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Behavior of Petrie Lines in Certain Edge-Transitive Graphs

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BEHAVIOR OF PETRIE LINES IN CERTAIN EDGE-TRANSITIVE GRAPHS

by

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Abstract

BEHAVIOR OF PETRIE LINES IN CERTAIN EDGE-TRANSITIVE GRAPHS

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We survey the construction and classification of one-, two- and infinitely-ended members of a class of highly symmetric, highly connected infinite graphs. In addition, we pose a conjecture concerning the relationship between the Petrie lines and ends of some infinitely-ended members of this class.

Chapter 1

Introduction

Interest in infinite graph theory can be traced back to the 1936 publication of Dénes König's *Theory of Finite and Infinite Graphs*([Kö90]), an event which is widely believed to mark the establishment of graph theory as a distinct discipline of mathematics. While many have concerned themselves with extending the formulations of finite graph theory to the infinite, it is often the case that infinite graphs are studied on their own. In 1964, Rudolf Halin defined an *end* of an infinite graph to be an equivalence class of rays under a choice equivalence relation ([Ha64]). We can describe an infinite graph by its number of ends. The simple subgraphs of a graph can also be divided into equivalence classes, to be defined fully in Chapter 3 of this thesis. In that chapter we also present Halin's discovery of a one-to-one correspondence between the ends of a graph and the equivalence classes of simple subgraphs ([Ha73]).

We then present results of Jack Graver and Mark Watkins, who set out to find all members of a class $\mathcal{G}$ of highly symmetric, highly connected graphs. Chapter 4 will reveal the comprehensive classification of graphs in $\mathcal{G}$ by the local action of their automorphism groups. In Chapters $6-8$ we detail methods of construction for the one-, two- and infinitely-ended members of this class. The method of construction for the one-ended graphs of $\mathcal{G}$ is attributed to Branko Grünbaum and G.C. Shephard in their 1973 publication [GS73]. The two-ended graphs, which take the form of quotient graphs of the integer lattice, were characterized by Watkins in 1991 ([Wa91]). Graver and Watkins developed a method called interleaving that produced the first known infinitely-ended members of $\mathcal{G}$, first published in 1997 in [GW97].

Graver and Watkins also define a Petrie walk, sometimes called a zig-zag walk. Petrie walks which are double rays are called *Petrie lines*. In Chapter 9 we present open problems and conjectures concerning the relationship between the Petrie lines and the ends of an infinitely-ended graph G in $\mathcal{G}$.

Chapter 2

Definitions and Preliminaries

A graph G is an ordered pair (V, E), where V is a set of *vertices* and E is a set of unordered pairs of vertices, called *edges*. If $e = \{u, v\} \in E$ we say that u and v are *adjacent*, that u and v are *neighbors*, or that u and v are *incident with* e. The edge set and vertex set of a graph G are denoted $E(G)$ and $V(G)$, respectively. The *degree* of a vertex v, denoted $\rho(v)$, is the number of neighbors of v. When $V(G)$ is infinite, it is possible for vertices to have infinite degree. We wish to exclude this possibility for the purposes of our discussion. A graph G is *locally finite* if $\rho(v) < \infty$ for all $v \in V(G)$.

A graph can be drawn by representing each vertex as a point in the plane and each edge as an arc joining incident vertices. A graph is *planar* if it can be drawn in such a way that the edge-arcs admit no intersections. Such a representation of a planar graph is called a *plane graph*.

We say that $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$. The *induced subgraph* of a set $U \subseteq V$ of G is the subgraph whose edges $\{u, v\}$ are all such pairs with $u, v \in U$. Let H and K be infinite subgraphs of a graph G. The subgraph H *terminates in* K if $H \setminus K$ is finite.

A *walk* is an ordered list of sequentially adjacent vertices. A subgraph H of a walk K is called a *subwalk* if H is also a walk. A *path* is a walk that has no repeated vertices. Often we denote a walk W with both its vertices and the edges that join them, as follows:

$$W = x_0, e_1, x_1, e_2, x_2, \ldots, e_{n-1}, x_{n-1}$$

A *circuit* is a walk $x_0, e_1, x_1, e_2, \ldots, e_n, x_n$ satisfying $x_0 = x_n$ and $\{x_0, x_1, \ldots, x_{n-1}\}$ are all distinct. The *length* of a path or walk is its number of edges. An infinite path U such that $\rho(v) = 2$ for all $v \in U$ is called a *double ray*. An infinite path $v_0, v_1, v_2, \ldots$ such that $\rho(v_0) = 1$ and $\rho(v_i) = 2$ when $i > 0$ is called a *ray*. If H is a ray or double ray and $K \subset H$ is a ray, then we say K is a *subray of H*.

The edges of a plane graph divide the plane into connected regions called *faces*. The set of all faces of a plane graph G will be denoted $F(G)$. Given a face $f \in F(G)$ the *covalence* of f is the length of the circuit enclosing f, denoted $\rho^*(f)$. If no such circuit exists then we say the covalence of f is infinite. If a vertex v or an edge e lies on the circuit or path around f then we say each of v and e are *incident with* f.

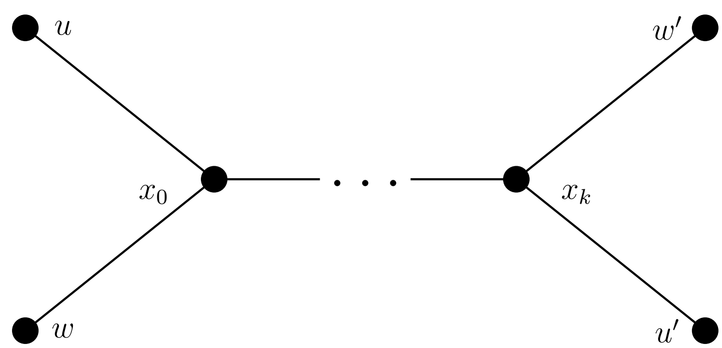

Figure 2.1: A crossing of paths U and V.

Given a plane graph G we can construct the *dual graph* G^*. Let $V(G^*) = F(G)$, and define

$$E(G^*) = \{\{f, g\} \mid f, g \in F(G) \text{ are incident with a common edge in } G\}$$

. If $G \cong G^*$, then G is said to be *self-dual*.

A *component* is a subgraph $K \subseteq G$ such that

(i) for all $u, v \in K$ there exists a path from u to v, and
(ii) whenever $w \notin K$ there is no path from u to w for all $u \in K$.

If a graph has only one component, then it is said to be *connected*.

Let $X = \{u, x_1, x_2, \ldots, x_n, v\}$ and $Y = \{u, y_1, y_2, \ldots, y_m, v\}$ be two paths joining vertices u and v. The paths X and Y are said to be *internally disjoint* if $V(X) \cap V(Y) \setminus \{u, v\} = \emptyset$. A graph G is *k-connected* if, for any pair of vertices $u, v \in V(G)$, there are k internally disjoint paths in G joining u and v. Let $H \subset V(G)$. If $G \setminus H$ has more than one component, then we call H a *vertex cut-set* of G. The *connectivity* $K(G)$ of a graph G is the size of the smallest vertex cut-set of G; the graph G is k-connected for any $k \leq K(G)$.

Let U and W be distinct paths in a plane graph G. Then $U \cap W$ will consist of one or more components, each of which will be paths. Let $V = \{x_0, x_1, \ldots, x_k\}$ be one such path. We will call V a *crossing* of U and W if

(i) neither x_0 nor x_k is 1-valent in U or W, and
(ii) in the clockwise labelings of the vertices around x_0 and x_k a vertex of V occurs after a vertex of U, but before a vertex of W (or the order is reversed in both labelings).

Figure 2.1 depicts a crossing.

An *automorphism* of a graph G is a permutation ϕ of $V(G)$ such that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(G)$. Up to isomorphism a graph is uniquely determined by the pair (V, E). The set of all automorphisms of a graph G is denoted $\mathrm{Aut}(G)$ and forms an algebraic group under function composition. We call $\mathrm{Aut}(G)$ the *automorphism group of* G. Let G be a graph and $\phi \in \mathrm{Aut}(G)$. Although ϕ acts on the vertices of G, ϕ induces a well-defined permutation group on each of $E(G)$ and $F(G)$. For this reason we denote by $\phi(e)$ and $\phi(f)$, the images of $e \in E$ and $f \in F$ under the induced permutation of E and F, respectively. A graph $G = (V, E)$ is said to be *edge-transitive* if, for every pair of edges e_1, $e_2 \in E(G)$, there exists an automorphism $\phi \in \mathrm{Aut}(G)$ such that $\phi(e_1) = e_2$. Vertex- and face-transitivity are defined similarly.

Let $G = (V, E)$ be a locally finite, edge-transitive, planar graph. A walk

$$\Pi = \ldots x_{i-2}, e_{i-1}, x_{i-1}, e_i, x_i, e_{i+1}, x_{i+1}, \ldots$$

in G with indices in $\mathbb{Z}$ or $\mathbb{Z}_m$ for some m is called a *Petrie walk* if the following three conditions are satisfied:

(i) Pairs of edges e_i and e_{i+1} are incident to a common face f_i for all i;
(ii) For each i the faces f_i and f_{i+1} are distinct;
(iii) No proper subwalk of Π satisfies conditions (i) and (ii).

Chapter 3

Simple Subgraphs and Ends

The study of infinite graphs allows us to examine subgraphs which are themselves infinite. There are often an uncountable number of these subgraphs, so it is useful to divide them into equivalence classes. In this chapter we define equivalence relations on two types of infinite subgraphs: rays and simple subgraphs. As discovered by Halin in his 1973 publication [Ha73], there is a fundamental correspondence between these two sets of equivalence classes.

Let G be an infinite graph and $U, V \subset G$ be rays. We define an equivalence relation $\sim_G$ on the rays of G by $U \sim_G V$ if there exists a ray $W \subset G$ such that both $U \cap W$ and $V \cap W$ are infinite. The equivalence classes under $\sim_G$ are called *ends*. A graph is k-ended if it has exactly k ends.

Example 1.

1. If a graph G is itself a ray then G is one-ended. If G is a double ray then G is two-ended.
2. If U and V are rays and $U \cap V$ is infinite, then $U \sim_G V$. Conversely, if U and V belong to different ends, then $U \cap V$ must be finite.

By definition alone it may appear that determining the equivalence of two rays requires the discovery of a third ray. However, another characterization of ends reveals that this is not the case. Instead we can consider the possible ways to "break up" an infinite graph into more than one component by the removal of a finite number of vertices, thus separating rays that belong to different ends.

Let G be a graph and $U, V \subset G$ be rays. Suppose there is some finite $T \subset G$ such that U and V terminate in different components of $G \setminus T$. Then we say T *separates* U and V and that T is an *ends-separating subgraph* of G. Furthermore, $U \sim_G V$ if and only if U and V terminate in the same component of $G \setminus T$ for every finite subgraph $T \subset G$. That is, it is impossible to separate U and V by removing a finite subgraph of G.

Example 2.

1. Let the *integer grid* be the graph G whose vertices are the points $(a, b) \in \mathbb{R}^2$ such that $a, b \in \mathbb{Z}$ and whose edges are all pairs of the form $\{(a, b), (a+1, b)\}$ or $\{(a, b), (a, b+1)\}$. For any finite subgraph $T \subset G$, the graph $G \setminus T$ will be connected. Therefore, G is one-ended.

2. An infinite k-valent tree T has an uncountable number of ends. Let v_0 be a vertex of T. Let U and V be distinct rays whose only common vertex is v_0. Then U and V belong to distinct components of $T \setminus v_0$ and thus different ends of T.

Now we define a simple subgraph, which is built upon the idea of breaking up a graph into components. An infinite subgraph H of a graph G is *simple* if H terminates in exactly one component of $G \setminus T$ for each finite $T \subset G$. For example, in any infinite graph a ray is a simple subgraph since the removal of a finite number of vertices will leave some subray intact. In general, simple subgraphs need not be connected; a simple subgraph can in fact be an infinite sequence of nonadjacent vertices.

If $G \setminus T$ is connected for any finite T, then G will be simple in itself. The converse of this statement is not necessarily true, however. If it is possible to divide G into more than one component, then G is simple when exactly one of those components is infinite. Furthermore, as proven by Halin in the following theorem, the removal of any finite subgraph of G must only ever result in a finite number of components.

Theorem 1 (Theorem 1 in [Ha73])**.** *G is simple in itself if and only if*

1. *G is one-ended and*
2. *for every finite $T \subset G$ the graph $G \setminus T$ has finitely many components.*

Proof. If G is simple, then it is at most one-ended; otherwise if G has more than one end then there exists some finite $T \subset G$ that separates one or more (infinite) rays. Hence if we can construct a ray in G then G will be one-ended.

Let v_0 be an arbitrary vertex of G. Then the graph $G \setminus v_0$ contains exactly one infinite component C_1 because G is simple. Since G is connected, we can find $v_1 \in C_1$ with $\{v_0, v_1\} \in E(G)$. Let C_2 be the infinite component of $C_1 \setminus v_1$, and pick v_2 such that $\{v_1, v_2\} \in E(G)$. Continuing in this fashion will produce a ray

$$v_0, v_1, v_2, \ldots .$$

Now suppose G is one-ended and that, for any finite $T \subset G$, the graph $G \setminus T$ has a finite number of components. Then G is infinite because it contains a ray. Let $\mathfrak{E}$ be the end of G. Choose an arbitrary finite subgraph T of G. Then there is exactly one component C of $G \setminus T$ in which all rays of $\mathfrak{E}$ terminate. Now we wish to show that $G \setminus C$ is finite. Toward contradiction suppose $G \setminus C$ is infinite. Then since there are only finitely many components of $G \setminus T$, there is an infinite component C' of $G \setminus (T \cup C)$. Let T' be a finite subgraph of C'. Then there are only finitely many components of $C' \setminus T'$. Otherwise $G \setminus (T \cup T')$ has infinitely many components. One of the components of $C' \setminus T'$ must be infinite and so by the method above we can find a ray in C', but then G has more than one end. This is a contradiction of our initial assumption that G be one-ended. Therefore, G must be simple. □

Given a simple subgraph H of a graph G we can easily find many more simple subgraphs of G by looking within H. Let T be a finite subgraph of G. By definition, all but finitely many vertices of H belong to exactly one component C of $G \setminus T$. If $H' \subset H$ is infinite, then only finitely many vertices of H' lie outside C, and thus H' also terminates in C. By this reasoning any infinite subgraph H' of a simple subgraph $H \subset G$ will also be simple in G.

Proposition 1 (Proposition 1 in [Ha73]). *If H is simple in G and $H' \subset H$ is infinite, then H' is simple in G.*

Let G be a graph and let H and H' be simple subgraphs of G. We can define an equivalence relation $\approx_G$ on the simple subgraphs of G by $H \approx_G H'$ if and only if $H \cup H'$ is also simple in G. That is, $H \approx_G H'$ if and only if both H and H' terminate in the same component of $G \setminus T$ for any finite $T \subset G$. In particular if H and H' are rays and $H \sim_G H'$, then $H \approx_G H'$.

As it is defined $\approx_G$ relies upon the consideration of $G \setminus T$ for every finite $T \subset G$. The following theorem offers an alternate characterization of this equivalence relation. Two simple subgraphs X and Y are equivalent under this relation if we can find an infinite set of disjoint paths, each of which joins a vertex of X with a vertex of Y.

Proposition 2 (Proposition 5 in [Ha73]). *Suppose $X = (x_i)_{i=1}^{\infty}$ and $Y = (y_i)_{i=1}^{\infty}$ are two simple, countable subgraphs of G. Then $X \approx_G Y$ if and only if there exist infinitely many disjoint paths $\{P_i\}_{i=1}^{\infty}$ in G such that each P_i connects an $x_j \in X$ with a $y_k \in Y$.*

Proof. Let X and Y be simple subgraphs of G such that $X \approx_G Y$ and suppose $\{P_i\}_{i=1}^{n}$ is a finite set of disjoint X, Y-paths in G. Then consider the subgraph $K = G \setminus (P_1 \cup P_2 \cup \ldots \cup P_n)$. Since $X \approx_G Y$, there exists a component C of K in which X and Y both terminate. Choose P_{n+1} in C. This path will be disjoint from each of $P_1, P_2, \ldots P_n$; inductively we can find infinitely many disjoint paths $\{P_i\}_{i=1}^{\infty}$.

Suppose on the other hand that infinitely many such paths exist, and let T be a finite subgraph of G. Then only finitely many paths P_i can have vertices in common with T, and only finitely many pairs $\{x_i, y_j\}$ can be separated by T. Thus there exist paths between infinitely many pairs $\{x_i, y_j\}$ in $G \setminus T$. It follows that X and Y terminate in the same component of $G \setminus T$. Therefore $X \approx_G Y$. □

The rays of a graph G are connected one-way infinite paths. By comparison many of the simple subgraphs of G may appear formless and scattered. In the following theorem Halin shows that while a simple subgraph may be disconnected, there is always a one-ended tree of which it is a subgraph.

Theorem 2 (Theorem 2 in [Ha73]). *Let $x_1, x_2, \ldots$ be an infinite sequence of distinct vertices of G. Then $X = \{x_1, x_2, \ldots\}$ is simple in G if and only if there exists a locally finite one-ended tree $H \subseteq G$ with $X \subseteq V(H)$.*

Proof. Suppose such a tree H exists. Then for each finite $T \subset G$ there is exactly one component of $G \setminus T$ in which H terminates, and so H is simple in G. Thus X is also simple in G.

Suppose X is simple in G. The desired tree will be constructed via a sequence of nonempty disjoint finite subtrees $H_1, H_2, \ldots$ of G and edges $e_1, e_2, \ldots \in E(G)$ satisfying the following:

(i) for all i, the edge e_i connects a vertex of H_i with a vertex of H_{i+1},

(ii) for every n, the vertices $\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} H_i$,

(iii) for every n, $G \setminus \left(\bigcup_{i=1}^{n} H_i \right)$ has exactly one component C_n with $C_n \cap X \neq \emptyset$. Hence since X is simple, $C_n \cap X$ must be infinite.

Suppose such a tree

$$H = \bigcup_{i=1}^{\infty} H_i \cup \{e_1, e_2, \ldots\}$$

was constructed. Then H is at least one-ended, because a ray exists which contains of all of the edges e_i. Since the H_i's were disjoint finite trees, any infinite connected subgraph of H must contain all of the edges e_i; hence any two distinct rays in H intersect at infinitely many vertices and thus lie in the same end.

To construct this tree consider the subgraph $G \setminus x_1$. Since X is simple there is exactly one component C_1 of $G \setminus x_1$ in which X terminates, and $V(X \setminus C_1)$ is finite. Since G is connected we can construct a finite tree H_1 in G containing x_1 and all vertices of $X \setminus C_1$, and there is an edge e_1 connecting x_1 with a vertex of C_1. Let C_2 be the component of $G \setminus (H_1 \cup \{x_2\})$ in which X terminates. The tree H_2 is built by connecting x_2 and all vertices of X which lie outside C_2. The edge e_2 will join x_2 and a vertex of C_2. (Note that if $x_2 \in H_1$, H_2 will be an empty tree.)

Assume $\{H_i\}_{i=1}^n$ have been constructed and satisfy (i), (ii) and (iii). Let i_n be the minimum integer i such that $x_i \notin \{H_i\}_{i=1}^n$. Then by condition (iii), x_{i_n} is in the infinite component C_n. Since G is connected, there exists an edge e_n between a vertex h of H_n and a vertex c of C_n. In C_n we can find a path P joining x_{i_n} and c. Since C_n is simple in G there is exactly one infinite component C_{n+1} of $C_n \setminus P$. Construct a finite tree H_{n+1} satisfying the following conditions:

(i) H_{n+1} lies in C_n,
(ii) $P \subseteq H_{n+1}$, and
(iii) H_{n+1} contains all vertices of X that lie inside $C_n \setminus C_{n+1}$.

Now let e_{n+1} be an edge connecting a vertex of P (and therefore H_{n+1}) with a vertex of C_{n+1}. □

The tree constructed thus establishes a relationship between the simple subgraphs of G and the rays of G. For every simple subgraph there is an associated ray, and every ray is itself a simple subgraph. Therefore we have a natural correspondence between the ends of a graph and the equivalence classes under $\approx_G$.

Proposition 3 (Proposition 4 in [Ha73])**.** *There is a one-to-one correspondence between ends of an infinite graph G and equivalence classes of simple subgraphs of G under* $\approx_G$.

This one-to-one correspondence motivates us to investigate the link between the ends of a graph G and subgraphs of G of another variety. In Chapter 5, we define a type of double ray called a Petrie line, and in Chapter 9 we present conjectures regarding the relationship between the ends and Petrie lines of highly connected, highly symmetric graphs.

Chapter 4

Classification of Locally Finite, Planar, Edge-Transitive Graphs

While the results in Chapter 3 apply to all infinite graphs, we from this point forward narrow our focus to a class of edge-transitive, highly connected graphs. In particular, the class $\mathcal{G}$ consists of all graphs that are locally finite, planar, 3-connected and edge-transitive. There are nine finite members of $\mathcal{G}$. As proven by Halin and Jung and reported in [GW97], the infinite members of $\mathcal{G}$ must possess exactly one, two or uncountable many ends. This provides a natural categorization of the class $\mathcal{G}$, which we use to organize our later chapters detailing the construction of many of these graphs.

In this chapter we are interested in a larger class $\mathcal{G}''$ that consists of all 2-connected, edge-transitive plane graphs that are not circuits of odd length. We expose a key result by Graver and Watkins, which provides a comprehensive classification of graphs in $\mathcal{G}''$ by the local action of their automorphism group. While representing the automorphism group of a graph $G \in \mathcal{G}''$ is potentially impossible, the symmetries of G can be catalogued by studying the local action of elements of $\mathrm{Aut}(G)$. We first consider the possible automorphisms that stabilize a given edge, vertex, face or Petrie walk in a graph $G \in \mathcal{G}''$. Then we present a theorem by Graver and Watkins which states that only fourteen combinations of these stabilizers are possible. Six of these classes are later determined to be empty.

To describe the stabilizers of the vertices, edges, and faces in a graph G, it is useful to define a triplet (v, e, f), called a *flag*, where $v \in V(G)$, $e \in E(G)$ and $f \in F(G)$ and all are mutually incident. We say G is flag-transitive if for each pair of flags (v, e, f) and (v', e', f') there exists a $\phi \in \mathrm{Aut}(G)$ that sends (v, e, f) to (v', e', f').

Lemma 1 (Lemma 3.1 in [GW97])**.** *Let $G \in \mathcal{G}''$. Then each automorphism of G is uniquely determined by its action on any flag of G.*

Proof. Suppose that $\phi, \psi \in \mathrm{Aut}(G)$ and that both ϕ and ψ map the flag (v, e, f) to the flag (v', e', f'). Let $\sigma = \phi \circ \psi^{-1}$ so that σ fixes (v, e, f). Then σ fixes the flag (v, e, f). Fixing the edge e, the vertex v and the face f also fixes the other edge d incident with v and f. Thus σ fixes the entire circuit that bounds f. Since G is a 2-connected plane graph, σ must also fix the bounding circuits of all faces that share a common edge with f. By this reasoning, all all vertices, edges and faces of the connected graph G must be fixed by σ. Thus σ is the identity automorphism, and $\phi = \psi$. □

For a graph $G \in \mathcal{G}''$, each edge is incident with exactly two faces and two vertices. Let ρ_0 and ρ_1 be the degrees of the vertices incident with e, and let ρ_0^* and ρ_1^* be the covalences of the faces incident with e. The *edge-symbol* of e is given by $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$. We say G has edge-symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$ if every edge of G has this edge-symbol, and in this

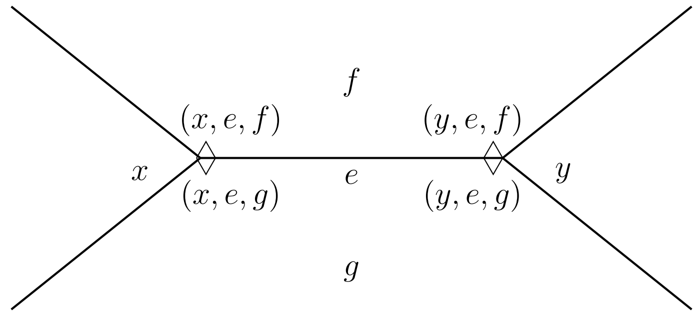

Figure 4.1: The four flags at the edge e.

case G is *edge-homogeneous*. As each graph in $\mathcal{G}''$ is edge-transitive, each is naturally edge-homogeneous.

An edge e of a graph $G \in \mathcal{G}''$ can be fixed by potentially four different automorphisms of G, which correspond to the four flags containing e (see Figure 4.1). Suppose e is incident with vertices x and y and faces f and g. Then e will be fixed by the following theoretical symmetries of G:

- ι, the identity map,
- λ_e, which maps (x, e, f) onto (y, e, f),
- τ_e, which maps (x, e, f) onto (x, e, g), and
- $\phi_e = \lambda_e \circ \tau_e$, which maps (x, e, f) onto (y, e, g).

While each of these four maps will theoretically fix the edge e, they may not all be present in the automorphism group of G. For example, if $\rho_0 \neq \rho_1$, then neither λ_e nor ϕ_e will be in $\mathrm{Aut}(G)$. The set of all automorphisms that fix e is a subgroup of $\mathrm{Aut}(G)$ called the *stabilizer of e*, denoted $\mathrm{stab}(e)$. A graph G is flag-transitive if and only if $\mathrm{stab}(e) = \langle \lambda_e, \phi_e \rangle$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

To fix a face $f \in F(G)$ via a possible symmetry of G, we may fix a vertex v incident with f while interchanging the two edges incident with both v and f. This map is denoted θ_{vf} and is a reflective symmetry across a line of symmetry of f. Let

$$v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_0$$

be the sequence of vertices and edges listed in cyclic order about f, where $n = \rho^*(f)$. Then f can also possibly be fixed by an automorphism σ_f, which maps the flag (x_i, e_i, f) to the flag (x_{i+1}, e_{i+1}, f) for all i. The map σ_f corresponds to a rotation of G about the center of f. The largest possible stabilizer of f, denoted $\mathrm{stab}(f)$, is $\langle \sigma_f, \theta_{vf} \rangle$, which is isomorphic to D_{2n}, the dihedral group of order $2n$. In some graphs, $\mathrm{stab}(f)$ will be isomorphic to a proper subgroup of D_{2n}, like in the case when $\rho_0 \neq \rho_1$. In this case σ_f cannot be an automorphism of G. However, σ_f^2 may be in $\mathrm{Aut}(G)$. If $\theta_{vf} \in \mathrm{Aut}(G)$ for every incident vertex-face pair (v, f), then G is called *ordinary*.

By a dual argument, the largest possible stabilizer of a vertex v, denoted $\mathrm{stab}(v)$, is isomorphic to the dihedral group of order $2m$, where $m = \rho(v)$. The map θ_{vf} will fix v for all faces f incident with v. If $e_0, e_1, \ldots, e_{m-1}$ is the list of edges incident with v, listed in cyclic order, then we can fix v by mapping e_i to e_{i+1} (modulo m) for each i via a rotation. We will denote this possible symmetry σ_v. If $\sigma_v \in \mathrm{Aut}(G)$, then $\sigma_v^m = \iota$.

Let G be a graph. A walk

$$\Pi = \ldots x_{i-2}, e_{i-1}, x_{i-1}, e_i, x_i, e_{i+1}, x_{i+1}, \ldots$$

in G with indices in $\mathbb{Z}$ or $\mathbb{Z}_m$ for some m is called a *Petrie walk* if the following three conditions are satisfied:

(i) Pairs of edges e_i and e_{i+1} are incident to a common face f_i for all i;
(ii) For each i the faces f_i and f_{i+1} are distinct.
(iii) No proper subwalk of Π satisfies conditions (i) and (ii).

Let Π be a Petrie walk of a graph $G \in \mathcal{G}''$, and write

$$\Pi = x_{-1}, e_0, x_0, e_1, \ldots, x_{i-1}, e_i, x_i, e_{i+1}, \ldots$$

Let $\gamma_{e_ie_{i+1}}$ be the map that sends (x_i, e_i, f_i) to $(x_{i+2}, e_{i+1}, f_{i+1})$, as in Figure 4.2.

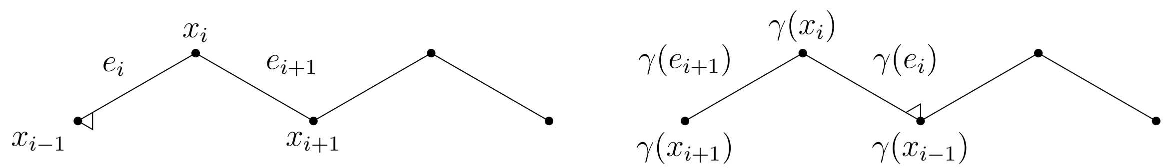

Figure 4.2: The action of $\gamma_{e_ie_{i+1}}$ on a short segment of a Petrie walk. For the purpose of brevity in the labeling, $\gamma_{e_ie_{i+1}}$ is simply labelled as γ.

If $\gamma_{e_ie_{i+1}} \in \mathrm{Aut}(G)$, then this automorphism fixes Π but does not fix any edges of Π. Furthermore, the direction of increasing indices of Π and $\gamma_{e_ie_{i+1}}(\Pi)$ will be oppositely oriented. Therefore, the map $\gamma_{e_ie_{i+1}}$ acts as a reflection composed with a translation along the Petrie walk Π. Given an edge $e_i \in \Pi$, the automorphism ϕ_{e_i} fixes e_i and reverses the direction of increasing indices along Π, acting as a rotation composed with a reflection. If (v, f) is the vertex-face pair incident to both e_i and e_{i+1} for some i, then θ_{vf} will also fix Π. The following identities reveal the relationship between these maps:

$$\theta_{vf} \circ \gamma_{e_ie_{i+1}} \circ \theta_{vf} = \phi_{e_i} \circ \gamma_{e_ie_{i+1}} \circ \phi_{e_i} = \gamma_{e_ie_{i+1}}^{-1}$$

This is the group presentation of a dihedral group whose order is twice that of $\gamma_{e_ie_{i+1}}$. We can write this group as either $\langle \theta_{vf}, \gamma_{e_ie_{i+1}} \rangle$ or $\langle \phi_{e_i}, \gamma_{e_ie_{i+1}} \rangle$.

The stabilizer of Π, denoted $\mathrm{stab}(\Pi)$, is a subgroup of $\langle \theta_{vf}, \gamma_{e_ie_{i+1}} \rangle$. By construction a given edge of G lies on at most two Petrie lines. Since G has exactly one edge orbit, there are at most two orbits of Petrie walks under the action of Aut G. These are called the *Petrie-orbits* of G.

The largest possible stabilizers of each of the edges, vertices, faces and Petrie walks are shown in Table 4.1.

Graph element	Possible stabilizer
edge	$\langle \tau_e, \lambda_e \rangle$
vertex	$\langle \theta_{vf}, \sigma_x \rangle$
face	$\langle \theta_{vf}, \sigma_f \rangle$
Petrie walk	$\langle \theta_{vf}, \gamma_{e_i e_{i+1}} \rangle \cong \langle \phi_{e_i}, \gamma_{e_i e_{i+1}} \rangle$

Table 4.1: The largest possible stabilizers of a given edge, vertex, face and Petrie walk of a graph in $\mathcal{G}''$

Given a graph $G \in \mathcal{G}''$, the presence of certain stabilizers in $\mathrm{Aut}(G)$ can determine the transitivity of the vertices, faces or Petrie walks of G. In this way, the local action of $\mathrm{Aut}(G)$ sheds light on its global action. The particular automorphisms and the resulting transitivity are presented in the following two lemmas.

Lemma 2 (Lemma 3.3 in [GW97]). *Let $G \in \mathcal{G}''$ and let (v, e, f) and (v, e', f) denote the two flags that are on both vertex v and face f. If $\sigma_v \in \mathrm{Aut}(G)$, then G is face-transitive; if $\sigma_f \in \mathrm{Aut}(G)$, then G is vertex-transitive; if $\gamma_{ee'} \in \mathrm{Aut}(G)$, then G is both vertex- and face-transitive.*

A graph can also be Petrie-transitive. Let Π be a Petrie walk of a graph $G \in \mathcal{G}''$. Then the *orbit of* Π, also called a *Petrie-orbit*, is the set of all Petrie walks $\Omega \subset G$ such that there exists some $\phi \in \mathrm{Aut}(G)$ for which $\phi(\Pi) = \Omega$. Since G is edge-transitive, there is exactly one edge orbit. Because of the way Petrie walks are constructed, each edge of G lies on exactly two Petrie walks. Thus, there can be at most two Petrie-orbits of G. If G only admits one Petrie-orbit, then G is *Petrie-transitive*.

Lemma 3 (Lemma 3.4 in [GW97]). *If $G \in \mathcal{G}''$, then $\mathrm{Aut}(G)$ acts on the set of Petrie walks, and under this action there are at most two Petrie-orbits. Furthermore, G is Petrie-transitive whenever $\mathrm{Aut}(G)$ includes an automorphism of the form $\sigma_v, \sigma_f, \lambda_e$ or τ_e.*

Graver and Watkins later prove the converse to the last statement of the above lemma. In addition, they develop three more lemmas that indicate the presence of certain automorphisms in $\mathrm{Aut}(G)$ if a graph $G \in \mathcal{G}''$ is vertex-, face- or Petrie-transitive (Lemma A1 in [GW97]). This allows them to add particular automorphisms and rule out others when finding the possible combinations of maps in $\mathrm{Aut}(G)$.

Given a graph $G \in \mathcal{G}''$ we can find the stabilizer of each of the four graph elements in Table 4.1. As listed by Graver and Watkins in [GW97], there are 14 possible combinations of these stabilizers. In Table 4.2 we present a summarized version of the complete classification of members of $\mathcal{G}''$ by the local action of their automorphism group, including some necessary restrictions on their edge-symbols.

Theorem 3 (Theorem 3.5 in [GW97]). *Let $G \in \mathcal{G}''$ have edge-symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$. Then at most 14 combinations of the edge-, vertex-, face- and Petrie-stabilizers in $\mathrm{Aut}(G)$ are possible. These combinations, including edge-symbol rules, are tabulated below.*

The authors describe their proof of this theorem as very technical in nature, as it exhausts all four cases of $\mathrm{stab}(e)$ for $e \in E(G)$, which corresponds to the four possible

Type	$\mathrm{stab}(e)$	$\mathrm{stab}(v)$	$\mathrm{stab}(f)$	$\mathrm{stab}(\Pi)$	Edge-symbol rules
1	$\langle\tau_e,\lambda_e\rangle$	$\langle\sigma_x,\theta_{vf}\rangle$	$\langle\sigma_f,\theta_{vf}\rangle$	$\langle\gamma_\Pi,\theta_{vf}\rangle$	$\rho_0=\rho_1$ and $\rho_0^*=\rho_1^*$
2	$\langle\tau_e\rangle$	$\langle\sigma_x,\theta_{vf}\rangle$	$\left\langle\sigma_f^2,\theta_{vf}\right\rangle$	$\langle\gamma_\Pi^2,\theta_{vf}\rangle$	$\rho_0\neq\rho_1$ and $\rho_0^*=\rho_1^*$ are even
2ex	$\langle\tau_e\rangle$	$\langle\sigma_x^2,\tau_e\rangle$	$\langle\sigma_f\rangle$	$\langle\gamma_\Pi\rangle$	$\rho_0=\rho_1$ are even and $\rho_0^*=\rho_1^*$
2^*	$\langle\lambda_e\rangle$	$\langle\sigma_x^2,\tau_e\rangle$	$\langle\sigma_f,\theta_{vf}\rangle$	$\langle\gamma_\Pi^2,\theta_{vf}\rangle$	$\rho_0=\rho_1$ are even
2^*ex	$\langle\lambda_e\rangle$	$\langle\sigma_x\rangle$	$\left\langle\sigma_f^2,\lambda_e\right\rangle$	$\langle\gamma_\Pi\rangle$	$\rho_0=\rho_1$ and $\rho_0^*=\rho_1^*$ are even
2^P	$\langle\phi_e\rangle$	$\langle\sigma_x^2,\tau_e\rangle$	$\left\langle\sigma_f^2,\theta_{vf}\right\rangle$	$\langle\gamma_\Pi,\theta_{vf}\rangle$	$\rho_0=\rho_1$ are even and $\rho_0^*=\rho_1^*$ are even
2^Pex	$\langle\phi_e\rangle$	$\langle\sigma_x\rangle$	$\langle\sigma_f\rangle$	$\langle\gamma_\Pi^2,\phi_e\rangle$	$\rho_0=\rho_1$ and $\rho_0^*=\rho_1^*$
3	$\langle\iota\rangle$	$\langle\sigma_x^2,\tau_e\rangle$	$\left\langle\sigma_f^2,\theta_{vf}\right\rangle$	$\langle\gamma_\Pi^2,\theta_{vf}\rangle$	ρ_0,ρ_1,ρ_0^* and ρ_1^* are all even
4	$\langle\iota\rangle$	$\langle\sigma_x\rangle,\langle\sigma_x^2,\tau_e\rangle$	$\left\langle\sigma_f^4,\theta_{vf}\right\rangle$	$\langle\gamma_\Pi^4,\theta_{vf}\rangle$	At least one of ρ_0,ρ_1 is even, and $\rho_0^*=\rho_1^*\equiv 0 \mod 4$
5	$\langle\iota\rangle$	$\langle\sigma_x\rangle$	$\left\langle\sigma_f^2\right\rangle$	$\langle\gamma_\Pi^2\rangle$	$\rho_0^*=\rho_1^*$ are even
4^*	$\langle\iota\rangle$	$\langle\sigma_x^4,\theta_{vf}\rangle$	$\langle\sigma_f\rangle,\left\langle\sigma_f^2,\theta_{vf}\right\rangle$	$\langle\gamma_\Pi^4,\theta_{vf}\rangle$	$\rho_0=\rho_1\equiv 0 \mod 4$ and at least one of ρ_0^*,ρ_1^* is even
5^*	$\langle\iota\rangle$	$\langle\sigma_x^2\rangle$	$\langle\sigma_f\rangle$	$\langle\gamma_\Pi^2\rangle$	$\rho_0=\rho_1$ are even
4^P	$\langle\iota\rangle$	$\langle\sigma_x^4,\theta_{vf}\rangle$	$\left\langle\sigma_f^4,\theta_{vf}\right\rangle$	$\langle\gamma_\Pi\rangle,\langle\gamma_\Pi^2,\theta_{vf}\rangle$	$\rho_0=\rho_1$, $\rho_0^*=\rho_1^*$ are all even
5^P	$\langle\iota\rangle$	$\langle\sigma_x^2\rangle$	$\left\langle\sigma_f^2\right\rangle$	$\langle\gamma_\Pi\rangle$	$\rho_0=\rho_1$, $\rho_0^*=\rho_1^*$ are all even

Table 4.2: The 14 possible types of graph in the class $\mathcal{G}''$

subgroups of $\mathbb{Z}_2\times\mathbb{Z}_2$. Each of these cases are then divided into subcases, depending on the transitivity of the vertices, faces and Petrie walks of G.

This complete categorization of the members of $\mathcal{G}''$ is often used by Graver and Watkins to investigate other properties of these graphs, such as their Petrie type (defined in Chapter 5). In some proofs, they determine the interactions among a subgraph H of a graph G and the automorphic images of H. In these instances, Theorem 3 provides the identity of particular automorphisms that act on H.

In Chapter 6 of [GW97] the authors reveal a relationship between the number of ends of a graph and its type under Theorem 3. In doing so, they found that five of the fourteen classes are empty, as no graph has automorphisms with the desired combined structure. As will be seen in Chapter 6, the type of a zero- or one-ended graph can be determined by its edge-symbol. The two-ended graphs in $\mathcal{G}''$ all possess the same edge-symbol and have essentially the same form; therefore they are of exactly one type (see Chapter 7). The conclusions about the infinitely-ended graphs follow conversely from results in Chapter 6 about one-ended graphs.

The class $\mathcal{G}'$ consists of the members of $\mathcal{G}''$ with no 2-valent vertices. The following theorem sorts the members of $\mathcal{G}'$ by their number of ends and then further by their type. To extend the results to $\mathcal{G}''$ the authors examine members of $\mathcal{G}''\setminus\mathcal{G}'$. A graph $G\in\mathcal{G}''\setminus\mathcal{G}'$ must have edge-symbol $\langle 2,\rho;\rho_0^*,\rho_1^*\rangle$. Furthermore, G must belong to one of five subclasses depending on the particular edge-symbol of G and other factors, such as the existence of a unique graph in $\mathcal{G}'$ from which G can be constructed by the subdivision of edges. The combination of these factors determines the type of G, and thus we can find the type of any graph in $\mathcal{G}''$.

Theorem 4 (Theorem 6.5 in [GW97])**.** *Let $G \in \mathcal{G}'$. If G is finite or 1-ended, then Aut G is of type 1, 2, 2^*, or 3. If G is 2-ended, then Aut G is of type 2^P. If G is infinitely-ended, then Aut G is of type 2^P, 3, 4, 4^*, 4^P, or 5^P.*

Corollary 1 (Corollary 6.7 in [GW97])**.** *If $G \in \mathcal{G}''$, then Aut G is of type 1, 2, 2^*, 2^P, 3, 4, 4^*, 4^P or 5^P.*

In Chapter 11 of [GW97], the authors prove that only eight of the 14 possible types are realizable as graphs in the class $\mathcal{G}$. In particular, there does not exist a graph in this class of type 5^P. These eight types provide a manageable list of possibilities for the symmetric properties of these infinite graphs.

Chapter 5

Petrie Walks

Much like rays and simple subgraphs, Petrie walks are another type of subgraph of an infinite graph G that indicate the properties and structure of G. Since there are an infinite number of Petrie walks, we narrow our focus to the crossings of Petrie walks in local sections of G. In order to discuss the crossings of Petrie walks, it is useful to create an indexing which tracks the vertices, edges and faces incident to a Petrie line. Let Π be a Petrie walk. We can write

$$\Pi = \ldots x_{i-2}, e_{i-1}, x_{i-1}, e_i, x_i, e_{i+1}, x_{i+1}, \ldots$$

For each integer j let (v_j, f_j) be the vertex-face pair incident with edges e_j and e_{j+1}. When traversing Π in the direction of increasing indices, if f_j lies on the righthand side of e_j whenever j is even, then the indexing $[v, e, f]$ is called a *righthand indexing* of Π. The *lefthand indexing* is the indexing that occurs if f_j is on the lefthand side of e_j as one moves in the the direction of increasing indices along Π. Given a particular indexing of Π, we will call the set of all edges with even indices the *even edges* of Π, denoted $E_0(\Pi)$. The *odd edges*, or the edges with odd indices, will be denoted $E_1(\Pi)$.

This indexing translates over easily to the dual of a graph. If $G \in \mathcal{G}'$ and $\Pi \subset G$ is a Petrie walk, then we can construct the *dual Petrie walk* $\Pi^* \subseteq G^*$ by interchanging the symbols x_i and f_i in the righthand (or lefthand) indexing of Π. The vertices of this path will now be labeled with $\{f_i\}$. The edges e_i and the faces x_i will now satisfy the conditions necessary for Π^* to be a Petrie walk.

If one begins at an arbitrary edge e and then constructs all Petrie walks containing e by adding the subsequent edge around either of the two faces incident to e, it is apparent that

1. every edge belongs to at most two Petrie walks, and
2. Petrie walks are allowed to cross themselves and/or one another.

In the following lemma, Graver and Watkins prove that there are rules that determine how pairs Petrie walks can cross one another and how individual Petrie walks can be indexed. By assigning indices to pairs of Petrie walks and ensuring that the conditions in the definition of Petrie walk are satisfied, they find that crossings of Petrie walks consist of exactly one edge.

Lemma 4 (Lemma 4.1 in [GW97])**.** *Let $G \in \mathcal{G}'$, and let Π and Ω be distinct Petrie walks in G.*

(a) The indexing $[v, e, f]$ is a righthand indexing of Π if and only if $[f, e, v]$ is a lefthand indexing of Π^*.
(b) The edge sets $E_0(\Pi) \cap E_0(\Omega) = \emptyset$ and $E_1(\Pi) \cap E_1(\Omega) = \emptyset$.
(c) Every edge common to Π and Ω belongs to a crossing of Π and Ω, and every crossing of Π and Ω has exactly one edge; its indices have opposite parities on Π and Ω.
(d) The edge sets $E_0(\Pi) \cap E_1(\Pi) = \emptyset$, i.e. each edge of Π is assigned exactly one index.

Proof. (a) Let $[v, e, f]$ be a righthand indexing of Π. Then the face f_0 is to the right of e_0 when moving in the direction of increasing indices along Π, and f_0 is incident to both e_0 and e_1. Now in Π^*, e_0 and e_1 are both incident to the vertex labeled f_0, which must be to the right of the face now labeled v_0. Therefore, the face v_0 is on the left side of both e_0 and e_1 when oriented in the direction of increasing vertices. Therefore, $[f, e, v]$ is a lefthand indexing of Π^*.

(b) Suppose $e_i \in \Pi$ and $d_j \in \Omega$ with $e_i = d_j \in G$ and $i \equiv j \mod 2$. Then the faces f_i and g_j are to the right of edges e_i and d_j in the direction of increasing indices. If $f_i = g_j$, then $e_{i+1} = d_{j+1}$, and $\Pi = \Omega$. Suppose $f_i \neq g_j$. Then either $f_i = g_{j-1}$ or $f_i = g_{j+1}$, and thus either $e_{i-1} = d_{j+1}$ or $e_{i+1} = d_{j-1}$, which implies that $\Pi = \Omega$. Since Π and Ω are distinct, $E_0(\Pi) \cap E_0(\Omega) = \emptyset$. An identical argument shows that $E_1(\Pi) \cap E_1(\Omega) = \emptyset$.

(c) Now again suppose $e_i = d_j$ for some $e_i \in \Pi, d_j \in \Omega$. Then by (b), i and j must have opposite parities, and the edges $e_{i-1}, e_{i+1}, d_{j-1}, d_{j+1}$ must all be distinct. Thus $e_i = d_j$ is a crossing of Π and Ω. This shows that every crossing of Π and Ω consists of exactly one edge.

(d) Suppose $e_0 = e_k \in \Pi$, and k is odd. Assume that $e_0, e_1, \ldots, e_{k-1}$ are all distinct. Then since $e_1 \neq e_{k-1}$, it follows that Π is the only Petrie walk through e_0. Since G is connected and planar, for any $d \in E(G)$ it is possible to find a path $e_0 = d_0, \ldots, d_n = d$ such that every pair of consecutive edges $\{d_i, d_{i+1}\}$ is incident to a common face. Note that this is similar to the condition necessary for a path to be a Petrie walk, but this particular $[d_0, d_n]$ path may contain three consecutive edges which are incident to a common face. We can say that each edge pair $\{d_i, d_{i+1}\}$ lies on some Petrie walk. In particular, the edges d_0, d_1 lie on exactly one Petrie walk, namely Π (the unique Petrie walk containing d_0). By edge transitivity, Π must be the only Petrie walk containing d_1, and so by induction, $d \in E(G) = E(\Pi)$. Therefore, each edge of G has two Π-indices of opposite parity, and $E_0(\Pi) = E_1(\Pi)$.

It also follows from edge transitivity that the two labels of a given edge of G are of the form e_j for some j and either e_{j-k} or e_{j+k}. Given a pair of edges e_i and e_{i+1} it cannot be the case that the alternative labels for these edges are of the form e_j and e_{j+1} respectively, for then i and j would have the same parity (by the definition of righthand indexing). Since $e_0 = e_k$, it follows that $e_j = e_{j+k}$ whenever j is even. Then when j is odd, $e_j = e_{j-k}$.

Thus the four edges $e_1 = e_{1-k}$, $e_{-1} = e_{-1-k}$, $e_{k+1} = e_{2k+1}$, $e_{k-1} = e_{2k-1}$ are all distinct, and so the single edge $e_0 = e_k$ is a crossing of Π with itself. Since G is edge-transitive, then every edge of G must be a crossing of Π with itself, and this crossing must be traversed exactly once by Π in either direction.

See Figure 5.1, and consider the edges $e_{-1} = e_{-k-1}$ and $e_1 = e_{-k+1}$. The edge $e_0 = e_k$ must occur after e_{-1} and before e_1 in the listed sequence of Petrie edges. We can label the vertices as in the picture so that this portion of Π can be listed as

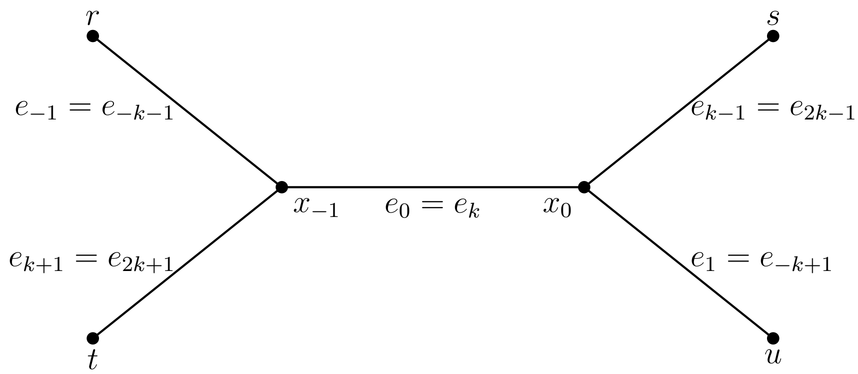

Figure 5.1: The crossing of Π with itself at $e_0 = e_k$.

$$\Pi = \ldots r, e_{-1}, x_{-1}, e_0, x_0, e_1, u, \ldots$$

Now let us consider the indices of the Petrie walk near the edges labeled e_{-k-1} and e_{-k+1}. Then the edge e_{-k} must be traversed after e_{-k-1} and before e_{-k+1}. However, now the direction of traversal must be opposite. Now this segment of Π is of the form

$$\Pi = \ldots x_{-1}, e_{-1-k}, r, e_{-k}, u, e_{1-k}, x_0, \ldots$$

In particular, the edge e_{-k} is incident with vertices r and u. Similarly, it can be shown that the edge e_{2k} must be incident to vertices s and t. This is impossible in any planar embedding of G. Since we have arrived at a contradiction, it must be the case that $E_0(\Pi) \cap E_1(\Pi) = \emptyset$, and every edge of Π receives exactly one label under its righthand indexing. $\square$

There is an important consequence of these restrictions on Petrie walk crossings and the uniqueness of indices along a given Petrie walk Π. Using the classification from Theorem 3, Graver and Watkins examined the crossings of Π with its automorphic images. They found that a Petrie walk in a graph $G \in \mathcal{G}''$ can assume one of two forms: a double ray or a circuit of even length (Theorem 4.2 in [GW97]). The double-ray variety of Petrie walks are called *Petrie lines*. Petrie walks that are circuits are called *Petrie circuits*.

In a given graph G if both Petrie-orbits consist of Petrie lines, G is said to be of *line type.* An example of a graph that is of line type is the one-ended edge-homogeneous infinite graph with edge-symbol $\langle 3, 3; 6, 6 \rangle$, depicted in Figure 5.2. If both orbits are Petrie circuits, G is of *circuit type*, as shown in Figure 5.3. If G contains both Petrie lines and Petrie circuits, G is of *mixed type*. An entire subclass of graphs in $\mathcal{G}$ of mixed type will be presented in Chapters 7 and 8.

The Petrie walk crossings of a graph $G \in \mathcal{G}'$ can reveal the number of Petrie-orbits in G. There are two configurations that indicate G is Petrie-transitive. The first is if G admits pairwise crossings of three distinct Petrie walks. Then second is if G contains a pair of Petrie walks that cross one another more than once. The following theorem and proof detail these results.

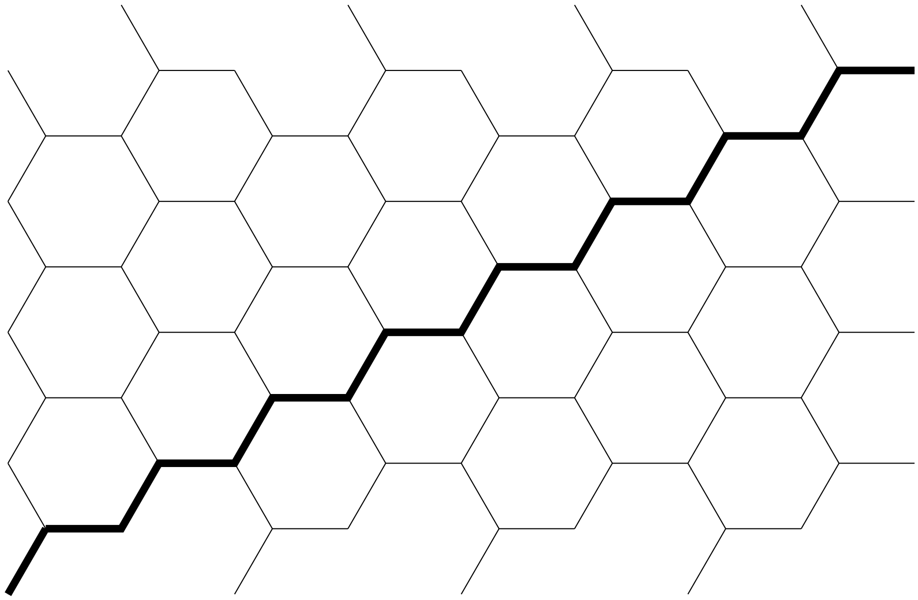

Figure 5.2: A local section of an edge-homogeneous graph of line type with edge-symbol $\langle 3, 3; 6, 6 \rangle$.

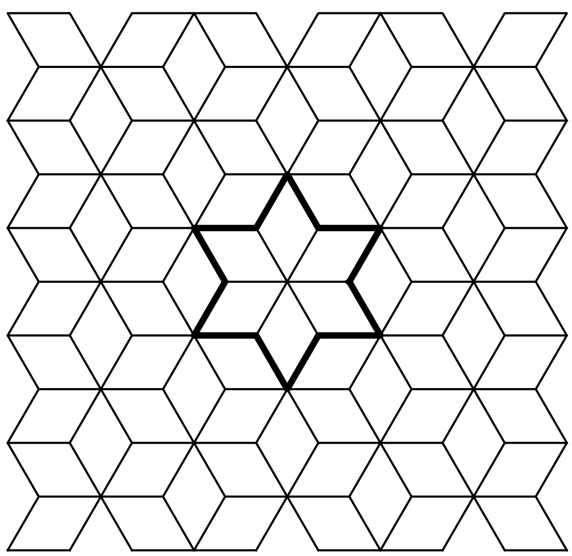

Figure 5.3: Part of an edge-homogeneous graph of circuit type with edge-symbol $\langle 3, 6; 4, 4 \rangle$.

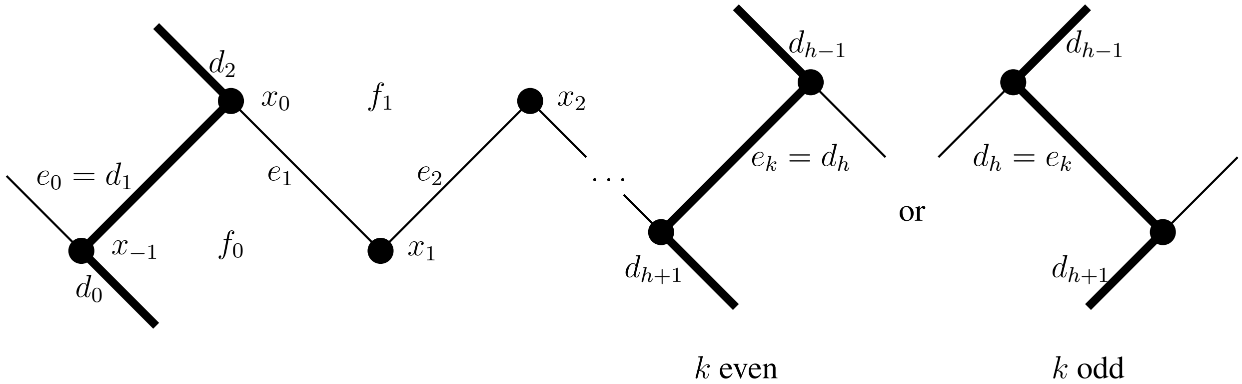

Figure 5.4: The eight possible configurations of the pair of crossings of Π and Ω.

Theorem 5 (Theorem 4.3 in [GW97]). *Let $G \in \mathcal{G}'$. If G contains either three distinct Petrie walks, every two of which have a common edge, or two distinct Petrie walks that have more than one common edge, then G is Petrie-transitive.*

Proof. First, note that an edge e belongs to at most two Petrie walks. In the case that e belongs to two distinct Petrie walks, either both Petrie walks are in the same orbit, or they are in different orbits. Hence, by edge-transitivity there are at most two distinct Petrie orbits in G. Now suppose toward contradiction that G contains three distinct Petrie walks, every two of which have a common edge. If G were not Petrie-transitive, then each of these three Petrie walks would belong to a distinct Petrie orbit, which is impossible. Thus G must be Petrie-transitive.

Now let Π and Ω be distinct Petrie walks having at least two edges in common. Label Π and Ω with righthand indexings $[x, e, f]$ and $[y, d, g]$ respectively such that $e_0 = d_1$ and $e_k = d_h$ and so that k is the smallest possible index in all pairs of Petrie walks that cross. Without loss of generality suppose $0 < k < h$. Note that $k > 1$, since we have established that Petrie walks must cross at exactly one edge.

Now suppose G is not Petrie-transitive. Then every edge induces two distinct Petrie walks which belong to two distinct Petrie orbits. Choose $\psi \in \mathrm{Aut}(G)$ such that $\phi(e_0) = e_1$. Then by assumption, $\psi(\Pi)$ is not the other Petrie walk through e_1, and so ψ must fix Π but not Ω. Now we have three distinct Petrie walks Π, Ω and $\psi(\Omega)$. We know that Π and Ω share a common edge and that $\Pi = \psi(\Pi)$ and $\psi(\Omega)$ share a common edge. Since we have assumed G is not Petrie-transitive, by the earlier part of this proof it follows that Ω and $\psi(\Omega)$ do not share a common edge.

There are eight possible configurations of Π and Ω at their crossings $e_0 = d_1$ and $e_k = d_h$. The (shortest) segment of Ω that connects d_1 and d_h contains either d_0 or d_2 and either d_{h-1} or d_{h+1} as shown in Figure 5.4. These four cases are treated differently depending on whether k is either even or odd.

The remainder of the proof relies upon the fact that $\phi \in \text{stab } \Pi$ and thus can only possibly be one of two automorphisms of G. Both cases are impossible due to the planarity of G and the fact that Ω and $\phi(\Omega)$ cannot cross. Now if ϕ is the automorphism $\theta_{x_0 f_0}$, then in all eight possible configurations of Π, the Petrie walk given by $\theta_{x_0 f_0}(\Omega)$ would cross Ω. Therefore $\theta_{x_0 f_0} \notin \mathrm{Aut}(G)$. By Theorem 3 the automorphism γ_Π must be in $\mathrm{Aut}(G)$. It is a rather tedious and technical matter to check that if Ω and $\gamma_\Pi(\Omega)$ do not cross, then it is impossible for d_0 to connect to d_{h-1} or for d_2 to join with d_{h+1} in any planar embedding. Assume without loss of generality that d_0 is joined with d_{h+1}. If $k > 2$ then $\gamma^2_\Pi(\Omega)$ and Ω would be distinct, and they would cross.

Suppose $k = 2$. Then $\gamma^2_\Pi(d_0) = d_{h+1}$ and $\gamma^2_\Pi(d_1) = d_h$. Now that Ω and $\gamma_\Pi(\Omega)$ agree on two consecutive edges, it must be the case that $\Omega = \gamma^2_\Pi(\Omega)$, and thus $x_{-1} = x_3$. This implies that Π is a cycle of length 4.

Now if we follow the same logic by finding an element of stab Ω and examining the action of this automorphism on Π, we can see that Ω must also be a Petrie circuit of length 4. Since Π and Ω were chosen arbitrarily, it follows that all Petrie walks in G are 4-cycles. However, there is only one graph with this property: the tetrahedron, a finite graph which is Petrie-transitive. □

While the previous results concern Petrie walks in general, there are several findings by Graver and Watkins which apply only to Petrie circuits. If two Petrie circuits have a crossing, they must have an even number of crossings, and thus graphs of circuit type are Petrie-transitive.

Corollary 2 (Corollary 4.4 in [GW97])**.** *Let $G \in \mathcal{G}'$. All Petrie circuits in G belong to the same Petrie-orbit; if G is of circuit type, then G is Petrie-transitive.*

Lemma 5 (Lemma 6.1 in [GW97])**.** *Let Π and Ω be two Petrie circuits of $G \in \mathcal{G}'$ that cross. Then one may label their common edges by $c_0, c_1, \ldots, c_n = c_0$, where n is even, so that c_i and c_{i+1} are the edges of consecutive crossings along both Π and Ω for $i = 0, \ldots, n-1$.*

While the statement of this lemma may appear obvious, it is the second part which is not obviously true. For it may be the case that the sequence $\{c_1, c_2, c_3, c_4\}$ appears in order of consecutive crossings on Π, but when traveling along Ω the order of crossings may be $\{c_1, c_3, c_2, c_4\}$. The lemma precludes this possibility. The order of crossings along Π must also be the order of crossings along Ω. We now include a proof of this lemma.

Proof. Since each crossing consists of exactly one edge (by Lemma 4), we can label the edges of the crossings along Π by $c_0, c_1, \ldots, c_n = c_0$. Then $\Pi \setminus \{c_1, c_2, \ldots, c_n\}$ consists of n components which we will call *segments*. Since the edges c_i represent crossings of Π, these n components lie on alternating sides of Π. Since the $[c_{n-1}, c_0]$-segment of Ω must lie on a particular side of Π, it follows that n is even.

Surely if $n = 2$, then c_1 and c_2 are consecutive crossings around each of Π and Ω regardless of ordering. Then suppose $n \geq 4$. Now call one side of Ω "inside" and the other "outside." Consider the set

$$C = \{\{c_i, c_j\} \mid c_i, c_j \text{ are joined by a segment that lies outside } \Omega\}$$

Choose a pair $\{c_i, c_j\} \in C$ so that $|i-j| \mod n$ is minimal. Without loss of generality we can assume that $i = 0$ and $i < j \leq \frac{n}{4}$. Since the $[c_i, c_j]$-segment of Ω lies on one particular side of Π, there must be an even number of crossings along Π between c_i and c_j, and thus j is odd.

Suppose $j \geq 3$. Then there exists a pair (k, m) such that $1 \leq k, m \leq j$ and the $[c_k, c_m]$ segment of Π lies outside Ω, contradicting the minimality of $|i - j|$. Thus c_0 and c_1 are consecutive on both Π and Ω. For a given i consider an automorphism which maps c_0 onto c_i. This automorphism will either fix or interchange Π and Ω. Therefore c_i and either c_{i-1} or c_{i+1} are consecutive crossings on both Π and Ω. If there exists a triplet c_{i-1}, c_i, c_{i+1} that are consecutive on Ω, then by edge-transitivity, a map which sends c_i to c_{i+1} must then send c_{i+1} to either c_{i-1} or c_{i+2}. By induction all crossings on Ω will be labeled consecutively.

We must now account for the possibility that for each i, the edges c_{2i} and c_{2i+1} are consecutive on Ω, but c_{2i} and c_{2i-1} are not consecutive on Ω. The segments of Ω that join consecutive crossings constitute exactly half of all the segments of Ω and thus must lie on the same side of Ω. Without loss of generality, suppose these segments are outside Ω. Now let

$$C' = \{\{c_i, c_j\} \mid c_i, c_j \text{ are joined by a segment that lies inside } \Omega\}.$$

Then choose a pair $\{c_i, c_j\} \in C'$ that minimizes $|i - j| \mod n$, and by the same reasoning applied to the pair chosen from C above, these crossings must occur consecutively along Ω. □

Suppose that a graph G in $\mathcal{G}''$ is of mixed type. Given an edge $e \in E(G)$, we have that e lies on some Petrie line Π and some Petrie circuit Ω. By Lemma 4, Π and Ω must cross exactly once at e. Therefore Ω separates two ends of Π, one which lies on the outside of Ω and another which lies on the inside. When this occurs we say Ω is an *ends-separating circuit*. Therefore G must have at least two ends.

Theorem 6 (Theorem 4.5 in [GW97])**.** *All graphs in $\mathcal{G}''$ of mixed type are multi-ended.*

Graver and Watkins determine that Petrie circuits are shortest ends-separating circuits in a multi-ended graph. The term "shortest" refers to the length of the circuit. In general, the converse to Theorem 6 is not true; however by adding a few conditions, the authors were able to use Theorem 3 to find symmetries that guarantee the existence of Petrie circuits in a multi-ended graph. Chapters 7 and 8 provide specific examples of multi-ended graphs of mixed type.

Theorem 7 (Theorem 5.3 in [GW97])**.** *If G is a multi-ended graph in $\mathcal{G}'$ which is not of type 4, 4^*, or 4^P, then G contains a Petrie circuit; moreover, every Petrie circuit of G is a shortest ends-separating circuit.*

In the next chapter, we will relate some universal properties of graphs of circuit type. These characteristics are a consequence of the above rules about Petrie circuit crossings. These results will also be used in Chapter 8 to show that graphs constructed via interleaving contain infinitely many ends.

Chapter 6

One-Ended Members of $\mathcal{G}'$

For the remainder of this thesis we will be presenting methods for the construction of the infinite members of $\mathcal{G}$. While we know how to generate all of the one- and two-ended graphs in $\mathcal{G}$, algorithms to produce all infinitely-ended members of $\mathcal{G}$ are not presently known. In this chapter we present a process that is used to produce all one-ended members of the larger class $\mathcal{G}'$.

The class of one-ended, locally finite, edge-transitive infinite graphs was studied exhaustively by Grünbaum and Shepherd in the context of tiling theory ([GS97]). An edge-homogeneous tiling can be identified uniquely (up to isomorphism) by an edge-symbol, which is of the form $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$, as defined in Chapter 4. If these four elements satisfy a set of conditions, then an edge-transitive graph with edge-symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$ exists. The following theorem provides these necessary conditions and describes a method for constructing the desired edge-transitive graph.

Theorem 8 (Theorem 1 in [GS87]). *A 3-connected edge-homogeneous planar graph G with symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$ exists if and only if $\rho_0, \rho_1, \rho_0^*, \rho_1^*$ are positive integers greater than or equal to 3, and if one of the following four mutually exclusive conditions is satisfied:*

- *(i)* $\rho_0, \rho_1, \rho_0^*, \rho_1^*$ *are all even;*
- *(ii)* $\rho_0 = \rho_1$ *is even and at least one of* ρ_0^*, ρ_1^* *is odd;*
- *(iii)* $\rho_0^* = \rho_1^*$ *is even and at least one of* ρ_0, ρ_1 *is odd;*
- *(iv)* $\rho_0 = \rho_1, \rho_0^* = \rho_1^*$ *and all are odd.*

Moreover, each such edge-homogeneous graph is edge-transitive.

Proof. Let G be an edge-homogeneous graph with symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$. The vertices with valence ρ_0 and ρ_1 must alternate in cyclic order around any given face of G. If $\rho_0 \neq \rho_1$, then both ρ_0^* and ρ_1^* must be even. Similarly, the covalences of the faces incident to a vertex must alternate in cyclic order around the vertex, so that if $\rho_0^* \neq \rho_1^*$, then both ρ_0 and ρ_1 must be even. If $\rho_0, \rho_1, \rho_0^*, \rho_1^*$ are all distinct, then they must all be even (Condition (i)).

Suppose $\rho_0^* \neq \rho_1^*$ and at least one of ρ_0^*, ρ_1^* is odd. Then, in order to satisfy edge homogeneity, it must be the case that $\rho_0 = \rho_1$ and both are even (Condition (ii)). Similarly, if one of $\rho_0 \neq \rho_1$ and at least one of ρ_0, ρ_1 is odd, then $\rho_0^* = \rho_1^*$ and both are even (Condition (iii)).

Now suppose $\rho_0 = \rho_1$ are both odd. Then $\rho_0^* = \rho_1^*$ (Conditions (ii)-(iv)).

Now suppose that any one of the conditions (i)-(iv) are satisfied by quantities $\rho_0, \rho_1, \rho_0^*, \rho_1^*$. Then an edge-homogeneous graph with the given symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$ will be constructed. Now let

$$s = \frac{1}{\rho_0} + \frac{1}{\rho_1} + \frac{1}{\rho_0^*} + \frac{1}{\rho_1^*}$$

There are three cases.

If $s > 1$, then it can be shown that there are exactly nine solutions which satisfy (i)-(iv). Each of these solutions generates a unique (up to isomorphism) finite edge-homogeneous graph. Each of these nine graphs correspond to the edges and vertices of a convex polyhedron in three-dimensional Euclidean space.

If $s = 1$ then there are exactly five distinct solutions which correspond to infinite edge-homogeneous plane graphs that can be drawn so that their faces are regular polygons. Two of these five graphs are shown in Figures 5 and 6. The remaining three are the graphs with edge-symbols $\langle 3, 3; 6, 6 \rangle$ and $\langle 4, 4; 4, 4 \rangle$ and their duals. (The latter is self-dual).

If $s < 1$, then there are infinitely many solutions for $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$ satisfying (i)-(iv). The construction of these graphs will require the use of hyperbolic geometry, in particular the Poincaré disk model of the hyperbolic plane H^2, which can be represented in E^2 as a circular disk D whose boundary we will call C. Lines in H^2 can be drawn as arcs of circles in D which intersect C orthogonally. Angles are preserved in the homeomorphism which maps E^2 to the Poincaré disk model representation of H^2.

In H^2, it is possible to construct a quadrangle with any given set of internal angles whose sum is less that 2π. Observe that since $s < 1$,

$$\frac{2\pi}{\rho_0} + \frac{2\pi}{\rho_1} + \frac{2\pi}{\rho_0^*} + \frac{2\pi}{\rho_1^*} < 2\pi,$$

so we can construct a convex quadrangle Q in H^2 whose internal angles are $\frac{2\pi}{\rho_0}, \frac{2\pi}{\rho_0^*}, \frac{2\pi}{\rho_1}, \frac{2\pi}{\rho_1^*}$ listed in clockwise order starting at a vertex labeled A. The other vertices will be labeled B, C, D in the same order, as shown in Figure 6.1.

Now suppose any of conditions (i), (ii), (iii) or (iv) are satisfied by $\rho_0, \rho_1, \rho_0^*, \rho_1^*$. Then in the case that $\rho_0^* = \rho_1^*$, it is possible to draw a line of symmetry through Q by bisecting angles drawing the straight line BD, which bisects angles B and D. In this case, the triangles BAD and BCD will be congruent.

In the case that ρ_0^* is even, it is possible to surround vertex A with ρ_0^* copies of Q by setting $Q = Q_1$, $B = B_1$, $C = C_1$, $D = D_1$ then obtaining Q_2 with similar labelings by reflecting Q_1 across the line AB_1. The next quadrangle Q_3 is produced by reflecting a copy of Q_2 across AD_2. Continuing in such a fashion, Q_i is the reflection of Q_{i-1} across the line AB_{i-1} when i is odd and AD_{i-1} when i is even. In this way we can construct ρ_0^* abutting copies of Q with common vertex A.

Now if ρ_0^* is odd, then necessarily $\rho_0 = \rho_1$, and triangles ABC and ADC are congruent. This congruence guarantees that rotation of Q_1 about A by $\frac{2\pi}{\rho_0^*}$ radians will result in the vertex D_2 of quadrangle Q_2 coinciding with B_1. Therefore we can surround the vertex A by ρ_0^* copies of Q in an edge-to-edge arrangement by rotating Q about A by angles $\frac{2\pi i}{\rho_1}$, $i \in \{1, 2, \ldots, \rho_0^* - 1\}$.

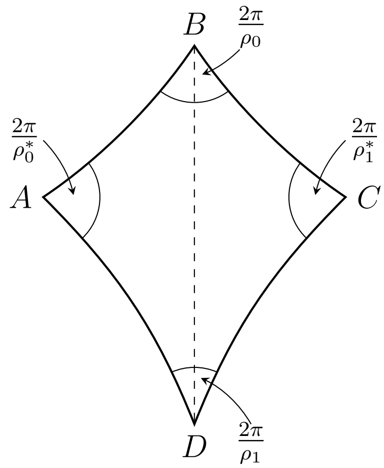

Figure 6.1: The quadrangle Q.

By following this process at the other vertices B, C and D, the quadrangle Q can be surrounded completely by copies of itself. Each of these copies can then be surrounded, and continuing inductively we obtain an infinite edge-to-edge tiling of H_2 by copies of Q, and following from the method of construction this tiling will be tile-transitive.

The representation of this tiling in the Poincaré disk model will be locally finite at every point except along the boundary C. Any neighborhood of a point of C contains infinitely many other points of C (we say such a point is an *accumulation point*). This does not pose a problem for the eventual local finiteness of our constructed graph, because the homeomorphism which maps the Poincare' disk to E^2 excludes the boundary C. If we apply this homeomorphism to the hyperbolic tiling we have just constructed, we can obtain an infinite tiling of E^2 by quadrangles whose vertices have valences $\rho_0, \rho_0^*, \rho_1, \rho_1^*$ in cyclic order, corresponding to the vertices $A, B, C,$ and D respectively in the original quadrangle Q. Let G be the plane graph whose vertices correspond to the vertices B and D in each copy of Q and whose edges correspond to the lines BD in each copy of Q. The graph G is the desired edge-transitive graph with edge-symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^* \rangle$. $\square$

Now we have a method for constructing *all* one-ended members of $\mathcal{G}$. Although it is not immediately apparent, we also know how to construct all graphs in $\mathcal{G}$ of circuit type.

Graver and Watkins ([Gw97]) determine that all graphs in $\mathcal{G}'$ of circuit type have at most one end and belong to a subclass with a particular family of edge-symbols. To show this the authors first presented the following lemma. The proof of this lemma is another example of the utility of Theorem 3. Rather than present the full rigorous proof, we include only an initial excerpt, in order to demonstrate the type of deduction enabled by the complete classification of graphs in $\mathcal{G}'$.

Lemma 6 (Lemma 6.2 in [GW97])**.** *If $G \in \mathcal{G}'$ is of circuit type, then either $\gamma_\Pi^2 \in \mathrm{Aut}(G)$ for every Petrie circuit Π of G or the edge-symbol of G or G^* is $\langle 4, 4; 3, h \rangle$ for some $h \geq 3$.*

Partial proof of Lemma 6: Suppose Π and Ω are distinct Petrie circuits of G that cross one another. Let $[v, e, f]$ and $[x, d, g]$ be righthand indexings of Π and Ω, respectively, so that $e_0 = d_1$ and $e_k = d_{m+1}$ are consecutive crossings on both Π and Ω. If $\gamma_\Pi^2 \notin \mathrm{Aut}(G)$ then G is of type 4 or 4^*, and thus all edge-stabilizers are trivial. Furthermore exactly one of the automorphisms $\theta_{v_0 f_0}$ and $\theta_{v_1 f_1}$ is in $\mathrm{Aut}(G)$. In the first case the $[d_1, d_{m+1}]$-segment of Ω and its image under $\theta_{v_0 f_0}$ lie on the same side of Π and cross one another an odd number of times. Since $\theta_{v_0 f_0}$ is a reflection, a given crossing will either be fixed or interchanged with another crossing under the action of $\theta_{v_0 f_0}$. It follows that one or more crossings (edges) are fixed by $\theta_{v_0 f_0}$. This contradicts the fact that all edge-stabilizers are trivial. Therefore $\theta_{v_1 f_1} \in \mathrm{Aut}(G)$; however, a similar argument yields another contradiction except in the case that $k = 3$. Now by Theorem 3 the stabilizer of f_0 must be $\langle \sigma_{f_0} \rangle$, so G must be of type 4^*.

In the remainder of the proof, Graver and Watkins consider the possible planar configurations of Θ and Ω at their crossing. Since we now know the automorphism type of G, the configuration must agree with the known identity of $\mathrm{stab}(e)$. This lemma, along with the results presented in Chapter 5 of this thesis about crossings of Petrie circuits, is used to prove that a graph $G \in \mathcal{G}'$ of circuit type is at most one-ended and has an edge-symbol belonging to a particular family of edge-symbols.

Theorem 9 (Theorem 6.3 in [GW97])**.**

1. *An infinite graph $G \in \mathcal{G}'$ is of circuit type if and only if G or G^* has edge-symbol $\langle 4, 4; 3, h \rangle$ with $h \geq 6$.*

2. *Every graph in $\mathcal{G}'$ of circuit type has at most one end.*

This brief chapter suffices to present all graphs in $\mathcal{G}'$ of circuit type and all one-ended graphs in $\mathcal{G}'$. The remainder of this thesis will detail a description of all two-ended members of $\mathcal{G}''$ and methods for the construction of some infinitely-ended members of $\mathcal{G}$. All graphs from this point forward will be of mixed type.

Chapter 7

Edge-Transitive Planar Strips

The presentation of the two-ended members of $\mathcal{G}''$ will follow the work of Watkins in his 1991 publication [Wa91], where he characterized all 2-connected, 2-ended, edge-transitive planar graphs. These graphs are a type of *strip*, which we will now define. For a subgraph C of a graph G, let

$$n(C) = \{v \in G \mid v \text{ is a neighbor of } c \text{ for some } c \in V(C)\}$$

The *neighborhood* of C, denoted $b(C)$, is the set of vertices $n(C) \setminus V(C)$. A connected graph G is called a *strip* if there exists a connected $C \subseteq V$ and an automorphism $\phi \in \mathrm{Aut}(G)$ such that $b(C)$ is nonempty and finite, $\phi(C \cup b(C)) \subseteq C$, and $C \setminus \phi(C)$ is finite.

Example 3. Let G be an infinite double ray, and write

$$G = \ldots x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$$

Then let C be the ray $x_0, x_1, x_2, \ldots$. Then $b(C) = \{x_{-1}\}$. If $\phi \in \mathrm{Aut}(G)$ is any translation of G along itself such that $\phi(x_0) = x_j$ and $j > 0$, then $\phi(C \cup \{x_{-1}\}) \subseteq C$, and $C \setminus \phi(C) = \{x_0, x_1, \ldots, x_{j-1}\}$. Therefore G is a strip. Also note that G is two-ended.

A double-ray, while edge-transitive and two-ended, is only 1-connected. As detailed by Watkins in [Wa91], there exists a class of planar strips that are 2-connected, 2-ended and edge-transitive. This class comprises *all* two-ended members of $\mathcal{G}''$. These graphs are described in the following theorem.

Theorem 10 (Proposition 2.5 in [GW97])**.** *For each $k \geq 2$, there exists a unique two-ended graph $G \in \mathcal{G}''$ with edge-symbol $\langle 4, 4; 4, 4\rangle$. It is the quotient graph of the planar tessellation with edge-symbol $\langle 4, 4; 4, 4\rangle$ (embedded as the integer lattice) by the identification $(x, y) \equiv (x + k, y + k)$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$.*

For a given $k \geq 2$, we can obtain a two-ended member of $\mathcal{G}''$ by wrapping the integer lattice around an infinitely long cylinder so that the vertices labeled (x, y) coincide with $(x + k, y + k)$ and are thus identified. Figure 7.1 depicts a local section of the quotient graph described in the previous theorem for $k = 3$. The two vertices labeled $(0, 0)$ are identified in the quotient graph, as are all pairs of identically labeled vertices.

To see that these graphs are two-ended and of mixed type, consider a graph G as described in Theorem 10. An example of a Petrie circuit Ω containing the origin is shown in Figure 7.1. The Petrie line Π containing the origin is of the form

$$\Pi = \ldots (i, -i+1), (i, -i), (i-1, -i), \ldots, (1, 0), (0, 0), (0, 1), \ldots, (-i, i-1), (-i, i), \ldots,$$

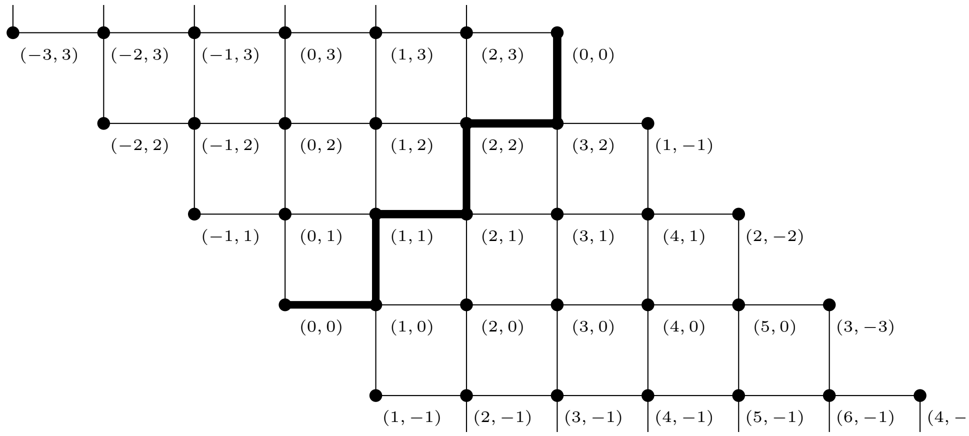

Figure 7.1: A planar embedding of an edge-transitive strip obtained by the identification $(x, y) \equiv (x + 4, y + 4)$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$.

The graph $G \setminus \Omega$ consists of two infinite components. In fact, the removal of any finite subgraph T results in at most two infinite components of $G \setminus T$, and thus G is two-ended. Since we have constructed a Petrie circuit and a Petrie line, it follows that graphs in this class of edge-transitive strips are of mixed type.

Chapter 8

Infinitely-Ended Plane Graphs of Mixed Type

Prior to the publication of [GW97], even the existence of infinitely-ended members of $\mathcal{G}$ had not been established. Graver and Watkins developed an inductive process called *interleaving* that yields many of them. A method due to B. Mohar ([Mo06]), called *tree amalgamation*, can also produce infinitely-ended members of $\mathcal{G}$. In this chapter we present the step-by-step methods of interleaving and tree amalgamation.

Modules

A graph Θ which is locally finite, 2-connected and edge-homogeneous with edge-symbol $\langle 2, \rho; \rho^*, \alpha\rangle$ is called a *module* if ρ, ρ^* and α are all even and greater than 3. We will say that two modules Θ_0 and Θ_1 with edge-symbols $\langle 2, \rho_0, \rho_0^*, \alpha\rangle$ and $\langle 2, \rho_1, \rho_1^*, \alpha\rangle$ are *compatible modules* if $\rho_0^* \neq \alpha$ and $\rho_1^* \neq \alpha$. Notice that graphs Θ_0 and Θ_1 both have faces with covalence α. We will call these faces the Z*-faces* of Θ_0 and Θ_1. The circuit enclosing a Z-face will be called a Z*-boundary*. The requirement that both $\rho_0^* \neq \alpha$ and $\rho_1^* \neq \alpha$ ensures a bipartition of $F(\Theta_0)$ and of $F(\Theta_1)$, allowing us to easily find the Z-faces of a module. See Figures 10 and 14 for examples of pairs of compatible modules whose Z-faces have been shaded in gray.

By interleaving two compatible modules Θ_0 and Θ_1 with edge-symbols $\langle 2, \rho_0; \rho_0^*, \alpha\rangle$ and $\langle 2, \rho_1; \rho_1^*, \alpha\rangle$ respectively, we can obtain an infinitely-ended graph $G \in \mathcal{G}'$ with edge-symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^*\rangle$. By varying α, we could potentially construct multiple non-isomorphic graphs with a given edge symbol. For this reason, we assign an *extended edge-symbol* $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^*; \alpha'\rangle$ to a graph attained by interleaving, where α' is the length of a shortest ends-separating circuit of G.

Interleaving: The Process

Let Θ_0 and Θ_1 be a pair of compatible modules. Initially, let G_0 be a copy of Θ_0. The desired graph will be the *limit graph*, $G = \bigcup G_i$. Now to obtain the graph G_1, for each Z-face in G_0 take a copy of Θ_1. Identify the Z-boundary of each Z-face of G_0 with a Z-boundary of a copy of Θ_1 so that every 2-valent vertex of G_0 is identified with a ρ_1-valent vertex of a copy of Θ_1 (and thus each ρ_0-valent vertex of G_0 is identified with a 2-valent vertex of a copy of Θ_1). We define the Z-faces of G_1 to be the Z-faces in each embedded copy of Θ_1 whose boundaries were excluded from the identification step. Examples of graphs G_1 produced by the first step of interleaving can be seen in Figures 8.2, 8.3 and 8.6.

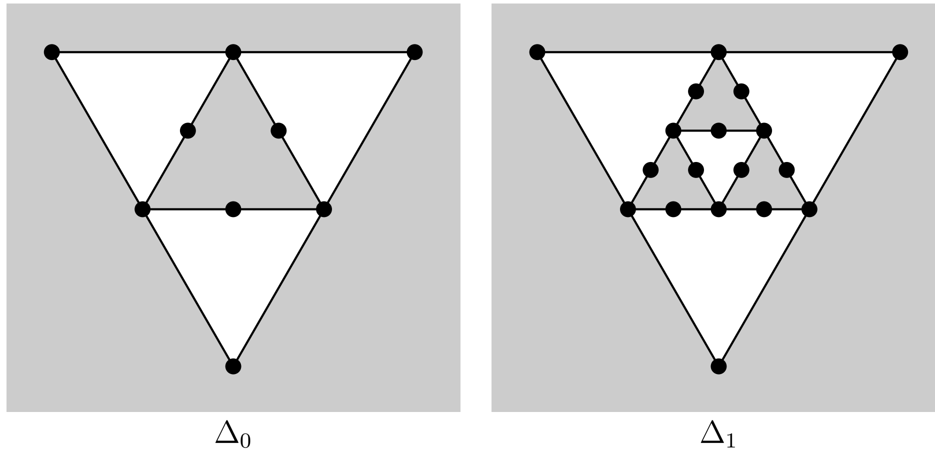

Figure 8.1: Compatible modules Δ_0 and Δ_1 with edge-symbols $\langle 2,4;4,6\rangle$ and $\langle 2,4;6,6\rangle$, respectively.

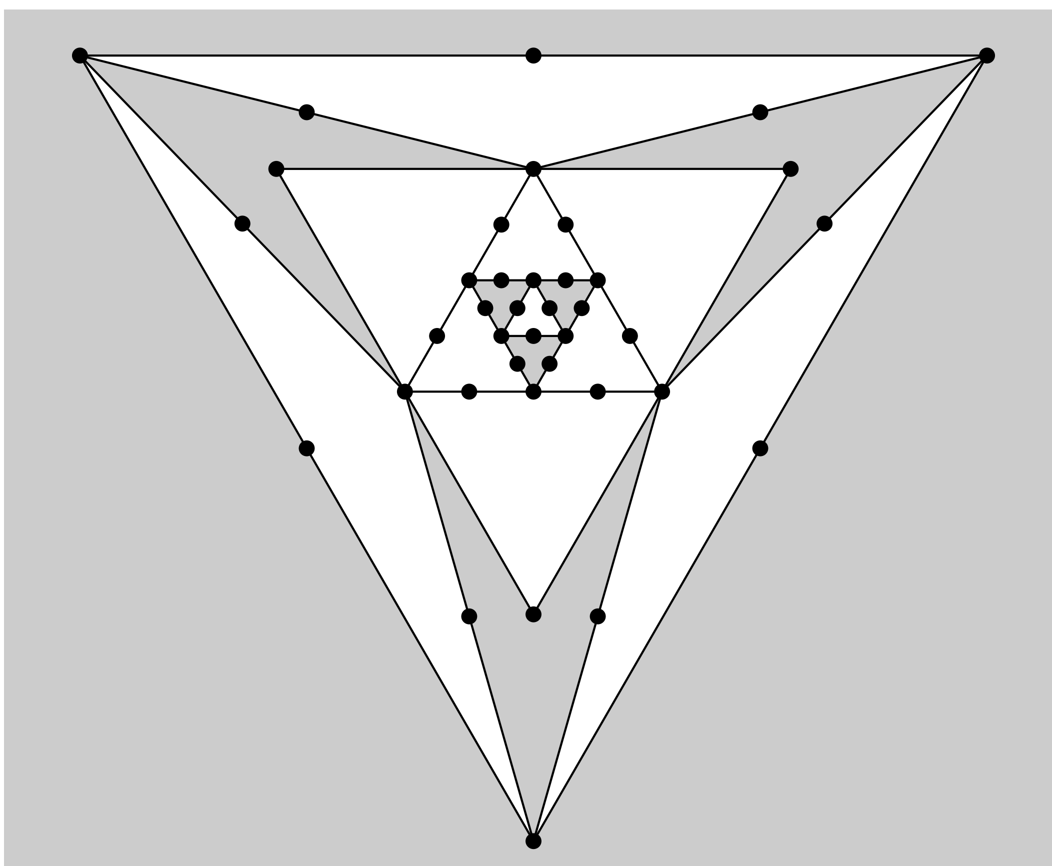

Figure 8.2: The graph G_1 obtained by interleaving Δ_0 and Δ_1 with $\Gamma_0 \cong \Delta_0$.

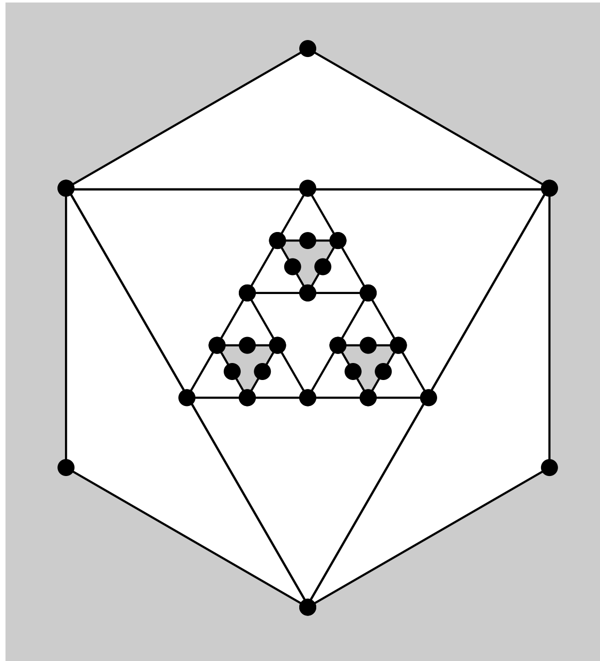

Figure 8.3: The graph G_1' obtained by interleaving compatible modules Δ_0 and Δ_1 with $G_0' \cong \Delta_1$

The i^{th} step of interleaving will produce a graph G_{i-1}, which will be a proper subgraph of the graph G_i. Suppose $i > 1$ and that G_{i-1} has already been constructed. Let $i \equiv j \mod 2$ and $i - 1 \equiv k \mod 2$. Then the Z-faces of G_{i-1} will be the Z-faces in each copy of Θ_k whose boundaries were excluded from the indentification in step $i-1$ of interleaving. Step $i + 1$ is as follows: Identify each Z-boundary in G_{i-1} with the Z-boundary of a copy of Θ_j so that the 2-valent vertices of Θ_j are identified with the ρ_k-valent vertices of G_{i-1}. Simultaneously, the ρ_j-valent vertices of Θ_j will be identified with the 2-valent vertices of G_{i-1}. The resulting graph is G_i. Continue inductively, and let G be the limit graph $\bigcup G_i$.

Properties of Graphs Constructed by Interleaving

Let $\mathcal{G}_{int}$ be the class of all graphs which have been constructed by interleaving pairs of compatible modules. Every graph $G \in \mathcal{G}_{int}$ possesses a list of properties, outlined in the following theorem of Graver and Watkins. Most of these characteristics are inherited from the compatible modules and are preserved during interleaving.

Lemma 7 (Lemma 3.1 in [GW97]). *A graph G obtained by interleaving two compatible modules Θ_0 and Θ_1 is locally finite, bipartite, 2-connected, multi-ended and edge-homogeneous with extended edge-symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^*; \alpha' \rangle$. Furthermore, the circuits in G corresponding to the Z-boundaries of Θ_0 and Θ_1 are Petrie circuits of length α in G (by Lemma 7.1, Watkins).*

Proof. First we prove that G is bipartite and 2-connected. Since Θ_0 and Θ_1 are bipartite

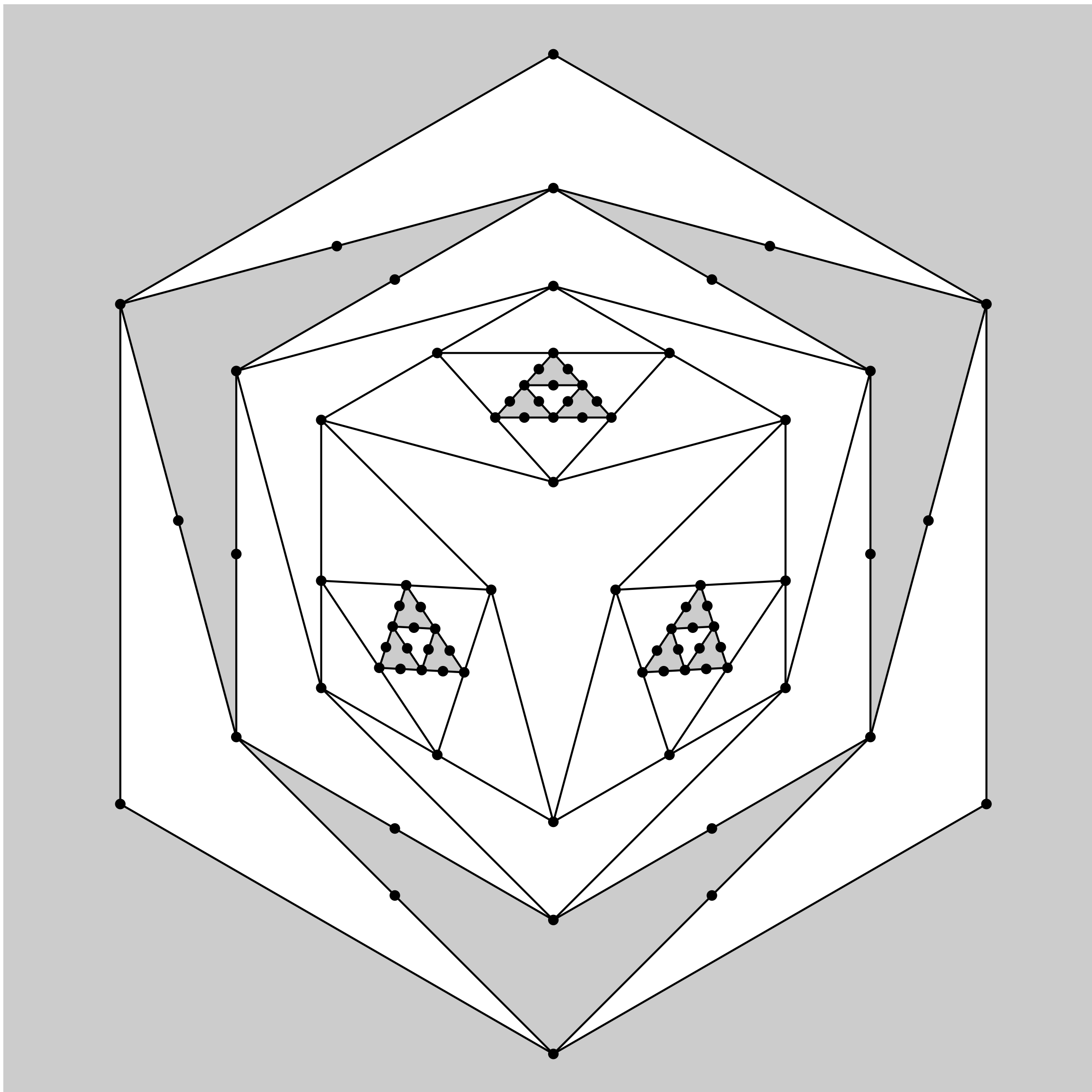

Figure 8.4: The graph G'_2 obtained by interleaving Δ_0 and Δ_1 with $G'_0 \cong \Delta_1$

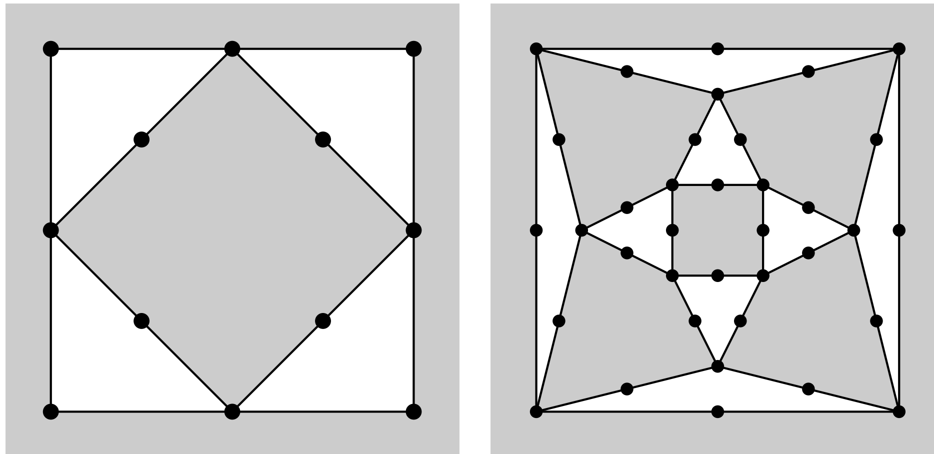

Figure 8.5: Compatible modules Θ_0 and Θ_1 with edge-symbols $\langle 2,4;4,8\rangle$ and $\langle 2,4;6,8\rangle$ respectively.

graphs, neither module contains a circuit of odd length. The i^{th} step of interleaving identifies a circuit in Θ_0 or Θ_1 with a circuit in the graph G_{i-1}. By induction, the limit graph G contains no circuit of odd length. Thus G is bipartite. Similarly, the connectivity of each G_i is extended to G, since the degree of any vertex $v \in V(G_i)$ is less than or equal to the degree of the corresponding vertex $v \in V(G)$.

To show G is edge-homogeneous, let $e \in E(G_i)$ for some step i of interleaving. Then $e \in E(G_j)$ whenever $j > i$, and in each instance e has edge symbol $\langle \rho_0, \rho_1; \rho_0^*, \rho_1^*\rangle$. From this, it straightforward to check that G is locally finite, since $\rho(v) \in \{\rho_0, \rho_1\}$ for all $v \in V(G)$.

We now demonstrate that the circuits in G corresponding to the Z-boundaries of Θ_0 and Θ_1 induce Petrie circuits of G. Consider such a circuit Ω in G. In particular, begin with a vertex $v_0 \in \Omega$ that corresponds to a 2-valent vertex in some copy of Θ_0, which was identified with a ρ_1-valent vertex in some copy of Θ_1. Let e_0, $e_1 \in E(G)$ be the edges incident to v_0 which correspond to the unique pair of edges in Θ_0 incident to v_0. During the interleaving process, a copy of Θ_0 (or Θ_1) was embedded into exactly one side of Ω, which we will call the "inside". Thus there is a unique face f_0 that lies outside Ω and is incident to both e_0 and e_1. Now since e_0 and e_1 are incident to a common face, there exists a Petrie walk Π containing both e_0 and e_1. Furthermore, each edge must necessarily be incident to distinct "inside" faces. Let v_1 be the other (not v_0) vertex on e_1. Since v_1 lies on Ω, v_1 corresponds to a ρ_0-valent vertex in Θ_0 that was identified with a 2-valent vertex in G_i for some i. It follows that v_1 (and therefore e_1) is incident to exactly one outside face f_1. Let e_2 be other edge incident to both v_1 and f_1. Then $e_2 \in \Pi$. Since all other faces incident to e_2 are inside faces, e_2 lies on Ω. Continuing in this fashion, we will alternate between vertices that correspond to 2-valent vertices in Θ_0 (thus guaranteeing a unique inside face) and vertices that correspond to 2-valent vertices in G_i (thus guaranteeing a unique outside face). The edges traversed in the Petrie walk are exactly the edges of the Ω.

Thus the cycles in G corresponding to the Z-boundaries of Θ_0 (or Θ_1) induce Petrie

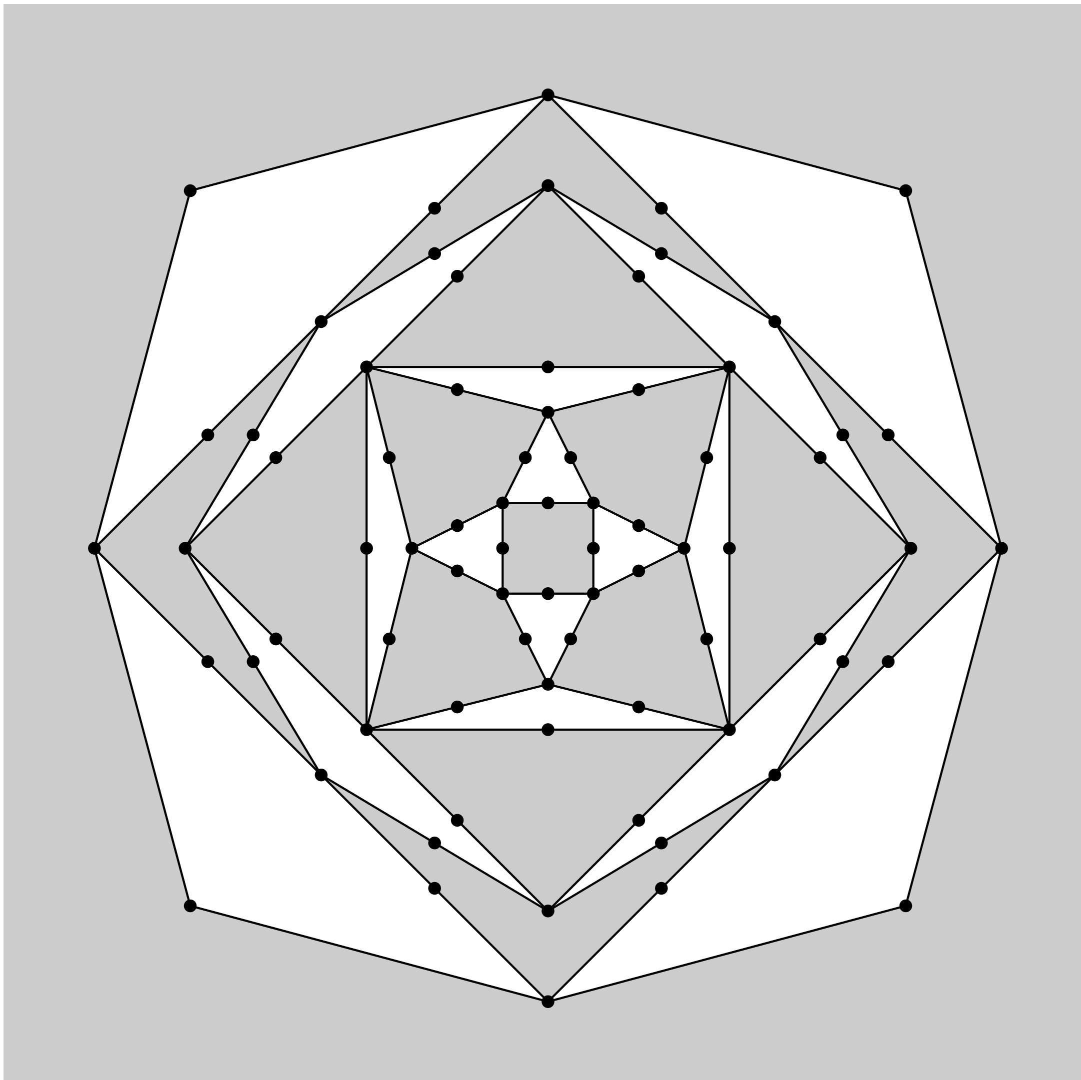

Figure 8.6: The graph Γ_1 obtained by interleaving compatible modules Θ_0 and Θ_1 with $\Gamma_0 \cong \Theta_1$.

cylces in G. Consider a Petrie line Π in G which crosses Ω. Then $G \setminus \Omega$ contains two components, each of which contains infinitely many vertices of Π. Therefore, G is multi-ended.

□

The previous theorem by Graver and Watkins outlines some properties of members of the class $\mathcal{G}_{int}$. At the end of this chapter, we present a conjecture concerning the relationship between the ends and the Petrie lines of such a graph $G \in \mathcal{G}_{int}$. As stated by the authors of [GW97], such a graph is of mixed type and has uncountably many ends. We will now prove this statement.

Theorem 11. *Let $G \in \mathcal{G}_{int}$. Then G is of mixed type and has uncountably many ends.*

Proof. By Theorem 9, since G is multi-ended and contains a Petrie circuit it must be of mixed type. Let Ω_0 and Ω_1 be sets of vertices corresponding to distinct Petrie circuits of G. Then $G \setminus (\Omega_0 \cup \Omega_1)$ contains three infinite components: the component outside both Ω_0 and Ω_1 and the components inside each of Ω_0 and Ω_1. Therefore the number of ends in G exceeds 2 and must therefore be uncountable (by theorems of R. Halin in [Ha73] and H.A. Jung. in [Ju81]). □

Tree Amalgamation

Tree amalgamation is a method developed by B. Mohar in [Mo06] for constructing infinitely-ended graphs. The process produces a *tree amalgamation* of graphs G_1 and G_2 over a connected infinite tree T.

Let p_1, p_2 be natural numbers. Then a (p_1, p_2)-*semi-regular tree* is an infinite tree T satisfying the following conditions:

(i) for all $v \in V(T)$, $\rho(v) \in \{p_1, p_2\}$;

(ii) when traversing any path $P \subset T$, the vertices of P alternate in degree.

For example, the (3,4)-semiregular tree is depicted in Figure 8.7.

To construct an infinitely-ended graph via tree amalgamation, we begin with a (p_1, p_2)-semiregular tree T. Let V_i denote the set of all p_i-valent vertices of T. Now we define a map c, which assigns to each edge of T an ordered pair (k, l) satisfying the following conditions:

(i) $0 \leq k < p_1$,

(ii) $0 \leq l < p_2$,

(iii) For each $v \in V_i$, all i^{th} coordinates of the set $\{c(e) \mid e \text{ is incident with } v\}$ are distinct.

Under this labeling, the first coordinates of the edge labels whose respective edges are incident with a common vertex $v \in V_1$ will exhaust all values in $\{0, 1, \ldots, p_1\}$.

In addition to the infinite tree T, we require two graphs G_1 and G_2. We find two families of subgraphs $\mathcal{S} = \{S_k \mid 0 \leq k < p_1\}$ of G_1 and $\mathcal{T} = \{T_l \mid 0 \leq l < p_2\}$ of G_2 such that all subgraphs in $\mathcal{S}$ and $\mathcal{T}$ have the same cardinality.

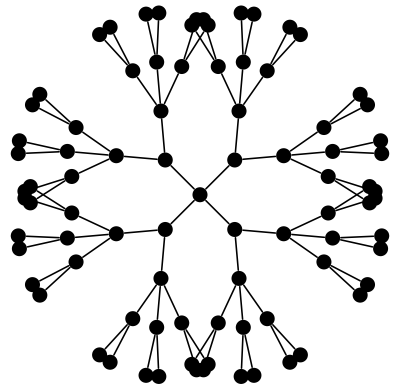

Figure 8.7: A local section of a (3,4)-semiregular tree

Next we define a set of *identifying maps*. For each label (k, l) define a bijection $\phi_{kl} : V(S_k) \to V(T_l)$. The label (k, l) determines the pair of subgraphs S_k and T_l that will be identified in the upcoming identification step. The map ϕ_{kl} specifies the pairs of vertices of S_k and T_l that will be identified.

Now we will assign to each vertex $v \in V(T)$ a copy of either G_1 or G_2. For each $v \in V_i$, let G_i^v be a copy of G_i. Label the copies of S_k or T_l in G_i^v by S_k^v or T_l^v. Then take the disjoint union of graphs G_i^v for all $v \in V(T)$.

We will join the graphs G_i^v by adding edges in the following way: For every edge $st \in E(T)$, where $s \in V_1$ and $t \in V_2$ with $c(st) = (k, l)$, identify each vertex $x \in S_k^s$ with the vertex $y = \phi_{kl}(x)$ in T_l^t. The resulting graph Y is called the *tree amalgamation of graphs G_1 and G_2 over the connecting tree T*.

Tree Amalgamation Meets Interleaving

A graph produced by tree amalgamation may not necessarily be possible to obtain via interleaving. For example, if the identification step of tree amalgamation requires the identification of some circuit S_k with some tree T_l, then the resulting graph Y may not be attainable via interleaving, since the identification step of interleaving identifies circuits with circuits. However, it is possible to add extra conditions to the identifying maps used in the tree amalgamation process so that the resulting graph is isomorphic to some graph produced by interleaving.

First, we start with two compatible modules Θ_0 and Θ_1 with edge-symbols $\langle 2, \rho_0; \rho_0^*, \alpha\rangle$ and $\langle 2, \rho_1; \rho_1^*, \alpha\rangle$, respectively. Let p_i be the number of Z-faces of Θ_i for $i = 1, 2$. Let

$$\mathcal{S} = \{S_k \ : \ 0 \le k < p_0\}$$

be the set of all Z-boundaries of Θ_0 and let

$$\mathcal{T} = \{T_l \ : \ 0 \le l < p_1\}$$

be the set of all Z-boundaries of Θ_1. Now let T be a (p_0, p_1)-semiregular tree whose edges have been labeled (k, l) by a map c in accordance with the conditions outlined in the general procedure above. For $i \in \{0, 1\}$, let V_i be the set of all p_i-valent vertices of T. For each $v \in V_i$, take a copy of Θ_i.

To ensure that we obtain a graph $G \in \mathcal{G}_{int}$, we must be careful with our identification map. For each (k, l), we require that the bijective map $\phi_{kl} : V(S_k) \to V(T_l)$ identifies the ρ_0-valent vertices of S_k with the 2-valent vertices of T_l (and thus the 2-valent vertices of S_k will be identified with the ρ_1-valent vertices of T_l.) By ensuring that this identification step follows the rules of interleaving, the resulting tree amalgamation G belongs to $\mathcal{G}_{int}$. This was proved by Adam McCaffery in his 2009 dissertation.

Theorem 12 ([Mc09). *] The tree amalgamation Y of compatible modules Θ_0 and Θ_1 over the connecting tree T by identifying maps which identify the appropriate vertices will be isomorphic to the graph G obtained by interleaving Θ_0 and Θ_1.*

Chapter 9

Petrie Lines and Ends

In a graph $G \in \mathcal{G}_{int}$, the presence of both an infinite number of ends and an infinite number of Petrie lines motivates a natural question. Just as there was a one-to-one correspondence between the equivalence classes of simple subgraphs and the ends of a graph, could there perhaps exist such a relationship between the ends and Petrie lines? That is, could there be a representative subray of a Petrie line for every end of an infinitely-ended graph in $\mathcal{G}$?

While a natural question, it was quickly answered in the negative. We discovered a graph G and subrays of two distinct Petrie lines of G that belong to the same end of G. We concluded that there could not be a one-to-one correspondence between the ends and Petrie lines of an arbitrary graph $G \in \mathcal{G}_{int}$. Let Π and Ω be Petrie lines. We say Π and Ω are *twin Petrie lines* of a graph G if there exist subrays $\Pi' \subset \Pi$ and $\Omega' \subset \Omega$ such that Π' and Ω' belong to the same end of G. Moreover, there is a sufficient condition on the form of the edge-symbol which guarantees the existence of twin Petrie lines. The result requires that the graph obtained is ordinary. The following lemma of [GW97] provides such ordinary graphs.

Lemma 8 (Lemma 7.2 in [GW97]). *Let Θ_0 and Θ_1 be compatible ordinary modules. Then the graph G obtained by interleaving Θ_0 and Θ_1 is ordinary.*

Proof. Let G_{i-1} be the graph obtained by the i^{th} step of interleaving. Since Θ_0 is ordinary, $\theta_{vf} \in \mathrm{Aut}(\Theta_0) \cong \mathrm{Aut}(G_0)$ for all incident vertex-face pairs $(v, f) \in G_0$, and this automorphism will permute the Z-faces of Θ_0. A given automorphism $\theta_{vf} \in \mathrm{Aut}(G_0)$ can be extended to G_1 by finding an automorphism $\theta_{xg} \in \mathrm{Aut}(\Theta_1)$ which will act on each copy of Θ_1 in G_1 in such a way that the resulting graph is $\theta_{vf}(G_1)$, where $\theta_{vf} \in \mathrm{Aut}(G_1)$. By this reasoning, given any vertex-face pair (v, f) in a graph G_i, θ_{vf} is an automorphism of G_i for all incident pair$(v, f) \in G_i$, and there exists an extension of θ_{vf} to $\mathrm{Aut}(G_{i+1})$. By induction, $\theta_{vf} \in \mathrm{Aut}(G)$ for all incident vertex-face pairs $(v, f) \in G$. $\square$

Theorem 13. *If $G \in \mathcal{G}_{int}$ is ordinary and has extended edge-symbol $\langle 4, 4; 4, k; m\rangle$, then G admits twin Petrie lines.*

Proof. Fix a 4-covalent face $f_0 \in G$. Choose consecutive edges e_0 and e_1 incident to f_0 so that f_0 lies to the right of e_0 when traversing the $[e_0, e_1]$-path, and so that the unique Petrie walk Π containing both e_0 and e_1 is a Petrie line. Preserve the labels of e_0, e_1 and f_0 and let $[v, e, f]$ be a righthand indexing of Π. Let Π' be a subray of Π whose indices are non-negative. Then the ray $\Omega = \theta_{v_1 f_0}(\Pi')$ is a subray of the Petrie line $\theta_{v_1 f_0}(\Pi)$. Furthermore, we claim that Ω belongs to the same end as Π'. For all even i, the edge pairs $\{e_i, e_{i+1}\}$ will be incident with a common 4-covalent face f_i. The map $\theta_{v_1 f_0}$ will map each of these edge pairs onto the other pair of edges incident with f_i while fixing the vertices with odd indices along Π. Thus when i is odd, $\theta_{v_1 f_0}(x_i) = x_i$. Therefore the rays Π' and $\theta_{v_1 f_0}(\Pi')$ have infinitely many vertices in common and belong to the same end. □

With these results, we have shown that not every end of a graph $G \in \mathcal{G}_{int}$ must possess a unique representative subray of a Petrie line. However, could it be true that every end of G contains at least one subray of a Petrie line? If so, for every pair of ends $\mathcal{E}_1$ and $\mathcal{E}_2$ of G, can we find a two-ended Petrie line with subrays in both $\mathcal{E}_1$ and $\mathcal{E}_2$? That is, we wonder if every *pair* of ends has a representative Petrie line.

In a plane graph $G \in \mathcal{G}_{int}$, there are uncountably many pairs of ends. We can track the rays (and therefore the ends) of G along the infinite connecting tree T used to construct the tree amalgamation G. Each edge of T indicates the identification of a pair of Z-boundaries. If $U = \{u_i\}_{i=1}^{\infty}$ is a ray of G, we can create an edge-sequence $\{t_i\}_{i=1}^{\infty}$ of T, where t_i is the edge of T whose corresponding identification resulted in the addition of u_i to G. This sequence of edges forms a path in T. Two rays U and V will belong to the same end of G if and only if their corresponding edge-sequences belong to the same end of T.

We conjecture that the Petrie lines of G induce edge-sequences that follow a predictable pattern along the connecting tree T. For this reason, it is possible that there is a pair of ends of G with no representative Petrie line. If the pair of ends strictly avoids the Petrie line pattern, it may not be possible to find a Petrie line belonging to both ends.

Conjecture 1. *There is a graph G with ends $\mathcal{E}_1$ and $\mathcal{E}_2$ such that no Petrie line belongs to both $\mathcal{E}_1$ and $\mathcal{E}_2$.*

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