Paper

Foundations of the theory of open waveguides

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Abstract — The theory of electromagnetic wave eigenmodes propagating on open dielectric and metallic waveguides has been reviewed. The main steps of different theoretical approaches to the problem are outlined and discussed. The unsolved problems and also directions of future development are pointed out.

Keywords — open waveguide theory, eigenvalue problems

The theory of natural waves (also known as normal waves, travelling waves, eigenmodes) able to propagate on open dielectric and metallic waveguides is far from a complete development. Commonly it is supposed that the field components of such a wave depend on the longitudinal coordinate z and time t as $e^{ihz-ikt/c}$, where h is the propagation constant (or modal wavenumber), k is the free-space wavenumber, and c is the free-space propagation velocity. Only in several simple cases, such as dielectric slab and coaxially-layered circular dielectric fiber, it is possible to study the waves in explicit form. Such a study has brought into consideration a variety of waves: surface-wave or guided modes, leaky waves, complex surface modes of lossless fiber, etc., differing by the field behavior as a function of coordinates. However, if the fiber cross-section is arbitrary, or if additional metal elements are present, as in the microstrip or slot lines, the theory meets great problems. There are many questions to be answered even for a mathematical formulation of the problem of natural waves. Clearly it should be a sort of eigenvalue problem for modal wavenumber h. However, what should be the domain of the variation of the eigenvalue parameter? In what class should one seek the modal field components as a function of coordinates in cross-section and along the waveguide? There have been several approaches to these problems; all of them appear to be not complete and should be re-examined.

The following is a summary of the results related to the theory of propagation of the time-harmonic waves on arbitrary-shaped electromagnetic open waveguides [1–14]. The main points of this study are as follows:

Start from the excitation problem

It is impossible to come to a reasonably general formulation of the modal eigenvalue problem from any other starting point than the problem of time-harmonic excitation $\left(\sim e^{ikt/c}, k = \mathrm{Re}k > 0\right)$ of an open waveguide by elementary electric and magnetic current sources. This is the same as determining the open waveguide Green functions. Here, a necessary assumption should be made that any field can be presented as a convolution with the Green functions.

Fourier transform

By virtue of infinite length of a regular waveguide along the z-axis, the Fourier transformation with the kernel e^{ihz} is a natural instrument of bringing the problem consideration to the two-dimesional (2D) space, for the field transforms as a function of cross-section r and integration parameter h. Here, another necessary assumption appears that the fields are no more than the slow-growth functions of z; hence, the Fourier integrals should be interpreted in terms of distributions. It is necessary to distinguish between the open waveguides, whose elements have a compact cross-section (embedded in free space) and those of non-compact cross-section, for example, compact open waveguides embedded into a flat-layered medium, whose cross-section has infinite boundaries. Although two cases have much in common, the latter one is more complicated. In the former case, Fourier transform enables one to reduce the excitation problem to a „conventional" one, for the Helmholtz operator $\Delta_t + k^2\varepsilon(r)\mu(r) - h^2$ in 2D open domain with the boundary and transmission conditions given at bounded curves. Further results relate only to this case.

Analytic continuation

Fourier transform approach naturally brings the analysis to a necessity of analytic continuation of the field transforms, from the real values of parameter h to the complex domain. This complex domain is uniquely determined and is common to all the open waveguides of compact cross-section: it is the infinite-sheet Riemann surface L of the function $\ln(k-h)(k+h)$.

Reichardt condition

On this Riemann surface, it is the Reichardt condition that serves as analytic continuation of the 2D Sommerfeld radiation condition (for $|h| < k$) and the exponential-decay condition (for $|h| > k$) from the real axis of the „physical" sheet. Due to this condition, but also due to the transmission-type conditions at the material boundaries - if they are present, these 2D problems for the analytic continuations of the field transforms are non-selfadjoint ones. Note that this condition permits the field transforms to grow exponentially with $r \to \infty$ if h is located at the sheets other than the „physical" sheet of L. Nevertheless, Reichardt condition guarantees the uniqueness of solution provided that h is not an eigenvalue.

Analytical regularization

For a wide class of open waveguides, 2D boundary-value problems for the Fourier transforms can be converted to the canonical Fredholm operator equations, $(I+A)X = B$, in some Hilbert space. Here, operators A are the meromorphic functions of $h \in L, k$, and all the geometrical and material parameters of the waveguide. Such a conversion is based commonly on the analytical regularization of the singular integral equations equivalent to original boundary-value problem. Here, Reichardt condition guarantees that arbitrary-source field can be represented as a convolution with the transforms of the Green functions for any complex $h \in L$.

Fredholm-Steinberg theorems

Once a regularization has been done, one can use the theory of Fredholm in the form generalized by Steinberg for the operators depending on parameter. The results are as follows: it is possible to prove the existence of the bounded resolvent, and hence, the existence of the Fourier transforms, as no more than meromorphic functions of $h \in L$. They have no finite accumulation points of the poles on L. The poles can be of only finite multiplicity. They are piece-wise-continuos functions of the geometry and piece-wise-analytic functions of k and material parameters. The continuity or analiticity can be lost only at such a value of parameter that two or more poles coalesce. The poles can appear or disappear only at the boundary of the domain of meromorphicity: at infinity and in the branch points $h = \pm k$. The residues of the poles of the Fourier transforms satisfy certain 2D source-free eigenproblem for the Helmholtz equation, in terms of the spectral parameter h located on L.

Generalized modal eigenvalue problems

The latter circumstance leads to a conclusion that the eigenproblems about the natural waves of an open waveguide can be studied independently of the excitation problem. However, in view of the above chain of considerations, it should be formulated in a generalized sense. Namely, it should include the Reichardt condition at the infinity in cross-section. In so doing one gets a universal framework to study all the types of known natural waves: both surface waves, and leaky, and complex surface ones, etc., and hence trace the transitions of each mode from one type to another with variation of non-spectral parameters.

Symmetry of spectrum

Some properties of the modal spectrum can be deduced directly from the formulation of generalized eigenproblem. It is shown that in any open waveguide the eigenwavenumbers h form symmetric pairs on L. Moreover, in lossles waveguides, they form conjugate quartets on L. Hence, it is enough to study them only in one quadrant of each Riemann sheet.

Free of spectrum domain

By using the vector Green formula, it is shown that on L there exists a non-empty domain, which is free of the spectrum of natural waves. This domain depends on the type of the waveguide. If the latter does not contain material (dielectric or magnetic) elements, this domain includes the whole „physical" sheet of L; in a lossless dielectric waveguide, it includes the intervals $|h| < k$ and $|h| \geq \sup \varepsilon^{1/2}$ of the real axis of the „physical" sheet; in the lossy case this whole real axis is free of spectrum, etc.

Discreteness of modal spectrum

For a wide class of virtually all the practical models of open waveguides, such generalized eigenproblems admit analytical regularization and are equivalently reducible to a homogeneous Fredholm operator equation $[I + A(h)]X = 0$. The set of eigenvalues of h on L forms the spectrum of the operator A and coinsides with the spectrum of generalized natural waves of the waveguide. As one can see, the latter is purely discrete on L. In particular, this enables one to conclude that surface-wave modes, whose wavenumbers are located on the finite interval $k < |h| < k \sup \varepsilon^{1/2}$ of the real axis of the „physical" sheet, can be only of finite number.

Existence of natural waves

Non-emptiness of the spectrum of generalized natural modes is the most hard point of the analysis. It can be proved „locally" based on the operator generalization of the Rusche theorem and explicit existence of eigenvalues of certain canonic open waveguides, as the zeros of well-known special functions. This once again needs a regularized form of the eigenvalue operator equation. However, to complete this proof to a „global" existence, one needs some guarancy that a finite change of nonspectral parameter cannot kick all the eigenvalues off to infinity or annihilate them in the branch-point. This proof needs additional work.

Orthogonality and power flux

Vector Green formula, applied to the eigenmode field, enables one to prove the orthogonality of the surface waves and the complex surface waves, in the power sense. If the modal wavenumber is not located on the „physical" sheet of L, this proof fails. The Green formula is also an instrument to study the properties of the power flux associated with a generalized natural mode. For example, it shows that any complex surface wave in a lossless open waveguide can be only hybrid (i.e., has all the six components of electromagnetic field) and does not carry power, as its total flux in cross-section in identical zero. Another important conclusion is that, in open waveguides, there is no necessity for a surface wave to carry the power strictly in the direction of its propagation; the opposite direction is allowed, although only for the hybrid modes. The analyticity of spectrum points as a function of k enables one to validate the concept of the group velocity.

Multiple poles and „associated" natural waves

Unlike hollow closed waveguides, for the open waveguides it is not possible to prove the simple character of the poles, and hence the eigenvalues. (This is also known for the impedance-wall and multilayer closed waveguides). Hence, multiple poles, of finite multiplicity M, can exist. If so, besides of the „parent" natural wave propagating as e^{ihz}, a finite chain of the „associated" natural waves appears that propagate as $z^m e^{ihz}, m = 1, 2, \ldots M-1$. This consideration validates the initial assumption, at the early stage of analysis of the excitation problem, that the field functions should be considered in the class of slow-growth functions of z.

Radiation condition

Strictly speaking, in the original 3D problem of the elementary-source excitation of open waveguide, the classical Sommerfeld condition of radiation is not valid for the extraction of unique solution. The reason is the presence of infinite along z boundaries, and hence possible presence of waves able to carry the power to infinity along the waveguide without attenuation. In view of the mentioned above results of study of the 2D problem for the field Fourier transforms, one can formulate an adequate condition of radiation, in the form of asymptotic request to the far-field behavior, that explicitly involves radiation, in the form of asymptotic request to the far-field behavior, that explicitly involves the surface waves. This condition guarantees uniqueness of the 3D problem solution and validates the early assumption that arbitrary-source field can be represented by a convolution with the Green functions. Here, one comes to a necessity of taking account of the direction of the power flux (or, equivalently, the sign of the group velocity) associated with each surface mode. The modified radiation condition enables one to formulate the Principle of Radiation as: *„No waves bringing power from infinity, in the scattered field"*. It is only if the losses are introduced in the waveguide elements that the modified radiation condition is reduced to the Sommerfeld one, as then no surface waves exist. The Principle of Radiation is then reduced to conventional form: *„No waves propagating from infinity, in the scattered field"*.

Defect of the model

A close view at the radiation condition reveals one intrincic defect of the original model of the time-harmonic excitation of a lossless infinite open waveguide. If the parameters of the lossless waveguide and k are such that „associated" surface-wave natural modes exist, then it appears not possible to apply even the modified condition of radiation. The reason is that in this simulation both the „parent" surface wave and the „associated" waves have zero total power flux, hence it is impossible to select proper sign of the modal wavenumber that ensures solution uniqueness.

Similarity between h and k as eigenparameters

From the formulation of the generalized eigenproblem, one can notice that the parameters h and k enter it in a very similar manner. Indeed, one can study this problem for k-eignenvalues, with $h = \mathrm{Re}h > 0$. Much of the above theory is valid in this case as well. E.g., the domain of analytic continuation in k is the same Riemann surface L. The same Reichardt condition and the same analytical regularization approach bring us to the conclusions about the discreteness of the k-spectrum on L and about the properties of eigenvalues as a function of h. However, the mentioned similarity is not total, hence the other properties of the spectrum of generalized natural oscillations are to be studied in more detail. Analysis of such a „dual" eigenproblem appears to be a natural stage of studying a more general problem of the excitation of open waveguide by a δ-pulse or time-dependent source having a fixed distribution along z as e^{ihz}, with $h = const$.

Extentions and unsolved problems

There are several possible directions of the extention of the developed theory, each of them being associated with a separate class of problems. The results obtained for the regular open waveguides can be generalized to the regular-periodic open waveguides. This calls for a generalization of the Fourier transform approach by using the Floquet-expansions in z, in transform domain. Then it is possible to see that the domain of analytic continuation in h is the Riemann surface of the function $\sum_{m=-\infty}^{\infty} \ln(k - h - 2\pi m/l)(k + h + 2\pi m/l)$, where l is the period along z. The other direction of work is the theory of open waveguides with non-compact cross-section, such as microstripline on infinitely wide dielectric substrate. Here, the approach of the double Fourier transform should be used. The study of the Green functions and radiation condition should apparently bring into consideration the surface waves of two types: those which stick to the strip and decay exponentially both in the air and in the substrate, and those which attenuate in the air but propagate in the substrate as cylindrical surface waves. Another interesting direction is the theory of the open waveguide bends and branchings. Here, the key problem is the one of a terminated (semi-infinite) open waveguide in free space.

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