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APPROXIMATE FORMULA FOR THE H/V RATIO OF RAYLEIGH WAVES IN INCOMPRESSIBLE ORTHOTROPIC HALF-SPACES COATED BY A THIN ELASTIC LAYER

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Abstract. This paper is concerned with the propagation of Rayleigh waves in an incompressible orthotropic elastic half-space coated with a thin incompressible orthotropic elastic layer. The main purpose of the paper is to establish an approximate formula for the Rayleigh wave H/V ratio (the ratio between the amplitudes of the horizontal and vertical displacements of Rayleigh waves at the traction-free surface of the layer). First, the relations between the traction amplitude vector and the displacement amplitude vector of Rayleigh waves at two sides of the interface between the layer and the half-space are created using the Stroh formalism and the effective boundary condition method. Then, an approximate formula for the Rayleigh wave H/V ratio of third-order in terms of dimensionless thickness of the layer has been derived by using these relations along with the Taylor expansion of the displacement amplitude vector of the thin layer at its traction-free surface. It is shown numerically that the obtained formula is a good approximate one. It can be used for extracting mechanical properties of thin films from measured values of the Rayleigh wave H/V ratio.

Keywords: Rayleigh waves, the Rayleigh wave H/V ratio, incompressible orthotropic elastic half-space, thin incompressible orthotropic elastic layer, approximate formula for the Rayleigh wave H/V ratio.

1. INTRODUCTION

As addressed by Junge et al. [1], the H/V ratio (the ratio between the amplitudes of the horizontal and vertical displacements of Rayleigh waves at the traction-free surface of half-spaces) is more sensitive than the Rayleigh wave velocity as an indicator of the state of stress, and, in contrast to the Rayleigh wave velocity, it is reference free. Therefore, formulas for the Rayleigh wave H/V ratio are a powerful tool for nondestructively evaluating prestress of structures before and during loading. The Rayleigh wave H/V ratio is also an important parameter which reflects fundamental properties of the elastic material [2]. It can be thus used for the nondestructive evaluation of the elastic constants of material [3], besides its well-known applications in seismology [4,5].

While a large number of formulas for the Rayleigh wave velocity have been derived, see for examples, [6–16], only few formulas for the Rayleigh wave H/V ratio have been obtained. They are, for example, the exact Rayleigh wave H/V ratio formula for a compressible layered half-space with traction-free surface [2], the exact and an approximate formula for that model of incompressible media [17]. However, these formulas are only for the isotropic media.

In this paper, an approximate formula of third order for the Rayleigh wave H/V ratio has been established for an incompressible orthotropic elastic half-space coated by a thin incompressible orthotropic elastic layer. This formula is derived by using the relations between the traction and displacement amplitude vectors of Rayleigh waves at two sides of the welded interface between the layer and the half-space, along with the Taylor expansion of the displacement amplitude vector of the thin layer at its traction-free surface. It is shown numerically that the obtained approximate formula is a good approximation.

2. RELATIONS BETWEEN THE TRACTION AND DISPLACEMENT AMPLITUDE VECTORS AT TWO SIDES OF THE INTERFACE

2.1. Basic equations for an incompressible orthotropic elastic layer in matrix form

Consider an elastic half-space $x_2 \geq 0$ coated by a thin elastic layer $-h \leq x_2 \leq 0$. Both the layer and half-space are assumed to be incompressible, orthotropic and they are in welded contact with each other. Note that same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a bar if pertaining to the layer. We are interested in the plain strain so that

$$u_i = u_i(x_1, x_2, t), \quad \bar{u}_i = \bar{u}_i(x_1, x_2, t), \quad i = 1, 2, \quad u_3 = \bar{u}_3 \equiv 0, \tag{1}$$

where t is the time. Since the material of the layer is incompressible and orthotropic, the strain-stress relations are [8]

$$\bar{\sigma}_{11} = -\bar{p} + \bar{c}_{11}\bar{u}_{1,1} + \bar{c}_{12}\bar{u}_{2,2}, \quad \bar{\sigma}_{22} = -\bar{p} + \bar{c}_{12}\bar{u}_{1,1} + \bar{c}_{22}\bar{u}_{2,2}, \quad \bar{\sigma}_{12} = \bar{c}_{66}(\bar{u}_{1,2} + \bar{u}_{2,1}), \tag{2}$$

where $\bar{\sigma}_{ij}$, $\bar{p}$ and $\bar{c}_{ij}$ are respectively the stress, the hydrostatic pressure associated with the incompressibility constraint and the material constants, commas indicate differentiation with respect to the spatial variables x_k. In the absence of body forces, the equations of motion are

$$\bar{\sigma}_{11,1} + \bar{\sigma}_{12,2} = \bar{\rho}\ddot{\bar{u}}_1, \quad \bar{\sigma}_{12,1} + \bar{\sigma}_{22,2} = \bar{\rho}\ddot{\bar{u}}_2, \tag{3}$$

where $\bar{\rho}$ is the mass density, a dot signifies differentiation with respect to the time t. The incompressibility gives

$$\bar{u}_{1,1} + \bar{u}_{2,2} = 0. \tag{4}$$

From (1)–(4) we have

$$\begin{bmatrix} \bar{\mathbf{u}}' \\ \bar{\mathbf{t}}' \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{t}} \end{bmatrix}, \tag{5}$$

where $\bar{\mathbf{u}} = \begin{bmatrix}\bar{u}_1 & \bar{u}_2\end{bmatrix}^T$, $\bar{\mathbf{t}} = \begin{bmatrix}\bar{\sigma}_{12} & \bar{\sigma}_{22}\end{bmatrix}^T$, the symbol "T" indicates the transpose of a matrix, the prime signifies the derivative with respect to x_2 and

$$M_1 = \begin{bmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{bmatrix},\ M_2 = \begin{bmatrix} \dfrac{1}{\bar{c}_{66}} & 0 \\ 0 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} -\bar{\delta}\partial_1^2 + \bar{\rho}\partial_t^2 & 0 \\ 0 & \bar{\rho}\partial_t^2 \end{bmatrix}, \tag{6}$$

here we use the notations $\partial_1 = \partial/\partial x_1$, $\partial_1^2 = \partial^2/\partial x_1^2$, $\partial_t^2 = \partial^2/\partial x_t^2$. Eq. (5) is called the matrix equation for an incompressible orthotropic elastic layer in plane strain.

2.2. Stroh formalism for an incompressible orthotropic elastic layer

Now we consider the propagation of a plane wave traveling in the x_1-direction with velocity c (> 0) and wave number k (> 0). Then, displacement components of the wave are sought in the form

$$\bar{u}_n = \bar{U}_n(y)e^{ik(x_1-ct)}, \quad \bar{\sigma}_{n2} = ik\bar{\Sigma}_n(y)e^{ik(x_1-ct)},\ n = 1,2,\ y = k\,x_2. \tag{7}$$

Substituting (7) into (5) yields

$$\xi' = iN\xi,\ y \in [-\varepsilon,\ 0],\ \varepsilon = kh, \tag{8}$$

where the prime signifies differentiation with respect to y and

$$\xi = \begin{bmatrix} \bar{U} \\ \bar{\Sigma} \end{bmatrix}, \quad \bar{U} = \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix}, \quad \bar{\Sigma} = \begin{bmatrix} \bar{\Sigma}_1 \\ \bar{\Sigma}_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \tag{9}$$

in which the matrices N_k are given by

$$N_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\ N_2 = \begin{bmatrix} \dfrac{1}{\bar{c}_{66}} & 0 \\ 0 & 0 \end{bmatrix},\ N_3 = \begin{bmatrix} -\bar{\delta} + \bar{\rho}c^2 & 0 \\ 0 & \bar{\rho}c^2 \end{bmatrix},\ N_4 = N_1, \tag{10}$$

where $\bar{\delta} = \bar{c}_{11} + \bar{c}_{22} - 2\bar{c}_{12}$. Eq. (8) is called the Stroh formalism [18] for an incompressible orthotropic elastic layer. From (8) it follows

$$\xi^{(n)} = i^n N^n \xi,\ N^n := \begin{bmatrix} N_1^{(n)} & N_2^{(n)} \\ N_3^{(n)} & N_4^{(n)} \end{bmatrix},\ y \in [-\varepsilon,\ 0]. \tag{11}$$

2.3. Relations between the traction and displacement amplitude vectors at two sides of the interface

Let $\varepsilon := kh$ be small (i.e., the layer is thin) and the surface $x_2 = -h$ of the layer is free of traction: $\bar{\Sigma}(-\varepsilon) = 0$. By expanding into Taylor series $\bar{\Sigma}(-\varepsilon)$ at $y = 0$ up to the third-order and using (11) along with $\bar{\Sigma}(-\varepsilon) = 0$ we arrive at the relation between the traction and displacement amplitude vectors at the layer-side of the interface $y = 0$, namely

$$A\bar{U}(0) + B\bar{\Sigma}(0) = 0, \tag{12}$$

where matrices A and B are given by

$$A = \begin{bmatrix} -i\left[\varepsilon(\bar{\rho}c^2-\bar{\delta})-\dfrac{\varepsilon^3}{6}\left(r_2+r_3\bar{\rho}c^2+\dfrac{\bar{\rho}^2c^4}{\bar{c}_{66}}\right)\right] & \dfrac{\varepsilon^2}{2}[2\bar{\rho}c^2-\bar{\delta}] \\ \dfrac{\varepsilon^2}{2}[2\bar{\rho}c^2-\bar{\delta}] & -i\left[\varepsilon\bar{\rho}c^2-\dfrac{\varepsilon^3}{6}\left\{3\bar{\rho}c^2-\bar{\delta}\right\}\right] \end{bmatrix},$$
$$B = \begin{bmatrix} 1-\dfrac{\varepsilon^2}{2}\left(r_1+\dfrac{\bar{\rho}c^2}{\bar{c}_{66}}\right) & i\varepsilon \\ i\varepsilon & 1-\dfrac{\varepsilon^2}{2} \end{bmatrix}, \tag{13}$$

where

$$r_1 = 1-\frac{\bar{\delta}}{\bar{c}_{66}}, \quad r_2 = \bar{\delta}\left(\frac{\bar{\delta}}{\bar{c}_{66}}-2\right), \quad r_3 = 2r_1+1. \tag{14}$$

According to Ogden and Vinh [8], the displacement components of Rayleigh waves in the half-space are given by

$$u_n = U_n(y)e^{ik(x_1-ct)}, \quad \sigma_{n2} = ik\Sigma_n(y)e^{ik(x_1-ct)}, \; n = 1,2 \tag{15}$$

where

$$U_1(y) = \alpha_1 B_1 e^{-b_1 y} + \alpha_2 B_2 e^{-b_2 y}, \quad U_2(y) = i(B_1 e^{-b_1 y} + B_2 e^{-b_2 y}), \tag{16}$$

$$\Sigma_1(y) = i(\beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}), \qquad \Sigma_2(y) = \gamma_1 B_1 e^{-b_1 y} + \gamma_2 B_2 e^{-b_2 y}, \tag{17}$$

here α_k, β_k, γ_k are defined by

$$\begin{aligned} &\alpha_k = b_k,\; \beta_k = c_{66}(1+\alpha_k^2),\; \gamma_k = (X-\delta+\beta_k)\alpha_k,\; k = 1,2 \\ &X = \rho c^2, \quad \delta = c_{11}+c_{22}-2c_{12}, \end{aligned} \tag{18}$$

and b_1, b_2 are to with positive real parts roots of the equation

$$c_{66}b^4 - (\delta - 2c_{66} - X)b^2 + (c_{66} - X) = 0. \tag{19}$$

From (19) we have

$$\begin{aligned} b_1^2 + b_2^2 &= \frac{\delta - 2c_{66} - X}{c_{66}} := S, \\ b_1^2.b_2^2 &= \frac{c_{66} - X}{c_{66}} := P. \end{aligned} \tag{20}$$

It is not difficult to verify that if a Rayleigh wave exists ($\rightarrow b_1, b_2$ having positive real parts), then

$$0 < X < c_{66},\; P > 0,\; S + 2\sqrt{P} > 0,\; b_1.b_2 = \sqrt{P},\; b_1 + b_2 = \sqrt{S + 2\sqrt{P}}. \tag{21}$$

Taking $x_2 = 0$ in (16) and (17) gives

$$\begin{aligned} &U_1(0) = \alpha_1 B_1 + \alpha_2 B_2,\; U_2(0) = i(B_1 + B_2), \\ &\Sigma_1(0) = i(\beta_1 B_1 + \beta_2 B_2),\; \Sigma_2(0) = \gamma_1 B_1 + \gamma_2 B_2. \end{aligned} \tag{22}$$

Eliminating B_1, B_2 from Eqs. (22) we have

$$\Sigma(0) = HU(0), \quad H = \begin{bmatrix} ih_{11} & h_{12} \\ -h_{12} & -ih_{22} \end{bmatrix}, \tag{23}$$

where

$$h_{11} = c_{66}(b_1 + b_2),\ h_{12} = c_{66}(1 - b_1 b_2),\ h_{22} = -c_{66} b_1 b_2 (b_1 + b_2), \tag{24}$$

and $b_1 b_2, b_1 + b_2$ are defined by (21). Eq. (23) is the relation between the traction and displacement amplitude vectors at the half-space-side of the interface $y = 0$.

3. AN APPROXIMATE FORMULAS FOR THE RAYLEIGH WAVE H/V RATIO

Suppose the layer and the half-space are perfectly bonded at the interface $y = 0$, then we have

$$\bar{U}(0) = U(0),\ \bar{\Sigma}(0) = \Sigma(0). \tag{25}$$

Expanding into Taylor series $\bar{U}(-\varepsilon)$ at $y = 0$ up to the third order yields

$$\bar{U}(-\varepsilon) = \bar{U}(0) - \varepsilon \bar{U}'(0) + \frac{\varepsilon^2}{2}\bar{U}''(0) - \frac{\varepsilon^3}{6}\bar{U}'''(0). \tag{26}$$

From (11), (25) and the relation between the traction and displacement amplitude vectors at the half-space-side of the interface (23), it follows

$$\begin{aligned} &\bar{U}'(0) = i(N_1 + N_2 H)U(0),\ \bar{U}''(0) = i^2(N_1^{(2)} + N_2^{(2)} H)U(0), \\ &\bar{U}'''(0) = i^3(N_1^{(3)} + N_2^{(3)} H)U(0). \end{aligned} \tag{27}$$

Substituting (27) into (26) leads to

$$\bar{U}(-\varepsilon) = QU(0). \tag{28}$$

Elements of matrix Q in (28) are defined by

$$\begin{aligned} Q_{11} &= 1 + \varepsilon\left\{\frac{h_{11}}{\bar{c}_{66}}\right\} - \frac{\varepsilon^2}{2}\left\{r_1 + \frac{\bar{\rho}c^2}{\bar{c}_{66}} + \frac{h_{12}}{\bar{c}_{66}}\right\} \\ &\quad - \frac{\varepsilon^3}{6}\left\{\left(\frac{1+r_1}{\bar{c}_{66}} + \frac{\bar{\rho}c^2}{\bar{c}_{66}^2}\right)h_{11}\right\}, \\ Q_{12} &= i\Big[\varepsilon\left\{1 - \frac{h_{12}}{\bar{c}_{66}}\right\} - \frac{\varepsilon^2}{2}\left\{\frac{h_{22}}{\bar{c}_{66}}\right\} \\ &\quad - \frac{\varepsilon^3}{6}\left\{r_1 + \frac{2\bar{\rho}c^2}{\bar{c}_{66}} - \left(\frac{1+r_1}{\bar{c}_{66}} + \frac{\bar{\rho}c^2}{\bar{c}_{66}^2}\right)h_{12}\right\}\Big], \\ Q_{21} &= i\Big[\varepsilon + \frac{\varepsilon^2}{2}\left\{\frac{h_{11}}{\bar{c}_{66}}\right\} - \frac{\varepsilon^3}{6}\left\{r_1 + \frac{\bar{\rho}c^2}{\bar{c}_{66}} + \frac{h_{12}}{\bar{c}_{66}}\right\}\Big], \\ Q_{22} &= 1 + \frac{\varepsilon^2}{2}\left\{-1 + \frac{h_{12}}{\bar{c}_{66}}\right\} + \frac{\varepsilon^3}{6}\left\{\frac{h_{22}}{\bar{c}_{66}}\right\}. \end{aligned} \tag{29}$$

On the other hand, using (12), (23), and taking into account (25) yield

$$ZU(0) = 0, \quad Z = A + BH. \tag{30}$$

Elements of matrix Z in (30) are defined by

$$\begin{aligned}
Z_{11} &= i\Big[h_{11} + \varepsilon\left\{\bar{\delta} - \bar{\rho}c^2 - h_{12}\right\} - \frac{\varepsilon^2}{2}\left\{\left(r_1 + \frac{\bar{\rho}c^2}{\bar{c}_{66}}\right)h_{11}\right\} \\
&\quad + \frac{\varepsilon^3}{6}\left\{r_2 + r_3\bar{\rho}c^2 + \frac{\bar{\rho}^2c^4}{\bar{c}_{66}}\right\}, \\
Z_{12} &= h_{12} + \varepsilon h_{22} + \frac{\varepsilon^2}{2}\left\{2\bar{\rho}c^2 - \bar{\delta} - \left(r_1 + \frac{\bar{\rho}c^2}{\bar{c}_{66}}\right)h_{12}\right\}, \\
Z_{21} &= -h_{12} - \varepsilon h_{11} + \frac{\varepsilon^2}{2}\left\{2\bar{\rho}c^2 - \bar{\delta} + h_{12}\right\}, \\
Z_{22} &= i\Big[-h_{22} + \varepsilon\left\{-\bar{\rho}c^2 + h_{12}\right\} + \frac{\varepsilon^2}{2}h_{22} + \frac{\varepsilon^3}{6}\left\{-\bar{\delta} + 3\bar{\rho}c^2\right\}\Big].
\end{aligned} \tag{31}$$

From (28) and (30), it follows

$$\chi := \left|\frac{\bar{u}_1(-h)}{\bar{u}_2(-h)}\right| = \left|\frac{\bar{U}_1(-\varepsilon)}{\bar{U}_2(-\varepsilon)}\right| = \left|\frac{Q_{12}Z_{11} - Q_{11}Z_{12}}{Q_{22}Z_{11} - Q_{21}Z_{12}}\right|, \tag{32}$$

where Z_{11}, Z_{12} are defined by (31), elements of matrix Q are defined by (29). Note that, due to $|Z| = 0$, therefore χ can be given by an alternative formula

$$\chi := \left|\frac{Q_{12}Z_{21} - Q_{11}Z_{22}}{Q_{22}Z_{21} - Q_{21}Z_{22}}\right|. \tag{33}$$

After some manipulations, we arrive at the desired approximate formula of third order for the H/V ratio, namely

$$\chi = \left|\frac{A_0 + A_1\varepsilon + A_2\frac{\varepsilon^2}{2} + A_3\frac{\varepsilon^3}{6} + O(\varepsilon^4)}{B_0 + B_1\varepsilon + B_2\frac{\varepsilon^2}{2} + B_3\frac{\varepsilon^3}{6} + O(\varepsilon^4)}\right|, \tag{34}$$

where

$$\begin{aligned}
&A_0 = -h_{12},\ A_1 = -h_{11} - h_{22}, \\
&A_2 = -\bar{\delta} + 2\left(1 + r_1 + \frac{\bar{\delta}}{\bar{c}_{66}}\right)h_{12} - \frac{1}{\bar{c}_{66}}(h_{12}^2 + h_{11}h_{22}), \\
&A_3 = 3\left(r_1 + \frac{\bar{\delta}}{\bar{c}_{66}}\right)h_{22} + \left(4r_1 + \frac{3\bar{\delta}}{\bar{c}_{66}} - \frac{\bar{\rho}c^2}{\bar{c}_{66}}\right)h_{11},
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
B_0 =& h_{11},\ B_1 = \bar{\delta} - \bar{\rho}c^2 - 2h_{12}, \\
B_2 =& -2h_{22} - \left(1 + r_1 + \frac{\bar{\rho}c^2}{\bar{c}_{66}}\right)h_{11}, \\
B_3 =& r_2 + (r_3 - 3)\bar{\rho}c^2 + \frac{\bar{\rho}^2c^4}{\bar{c}_{66}} + \left(3 + 4r_1 + \frac{3\bar{\delta}}{\bar{c}_{66}} + \frac{\bar{\rho}c^2}{\bar{c}_{66}}\right)h_{12} \\
&- \frac{2}{\bar{c}_{66}}(h_{12}^2 + h_{11}h_{22}).
\end{aligned} \tag{36}$$

In the dimensionless form Eq. (34) is of the form

$$\chi = \left| \frac{\bar{A}_0 + \bar{A}_1\varepsilon + \bar{A}_2\dfrac{\varepsilon^2}{2} + \bar{A}_3\dfrac{\varepsilon^3}{6} + O(\varepsilon^4)}{\bar{B}_0 + \bar{B}_1\varepsilon + \bar{B}_2\dfrac{\varepsilon^2}{2} + \bar{B}_3\dfrac{\varepsilon^3}{6} + O(\varepsilon^4)} \right|, \tag{37}$$

in which the coefficients $\bar{A}_k, \bar{B}_k (k = 0, 1, 2, 3)$ are given by (47) in Appendix A and they depend on the following dimensionless parameters

$$e_\delta = \frac{\delta}{c_{66}}, \quad \bar{e}_\delta = \frac{\bar{\delta}}{\bar{c}_{66}}, \quad r_\mu = \frac{\bar{c}_{66}}{c_{66}}, \quad r_v = \frac{c_2}{\bar{c}_2}, \quad x = \frac{X}{c_{66}}, \tag{38}$$

where $c_2 = \sqrt{c_{66}/\rho}$, $\bar{c}_2 = \sqrt{\bar{c}_{66}/\bar{\rho}}$. It is clear that the H/V ratio χ depends on 5 dimensionless parameters: e_δ, $\bar{e}_\delta$, r_μ, r_v and ε which are subjected the inequalities [19]

$$r_\mu > 0, \quad r_v > 0, \quad e_\delta > 0, \quad \bar{e}_\delta > 0, \quad \varepsilon > 0. \tag{39}$$

Note that the Rayleigh wave H/V ratio χ depends on the dimensionless Rayleigh wave velocity x that is a solution of the secular equation (3.14) in [19] and it depends also on 5 dimensionless parameters mentioned above.

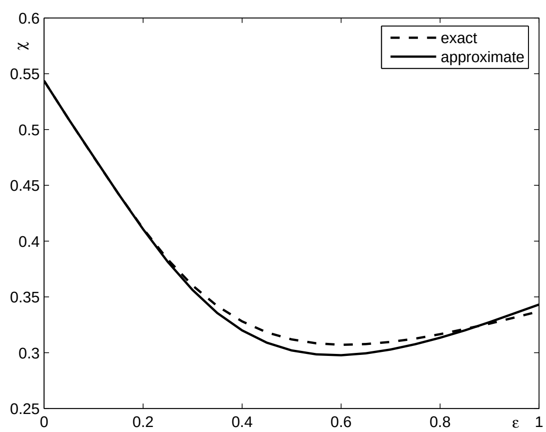

Fig. 1. Dependence of the H/V ratio on $\varepsilon \in [0, 1]$ that is calculated by the exact formula (dashed line), by the third-order approximate formula (37) (solid line). Here we take: $e_\delta = \bar{e}_\delta = 4$, $r_\mu = 1$, $r_v = 3$

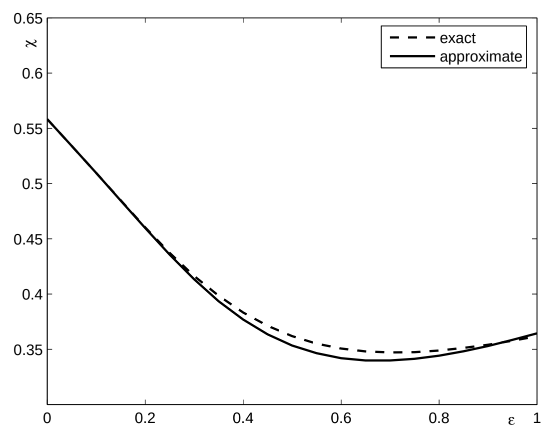

Fig. 2. Dependence of the H/V ratio on $\varepsilon \in [0, 1]$ that is calculated by the exact formula (dashed line), by the third-order approximate formula (37) (solid line). Here we take: $e_\delta = 3.8$, $\bar{e}_\delta = 3.7$, $r_\mu = 1$, $r_v = 2.5$

Figs. 1, 2 present the third-order approximate curves and the exact curves of the Rayleigh wave H/V ratio χ depending on ε in the interval $[0, 1]$. The approximate curves (solid lines) are calculated by Eq. (37) and the exact curves (dashed lines) are computed by using the exact secular equation (66) in Ref. [20], the exact expressions of displacements in Ref. [21] and the incompressible limit method [20]. It is shown from Figs. 1, 2 that the the third-order approximate formula (37) is a good approximation. Note that, an elegant approximate formula of H/V ratio was obtained in [17] for the incompressible media.

However, this formula is only good for high frequency, not for small frequency as shown in Fig. 5 of [2].

Special cases:

When $\varepsilon = 0$, from (34) we have

$$\chi = \left|\frac{\bar{A}_0}{\bar{B}_0}\right| = \left|\frac{h_{12}}{h_{11}}\right|. \tag{40}$$

On the other hand, from (23) and $\Sigma(0) = 0$ we have $|H| = 0$, i.e.,

$$h_{12}^2 + h_{11}h_{22} = 0. \tag{41}$$

From (40) and (41) it follows

$$\chi^2 = \frac{h_{12}^2}{h_{11}^2} = -\frac{h_{22}}{h_{11}} = b_1 b_2 = \sqrt{1-x}. \tag{42}$$

When the layer and the half-space are both isotropic

$$c_{11} = c_{22},\ c_{66} = \mu,\ \bar{c}_{11} = \bar{c}_{22},\ \bar{c}_{66} = \bar{\mu},\ c_{11} - c_{12} = 2c_{66},\ \bar{c}_{11} - \bar{c}_{12} = 2\bar{c}_{66}. \tag{43}$$

With the help of (43) and (38) one can see that

$$\begin{aligned} &x = \frac{\rho c^2}{\mu},\ e_\delta = \bar{e}_\delta = 4,\ c_2 = \sqrt{\frac{\mu}{\rho}},\ \bar{c}_2 = \sqrt{\frac{\bar{\mu}}{\bar{\rho}}},\ r_v = \frac{c_2}{\bar{c}_2},\ r_\mu = \frac{\bar{\mu}}{\mu}, \\ &\bar{x} = r_v^2 x,\ S = 2 - x,\ P = 1 - x,\ b_1 = 1,\ b_2 = \sqrt{1-x}. \end{aligned} \tag{44}$$

Then, the Rayleigh wave H/V ratio is defined by Eq. (37) with $\bar{A}_i$ and $\bar{B}_i$ ($i = 0, 1, 2, 3$) are given by

$$\begin{aligned} \bar{A}_0 &= \frac{\sqrt{1-x} - 1}{r_\mu},\ \bar{A}_1 = -\frac{x}{r_\mu}, \\ \bar{A}_2 &= -4 + \frac{4(1 - \sqrt{1-x})}{r_\mu} - \frac{x + (x-4)\sqrt{1-x}}{r_\mu^2}, \\ \bar{A}_3 &= -\frac{1 + \sqrt{1-x}}{r_\mu}(r_v^2 x + 3\sqrt{1-x}), \end{aligned} \tag{45}$$

and

$$\begin{aligned} \bar{B}_0 &= \frac{1 + \sqrt{1-x}}{r_\mu},\ \bar{B}_1 = 4 - r_v^2 x - \frac{2(1 - \sqrt{1-x})}{r_\mu}, \\ \bar{B}_2 &= \frac{1 + \sqrt{1-x}}{r_\mu}(2 - r_v^2 x + 2\sqrt{1-x}), \\ \bar{B}_3 &= 8 - 8r_v^2 x + r_v^4 x^2 + \frac{(3 + r_v^2)(1 - \sqrt{1-x})}{r_\mu} - \frac{2(x + (x-4)\sqrt{1-x})}{r_\mu^2}. \end{aligned} \tag{46}$$

4. CONCLUSIONS

In this paper, the propagation of Rayleigh waves in an incompressible orthotropic elastic half-space coated by a thin incompressible orthotropic elastic layer is investigated. An approximate formula for the Rayleigh wave H/V ratio of third-order in terms of dimensionless thickness of the layer has been established by using the relations between the traction and displacement amplitude vectors of Rayleigh waves at two sides of the welded interface between the layer and the half-space. It is shown numerically that the obtained approximate formula is a good approximation. The obtained approximate formula can be employed as theoretical base for evaluating mechanical properties of thin films from measured values of the Rayleigh wave H/V ratio.

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APPENDIX A
The expressions of $\bar{A}_k, \bar{B}_k$ $(k = 0, 1, 2, 3)$

$$
\begin{aligned}
&\bar{A}_0 = \frac{b_1 b_2 - 1}{r_\mu},\ \bar{A}_1 = \frac{(b_1+b_2)(b_1 b_2 - 1)}{r_\mu},\\
&\bar{A}_2 = -\bar{e}_\delta + \frac{4(1-b_1b_2)}{r_\mu} - \frac{(1-b_1b_2)^2 - b_1b_2(b_1+b_2)^2}{r_\mu^2},\\
&\bar{A}_3 = \frac{(b_1+b_2)(4-\bar{e}_\delta - r_v^2 x - 3b_1b_2)}{r_\mu},\\
&\bar{B}_0 = \frac{b_1+b_2}{r_\mu},\ \bar{B}_1 = \bar{e}_\delta - r_v^2 x - \frac{2(1-b_1b_2)}{r_\mu},\\
&\bar{B}_2 = \frac{(b_1+b_2)(2b_1b_2 - 2 + \bar{e}_\delta - r_v^2 x)}{r_\mu},\\
&\bar{B}_3 = \bar{e}_\delta^2 - 2\bar{e}_\delta - 2\bar{e}_\delta r_v^2 x + r_v^4 x^2 + \frac{(7-\bar{e}_\delta + r_v^2 x)(1-b_1b_2)}{r_\mu}\\
&\qquad - 2\frac{(1-b_1b_2)^2 - b_1b_2(b_1+b_2)^2}{r_\mu^2}.
\end{aligned}
\tag{47}
$$