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Mixed Degree Number Field Computations

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ABSTRACT. We present a method for computing complete lists of number fields in cases where the Galois group, as an abstract group, appears as a Galois group in smaller degree. We apply this method to find the twenty-five octic fields with Galois group $\mathrm{PSL}_2(7)$ and smallest absolute discriminant. We carry out a number of related computations, including determining the octic field with Galois group $2^3\!:\mathrm{GL}_3(2)$ of smallest absolute discriminant.

1. INTRODUCTION

1.1. **Overview.** Number theorists have computed number fields with minimal absolute discriminants for each of the thirty possible Galois groups in degrees at most 7. In degrees 8 and 9, the minimal fields are known for the seventy-five solvable Galois groups. All these minimal fields are available, together with references to sources, on the Klüners-Malle database [KM01]. The minimal fields are also available, typically as first elements on long complete lists, at the online databases [JR14a, LMF18].

The Klüners-Malle paper [KM01] also gives smallest known absolute discriminants for the five nonsolvable octic groups and the four nonsolvable nonic groups. Despite the many years that have passed since its publication, rigorous minima have not been established for these nine groups. In this paper, we address two of the nine cases, proving that the absolute discriminants $3^8 7^8$ and 5717^2 presented in [KM01] for the octic groups $\mathrm{PSL}_2(7)$ and $2^3\!:\mathrm{GL}_3(2)$ are indeed minimal. These cases are related through the exceptional isomorphism $\mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$.

One element of our approach for finding the $\mathrm{PSL}_2(7)$ minimum was suggested already in [KM01]: any octic $\mathrm{PSL}_2(7)$ field K_8 has the same Galois closure as two septic $\mathrm{GL}_3(2)$ fields K_{7a} and K_{7b}. As the discriminants satisfy $D_{7a} = D_{7b} \mid D_8$, one can in principle establish minimality of the octic discriminant 21^8 by conducting a search of all septic fields with absolute discriminant $\leq 21^8$. We combine this with the method of targeted Hunter searches, which requires us to analyze, on a prime-by-prime basis, how discriminants either stay the same or increase when one passes from septic to octic fields. This targeting based on discriminants makes the computation feasible. We add several smaller refinements to make the computation run even faster.

Our title refers to the general method of carrying out a carefully targeted search in one degree to obtain a complete list of fields in a larger degree. Section 2 gives background and then Section 3 describes the general method, using our case where the two degrees are 7 and 8 as an illustration. Section 4 presents our minimality

result for $\mathrm{PSL}_2(7)$, improved in Theorem 1 to the complete list of twenty-five octic $\mathrm{PSL}_2(7)$ fields with discriminant $\leq 30^8$. This section also presents corollaries giving minimal absolute discriminants for certain related groups in degrees 16, 24, and 32.

Section 5 gives a second illustration of the mixed degree method, now with degrees 5 and 6. Here we use the exceptional isomorphisms $A_5 \cong \mathrm{PSL}_2(5)$ and $S_5 \cong \mathrm{PGL}_2(5)$ and Theorem 2 considerably extends the known list of sextic $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$ fields. We also explain in this section potential connections with asymptotic mass formulas and Artin representations.

Our final section returns to groups related to the septic group $\mathrm{GL}_3(2)$. Theorem 3 finds all alternating septics with discriminant $\leq 12^7$. The long runtime of this search makes clear the importance of targeting for Theorem 1. However just the bound 12^7 is sufficient for our last corollary, which confirms minimality of the $2^3{:}\,\mathrm{GL}_3(2)$ field with discriminant 5717^2.

1.2. **Notation and conventions.** We denote the cyclic group of order n by C_n. We use $N{:}H$ to denote a semi-direct product with normal subgroup N and complement H.

A number field is a finite extension of $\mathbb{Q}$, which we consider up to isomorphism. If $K/\mathbb{Q}$ is such an extension with degree n, then its normal closure, K^g, is Galois over $\mathbb{Q}$. Moreover, $\mathrm{Gal}(K^g/\mathbb{Q})$ comes with a natural embedding into S_n, which is well-defined up to conjugation. We denote the image of such an embedding, which is a transitive subgroup of S_n, by simply $\mathrm{Gal}(K)$.

Transitive subgroups of S_n, considered up to conjugation, have been classified and indexed for small n, and is available through *Magma* [BCP97] and *Pari* [PAR15], as well as though the Galois groups section of the LMFDB web site [LMF18], which provides information on each such conjugacy class of subgroups for $n < 24$. Here, we denote the jth subgroup by nTj. In §4.2 we use the classification of nearly 3 million transitive subgroups of S_{32} which was completed more recently in [CH08].

When several non-isomorphic fields have the same splitting field, we refer to them as siblings. For example, fields K_{7a}, K_{7b}, and K_8 as in the overview are siblings. If K_1 and K_2 are siblings, then $\mathrm{Gal}(K_1) \cong \mathrm{Gal}(K_2)$ as abstract groups, but as described above, the two Galois groups typically come with different embeddings into S_n, possibly even for different n. We aim to exploit this difference where possible for the computations in this paper.

While the nTj notation specifies both the degree n of the stem field and the conjugacy class of the subgroup in S_n, the numeric identifier conveys no information on the structure of the group. We use group names in the spirit of [CHM98], which assigns standard names for the groups nTj indicating their structures as permutation groups for $n < 16$. For example, $7T5 = \mathrm{GL}_3(2)$ and $8T37 = \mathrm{PSL}_2(7)$ have natural transitive actons on the projective spaces $\mathbb{P}^2(\mathbb{F}_2)$ and $\mathbb{P}^1(\mathbb{F}_7)$ of orders 7 and 8 respectively. The third group mentioned in the overview is the group of affine transformations of $\mathbb{F}_2^3$. Our notation emphasizes its semidirect product structure: $8T48 = 2^3{:}\,\mathrm{GL}_3(2)$.

It is often enlightening to shift the focus from the absolute discriminant $|D|$ of a degree n number field K to the corresponding root discriminant $\mathrm{rd}(K) = \delta = |D|^{1/n}$. We generally try to indicate both, as in the numbers 21^8 and 30^8 of the overview.

2. Background

Our method of mixed degree targeted Hunter searches is built on well-established methods of searching for number fields, which we now briefly explain.

2.1. **Hunter searches.** A Hunter search, named for J. Hunter, is a standard technique for computing all primitive number fields of a given degree with absolute discriminant less than a given bound [Coh00]. Here a number field K is primitive if it has exactly two subfields, itself and $\mathbb{Q}$. When the degree n is prime, as in our cases $n = 7$ and $n = 5$ here, the primitivity assumption is automatically satisfied.

The only two inputs for a standard Hunter search are the degree and the discriminant bound. Some implementations optimize for a particular signature which can then be thought of as a third input, but here we search all signatures simultaneously. The computation itself is an exhaustive search for polynomials with integer coefficients bounded by various inequalities.

2.2. **Targeted Hunter searches.** Targeting, introduced in [JR99], and refined in [JR03], allows one to search for fields with particular large discriminants. One carries out a Hunter search, but only for fields which match a given combination of local targets. The targets, described below, determine both the discriminant and the local behavior of the field at ramifying primes p. This latter information forces a defining polynomial to satisfy congruences modulo several prime powers, and these congruences greatly reduce the number of polynomials one needs to inspect.

To describe the targets more precisely, let p be a prime number and let K be a degree n number field. Then $K \otimes \mathbb{Q}_p \cong \prod_{i=1}^{g} K_{p,i}$ where each $K_{p,i}$ is a finite extension of $\mathbb{Q}_p$. At its most refined level, a local target may be a single p-adic algebra, up to isomorphism. In a few situations, we do work at this level. However typically, one wants to treat natural collections of p-adic algebras as a single target.

Let $\mathbb{Q}_p^{\mathrm{unr}}$ be the unramified closure of $\mathbb{Q}_p$. Then, similar to the decomposition above we have

$$(1) \qquad K \otimes \mathbb{Q}_p^{\mathrm{unr}} \cong \prod_{j=1}^{t} L_{p,j},$$

where each $L_{p,j}$ is a finite extension of $\mathbb{Q}_p^{\mathrm{unr}}$. Let e_j be the ramification index of the field $L_{p,j}$, and $(p)^{c_j}$ its discriminant ideal. We note that the e_j give the sizes of the orbits of the p-inertia subgroup acting on the roots of an irreducible defining polynomial for K. A typical local target is then a pair $((e_1, \ldots, e_t), \sum_{j=1}^{t} c_j)$ with the list of e_j weakly decreasing. The ramification indices $(e_1, \ldots, e_t)$ give a partition of $[K : \mathbb{Q}]$, and the discriminant of the local algebra, which equals the p-part of the discriminant of K, is $(p)^{\sum_j c_j}$.

A local target at p determines a list of congruences modulo some fixed power of p. For tamely ramified primes we work modulo p, while we use higher powers for wildly ramified primes. When there is more than one ramifying prime, the lists of congruences are simply combined via the Chinese remainder theorem.

3. Mixed degree targeted Hunter searches

In a mixed degree targeted search, one has a Galois group G and transitive permutation representations of two different degrees $n < m$. Each degree n field K_n with Galois group $G \hookrightarrow S_n$ determines a degree m field K_m with Galois group

$G \hookrightarrow S_m$. Targets are triples $((e_1, \ldots, e_t), c_n, c_m)$ where c_n is the local discriminant exponent for the small degree fields searched, while c_m is the local discriminant exponent of the larger degree fields actually sought. One uses the values p^{c_m} to decide which combinations of targets to search, and $((e_1, \ldots, e_t), c_n)$ to carry out the actual search in degree n. We describe how one deals with the two degrees here, often by using our first case with $n = 7$ and $m = 8$ as an example. Once we have the degree n polynomials in hand, we compute the corresponding degree m polynomials as resolvents using *Magma* [BCP97].

3.1. **Tame ramification.** The behavior of tame ramification under degree changes is straightforward. Let K be a degree n number field, G its Galois group, and p a tamely ramified prime. The inertia subgroup I for a prime above p is cyclic; let τ be a generator. Via the given inclusion $G \subseteq S_n$, we let $e_1, e_2, \ldots, e_t$ be the cycle type of τ. These match the e_j of the local target described above. The exponent of p in the discriminant of K is then given by

$$(2) \qquad c_n = \sum_{j=1}^{t}(e_j - 1) = n - t.$$

When one is considering also a second degree m, one just runs through the above procedure a second time. In our first case, $G \cong \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$, each row of Table 1 represents a candidate for τ. The row then gives the corresponding pair of partitions (λ_7, λ_8) and pair of discriminant exponents (c_7, c_8). These discriminant exponents are computed from the partitions via formula (2).

TABLE 1. Cycle types and discriminant exponents for $\mathrm{GL}_3(2) \cong$ $\mathrm{PSL}_2(7)$ in degrees 7 and 8.

λ_7	λ_8	c_7	c_8
7	7,1	6	6
4,2,1	4,4	4	6
3,3,1	3,3,1,1	4	4
2,2,1,1,1	2,2,2,2	2	4

Note that if a prime p is tamely ramified in our pair of fields (K_7, K_8), then its minimal contribution to the discriminant of the octic is p^4. Thus, when searching for octic fields with absolute discriminant $\leq B$, we need only consider primes $p \leq \sqrt[4]{B}$. Our largest search used $B = 30^8$, so $p \leq 900$.

The relation $D_7 \mid D_8$ mentioned in the introduction is due to the fact that we always have $c_7 \leq c_8$, and that this inequality also holds for wildly ramified primes. In two of the tame cases, one has equality, but in the other two tame cases one has strict inequality. Our method using targeted searches makes use of the strictness of these latter inequalities.

3.2. **Wild ramification.** An explicit description of the behavior of wild p-adic ramification under degree changes becomes rapidly more complicated as $\mathrm{ord}_p(G)$ increases. We describe just our case $G \cong \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$ here, as this case represents the basic nature of the general case well.

Since $|G| = 2^3 \cdot 3 \cdot 7$, the only primes which can be wildly ramified in a G extension are 2, 3, and 7. For a subgroup of G to be an inertia group for a wildly ramified prime p, it must be an extension of a cyclic group of order prime to p by a non-trivial p-group. The candidates for a $G = \mathrm{PSL}_2(7)$ extension are given in Table 2. They run over all of the non-trivial proper subgroups of $\mathrm{PSL}_2(7)$ up to conjugation,

TABLE 2. Wild ramification data for $\mathrm{PSL}_2(7)$.

p	I	D	λ_7		λ_8	(c_7, c_8)
7	$C_7{:}C_3$	$C_7{:}C_3$	7		7,1	$(8,8), (10,10)$
	C_7	$C_7, C_7{:}C_3$	7		7,1	$(12,12)$
3	S_3	S_3	3,3,1		6,2	$(6,8), (10,12)$
	C_3	C_3, S_3	3,3,1		3,3,1,1	$(8,8)$
2	A_4	S_4	4,3	6,1	4,4	$(6,8), (10,16)$
	D_4	D_4	4,2,1		8	$(12,22), (14,24)$
	C_4	C_4, D_4	4,2,1		4,4	$(14,22)$
	V	V, D_4, A_4	2,2,2,1	4,1,1,1	4,4	$(6,12), (8,16)$
	C_2	C_2, V, C_4	2,2,1,1,1		2,2,2,2	$(4,8), (6,12)$

with the exception of two conjugacy classes of subgroups isomorphic to S_4.

Each subgroup in the table is a candidate for being the inertia group for a wild prime for only one prime. The horizontal lines separate the subgroups according to this prime. The second column gives the isomorphism type of the candidate for inertia, and the third column gives corresponding candidates for the decomposition group. Over other 2-adic ground fields, $A_4 = I = D$ is possible, but not over $\mathbb{Q}_2$ since there is no ramified C_3 extension of $\mathbb{Q}_2$.

The columns labeled λ_7 and λ_8 show the orbit sizes of the actions of I in the degree seven and eight representations respectively. There are two possibilities for inertia group A_4 and V in degree 7, so we give both. The orbit sizes are helpful in determining the data c_7 and c_8 in each case, and λ_7 is the partition of 7 needed for carrying out the targeted Hunter search.

In most cases, it is clear from Galois theory how to interpret the orbit sizes. For example, inside a Galois A_4 field, there are unique subfields of degrees 3 and 4 up to isomorphism. So the 4 in the first A_4 entry is for the usual quartic representation, and the 3 is its resolvent cubic. More detailed computations with the groups allow us to resolve the two ambiguities, which are as follows.

- A Galois D_4 field has three quadratic subfields and three quartic subfields (up to isomorphism). In the degree 7 partition 4,2,1, the 4 represents a quartic stem field, say defined by a polynomial f, and then the 2 represents the field obtained from a root of $x^2 - \mathrm{Disc}(f)$.
- A V field has three quadratic subfields. In the line for V, the 2,2,2,1 represents the product of these quadratic fields and $\mathbb{Q}_2$.

The last column gives a list of candidate pairs (c_7, c_8), coming by analyzing the corresponding local extensions. Some cases can be done using just Galois theory and general properties of extensions of local fields. A simple approach, however, which applies to all cases is to make use of the complete lists of the relevant local fields [JR06, LMF18].

For example, suppose 2 is wildly ramified with inertia subgroup isomorphic to A_4. Then, the decomposition group is S_4 and from the tables of degree 4 extensions of $\mathbb{Q}_2$, we see that there are three possibilities for a Galois S_4 field with A_4 as its inertia subgroup. As degree 4 fields, one S_4 has discriminant 2^4 and the other two have discriminant 2^8. Computing the resolvents of these fields in degrees 3, 4, and 6, we see that the quartic field with discriminant 2^4 gives $(c_7, c_8) = (6, 8)$ and the two quartic fields with discriminant 2^8 give $(c_7, c_8) = (10, 16)$. We note that in both cases, the values for c_7 do not depend on the partition λ_7, which is relevant in §3.3.4.

The list of targets for each prime is fairly straightforward to read off Table 2. For example, for $p = 3$ we have only $((3, 3, 1), 6, 8)$, $((3, 3, 1), 8, 8)$, and $((3, 3), 10, 12)$. With $p = 2$, one has $((4, 2, 1), 14, 24)$ from $I = D_4$ and $((4, 2, 1), 14, 22)$ from $I = C_4$; however, only the latter gets used since it has the same partition and c_7 and a smaller value of c_8.

3.3. **Further savings.** Various techniques can reduce the number of polynomials that it is necessary to search. Again we illustrate these reductions by our first case with $G \cong \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$. The first and fourth reductions below simply eliminate some local targets $((e_1, \ldots, e_t), c_7, c_8)$ from consideration. The second and third let us reduce the size of some of the local targets.

3.3.1. *Savings from evenness at half the tame primes.* Our first savings comes from $\mathrm{GL}_3(2)$ being an even subgroup of S_7, i.e., from the inclusion $\mathrm{GL}_3(2) \subset A_7$. To obtain this savings, we make use of the following general lemma.

Lemma 1. *Suppose n and p are distinct primes, K is a degree n number field whose Galois group G is contained in A_n, and p is totally ramified in K. Then p is a quadratic residue modulo n.*

Proof. Let D be the decomposition group for a prime above p. Tame Galois groups over $\mathbb{Q}_p$ are 2-generated by σ and τ where $\sigma\tau\sigma^{-1} = \tau^p$ (see [Iwa55]). Here, τ is a generator of the inertia subgroup and σ is a lift of Frobenius. Thus D is isomorphic to $(\mathbb{Z}/n){:}(\mathbb{Z}/f)$ where f is the order σ, and the action of σ on $\mathbb{Z}/n$ is multiplication by p. The Galois group is a subgroup of A_n which forces σ to be an even permutation, which in turn implies that p is a square modulo n. $\square$

In our case of $n = 7$, the lemma says that if $p \neq 7$ is totally ramified, then p must be congruent to 1, 2, or 4 modulo 7, eliminating "totally ramified" as a target for approximately half of the primes.

3.3.2. *Savings from evenness at $p = 7$.* We can achieve a savings from the fact that $\mathrm{GL}_3(2) \subset A_7$ at $p = 7$ as well. It is evident from the complete lists of degree 7 extensions of $\mathbb{Q}_7$ in [JR06, LMF18] that having discriminant $(7)^c$ with c even is not sufficient to ensure that the Galois group is even. In fact, for each even value of c, only half of the fields, counted by mass, have even Galois group. We computed lists of congruences for each 7-adic septic field with even Galois group and then merged the lists of congruences.

To target a specific 7-adic field, we start with a defining polynomial such that the power basis formed from a root α will generate the ring of integers over $\mathbb{Z}_7$. We then consider a generic element $\beta = a_0 + a_1\alpha + \cdots + a_6\alpha^6$ and compute its characteristic polynomial in $\mathbb{Z}_7[a_0, \ldots, a_6][x]$. Working modulo 7^2 we then enumerate all possibilities for the polynomial.

3.3.3. *Savings from* $\mathrm{ord}_3(G) = 1$ *at* $p = 3$. Cases when 3 is wildly ramified also offer an opportunity to reduce the search time by more refined targeting. As can be seen from the two relevant lines of Table 2, the decomposition subgroup is isomorphic to C_3 or S_3. In either case, the orbit partition for the decomposition group is $(3, 3, 1)$. Thus, a defining polynomial factors as the product of two cubics times a linear polynomial over $\mathbb{Z}_3$.

The savings comes from the fact that the two cubics have to define the same 3-adic field. So, the procedure here starts with computing possible polynomials for each ramified cubic extension of $\mathbb{Q}_3$ modulo some 3^r. We take all products of the form $(x+a)g_1g_2$ where the g_i come from the list for a given field and a runs through all possibilities in $\mathbb{Z}/3^r$. In our actual search, we worked modulo 3^2.

The resulting local targets are considerably smaller. For example, a target $((3, 3, 1), 8)$ from our general method includes cases where the two cubic factors define non-isomorphic fields with discriminant ideal $(3)^4$ and also cases where the cubics have discriminant ideals $(3)^3$ and $(3)^5$. All these possibilities are not searched in our refinement.

3.3.4. *Exploiting arithmetic equivalence at* $p = 2$. The final refinement we use exploits the fact that for each octic field sought, we need to find just one of its two siblings in degree 7. These pairs of septic fields are examples of arithmetically equivalent fields. The two fields K_{7a} and K_{7b} have the same Dedekind zeta function, the same discriminant, and the same ramification partition at all odd primes. However, at $p = 2$ one can have $\lambda_{7a} \neq \lambda_{7b}$.

In Table 2, there are two orbit partitions for the inertia group A_4, and again two orbit partitions for V. For each of these cases, if a septic $\mathrm{GL}_3(2)$ field has inertia subgroup I and one orbit partition, its sibling has the other orbit partition for I. We save by targeting 4,3 and 4,1,1,1, but not their transforms 6,1 and 2,2,2,1.

3.3.5. *Savings from global root numbers being* 1. As we mentioned in §2.1, our code does not distinguish signatures. If it did, there would be an opportunity for yet further savings as follows. A separable algebra K_v over $\mathbb{Q}_v$ has a local root number $\epsilon(K_v) \in \{1, i, -1, -i\}$. For $v = \infty$, one has $\epsilon(\mathbb{R}^r\mathbb{C}^s) = (-i)^s$. For v a prime p, one has $\epsilon(K_p) = 1$ unless the inertia group I_p has even order. Further information about local root numbers is at [JR06, §3.3], with many root numbers calculated on the associated database. The savings comes from the reciprocity relation $\prod_v \epsilon(K_v) = 1$, so that the signature is restricted by the behavior at ramifying primes.

While we are not using local root numbers in our searches, we are using them in our interpretation of the output of our first case. Interesting facts here include the general formulas $\epsilon(K_{7a,v}) = \epsilon(K_{7b,v})$ and $\epsilon(K_{8,v}) = 1$. Also, from Tables 1 and 2, one has equality of discriminant exponents $c_7 = c_8$ at a prime p if and only if $|I_p|$ is odd; so in this case the septic sign $\epsilon(K_{7a,p}) = \epsilon(K_{7b,p})$ is 1.

4. Results for $\mathrm{PSL}_2(7)$ and related groups

4.1. **A complete list of** $\mathrm{PSL}_2(7)$ **octics.** Our search for $\mathrm{PSL}_2(7)$ fields with $\mathrm{rd}(K) \leq 21$ took 41 CPU-hours and confirmed that the discriminant 21^8 given in [KM01] is indeed the smallest. The extended search through $\mathrm{rd}(K) \leq 30$ took approximately four CPU-months. In this extended search, we combined targets for a given prime in a subsearch whenever the contribution to the octic field discriminant is the same. In this sense, the computation consisted of 1471 subsearches of

varying difficulty. The fastest 380 cases took at most 10 seconds each, the median length case took 6.5 minutes, and the slowest ten cases took from 20 to 35 hours each. The slowest cases all involved searches where $c_7 = c_8$ for every ramifying prime. This larger search found twenty-five fields.

Theorem 1. *There are exactly* 25 *octic fields with Galois group* $PSL_2(7)$ *and discriminant* $\leq 30^8$, *as given in Table 3. The smallest discriminant of such a field is* 21^8.

The full list of fields is also available in a computer-readable format by searching the websites [JR14a, LMF18]. The two septic siblings of the first octic $PSL_2(7)$ field are given by $x^7 - 7x^4 - 21x^3 + 21x^2 + 42x - 9$ and $x^7 - 7x + 3$, the latter being the famous Trinks polynomial [Tri68].

#	K_8	δ_8	δ_7
1	$x^8 - 4x^7 + 7x^6 - 7x^5 + 7x^4 - 7x^3 + 7x^2 + 5x + 1$	21.00	23.70
2	$x^8 - x^7 + 7x^6 - x^5 + 33x^4 + x^3 + 61x^2 + 13x + 58$	21.21	23.97
3	$x^8 - 4x^7 + 14x^6 - 24x^5 + 29x^4 - 32x^3 + 18x^2 - 16x + 17$	21.54	18.44
4	$x^8 - 3x^7 + 4x^6 + 2x^5 - 10x^4 + 16x^3 - 20x + 28$	22.37	25.48
5	$x^8 - 2x^7 + 10x^6 - 17x^5 + 28x^4 - 38x^3 + 34x^2 - 17x + 10$	22.45	25.58
6	$x^8 - 2x^7 + 2x^6 - 8x^5 + 16x^4 - 16x^3 + 14x^2 - 10x + 4$	23.16	26.50
7	$x^8 - 3x^7 + 9x^6 - 21x^5 + 44x^4 - 69x^3 + 84x^2 - 84x + 73$	23.39	26.81
8	$x^8 - 4x^7 + 10x^6 - 12x^5 - 7x^4 + 44x^3 - 46x^2 - 4x + 95$	24.16	21.02
9	$x^8 - 3x^7 + x^6 + 9x^5 - 3x^4 - 57x^3 + 133x^2 - 132x + 76$	24.23	27.91
10	$x^8 - 4x^7 + 14x^6 - 28x^5 + 49x^4 - 56x^3 + 56x^2 - 14x + 7$	24.25	22.92
11	$x^8 - x^7 + 5x^6 - 19x^5 + 31x^4 - 47x^3 + 47x^2 - 17x + 4$	25.14	29.11
12	$x^8 - x^7 + 14x^4 - 28x^3 + 28x^2 - 14x + 14$	26.32	26.52
13	$x^8 - 4x^7 + 11x^6 - 17x^5 + 37x^4 - 78x^3 + 132x^2 - 153x + 72$	26.78	31.29
14	$x^8 - x^7 - 7x^6 - 7x^5 + 7x^4 + 49x^3 + 77x^2 + 31x + 4$	26.84	31.37
15	$x^8 - x^7 + x^6 - 11x^5 + 11x^4 + 35x^3 + 45x^2 + 35x + 10$	26.97	27.26
16	$x^8 - 2x^7 + 4x^6 + 2x^5 + 27x^4 - 46x^3 + 84x^2 - 10x + 59$	27.01	22.41
17	$x^8 - 3x^7 + 6x^6 + 2x^5 + 6x^3 + 4x^2 + 6x + 6$	27.17	31.82
18	$x^8 - 2x^7 + 2x^6 - 14x^5 + 46x^4 - 86x^3 + 126x^2 - 118x + 49$	27.35	18.14
19	$x^8 - 4x^7 + 14x^6 - 28x^5 + 49x^4 - 56x^3 + 56x^2 - 24x + 5$	28.00	20.41
20	$x^8 + 28x^4 + 112x^2 - 32x + 84$	28.00	24.88
21	$x^8 - 4x^7 + 14x^6 - 28x^5 + 63x^4 - 84x^3 + 98x^2 - 52x + 19$	28.00	24.88
22	$x^8 - 4x^7 + 49x^4 - 42x^3 + 77x^2 - 31x + 19$	28.86	20.77
23	$x^8 - 4x^7 + 9x^6 - 12x^5 + 4x^4 - 4x^3 + 13x^2 + 4x + 1$	29.05	31.64
24	$x^8 + 10x^6 + 12x^4 - 28x^3 - 58x^2 + 28x + 51$	29.22	27.14
25	$x^8 - x^7 - 4x^6 - 12x^5 + 8x^4 + 64x^3 + 16x^2 - 32x + 24$	29.94	9.39

TABLE 3. The 25 octic fields with Galois group $PSL_2(7)$ and root discriminant ≤ 30, along with their root discriminants, and the root discriminants of the corresponding septic siblings.

Figure 1 gives a visualization of how the ordering by discriminant of septic fields and the ordering by discriminant of octic fields, all having the simple group of order 168 as their Galois group, seem to have little to do with one another. The 25 $PSL_2(7)$ octics of Theorem 1 give the 24 points beneath the $\delta_8 = 30$ line, the

drop of one coming from the fact that the point $(\delta_7, \delta_8) \approx (24.88, 28.00)$ comes from two fields. Theorem 3 likewise says that there are only 23 points to the left of the $\delta_7 = 12$ line. The rectangle where both $\delta_7 \leq 12$ and $\delta_8 \leq 30$ contains just the single point coming from the last field on Table 3.

Another way of making clear how the ordered lists differ sharply is to consider first fields. As noted above, the first octic field, highlighted in Theorem 1, is a sibling of the Trinks field. Recall also that septic $\mathrm{GL}_3(2)$ fields come in sibling pairs with the same discriminant. In the list of septic $\mathrm{GL}_3(2)$ fields ordered by root discriminant at [JR14a, LMF18], the Trinks field is currently in the 1009th pair, with root discriminant about 23.70. Conversely, the octic sibling of the first septic $\mathrm{GL}_3(2)$ pair currently ranks 66th in the corresponding list of octic fields at [JR14a, LMF18].
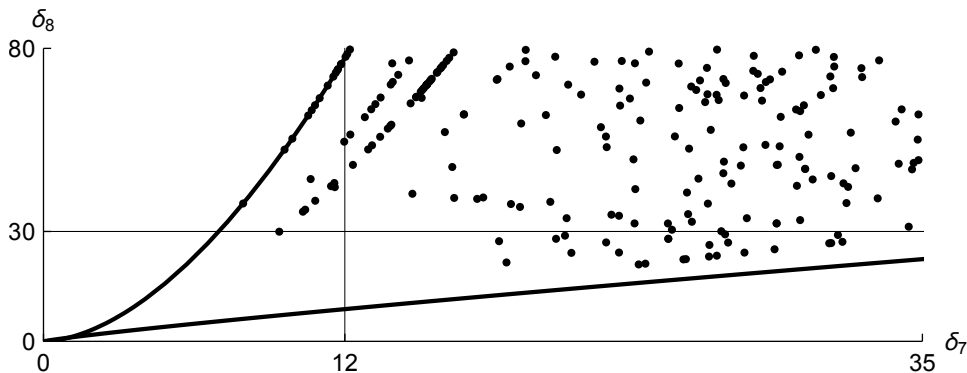

FIGURE 1. Root discriminant pairs (δ_7, δ_8) associated to $\mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$, including all pairs with $\delta_7 \leq 12$ from Theorem 3 and all pairs with $\delta_8 \leq 30$ from Theorem 1.

Figure 1 also gives some sense of the geography of sibling triples (K_{7a}, K_{7b}, K_8). The upper bound corresponds to the extreme case $D_8 = D_7^2$, graphed as $\delta_8 = \delta_7^{7/4}$. The lower bound likewise corresponds to the extreme case $D_8 = D_7$, graphed as $\delta_8 = \delta_7^{7/8}$.

As one example of the details visible in this geography, note that the figure shows no point on the lower bound. This is because a necessary and sufficient condition to be on the lower bound is that all inertia groups have odd size, by Tables 1 and 2. As mentioned in §3.3.5, this condition forces the septic p-adic local signs to all be 1. Reciprocity then forces the infinite local sign to be 1 as well, which forces the fields to be totally real. We expect that the first instance of such a totally real sibling triple is well outside the window of Figure 1. There is just one such sibling triple currently on the Klüners-Malle database [KM01], with $D_7 = D_8 = 2^4 13^4 131^4$, and thus $(\delta_7, \delta_8) = (104.33, 58.36)$. On the other hand, the figure shows twelve points seemingly on a curve just above the lower bound. These points are the ones with $D_8/D_7 = 9$, and thus indeed lie on the curve $\delta_8 = 3^{1/4}\delta_7^{7/8}$. They come from the twelve fields having the prime 3 in bold in Table 4.

4.2. **Class groups and class fields.** The class groups of all twenty-five fields in Theorem 1 can be computed unconditionally by either *Magma* [BCP97] or *Pari*

[PAR15]. They are all cyclic and Table 4 gives their orders. The fact that all but three of these class groups are non-trivial is already remarkable. By way of contrast, the first 620 S_5 quintic fields ordered by absolute discriminant all have trivial class group.

Using Theorem 1, one can get complete lists of fields with root discriminant ≤ 30 for many other Galois groups via class field theory. We restrict ourselves to four Galois groups, chosen because they interact interestingly with the class groups in Table 4. Each Galois group is even so absolute discriminants coincide with discriminants. Fields in Corollaries 1, 2, 3, and 4 correspond to entries in columns t, u, v, and w respectively in Table 4, these entries indicating ramification over the octic base. Most entries are 1, indicating that these class fields correspond to quotient groups of the class groups. However each column has entries larger than 1, so that each complete list of fields also includes ramified extensions not seen from class groups.

To get the first two Galois groups, consider the unramified tower $K_{32}/K_{16}/K_8$ coming from the first field in Table 4, K_8, and its cyclic class group of order four. Defining equations can be computed using *Magma* [BCP97] or *Pari*'s [PAR15] class field theory commands. Following the conventions of §1.2, let $G_{16} = \mathrm{Gal}(K_{16})$ and $G_{32} = \mathrm{Gal}(K_{32})$ be the corresponding Galois groups. Then K_{16} and K_{32} are the unique fields of degree 16 and 32 respectively with smallest discriminant for these Galois groups. In fact, G_{16} is just the Cartesian product $\mathrm{PSL}_2(7) \times C_2 = 16T714$. More interestingly, $G_{32} = 32T34620$ is a non-split double cover of G_{16}, having $\mathrm{SL}_2(7) = 16T715$ as a subgroup with quotient group C_2.

One can carry out a similar analysis for all twenty-five base fields, allowing ramified towers $K_{32}/K_{16}/K_8$ as well, with discriminants denoted D_{32}, D_{16}, and D_8. Using *Pari* [PAR15], we first compute those extensions with root discriminant ≤ 30. There are 56 possible K_{16} in all and 163 such K_{32}. As the corollaries indicate, these numbers are reduced when we extract the fields with the Galois groups sought. To use small numbers only to describe ramification, write $D_{16} = D_8^2 t^2$ and $D_{32} = D_{16}^2 u^4$. This analysis gives the following two consequences of Theorem 1.

Corollary 1. *There are exactly* 25 *number fields with Galois group* $\mathrm{PSL}_2(7) \times C_2 = 16T714$ *and discriminant* $\leq 30^{16}$. *Base octics and ramification invariants* t *are given in Table 4. The smallest discriminant of such a field is* 21^{16} *and the field has defining polynomial*

$$x^{16} - 4x^{15} + 9x^{14} - 14x^{13} + 14x^{12} - 14x^{10} + 8x^9 + 45x^8 - 82x^7$$
$$+ 49x^6 + 63x^5 - 112x^4 + 49x^3 + 99x^2 - 130x + 100.$$

Corollary 2. *There are exactly* 14 *number fields of degree* 32 *with Galois group* $32T34620$ *and discriminant* $\leq 30^{32}$. *Base octics and ramification invariants* u *are given in Table 4. The smallest discriminant of such a field is* 21^{32} *and the field has defining polynomial*

$$x^{32} - x^{31} + 2x^{30} + x^{29} + 8x^{28} - 7x^{27} + 21x^{26} - 9x^{25} - 12x^{24} + 248x^{23}$$
$$- 548x^{22} - 65x^{21} + 2653x^{20} - 4879x^{19} + 2564x^{18} + 4198x^{17} - 7780x^{16}$$
$$+ 3593x^{15} + 4020x^{14} - 7014x^{13} + 4935x^{12} - 2042x^{11} + 929x^{10} - 787x^9$$
$$+ 695x^8 - 215x^7 + 70x^6 - 42x^5 + 15x^4 - 15x^3 + 2x^2 + x + 1.$$

#	D_8	D_7	h	$=$	a	ℓ	c	e	t	u	v	w
1	$\mathbf{3}^8 7^8$	$\mathbf{3}^6 7^8$	4	=	2	2			1,7,7	1,7		
2	$2^6 \mathbf{3}^4 53^4$	$2^6 \mathbf{3}^2 53^4$	4	=	2	2			1	1		$2,2,3,3$
3	$2^{16} 29^4$	$\mathbf{2}^{10} 29^4$	1						4			
4	$2^6 \mathbf{3}^4 59^4$	$2^6 \mathbf{3}^2 59^4$	4	=	2	2			1	1		
5	$\mathbf{3}^6 97^4$	$\mathbf{3}^4 97^4$	2	=	2				1			
6	$2^6 \mathbf{3}^6 11^6$	$2^6 \mathbf{3}^4 11^6$	6	=	2		3		1	3	1	
7	$\mathbf{3}^4 5^4 11^6$	$\mathbf{3}^2 5^4 11^6$	4	=	2	2			1	1		
8	$2^{16} 11^6$	$\mathbf{2}^{10} 11^6$	1						4	3	2	
9	$\mathbf{3}^6 113^4$	$\mathbf{3}^4 113^4$	2	=	2				1	3		
10	$2^8 \mathbf{3}^4 7^8$	$2^6 \mathbf{3}^2 7^8$	2	=	2				1	4		
11	$2^6 \mathbf{3}^6 43^4$	$2^6 \mathbf{3}^4 43^4$	2	=	2				1	3		
12	$2^6 \mathbf{5}^4 7^8$	$2^6 \mathbf{5}^2 7^8$	2	=	2				1			
13	$\mathbf{3}^4 239^4$	$\mathbf{3}^2 239^4$	4	=	2	2			1	1		
14	$2^6 \mathbf{3}^6 7^8$	$2^6 \mathbf{3}^4 7^8$	2	=	2				1			
15	$2^6 \mathbf{5}^6 23^4$	$2^6 \mathbf{5}^4 23^4$	8	=	4	2			1			
16	$2^8 \mathbf{5}^4 11^6$	$2^6 \mathbf{5}^2 11^6$	2	=	2				1			
17	$2^6 \mathbf{3}^8 29^4$	$2^6 \mathbf{3}^6 29^4$	4	=	2	2			1	1		
18	$2^8 \mathbf{11}^4 17^4$	$2^6 \mathbf{11}^2 17^4$	2	=	2				1			
19	$\mathbf{2}^{16} 7^8$	$\mathbf{2}^8 7^8$	2	=	2				1			
20	$2^{16} 7^8$	$\mathbf{2}^{10} 7^8$	1									
21	$2^{16} 7^8$	$\mathbf{2}^{10} 7^8$	2	=			2					1
22	$7^8 \mathbf{17}^4$	$7^8 \mathbf{17}^2$	2	=	2				1			
23	$\mathbf{2}^8 211^4$	$\mathbf{2}^4 211^4$	4	=	2	2			1	1		
24	$2^4 \mathbf{7}^4 61^4$	$2^4 \mathbf{7}^2 61^4$	4	=	2	2			1	1		
25	$2^6 \mathbf{317}^4$	$2^6 \mathbf{317}^2$	6	=	2		3		1		1	

TABLE 4. Discriminants of the octic $\mathrm{PSL}_2(7)$ fields of Theorem 1 and their septic siblings, the class number h of each octic, and ramification invariants t, u, v, and w of abelian extensions. Field numbers are the same as in Table 3. The boldface conventions and the factorization $h = a\ell c e$ are explained in the paragraph containing equation (3).

The first line of Table 4 indicates two degree 32 fields. The one highlighted in Corollary 2 has ramification invariants $(t, u) = (1, 1)$ and the other one has ramification invariants $(t, u) = (1, 7)$.

To get a third Galois group, note that the sixth field K_8 has class number divisible by 3, yielding an extension K_{24}/K_8. Let $G_{24} = \mathrm{Gal}(K_{24})$. The Galois group G_{24} is in fact $24T284$, which is $\mathrm{PSL}_2(7)$ itself, but now in its action on cosets of C_7. So $\mathrm{PSL}_2(7)$ octics are in bijection with $24T284$ fields via an elementary resolvent construction. In this case, we relate discriminants via $D_{24} = D_8^3 v^4$. Inspecting the twenty-five $24T284$ fields coming from the twenty-five octics says in particular that the above class field K_{24} in fact has the minimal discriminant:

Corollary 3. *There are exactly three number fields with Galois group $24T284$ and discriminant $\leq 30^{24}$. Base octics and ramification invariants are given in Table 4. The smallest discriminant of such a field is $(66^{3/4})^{22} \approx 23.16^{22}$ and the field has*

defining polynomial

$$x^{24} - 6x^{23} + 14x^{22} - 8x^{21} - 26x^{20} + 34x^{19} + 72x^{18} - 204x^{17} + 109x^{16}$$
$$+ 162x^{15} - 148x^{14} - 260x^{13} + 496x^{12} - 248x^{11} + 18x^{10} - 216x^{9}$$
$$+ 484x^{8} - 402x^{7} + 156x^{6} - 74x^{5} + 102x^{4} - 76x^{3} + 22x^{2} - 2x + 1.$$

Our discussion so far is related to three well-defined factors of class numbers of $\mathrm{PSL}_2(7)$ octics $K = K_8$. We call these the abelian, lifting, and cubic factors, and they give three columns in Table 4:

$$(3) \qquad a = [AK : K], \qquad \ell \in \{1, 2\}, \qquad c \in \{1, 3\}.$$

For the abelian factor, A is the largest cyclotomic field such that the composite field AK is unramified over K. In the table, the primes for which $A/\mathbb{Q}$ is ramified are put in bold in the D_8 column of Table 4. To identify the lifting factor, we use the septic local signs ϵ_v of §3.3.5. A prime p is put in bold in the D_7 column exactly when $\epsilon_p = -1$. If, for every such odd p, the inertia group I_p has order divisible by $2^{\mathrm{ord}_2(p-1)}$, and if also an analogous condition at 2 holds if 2 is in bold, then AK has an unramified quadratic extension $\widetilde{AK}$ with $\widetilde{AK}/A$ having Galois group $\mathrm{SL}_2(7)$. The lifting factor is $\ell = 2$ in this case and otherwise $\ell = 1$. The cubic factor is $c = 3$ if the canonical extension K_{24}/K_8 as above is unramified, and $c = 1$ if it is ramified.

In the general analysis of $\mathrm{PSL}_2(7)$ number fields K, let e denote the rest of the class number h, meaning $e = h/(a\ell c)$. In the table, $e > 1$ only once. In this case, computation shows that the Hilbert class field K_{16} has defining polynomial

$$(4) \quad x^{16} + 40x^{14} + 588x^{12} + 3808x^{10} + 12236x^{8} + 9856x^{6} + 3248x^{4} + 384x^{2} + 16.$$

Its Galois group $\mathrm{Gal}(K_{16}) = 16T1506$ has order $2^4 \cdot |\mathrm{PSL}_2(7)|$. However the resulting field, with discriminant 28^{16}, is not minimal, as it is undercut by four ramified extensions of the second octic in Table 4:

Corollary 4. *There are exactly five number fields of degree* 16 *with Galois group* $16T1506$ *and discriminant* $\leq 30^{16}$. *The smallest two discriminants* $2^{16}3^{8}53^{8} \approx 25.22^{16}$ *and* $2^{12}3^{12}53^{8} \approx 27.91^{16}$ *each arise from two fields. These fields respectively have have defining polynomials* $f(x^2)$, $f(-3x^2)$, $g(x^2)$ *and* $g(-3x^2)$ *where*

$$f(x) = x^8 + 8x^7 + 32x^6 + 44x^5 + 382x^4 + 496x^3 + 656x^2 - 20x + 1,$$
$$g(x) = x^8 + 21x^7 + 177x^6 + 585x^5 + 1071x^4 + 1215x^3 + 27x^2 - 1053x + 324.$$

In this case, we write the discriminant of the degree sixteen field as $D_8^2 w^4$, so that the invariants w are as in Table 4.

5. Results for $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$

5.1. Complete lists of $\mathrm{PSL}_2(5)$ **and** $\mathrm{PGL}_2(5)$ **sextics.** Our second illustration of the method of mixed degree searches comes from the sextic groups $\mathrm{PSL}_2(5) = 6T12$ and $\mathrm{PGL}_2(5) = 6T14$. The fields with smallest root discriminant were obtained by a direct sextic search in [FP92] and [FPDH98]. These root discriminants are $2^1 67^{1/3} \approx 8.12$ and $2^1 3^{1/3} 7^{1/2} \approx 11.01$.

In this second illustration, the smaller degree is 5, via the isomorphisms $A_5 \cong \mathrm{PSL}_2(5)$ and $S_5 \cong \mathrm{PGL}_2(5)$. Table 5 gives an analysis of tame ramification, with the bottom three lines being relevant to $S_5 \cong \mathrm{PGL}_2(5)$ only. As before, each partition

TABLE 5.   Tame ramification data for $A_5 \cong \mathrm{PSL}_2(5)$ and $S_5 \cong \mathrm{PGL}_2(5)$ in degrees 5 and 6.

λ_5	λ_6	c_5	c_6
5	5,1	4	4
3,1,1	3,3	2	4
2,2,1	2,2,1,1	2	2
4,1	4,1,1	3	3
3,2	6	3	5
2,1,1,1	2,2,2	1	3

λ_n determines the corresponding discriminant exponent c_n via formula (2). The behavior of wild ramification in this 5-to-6 context is analogous to the case of $\mathrm{GL}_3(2)$ and $\mathrm{PSL}_2(7)$ discussed earlier and so we omit the detailed analysis. From [JR14b], one knows that the bounds suggested by the tame table hold in general: $D_5 \le D_6 \le D_5^2$ for $A_5 \cong \mathrm{PSL}_2(5)$ and $|D_5| \le |D_6| \le |D_5|^3$ for $S_5 \cong \mathrm{PGL}_2(5)$.

As pointed out in [JR14b], the first bound $|D_5| \le |D_6|$ implies that a complete table of quintic fields up through discriminant bound B determines the corresponding complete table of sextic fields up through B. In contrast to the situation for $\mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$, this observation and existing tables of fields give non-empty complete lists of fields in the larger degree. In fact, taking $B = 12{,}000{,}000$ from our extension [JR14a] of [SPDyD94], one gets 78 $\mathrm{PSL}_2(5)$ sextics and 34 $\mathrm{PGL}_2(5)$ sextics with root discriminant at most $B^{1/6} \approx 15.13$.

In Table 5 there are three instances when $c_5 < c_6$. Accordingly, we can use targeting to substantially reduce the quintic search space for the sextic fields sought. The result for root discriminant $\delta_6 \le 35$ is as follows.

Theorem 2. *Among sextic fields with absolute discriminant $\le 35^6$, exactly 2361 have Galois group $\mathrm{PSL}_2(5)$ and 3454 have Galois group $\mathrm{PGL}_2(5)$.*

Lists of fields can be retrieved by searching the websites [JR14a, LMF18].

In parallel with the figure for our first case, Figure 2 illustrates our second case. The regularity near the bottom boundary $\delta_6 = \delta_5^{5/6}$ is easily explained, as follows. For any pair (K_5, K_6), the ratio D_6/D_5 is always a perfect square r^2. The pair gives rise to a point on the curve $\delta_6 = r^{1/3}\delta_5^{5/6}$. In the $S_5 \cong \mathrm{PGL}_2(5)$ case, the curves corresponding to $r = 1, 2, \ldots, 14, 15$ are all clearly visible. The first "missing curve," clearly visible as a gap, corresponds to $r = 16 = 2^4$. This curve is missing because none of the 2-adic possibilities for (c_5, c_6) satisfy $c_6 - c_5 = 8$. In the $A_5 \cong \mathrm{PSL}_2(5)$ case, there are fewer 2-adic possibilities and the first four visible gaps correspond to $r = 4, 8, 12,$ and 16.

5.2. Connections with expected mass formulas.

Let $NF_n(G, x)$ denote the set of isomorphism classes of degree n number fields K with $\mathrm{Gal}(K) = G \subseteq S_n$ and root discriminant at most x. Here G is well-defined, as a subgroup of S_n, up to conjugation. From the quintic search in [JR14a], one knows

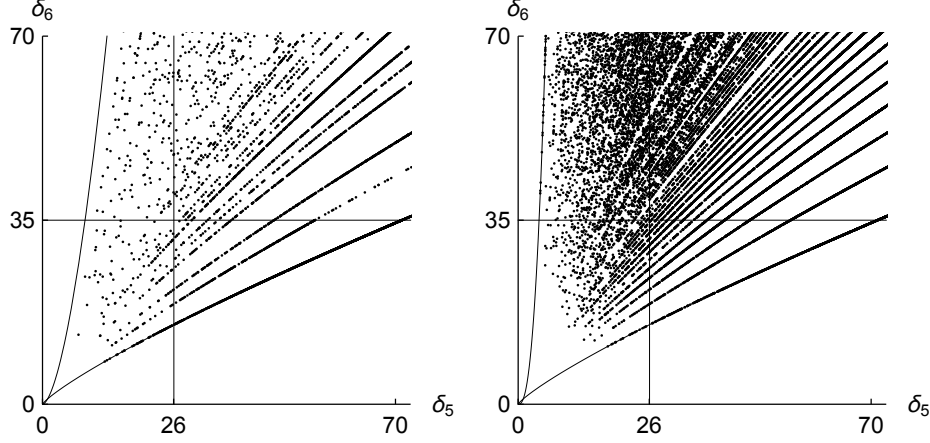$$(5) \qquad |NF_5(A_5, 26)| = 539, \qquad\qquad |NF_5(S_5, 26)| = 726862.$$

FIGURE 2.   Root discriminant pairs (δ_5, δ_6) associated to $A_5 \cong$ $\mathrm{PSL}_2(5)$ (left) and $S_5 \cong \mathrm{PGL}_2(5)$ (right), including all pairs in the window with $\delta_5 \leq 26$ from [JR14a, LMF18], and all pairs with $\delta_6 \leq 35$ from Theorem 2.

The ratio $539/726862 \approx 0.00074$ is an instance of the familiar informal principle "S_n fields are common but A_n fields are rare." In this light, the much larger ratio $2361/3454 \approx 0.68356$ from Theorem 2 is surprising.

However, the fact that $NF_6(\mathrm{PSL}_2(5), 35)$ and $NF_6(\mathrm{PGL}_2(5), 35)$ have such similar sizes can be explained as follows. For $g \in S_n$ with cycle type $n_1, n_2, \ldots, n_k$, let $\epsilon_g = \sum_{j=1}^{k}(n_j - 1)$. For a transitive permutation group $G \subseteq S_n$ and a conjugacy class $C \subseteq G$, define ϵ_C to be ϵ_g for any $g \in C$. Define a_G to be the reciprocal of the minimum of the ϵ_C over non-identity conjugacy classes C, and define b_G to be the number of classes obtaining this minimum. Then Malle conjectured an asymptotic growth rate

$$|NF_n(G, x)| \sim c_G x^{na_G} \log(x)^{b_G - 1},$$

for some constant c_G [Mal04]. Note that we are presenting Malle's conjecture in a renormalized form, since here x is a bound on root discriminant while in [Mal04] it is a bound on absolute discriminant. While Klüners [Klü05] has found a counterexample to the general statement of the conjecture, the problem is related to the presence of roots of unity in subfields, and Malle's conjecture is still expected to hold in the cases considered here.

For A_n, the two minimizing classes have cycle types $2^2, 1^{n-4}$ and $3, 1^{n-3}$, while for S_n the unique minimizing class is $2, 1^{n-2}$. Thus, consistent with numerical data like (5), one expects very different growth rates:

$$(6) \qquad |NF_n(A_n, x)| \sim c_{A_n} x^{n/2} \log x, \qquad\qquad |NF_n(S_n, x)| \sim c_{S_n} x^n.$$

For $n \leq 5$, this growth rate is proved for S_n with identified constants, and it is known that the growth for A_n is indeed slower; see [Bha10] for S_5 and [BCT15] for A_5.

But now, for $G \in \{\mathrm{PSL}_2(5), \mathrm{PGL}_2(5)\}$, the unique minimizing class is 2,2,1,1. Thus, consistent with Theorem 2, these two groups should have the same asymptotics up to a constant, both having the form $|NF_6(G, x)| \sim c_G x^3$. There are similar comparisons associated to our two other theorems. For the group of Theorem 1, one should have $|NF_8(\mathrm{PSL}_2(7), x)| \sim c_{\mathrm{PSL}_2(7)} x^2 \log x$ from 2,2,2,2 and 3,3,1,1. This growth rate is substantially less than the expected $|NF_8(\mathrm{PGL}_2(7), x)| \sim c_{\mathrm{PGL}_2(7)} x^{8/3}$ from 2,2,2,1,1. Indeed, there are twenty-six known $\mathrm{PGL}_2(7)$ octics with root discriminant ≤ 21 [JR14a, LMF18], and only one such $\mathrm{PSL}_2(7)$ octic from Theorem 1. For the groups of Theorem 3, $|NF_7(\mathrm{GL}_3(2), x)| \sim c_{\mathrm{GL}_3(2)} x^{7/2}$ should grow more slowly than $|NF_7(A_7, x)| \sim c_{A_7} x^{7/2} \log x$. The search underlying Theorem 3 is complete through discriminant 12^7. It can be expected to be near-complete for x substantially past 12^7, and these extra fields do indicate a general increase in $|NF_7(A_7, x)|/|NF_7(\mathrm{GL}_3(2), x)|$.

5.3. **Complete lists of Artin representations.** Theorems 1 and 2 can each be viewed from a different perspective. Computing octic $\mathrm{PSL}_2(7)$ fields is essentially the same as computing Artin representations for the irreducible degree 7 character of $\mathrm{GL}_3(2)$; computing sextic $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$ fields is equivalent to computing Artin representations for certain irreducible degree 5 characters of A_5 and S_5. In each case, discriminants match conductors.

In [JR17], we raised the problem of constructing lists of general Artin representations which are complete out through some conductor cutoff. For example, $\mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(5)$ each have two conjugate three-dimensional representations. These representations are particularly interesting because of their low degree, which means that the associated L-functions are relatively accessible to analytic computations. In Tables 8.3 and 8.1 of [JR17], we obtained the first pair of representations for $\mathrm{PSL}_2(7)$ and the first eighteen pairs for $\mathrm{PSL}_2(5)$. The method of targeting improves on our method in [JR17], and would allow one to substantially extend these lists.

6. RESULTS FOR $\mathrm{GL}_3(2)$, A_7, AND RELATED GROUPS

6.1. **Extending the known lists for $\mathrm{GL}_3(2)$ and A_7.** The first pair of fields for $\mathrm{GL}_3(2)$ and first field for A_7 were determined by Klüners and Malle in [KM01]. We extend the complete lists of such fields by employing the same technique as [KM01]: a standard Hunter search modified to select polynomials with Galois group contained in A_7. The latter condition comes into play as the first step when inspecting a polynomial in the search to see if it is suitable. Testing if the polynomial discriminant is a square can be done very quickly and filters out all the polynomials with odd Galois group.

Theorem 3. *Among septic fields with discriminant $\leq 12^7$, exactly 46 have Galois group $\mathrm{GL}_3(2)$ and 17 have Galois group A_7.*

Carrying out the computation up to discriminant 12^7 took six and a half months of CPU time and inspected roughly 10^{12} polynomials. As in other cases, defining polynomials and other information for these fields can be obtained from the websites [JR14a, LMF18].

The computation establishing Theorem 3 sheds light on the list of $\mathrm{PSL}_2(7)$ fields established by Theorem 1. The runtime of a Hunter search in degree n with discriminant bound B is proportional to $B^{(n+2)/4}$ [JR98]. Using this, our estimate

for the runtime for confirming the first $\mathrm{PSL}_2(7)$ octic field by computing septic fields without targeting is approximately $(21^8/12^7)^{9/4}6.5/12 \approx 3$ million CPU-years. To get Theorem 1's complete list through discriminant 30^8 would then take $(30^8/12^7)^{9/4}6.5/12 \approx 2$ billion CPU-years, as opposed to the four CPU-months with mixed degree targeted searching.

6.2. **The first $2^3\!:\mathrm{GL}_3(2)$ field.** As observed in [KM01], a sufficiently long complete list of septic $\mathrm{GL}_3(2)$ fields can be used to determine the first octic $2^3\!:\mathrm{GL}_3(2)$ field. The splitting field of a $2^3\!:\mathrm{GL}_3(2)$ polynomial contains (up to isomorphism), two subfields K_{7a} and K_{7b} of degree 7, two subfields K_{14a} and K_{14b} of degree 14 and Galois group $14T34$, and two subfields K_{8a} and K_{8b} of degree 8 and Galois group $8T48$. One of the septic fields is contained in both K_{14a} and K_{14b}, and the other is contained in neither.

Since the septic fields are arithmetically equivalent, they have the same discriminants, i.e., $D_{7a} = D_{7b}$. The other indices can be adjusted so that

(7) $$D_{7x}D_{8x} = D_{14x} \quad \text{for } x \in \{a, b\}.$$

This comes from a character relation on the relevant permutation characters. Because of the asymmetry in the field inclusions described above, the fact that $\mathrm{GL}_3(2)$ fields come in arithmetically equivalent pairs does not play a role in our computations, and so we have forty-six septic ground fields to consider separately. Accordingly, we drop x from the notation, always taking the correct octic resolvent so that (7) holds.

Of the forty-six septic $\mathrm{GL}_3(2)$ fields with discriminant $\leq 12^7$, only one has a non-trivial narrow class group. This field has narrow class number two, and is $K_7 = \mathbb{Q}[x]/f(x)$ with

(8) $$f(x) = x^7 - x^6 - x^5 - 2x^4 - 7x^3 - x^2 + 3x + 1.$$

The unramified quadratic extension turns out to be simply $K_{14} = \mathbb{Q}[x]/f(-x^2)$, which has Galois group $14T34$. So K_{14} and K_7 both have root discriminant $5717^{2/7} \approx 11.84$. By (7), the sibling K_8 of K_{14} has the even smaller root discriminant $5717^{1/4} \approx 8.70$.

In general, given all septic $\mathrm{GL}_3(2)$ fields up to some discriminant bound B, one can get all $14T34$ fields up to the discriminant bound B^2 via quadratic extensions. If $\mathfrak{d}$ is the relative discriminant of K_{14}/K_7, then $D_{14} = D_7^2 N_{K_7/\mathbb{Q}}(\mathfrak{d})$, and so $N_{K_7/\mathbb{Q}}(\mathfrak{d})$ must be at most B^2/D_7^2. In the same way, the stronger bound $N_{K_7/\mathbb{Q}}(\mathfrak{d}) \leq B/D_7$ is necessary and sufficient for the resolvent $8T48$ field to have discriminant $\leq B$. Taking $B = 12^7$ now, the quotient B/D_7 decreases from $12^7/(13^2109^2) \approx 17.85$ for the first two ground fields K_7 to $12^7/(2^6743^2) \approx 1.01$ for the last two ground fields. For the first twenty-six ground fields, computation shows that there are no $14T34$ overfields satisfying $N_{K_7/\mathbb{Q}}(\mathfrak{d}) \leq B^2/D_7^2$. For the last twenty fields, already $B/D_7 < \sqrt{2}$ and so the lack of overfields, except for the unramified one above, follows from the narrow class numbers being 1. Hence we have the following corollary of Theorem 3.

Corollary 5. *The field $\mathbb{Q}[x]/f(-x^2)$ from (8) is the only degree fourteen field with Galois group $14T34$ and discriminant $\leq 12^{14}$. Its sibling, with defining polynomial*

$$x^8 - 4x^7 + 8x^6 - 9x^5 + 7x^4 - 4x^3 + 2x^2 + 1,$$

is the only the octic field with Galois group $2^3\colon\mathrm{GL}_3(2) = 8T48$ and discriminant $\leq 12^7$. These two fields have discriminants $5717^4 \approx 11.84^{14}$ and $5717^2 \approx 8.70^8$ respectively.

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