Paper

Error probability and error stream properties in channel with slow Rician fading

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Abstract — In a radio communication channel wave parameters fluctuate randomly. The signal envelope undergoes deep fades. When binary information is transmitted through such a channel, fading causes random variation of probabilities of error associated with the detection of individual elementary signals, which produces a clustering of errors. The paper presents an analytical description of the probability of bit error in the channel with very slow Rician fading and Gaussian noise for noncoherent and coherent detection. Digital systems employing error detection or error correction coding are generally based on the transmission of blocks of *N* sequential bits. Expressions are given for the probability of *n* errors occurring in *N* bits (weighted spectrum of errors) and the probability of more than *n* errors in a block of *N* bits (block error probability) for noncoherent frequency shift keying (NCFSK). Also the calculations are presented graphically.

Keywords — ***Rician fading channels, multipath propagation, bit error probability, stream of errors, weighted spectrum of errors, block error probability.***

1. Introduction

Transmission of signal in digital radio communication systems takes place in the presence of random additive and random multiplicative disturbances; the multiplicative disturbances called ordinary as fading. We assume that the additive disturbances are represented by white Gaussian noise with zero mean value. The fading is considered as nonfrequency selective. This is valid for most cases of mobile data communications with moderate bit rates. The fading process is assumed to be stationary and slowly varying compared with the N bits duration; it is constant during data block duration. We assume that the fading is described by the Rician distribution. It is one of the double-parameter distribution of the signal envelope allowing to describe propagation conditions existing in radio channel in greater detail than a simpler and more frequently applied single parameter Rayleigh distribution. The Rayleigh and Rician distribution are only special case solutions of the random vector problem. The Nakagami distribution provides a more general solution.

The Rician distribution is an analytical model sufficient for a channel where the useful signal $s(t)$ is a sum of the stationary diffuse Gausssian signal $x(t) = a_x \cos[\omega_0 t + \varphi_x(t)]$ with zero mean value and the direct harmonic signal $A\cos_0 t$, i.e.:

$$s(t) = A\cos\omega_0 t + a_x \cos\left[\omega_0 t + \varphi_x(t)\right] = \\ = r\cos\left[\omega_0 t + \varphi_s(t)\right] . \qquad (1)$$

The Rician model covers the superposition of a random Rayleigh signal with the fixed nonrandom signal. The Rician distribution can also be closely approximated by the Nakagami distribution. The Rician fading model applies in microcellular and satellite radio communication channels.

In radio communication channel radio waves parameters fluctuate randomly. Data transmission from and to mobile terminals suffers from fading effects caused by multipath propagation. The signal envelope undergoes deep fades. When binary information is transmitted through such a channel, fading causes random variation of probabilities of error associated with the detection of individual elementary signals. Deep fades cause bursts of bit errors in the transmitted data, i.e. errors in digital transmission over fading channels occur in bursts. In digital communication an important quantity is the bit error probability.

This paper presents an expression for the average probability of bit error for binary transmission in Rician channel with noncoherent frequency shift keying (NCFSK), differentially coherent phase shift keying (DPSK), coherent frequency shift keying (CFSK) and coherent phase shift keying (CPSK). Formulas for the average weighted spectrum of errors and the average block error probability for NCFSK are also presented.

2. Average probability of bit error

The static bit error probability for several common binary modulation schemes with optimum detection of nonfading signals in Gaussian noise is given by the formula

$$P_s(\rho) = \begin{cases} \dfrac{\exp(-\alpha\rho)}{2} & \text{for NCFSK, DPSK} \\ \dfrac{\operatorname{erfc}(\sqrt{\alpha\rho})}{2} & \text{for CFSK, CPSK} \end{cases} \qquad (2)$$

where: ρ is the instantaneous signal-to-noise power ratio (SNR), $\operatorname{erfc}(\sqrt{x})$ denotes the error function [4], $\alpha = 0.5$ for NCFSK, CFSK and $\alpha = 1$ for DPSK, CPSK.

Since in the Eq. (2) the argument of the error function appears in the lower limit of the integral, it is analytically difficult to perform averages of this equation. Another form for static bit error probability is presented in [2]:

$$P_s(\rho) = \frac{(a\rho)^b}{\Gamma(b)} \int\limits_0^{\Pi/2} \frac{\cos\varphi}{(\sin\varphi)^{2b+1}} \exp\left(\frac{-a\rho}{\sin^2\varphi}\right) d\varphi\,, \qquad (3)$$

where a and b are the coefficients which depend on the particular form of modulation and detection, i.e. $a = b = 0.5$ for CFSK, $a = 0.5$, $b = 1$ for NCFSK, $a = 1$, $b = 0.5$ for CPSK, $a = b = 1$ for DPSK.

Assuming Rician fading, the envelope r of the useful signal $s(t)$, described by Eq. (1), has the probability density function:

$$p(r) = \frac{r}{\sigma_x^2} \exp\left(-\frac{r^2 + A^2}{2\sigma_x^2}\right) I_0\left(\frac{Ar}{\sigma_x^2}\right); \quad r \geq 0\,, \qquad (4)$$

where: A is the envelope of the direct component of the useful signal $s(t)$, σ_x^2 is the variance of the diffused component $x(t)$ of the useful signal $s(t)$, $I_0(\alpha)$ is the zero order modified Bessel function of the first kind [4].

Probability density function of the square of the envelope $r(t)$ of the useful signal is given by the formula:

$$p(r^2) = \frac{1}{2\sigma_x^2} \exp\left(-\frac{r^2 + A^2}{2\sigma_x^2}\right) I_0\left(\frac{Ar}{\sigma_x^2}\right); \quad r \geq 0\,. \qquad (5)$$

The average value of the square of the envelope $r(t)$ can be written as

$$E(r^2) = A^2 + 2\sigma_x^2\,, \qquad (6)$$

where $E(x)$ denotes the expected value of the argument.

From Eq. (5) we can describe the distribution of random variable, which represents ρ defined as the instantaneous ratio power of the useful signal to average power N_0 of the additive Gaussian noise, i.e. the instantaneous signal-to-noise power ratio:

$$\rho = \frac{r^2}{2N_0}\,. \qquad (7)$$

The probability density function of ρ in Rician channel is denoted by the formula:

$$p(\rho) = \frac{N_0}{\sigma_x^2} \exp\left(-\frac{2N_0\rho + A^2}{2\sigma_x^2}\right) I_0\left(\frac{A\sqrt{2N_0\rho}}{\sigma_x^2}\right); \quad \rho \geq 0\,. \qquad (8)$$

With sufficiently slow fading, the average probability of bit error $P_D(A, \sigma_x, N_0)$, i.e. the dynamic probability of bit error, equals to $P_s(\rho)$ averaged over the distribution of SNR ρ:

$$P_D(A, \sigma_x, N_0) = \int\limits_0^\infty P_s(\rho)\, p(\rho)\, d\rho = E\left[P_s(\rho)\right]. \qquad (9)$$

Inserting Eqs. (2) and (8) into (9) gives the average probability of bit error in Rician channel [5]:

$$P_D(A, \sigma_x, N_0) = \frac{1}{2\Gamma(b)} \exp\left(\frac{-A^2}{2\sigma_x^2}\right) \times \times \sum_{k=0}^{\infty} \left(\frac{A^2}{2\sigma_x^2}\right)^k \frac{\Gamma(b+k+1)}{(k!)^2} B_g(k+1, b)\,, \qquad (10)$$

where: $\Gamma(b)$ is the gamma function [4], $g = \frac{N_0}{a\sigma_x^2 + N_0}$ and $B_g(x, y)$ is the incomplete beta function [4].

Let us introduce additional factors, which describe Rician channel and can be defined as

$$\rho_1 = \frac{A^2}{2\sigma_x^2}; \quad \rho_2 = \frac{\sigma_x^2}{N_0}; \quad \rho_3 = \frac{A^2}{2N_0} = \rho_1\rho_2\,. \qquad (11)$$

Then, the expected value of the SNR ρ we can express as

$$E(\rho) = \rho_0 = \rho_3 + \rho_2\,. \qquad (12)$$

In literature the factor ρ_1 (direct signal to diffused signal power ratio) sometimes is denoted as K and ρ_0 (average SNR) is denoted as Γ [1].

In the end, the formula for the average probability of bit error in Rician channel can be written as

$$P_D(\rho_1, \rho_2) = = \frac{1}{2\Gamma(b)} \exp(-\rho_1) \sum_{k=0}^{\infty} \rho_1^k \frac{\Gamma(b+k+1)}{(k!)^2} B_{g1}(k+1, b)\,, \qquad (13)$$

where $g1 = \frac{1}{a\rho_2 + 1}$ or using the relation (12)

$$P_D(\rho_1, \rho_0) = \frac{1}{2\Gamma(b)} \exp\left(-\rho_1\right) \times \times \sum_{k=0}^{\infty} \rho_1^k \frac{\Gamma(b+k+1)}{(k!)^2} B_{g2}(k+1, b)\,, \qquad (14)$$

where $g2 = \frac{1+\rho_1}{a\rho_0 + 1 + \rho_1}$.

The error-rate formulas (10), (13), (14) are valid for four principal cases of binary modulation in Rician channels. The result over a Rayleigh channel can be obtained from our results when $A = 0$. However, in the border case when $\rho_2 = 0$, we have the formula for bit error probability in channel without fading. Figure 1 presents the average bit error probability for binary NCFSK modulation in Rician channel as a function of ρ_0 for various values ρ_1.

For detection in the noncoherent systems (when $b = 1$) the formula for the average probability of bit error in Rician channel can be expressed as

$$P_D(A, \sigma_x, N_0) = \frac{N_0}{2(\alpha\sigma_x^2 + N_0)} \exp\left[\frac{-A^2\alpha}{2(\alpha\sigma_x^2 + N_0)}\right] \qquad (15)$$

or

$$P_D(\rho_1, \rho_2) = \frac{1}{2(\alpha\rho_2 + 1)} \exp\left(-\frac{\alpha\rho_1\rho_2}{\alpha\rho_2 + 1}\right) \qquad (16)$$

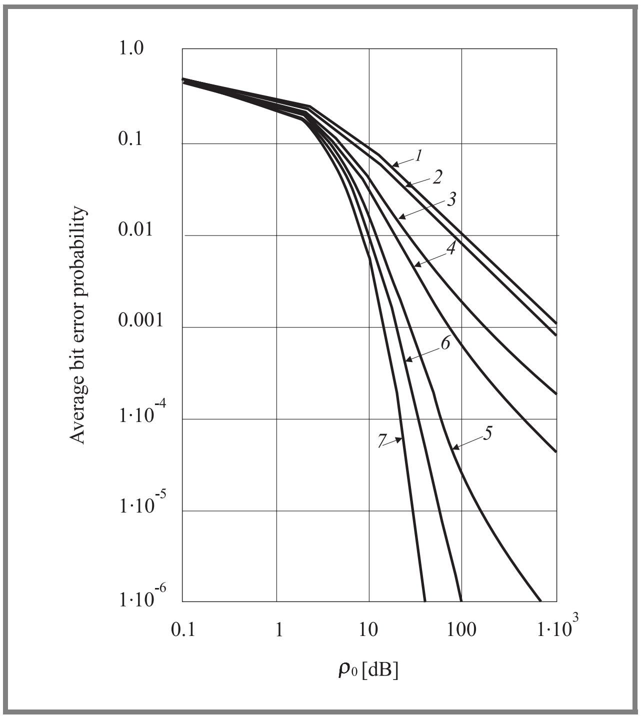

Fig. 1. The average bit error probability for NCFSK in channel with Rician fading: *1* – $\rho_1 = -10$; *2* – $\rho_1 = 0$; *3* – $\rho_1 = 5$; *4* – $\rho_1 = 7$; *5* – $\rho_1 = 10$; *6* – $\rho_1 = 12$; *7* – $\rho_1 = 16$ (*all values in dB*).

and after using the relation (12) we can rewrite it as

$$P_D(\rho_1, \rho_0) = \frac{1+\rho_1}{2(\alpha\rho_0+\rho_1+1)} \exp\left(-\frac{\alpha\rho_1\rho_0}{\alpha\rho_0+\rho_1+1}\right) \tag{17}$$

where $\alpha = 0.5$ for NCFSK and $\alpha = 1$ for DPSK.

3. Average weighted spectrum of errors

In case of digital transmission over a fading channel, time variation causes the change of bit error probability with the effect of clustering errors in the received signal. Forward error correction is often used to break up the clustering [3, 7, 8]. Digital systems employing error detection or error correction coding are generally based on the transmission of blocks of N bits. The average probability of bit error specified by Eq. (10) neither describes the number of errors nor their placement in the stream of errors. In a communication system which transmits data in blocks on N bits the probability of n errors in a block and the probability of more than n errors in a block are an important quantities which describe the stream of errors in channel with fading.

The weighted spectrum of errors, i.e., the probability of n errors occurring in a transmission of N bits, for independent bit errors is given by the binomial distribution

$$P(L_N = n) = \binom{N}{n} P_s^n(\rho)\,[1-P_s(\rho)]^{N-n} \qquad n = 0, 1, \ldots, N\,. \tag{18}$$

The average probability of n errors in a block of N bits (average weighted spectrum of errors) is denoted as [5, 6]

$$P_D(L_N = n) = \int_0^\infty P(L_N = n)p(\rho)\mathrm{d}\rho = \binom{N}{n}\sum_{j=0}^{N-n}\binom{N-n}{j}(-1)^j E\left[P_s^{j+n}(\rho)\right]. \tag{19}$$

The average in Eq. (19) is formed over the instantaneous SNR ρ which has the probability density function $p(\rho)$ described by Eq. (8).

Assuming that the Rician fading is very slow, nonselective and independent, then the instantaneous SNR remains the same over a block of N bits. For noncoherent FSK in channel with Rician fading the average weighted spectrum of errors is given by

$$P_D(L_N = n) = 2\binom{N}{n} N_0 \sum_{i=0}^{N-n}\binom{N-n}{i}(-1)^i \times \left(\frac{1}{2}\right)^{i+n}\frac{1}{(n+i)\sigma_x^2+2N_0}\exp\left(\frac{-A^2(n+i)}{2\left[\sigma_x^2(n+i)+2N_0\right]}\right). \tag{20}$$

Using the relations (11) and (12) we can rewrite formula (20) as

$$P_D(L_N = n) = 2\binom{N}{n}\sum_{i=0}^{N-n}\binom{N-n}{i}(-1)^i\left(\frac{1}{2}\right)^{i+n} \times \frac{1+\rho_1}{(n+i)\rho_0+2(1+\rho_1)}\exp\left(\frac{-\rho_1\rho_0(n+i)}{\rho_0(n+i)+2(1+\rho_1)}\right). \tag{21}$$

Equation (21) has been used to calculate the average probability of n errors in N bits. Figure 2 shows the $P_D(L_N = n)$ for $N = 30$ and for various values of ρ_1 and ρ_0 ; the values of ρ_1 and ρ_0 are as taken as $P_D(\rho_1, \rho_0) = 10^{-3}$.

Additional, Fig. 3 shows the average weighted spectrum of errors in N bits for NCFSK for different values of N, ρ_1 and ρ_0.

The probability $P_D(L_N = n)$ takes into account only the total number of errors and disregards their distribution. It is useful only for the performance evaluation of random

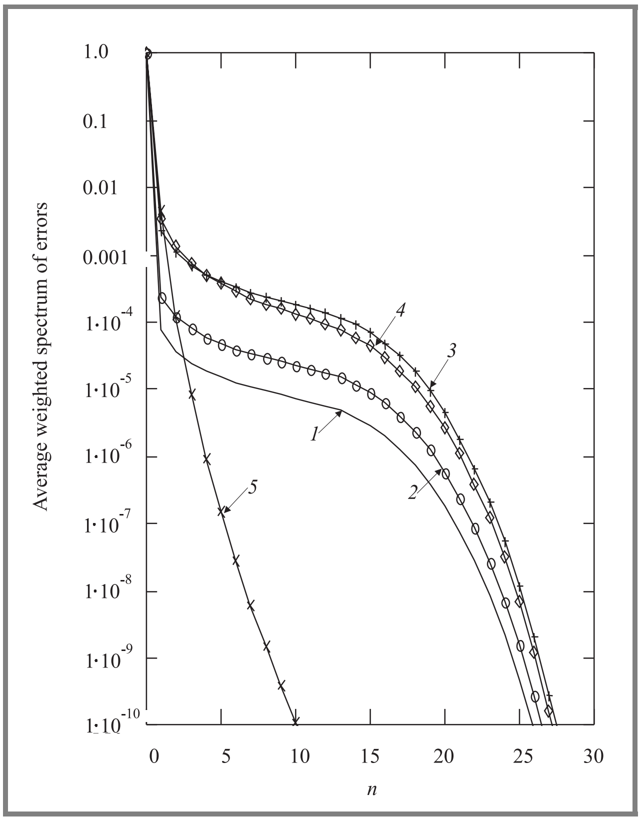

Fig. 2. Average probability of n errors in $N = 30$ bits for NCFSK: *1* – $\rho_1 = 0$, $\rho_0 = 43$; *2* – $\rho_1 = 1$, $\rho_0 = 38$; *3* – $\rho_1 = 5$, $\rho_0 = 23$; *4* – $\rho_1 = 7$, $\rho_0 = 19$; *5* – $\rho_1 = 16$, $\rho_0 = 13$ (*all values in dB*).

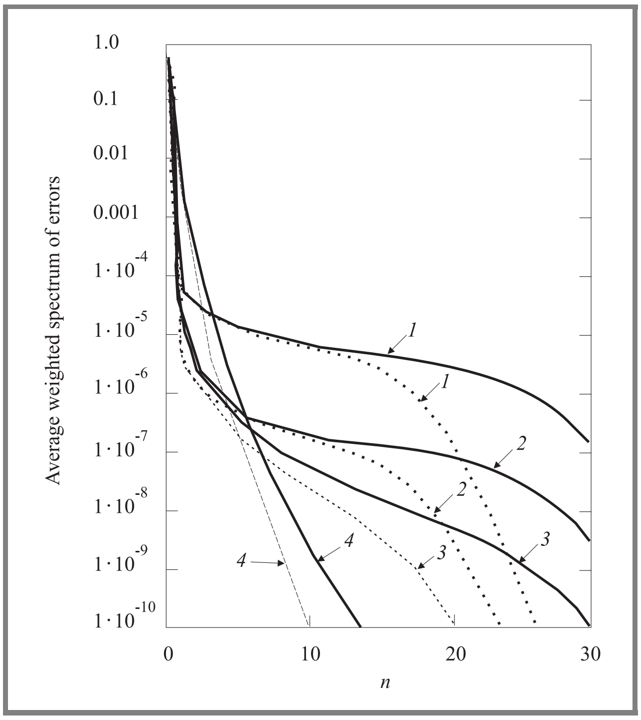

Fig. 3. Average probability of n errors in N bits for NCFSK; solid line $N = 48$, dot line $N = 30$: *1* – $\rho_1 = 0$, $\rho_0 = 43$; *2* – $\rho_1 = 10$, $\rho_0 = 28$; *3* – $\rho_1 = 12$, $\rho_0 = 20$; *4* – $\rho_1 = 16$, $\rho_0 = 13$ (*all values in dB*).

error-correcting codes. It cannot be used for burst-error-correcting codes. If the random error-correcting code is used, with the code capable of correcting up to n random errors, then the probability of correct decoding is $\sum_{i=0}^{n} P_D(L_N = i)$.

4. Average block error probability

When the system with burst-error correcting code is used the probability of correct decoding cannot be expressed only in terms of $P_D(L_N = n)$. The burst-error-correction code can correct all error vectors with length less than or equal to n. In this case the important quantity is the average probability of more than n errors in a block of N bits, i.e., average block error probability. It is denoted by

$$P_D(L_N > n) =$$

$$= 1 - \sum_{i=0}^{n} \binom{N}{i} \int_0^\infty P_s^i(\rho)\,[1 - P_s(\rho)]^{N-i}\, p(\rho)\mathrm{d}\rho =$$

$$= 1 - \sum_{i=0}^{n} \binom{N}{i} \sum_{j=0}^{N-i} \binom{N-i}{j} (-1)^j E\left[P_s^{j+i}(\rho)\right]. \quad (22)$$

Thus, from Eqs. (22), (2), (8), the average block error probability for NCFSK we can expressed as

$$P_D(L_N > n) = 1 - 2N_0 \sum_{j=0}^{n} \binom{N}{j} \sum_{i=0}^{N-j} \binom{N-j}{i} (-1)^i \times$$

$$\times \left(\frac{1}{2}\right)^{i+j} \frac{1}{(j+i)\sigma_x^2 + 2N_0} \exp\left(\frac{-A^2(j+i)}{2[\sigma_x^2(j+i) + 2N_0]}\right) \quad (23)$$

or in the form

$$P_D(L_N > n) = 1 - 2\sum_{j=0}^{n} \binom{N}{j} \sum_{i=0}^{N-j} \binom{N-j}{i} (-1)^i \left(\frac{1}{2}\right)^{i+j} \times$$

$$\times \frac{1+\rho_1}{(j+i)\rho_0 + 2(1+\rho_1)} \exp\left(\frac{-\rho_1\rho_0(j+i)}{\rho_0(j+i) + 2(1+\rho_1)}\right). \quad (24)$$

Figure 4 shows the average block error probability $P_D(L_{30} > 0)$ for NCFSK, i.e., the probability of at least one error in $N = 30$ bits.

When we plotted the expression (4) presented in references [1] for $D = 1$, i.e. for no diversity case, we become the same result as is shown in Fig. 4.

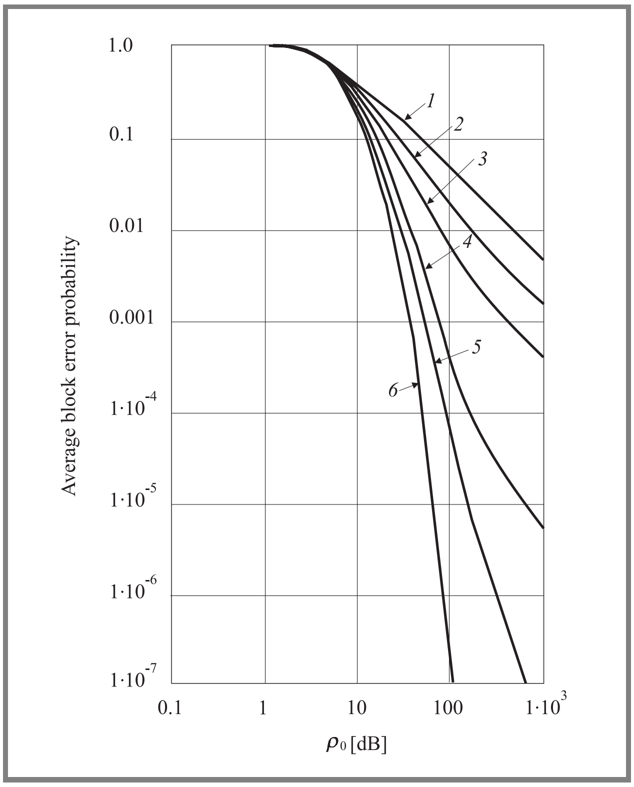

Fig. 4. Average block error probability $P_D(L_{30} > 0)$ for NCFSK and $N = 30$: *1* – $\rho_1 = 1$; *2* – $\rho_1 = 5$; *3* – $\rho_1 = 7$; *4* – $\rho_1 = 10$; *5* – $\rho_1 = 12$; *6* – $\rho_1 = 14$ (*all values in dB*).

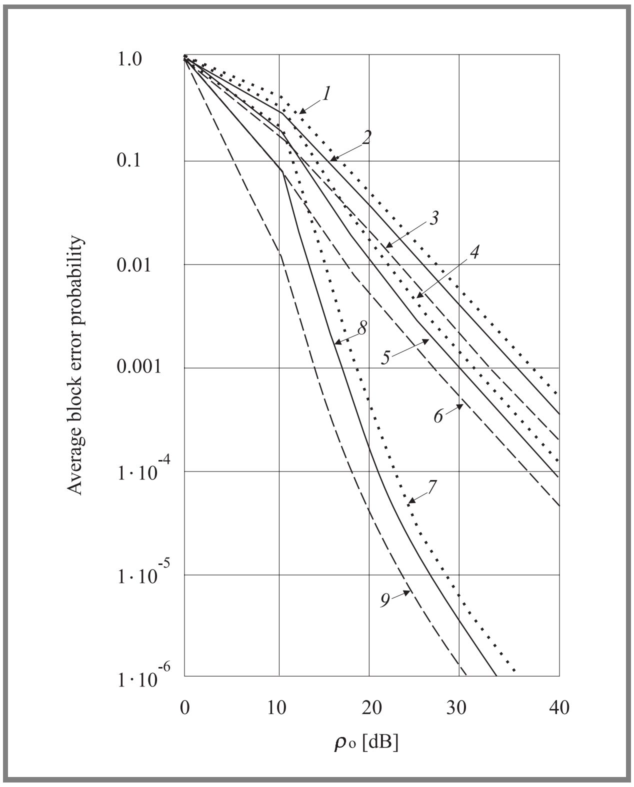

Fig. 5. Average block error probability $P_D(L_{30} > n)$ for NCFSK and $N = 30$: *1* – $\rho_1 = 0$, $n = 0$; *2* – $\rho_1 = 0$, $n = 1$; *3* – $\rho_1 = 0$, $n = 4$; *4* – $\rho_1 = 5$, $n = 0$; *5* – $\rho_1 = 5$, $n = 1$; *6* – $\rho_1 = 5$, $n = 4$; *7* – $\rho_1 = 10$, $n = 0$; *8* – $\rho_1 = 10$, $n = 1$; *9* – $\rho_1 = 10$, $n = 4$ (*all values in dB*).

Also Fig. 5 shows the average block error probability $P_D(L_{30} > 0)$ in a block of $N = 30$ bits for NCFSK and for various values of ρ_1 and n.

5. Conclusion

The presented equations are valid for nonselective, very slow and independent Rician fading. Since the presented expressions include results for the cases of Rayleigh fading and no fading, they can be widely used to evaluate the performance of error control techniques for mobile radio.

Presented results can be useful for error detection or error correction coding. In a communication system that transmits data in blocks of N bits an important quantity is the probability of more than n errors in block. If a simple automatic repeat request scheme is used, the throughput can be determined from $P_D(L_N > 0)$, i.e. from the probability of at least one error in blocks. However, if the use of forward error correction is to be investigated, the knowledge of $P_D(L_N > n)$ is required. In case of error detection a block is received correctly only if all N bits are received without error.

In Figs. 1 – 5 numerical results are presented, the influence of fading for error detection is presented. The obtained expression can easily be programmed using standard mathematical software package such as Mathcad.

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