Paper

Error statistics for concatenated systems on non-renewal time-varying channels

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Abstract — The statistics of the error process generated by a discrete super channel formed by the concatenation of a constrained encoder, a non-renewal finite state channel (FSC), and a constrained decoder is studied in this paper. First, recursions are developed for the error weight distribution. This statistics is relevant to the design of coding schemes and interleaving in concatenated systems. We also study the renewal nature of the residual error process as modified by the constrained decoder. Ferreira *et al.* conjectured that if the channel model is a renewal FSC, the super channel can be modeled as a similar renewal model. We use a statistics called the multigap distribution to analytically disprove this hypothesis. Furthermore, the effect of interleaving is investigated from a new perspective using the variance of the multigap distribution.

Keywords — finite state channels, multigap distribution, constrained codes, concatenated systems, Gilbert-Elliott channels, combinatorial methods.

1. Introduction

Finite state channels are mathematical models based on probabilistic or deterministic function of Markov chains that are capable of modeling the correlated error sequence produced by a broad class of communication channels with memory. Thus, for example, in a binary system the error sequence is the pairwise modulo two sum of the input $\{x_k\}_{k=1}^{\infty}$ and output $\{y_k\}_{k=1}^{\infty}$ sequence of the channel defined as follows. At the kth time interval, the error bit e_k is equal to zero (indicating no error) if $x_k = y_k$, or e_k is equal to one (indicating an error) if $x_k \neq y_k$.

In several applications, such as magnetic and optical storage systems, the communication system uses the concatenation of forward error correcting codes and constrained codes. The constraints introduced into the transmitted sequence offer the possibility of achieving a desired spectral shaping, reducing the intersymbol interference and improving the synchronization ability of communication systems. The process of selecting an appropriate error control coding scheme should take into account the error statistics of the super channel in order to correct or detect a specific set of most probable error sequences produced by the super channel.

Ferreira *et al.* [1] introduced the problem of calculating a statistics known as error-free run distribution for a super channel comprised of a certain class of constrained codes and a burst channel modeled according to a renewal Fritchman channel with one error state. In such renewal channels, sequences of zeros (also known as gap intervals) before and after an error are statistically independent and identically distributed random variables. The renewal assumption simplifies the analysis of the model, but it can be demonstrated that many communication systems present some dependence in the occurrence of successive gap intervals [2, 3]. The Gilbert-Elliott channel and Fritchman channels with more than one error state are, for example, non-renewal FSC models. The characterization of the residual error sequence at the output of the super channel with embedded non-renewal FSC models is of interest in this paper.

The development of mathematical tools to use with general non-renewal FSC models for system performance evaluation has been considered in the literature [4 – 6]. In this paper, we extend the general framework proposed in [6] to compute the statistics of the error sequence at the output of the super channel as a function of the inner model parameters. The main idea is to find a fractional generating series whose coefficients of its series expansion yields the statistics of interest. This approach allows the development of generating series and recursions for certain statistics for which no previous methods existed.

This paper is organized as follows. Section 2 contains a brief review of FSC models. The characterization of the residual error sequence at the output of the super channel is the main subject of Section 3. First, we derive recursion for a statistics called error weight distribution, that is, the probability of m errors in a block of length n. This statistics is relevant to the design of coding schemes and interleaving in concatenated systems. Next, we study the multigap distribution of the super channel. This statistics has been used as a test of non-renewalness of the error process and is capable of revealing interesting properties of burst channels, such as, the gap length spread, and the correlation coefficient between gap intervals. It is analytically proved in this section that the pair constrained encoder/decoder may convert a renewal burst channel into a non-renewal super channel. Conclusions are summarized in Section 4.

We adopt the following notation throughout this work. The matrices $\mathbf{I}$ and $\mathbf{1}$ stand for the identity matrix and a column vector of ones. $\mathbf{E}\{E_k\}$ stands for the expected value of the random variable E_k. If s and z are commutative indeterminates, $[s^k z^n]P(s,z)$ denotes the coefficient of $s^k z^n$ in the formal power series $P(s,z)$.

2. Model description

Consider $\{S_k\}_{k=0}^{\infty}$ a stationary Markov chain with state space $\mathscr{N}_N = \{0, 1, \ldots, N-1\}$, transition probability matrix $\mathbf{P}$, and stationary probability row vector $\boldsymbol{\pi}$. At the kth time interval, the chain makes a transition from state $S_{k-1}=i$ to $S_k = j$ with probability $p_{i,j}$ and generates an error bit e_k (independent of i), with probability $P(E_k = e_k \mid S_k = j)$. Define two $N \times N$ matrices, $\mathbf{P}(e_k)$, $e_k \in \{0,1\}$, whose (i,j)th entry is $P(E_k = e_k \mid S_k = j)P(S_k = j \mid S_{k-1} = i)$, which is the probability that the output symbol is e_k when the chain makes a transition from state i to j. The probability of an error sequence of length n, may be expressed in a matrix form as:

$$P(e_1 e_2 \cdots e_n) = \boldsymbol{\pi} \left(\prod_{k=1}^{n} \mathbf{P}(e_k) \right) \mathbf{1}.$$

The matrices $\mathbf{P}(0)$, $\mathbf{P}(1)$ and $\boldsymbol{\pi}$ for the Gilbert-Elliott channel (GEC) [7] illustrated in Fig. 1 are given below:

$$\mathbf{P}(0) = \begin{bmatrix} (1-Q)(1-g) & Q(1-b) \\ q(1-g) & (1-q)(1-b) \end{bmatrix}; \quad (1)$$

$$\mathbf{P}(1) = \begin{bmatrix} (1-Q)g & Qb \\ qg & (1-q)b \end{bmatrix}; \quad (2)$$

$$\boldsymbol{\pi} = [\pi_0 \; \pi_1] = \left[\frac{q}{q+Q} \; \frac{Q}{q+Q} \right]. \quad (3)$$

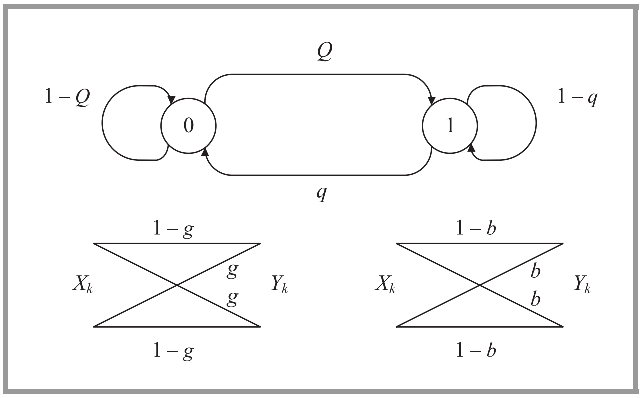

Fig. 1. Gilbert-Elliott model for burst channels.

An FSC model can also be described as a deterministic function of a Markov chain. In the Fritchman channel [8], the state space $\mathscr{N}_N$ is partitioned into two disjoint subsets, $\mathscr{A}_0 = \{0, \ldots, N-2\}$ (the good states), and $\mathscr{A}_1 = \{N-1\}$ (the bad state). The error bit, e_k, is a *deterministic* function of the current state S_k, and assumes the value $e_k = 0$ (no error) if $S_k \in \mathscr{A}_0$ or $e_k = 1$ (error) if $S_k \in \mathscr{A}_1$. A model with K good states and 1 bad state is denoted by $(K,1)$-FC. In particular, the matrices $\mathbf{P} = \mathbf{P}(0) + \mathbf{P}(1)$, $\mathbf{P}(0)$, $\mathbf{P}(1)$, for the $(2,1)$-FC model are given by:

$$\mathbf{P} = \left[\begin{array}{cc|c} \lambda_0 & 0 & 1-\lambda_0 \\ 0 & \lambda_1 & 1-\lambda_1 \\ \hline p_{2,0} & p_{2,1} & 1-p_{2,0}-p_{2,1} \end{array} \right]; \quad (4)$$

$$\mathbf{P}(0) = \left[\begin{array}{cc|c} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ \hline p_{2,0} & p_{2,1} & 0 \end{array} \right]; \quad (5)$$

$$\mathbf{P}(1) = \left[\begin{array}{cc|c} 0 & 0 & 1-\lambda_0 \\ 0 & 0 & 1-\lambda_1 \\ \hline 0 & 0 & 1-p_{2,0}-p_{2,1} \end{array} \right]. \quad (6)$$

Tsai modeled a fading HF communication channel using the following (2,1)-FC model [8]:

$$\mathbf{P} = \left[\begin{array}{cc|c} 0.99911 & 0 & 0.00089 \\ 0 & 0.73644 & 0.26356 \\ \hline 0.36258 & 0.58510 & 0.05232 \end{array} \right]. \quad (7)$$

3. Error statistics

Two statistics of the residual error sequence produced by the super channel are derived in this section. The error weight distribution and the multigap distribution.

3.1. Error weight distribution

The error weight distribution is the probability of an FSC generates exactly m consecutive errors in a block of length n. This probability is denoted by $P(m,n)$. An expression for $P(m,n)$ can be obtained by first finding the following bivariate generating series:

$$H_P(s,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P(m,n) s^m z^n, \quad (8)$$

where the indeterminates s and z mark the number of ones and the length of the error sequence, respectively. Our interest in this series relies on the fact that the coefficient of $s^m z^n$ is the quantity of interest, that is, $P(m,n) = [s^m z^n] H_P(s,z)$. The generating series $H_P(s,z)$ for the error process $\{e_k\}_{k=1}^{\infty}$ of FSC models is given in [6] as:

$$H_P(s,z) = \boldsymbol{\pi}(\mathbf{I} - \mathbf{P}(1)sz - \mathbf{P}(0)z)^{-1}\mathbf{1}. \quad (9)$$

Upon substitution of the matrices (1)-(3) into (9), we express $H_P(s,z)$ for the GEC as the ratio of two polynomials in s and z:

$$H_P(s,z) = \frac{1 + c_{1p}z + c_{2p}zs}{1 + c_1 z + c_2 zs + c_3 z^2 + c_4 z^2 s + c_5 z^2 s^2}, \quad (10)$$

where

$$\begin{aligned} c_{1p} &= [qQ(2-b-g) - q(1-q)(1-b) + \\ &\quad -Q(1-Q)(1-g)]/(q+Q); \\ c_{2p} &= [qQ(b+g) - Q(1-Q)g - q(1-q)b]/(q+Q), \end{aligned}$$

and

$$\begin{aligned} c_1 &= (Q(1-g)+q(1-b)-(2-g-b)); \\ c_2 &= -(b(1-q)+g(1-Q)); \\ c_3 &= (1-b)(1-g)(1-q-Q); \\ c_4 &= (1-q-Q)(b+g-2gb); \\ c_5 &= (1-q-Q)gb. \end{aligned} \tag{11}$$

It is simple go from a generating series to recurrence formulas, which provides a rapid computational scheme for the problem. The denominator polynomial of $H_p(s,z)$ is responsible for the recurrence relation, while the numerator polynomial defines the initial conditions. From the generating series (10) we derive the following 6-term recurrence formula for $P(m,n)$ for the GEC:

$$\begin{aligned} P(m,n) = &-c_1 P(m,n-1) - c_2 P(m-1,n-1) + \\ &-c_3 P(m,n-2) - c_4 P(m-1,n-2) + \\ &-c_5 P(m-2,n-2), \end{aligned} \tag{12}$$

for $m \geq 0$, $n > 1$, where the coefficients $\{c_i\}_{i=1}^{5}$ are given by (11). The initial conditions are:

$$\begin{aligned} P(m,n) &= 0, \text{ for } m,n<0,\ m>n;\ \ P(0,0)=1; \\ P(0,1) &= \boldsymbol{\pi}\mathbf{P}(0)\mathbf{1} = c_{1p} - c_1; \\ P(1,1) &= \boldsymbol{\pi}\mathbf{P}(1)\mathbf{1} = c_{2p} - c_2. \end{aligned}$$

We now turn to the calculation of $P(m,n)$ for the error sequence produced by the super channel. This statistics is sensitive to the operation of the constrained decoder. The next examples consider specific constrained codes.

Example 1 (1/2-rate NRZ Miller code). The encoder and decoder operations of the 1/2-rate nonreturn-to-zero (NRZ) Miller code are explained in [1]. At the kth interval, the bit c_k is the input to the constrained encoder yielding two output bits, $x_{k,1}x_{k,2}$, which are the inputs to the FSC model. The two output bits of the channel, $y_{k,1} = x_{k,1} \oplus e_{k,1}$ and $y_{k,2} = x_{k,2} \oplus e_{k,2}$, where $\oplus$ is modulo 2 addition, are the input to the constrained decoder. The decoder maps these two bits onto the symbol r_k, according to the following rule:

$$\begin{aligned} r_k = y_{k,1} \oplus y_{k,2} &= (x_{k,1} \oplus x_{k,2}) \oplus (e_{k,1} \oplus e_{k,2}); \\ &= c_k \oplus z_k, \end{aligned}$$

where $z_k = e_{k,1} \oplus e_{k,2}$ is the the binary error sequence for the super channel. It is clear that z_k is equal to 0, that is $r_k = c_k$, if $e_{k,1}e_{k,2}$= 00 or 11. Otherwise, z_k is equal to 1, that is $r_k \neq c_k$, if $e_{k,1}e_{k,2}$ = 01 or 10. From this explanation, we conclude that the probability $P(m,n)$ for the error sequence $\{z_k\}_{k=1}^{\infty}$ for the 1/2-rate NRZ Miller code is expressed in a form similar to (9):

$$P(m,n) = [s^m z^n]\boldsymbol{\pi}(\mathbf{I} - \mathbf{P}'(1)sz - \mathbf{P}'(0)z)^{-1}\mathbf{1}, \tag{13}$$

where the matrices $\mathbf{P}'(0)$ and $\mathbf{P}'(1)$ are:

$$\begin{aligned} \mathbf{P}'(0) &= \mathbf{P}(0)\,\mathbf{P}(0) + \mathbf{P}(1)\,\mathbf{P}(1); \\ \mathbf{P}'(1) &= \mathbf{P}(0)\,\mathbf{P}(1) + \mathbf{P}(1)\,\mathbf{P}(0). \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} P(m,n) = [s^m z^n]\boldsymbol{\pi}\{\, &\mathbf{I} - [\,\mathbf{P}(0)\,\mathbf{P}(1) + \mathbf{P}(1)\,\mathbf{P}(0)]\,sz + \\ &-[\,\mathbf{P}(0)\,\mathbf{P}(0) + \mathbf{P}(1)\mathbf{P}(1)]\,z\}^{-1}\mathbf{1}, \end{aligned} \tag{15}$$

yielding the same recursion formula given by (12), but with the following coefficients:

$$\begin{aligned} c_1 &= -(1-q)^2(1-2b+2b^2) - (1-Q)^2(1-2g+2g^2) + \\ &\quad -2qQ(1-b-g+2gb); \\ c_2 &= -2(1-Q)^2(1-g)g - 2qQ(b+g-2gb) + \\ &\quad -2(1-q)^2(1-b)b; \\ c_3 &= (1-Q)^2(1-q)^2(1-2b+2b^2)(1-2g+2g^2) + \\ &\quad -qQ(1-q)(1-Q)[b^2(1-g)^2 + 2g^2b^2 + \\ &\quad +2(1-g)^2(1-b)^2 + 2(1-g)(1-b)gb + \\ &\quad +g^2(1-b)^2] + q^2Q^2(1-b-g+2gb)^2; \\ c_4 &= 2[b(1-b)(1-2g+2g^2) + \\ &\quad +g(1-g)(1-2b+2b^2)](1-q-Q)^2; \\ c_5 &= 4(1-Q)^2(1-g)g(1-q)^2(1-b)b + \\ &\quad +q^2Q^2(b+g-2gb)^2 + \\ &\quad -(1-Q)Q(1-q)q[6(1-g)(1-b)bg + \\ &\quad +(1-g)^2b^2 + g^2(1-b)^2], \end{aligned}$$

and initial conditions:

$$\begin{aligned} P(m,n) &= 0, \text{ for } m,n<0,\ m>n;\ \ P(0,0)=1; \\ P(0,1) &= \boldsymbol{\pi}\mathbf{P}'(0)\mathbf{1}; \\ P(1,1) &= \boldsymbol{\pi}\mathbf{P}'(1)\mathbf{1}, \end{aligned}$$

where the matrices $\mathbf{P}'(0)$ and $\mathbf{P}'(1)$ are given by (14).

Example 2 (Systematic 1/2-rate Miller code). We consider now a systematic 1/2-rate Miller code [1]. At each interval, the constrained decoder receives a pair of bits, $y_{k,1}y_{k,2}$, and the decoded bit r_k is read directly from the second bit $y_{k,2}$. The first received bit $y_{k,1}$ is discarded in the decoding process. The error bit for the super channel z_k is equal to one if $e_{k,1}e_{k,2} = (0 \cup 1)1$, or z_k is equal to zero if $e_{k,1}e_{k,2} = = (0 \cup 1)0$. $P(m,n)$ is given by formula (13), where

$$\begin{aligned} \mathbf{P}'(0) &= \mathbf{P}\mathbf{P}(0); \\ \mathbf{P}'(1) &= \mathbf{P}\mathbf{P}(1). \end{aligned} \tag{16}$$

Thus

$$P(m,n) = [s^m z^n]\boldsymbol{\pi}(\mathbf{I} - \mathbf{P}\mathbf{P}(1)sz - \mathbf{P}\mathbf{P}(0)z)^{-1}\mathbf{1}. \tag{17}$$

The recursion formula for $P(m,n)$ for the GEC is given by (12) with the following coefficients:

$$\begin{aligned} a_1 &= -(1-q)^2(1-b) - qQ(2-b-g) + \\ &\quad -(1-Q)^2(1-g); \\ a_2 &= (2-q-Q)(Qg+qb) - (b+g); \\ a_3 &= -2(1-Q)Q(1-g)(1-q)q(1-b) + \\ &\quad +(1-Q)^2(1-g)(1-q)^2(1-b) + \\ &\quad +Q^2q^2(1-b)(1-g); \\ a_4 &= (b+g-2gb)[-2Q(1-Q)q(1-q) + \\ &\quad +(1-Q)^2(1-q)^2 + Q^2q^2]; \\ a_5 &= bg[(1-q)^2 - Q(2-2q-Q)], \end{aligned} \tag{18}$$

and initial conditions:

$$\begin{aligned} P(m,n) &= 0, \text{ for } m,n<0,\ m>n;\ \ P(0,0)=1; \\ P(0,1) &= \boldsymbol{\pi}\mathbf{P}'(0)\mathbf{1}; \\ P(1,1) &= \boldsymbol{\pi}\mathbf{P}'(1)\mathbf{1}, \end{aligned} \tag{19}$$

where the matrices $\mathbf{P}'(0)$ and $\mathbf{P}'(1)$ are given by Eqs. (16). To exemplify an application of recursion given by formulae (12), (18) and (19) we consider a situation where an error correcting (15,7) BCH code with error correction capability $t=2$ is used to correct the errors produced by a super channel. Figure 2 shows the probability of codeword error, PCE = $\sum_{m=t+1}^{n} P(m,n)$, as a function of the average burst length of the GEC, $\lambda = 1/q$. PCE for a GEC alone is also shown in the figure. The following channel parameters are held fixed $q/Q = 20$, $b = 0.3$, $g = 1\cdot 10^{-3}$.

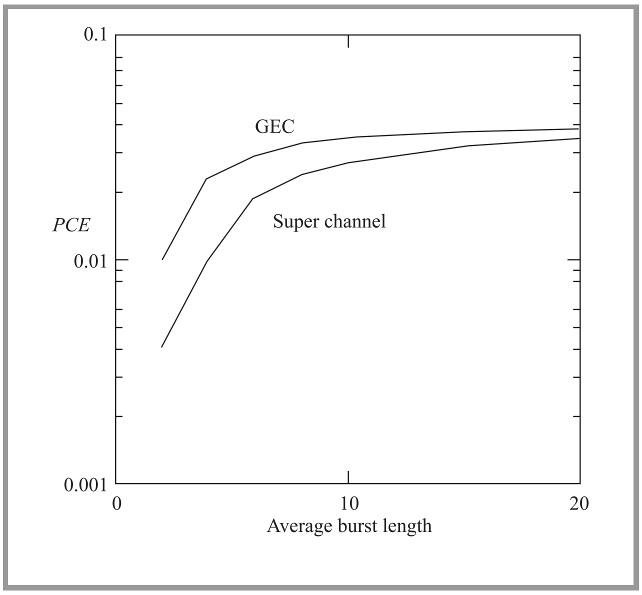

Fig. 2. Probability of codeword error as a function of the average burst length for a (15,7) BCH code on a super channel. The super channel is the concatenation of the systematic 1/2-rate Miller code and the GEC model.

3.2. The multigap distribution

It was conjectured in [1] that the error sequence $\{z_k\}_{k=1}^{\infty}$ generated by a super channel comprised of an 1/2-rate NRZ Miller code and a renewal Fritchman channel can be modeled as a renewal Fritchman channel. In this section, we develop a statistics called the variance of the multigap distribution to disprove this hypothesis analytically, by showing the non-renewal nature of this super channel.

The length of a gap is the number of zeros between two errors plus one (the last error is included). The error process $\{E_k\}_{k=1}^{\infty}$ can be regarded as a sequence of gaps $\{G_k\}_{k=1}^{\infty}$, where G_k is the length of the kth gap. For example, the error sequence of length 9, $\mathbf{E}_9 = 110000001$ corresponds to the gap sequence $\mathbf{G}_9 = 117$ (by assumption, $E_0 = 1$).

Let the random variable $G^r = \sum_{i=k}^{k+r-1} G_k$ be the sum of r consecutive gap lengths. The multigap length distribution, denoted as $M(r,l)$, is defined as $M(r,l) = P(G^r = l)$.

If the error process is renewal, this means that $\{G_k\}_{k=1}^{\infty}$ are independent random variables, then the variance of G^r is $Var(G^r) = r\,Var(G_1)$. Therefore, a linear growth of $Var(G^r)$ with r is sufficient to show the renewal nature of the channel. The problem of finding an expression for the multigap distribution for FSC models was considered in [6]. It is shown in [6] that the generating series $H_M(s,z)$ for $M(r,l)$ is:

$$\begin{aligned} H_M(s,z) &= \sum_{r,l} M(r,l)\, s^r z^l; \\ &= \tfrac{1}{P(1)}\, \boldsymbol{\pi}\mathbf{P}(1)(\mathbf{I}-(\mathbf{I}-\mathbf{P}(0)z)^{-1}\mathbf{P}(1)sz)^{-1}\mathbf{1}. \end{aligned} \tag{20}$$

The variance of G^r, denoted as $Var(G^r)$, can be expressed as:

$$Var(G^r) = [s^r]\left\{\frac{\partial^2 H_M(s,z)}{\partial z^2}\right\}_{z=1} + \mathbf{E}\{G^r\}(1-\mathbf{E}\{G^r\}). \tag{21}$$

To study the renewal property of communication channels, Adoul [9] defined a quantity called variation coefficient $K(r)$ as:

$$Var(G^r) = K(r)\, Var_{BSC}(G^r), \tag{22}$$

where $Var_{BSC}(G^r) = r(1-P(1))/P(1)^2$ is the variance of G^r for the memoryless binary symmetric channel (BSC) channel with crossover probability $P(1)$. It is important to notice that for renewal processes $K(r) = K(1)$, for all r, that is, the curve $K(r)$ versus r is a constant for all r. An expression for $K(r)$ for the renewal (2,1)-FC model is given below:

$$\begin{aligned} K(r) = {} & (1+\lambda_0)(1+\lambda_1)[p_{2,1}(1-\lambda_0)+(1-\lambda_1)p_{2,0}] + \\ & -[(1-\lambda_0)p_{2,1}+(1-\lambda_1)p_{2,0}]^2 / [(1-\lambda_0)(1-\lambda_1) + \\ & +p_{2,0}(1-\lambda_1)+p_{2,1}(1-\lambda_0)][p_{2,0}(1-\lambda_1)+p_{2,1}(1-\lambda_0)]. \end{aligned}$$

We now investigate the renewal property of the error process $\{z_k\}_{k=1}^{\infty}$ of the super channel. To address this problem for a specific constrained code, we calculate the following generating series:

$$H_V(s) = \sum_{r=0}^{\infty} Var(G^r)\, s^r, \tag{23}$$

where $Var(G^r)$ is calculated for the process $\{z_k\}_{k=1}^{\infty}$ using Eqs. (20) and (21) with the matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$ replaced by $\mathbf{P}'(0)$ and $\mathbf{P}'(1)$ given by Eqs. (14) and (16). If the process is renewal, we have:

$$H_V(s) = Var(G^1)\frac{s}{(1-s)^2} = Var(G^1)\sum_{r=0}^{\infty} r\, s^r. \tag{24}$$

We consider first the super channel comprised of an 1/2-rate NRZ Miller code and a (2,1)-FC with matrices $\mathbf{P}(0)$ and

$\mathbf{P}(1)$ given by Eqs. (5) and (6). Using the matrices $\mathbf{P}'(0)$ and $\mathbf{P}'(1)$ given by Eqs. (14) we found that $H_V(s)$ for this super channel is of the form:

$$H_V(s) = \frac{b_1 s}{(1-s)^2} \frac{(b_2 + b_3 s)}{(b_4 + b_5 s)}, \qquad (25)$$

where $\{b_i\}_{i=1}^{5}$ are distinct constants depending on the FSC parameters. Therefore, we proved that the super channel under consideration is non-renewal. In [1] a renewal (2,1)-FC model was proposed to represent this super channel when the inner channel is the Tsai's (2,1)-EFC model given by Eq. (7). Numerical values for $K(r) \cdot r$ for this super channel is shown in Table 1. The values of $K(r)$ in the table differ only in the third significant digit, indicating that the renewal approximation is valid for these specific parameters.

Table 1
Variation coefficient $K(r)$ versus the number of consecutive gap lengths r. The super channel is the concatenation of the 1/2-rate NRZ Miller code and the Tsai's (2,1)-EFC model given by Eq. (7)

r	$K(r)$
1	4.21862
2	4.21791
3	4.21764
4	4.21750
10	4.21728
20	4.21718
30	4.21715

When we consider the systematic 1/2-rate Miller code with matrices $\mathbf{P}'(0)$ and $\mathbf{P}'(1)$ given by Eqs. (16) the generating series $H_V(s)$ satisfies Eq. (24) and this super channel is renewal. Noticed that, in this case, the process $\{z_k\}_{k=1}^{\infty}$ is a sample of the error sequence $\{e_k\}_{k=1}^{\infty}$, or $z_k = e_{2k}$. In general, we can show that if the samples are spaced l intervals apart, or $z_k = e_{lk}$, the renewal property is maintained. Table 2 shows that values of $K(1)$ versus l for the sampled process $z_k = e_{lk}$, when the inner channel is the (2,1)-FC model given by Eq. (7).

Adoul [9] defined a process whose $K(1)$ is greater than one as a *more variable* process, in the sense that the gap lengths spread widely from their mean value (errors have a trend to be clustered). The further $K(1)$ is from 1, the more pronounced is this trend. Table 2 shows that $K(1) = 4.443$ for the (2,1)-FC model ($l = 1$), and $K(1)$ decreases to 2.761 for the super channel with the systematic 1/2-rate NRZ Miller code ($l = 2$). As the value of l increases, the process $\{z_k\}_{k=1}^{\infty}$ tends to become memoryless and $K(1)$ tends to 1. If we encompass the FSC model with an interleaving and a deinterleaving with finite interleaving depth l, we can regard the sampled process $z_k = e_{lk}$ as the error sequence at each row of the deinterleaver. We can use the results in Table 2 to investigate the ideal value of l that renders the channel memoryless. Numerical values for $K(r) \cdot r$

Table 2
$K(1)$ versus interleaving depth l.
The inner channel is the (2,1)-FC model given by Eq. (7)

l	$K(1)$
1	4.443
2	2.761
3	2.100
4	1.790
5	1.600
10	1.235
20	1.068
30	1.025
40	1.0099

for the GEC and the concatenation of the 1/2-rate NRZ Miller code and the GEC are shown in the solid curves of Fig. 3. The channel parameter considered are $Q = 4 \cdot 10^{-6}$, $q = 4.7 \cdot 10^{-4}$, $b = 0.3$ and $g = 1 \cdot 10^{-3}$. The positive increment $K(2) - K(1)$ indicates that both channels have positive correlation between gap intervals. The analysis of the super channel may be simplified if we define a new FSC model that characterizes the error structure of the super channel. The triangle symbols in the figure shows the behavior of the $K(r)$ for a GEC that represents the super channel under consideration, where we found that the parameters of the new GEC are $Q = 8 \cdot 10^{-6}$, $q = 9.3 \cdot 10^{-4}$, $b = 0.42$ and $g = 2 \cdot 10^{-3}$. These results differed from the solid curves in the third significant digit.

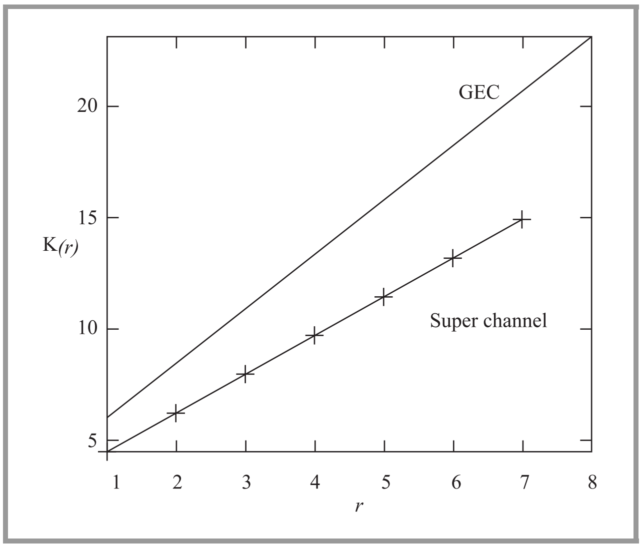

Fig. 3. Variation coefficient $K(r)$ as a function of the number of consecutive gap lengths r. The super channel is the concatenation of the 1/2-rate NRZ Miller code and the GEC model.

Equally accurate results were obtained for the $P(m,n)$ statistics. From this analysis, we conclude that the GEC represents the super channel with high accuracy.

4. Conclusions

We have studied the statistics of a binary error process generated by a super channel comprised of a specific constrained code and a general non-renewal FSC model. The theory proposed in [6] is extended to compute the generating series for the error weight distribution in terms of the channel matrices $\boldsymbol{\pi}$, $\mathbf{P}$, $\mathbf{P}(0)$ and $\mathbf{P}(1)$, for two specific constrained codes. The variance of the multigap distribution was employed to prove that the incorporation of a constrained code into the system may convert a renewal burst channel into a non-renewal super channel. This analysis was applied to a sampled error process in order to investigate the ideal value of the interleaving depth. Finally, our results suggest that an FSC could be directly used to model a super channel, simplifying the analysis of the overall system.

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