} = K_{9y}^{\text{gal}}$.

Finally for the five $b = 4$ cases, we can use again the nonic resolvent K_{9x} and the exact sequence (15). The slopes of K are s_1, s_2, s_3, and s_4. The 3-group T_{17} has just one normal subgroup of order 3, and hence the normal overgroups T_{20}, T_{21}, T_{22}, and T_{24} also have just one normal subgroup of order 3. So the slopes of K_{9x} must be s_1, s_2, and s_3, which are all identified by the previous paragraph. The highest slope s_4 is identified too, so all slopes of K have been identified. Again this computation would work equally well with K_{9x} replaced by K_{9y}.

6 Global Applications

The last section of [10] concerns degree 13 fields with Galois group $PSL_3(3)$ and illustrates how ramification can be analyzed completely, with the analysis at 2 requiring octic 2-adic fields and the analysis at 3 requiring nonic 3-adic fields. We are presently pursuing other applications which similarly require a number of group-theoretic preliminaries to describe globally. Here we will stay in a setting where the only group-theory we need is what we have set up in previous sections.

Proposition 6.1 **A.** *There are exactly thirteen isomorphism classes of solvable nonic number fields with discriminant of the form $\pm 3^b$, namely the fields $K = \mathbf{Q}[x]/f(x)$ with $f(x) = \sum a_{9-i}x^i$ as given in Table 6.2.*

B. *Assuming that Odlyzko's GRH lower bounds on discriminants hold, there are no non-solvable nonic number fields with discriminant of the form $\pm 3^b$.*

To establish Part A, we need to use the group theory set up in Sections 2 and 3, but not the specifically 3-adic information of Sections 4 and 5. First, one knows that there are no quartic number fields with discriminant $\pm 3^b$. This implies that there are no primitive nonic solvable number fields of discriminant $\pm 3^b$ because all seven solvable primitive groups have a quotient group of the form V_4, C_4, A_4, or S_4, according to Table 3.3. For the same reason, an imprimitive group G can only appear if its size has the form 3^b or $2^1 3^b$.

Table 6.2. The thirteen nonic solvable number fields with discriminant of the form $\pm 3^b$, sorted by increasing top slope

c	G	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	s_1	s_2	s_3	s_4
19	4	1	0	0	−3	0	0	−6	0	0	−1	**2**	**2.5**		
21	13	1	0	0	−3	0	0	0	0	0	1	**2**	2.5	**2.833**	
22	11	1	0	0	−3	0	0	3	0	0	8	2	**2.5**	**2.833**	
22	1	1	0	−9	0	27	0	−30	0	9	1	**2**	**3**		
23	22	1	0	0	−6	0	0	9	0	0	−3	**2**	2.5	2.833	**3.167**
23	22	1	0	0	−3	0	0	0	0	0	3	**2**	2.5	2.833	**3.167**
23	22	1	0	0	−3	0	0	−9	0	0	3	**2**	2.5	2.833	**3.167**
25	20	1	0	−9	−6	27	36	−24	−54	−9	22	2	**2.5**	2.833	**3.333**
25	20	1	0	−9	−3	27	18	−24	−27	−9	23	2	**2.5**	2.833	**3.333**
25	20	1	0	−9	−3	27	18	−15	−27	−36	−4	2	**2.5**	2.833	**3.333**
26	3	1	0	0	−9	0	0	27	0	0	−3	**2.5**	**3.5**		
26	10	1	0	0	0	0	0	0	0	0	−3	2	**2.5**	**3.5**	
26	10	1	0	0	−9	0	0	27	0	0	−24	2	**2.5**	**3.5**	

It is also known that there are exactly two cubic fields with discriminant $\pm 3^b$, namely $\mathbf{Q}[x]/(x^3 - 3x - 1)$ with Galois group C_3 and $\mathbf{Q}[x]/(x^3 - 3)$ with Galois group S_3. So we need look only at cubic extensions of these fields, using the exhaustive method described in Chapter 5 of [3]. Since neither of the two cubic fields contains cube roots of unity, the method requires us to adjoin cube roots of unity to get sextic fields K_6, and look within degree eighteen overfields of these to get the desired nonic fields. The method requires that K_{18}/K_6 be abelian, but abelianness is ensured by $\mathrm{ord}_2(|G|) \leq 1$. These computations, which of course we have only briefly sketched here, establish Part A.

Before moving on to establishing Part B, we will comment on some ways that Table 6.2 illustrates our previous sections. The T_{11} field and the T_{13} field form a twin pair. Similarly, the three T_{20} fields and the three T_{22} fields each form a triplet. Triplets have a cyclic order and if we call the top-listed field in each triplet K_a, then the next is K_b and the final one is K_c. Slopes are given in the same format as Tables 5.1 and 5.2. The fact that the small visible slope can change within twins but not triplets is illustrated.

Part B is similar to some other non-existence statements in the literature, for example the statement in [16], which says in particular that there are no $PSL_2(8)$ nonics with discriminant of the form $\pm 2^a$. To establish Part B, we use the 3-adic analysis of Sections 4 and 5, including the determination of hidden slopes. If K is a nonic 3-adic field with slopes $s_1 \leq \cdots \leq s_b$ and $[K^{\mathrm{gal},t} : K^{\mathrm{gal},u}] = t$, then the root discriminant of K^{gal} is 3^β with

$$\beta = \frac{2}{3}s_b + \cdots + \frac{2}{3^b}s_1 + \frac{1}{3^b}\frac{t-1}{t}. \tag{16}$$

This type calculation is explained further in [10]. From Table 5.1, one sees that in the cases $b = 2$, $b = 3$, and $b = 4$, the largest that β can be is respectively $53/18 = 2.9\overline{4}$, $55/18 = 3.0\overline{5}$, and $511/162 \approx 3.15432$, these bounds all being realized in the largest discriminant case $c = 26$, always with $t = 2$. The corresponding 3^β are then approximately 25.40, 28.70, and 31.99. These numbers are thus upper bounds for the root discriminant of a Galois number field with discriminant $\pm 3^b$ and Galois group $PSL(2, 8)$, $P\Sigma L(2, 8)$ and (A_9 or S_9) respectively. However Odlyzko's GRH bounds say that a field with root discriminant at most 25.40, 28.70, and 31.99 respectively must have degree at most 380, 1000, and 4400 [12]. These numbers are respectively less than $|PSL(2, 8)| = 504$, $|P\Sigma L(2, 8)| = 1512$, and $|A_9| = 9!/2$, giving Part B.

References

[1] G. Butler and J. McKay. The transitive groups of degree up to eleven. *Comm. Algebra*, 11(8):863–911, 1983.

[2] H. Cohen. *A course in computational algebraic number theory*. Springer-Verlag, Berlin, 1993.

[3] ——— *Advanced topics in computational number theory*, volume 193 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[4] A. Colin. Relative resolvents and partition tables in Galois group computations. In *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation (Kihei, HI)*, pages 78–84 (electronic), New York, 1997. ACM.

[5] J. H. Conway, A. Hulpke, and J. McKay. On transitive permutation groups. *LMS J. Comput. Math.*, 1:1–8 (electronic), 1998.

[6] H. Darmon and D. Ford. Computational verification of M_{11} and M_{12} as Galois groups over **Q**. *Comm. Algebra*, 17(12):2941–2943, 1989.

[7] Y. Eichenlaub. *Problèmes effectifs de théorie de Galois en degrés 8 à 11.* PhD thesis, Université Bordeaux I, 1996.

[8] A. Hulpke. Techniques for the computation of Galois groups. In *Algorithmic algebra and number theory (Heidelberg, 1997)*, pages 65–77. Springer, Berlin, 1999.

[9] J. W. Jones and D. P. Roberts. Octic 2-adic fields. In preparation.

[10] ——— A database of local fields. Submitted, 2003.

[11] M. Krasner. Remarques au sujet d'une note de J.-P. Serre: "Une 'formule de masse' pour les extensions totalement ramifiées de degré donné d'un corps local". *C. R. Acad. Sci. Paris Sér. A-B*, 288(18):A863–A865, 1979.

[12] J. Martinet. Petits discriminants des corps de nombres. In *Number theory days, 1980 (Exeter, 1980)*, volume 56 of *London Math. Soc. Lecture Note Ser.*, pages 151–193. Cambridge Univ. Press, Cambridge, 1982.

[13] P. Panayi. *Computation of Leopoldt's p-adic regulator.* PhD thesis, University of East Anglia, 1995.

[14] S. Pauli and X.-F. Roblot. On the computation of all extensions of a p-adic field of a given degree. *Math. Comp.*, 70(236):1641–1659 (electronic), 2001.

[15] J.-P. Serre. Une "formule de masse" pour les extensions totalement ramifiées de degré donné d'un corps local. *C. R. Acad. Sci. Paris Sér. A-B*, 286(22):A1031–A1036, 1978.

[16] J. Tate. The non-existence of certain Galois extensions of $\mathbf{Q}$ unramified outside 2. In *Arithmetic geometry (Tempe, AZ, 1993)*, volume 174 of *Contemp. Math.*, pages 153–156. Amer. Math. Soc., Providence, RI, 1994.