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DYNAMIC ANALYSIS OF PRESTRESSED BERNOULLI BEAMS RESTING ON TWO-PARAMETER FOUNDATION UNDER MOVING HARMONIC LOAD

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Abstract. This paper describes the dynamic analysis of prestressed Bernoulli beams resting on a two-parameter elastic foundation under a moving harmonic load by the finite element method. Using the cubic Hermitian polynomials as interpolation functions for the deflection, the stiffness of the Bernoulli beam element augmented by that of the foundation support and prestress is formulated. The nodal load vector is derived using the polynomials with the abscissa measured from the left-hand node of the current loading element to the position of the moving load. Using the formulated element, the dynamic response of the beams is computed with the aid of the direct integration Newmark method. The effects of the foundation support, prestress as well as excitation frequency, velocity and acceleration on the dynamic characteristics of the beams are investigated in detail and highlighted.

1. INTRODUCTION

The dynamic analysis of beams under moving loads plays an important role in the field of railway and bridge engineering, and has attracted much attention from researchers for many years. The early work on the topic has been described by Timoshenko *et al.* in [1], where the governing equation for a uniform Bernoulli beam subjected to moving harmonic force with constant velocity was solved by the mode superposition method. In [2], Fryba presented a solution for vibration of simply supported beam under moving loads and axial forces. Employing the traditional 2D Bernoulli beam element, Thambiratnam and Zhuge [3] computed the dynamic amplification factor for beams resting on a Winkler elastic foundation subjected to moving loads. Chen *et al.* [4] investigated the response of infinite Timoshenko beam on a viscoelastic foundation to a moving harmonic load by deriving the dynamic stiffness matrix for the beam. The natural frequencies and mode shapes of Bernoulli-type beams subjected to moving loads with variable velocity have been investigated by Dugush and Eisenberger [5] by both the modal and direct integration methods. Using the Fourier transform method, Kim [6] obtained the steady-state response to moving loads of axial loaded beams resting on a Winkler elastic foundation. Adopting polynomials as trial function for the deflection in the Lagrangian equations, Kocatürk and Şimşek [7] investigated the vibration of viscoelastic beams subjected to an eccentric

two-parameter elastic foundation under a moving concentrated harmonic load is conducted using the finite element method. The prestress is assumed to be resulted from initially loaded by axial forces, and velocity of the moving load is considered variable. To this end, a finite element taking the effect of both prestress, foundation support is formulated. In the formulation, a two-parameter foundation model taking the interaction between springs of the traditional Winkler foundation, previously employed by the first author in [8] is adopted. The two-parameter foundation model shows some advantage including the accuracy in modelling the effect of the foundation support on structures [9, 10]. Using the formulated element, the dynamic response of the beams is computed using the direct integration Newmark method. The influence of the prestress, foundation support, external load parameters on the dynamic characteristics of the beams is investigated in detail.

2. ELEMENT FORMULATION

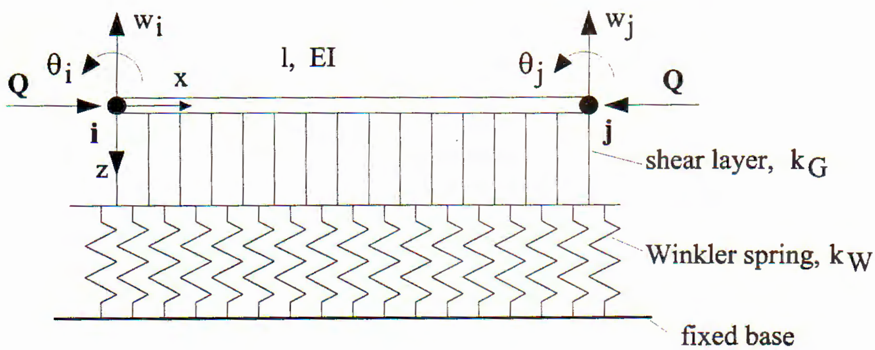


Fig. 1. A two-node prestressed Bernoulli beam element resting on two-parameter foundation

Consider a two-node beam element resting on a two parameter foundation as shown in Fig. 1. In addition to the conventional Winkler springs, a shear layer is introduced in the foundation model to take the interaction between the springs into account. In the figure, l and EI denote the length and the bending rigidity of the element, respectively. The element is initially stressed by an axial force Q. At each node the element has two degrees of freedom, namely a lateral translation and a rotation about an axis normal to the plane of the paper. Thus, the vector of nodal displacements contains four components as

$$\mathbf{d} = \{w_i \quad \theta_i \quad w_j \quad \theta_j\}^T, \tag{2.1}$$

where and afterwards the superscript ^{T} denotes the transpose of a vector or a matrix. The total potential energy of a prismatic beam element is stemming from the beam bending, foundation deformation, and the potential of the axial load as [11,12]

$$\begin{aligned} U &= U_B + U_W + U_G + U_Q \\ &= \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^l k_W w^2 dx + \frac{1}{2}\int_0^l k_G\left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^l Q\left(\frac{\partial w}{\partial x}\right)^2, \end{aligned} \tag{2 2}$$

where k_W (unit of force/lenght^2) and k_G (unit of force) denote the stiffness of the Winkler springs and the shear layer, respectively. Following standard approach of the finite element method, we adopted here with the Hermitian polynomials as interpolation scheme for the deflection w

$$N_{w1} = 2\frac{x^3}{l^3} - 3\frac{x^2}{l^2} + 1 \quad ; \qquad N_{w2} = \frac{x^3}{l^2} - 2\frac{x^2}{l} + x;$$
$$N_{w3} = -2\frac{x^3}{l^3} + 3\frac{x^2}{l^2}; \qquad N_{w4} = \frac{x^3}{l^2} - \frac{x^2}{l}. \tag{2.3}$$

Substituting (2.3) into Eq. (2.2), one get

$$\begin{aligned} U =&\frac{2}{l^3}EI\Big[3(w_i - w_j)^2 + 3l(w_i - w_j)(\theta_i + \theta_j) + l^2(\theta_i^2 + \theta_i\theta_j + \theta_j^2)\Big] \\ &\frac{l}{420}k_W\Big[78(w_i^2 + w_j^2) + 54w_iw_j + 2l^2(\theta_i^2 + \theta_j^2) - 3l^2\theta_i\theta_j + 22l(w_i\theta_i - w_j\theta_j) \\ &- 13l(w_i\theta_j - w_j\theta_i)\Big] + \frac{1}{30l}(k_G + Q)\Big[18(w_i - w_j)^2 + 3l(w_i - w_j)(\theta_i + \theta_j) \\ &+ l^2(2\theta_i^2 - \theta_i\theta_j + 2\theta_j^2)\Big] \end{aligned} \tag{2.4}$$

The element stiffness is obtained by twice differentiating the strain energy respective the nodal displacements

$$\mathbf{k} = \left[\frac{\partial^2 U}{\partial \mathbf{d}^2}\right] = \mathbf{k}_B + \mathbf{k}_W + \mathbf{k}_G + \mathbf{k}_Q, \tag{2.5}$$

where

$$\mathbf{k}_B = \frac{1}{l^3}EI\begin{bmatrix} 12 & & sym. & \\ 6l & 4l^2 & & \\ -12 & -6l & 12 & \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}; \quad \mathbf{k}_W = \frac{1}{420}k_W\begin{bmatrix} 156l & & sym. & \\ 22l^2 & 4l^3 & & \\ 54l & 13l^2 & 156l & \\ -13l^2 & -3l^3 & -22l^2 & 4l^3 \end{bmatrix}$$
$$\mathbf{k}_G = \frac{1}{30l}k_G\begin{bmatrix} 30 & & sym. & \\ 3l & 4l^2 & & \\ -30 & -3l & 30 & \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}; \quad \mathbf{k}_Q = \frac{1}{30l}Q\begin{bmatrix} 30 & & sym. & \\ 3l & 4l^2 & & \\ -30 & -3l & 30 & \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \tag{2.6}$$

For a dynamic analysis, a mass matrix is required, and a consistent mass matrix based on the interpolation function (2.3) presented in [13] is adopted in the present paper. The formulation of the consistent mass matrix is as follows

$$\mathbf{m} = \frac{m}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}, \tag{2.7}$$

where $m = \rho l A$ (ρ is mass density, and A is the cross-sectional area) is the total element mass. It is noted that expression for the mass matrix (2.7) is similar to that of stiffness matrix $\mathbf{k}_W$ in Eq. (2.6).

3. GOVERNING EQUATION AND NUMERICAL ALGORITHM

Consider a simply supported prestressed beam with the length of L resting on a two-parameter foundation with a moving concentrated harmonic load, $F = P\cos(\Omega t)$, travelling along the beam from left to right as shown in Fig. 2. Denoting x_F is the current position of the moving load, measured from the left-hand end of the beam. Assuming at time $t = 0$ the load F is at the left-hand support and having a velocity v_o, it then travels with a constant acceleration to the right, and its velocity at the right-hand support is v_f. Following the standard procedure of the finite element method, the beam is discretized into a number of finite elements. The equation of motion in terms of the finite element method when ignoring the damping effect for the beam can be written in the form [13]

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{K}\mathbf{D} = P\cos(\Omega t)\mathbf{N}, \tag{3.1}$$

where $\mathbf{M}$ and $\mathbf{K}$ respectively are structural mass and stiffness matrices, obtained by assembling the element matrices $\mathbf{k}$ and $\mathbf{m}$, formulated in Section 2 in the standard way of the finite element method; $\mathbf{D}$ and $\ddot{\mathbf{D}} = \partial^2\mathbf{D}/\partial t^2$ are the structural nodal displacements and accelerations, respectively; $\mathbf{N}$ is the vector of shape functions for the beam, and having the form

$$\mathbf{N} = \{0\;0\;0 \ldots\; N_{w1}\; N_{w2}\; N_{w3}\; N_{w4}\; 0\;0\;0 \ldots\; 0\;0\;0\}^T, \tag{3.2}$$

where N_{w1}, N_{w2}, N_{w3}, N_{w4} are defined by Eq. (2.3), in which the abscissa x is measured from the left-hand node of the current loading element, and for the case of equal-element mesh is computed as (see Fig. 2)

$$x = x_F - (n-1)l = \frac{v_f - v_o}{2\Delta t}t^2 + v_o t - (n-1)l, \tag{3.3}$$

with l, as before is the element length, and n denotes the number of the element on which the load is acting; t is the current time, and Δt is the total time needed for the load to move completely from the left-hand support to the right-hand support.

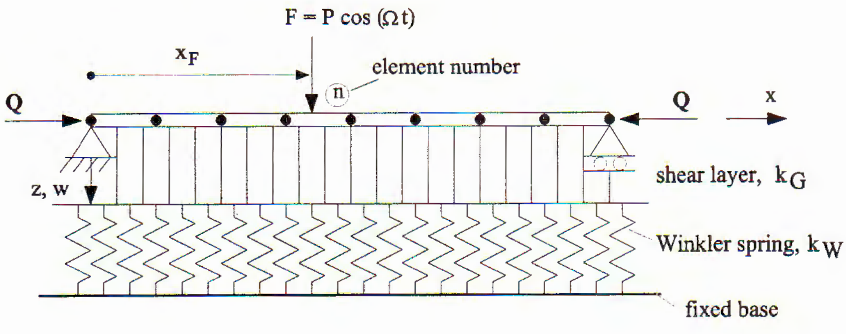


Fig. 2. Simply supported prestressed beam resting on a two-parameter elastic foundation foundation subjected to a moving harmonic load $F = P\cos(\Omega t)$

Eq. (3.1) is solved by the step-by-step direct integration Newmark method, in which the nodal displacements and velocities at a new time t_{n+1} are implicitly computed as [13]

$$\begin{aligned}\mathbf{D}_{n+1} &= \mathbf{D}_n + h\dot{\mathbf{D}}_n + h^2\left(\frac{1}{2}-\beta\right)\ddot{\mathbf{D}}_n + h^2\beta\ddot{\mathbf{D}}_{n+1} \\ \dot{\mathbf{D}}_{n+1} &= \dot{\mathbf{D}}_n + (1-\gamma)h\ddot{\mathbf{D}}_n + \gamma h\ddot{\mathbf{D}}_{n+1},\end{aligned} \tag{3.4}$$

where $h = (t_{n+1} - t_n)$ is the time step; β and γ are constants; $\dot{\mathbf{D}} = \partial\mathbf{D}/\partial t$ is the nodal velocities. Choosing $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$ (as in this paper), Eq. (3.4) leads to the average constant acceleration formula, which unconditional numerically stability. As seen from Eq. (3.4), in order to compute $\mathbf{D}_{n+1}$ and $\dot{\mathbf{D}}_{n+1}$, the acceleration at time t_{n+1} is needed, that is an implicit time-integration method is required.

4. NUMERICAL RESULTS AND DISCUSSIONS

Using the finite element formulated in Section 2, a computer code based on the direct integration Newmark method is developed for solving Eq. (3.1). To investigate the dynamic response of beams to a moving load, a beam employed by Kocatürk and Şimşek [7] with the following geometry and material data is adopted herewith

$$L = 20 \text{ m};\; I = 0.08824 \text{ m}^4;\; \rho A = 1000 \text{ kg/m};\; E = 3\times 10^9 \text{ N/m}^2;\; P = 100 \text{ kN}$$

where L denotes the total beam length, and P is the amplitude of the moving load. The numerical results reported below are obtained by a mesh of 20-equal elements.

4.1. Methodology verification

This Subsection aims to verify the formulation and the developed computer code by comparing the numerical results to some published work. To this end, following the work in [11], we introduce herewith the dimensionless parameters

$$k_1 = \frac{L^4}{EI}k_W \quad ; \quad k_2 = \frac{L^2}{\pi^2 EI}k_G, \tag{4.1}$$

which represented the stiffness of the Winkler springs and shear layer, respectively. We also introduce the so-called frequency parameter, defined as

$$\mu = \left(\frac{\rho A L^4}{EI}\omega^2\right)^{1/4}, \tag{4.2}$$

where ω is the the fundamental frequency of the beams.

Table 1. The first three natural frequencies (rad/s) of simply supported beam without foundation support and at various value of axial force

	present work			Ref. [7]*		
	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
$Q = 0$	42.7366	170.9477	384.6428	42.7362	170.9466	384.7235
$Q = -1\times 10^3\,kN$	42.4470	170.6587	384.3540	42.4466	170.6573	384.4354
$Q = \;\; 1\times 10^3\,kN$	43.0244	171.2361	384.9314	43.0242	171.2374	385.0123

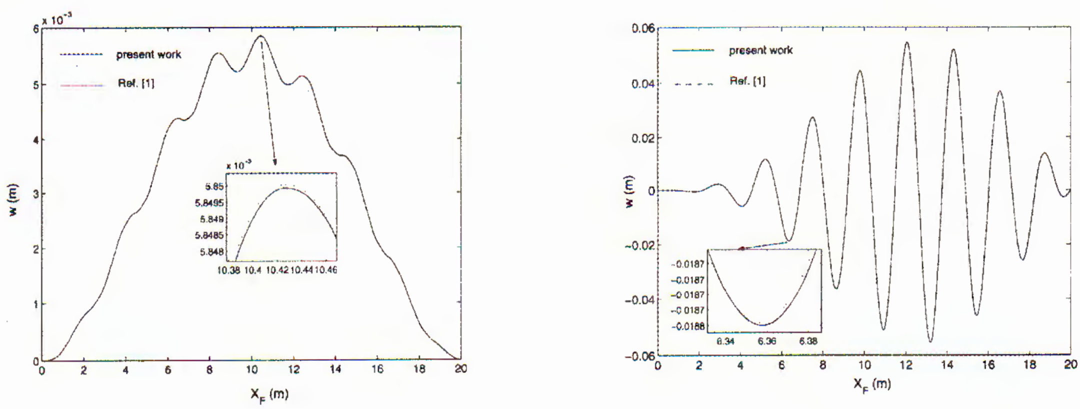
* with 10 terms in the trial deflection shape.

Table 2. The frequency parameter μ of prestressed simply supported beam at various value foundation parameters (k_1, k_2)

	$Q = 0$ present work	$Q = 0$ Ref. [11]*	$Q = -1 \times 10^3$ (kN)	$Q = 1 \times 10^3$ (kN)
$(k_1, k_2) = (1, 0)$	3.1496	3.1496	3.1390	3.1601
$(k_1, k_2) = (100, 0)$	3.7483	3.7483	3.7421	3.7546
$(k_1, k_2) = (1, 0.5)$	3.4827	3.4826	3.4749	3.4904
$(k_1, k_2) = (100, 0.5)$	3.9608	3.9608	3.9555	3.9661
$(k_1, k_2) = (1, 1)$	3.7408	3.7407	3.7345	3.7471
$(k_1, k_2) = (100, 1)$	4.1437	4.1437	4.1391	4.1483
$(k_1, k_2) = (1, 2.5)$	4.3002	4.3001	4.2960	4.3043
$(k_1, k_2) = (100, 2.5)$	4.5824	4.5824	4.5789	4.5858

* available for the case $Q = 0$ only.

Table 1 list the first three natural frequencies of the simply supported beam without foundation support with various values of the axial force. The frequency parameter defined by Eq. (4.2) at various values of the foundation parameters and axial force is given in Table 2. The corresponding frequencies and parameter respectively reported in Refs. [7] and [11] are also listed in the tables. It is noted from the tables that the frequencies obtained in the present work are in excellent agreement with that reported in Refs. [7] and [11].



(a) $\Omega = 0$ (b) $\Omega = 40$ rad/s

Fig. 3. Deflection under the moving load for constant velocity $v = 15\, m/s$ and at different excitation frequencies

Fig. 3 shows the deflection under moving load of the simply supported beam without foundation support at a constant velocity $v = v_o = v_f = 15$ m/s, and at different excitation frequencies $\Omega = 0$ and $\Omega = 40$ rad/s. It is noted that the excitation frequency $\Omega = 40$ rad/s is very near the first natural frequency of the beam (confirm Table 1), so that the deflection of the beam shown in Fig. 3.b is much larger than that of Fig. 3.a due to the resonance phenomenon. For the purpose of comparison, the figure also shows the analytical solution obtained from the mode superposition method by Timoshenko *et al.* [1], where the defection of the beam under a moving harmonic load with constant velocity

v is given by

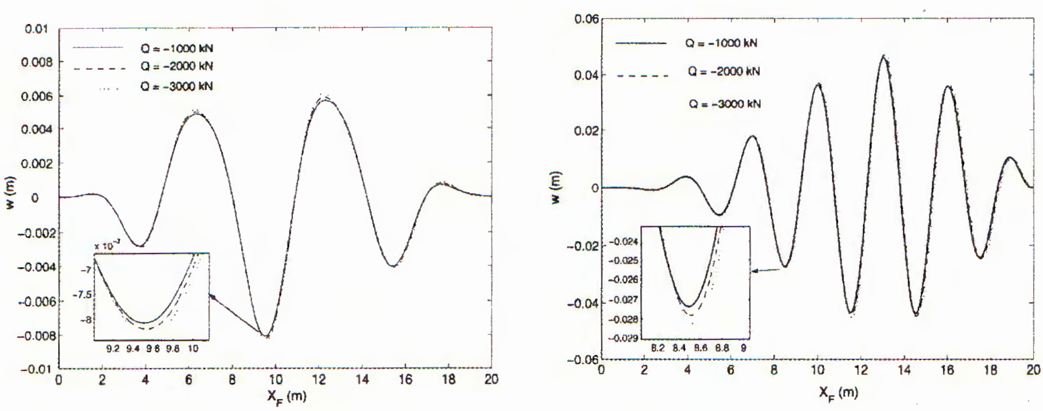
$$w = \frac{PL^3}{EI\pi^4} \sum_{i=1}^{\infty} \sin \frac{i\pi x}{L} \left[\frac{\sin\left(\frac{i\pi v}{L} + \Omega\right)t}{i^4 - (\beta + i\alpha)^2} + \frac{\sin\left(\frac{i\pi v}{L} - \Omega\right)t}{i^4 - (\beta - i\alpha)^2} - \frac{\alpha}{i} \left(\frac{\sin \frac{i^2\pi^2}{L^2} at}{-i^2\alpha^2 + (i^2 - \beta)^2} + \frac{\sin \frac{i^2\pi^2}{L^2} at}{-i^2\alpha^2 + (i^2 + \beta)^2} \right) \right] \tag{4.3}$$

where $\alpha = vL/\pi a$, $(a = \sqrt{EI/\rho})$ is the ratio of the period $\tau = 2L^2/\pi a$ of the fundamental type of vibration of the beam to twice the total time Δt needed for the load completely passing the beam; $\beta = \tau/\tau_o$ is the ratio of the period of the fundamental type of vibration of the beam to the period $\tau_o = 2\pi/\Omega$ of the harmonic load. It is noted that some notations in Eq. (4.3) have been modified, so that they are in consistent with the notations of the present paper.

As seen from Fig. 3, the deflection obtained in the present study is in excellent agrement with the result of Timoshenko *et al.* in the case of moving load ($\Omega = 0$) and in the case of moving harmonic load ($\Omega = 40$ rad/s). Even with zoomed parts (shown by small boxes inside the figures), we hardly realize the difference between the curves of Timoshenko *et al.* and that of present work. Thus, from above comparison, we can conclude from his Subsection that the element formulation and computer code developed in this study are very accurate in dynamic analysis of beams under the moving load.

4.2. Response with different parameters

This Subsection aims to investigate the response of the prestressed beam at various values of the axial force, foundation parameters, frequency and velocity of the moving load. The beam with the geometric and material data as in Subsection 4.1 is again adopted herewith for the investigation.



(a) $\Omega = 20$rad/s (b) $\Omega = 40$ rad/s

Fig. 4. Response of simply supported beam without foundation support to moving harmonic load at various values of axial force and at a velocity $v = 20\,m/s$

Fig. 6 shows the response of the simply supported beam to the moving load with constant velocity $v = 20\,\mathrm{m/s}$ computed with various values of the compressive force Q and at excitation frequencies of $20\,\mathrm{rad/s}$ and $40\,\mathrm{rad/s}$. It is seen from the figure that the response of the beam is not very much affected by the axial forces, regardless of the excitation frequencies. The dynamic deflection in Fig. 4.b is much higher than that in Fig. 4.a since Fig. 4.b computed at an excitation frequency near the resonance frequency as above remark.

Fig. 5 shows the response of the prestressed beam to the moving load with different velocities, different excitation frequencies and at an axial compressive force $Q = -2000\,\mathrm{kN}$. The response of the beam is very much influenced by the speed of the moving load, and the excitation frequency also. In Fig. 5.a, the dynamic deflection of the beam gradually increases with an increment in the velocity, and it then decreases. The critical velocity is governed by the excitation frequency, e.g. $v_{max} = 60\,m/s$ in Fig. 5.a, while in 5.b $v_{max} = 20$ m/s. Again, the dynamic deflection of the curves in Fig. 5b is much larger than that of the corresponding curves in Fig. 5.a.

The effects of the foundation parameters on the response of the beam are shown in Fig. 6 and Fig. 7. The curves shown in the figures are obtained for the case $v = 20\,\mathrm{m/s}$, $\Omega = 20\,\mathrm{rad/s}$ and $Q = -2000\,\mathrm{kN}$. As seen from the figures, the foundation support remarkably reduces the dynamic deflection of the beam, and the inclusion of the Winkler spring interaction (represented by k_2) into consideration has a similar effect.

4.3. Maximum dynamic deflection

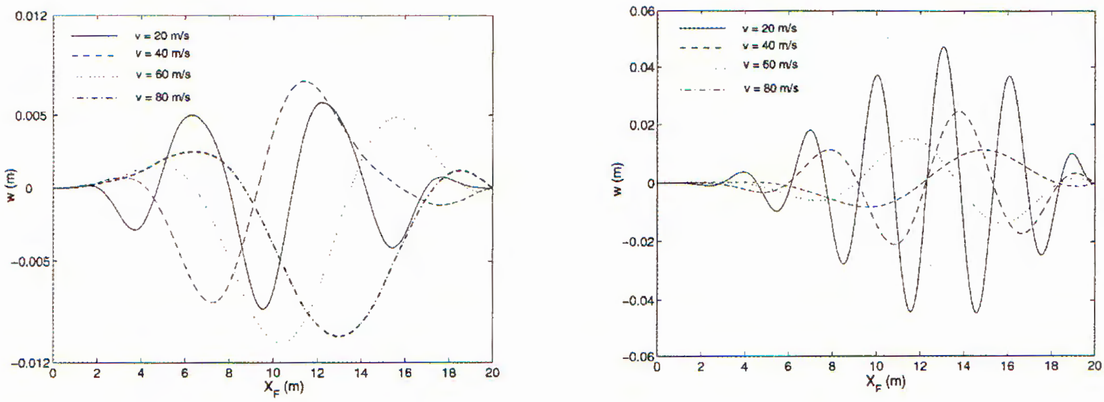
The dependence of the maximum dynamic deflections of simply supported prestressed beam on the load velocity and the foundation parameters is shown in Fig. 8 and Fig. 9 for the cases of excitation frequency $\Omega = 20\,\mathrm{rad/s}$ and $\Omega = 40\,\mathrm{rad/s}$, respectively. It is clearly seen from the figures, there is the so-called critical velocity at which the maximum dynamic deflection attains a extreme value, and this critical velocity depends on the foundation stiffness and the excitation frequency as well. The influence of the velocity on the maximum dynamic deflection is considerably changed with the presence of the second foundation stiffness parameter k_2, as seen in Figs. 8.b and 9.b in comparison with Figs. 8.a and Fig. 9.a. In addition, the change in the maximum dynamic deflection is much sharper for case of the excitation frequency near the resonant frequency than that far from the resonant frequency.

4.4. Effect of acceleration

In the above discussion we assume the velocity of the moving load is constant. This assumption is now relieved, and the effect of acceleration on the dynamic response of the beam is investigated in this Subsection. For the sake of simplicity, the acceleration is considered constant, and it is represented through the difference between the velocities of the moving load at right-hand and left-hand ends of the beam. In this regard and recalling the notations in Eq. (3.3), the computation is performed with $v_o = 10\,m/s$, various values of $v_f = 10$, 20, 30 and 40 m/s, and for different cases of foundation support and excitation frequency.

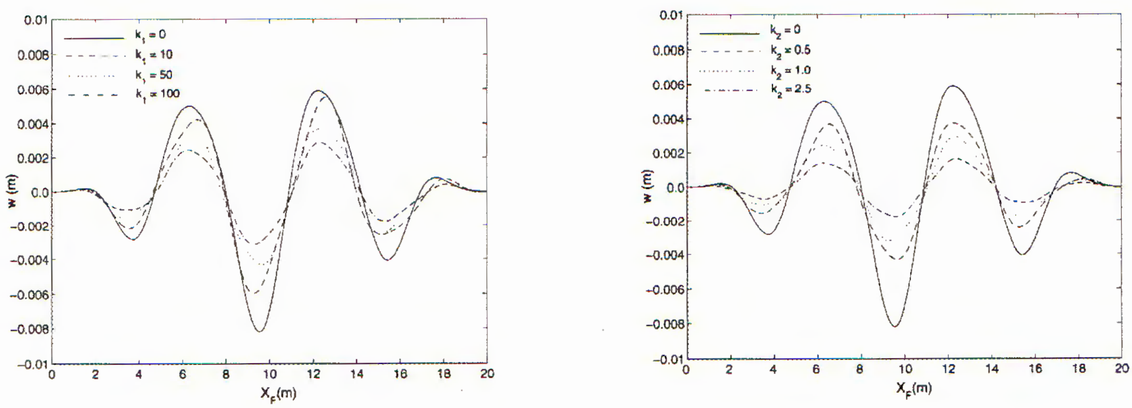
The effect of acceleration on response of prestressed simply supported beams for different case of the foundation stiffness and excitation frequencies is shown in Figs. 10 -

12. The dynamic deflection of the beam is somehow affected by the acceleration, and the increment in the deflection by the acceleration or not is depended on the foundation stiffness, regardless of the excitation frequencies. For any case of the foundation support, the period of the dynamic response is considerably reduced by the acceleration support, regardless of the excitation frequencies. The second foundation parameter k_2 also contributes to the reduction in the period of the dynamic response of the beam, as clearly shown by the difference between Fig. 11 and Fig. 12.



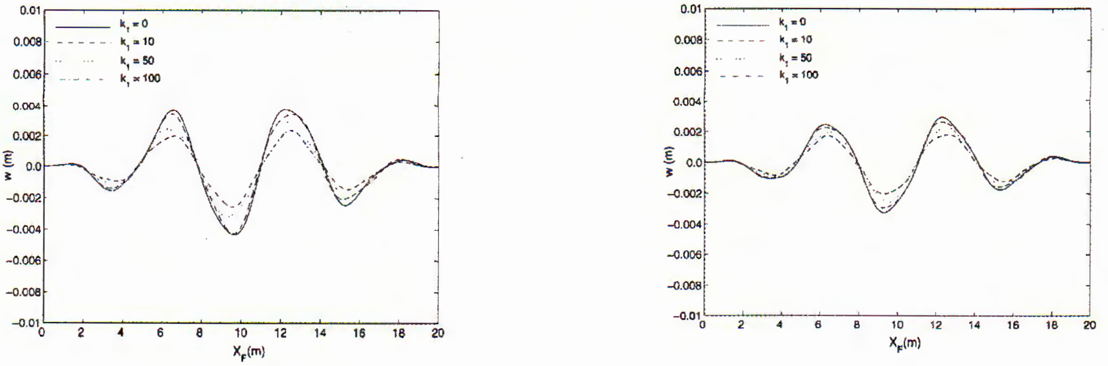
(a) $\Omega = 20$ rad/s (b) $\Omega = 40$ rad/s

Fig. 5. Response of simply supported beam without foundation support to moving harmonic load with various velocities and at an axial force $Q = -2000\,kN$



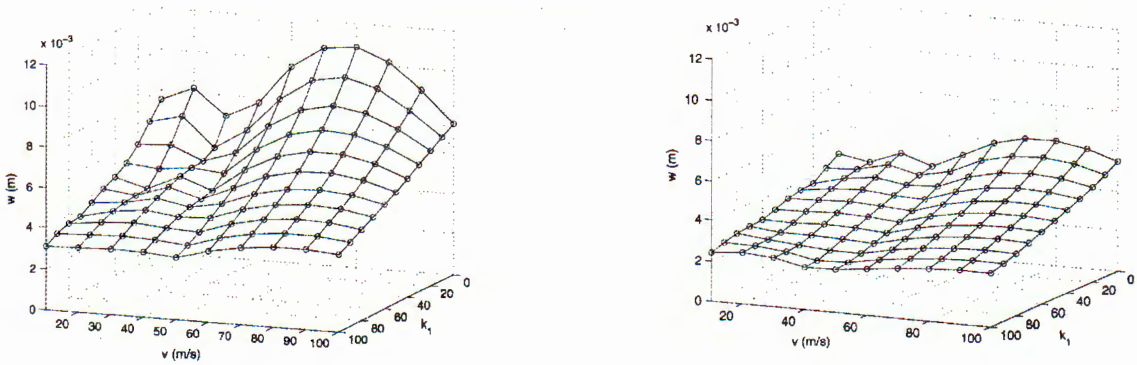
(a) different k_1, $k_2 = 0$ (b) different k_2, $k_1 = 0$

Fig. 6. Response of simply supported beam without foundation support to moving harmonic load at different values of foundation parameters and at $v = 20\,m/s$, $\Omega = 20\,rad/s$, $Q = -2000\,kN$



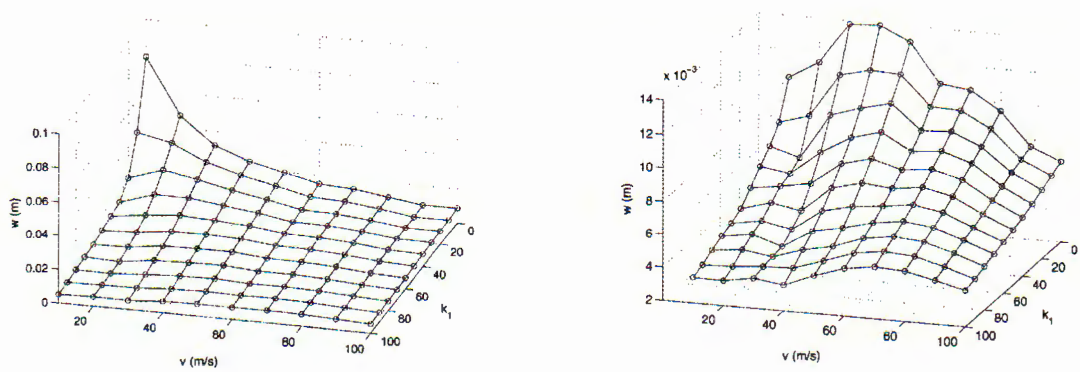
(a) different k_1, $k_2 = 0.5$ (b) different k_2, $k_1 = 1$

Fig. 7. Response of simply supported beam without foundation support to moving harmonic load at different values of foundation parameters and at $v = 20\,m/s$, $\Omega = 20\,rad/s$, $Q = -2000\,kN$



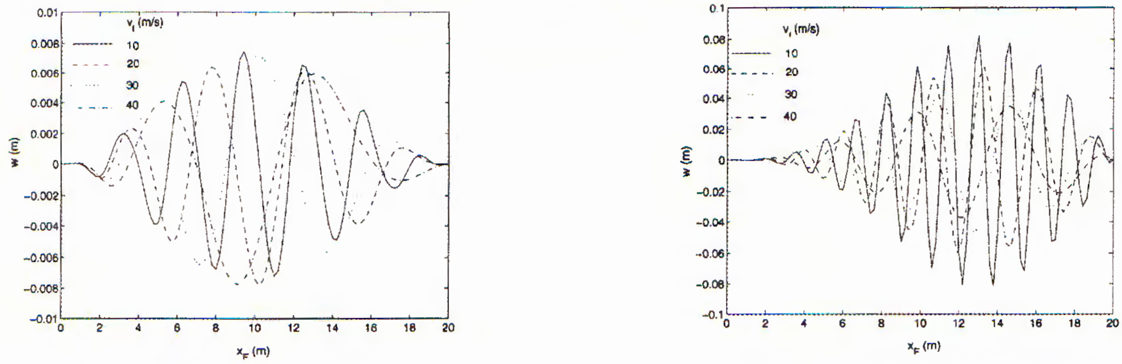
(a) $k_2 = 0$, $Q = -2000$ kN (b) $k_2 = 0.5$, $Q = -2000$kN

Fig. 8. Dependence of maximum dynamic deflections of simply supported prestressed beam on the load velocity and the foundation parameters for the case excitation frequency $\Omega = 20\,rad/s$



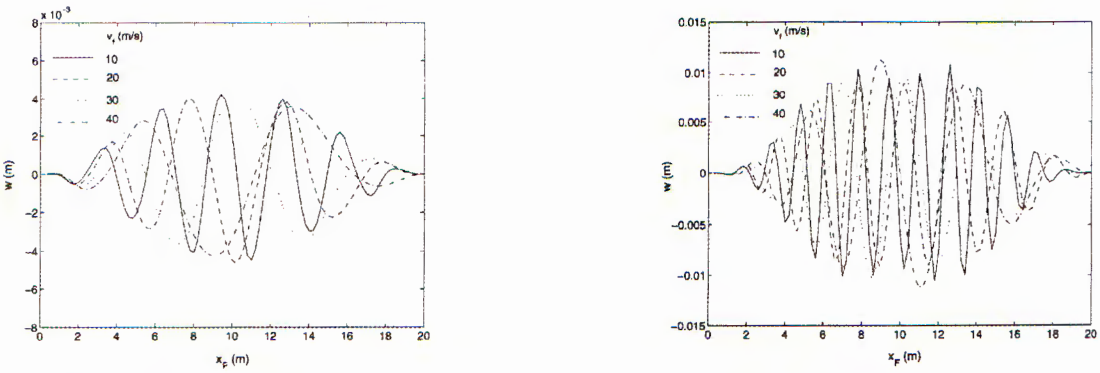
(a) $k_2 = 0$, $Q = -2000$ kN (b) $k_2 = 0.5$, $Q = -2000$kN

Fig. 9. Dependence of maximum dynamic deflections of simply supported prestressed beam on the load velocity and the foundation parameters for the case excitation frequency $\Omega = 40\,rad/s$



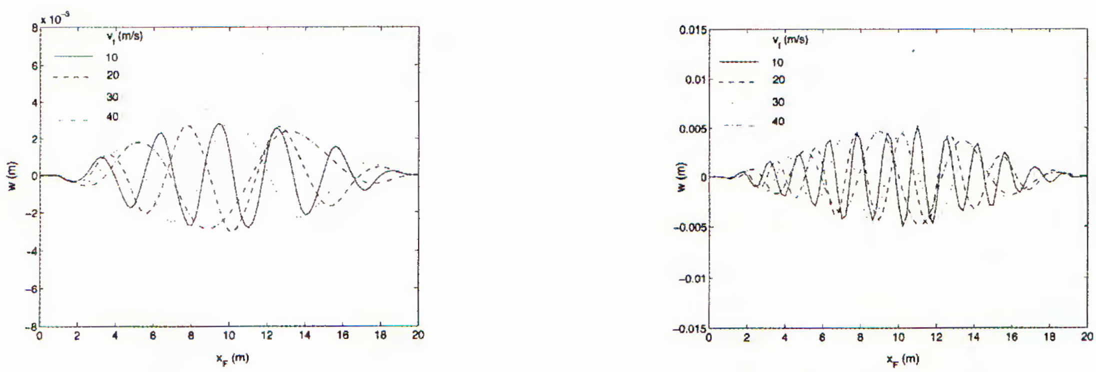
(a) $\Omega = 20$ rad/s, different v_f (b) $\Omega = 40$ rad/s, different v_f

Fig. 10. Effect of acceleration on response of prestressed beam without foundation support for the case $v_o = 10\,m/s$ and $Q = -2000\,kN$



(a) $\Omega = 20$ rad/s, different v_f (b) $\Omega = 40$ rad/s, different v_f

Fig. 11. Effect of acceleration on response of prestressed beam resting on elastic foundation for the case $k_1 = 50$, $k_2 = 0$, $v_o = 10\,m/s$ and $Q = -2000\,kN$



(a) $\Omega = 20$ rad/s, different v_f (b) $\Omega = 40$ rad/s, different v_f

Fig. 12. Effect of acceleration on response of prestressed beam resting on elastic foundation for the case $k_1 = 50$, $k_2 = 0.5$, $v_o = 10\,m/s$ and $Q = -2000\,kN$

5. CONCLUDING REMARKS

The dynamic analysis of prestressed beams resting on a two-parameter elastic foundation under a moving harmonic concentrated load by the finite element method has been described on the paper. Using the cubic Hermitian polynomials as interpolation functions for the displacement field, the stiffness of the Bernoulli beam element augmented by that of the foundation and prestress was formulated and employed in computing the natural frequencies and response of the beams. The nodal load vector was derived using the polynomials with the abscissa measured from the element left-hand node to the current position of the moving load. Using the formulated element and nodal load vector, the dynamic analysis of the beam with different values of the foundation parameters, axial force, excitation frequencies and velocities has been performed. The effects of the parameters on the natural frequencies and dynamic response of the beam were investigated and described in detail.

It is noted that the numerical investigations presented in Section 4 are just described for the case of simply supported beam, but in regard of the finite element method used in the present paper, the extension to other case of boundary conditions is a trivial task.

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PHÂN TÍCH ĐỘNG HỌC DẦM BERNOULLI DỰ ỨNG LỰC NẰM TRÊN NỀN ĐÀN HỒI HAI THAM SỐ CHỊU TẢI TRỌNG DI ĐỘNG

Bài báo trình bày phân tích động học của dầm Bernoulli dự ứng lực nằm trên nền đàn hồi hai tham số chịu tải trọng di động bằng phương pháp phần tử hữu hạn. Ma trận độ cứng của phần tử dầm Bernoulli bổ xung bởi ma trẫn độ cứng của nền đàn hồi và dự ứng lực được xây dựng trên cơ sở các đa thức Hermite. Véc-tơ lực nút được thiết lập từ các đa thức Hermite, trong đó hoành độ được tính từ nút trái phần tử có lực tác động tới vị trí hiện tại của lực. Sử dụng công thức phần tử, ứng xử động học của dầm được thu nhẫn bằng phương pháp tích phân trưc tiếp Newmark. Ảnh hưởng của nền đàn hồi, dự ứng lực, tần số, vận tốc và gia tốc của lực ngoài tới các đặc trưng động học của dầm được nghiên cứu chi tiết.