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A Survey of Graphs of Minimum Order with Given Automorphism Group

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A SURVEY OF GRAPHS OF MINIMUM ORDER WITH GIVEN AUTOMORPHISM GROUP

by

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Abstract

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We survey vertex minimal graphs with prescribed automorphism group. Whenever possible, we also investigate the construction of such minimal graphs, confirm minimality, and prove a given graph has the correct automorphism group.

Chapter 1

Introduction

In 1939, Roberto Frucht proved a highly significant graph theoretic conjecture: for every finite group, there exists a graph whose automorphism group is isomorphic to that finite group [3]. Numerous authors launched investigations into determining the possible constructions for such graphs given a particular finite group, and, consequently, questions arose concerning the extremal properties of these graphs, either in regard to vertices or edges (or both simultaneously). Here, we restrict our survey to the consideration of graphs with a given automorphism group and the least possible number of vertices.

An important result of this nature was established by Babai in 1974 [2]. Specifically, he found an upper bound for the minimum number of vertices in a graph with automorphism group isomorphic to a particular finite group. Excluding the cyclic groups order 3, 4, and 5, this upper bound is less than or equal to twice the order of the given finite group. Based on Babai's conclusions, several authors have successfully narrowed this lower bound (or found the exact least number of vertices) for various finite groups. For example, the minimum number of vertices is known for the three aforementioned exceptions and is larger than this upper bound in each case. See Chapter 2 for full details.

In Chapters 2 through 4, the finite groups which we discuss are the cyclic, dihedral, and generalized quaternion groups, respectively. In Chapter 5, we include a brief analysis of the known results for the hyperoctahedral, symmetric, and alternating groups.

1.1 Terminology

To begin, we introduce the definitions and notation which are used throughout: A *graph* Γ is the ordered pair (V, E), where V is a finite set of vertices and E is a set of edges where $E \subseteq \{ \{x, y\} : x, y \in V\}$. We denote these sets by $V(\Gamma)$ and $E(\Gamma)$, respectively. In addition, two vertices $x, y \in V(\Gamma)$ are *adjacent* if and only if $\{x, y\} \in E(\Gamma)$.

Equivalently, we also say x is a *neighbor* of y (and vice versa, y is a neighbor of x) if and only if $\{x, y\} \in E(\Gamma)$. For $x \in V(\Gamma)$, the *neighborhood* of x is given by $N(x) = \{y \in V(\Gamma) : \{x, y\} \in E(\Gamma)\}$. Further, we say the *degree* of x is equal to $|N(x)|$ and denote this as $\rho(x)$.

Graph automorphisms are the set of adjacency preserving bijections on $V(\Gamma)$. This set forms a group which we call the *automorphism group* of Γ and is denoted $\mathrm{Aut}(\Gamma)$. In particular, suppose φ is a permutation of $V(\Gamma)$. Then $\varphi \in \mathrm{Aut}(\Gamma)$ if and only if $\{\varphi(x), \varphi(y)\} \in E(\Gamma)$ precisely when $\{x, y\} \in E(\Gamma)$.

Within each chapter, we designate the finite group under consideration by G. We include all known values of $\alpha(G)$, the minimum number of vertices of a graph Γ having $\mathrm{Aut}(\Gamma) \cong G$. Moreover, we call such graphs *minimal*, and, whenever possible, we discuss the known constructions for such graphs. Notationally, we say G*-graphs* denote graphs with $\mathrm{Aut}(\Gamma) \cong G$, where G is current group being discussed.

Chapter 2

Finite Cyclic Groups

In each section of this chapter, the group G is considered to be an embedding of some cyclic group $\mathbb{Z}_n$ in a symmetric group S_k. For instance in Section 2.2.1, we show

$$\mathbb{Z}_4 \cong \langle (1\ 2)(1'\ 2'\ 3'\ 4')(1''\ 2''\ 3''\ 4'') \rangle = G$$

is the embedding of $\mathbb{Z}_4$ into the smallest set of symbols such that it is an automorphism group of a graph. Notice that G is isomorphic to a subgroup of S_{10}. We use the symbol Γ to refer to a graph whose automorphism group $\mathrm{Aut}(\Gamma)$ is isomorphic to G. Likewise, the group G of $\alpha(G)$ corresponds to the particular cyclic group embedding regarded in each section.

In 1985, Arlinghaus completed a comprehensive treatise regarding minimal graphs with finite abelian automorphism group. His memoir builds upon Meriwether's unpublished 1963 investigation of minimal graphs with finite cyclic automorphism group. Arlinghaus extends Meriwether's results to all finite abelian groups and determines many of the constructions for minimal graphs with these groups [1].

However, due to the presence of 2-, 3-, and/or 5-cycles in the elements of $\mathrm{Aut}(\Gamma)$, exceptional structures arise in certain $\mathbb{Z}_n$ graphs, preventing a straightforward determination of $\alpha(G)$ for most graphs with non-prime power order cyclic group; likewise, this forces even more complex structures in graphs with finite abelian group. Due to its length and complexity, we do not include Arlinghaus's algorithm for determining $\alpha(G)$ for all finite abelian groups and verification that the values obtained are minimal; further, we omit discussion of most of his constructions. Full details of his results may be found in his memoir [1].

Herein, we consider Arlinghaus's determination of $\alpha(G)$ for all cyclic groups and include constructions of their corresponding minimal graphs when necessary. We also discuss some examples of particular interest. In 1958, Sabidussi incorrectly addressed the cyclic cases [14]. Based upon this work, Harary and Palmer published a short paper in 1965, the results of which are dependent upon false conclusions [9]. Two of their graphs, however, are in fact minimal. In addition, Sabidussi's 1966 review partially corrects his results, and he quotes Meriwether's (unpublished) work on graphs with cyclic automorphism group. We discuss these constructions and also include the correct minimal construction for $\mathbb{Z}_4$ as originally constructed by Meriwether.

2.1 Preliminaries

For the known minimal constructions of particular graphs, we show that the automorphism group for each graph is indeed isomorphic to the finite cyclic group in question. We briefly overview this method in Section 2.1.3, specifically in Lemma 2.1.3.3.

We assume that a graph Γ has the desired automorphism group, i.e. $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_n$, and prove that $\alpha(G)$ is indeed minimal. We rely upon the strategy employed by Arlinghaus: we construct a graph Γ' where $\mathrm{V}(\Gamma') < V(\Gamma)$ and suppose $\mathrm{Aut}(\Gamma') \cong \mathbb{Z}_n$. Then we show that under these conditions, $\mathrm{Aut}(\Gamma')$ is forced to contain noncentral elements, a contradiction. That is, since $Z(\mathrm{Aut}(\Gamma')) \neq \mathrm{Aut}(\Gamma')$, the group $\mathrm{Aut}(\Gamma')$ is nonabelian. Hence, it cannot be isomorphic to $\mathbb{Z}_n$ [1].

The subsequent lemmas address commutativity and include several exceptional cases. Their proofs are very similar in structure and detail. The techniques to deconstruct the permutation structure of $\mathrm{Aut}(\Gamma)$ are nearly identical but necessary for the determination of $\alpha(G)$. In order to streamline this process, Arlinghaus introduces some notation of his own; e.g. he defines mappings which take a cycle contained in a permutation of $\mathrm{Aut}(\Gamma)$ and decomposes it into a product of transpositions. He discards minor details and only provides a few examples of his exact computations.

For the purpose of this chapter, we introduce and discuss a few of these commutativity lemmas in depth. Later lemmas and theorems involving $\alpha(G)$ and the corresponding minimal graphs require these lemmas.

While Arlinghaus' notation is not self-evident, the opacity of his method is made up for in the efficiency of expressing his computational arguments [1]. We generally follow his notation but deviate from it when additional clarity is desired.

2.1.1 Notation

Let Γ be a graph and $\varphi \in \mathrm{Aut}(\Gamma)$, where φ is a permutation. In keeping with Arlinghaus, we write $x\varphi = y$ and read this operation as "replace x with y under the operation of φ." In effect, we are relabeling a vertex of Γ under right multiplication. The operations that we define are either cyclic permutations or involutions of the described vertex set of Γ.

Now suppose $\varphi \in \mathrm{Aut}(\Gamma)$ and this permutation contains the cycle σ. We use O and a subscript corresponding to a given cycle in order to denote the orbit made up of the labels moved by the named orbit. For example, let $\phi = (1234)(5678)(9\,10)$ and $\sigma = (5678)$. Then $O_\sigma = \{5, 6, 7, 8\}$.

2.1.2 Commutativity Lemmas

Lemma 2.1.2.1. Let σ be a cycle of length n and x the element in the first position of σ:

a) Define χ_σ as the product of transpositions

$$\chi_\sigma = \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} (x\sigma^i, x\sigma^{n-i-1}),$$

where $\lfloor \cdot \rfloor$ is the floor function. If n is odd, the element in the $\lfloor \frac{n}{2} \rfloor$ position of σ is fixed by χ_σ.

b) Define λ_σ as the product of transpositions

$$\lambda_\sigma = \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} (x\sigma^i, x\sigma^{n-(i+1)-1}),$$

where $\lfloor \cdot \rfloor$ is the floor function. If n is odd, the element in the last position σ is fixed. Whereas if n is even, both the elements in the last position and $\lfloor \frac{n}{2} \rfloor$ position of σ are fixed by λ_σ.

Example. Let $\sigma = (1\ 3\ 4\ 6\ 2\ 5)$.

a) Then $\chi_\sigma = (1\ 5)(3\ 2)(4\ 6)$. If a particular cycle is known for a given $\mathrm{Aut}(\Gamma)$, the process for determining is relatively simple. However, since most of our operations will involve arbitrary cycles (with fixed length) of permutations, it is illustrative to include a precise computation of the above decomposition of σ.

$x\sigma^i$	$x\sigma^{n-i-1}$	Transpositions of χ_σ
$1\sigma^0 = 1$	$1\sigma^5 = 5$	$(1\ 5)$
$1\sigma^1 = 3$	$1\sigma^4 = 2$	$(3\ 2)$
$1\sigma^2 = 4$	$1\sigma^3 = 6$	$(4\ 6)$

b) Then $\lambda_\sigma = (1\ 2)(3\ 6)$. As with χ_σ, the computation to find λ_σ is straightforward for a particular cycle σ. Once again, we include the computation:

$x\sigma^i$	$x\sigma^{n-(i+1)-1}$	Transpositions of λ_σ
$1\sigma^0 = 1$	$1\sigma^4 = 2$	$(1\ 2)$
$1\sigma^1 = 3$	$1\sigma^3 = 6$	$(3\ 6)$
$1\sigma^2 = 4$	$1\sigma^2 = 4$	$(4\ 4)$

The defining characteristic of graph automorphisms is that they are adjacency preserving bijections: $\{x, y\} \in E(\Gamma)$ if and only if $\{x\varphi, y\varphi\} \in E(\Gamma)$.

By extension, if $\varphi \in \mathrm{Aut}(\Gamma)$ and its the disjoint cycle decomposition contains cycles σ and τ such that $x \in O_\sigma$ and $y \in O_\tau$, then the existence of an edge $\{x, y\} \in E(\Gamma)$ implies the existence of many other edges; namely, $\{x\sigma^k, y\tau^k\} \in E(\Gamma)$ for all integers $0 \le k < |\sigma\tau|$, where $|\cdot|$ denotes the order of a permutation. As a consequence, the number of divisors shared among the orders of the cycles within an automorphism directly affects the number of adjacencies which are present in a graph. We illustrate this fact through the following lemma.

Lemma 2.1.2.2. Suppose $\varphi \in \mathrm{Aut}(\Gamma)$ and that σ and τ are distinct cycles in the disjoint cycle decomposition of φ with $|\sigma| = m$ and $|\tau| = n$' Let $d = \gcd(m, n)$. Further, suppose $x \in O_\sigma$ and $y \in O_\tau$. Then $\{x, y\} \in E(\Gamma)$ if and only if $\{x\sigma^{id}, y\tau^{jd}\} \in E(\Gamma)$ for all $i, j \in \mathbb{Z}$.

Proof. Since any permutation from $\mathrm{Aut}(\Gamma)$ preserves edges, it suffices to show that $\{x, y\}\varphi = \{x'y'\}$ for some $\varphi \in \mathrm{Aut}(\Gamma)$. Suppose we have the conditions listed above. By Cayley's Theorem, we know that since $\mathrm{Aut}(\Gamma)$ is a subgroup of the symmetric group and likewise closed under multiplication, if $\varphi \in \mathrm{Aut}(\Gamma)$, then $\varphi^i \in \mathrm{Aut}(\Gamma)$ for all $i \in \mathbb{Z}$. Choose $r, s \in \mathbb{Z}$ such that $rm + sn = d$. Then for $i, j \in \mathbb{Z}$, $\varphi^{jrm+isn} \in \mathrm{Aut}(\Gamma)$. Now consider the following computation:

$$\{x, y\}\varphi^{jrm+isn} = \{x, y\}\varphi^{jrm}\varphi^{isn} = \{x\sigma^{jrm}, y\tau^{jrm}\}\varphi^{isn}$$

We apply right multiplication with φ^{jrm} to the vertices of edge $\{x, y\}$. Since $x \in O_\sigma$ and $y \in O_\tau$, each vertex is only moved by σ and τ, respectively.

$$\begin{aligned} &= \{x(\sigma^m)^{jr}, y\tau^{j(d-sn)}\}\varphi^{isn} = \{x(1), y\tau^{jd}\tau^{-jsn}\}\varphi^{isn} \\ &= \{x, y\tau^{jd}(\tau^n)^{-js}\}\varphi^{isn} = \{x, y\tau^{jd}(1)\}\varphi^{isn} = \{x, y\tau^{jd}\}\varphi^{isn} \end{aligned}$$

Note that in the steps above we simply applied the fact that $rm + sn = d$ and the order of the cycles. That is, $|\sigma| = m$ and $|\tau| = n$, so $\sigma^m = 1$ and $\tau^n = 1$.

$$\begin{aligned} &= \{x\sigma^{isn}, y\tau^{jd}\tau^{isn}\} = \{x\sigma^{i(d-rm)}, y\tau^{jd}\} \\ &= \{x\sigma^{id}\sigma^{-rm}, y\tau^{jd}\} = \{x\sigma^{id}, y\tau^{jd}\} = \{x', y'\} \end{aligned}$$

Since we have formed this argument with a chain of equalities, we can easily see that the reverse direction also holds. Therefore, $\{x, y\} \in E(\Gamma)$ if and only if $\{x', y'\} \in E(\Gamma)$. □

Corollary 2.1.2.1. Let $\varphi \in \mathrm{Aut}(\Gamma)$. Suppose σ and τ are distinct cycles in the disjoint cycle decomposition of φ such that $\gcd(|\sigma|, |\tau|) = 1$. Define $E_{\sigma,\tau} = \{\{x, y\} : x \in O_\sigma, y \in O_\tau\}$. Then either $E_{\sigma,\tau} \subseteq E(\Gamma)$ or $E_{\sigma,\tau} \cap E(\Gamma) = \emptyset$.

Proof. Suppose $x \in O_\sigma, y \in O_\tau$ and $\{x, y\} \in E(\Gamma)$. Then $\{x\sigma^i, y\tau^j\} \in E(\Gamma)$ for all $i, j \in \mathbb{Z}$ by the previous lemma. But notice that $E_{\sigma,\tau} = \{\{x\sigma^i, y\tau^j\} : i, j \in \mathbb{Z}\}$, and the result holds. □

We demonstrate the application of this corollary, as well as the fact that relatively prime cycles have little to no effect on one another with a brief example. Further, we note that this corollary is usually used to show that a given graph *does not* have cyclic automorphism group, as shown in the next example.

Example. Suppose $\varphi \in \mathrm{Aut}(\Gamma)$ containing cycles $\sigma = (1\ 2\ 3)$ and $\tau = (4\ 5)$. Clearly the lengths of σ and τ are relatively prime.

By the lemma, if $x \in O_\sigma, y \in O_\tau$, then $\{x, y\} \in E(\Gamma)$ if and only if $\{x\sigma^i, y\tau^j\} \in E(\Gamma)$. Then $\sigma = (123)$ and $\tau = (45)$. The powers of σ are $\sigma^0 = ()$, $\sigma^1 = (123)$, $\sigma^2 = (132)$; the powers of τ are $\tau^0 = ()$ and $\tau^1 = (45)$. Thus, $(\sigma,\tau)^0 = ()$, $(\sigma,\tau)^1 = \sigma\tau$, $(\sigma,\tau)^2 = \sigma^2$, $(\sigma,\tau)^3 = \tau$, $(\sigma,\tau)^4 = \sigma$, and $(\sigma,\tau)^5 = \sigma^2\tau$.

We have already assumed that φ is an automorphism. Suppose $\{1, 4\}$ is an edge of Γ. We consider this edge under the action of φ. Observe that $1 \in O_\sigma$ and $4 \in O_\tau$. Then the following edges must be present:

Action of φ on $\{1,4\}$	
$\{1,4\}() = \{1,4\}$	$\{1,4\}\varphi^3 = \{1,4\tau\} = \{1,5\}$
$\{1,4\}\varphi^1 = \{1\sigma,4\tau\} = \{2,5\}$	$\{1,4\}\varphi^4 = \{1\sigma,4\} = \{2,4\}$
$\{1,4\}\varphi^2 = \{1\sigma^2,4\} = \{3,4\}$	$\{1,4\}\varphi^5 = \{1\sigma^2,4\tau\} = \{3,5\}$

Therefore, the edge orbit of $\{1, 4\}$ under this graph automorphism is

$$O_{\{1,4\}} = \{\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\},\{3,5\}\},$$

and so φ acting on $\{1, 4\}$ forms the induced subgraph K_{312}. Hence if $\{1, 4\} \in$ and $\varphi \in \mathrm{Aut}(\Gamma)$, then Γ cannot have cyclic automorphism group.

The presence of one edge between the cycles σ and τ implied the presence of at least five distinct edges in Γ. In fact, adjacencies occured between every possible pair of vertices from each of the cycles.

2.1.3 Permutation Lemmas

The lemmas in the preceding section concern the permutation structure of a graph Γ with a given $\mathrm{Aut}(\Gamma)$. We wish to apply these in the following manner: if we provide a Γ such that a given permutation representation $G \le S_k$ with $G \cong \mathbb{Z}_n$ has $G \le \mathrm{Aut}(\Gamma)$, then there exists $\gamma \notin Z(\mathrm{Aut}(\Gamma))$. We use these lemmas to eliminate the possible permutation structures of Γ which are not cyclic.

Once we construct Γ with the correct permutation structure, the desired $\mathrm{Aut}(\Gamma)$ is found. The upper bound for $\alpha(G)$ is determined by the structure of Γ imposed by $\mathrm{Aut}(\Gamma)$. Under the guidance of Arlinghaus, we use the permutation lemmas of this section to show that particular automorphisms of $\mathrm{Aut}(\Gamma)$ must contain additional cycles, since more vertices must be available to be permuted. By necessity, the vertex set of Γ is forced to be larger: $|V(\Gamma)| \le \alpha(G)$.

We confirm the reverse inequality by showing that there exists an automorphism $\phi \in \mathrm{Aut}(\Gamma)$ such that $\langle \phi \rangle = \mathrm{Aut}(\Gamma)$ is required to have the permutation structure that meets the conditions for which $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_n$; no automorphisms on fewer symbols suffice.

We provide a discussion of his arguments detailing both directions in establishing $\alpha(G)$ and include the main lemmas (and corollaries) which are used to justify the theorems. contained in the last two sections. Due to the involved nature these cyclic cases and Arlinghaus's (necessarily) lengthy proofs, we omit complete descriptions when possible.

Notably, we only include the lemmas directly necessary to the calculation of $\alpha(\mathbb{Z}_n)$. For full details, see Arlinghaus's memoir which completely discusses the original nine-part lemma [1].

Lemma 2.1.3.1. Let Γ be a graph and $\phi \in \mathrm{Aut}(\Gamma)$. Suppose the cycle decomposition of ϕ contains one and only one cycle σ such that one of the following conditions is true:

1. σ has cycle length $n > 2$ and the cycle decompositions of all other automorphisms in $\mathrm{Aut}(\Gamma)$ only contain cycles of length two or coprime to n.

2. σ has cycle length $2n > 4$ and the cycle decompositions of all other automorphisms in $\mathrm{Aut}(\Gamma)$ only contain cycles of length two or coprime to n. .

3. σ has cycle length 3^n for $n \geq 1$, there exists a cycle τ in some automorphism of A of length $3m$ such that $\gcd(m, 3) = 1$ and $m \geq 1$, and the cycle decompositions of all other automorphisms in $\mathrm{Aut}(\Gamma)$ only contain cycles of length two or coprime to $3n$.

4. σ has cycle length 5^n for $n \geq 1$, there exists a cycle τ in some automorphism of A of length $5m$ such that $\gcd(m, 5) = 1$ and $m \geq 1$, and the cycle decompositions of all other automorphisms in $\mathrm{Aut}(\Gamma)$ only contain cycles of length two or coprime to $5n$.

Then there exists an automorphism $\psi \in \mathrm{Aut}(\Gamma)$ such that ψ and ϕ do not commute. Further, $|\psi| = 2$ (note that this particular result does not hold in all cases of the original lemma given by Arlinghaus).

Proof. Arlinghaus only includes a thorough proof for one case (listed third in the lemma above) of the original lemma, omitting nearly all details for the remaining cases. He first indicates the necessary permutation structure for ψ and then confirms $\psi \in \mathrm{Aut}(\Gamma)$.

Given the extent of the arguments involved, we only include the description of each particular ψ which belongs to $\mathrm{Aut}(\Gamma)$ and does not commute with ϕ under the prescribed conditions:

Let π denote the product of transpositions (possibly) contained in the given automorphism of $\mathrm{Aut}(\Gamma)$. Recall the functions χ and λ as defined in the previous section.

1. If n is odd, then $\psi = \chi_\sigma$. Otherwise, $\psi = \chi_\sigma \pi$

2. $\psi = \lambda_\sigma$

3. If n is odd, then $\psi = \chi_\sigma\chi_\tau$. Otherwise, $\psi = \chi_\sigma\chi_\tau\pi$

4. If every symbol of σ is adjacent to one or none of the first five symbols in τ, then $\psi = \chi_\sigma\chi_\tau\pi$. If every symbol of σ is adjacent to two of the first five letters of τ, then $\psi = \chi_\sigma\lambda_\tau$

□

We can delineate the permutation structure of $\mathrm{Aut}(\Gamma)$ on the basis of even less restrictive conditions. We present the following lemma:

Lemma 2.1.3.2. Let Γ be a graph and $\phi \in \mathrm{Aut}(\Gamma)$. Suppose the cycle decomposition of ϕ contains the cycle types as listed in one of the following:

1. A cycle of length 2^n where $n \geq 1$, a cycle of length $4m$ where such that $\gcd(m, 2) = 1$ and $m \geq 1$, and other cycles of length coprime to $2m$.

2. A cycle of length $4n$ where $n \geq 1$, a cycle of length $4m$ where such that $\gcd(m, 2n) = 1$ and $m \geq 1$, and other cycles of length coprime to $2mn$.

Then there exists an automorphism $\psi \in \mathrm{Aut}(\Gamma)$ such that ψ and ϕ do not commute. Further, $|\psi| = 2$.

In the next lemma, we consider three graph constructions. The first has $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{p^k}$ for $p \geq 7$, the second has $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{p^k}$ for $p = 3$ or $p = 5$, and the third has $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{2^k}$ for $p = 2$ when $k > 1$. As we prove later, the third construction has $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_4$ when $k = 2$. The lower bound of $\alpha(G)$ for each such group is stated as a corollary.

Lemma 2.1.3.3. Let Γ be a graph. Suppose p is a prime and $k \geq 1$.

1. If $p \geq 7$, then $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{p^k}$ when Γ is defined as follows:

   (a) $V(\Gamma)$ is given by the union $X(p) \cup X'(p)$, where

      i. $X(p^k) = \{1, 2, \ldots, p^k\}$

      ii. $X'(p) = \{1', 2', \ldots, p'\}$

   (b) Let $i \in X(p^k)$ and $j \in X'(p)$. $E(\Gamma)$ is designated as follows:

$$\begin{cases} \{i, i+1\} & \text{for all } i, \text{ addition} \mod (p^k) \\ \{j', (j+1)'\} & \text{for all } j, \text{ addition} \mod (p) \\ \{i, (j+m)'\} & \text{for } m = -1, 0, 2 \text{ and } i \equiv j \mod \gcd(p^k, p) \end{cases}$$

2. If $p \geq 3$, then $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{p^k}$ when Γ is defined as follows:

   (a) $V(\Gamma)$ is given by the union $X(p^k) \cup X'(p) \cup X''(p)$, where

i. $X(p^k) = \{1, 2, \ldots, p^k\}$

ii. $X'(p) = \{1', 2', \ldots, p'\}$

iii. $X''(p) = \{1'', 2'', \ldots, p''\}$

(b) Let $i \in X(p^k)$, $j \in X'(p)$, and $r \in X''(p)$. $E(\Gamma)$ is designated as follows:

$$\begin{cases} \{i, i+1\}^* & \text{for all } i, \text{ addition} \mod (p^k) \\ \{j', (j+1)'\} & \text{for all } j, \text{ addition} \mod (p) \\ \{i, j'\} & \text{for } i \equiv j \mod \gcd(p^k, p) \\ \{i, r''\} & \text{for } i \equiv r \mod \gcd(p^k, p) \\ \{j', (r+m)''\} & \text{for } m = 0, 1 \text{ and } j \equiv r \mod p \end{cases}$$

3. If $p = 2$ and $k \geq 2$, then $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{2^k}$ when Γ is defined as follows:

(a) $V(\Gamma)$ is given by the union $X(p) \cup X'(p^k) \cup X''(p^k)$, where

i. $X(2) = \{1, 2\}$

ii. $X'(p^k) = \{1', 2', \ldots, (p^k)'\}$

iii. $X''(4) = \{1'', 2'', 3'', 4''\}$

(b) Let $i \in X(2)$, $j \in X'(p^k)$, and $r \in X''(4)$. $E(\Gamma)$ is designated as follows:

$$\begin{cases} \{j', (j+1)'\} & \text{for all } j, \text{ addition} \mod (p^k) \\ \{i, j'\} & \text{for } i \equiv j \mod \gcd(2, p^k) \\ \{i, r''\} & \text{for } i \equiv r \mod 2 \\ \{j', (r+m)''\} & \text{for } m = 0, 1 \text{ and } j \equiv r \mod \gcd(4, p^k) \end{cases}$$

* These edges need not be included when $k = 1$ and $p = 3$ or $p = 5$.

Proof. Sabidussi, Meriwether (unpublished), and Arlinghaus prove 1 [1, 14]. Arlinghaus states that 2 is a generalization of the constructions given by Sabidussi as well as Harary and Palmer [1, 9, 14]. We omit these arguments and prove 3 instead, observing that all such proofs would be extremely similar. Assume we have the construction of Γ given in 3. First, we show $\mathbb{Z}_4 \leq \mathrm{Aut}(\Gamma)$.

Consider an embedding of $\mathbb{Z}_4$ into S_{10}: $\mathbb{Z}_4 \cong \langle (12)(1'2'3'4')(1''2''3''4'') \rangle \leq S_{10}$(omitted here, this fact is easily checked), and observe that this cyclic group is the set

$$\{1,\ (12)(1'2'3'4')(1''2''3''4''),\ (1'3')(2'4')(1''3'')(2''4''),\ (12)(1'4'3'2')(1''4''3''2'')\}.$$

Recall Lemma 2.1.2.2. If $\varphi \in \mathrm{Aut}(\Gamma)$, then $\{x, y\} \in E(\Gamma)$ if and only if $\{x\varphi, y\varphi\} \in E(\Gamma)$. This adjacency preserving bijection holds even when the vertices $x, y \in V(\Gamma)$ occur within separate cycles of the decomposition of φ. Notice that the graph Γ has four distinct sets of edges.

Under any action of $\mathbb{Z}_4$ on $V(\Gamma)$, $E(\Gamma)$ is partitioned into four full edge orbits of Γ; edges are mapped to edges and non-edges to non-edges. Furthermore, each of these edge orbits corresponds to exactly one distinct set of edges of Γ.

The adjacencies of $E(\Gamma)$ are necessarily preserved because the permutation structure of the given embedding of $\mathbb{Z}_4$ respects the given construction of Γ. Hence,

$$\mathbb{Z}_4 \cong \langle (12)(1'2'3'4')(1''2''3''4'') \rangle \leq \mathrm{Aut}(\Gamma).$$

Now we show the reverse inequality. Recall that for a vertex $v \in V(\Gamma)$, we define the degree of v, denoted $\rho(v)$, as the total number of its neighbors. If we pick $v \in V(\Gamma)$, then either $\rho(v) = 3$, $\rho(v) = 4$, or $\rho(v) = 5$. Notice that since Γ contains three different degree types, the vertices of Γ are partitioned by degree; that is, each of these sets is invariant under any automorphism of $\mathrm{Aut}(\Gamma)$.

We use right multiplication in keeping with Arlinghaus. Again, we apply Lemma 2.1.2.2. Since $\mathbb{Z}_4 \cong \langle (12)(1'2'3'4')(1''2''3''4'') \rangle \leq \mathrm{Aut}(\Gamma)$, there exists $\varphi \in \mathrm{Aut}(\Gamma)$ such that $i\varphi = j$ for $i, j \in X$, $i, j \in X'$, or $i, j \in X''$. Suppose there exists $\psi \in \mathrm{Aut}(\Gamma)$ such that $\psi \neq \varphi$ and $i\psi = j$. Thus, $i\varphi\psi^{-1} = i$. However, we assert (and it suffices to show) that *only* the trivial automorphism of $\mathrm{Aut}(\Gamma)$ fixes a vertex of Γ. That is, we have $\varphi\psi^{-1} = 1$ so $\varphi = \psi$, implying $\psi \in \mathbb{Z}_4$ and, certainly, $\mathrm{Aut}(\Gamma) \leq \mathbb{Z}_4$.

Let $\psi \in \mathrm{Aut}(\Gamma)$. Without loss of generality, we consider the action of ψ on $1'$: $1'\psi \in \{1', 2', 3', 4'\}$. Then there is some $k \in \{0, 1, 2, 3\}$ such that $1'\psi\varphi^k = 1'$ with

$$\varphi = (12)(1'2'3'4')(1''2''3''4'').$$

We know that $\psi\varphi^k = ()$, and so $\psi = \varphi^{-k} \in \langle (12)(1'2'3'4')(1''2''3''4'') \rangle$, which implies $\psi \in \langle (12)(1'2'3'4')(1''2''3''4'') \rangle \cong \mathbb{Z}_4$. Hence, we must show that $\psi\varphi^{-k} = ()$ is the trivial automorphism for our reasoning to hold.

Consider the following neighbors of $1'$: 1, $1''$, and $2''$. It follows that both 1 and 2 are fixed as $\rho(1) = \rho(2) = 4$, and only this pair of vertices has this degree. Then $1''$, which is adjacent to 1, and $2''$, which is adjacent to 2, are fixed. Consequently, $2'$ and $4'$, both adjacent to 2, are fixed. The remaining three vertices, $3'$, $3''$, and $4''$, are similarly fixed.

Thus, if an automorphism fixes a vertex, the entire graph Γ is fixed. That is, $\psi\varphi^{-1}$ is the trivial automorphism and we have the reverse inequality as desired, i.e. $\mathrm{Aut}(\Gamma) \leq \mathbb{Z}_4$. Hence, $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_4$. Implicitly, we have also established an upper bound for minimality: $\alpha(G) \leq 10$. □

Corollary 2.1.3.1. Using the graphs constructed in the preceding lemma,

1. $\alpha(G) \leq p^k + p$ when $p \geq 7$ for prime p and $k \geq 1$

2. $\alpha(G) \leq p^k + 2p$ when $p = \in 3, 5$ and $k \geq 1$

3. $\alpha(\mathbb{Z}_{2^k}) \leq 2^k + 6$ when $k \geq 2$

In lieu of a formal proof, we remark that considering the constructions for Γ and appropriately applying the conditions listed in Lemma 2.1.3.1 and Lemma 2.1.3.2 provide the basis for the upper bound given above.

2.2 Cyclic Groups of Prime Power Order

As we stated in the introduction, the presence of certain cycle types in the automorphisms of $\mathrm{Aut}(\Gamma)$ prohibit the development of a concise theorem concerning $\alpha(G)$ for all finite abelian groups. In particular, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$ are excluded from several theorems regarding $\alpha(G)$ as well as minimal (vertex or edge) graphs. For example, Babai's theorem finding an upper bound for $\alpha(G)$ [2].

As we demonstrate later (and as we showed above), some generalizations can be made regarding cyclic groups of prime power order $n \geq 7$, where $n = p^k$ for prime p and $k \geq 1$. However, the exceptions forced by these problematic cycle types complicate such determinations for groups not of prime order.

We discuss these exceptions stemming from the structure of a $\mathbb{Z}_n$ graph when n has prime power order divisible by 2, 3, or 5.

2.2.1 Minimal Graphs for Cyclic Groups of Order 2^k, 3^k, and 5^k

First, we include a general theorem determining $\alpha(G)$ for $\mathbb{Z}_n$ graphs of this type; then we examine the specific minimal constructions of such graphs when $p^k = 3, 4$, or 5.

Theorem 2.2.1.1. Let Γ be a graph with $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{p^k}$. Suppose $p \in \{2, 3, 5\}$ and $k \geq 1$ is an integer. Then

$$\alpha(\mathbb{Z}_{p^k}) = \begin{cases} 2 & \text{if } p = 1,\ k = 1 \\ p^k + 2p & \text{if } p \neq 2,\ k \geq 1 \\ p^k + 6 & \text{if } p = 2,\ k > 1 \end{cases}$$

Proof. The result for $\mathbb{Z}_2$ is clear. Considering the second case, assume we have the stated conditions. The upper bound $\alpha(\mathbb{Z}_{p^k}) \leq p^k + 2p$ holds from from Corollary 2.1.3.4. Let Γ be a graph with $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_{p^k}$ such that $\varphi \in \mathrm{Aut}(\Gamma)$ and $\langle\varphi\rangle \cong \mathbb{Z}_{p^k}$. Then the disjoint cycle notation of φ must contain at least one cycle of length p^k, and all remaining cycles have length of some power of p. Further, applying condition 1 of Lemma 2.1.3.1, the decomposition of φ must contain another nontrivial cycle. If this cycle is length either $p = 3$ or $p = 5$, then conditions 3 and 4, respectively, of Lemma 2.1.3.1 forces a third nontrivial cycle. As a result, $|V(\Gamma)| \geq p^k + p + p = p^k + 2p$, and so $\alpha(G) \geq p^k + 2p$. On the other hand, if this second nontrivial cycle has length p^m with $m \geq 2$, then clearly $|V(\Gamma)| \geq p^k + p^m > p^k + 2p$. Thus, $\alpha(G) \geq p^k + 2p$, proving equality.

For the third case, assume we have the stated conditions and $n = 2^k$ for $k \geq 2$. The upper bound $\alpha(\mathbb{Z}_{2^k}) \leq p^k + 6$ holds from from Corollary 2.1.3.4. Let Γ be a graph with $\text{Aut}(\Gamma) \cong \mathbb{Z}_n$ such that $\varphi \in \text{Aut}(\Gamma)$ and $\langle\varphi\rangle \cong \mathbb{Z}_n$. Then the disjoint cycle notation of φ must contain at least one cycle of length n, and all remaining cycles have length of some power of 2. Since Γ and φ meet condition 1 of Lemma 2.1.3.1, another cycle must exist in the decomposition of φ, which is neither trivial nor a transposition. If the length of this cycle greater than 4, $|V(\Gamma)| \geq 2^k + 8$ and the result holds from the second case of the theorem. Otherwise, the length of this cycle is 4, in which case by Lemma 2.1.3.1 once again, a third nontrivial is forced to exist, i.e. $|V(\Gamma)| \geq 2^k + 4 + 2 = 2^k + 6$. Thus, $\alpha(\mathbb{Z}_{2^k}) \geq p^k + 6$, proving equality. □

Recall that F. Harary and E. Palmer published a paper regarding $\mathbb{Z}_n$ graphs that are both vertex and edge minimal, including graph constructions for the three special cases given in this section's first paragraph. While some of their results are actually specious [13], the minimal constructions for $\mathbb{Z}_3$ and $\mathbb{Z}_5$ are not in dispute and are verified by these authors, as well as Meriwether and Arlinghaus [1, 9].

In fact, Harary and Palmer successfully reduced the total number edges within Sabidussi's 1959 construction of a graph with $\text{Aut}(\Gamma) \cong \mathbb{Z}_3$. As Arlinghaus remarks in his memoir, their construction is a specific example of a graph with automorphism group $\mathbb{Z}_{p^n}$ and $p^n + 2p$ vertices for prime $p \geq 3$ and $n \geq 1$ [1]. We note, however, and subsequently show that such a construction is only minimal for $p = 3$ or $p = 5$.

Harary and Palmer's paper was reviewed by Sabidussi, who did not correct the main error in their construction of $\mathbb{Z}_4$. That is, $\alpha(G) = 10$, not 12, which Meriwether proved in 1963. Arlinghaus reaffirms this fact and we prove it here. His review only addresses the results based on two (false) theorems from his own paper, indicating that their conclusions based on his assumptions are questionable [13].

We now discuss the minimal constructions for the special cases $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$, specifically proving minimality for $\mathbb{Z}_4$. While the proof provided in Theorem 2.2.1.1 suffices to verify minimality, few authors have provided explicit details regarding these particular cases.

Theorem 2.2.1.2. Let Γ be a graph with $\text{Aut}(\Gamma) \cong \mathbb{Z}_n$ corresponding to one of the constructions given in Lemma 2.1.3.3, then

1. $\alpha(\mathbb{Z}_3) = 9$.

2. $\alpha(\mathbb{Z}_4) = 10$.

3. $\alpha(\mathbb{Z}_5) = 15$.

Proof. We verify the second statement; the other cases are proven in a similar manner. In Lemma 2.1.3.3 part 3, we confirm that the construction given has $\text{Aut}(\Gamma) \cong \mathbb{Z}_4$ and

$\alpha(G) \leq 10$; only the reverse inequality for minimality remains to be shown. We consider the other possible embeddings of $\mathbb{Z}_4$ into a subgroup of the symmetric group S_k for which $k \leq 10$: $\langle(1234)\rangle \leq S_4$, $\langle(1234)(56)\rangle \leq S_6$, $\langle(1234)(56)(78)\rangle \leq S_8$, $\langle(1234)(5678)\rangle \leq S_8$, and $\langle(1234)(56)(78)(9\,10)\rangle \leq S_{10}$. The remaining subgroup of S_{10} isomorphic to $\mathbb{Z}_4$ is what we show to be the correct embedding.

First, let Γ be a graph such that $\varphi \in \mathrm{Aut}(\Gamma)$ and $\langle\varphi\rangle = \langle(1234)\rangle$. If Γ is a square, then $\mathrm{Aut}(\Gamma) \cong D_8$, and, as its complement, Γ' must have the same automorphism group, $\mathrm{Aut}(\Gamma') \cong D_8$. If Γ is complete, $\mathrm{Aut}(\Gamma) \cong S_4$; likewise, $\mathrm{Aut}(\Gamma') \cong S_4$. We have exhausted all possible graphs on four symbols, none of which have the correct automorphism group, demonstrating $\alpha(G) > 4$.

When Γ is a graph on the stated number of symbols, we observe that the rest of the cyclic representations of $\mathbb{Z}_4$ meet the first condition of Lemma 2.1.3.1. Notice that Lemma 2.1.3.2 may also be similarly applied in some of these cases. As a result, an automorphism group which contains a permutation of the type given above must also contain another permutation which does not commute with the first, forcing $\mathrm{Aut}(\Gamma)$ to be non-abelian and $\alpha(G) \geq 10$.

Hence, for a graph Γ with $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_4$, the smallest possible embedding of $\mathbb{Z}_4$ into a subgroup of S_k is S_{10}. That is, $\mathbb{Z}_4 \cong \langle(12)(1'2'3'4')(1''2''3''4'')\rangle \leq S_{10}$ and, thus, $\alpha(G) = 10$. □

2.2.2 Minimal Graphs for Cyclic Groups of Order p^k

Finally, we exhibit the case which resolves all cyclic groups of prime power order.

Theorem 2.2.2.1. Let $p \geq 7$ be prime and $k \geq 1$ an integer. Then

$$\alpha(\mathbb{Z}_{p^k}) = p^k + p$$

We omit the proof; for similar arguments justifying the upper bound for $\alpha(G)$ in this case, see the proof provided for Theorem 2.2.1.1.

2.3 Cyclic Groups not of Prime Power Order

We provide the main theorem for determining $\alpha(G)$ in cyclic groups of nonprime power order, omitting Arlinghaus's extensive proof. Further, we note that the minimal $\mathbb{Z}_n$-graphs for composite n are often unions of graphs corresponding to the prime factors of n. For instance, $\mathbb{Z}_{24} = \mathbb{Z}_8 \times \mathbb{Z}_3$, so the minimal graph with $\mathbb{Z}_{24}$ automorphism group is the union of the graphs $\mathbb{Z}_{2^3}$ and $\mathbb{Z}_3$.

2.3.1 Minimal Graphs for Cyclic Groups not Prime Power Order

As we previously observed, powers of 2, 3 and 5 are the exceptions which prevent the formulation of general theorem to calculate $\alpha(\mathbb{Z}_n)$ for each n. The conditional structure of the theorem addresses all possible such decompositions of nonprime power orders and the resulting computational effects on $\alpha(G)$.

For the purposes of this theorem, we employ the following notation:

1. Let $n = 2^a 3^b 5^c p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}$ with $5 < p_i < p_j$ if $i < j$, p_i is a prime power for each i, and $1 \leq k_i$ for each i.

2. Let $T = \sum_{i=1}^{s} \alpha(\mathbb{Z}_{p_i^{k_i}})$ such that $\alpha(\mathbb{Z}_{p_i^{k_i}})$ is $\alpha(G)$ for each cyclic group order $p_i^{k_i}$ where $1 \leq i \leq s$.

Theorem 2.3.1.1. Let Γ be a graph with $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_n$, keeping all notation as defined above. Then

$$\alpha(G) = \begin{cases} T - 4 & \text{if } a = 2,\ b \geq 1,\ c = 1 \\ T - 3 & \text{if } a \neq 2,\ b \geq 1,\ c = 1 \\ T - 1 & \text{if } a = 2,\ b \geq 1,\ c \neq 1 \\ T - 1 & \text{if } a \geq 2,\ b = 1,\ c \neq 1 \\ T & \text{otherwise} \end{cases}$$

Briefly, we remark that Arlinghaus devotes Chapter 6 of his memoir to the statement and proof of this theorem. His extension of the cyclic results to all finite abelian groups constitutes the remainder of his memoir [1].

Chapter 3

Dihedral Groups

In each section of this chapter, the group G is considered to be an embedding of some dihedral group D_{2n} in a symmetric group S_k. Note that the dihedral groups under consideration have order $2n$. The symbol Γ refers to a graph whose automorphism group $\mathrm{Aut}(\Gamma)$ is isomorphic to G. Likewise, the group G of $\alpha(G)$ corresponds to the particular dihedral group embedding regarded in each section.

The problem of finding the fewest number of vertices of a graph with $\mathrm{Aut}(\Gamma) \cong D_{2n}$ was thought to have been solved by G. Haggard in 1973 [5]. However, a 1979 paper by D. McCarthy asserts that Haggard's results are valid for $\alpha(G)$ only when $n < 7$, $n \geq 5$ is a prime power, and possibly for $n = 12, 15, 20, 24, 30$. When his paper was published, the precise determination of these five cases was unknown [11]. A manuscript including these is soon to be published; see second paragraph below.

For $n \geq 5$, $\alpha(G) = n$ was erroneously assumed to hold for all values of n, based upon Haggard's claim that a set of n vertices exists within a corresponding graph of dihedral automorphism group whereby the set is cyclically permuted by any such $\varphi \in \mathrm{Aut}(\Gamma)$ with $|\varphi| = n$. Observe that $\langle\varphi\rangle$ generates the rotational subgroup of D_{2n}. Nonetheless, this "special set of generators" only exists when n is a prime power or twice an odd prime power[6, 11]. Other values of n do not force the inclusion of an n-vertex set in a dihedral graph, allowing $\alpha(G) \leq n$. Examples include these values of n: if $n = 77$, then $\alpha(D_{154}) = 36$; if $n = 1001$, then $\alpha(D_{2002}) = 62$.

Therefore, we present Haggard's determination of $\alpha(G)$ when $n = 3, 4$, or 6; $n \geq 5$ is a prime power; and $n \geq 8$ is twice an odd prime power. Secondly, we describe McCarthy's results for n not a prime power and not divisible by 2, 3, or 5. We also note that because of the importance of McCarthy's findings, the bulk of this chapter will be devoted to this case of n.

The remaining cases, however, have yet to be considered. We remark that authors C. Graves, S. Graves, and L.-K. Lauderdale have submitted a paper solving the case for D_{2n} when $4 \nmid n$. In addition, they are preparing for submission a manuscript that solves the case when $4|n$. Hence, all possible values of n for D_{2n} and the corresponding values of $\alpha(G)$ are investigated.

The authors of each of these works construct minimal graphs for their found values of $\alpha(G)$ (if correct), including proof of the graphs having the desired automorphism group.

While edge minimality (and maximality) is the main topic of Haggard's and McCarthy's papers [5, 11], we will only focus on the least number of vertices possible for a graph having $\mathrm{Aut}(\Gamma) \cong D_{2n}$. This is due in part to the difficulty of determining edge minimal graphs for a given $\alpha(G)$, especially when $\alpha(G) < n$ for D_{2n}. Thus, exhibiting separate cases for the possible values of n, we discuss $\alpha(G)$ for the dihedral group and the corresponding minimal graphs with $\mathrm{Aut}(\Gamma) \cong D_{2n}$. Moreover, we note that graphs discussed within these sections are *not* necessarily edge minimal.

3.1 Minimal Graphs for D_{2n} when $n \leq 6$

Theorem 3.1.0.1. For D_{2n}, let n equal $3, 4$, and 6, respectively. Then $\alpha(D_6) = 3$, $\alpha(D_8) = 4$, and $\alpha(D_{12}) = 5$.

As stated by Haggard, the result for D_6, which is isomorphic to S_3, follows from the 1968 work of L. Quintas on graphs with symmetric automorphism group. We note that the construction of a minimal graph having $\mathrm{Aut}(\Gamma) \cong D_6$ is either totally disconnected or complete [12] (see Chapter 6 for details on graphs with $\mathrm{Aut}(\Gamma) \cong S_n$).

Haggard shows that for D_{2n} where $4 \leq n \leq 10$, a graph with $\mathrm{Aut}(\Gamma) \cong D_8$ can be constructed by the union of two graphs, say Δ and Δ' (observe that $\langle (1234), (13) \rangle \cong D_8$. Assuming $\mathrm{Aut}(\Delta) \cong D_8$, then Δ' will be null, totally disconnected, or a 6-vertex asymmetric graph. Varying certain properties of Δ, e.g. connectedness, as well as its union with one of the three possible Δ' determines the number of edges for the construction of the desired D_8-graph. However, the least number of vertices for any such possible graph is 4, as listed in the theorem above.

Assuming a graph has D_{12} automorphism group, Haggard notes that it must be constructed on at least 5 vertices; otherwise, a D_{12} does not exist. More specifically, $D_{12} \cong S_2 \times S_3$. Having established $\alpha(D_{12}) = 5$, we further acknowledge that a 5-vertex totally disconnected graph has S_5 automorphism group, a conclusion again based on Quintas [12]. Thus, the construction of a minimal D_{12}-graph must have at least 1 edge, which Haggard shows is, in fact, exactly 1.

3.2 Minimal Graphs for D_{2n} when $n \geq 5$ is Prime Power

Recall that we have defined Γ to be a graph with dihedral automorphism group and that the the number of vertices for a minimal graph Γ is $\alpha(G)$.

Theorem 3.2.0.1. If $n = p^k$ for $p \geq 5$ is prime and $k \in \mathbb{N}$. Then $\alpha(D_{2n}) = n$.

For these values of n, D_{2n} is "directly indecomposable": a group which cannot be decomposed into the direct product of proper subgroups [5]. Under these conditions, Haggard shows that a Γ with $\mathrm{Aut}(\Gamma) \cong D_{2n}$ must contain a unique set of n vertices. Moreover, this set is cyclically permuted by any automorphism which can generate the rotational subgroup of D_{2n}, as we discussed in the introduction of this chapter. Since any construction of such a Γ must contain an n-vertex set, $\alpha(G)$ is at least n. Assuming the correct automorphism group, a minimal graph has $\alpha(G) = n$.

3.3 Minimal Graphs for D_{2n} when $n \geq 8$ and $n = 2p^k$ for Odd Prime p

As previously mentioned, any notational differences between our values for $\alpha(G)$ and McCarthy's arise from denoting the dihedral group as D_{2n} rather than D_n.

Theorem 3.3.0.1. Let $n = 2p^k$ where p is an odd prime and $n \geq 8$. Then $\alpha(G) = \dfrac{n}{2} + 2$.

For these values of n, Haggard notes that $D_{2n} = D_{2\cdot 2p^k} = D_{2p^k} \times \mathbb{Z}_2$. Of course, as we discussed in the former section, D_{2p^k} is directly indecomposable and Γ, again, must contain an n-vertex set. We also remark that $\alpha(\mathbb{Z}_2) = 2$ [1] (see Chapter 1 for details about graphs with cyclic automorphism group). As before, Haggard makes similar arguments for the construction of Γ but with an additional two points to account for $\mathbb{Z}_2$. If we asssume Γ has the given automorphism group, then a minimal graph must have $\alpha(G) = p^k + 2 = \dfrac{n}{2} + 2$.

3.4 Minimal Graphs for D_{2n} when n is not Prime Power and $2, 3, 5 \nmid n$

For the remainder of this section we assume n is not a prime power and its prime divisors are greater than 5. McCarthy determines $\alpha(G)$ for such n and further constructs a graph on $\alpha(G)$ vertices which indeed has $\mathrm{Aut}(\Gamma) \cong D_{2n}$. To aid in his proofs, many of which are combinatorial in nature, he defines an arithmetic function that we include here, $\omega(n)$, deviating slightly from his original notation:

$$\begin{aligned}
&\omega(rs) = \omega(r) + \omega(s), \text{where r, s are relatively prime} \\
&\omega(p) = 2p \\
&\omega(p^k) = p^k + 2p, \text{for a prime } p \text{ and k} > 1.
\end{aligned}$$

McCarthy then conducts the following procedure to find $\alpha(G)$: constructs a graph Γ on $\omega(n)$ vertices, whilst verifying $\mathrm{Aut}(\Gamma) \cong D_{2n}$; establishes $\alpha(G) \leq \omega(n)$; and finally confirms the reverse inequality to prove $\alpha(G) = \omega(n)$.

We summarize his results. Note that notational differences are intended for clarity. To build the graph Γ, McCarthy first constructs what he deems as "building blocks": i.e. several smaller graphs, denoted by Δ, and defined below.

Definition. Let $d, m > 5$ and neither d nor m are divisible by 2, 3, or 5. Let $d|m$ and suppose Δ a graph. Then

1. $\Delta(m, d)$ is a graph on $m + 2d$ vertices and $7m$ edges.

   (a) The vertex set of $\Delta(m, d)$ is given by the union $X(m) \cup X'(d) \cup X''(d)$, where

i. $X(m) = \{1, 2, \ldots, m\}$

ii. $X'(d) = \{1', 2', \ldots, d'\}$

iii. $X''(d) = \{1'', 2'', \ldots, d''\}$

(b) Let $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_d$. The edge set of $\Delta(m, d)$ is designated as follows:

$$\begin{cases} \{i, i+1\} & \text{for all } i,\ j \\ \{i, j'\} & \text{when } i \equiv\ j,\ j+1 \text{ or } j-2 \mod d \\ \{i, j''\} & \text{when } i \equiv\ j,\ j-1, \text{ or } j+2 \mod d \end{cases}$$

Under the stated conditions, the edges between the elements of $X(m)$ form a simple cycle of length m. We also note that the vertices of $X'(d) \cup X''(d)$ are only adjacent to vertices from $X(m)$. Moreover, each vertex $v \in X(m)$ has degree $\rho(v) = 8$ (i.e. 2 neighbors per vertex within the simple cycle and 3 additional neighbors each from $X'(d)$ and $X''(d)$). Each vertex v of either $X'(d)$ or $X''(d)$ has $\rho(v) = 3\dfrac{m}{d}$ (since $d|m$ and we are considering three equivalences for each i).

2. $\Delta(d)$ is a graph on $2d$ vertices and $5d$ edges.

(a) The vertices of $\Delta(d)$ belong to the union $X'(d) \cup X''(d)$ (as the two sets defined above).

(b) Let $i, j \in \mathbb{Z}_d$. The edges between vertices i and j of $\Delta(d)$ are then

$$\begin{cases} \{j', (j+1)'\} & \text{for all } j \\ \{j'', (j+1)''\} & \text{for all } j \\ \{i', j''\} & \text{when } i \equiv\ j,\ j-1, \text{ or } j+2 \mod d \end{cases}$$

Here, the conditions yield that $X'(d)$ and $X''(d)$ both form respective simple cycles of length d, and, given this construction, all vertices v of $\Delta(d)$ have degree $\rho(v) = 5$.

Consider an automorphism of Δ. If $\gamma \in \mathrm{Aut}(\Delta(m, d))$ exists such that $\gamma(x) \in X''(d)$ for all $x \in X'(d)$ and $\gamma(y) \in X'(d)$ for all $y \in X''(d)$, $X(m)$ must remain invariant. If neither X' nor X'' are interchanged under an automorphism, then both sets are invariant within themselves (regardless to type of Δ-graph). Of course, this invariance must respect the degree of each type of neighbor. For example, suppose v is a vertex of Δ under such an automorphism. If $\Delta = \Delta(m, d)$, v is either degree $\rho(v) = 8$ or $\rho(v) = 3\dfrac{m}{d}$. If the former case, the neighbors of v consist of two vertices of degree 8 and six vertices of degree $3\dfrac{m}{d}$, three each from X' and X''; the pair may be exchanged under the automorphism and likewise each set of three permuted among themselves. If the latter case, every neighbor of v is degree 8 (i.e. belongs to $X(m)$) and may be mapped to any other neighbor of v.

Similarly, every vertex v of $\Delta(d)$ are has $\rho(v) = 5$ and invariance holds in an identical manner (respecting X' and X'').

Concerning the second type of automorphsim, we state a lemma which will aid in showing $\mathrm{Aut}(\Gamma) \cong D_{2n}$ once Γ is constructed by a carefully selected set of Δ graphs.

Lemma 3.4.0.1. Let $d > 5$. If $\varphi \in \mathrm{Aut}(\Delta(m, d))$ and $\varphi(1) = 1$, then $\varphi = () \in \mathrm{Aut}(\Delta(m, d))$. If $\varphi \in \mathrm{Aut}(\Delta(d))$ and $\varphi(1') = 1'$, then $\varphi = () \in \mathrm{Aut}(\Delta(d))$.

Proof. Let u be a vertex in Δ and suppose $N(v)$ denotes the set of neighbors of u. Recall the definition of $N(v)$ from the introduction: $N(v) = \{u \in V : \{u, v\} \in E\}$. Since $\varphi \in \mathrm{Aut}(\Gamma)$ must preserve adjacencies, $N(v)$ is invariant under φ; that is, $\{\varphi(v) : u \in N(v)\} = N(v)$.

Now we observe the effects of φ on a specific vertex. If three consecutive vertices of a circuit in $\Delta(m, d)$ or $\Delta(d)$ are fixed, all vertices of a circuit are fixed. In order to show this, we exploit the fact that the intersection or union of an invariant set is itself invariant.

Suppose φ fixes $1 \in X(m)$ for $1 \in X(m)$ in $\Delta(m, d)$. We know from the previous paragraph that $N(1)$ is invariant under φ; the set $X(m)$ is invariant because all of its vertices are degree 8. Moreover, 2 and m are also invariant since $N(1) \cap X(m) = \{2, m\}$.

Now we consider the neighbors of 1, 2, and m which belong to $X''(d)$. Call each of these sets $N''(1)$, $N''(2)$, and $N''(m)$, respectively. Then

$$\begin{cases} N''(1) = N(1) \cap X''(d) = \{1'', 2'', (d-1)''\} \\ N''(2) = N(2) \cap X''(d) = \{2'', 3'', (d+2)''\} \\ N''(m) = N(m) \cap X''(d) = \{d'', 1'', (d-2)''\} \end{cases}$$

Again, we know $N''(1)$ is invariant under φ since both $N(1)$ and $X''(d)$ are invariant. If φ also fixes 2 and m, $N''(2)$ and $N''(m)$ are invariant. Otherwise, φ interchanges 2 and m, interchanging the sets $N''(2)$ and $N''(m)$ as well.

Further, we can apply this line of reasoning to $N''(1) \cap N''(2)$ and $N''(1) \cap N''(m)$. When these intersections are distinct, the sets are either invariant (if $\varphi(2) = 2$ and $\varphi(m) = m$) or interchanged (if $\varphi(2) = m$ and $\varphi(m) = 2$). However, interchanging 2 and m also forces their neighbors 3 and $m - 1$ to be interchanged under φ. Therefore the sets $N''(1) \cap N''(3)$ and $N''(1) \cap N''(m-1)$ must be interchanged.

Recall that based upon our choice of n for D_{2n}, we must have $d > 5$. Hence, the above intersections each contain a single point:

$$\begin{cases} N''(1) \cap N''(2) = \{2''\} \\ N''(1) \cap N''(m) = \{1''\} \\ N''(1) \cap N''(3) = \{1''\} \\ N''(1) \cap N''(m-1) = \{(d-1)''\} \end{cases}$$

Consequently, φ cannot simultaneously exchange $1''$ with both $2''$ and $(d-1)''$, forcing

φ to fix 2 and m. As another result, $1''$ and $2''$ are also fixed since $N''(1) \cap N''(2) = \{2''\}$ and $N''(1) \cap N''(m) = \{1''\}$.

The argument continues in this fashion: having examined the effects of fixing 1 under φ, one consequence of which was $\varphi(2) = 2$, we examine the neighbors of the fixed neighbors of 1. For example, we can recognize that because 2 is fixed, 3 and $3''$ are forced to be fixed and so on. Therefore, all vertices of $X(m)$ and $X''(d)$ are fixed by φ. McCarthy remarks that the reasoning for $X'(d)$ is extremely similar to what is given above and likewise for $\Delta(d)$; thus we omit the second argument and the nearly identical proof for $\Delta(d)$.

With confidence, we can conclude that if φ fixes either $1 \in X(m)$ when $\Delta = \Delta(m, d)$ or $1' \in X'(d)$ when $\Delta = \Delta(d)$, then φ fixes all vertices of Δ. $\square$

Haggard claims that one immediate consequence follows as a result of the preceding lemma. We state his assertion in the next theorem:

Theorem 3.4.0.1. Let $d > 5$. Then $\mathrm{Aut}(\Delta(m, d)) \cong D_{2m}$ and $\mathrm{Aut}(\Delta(d)) \cong D_{2d}$.

We omit the proof of this theorem. However, we prove similar such examples in Chapters 2 and 4. We now have the ability to construct Γ with $\mathrm{Aut}(\Gamma) \cong D_{2n}$. McCarthy contends that arguments analogous to those required for the theorem above justify his given construction. For $\alpha(D_{2n})$, recall the arithmetic function $\omega(n)$ as defined by McCarthy. In a simplification McCarthy's notation, these are the definitions for the Δ graphs that form Γ:

Definition. Let p be a prime.

1. When $k > 1$, $\Delta'(p^k) = \Delta(p^k, p)$
2. When $k = 1$, $\Delta'(p) = \Delta(p)$. Note $\Delta'(p)$ has $w(p)$ vertices and $\mathrm{Aut}(\Delta'(p)) \cong D_{2p}$ when $p > 5$.

To construct Γ, we let $n = p_1^{k_1} p_2^{k_2} \dots p_t^{k_t}$ and each $p_s^{k_s}$ be a distinct prime for $k_s > 0$ and $1 \leq s \leq t$; we assume every $p_s > 5$ and $t > 1$. For $1 \leq s \leq t$, we consider $\bigcup\limits_{s=1}^{t} \Delta'(p_s^{k_s})$. The vertex sets contained in the union of these copies of $\Delta'(p_s^{k_s})$ are disjoint between graphs. Now we create additional edges between the following sets: $X'(p_s)$ and $X'(p_r)$; $X''(p_s)$ and $X''(p_r)$, for $s \neq r$, $1 \leq s$, and $r \leq t$. An edge is added so that every element of the first set is now adjacent to every element in the second set. Together with the union above and additional edges between vertex sets, we establish the graph Γ.

Based on this construction, we can detail several important features of Γ, leading up to the proof that $\mathrm{Aut}(\Gamma) \cong D_{2n}$. Let R denote the set of all subscripts such that $k_r > 1$ and $1 \leq r \leq t$. Then each vertex v of $X'(p_s) \cup X''(p_s)$ has degree

$$\rho(v) = \begin{cases} 5 - p_s + \sum p_s, & s \notin R \\ 3\dfrac{p_s^{k_s}}{p_s} - p_s \sum p_s & s \in R \end{cases}$$

and every vertex in $X(p_s^{k_s})$ has degree 8 whenever $s \in R$. Additionally, the vertices of Γ total $w(n) = \sum w(p_s^{k_s}) = \sum p_s^{k_s} + 2p_s$.

Thus, for each $r \in \{1, \ldots, t\}$ such that $k_r > 1$, the vertex set of Γ comprises $X = \bigcup_{r \in R} X(p_s^{k_s})$ and $X' \cup X'' = X'(p_s) \cup X''(p_s)$ for $1 \leq s \leq t$. Notice that X contains vertices of exactly degree 8 because $p_s > 5$ and all elements not contained in X must have a larger degree. As a direct result, X and its complement $X' \cup X''$ are invariant under every automorphism of Γ. We now present McCarthy's second lemma which will enable us to show $\mathrm{Aut}(\Gamma) \cong D_{2n}$.

Lemma 3.4.0.2. Suppose $\varphi \in \mathrm{Aut}(\Gamma)$. Then the subset of the vertices of Γ X is invariant under φ, and, if $r \in R$, $X(p_r^{k_r})$ is also invariant. Additionally, one of the following is true:

1. X' and X'' are interchanged.

2. X' and X'' are invariant, and, if $r \in R$, $X'(p_s)$ and $X''(p_s)$ are invariant for all s.

While omitted here, full details of this lemma's proof may be found in McCarthy's paper [11]. Briefly, we remark that the effect of φ acting Γ is readily apparent, given the construction of such a graph. That is, the vertex set X of Γ comprises various disjoint unions of simple circuits, each of which remains invariant under φ. Moreover, the action of φ on the sets X' and X'', which are either interchanged or invariant, translates directly to the action of φ on the disjoint unions which make up each of these vertex sets.

We now define two particular permutations, φ and χ, as given by McCarthy. Notice that the mappings describe the behavior of φ and χ on the vertices of each $\Delta'(p_s^{k_s})$ contained in Γ under the respective permutation:

$$\begin{aligned}
&\varphi(i') = (i+1)',\ \varphi(i'') = (i+1)'' \\
&\chi(i') = (p_s - i)'',\ \chi(i'') = (p_s - i)' \ \text{ for all } i \in \mathbb{Z}_{p_s} \\
&\varphi(j) = j+1,\ \chi(j) = (p_s^{k_s} - j) \ \text{ for all } j \in \mathbb{Z}_{p_s^{k_s}} \text{ when } s \in R
\end{aligned}$$

As defined the powers of φ and χ act as rotations and reflections, respectively, of the given circuits of vertices. We note that $|\varphi| = n$, $|\chi| = 2$, and $\chi^{-1}\varphi\chi = (\chi^{-1}\chi)\varphi^{-1} = \varphi^{-1}$. Furthermore, each vertex set contained in Γ (i.e. X, X', and X'') remains invariant under φ, whereas under χ, X is invariant and either X' and X'' are interchanged or invariant.

Consider the subgroup of $\mathrm{Aut}(\Gamma)$ generated by φ and χ, which McCarthy denotes as A. We assert that $A \cong D_{2n}$ because of the relations given above and since every element of A is written as φ^k or $\varphi^k\chi$ with $k \in \{0, 1, 2, \ldots, n-1\}$ [8].

Applying the two previous lemmas, we can show that A is actually the entire automorphism group, i.e. $A = \mathrm{Aut}(\Gamma)$, thereby completing the proof that Γ has $\mathrm{Aut}(\Gamma) \cong D_{2n}$.

Theorem 3.4.0.2. Let Γ have the construction as given above. Then Γ has dihedral automorphism group.

Proof. Let φ and χ be defined as the automorphisms listed above. Then $\langle \varphi, \chi \rangle = A$, $A \cong D_{2n}$, and $A \leq \mathrm{Aut}(\Gamma)$. It remains to be shown that for any $\gamma \in \mathrm{Aut}(\Gamma)$, $\gamma \in A$.

Following the argument given by McCarthy, we consider $\chi^{-k}\gamma$. If X' and X'' are invariant under γ, let $k = 0$. Otherwise, γ interchanges X' and X'' and we let $k = 0$ (observing that χ also interchanges these two sets). Thus, by the second lemma, X' and X'' are invariant under $\chi^{-k}\gamma$. Similarly, $\chi^{-k}\gamma$ leaves $X'(p_s)$ and $X''(p_s)$ invariant. By the properties of φ (see definition above), this invariance also holds for $\varphi^{-i}\chi^{-k}\gamma$ for all $i \in \{0, 1, \ldots, n-1\}$.

Now, McCarthy states that if there exists an i of $\varphi^{-i}\chi^{-k}\gamma$ such that the vertices $1 \in X(p_s^{k_s})$ and $1' \in X'(p_s)$ are fixed for each $s \in R$ and $s \notin R$, respectively, then $\varphi^{-i}\chi^{-k}\gamma$ fixes Γ by the first lemma. In other words, if $\varphi^{-i}\chi^{-k}\gamma = 1$ then $(\varphi^i\chi^k\varphi^{-i}\chi^{-k})\gamma = \varphi^i\chi^k$.

Hence, $\gamma = \varphi^i\chi^k$ where $i \in \{0, 1, \ldots, n-1\}$ and $k \in \{0, 1\}$. Thus, $\gamma \in A$, implying $\mathrm{Aut}(\Gamma) \leq A$.

To find such an integer, we examine the necessary requirements. Under $\chi^{-k}\gamma$, $X(p_s^{k_s})$ is invariant, so $\chi^{-k}\gamma(1) = v_s \in X(p_s^{k_s})$. Now let v_s correspond to the smallest positive integer such that $\chi^{-k}\gamma(1') = v_s \in X'(p_s)$ when $s \notin R$. Then let i correspond to $i \cong v_s$ mod $p_s^{k_s}$. This integer i, McCarthy explains, exists according to the Chinese Remainder Theorem, since $p_1^{k_1}, p_2^{k_2}, \ldots, p_t^{k_t}$ are mutually coprime. Therefore, $\varphi^i(1) = v_s \in X(p_s^{k_s})$ and $\varphi^i(1') = v_s \in X'(p_s)$ whenever $s \in R$ and $s \notin R$, respectively, and, moreover, $\varphi^{-i}\chi^{-k}\gamma$ acts on the vertex sets of Γ in the desired way.

Hence, $\gamma \in A$ so $\mathrm{Aut}(\Gamma) = A \cong D_{2n}$. $\square$

Altogether, we have shown that Γ has the desired automorphism group, and, furthermore, we have that $\alpha(G) \leq \omega(n)$ since Γ has $\omega(n)$ vertices. The reverse inequality must hold true to establish $\alpha(G) = \omega(n)$. Because of the length and technical nature of McCarthy's proofs (most of which are not graph-theoretic), we summarize his results on this matter.

First, we state a general result: For any graph Γ with $A \leq \mathrm{Aut}(\Gamma)$, if $\varphi \in \mathrm{Aut}(\Gamma)$ such that each nontrivial orbit of A is left invariant, then there exists a $\varphi' \in \mathrm{Aut}(\Gamma)$ such that each nontrivial orbit of A is fixed but agrees everywhere else with φ.

Recall the definition of a directly indecomposable group; the dihedral group is directly indecomposable for all n not twice an odd prime power. Suppose Γ is a graph with $\mathrm{Aut}(\Gamma) \cong D_{2n}$. McCarthy verifies that if the rotational subgroup of D_{2n} is decomposable on the vertex set of Γ (say, into a direct sum of permutational subgroups A_1 and A_2 where $|A_1|, |A_2| > 2$), then there exists a nontrivial orbit of A_1 not invariant under φ. Hence, the number of vertices of Γ must be of the form given by $\omega(n)$.

Finally, McCarthy establishes the desired inequality, i.e. $\alpha(G) \geq \omega(n)$. Suppose A is a cyclic group of permutations acting on a set of n elements. Suppose $|A| = n'$ where $n' = p_1^{k_1}p_2^{k_2}\ldots p_r^{k_r}$ for $r > 1$ and each p_i is a distinct prime for all $i \in \{1, 2, \ldots, r\}$. If A does not have any direct summands of permuational subgroups of order $p_s^{k_s}$, then $n > \omega(n')$.

Therefore, McCarthy's construction of Γ has $\mathrm{Aut}(\Gamma) \cong D_{2n}$ with $\alpha(G) = \omega(n)$ when n is not a prime power and $2, 3, 5 \nmid n$.

Chapter 4

Quaternion Groups

In each section of this chapter, the group G is considered to be an embedding of the generalized quaternion group Q_{2^n} in a symmetric group S_k. The symbol Γ refers to a graph whose automorphism group $\mathrm{Aut}(\Gamma)$ is isomorphic to G. Likewise, the group G of $\alpha(G)$ corresponds to the embedding of the generalized quaternion group regarded in each section. Additionally we say that σ and τ are the generators of Q_{2^n}, whereas each had previously represented an individual cycle of a permutation belonging to $\mathrm{Aut}(\Gamma)$.

The authors Christina Graves, Stephen Graves, L.-K. Lauderdale have a manuscript in which they determine that 2^{n+1} is the minimum number of vertices for Γ when $n > 3$ [4]. Although beyond the scope of this survey, the authors also constructed a "smallest graph," a minimal graph which is constructed on the fewest number of edges. We note that a special case of smallest graph arises when $n = 3$ and is treated independently from all $n > 3$. Full details will be available upon publication.

4.1 Finding $\alpha(G)$

In order to provide a proof of the main results, we must first detail a few lemmas concerning the properties of Q_{2^n}.

Lemma 4.1.0.1. Suppose σ and τ are generators of Q_{2^n} as given in the following presentation: $Q_{2^n} = \langle \sigma, \tau : \sigma^{2^{n-1}} = 1 = \tau^4, \tau\sigma\tau^{-1} = \sigma^{-1}, \sigma^{2^{n-2}} = \tau^2 \rangle$. Then the only element of order two is $\sigma^{2^{n-2}} = \tau^2$, and each element in the set $Q_{2^n} \backslash \langle \sigma \rangle$ has order four.

Proof. Let $k \in \{0, 1, \dots, 2^{n-2} - 1\}$. Now every element in the set $Q_{2^n} \backslash \langle \sigma \rangle$ has the form $\sigma^k \tau$. We observe that because $(\sigma^k\tau)^2 \neq 1$ and $(\sigma^k\tau)^2 = \sigma^k\tau\sigma^k\tau = \sigma^k\sigma^k\tau\tau^{-1}\tau = \sigma^k\sigma^{-k}\tau\tau = \tau^2$, this element has neither order one nor two. Moreover, $\sigma^k\tau$ must have order four since $(\sigma^k\tau)^4 = ((\sigma^k\tau)^2)^2 = (\tau^2)^2 = \tau^4 = 1$.

To continue, we note that $|\sigma| = 2^{n-1}$, and, as the order of a cyclic group is equal to the order of its generator, we have $|\langle\sigma\rangle| = 2^{n-1}$. Thus, the subgroup $\langle\sigma\rangle$ of Q_{2^n} has order 2^{n-1}. Each element of $\langle\sigma\rangle$ is distinct, which indicates that the only element of order two within this subgroup must be $\sigma^{2^{n-2}}$, since $\sigma^{2^{n-2}} = \tau^2$.

As we have shown above, every element in $Q_{2^n} \backslash \langle \sigma \rangle$ must have order four, and since there is only one element in $\langle\sigma\rangle$ of order two, namely $\sigma^{2^{n-2}}$, we can conclude that the only element of order two in Q_{2^n} is $\sigma^{2^{n-2}}$. $\square$

Babai's result states $\alpha(G) \leq 2|G|$ for a finite group G other than cyclic groups of order 3, 4, or 5 [2]. In general, a vertex minimal graph with $\mathrm{Aut}(\Gamma) \cong Q_{2^n}$ will have at most 2^{n+1} vertices. In their manuscript, Graves et al. confirm that this bound is sharp.

For every automorphism group, there is some k so that the group can be written as a subgroup of S_k. The following lemma establishes a lower bound for such a k when the given automorphism group is isomorphic to Q_{2^n}; the result is due to Graves et al.

Lemma 4.1.0.2. If $k < 2^n$, then Q_{2^n} is not isomorphic to a subgroup of S_k.

Proof. Suppose for sake of contradiction that the statement above is false. Then there exists a faithful homomorphism $\phi : Q_{2^n} \to S_k$, where $k < 2^n$. In other words, the kernel of ϕ, $\ker_\phi$, must be trivial by the first isomorphism theorem:

$$\frac{Q_{2^n}}{\ker_\phi} = \frac{Q_{2^n}}{1} = Q_{2^n} \cong \phi(Q_{2^n}) = S_k.$$

By this homomorphism ϕ, Q_{2^n} acts on a set of symbols, say A, where $|A| = k$.

Now for each $a \in A$, consider the set

$$\mathrm{stab}_{Q_{2^n}}(a) = \{g \in Q_{2^n} : g \cdot a = a\},$$

and note that $\mathrm{stab}_{Q_{2^n}} \leq Q_{2^n}$. When we apply the Orbit-Stabilizer Theorem, we find

$$[Q_{2^n} : \mathrm{stab}_{Q_{2^n}}(a)] = \frac{|Q_{2^n}|}{|\,\mathrm{stab}_{Q_{2^n}}(a)|} = |\,\mathrm{orb}_{Q_{2^n}}(a)|.$$

However, since $|\,\mathrm{orb}_{Q_{2^n}}| < 2^n$, it is not a trivial subgroup, which implies that

$$\frac{|Q_{2^n}|}{|\,\mathrm{stab}_{Q_{2^n}}(a)|} \neq 2^n.$$

Moreover, $|\,\mathrm{stab}_{Q_{2^n}}(a)| \neq 1$. Thus, $\mathrm{stab}_{Q_{2^n}}(a)$ is also not a trivial subgroup of Q_{2^n}.

Furthermore, we state an important fact of the generalized quaternion group: every subgroup of Q_{2^n} is either cyclic or generalized quaternion. We have already shown that every generalized quaternion group contains a unique element of order two. Thus, according to Lagrange's Theorem, we know that every cyclic subgroup of Q_{2^n} must have even order. Moreover, as a general fact of cyclic groups, all cyclic groups of even order contain a unique element of order two.

Hence, the involution $\sigma^{2^{n-2}} = \tau^2$ is contained in every subgroup of Q_{2^n}, and so $\sigma^{2^{n-2}} \in \mathrm{stab}_{Q_{2^n}}(a)$ for each $a \in A$. Therefore, $\langle \sigma^{2^{n-2}} \rangle \in \ker_\phi$, meaning the kernel is not trivial. By definition, ϕ cannot not be faithful, which contradicts our original assumption. $\square$

Combining this result with Babai's, we have $2^n \leq \alpha(Q_{2^n}) \leq 2^{n+1}$. However, Graves et al. show that if we represent Q_{2^n} as a subgroup G of S_k for $2^n \leq k < 2^{n+1}$, and assume that

Γ is a graph with $G \leq \mathrm{Aut}(\Gamma)$. Then there is some $\gamma \in \mathrm{Aut}(\Gamma) \backslash G$. Hence, $\alpha(Q_{2^n}) = 2^{n+1}$. The graph constructed by the quaternion authors has the same vertex order as a graph produced by Babai's construction, but is in fact shown to also be edge minimal.

4.2 Constructing Γ with $\mathrm{Aut}(\Gamma) \cong Q_{2^n}$

Having established $\alpha(G)$, we reproduce here the construction of an edge minimal graph Γ with $\mathrm{Aut}(\Gamma) \cong Q_{2^n}$. Following Graves et al., we prove Γ it has the desired automorphism group. In order to discuss this proof, however, we first include the authors' construction.

Letting $n \geq 4$, suppose the graph Γ_1 has the vertex set

$$\mathrm{V}(\Gamma_1) = Q_{2^n} = \{1, x, \ldots, x^{2^{n-1}-1}, y, xy, \ldots, x^{2^{n-1}-1}y\}$$

and edge set

$$\mathrm{E}(\Gamma_1) = \{\{g, gy\} : g \in Q_{2^n}\}.$$

The map $\phi : Q_{2^n} \to Q_{2^n}$ defined by $\phi(x) = a$ and $\phi(y) = b$ is an isomorphism. Thus, $\phi(Q_{2^n})$ is an isomorphic copy of Q_{2^n} under ϕ. Letting $\overline{1} = a^0$, suppose the graph Γ_2 has the vertex set

$$\mathrm{V}(\Gamma_2) = \phi(Q_{2^n}) = \{\overline{1}, a, \ldots, a^{2^{n-1}-1}, b, ab, \ldots, a^{2^{n-1}-1}b\}$$

and an empty edge set. Finally, let Γ be the graph with the vertex set

$$V(\Gamma) = \mathrm{V}(\Gamma_1) \cup \mathrm{V}(\Gamma_2)$$

and edge set

$$E(\Gamma) = \mathrm{E}(\Gamma_1) \cup \{\{g, hc\} : g \in \langle x, y\rangle, \phi(g) = h, c \in \{\overline{1}, a, b\}\}.$$

having 2^{n+1} vertices and $4 \cdot 2^n$, i.e. 2^{n+2}, edges [4].

As before, they embed Q_{2^n} into a symmetric group, now of 2^{n+1} symbols, and define its generators σ and τ, which will permute the vertices of Γ. The authors then prove Γ has $\mathrm{Aut}(\Gamma) \cong Q_{2^n}$, showing that Γ is vertex minimal.

We include this theorem and its proof, with added detail, here.

Theorem 4.2.0.1. The graph Γ as defined in the construction above has $\mathrm{Aut}(\Gamma) \cong Q_{2^n}$.

Proof. Suppose Γ is a graph with $V(\Gamma)$ and $E(\Gamma)$ as listed above. We have previously stated that σ and τ are generators of Q_{2^n} as defined by the quaternion authors. It suffices to show that Q_{2^n} is a subgroup of $\mathrm{Aut}(\Gamma)$ and, likewise, that any element of $\mathrm{Aut}(\Gamma)$ can be written as an element of a set isomorphic to Q_{2^n}.

First, let $\omega \in Q_{2^n}$ and the map $\pi_\omega : V(\Gamma) \to V(\Gamma)$ be given by $\pi_\omega(v) = \omega(v)$ for all $v \in V(\Gamma)$. Because of this equivalence, each π_ω must have the same permutation structure as the elements of the embedded quaternion group $Q_{2^n} = \langle \sigma, \tau \rangle$. Observe, then, that the set

$$\{\pi, \pi_\sigma, \ldots, \pi_{\sigma^{2^{n-1}-1}}, \pi_\tau, \pi_{\sigma\tau}, \ldots, \pi_{\sigma^{2^{n-1}-1}\tau}\} = \{\pi_\omega : \omega \in Q_{2^n}\}$$

is clearly an an isomorphic copy of Q_{2^n}.

We also claim π_ω preserves the adjacency relations of $E(\Gamma)$. That is, $\pi_\omega(v)$ is an automorphism for all $v \in V(\Gamma)$. To demonstrate this fact, we include a table of the vertex types and their neighbors within Γ based on the given construction, letting $k \in \{0, 1, \ldots 2^{n-1} - 1\}$ and d represent integers modulo 2^{n-1}:

Vertex Type	Neighbors
x^k	$x^k y,\ x^{(k-2^{n-2})d} y,\ a^k,\ a^{(k+1)d},\ a^k b$
$x^k y$	$x^k,\ x^{(k-2^{n-2})d},\ a^{(k-2^{n-2})d},\ a^k b,\ a^{(k-1)d} b$
a^k	$x^k,\ x^{(k-1)d},\ x^{(k-2^{n-2})d} y$
$a^k b$	$x^k,\ x^k y,\ x^{(k+1)d} y$

Recall that according to the construction of Γ, $x^k,\ x^k y \in \mathrm{V}(\Gamma_1)$ and $a^k,\ a^k b \in \mathrm{V}(\Gamma_2)$. For all $v \in \mathrm{V}(\Gamma_1)$, $\rho(v) = 5$ (i.e. each v is degree 5), and every $v \in \mathrm{V}(\Gamma_2)$ has $\rho(v) = 3$. Thus, if π_ω is an automorphism of Γ, then $\mathrm{V}(\Gamma_1)$ and $\mathrm{V}(\Gamma_2)$ must be invariant under any $\omega \in Q_{2^n}$. We demonstrate this property by now including the full definition of σ and τ as given by the quaternion authors [4]. Note that all exponents of the symbols contained in the cycles of τ are taken modulo 2^{n-1}:

$$\sigma = (1,\ x,\ \ldots,\ x^{2^{n-1}-1})(y,\ xy,\ \ldots,\ x^{2^{n-1}-1}y)(1,\ a,\ \ldots,\ a^{2^{n-1}-1})(b,\ ab,\ \ldots,\ a^{2^{n-1}-1}b)$$

and

$$\tau = \prod_{i=0}^{2^{n-2}-1} (x^i,\ x^{-i}y,\ x^{2^{n-2}+i},\ x^{2^{n-2}-i}y)(a^i,\ a^{-i}b,\ a^{2^{n-2}+i},\ a^{2^{n-2}-i}b).$$

Since all of the cycles within σ and τ are disjoint with respect to each alphabet, vertices of degree 5 will only be permuted with vertices of degree 5 and, likewise, for degree 3 vertices. Thus, $\mathrm{V}(\Gamma_1)$ and $\mathrm{V}(\Gamma_2)$ are invariant under π_ω.

Lastly, π_ω must preserve the adjacency relations of Γ when permuting vertices within these invariant sets. We demonstrate this property by briefly describing how a vertex, say x^k, and its neighbors are mapped under σ and τ. Again, $k \in \{0, 1, \ldots 2^{n-1} - 1\}$ and all powers are taken $\mod 2^{n-1}$.

The table above lists the neighbors of x^k. Under σ, x^k is sent to x^{k+1}. Of course, then, we also have $\sigma(x^k y) = x^{k+1} y$; $\sigma(x^{k-2^{n-2}} y) = x^{(k+1)-2^{n-2}} y$; $\sigma(a^k) = a^{k+1}$; $\sigma(a^{k+1}) = a^{k+2}$; and $\sigma(a^k b) = a^{k+1} b$, which were precisely the neighbors of x^{k+1} before the permutation. Thus, all of the adjacencies of x^k have been preserved. Similarly, any power of σ will preserve adjacencies in this manner; a fact which is easily checked since each vertex will be moved in increasing order of exponent modulo 2^{n-1}.

Under τ, x^k is sent to $x^{-k}y$, so $\tau(x^ky) = x^{-(k+2^{n-2})}$; $\tau(x^{k-2^{n-2}}y) = x^{-k}$; $\tau(a^k) = a^{-k}b$; $\tau(a^{k+1}) = a^{-(k+1)}b$; and $\tau(a^kb) = a^{-(k+2^{n-2})}$. From the table, we can see that x^{-k} is a vertex of the form x^ky, and plugging in $-k$ yields the same neighbors of x^k which have just been found under τ.

Finally, similar arguments can be made for all $\sigma^k\tau$ where $k \in \{1, 2, \ldots, 2^{n-1}-1\}$, as τ permutes the vertices of Γ in the way shown above and then σ shifts the vertices again according to their exponents. Both permutations, therefore, move a vertex and all of its corresponding neighbors so that the adjacency relations of $E(\Gamma)$ are maintained.

To summarize, for any $\omega \in Q_{2^n}$, π_ω acts as an automorphism of Γ for all $v \in V(\Gamma)$. Thus, we have that $\{\pi_\omega : \omega \in Q_{2^n}\} \cong Q_{2^n}$ is a subgroup of $\mathrm{Aut}(\Gamma)$.

For the second half of this proof, we show that an element of $\mathrm{Aut}(\Gamma)$ can be written as π_ω for some $\omega \in Q_{2^n}$, thereby confirming $Q_{2^n} \cong \mathrm{Aut}(\Gamma)$. We have already confirmed that $\mathrm{V}(\Gamma_1)$ and $\mathrm{V}(\Gamma_2)$ are invariant under any chosen automorphism of $\mathrm{Aut}(\Gamma)$. Moreover, since Q_{2^n} is transitive, a property of the generalized quaternion group, and $Q_{2^n} \leq \mathrm{Aut}(\Gamma)$, there must exist an automorphism between any two vertices of either $\mathrm{V}(\Gamma_1)$ or $\mathrm{V}(\Gamma_2)$. Simply put, for any $v, v' \in \mathrm{V}(\Gamma_1)$, or $\mathrm{V}(\Gamma_2)$, there exists $\phi \in \mathrm{Aut}(\Gamma)$ where $\phi(v) = v'$.

Suppose without loss of generality, we have $\phi \in \mathrm{Aut}(\Gamma)$ as given above and acting on $v, v' \in V(\Gamma)$ as stated. Further, suppose there exists $\psi \in \mathrm{Aut}(\Gamma)$ such that $\psi \neq \phi$ and $\psi(v) = v'$. Thus, $\psi^{-1}\phi(v) = v$. As with the quaternion authors, however, we assert that only the trivial automorphism of $\mathrm{Aut}(\Gamma)$ fixes a vertex of Γ, which forces $\psi = \phi$, implying $\psi \in Q_{2^n}$ and, certainly, $\mathrm{Aut}(\Gamma) \leq Q_{2^n}$.

The rest of the proof follows exactly from C. Graves, S. Graves, and L.-K. Lauderdale. In short the authors show that if an automorphism, say χ, fixes any vertex of Γ, then all vertices of Γ are fixed, i.e. χ must be the trivial automorphism. This consequence arises from the properties of vertices of the induced subgraphs of Γ. Examples of such subgraphs are also featured in the quaternion authors' proof.

Consider the subgraphs located sequentially outward from a given fixed vertex. Within each of these subgraphs, the vertices are either forced to be fixed or lie in an invariant set. Proceeding outward in this fashion from the original fixed vertex, more and more vertices of the invariant sets become necessarily fixed, until, ultimately, the entire graph of Γ is fixed.

Therefore, $\psi^{-1}\phi(v) = \chi(v) = v$, implying $\psi = \phi$, and finally, $\mathrm{Aut}(\Gamma) \cong Q_{2^n}$. □

For $n \geq 4$, minimal graphs will have the construction given above on 2^{n+1} vertices.

We note, however, that the construction differs slightly for the case $n = 3$: a minimal graph with $\mathrm{Aut}(\Gamma) \cong Q_{2^n}$ where $n = 3$ has 16 vertices and greater than 2^{n+2} edges. In particular, 44 edges, rather than 32 [4]. The full construction is featured in the last section of the quaternion paper, along with a full proof utilizing the method of exhaustion.

Chapter 5

Hyperoctahedral, Symmetric, and Alternating Groups

We devote this chapter to the remaining groups which have not yet been discussed and for whom the values of $\alpha(G)$ are known.

Hence for each section of this chapter, the group G is considered to be an embedding of the given group. in a symmetric group S_k. The symbol Γ refers to a graph whose automorphism group $\mathrm{Aut}(\Gamma)$ is isomorphic to G. Likewise, the group G of $\alpha(G)$ corresponds to the particular group embedding regarded in each section.

We denote each group as follows: the hyperoctahedral group, $\mathbb{Z}_2 \wr \mathrm{S}_n$, of order $2^n n!$ is H_n; the symmetric group of order $n!$ is S_n; and the alternating group of order $\dfrac{n!}{2}$ is A_n.

For the first section, we consider $\alpha(G)$ for graphs having hyperoctahedral automorphism group, which follows as a consequence of G. Haggard, D. McCarthy, and A. Wohlgemuth's results concerning "extremal edge problems" for graphs of this type.

Next, we summarize L. Quintas's findings as they relate to $\alpha(G)$ for Γ with $\mathrm{Aut}(\Gamma) \cong S_n$ and include a brief proof concerning the construction of such a graph. While the focus of his paper centers on determining edge minimality for graphs with symmetric automorphism group, vertex minimality, though not directly mentioned, can be construed from his work.

In the last section of the chapter, we condense a paper by M. Liebeck. He provides a full examination of graphs having either alternating or a particular finite classical automorphism group. However, we remark that Liebeck, in the cases of the set of finite classical groups, only establishes a lower bound for $\alpha(G)$ and does not attempt the construction of a graph, citing the difficulty of the problem. For this reason, we only include his results for A_n.

5.1 Minimal Graphs with $\mathrm{Aut}(\Gamma) \cong H_n$

As a result of Frucht, we know a graph with hyperoctahedral automorphism group exists [3]. Haggard et al. do not construct such a graph; rather, they assume a graph Γ has $\mathrm{Aut}(\Gamma) \cong \mathrm{S}_n \wr \mathbb{Z}_p$, for a prime p, and examine the structure imposed on Γ by $\mathrm{Aut}(\Gamma)$ [7]. The framework of Γ is necessitated by the properties of the automorphism group acting on its vertex set.

Restricting $p = 2$, we extrapolate $\alpha(G)$ from their results, noting $H_n = \mathrm{S}_n \wr \mathbb{Z}_2$.

Lemma 5.1.0.1. Let Γ be a graph with $\mathrm{Aut}(\Gamma) \cong \mathrm{S}_n \wr \mathbb{Z}_p$, where $n > 1$ and p is prime. Then $|V(\Gamma)| \geq np$.

Proof. A lemma stated and proved by Haggard et al. affirms that if $\mathrm{S}_n \wr \mathbb{Z}_p$ acts faithfully on a set, then the set must contain at least np elements [7].

Each action of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$ induces a permutation representation of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$. If the permutation representation associated with an action is injective, the action is faithful.

Since only the trivial automorphism can fix the graph Γ, the kernel of an action of $\mathrm{S}_n \wr \mathbb{Z}_p$ on $V(\Gamma)$ is trivial, i.e. faithful. Thus, given $\mathrm{Aut}(\Gamma) \cong \mathrm{S}_n \wr \mathbb{Z}_p$, we have $|V(\Gamma)| \geq np$.

Full graph theoretic details and an alternate arithmetic argument are provided in the Haggard et al. paper [7]. □

Haggard et al. state that as an "immediate consequence" of the above lemma, a graph with $\mathrm{S}_n \wr \mathrm{Z}_p$ automorphism group cannot exist on fewer than np vertices, and, likewise, an H_n-graph cannot have less than $2n$ vertices [7].

Theorem 5.1.0.1. *Let Γ be a graph with $\mathrm{Aut}(\Gamma) \cong H_n$. Then $\alpha(G) = 2n$.*

Proof. By the previous lemma, $|V(\Gamma)| \geq 2p$ and no graph with hyperoctahedral group exists for $V(\Gamma) < 2n$. Hence, $\alpha(G) = 2n$, the minimum value for which an H_n-graph can exist. Thus, $\alpha(G) = 2n$ for a graph Γ with $\mathrm{Aut}(\Gamma) \cong H_n$. □

Interestingly, we remark that the results for $n = 2$ and $n = 3$, which comply with the value given above, also represent unique cases of the hyperoctahedral group, since $\mathrm{H}_2 \cong \mathrm{D}_8$ and $\mathrm{H}_3 \cong \mathrm{S}_2 \times \mathrm{S}_4$. Observe that $\alpha(\mathrm{H}_2) = \alpha(\mathrm{D}_8) = 4$ (see chapter 3) and $\alpha(\mathrm{H}_3) = \alpha(S_2) + \alpha(\mathrm{S}_4) = 6$ (see succeeding section).

5.2 Minimal Graphs with $\mathrm{Aut}(\Gamma) \cong S_n$

As we remarked in the introduction of this chapter, L. Quintas indirectly determines $\alpha(G)$ for a graph with symmetric automorphism group. Within the proof of his main theorem on edge minimal graphs of symmetric automorphism group, he explains that no S_n graph exists on fewer than n vertices and mentions that the only S_n graphs possible on n vertices are totally disconnected or complete.

We present his results here as formal statement and proof, including $\alpha(G)$ as a corollary.

Theorem 5.2.0.1. *A graph Γ on n vertices has $\mathrm{Aut}(\Gamma) \cong S_n$ if and only if Γ is either totally disconnected or complete.*

Proof. For $n \leq 3$, the assertion above clearly holds. Let Γ be a graph on $n > 3$ vertices with symmetric automorphism group. Of course, every element of S_n is a permutation of $V(\Gamma)$ which must preserve the adjacencies of $E(\Gamma)$. Suppose for sake of contradiction that Γ is neither totally disconnected nor complete.

Then there exist vertices $v, v', v'' \in V(\Gamma)$ for which $\{v, v'\} \in E(\Gamma)$ but $\{v, v''\} \notin E(\Gamma)$. However, S_n is transitive, containing $\binom{n}{2}$ transpositions which fix all but two elements. That is, there exists a unique $\phi \in S_n$ such that $\phi(v'') = v'$ and all other vertices of $V(\Gamma)$ are fixed. Since we assumed $\{v, v''\} \notin E(\Gamma)$, we have a contradiction. Thus, Γ is either totally disconnected or complete.

Now we proceed with the second half of the proof. Let Γ be a graph on $n > 3$ vertices.

Case 1: Suppose Γ is totally disconnected. Since $E(\Gamma) = \emptyset$, no adjacencies occur, so adjacencies are preserved under all permutations of $V(\Gamma)$. Thus, all possible permutations of $\mathrm{Aut}(\Gamma)$ may be written in the form of an element of S_n. Further, $|\mathrm{Aut}(\Gamma)| = n! = |S_n|$. We conclude $\mathrm{Aut}(\Gamma) \cong S_n$.

Case 2: Suppose Γ is complete. The argument follows nearly identically to one given above, with proper modifications to the edge set of Γ.

Therefore, an totally disconnected or complete graph on n vertices has S_n automorphism group. ☐

Corollary 5.2.0.1. A minimal graph with symmetric automorphism group has $\alpha(G) = n$.

Proof. First, an S_n graph cannot exist on fewer than n vertices [1]: too few vertices are present in order to attain the required amount of permutations [12]. We can thereby establish $\alpha(G) \geq n$.

Moreover by the theorem above, a graph with symmetric group exists on exactly n vertices.

Hence, the minimal number of vertices possible for an S_n graph is n. ☐

5.3 Minimal Graphs with $\mathrm{Aut}(\Gamma) \cong A_n$

For $n \geq 13$, Liebeck constructs minimal graphs having alternating automorphism group [10]. We summarize his findings and refer the reader to Liebeck's paper for full details of the construction of these graphs and proof of their minimality [10].

We note that Liebeck first corroborates (for $n \geq 23$) a conclusion from Babai: A_n graphs have $\alpha(G) \geq c^n$ for some constant $c > 1$. In particular, Liebeck concludes

$$\alpha(G) \geq \frac{1}{2}\binom{n}{\lfloor n/2 \rfloor},$$

where $\lfloor \cdot \rfloor$ is the floor function. He indicates that his lower bound for $\alpha(G)$ follows easily from Babai's assertion, since an application of Stirling's approximation yields

$$\frac{1}{2}\binom{n}{\lfloor n/2 \rfloor} \sim \frac{2^n}{\sqrt{2\pi n}},$$

and, clearly, $2^n/\sqrt{2\pi n} \geq c^n$ where $c > 1$.

Although he constructs Γ with $\mathrm{Aut}(\Gamma) \cong A_n$ for all n larger than 7, Liebeck only proves minimality for $n \geq 13$. As such, we only include these values of n in the following theorem:

[1] Note that $|S_n| = n!$.

Theorem 5.3.0.1. Let Γ be a graph with alternating automorphism group. Then for $n \geq 13$, we have

$$\alpha(G) = \begin{cases} 2^n - n - 2 & \textit{when } n \equiv 0, 2 \mod 4 \\ 2^n + \binom{n}{n/2} - n - 2 & \textit{when } n \equiv 1 \mod 4 \\ 2^n + 2\binom{n}{(n-1)/2} - n - 2 & \textit{when } n \equiv 3 \mod 4 \end{cases},$$

taking all values of n modulo 4 [10].

Chapter 6

Conclusion

Many finite groups have yet to be thoroughly investigated (or considered at all). For example, the author M. Liebeck only establishes a lower bound of $\alpha(G)$ for several finite classical groups. However, as the main goal of our survey, we have explored all known values of $\alpha(G)$ for the six finite groups found in the preceeding chapters. Again, when possible, we included the construction of the minimal graph Γ having $\mathrm{Aut}(\Gamma) \cong G$.

Given the extent of the lemmas and theorems required, we cannot provide a concise synopsis for these results. However, as an aid to those readers only concerned with the conclusions (i.e. $\alpha(G)$ and minimal graphs) for a particular finite group (or groups), we include a table. For this table, we use commas between pages to denote separate results. Let G be the given finite group of the respective chapter and Γ be a graph with $\mathrm{Aut}(\Gamma) \cong G$:

G	page(s) with $\alpha(G)$ values	page(s) with construction of minimal Γ
$\mathbb{Z}_n$	pgs. 13, 15, 16	pgs. 10-12, 14
D_{2n}	pgs. 18-19, 25	pgs. 20-21, 22-23, 24-25
Q_{2^n}	pg. 29	pgs. 29-32, 32
H_n	pg. 34	N/A
S_n	pg. 35	pg. 35
A_n	pg. 36	see [10]

As mentioned in previous chapters, some authors have also considered smallest graphs, a combined notion of vertex and edge minimality within a graph of given automorphism group. With the knowledge of $\alpha(G)$ (typically a prerequisite) now known for a number of finite groups, research regarding smallest graphs may progress further.

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