Paper

Weight versus reference point multiple criteria decision making methods – analogies and differences

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Abstract — In this work we shall be concerned with interactive multiple criteria decision making methods. We show how on the technical level the class of reference point methods can be reduced to the class of weight methods. Though methods from these two classes represent two different interactive decision making paradigms, the equivalence observed opens a way for a joint implementation of a pair of methods each representing a different class. This would establish a firm ground for systematic comparison of both classes of methods as well as for hybrid schemes mixing decisional tools specific for each class.

Keywords* — *multiple criteria decision making, weight methods, reference point methods.

1. Introduction

A rough taxonomy of interactive multiple criteria decision making (MCDM) methods distinguishes three major classes, namely *weight methods, reference point methods*, and *constraint methods*. All methods of these classes amount to a partial, decision maker (DM) guided search of the set of *efficient* decisions. The trichotomy is based on which elements are manipulated to capture DM's preferences: weights, reference points, or constraints. In this presentation we shall confine ourselves to the first two classes, which are believed to capture DM's preferences in a favourable manner.

In weight methods the DM articulates his partial preferences pointing to preferred decisions in pairwise comparisons. Partial preferences are translated next into relations expressed in terms of weights. In some methods weights are provided explicitly by the DM. In reference point methods the DM articulates his preferences by pointing to reference points which can be any elements of the space of criteria.

In weight methods *the set of weights* is systematically searched and reduced according to DM articulated preferences. The volume of the set of weights is a natural measure of progress and convergence of the decision making process. Reference point methods lack such a systemic convergence indicator.

The purpose of this paper is to show that in technical terms reference point methods can be reduced to weight methods. With such an interpretation provided it is possible to implement methods of these two classes in the same technical framework. This would establish a firm ground for systematic comparison of both classes of methods as well as for hybrid schemes mixing decisional tools specific for each class. Moreover, a convergence indicator is then available for either class of methods.

The plan of the paper is as follows. In Section 2 we recall all the relevant definitions and formulations. In Section 3 we recall characterizations of the set of properly efficient decisions, namely the characterization by weight manipulations and the characterization by reference point manipulations, and in Section 4 we recall how these characterizations are used in the two classes of MCDM methods considered. In Section 5 we show that under a restriction of reference point methods, weight methods and reference point methods are technically equivalent. In Section 6 we discuss practical significance of such an equivalence. Section 7 concludes.

2. Preliminaries

In the multiple criteria decision making framework a decision problem is formalized as follows:

$$\text{choose “}\textit{the most preferred}\text{” vector } f(x),\ x \in X_0 \subseteq X\,, \tag{1}$$

where X is the space of decisions, X_0 is the set of feasible decisions, $f : X \to \mathcal{R}^k$ is the criteria map, where $f = (f_1, \ldots, f_k)$ and $f_l : X \to \mathcal{R}$, $l = 1, \ldots, k$, are criteria functions. We assume that all criteria are of the type “better if more”.

From the algorithmic point of view the above problem is ill-defined. As long as we do not know what “the most preferred” means precisely we are not in a position to propose a problem solving method. The only source of supplementary information to those already given in (1) can be the decision maker (DM). The underlying assumption of MCDM is that this information cannot be acquired from the DM at once.

A formal model for MCDM is offered by the *vector optimization problem*, namely

$$vmax\, f(x)\,,\ x \in X_0 \subseteq X\,, \tag{2}$$

where *vmax* stands for the identification of all *efficient* decisions of X_0. This problem is well-defined which means that under minor assumptions, satisfied in practical problems, the solution to (2) always exists.

Decisions are represented by their criteria values. With this in mind, from now on we shall be dealing with elements $f(x)$ of set $f(X_0)$ and for the sake of simplicity we shall use the notation

$$y = f(x) \text{ and } Z = f(X_0).$$

Elements of set Z we shall call *outcomes*. Under this convention, for given feasible decision x, $y_l = f_l(x)$ is the value of lth component of outcome $y = f(x)$. Thus, y_l is the value of lth criterion.

All properties of decisions we shall need throughout this paper can be defined in terms of outcomes. The notation x, X_0, $f(x)$, $f(X_0)$ has to be used only when one is to operationalize an implicit (i.e. in the form of constraints) feasible decision representation.

The element $\hat{y}$ representing the hypothetical decision which maximizes all objective functions, called *utopian* element, is calculated as

$$\hat{y}_l = max_{y\in Z} y_l, \; l = 1,...,k.$$

Definition 1. *The outcome* $\bar{y} \in Z$ *is* ***efficient*** *if* $y_l \geq \bar{y}_l$, $l = 1,...,k$, $y \in Z$, *implies* $y = \bar{y}$.

For clarity of presentation and without loss of generality, in this paper we confine ourselves exclusively to a subset of efficient outcomes, namely to *properly efficient* outcomes.

Definition 2 [3]. *The outcome* $y \in Z$ *is* ***properly efficient*** *if it is efficient and there exists a finite number* $M > 0$ *such that for each* i *we have*

$$\frac{y_i - \bar{y}_i}{\bar{y}_j - y_j} \leq M$$

for some j *such that* $y_j < \bar{y}_j$, *whenever* $y \in Z$ *and* $y_i > \bar{y}_i$.

The set of all properly efficient outcomes we shall denote by $\mathcal{P}$. The distinction between efficient and properly efficient outcomes, important in formal considerations, is of little importance in practical MCDM problems. It is enough to recall that in the case set Z is polyhedral or finite all efficient outcomes are properly efficient.

3. Pareto set characterizations

A corner stone for every interactive MCDM method is the ability to derive properly efficient outcomes. Every properly efficient outcome should be potentially derivable. The so-called characterizations of $\mathcal{P}$ are useful for this purpose. Bellow we recall two types of characterizations prized for their generality and therefore often exploited in MCDM methods, namely:

- the characterization by weight manipulations,
- the characterization by reference point manipulations.

Any of the above characterizations represents a parametric family of optimization problems.

3.1. Characterization by weight manipulations

The idea of characterizing the Pareto set by weight manipulations consists in constructing a *surrogate objective function* parameterized by k parameters – *weights*. A surrogate objective function when maximized (or minimized – depending on the surrogate objective function form) over Z yields properly efficient outcome of vector optimization problem (2) (cf. Fig. 1). By changing weights and solving the resulting optimization problems one derives different properly efficient outcomes.

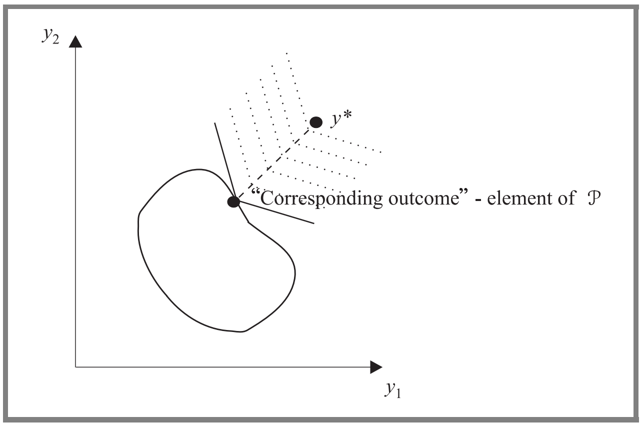

Fig. 1. Contours of a surrogate objective function and the properly efficient outcome "corresponding" to the selected vector of weights.

Below we shall make use of a selected element of the criteria space denoted y^* and defined as

$$y^*_l = \hat{y}_l + \varepsilon, \; l = 1,...,k,$$

where ε is any positive number and $\hat{y}$ is the utopian element defined in the previous chapter.

Sufficient condition for proper efficiency. An outcome which solves the optimization problem

$$\min_{y\in Z}\max_{l} \lambda_l((y^*_l - y_l) + \rho e^k(y^* - y)), \tag{3}$$

or the problem

$$\min_{y\in Z}\max_{l} \lambda_l(y^*_l - y_l) + \rho e^k(y^* - y), \tag{4}$$

where $\lambda_l > 0$, $l = 1,...,k$, $\rho > 0$, and e^k is the k-dimensional row vector with all components equal to one, is **properly efficient** [1, 4, 11–13].

The surrogate functions (3) and (4) are the most general forms of functions used in weight manipulation methods.

3.2. The characterization by reference point manipulations

The idea of characterizing the Pareto set by reference point manipulations consists in constructing a surrogate objective function parameterized by an element y of $\mathcal{R}^k$. A surrogate objective function when minimized over Z yields a properly efficient outcome of vector optimization problem (2) (cf. Fig. 2). By changing reference points and solving the resulting optimization problems one derives different properly efficient outcomes.

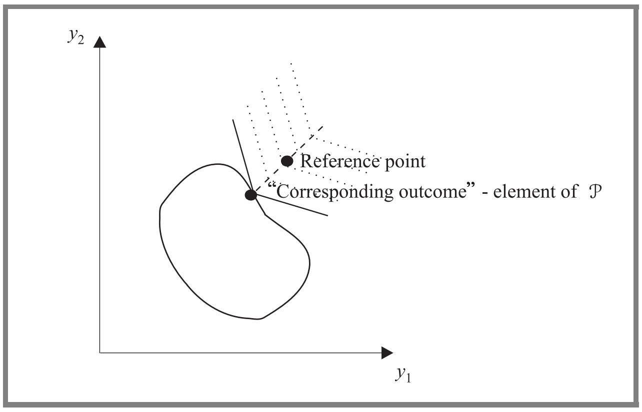

Fig. 2. Contours of a surrogate objective function and the properly efficient outcome "corresponding" to the selected reference point.

A continuous function $s_{\bar{y}}(y) : \mathcal{R}^k \to \mathcal{R}$, where $\bar{y} \in \mathcal{R}^k$, ($\bar{y}$ – a *reference point*), is called an *achievement function*. In the context of this paper it is required that an achievement function is *ε-strongly increasing* [10].

We define the following optimization problem:

$$\min_{y \in Z} s_{\bar{y}}(y) \,. \tag{5}$$

Let outcome $\check{y}$ be a solution of problem (5), i.e.

$$\check{y} = \arg\min_{y \in Z} s_{\bar{y}}(y) \,.$$

Sufficient condition for proper efficiency. If $s_{\bar{y}}$ is ε-strongly increasing, then outcome $\check{y}$ is **properly efficient** [10].
Functions (3) and (4) for each $\lambda,\ \lambda_l > 0,\ l = 1,...,k$, are ε-strongly increasing; they both are achievement functions with $\bar{y} = y^*$. Various other forms of achievement functions exists but for the properties required achievement functions (3) and (4) posses the simplest form.

4. Methods

4.1. Weight methods

In weight methods ([2, 8, 9, 14] to name just a few, the reader is referred to e.g. [7] for a more complete list of references) the DM articulates his preferences by pointing (directly or indirectly) to a vector of weights. Then a properly efficient outcome which "corresponds" to the selected vector of weights is determined with the help of a surrogate objective function (cf. Section 3.1). The notion of correspondance is intuitively explained in Fig. 1. By manipulating weights the DM is able to determine a subset of $\mathcal{P}$ set and from this subset select the most preferred outcome.
In that manner the set of weights is systematically searched and reduced. Search can be organized in the form of *weight cuts* (the Zionts-Wallenius method and the Dell-Karwan method) or *weight zooming* (the Tchebycheff method by Steuer). Reductions of the set of weights give rise to a natural stopping rule: search is terminated if the set of weights is so small that outcomes corresponding to weights from this set differ insignificantly. Other usual stopping rules such as limit of the elapsed time or limit of iterations are of purely technical nature.

4.1.1. Weight cut methods

In weight cut methods it is assumed that the surrogate objective function used approximates locally DM's implicit utility function. With such an assumption in place a pair of outcomes subjected to DM's evaluation yields a weight cut. With the surrogate function (3) and with two outcomes y^a, y^b we have

$$\max_l \lambda_l((y^*_l - y^a_l) + \rho e^k(y^* - y^a)) < \max_l \lambda_l((y^*_l - y^b_l) + \\ + \rho e^k(y^* - y^b)) \tag{6}$$

if the DM prefers y^a to y^b, and

$$\max_l \lambda_l((y^*_l - y^a_l) + \rho e^k(y^* - y^a)) > \max_l \lambda_l((y^*_l - y^b_l) + \\ + \rho e^k(y^* - y^b)) \tag{7}$$

otherwise.

The cut (6) or (7) reduces the set of vectors λ. Vectors λ from the reduced set are selected and problem (3) is solved to derive elements of $\mathcal{P}$ for successive DM evaluations.

4.1.2. The Tchebycheff method

The so-called Tchebycheff method exploits problem (3) to determine properly efficient outcomes (in the original version of the method $\rho = 0$).
The method consists of the following operations: selecting a number of vectors $\lambda,\ \lambda_l > 0,\ l = 1,...,k$, and then, iteratively:

– solving problem (3) for all selected λ to derive a number of properly efficient outcomes,

– selecting by the DM the most preferred outcome $\tilde{y}$,

– selecting a number of vectors λ, $\lambda_l > 0$, $l = 1,...,k$, in a neighborhood of $\tilde{\lambda}$ corresponding to the most preferred outcome $\tilde{y}$.

The above process has an effect of "zooming" in the set of weights in a quest for weights which yield a sequence of increasingly (or at least non decreasingly) preferred outcomes.

4.2. Reference point methods

In the simplest version of reference point methods the DM articulates his preferences by pointing to a reference point. The reference point can be an outcome, i.e. an element of Z, or any other element of $\mathcal{R}^k$. Then a properly efficient outcome which "corresponds" to the reference point and the achievement function used (cf. Section 3.2) is determined. The notion of correspondence is intuitively explained in Fig. 2. By manipulating reference points the DM is able to determine a subset of $\mathcal{P}$ set and from this subset select the most preferred outcome.

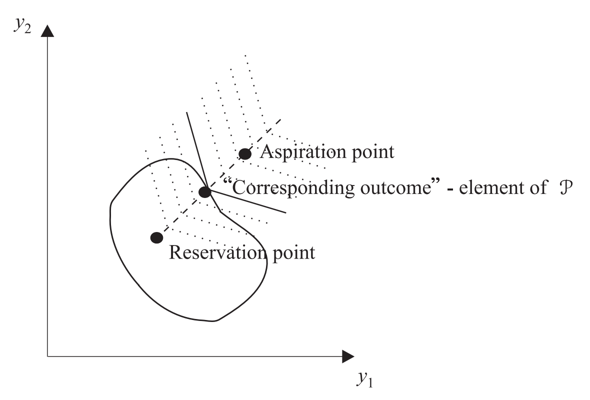

Fig. 3. Contours of a surrogate objective function and the properly efficient outcome "corresponding" to the selected pair of reservation-aspiration points.

A variant of reference point methods admits also DM pointing to a pair of reference points; a point y^{res} called a *reservation point* and a point y^{asp}, $y^{asp} \in y^{res} + int(\mathcal{R}^k_+)$, called an *aspiration point*, where $int(\cdot)$ denotes the interior of a set. It is quite natural to assume that $y^{res} \in Z$ and $y^{asp} \notin Z$ provided such points are easily identifiable. In general, the condition $y^{asp} \in y^{res} + int(\mathcal{R}^k_+)$ is sufficient. It is possible then to construct an achievement function such that an outcome y which minimizes that function over Z is an element of $\mathcal{P}$ farthest from the reservation point and at the same time closest to the aspiration point. One such an achievement function is the function (3), where

$$\lambda_l = \frac{1}{y_l^{asp} - y_l^{res}}, \; l = 1,...,k.$$

This is schematically illustrated in Fig. 3.
In reference point methods no explicit evaluations (comparisons) of outcomes take place.

5. Weight versus reference point methods

5.1. Weight versus reference point methods – methodological level

On the methodological level weight methods and reference point methods represent two entirely different decision making paradigms.
In weight methods it is assumed (assumption A), often implicitly, that at each iteration of the interactive decision making process the DM is able to express his partial preferences by pointing to a preferred outcome (and hence decision) from a handful of outcomes presented to him. Then his preference is translated into relations in terms of vectors λ.
In reference point methods it is assumed (assumption B) that at each iteration of an interactive decision making process the DM is able to express his partial preferences by pointing to a reference point representing his preferred decisional pattern, or, as in the variant of the reference point methods, by pointing to a pair of reservation-aspiration points.
There is no decisive evidence which assumption is better justified. Quite evidently assumption A is better justified than assumption B when the DM posses some analytical capabilities. In turn, assumption B seems to be better justified than assumption A when the DM acts intuitively and tends to present his preferences in a holistic manner. Pointing to a reference point is a holistic form of expressing preferences.

5.2. Weight versus reference point methods – technical level

Let us observe that in weight methods selecting at each iteration a vector $\lambda > 0$ amounts in fact to selecting a half-line starting from y^* along which the apexes of the contours of the function (3) lie (cf. Fig. 1). This line has the form

$$s = y^* - t\tau,$$

where $t > 0$ and $\tau = (\tau_1,...,\tau_k)$, $\tau_l = \frac{1}{\lambda_l}$, $l = 1,...,k$. In course of iterations one gets a "fan" of half-lines all starting at y^* (Fig. 4).
In reference point methods the DM specifies at each iteration a reference point y^{ref}, what amounts in fact to selecting

(recall we have assumed that (3) is the achievement function) a half-line starting from y^{ref}, i.e.

$$s' = y^{ref} - t\tau,$$

where $t > 0$ and $\tau = (\tau_1, ..., \tau_k)$, $\tau_l = \frac{1}{\lambda_l}$, $l = 1, ..., k$. In course of iterations one gets a "forest" of parallel half-lines (the vector λ is fixed) (Fig. 5).

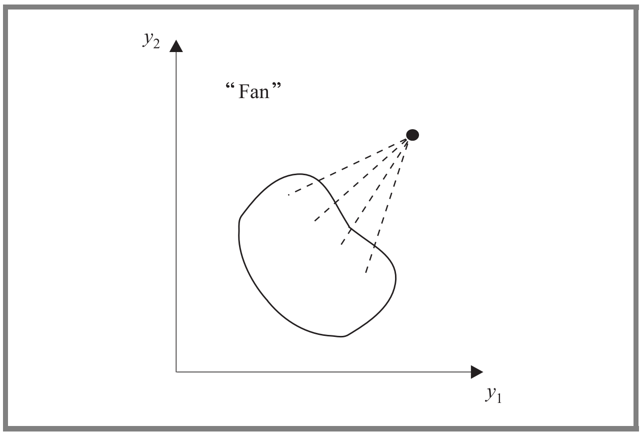

Fig. 4. A fan-type interactive decision making process.

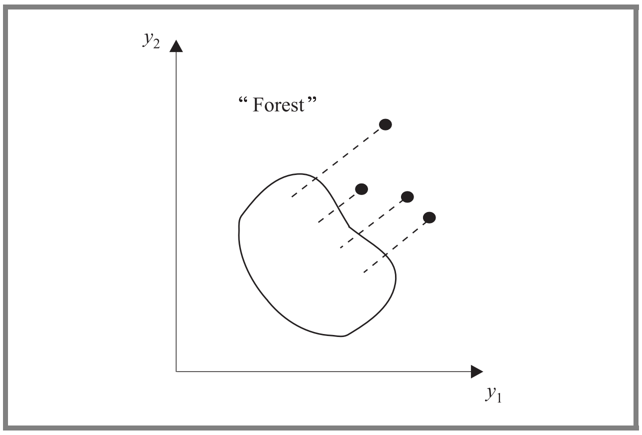

Fig. 5. A forest-type interactive decision making process.

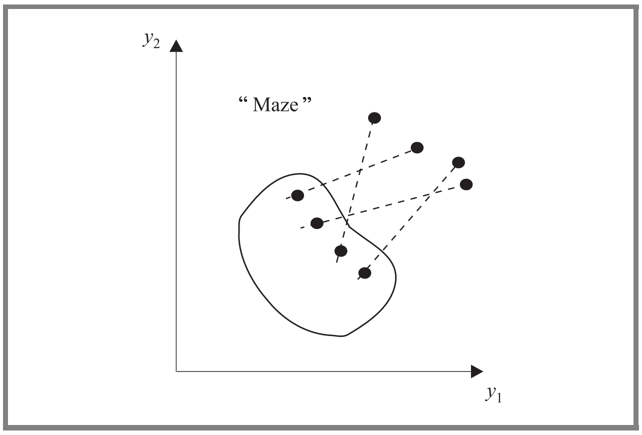

Fig. 6. A maze-type interactive decision making process.

In the variant of reference point methods the DM specifies at each iteration a reservation point y^{res} and an aspiration point y^{asp}, what amounts in fact to selecting (recall we have assumed that (3) is the achievement function) a half-line starting from y^{asp} and passing through y^{res}, i.e.

$$s'' = y^{asp} - t\tau,$$

where $t > 0$ and $\tau = (\tau_1, ..., \tau_k)$, $\tau_l = \frac{1}{y_l^{asp} - y_l^{res}}$, $l = 1, ..., k$. In course of iterations one gets a "maze" of half-lines (Fig. 6).

Table 1 summarizes the mechanics of the methods.

Table 1
Mechanics of considered interactive decision making methods

Methods	Decisional item	
	fixed	to be selected
Weight methods	y^*	τ
Reference point methods	τ	y^{ref}
Reference point methods – the variant	–	y^{res}, y^{asp}

From Table 1 we see that though weight methods and reference point methods represent totally different decision making (searching) methodologies, technically they are very similar. Indeed, in each method to proceed to the next iteration, i.e. to derive a subsequent trial outcome, a combination of two decisional items is required: either two elements of $\mathcal{R}^k$ or a direction and an element of $\mathcal{R}^k$. The methods differ in presence or absence of fixed items and in which item is an active toll to search over the set of efficient outcomes.

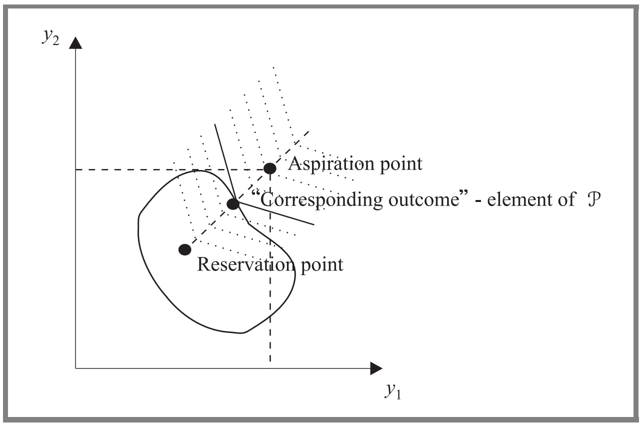

Fig. 7. Aspiration points take the role of y^* in appropriately constrained set Z.

The most flexible method is the variant of reference point methods since no decisional item is fixed a priori. Let us note, however, that a great extent of flexibility is not necessarily always plausible.

We can make weight methods and the variant of reference point methods *technically equivalent* with the following restriction of the latter. Let us assume $y^{asp} = y^*$, i.e. the aspiration point is fixed. Then DM changes only reservation points what amounts in fact to selecting a half-line starting from y^* and passing through y^{res}, i.e.

$$s''' = y^* - t\tau,$$

where $t > 0$, and $\tau = (\tau_1, ..., \tau_k)$, $\tau_l = \frac{1}{y_l^* - y_l^{res}}$, $l = 1, ..., k$. If it is the case, we can say that both methods rely on a direction selection mechanism.

Let us observe that y^* plays the same role to Z as y^{asp} (y^{ref}) to the set

$$Z \cap \{y \,|\, y_l \leq y_l^{asp} - \varepsilon, \; l = 1, ..., k\}$$

$$(Z \cap \{y \,|\, y_l \leq y_l^{ref} - \varepsilon, \; l = 1, ..., k\}),$$

where $\varepsilon > 0$ is the value we used in Section 3 to define y^* (cf. Fig. 7).

To decide if the proposed restriction of the variant of reference point methods is methodologically justified a vast practical experience with applications of these methods is required and this is lacking. At this stage we can only note that the technical equivalence of weight methods and the variant of reference point methods we have just shown has some interesting practical consequences.

6. Discussion

There are two major practical consequences of the technical equivalence of weight methods and the variant of reference point methods.

The first consequence is that with the equivalence shown two methods, one representing the class of weight methods and other representing the class of reference point methods, can be implemented jointly with the same computing (optimizing) software and an interface admitting the DM to select which of these two methods he would like to work. This would establish a firm ground for systematic comparison of these methods in the same technical environment. This also would open a way for some hybrid type decision processes mixing elements of the two methods.

The second consequence is as follows. In weight methods the principle of weight set reduction gives rise to a natural convergence measure. Namely, convergence can be controlled (and a stopping rule invoked) basing on the "volume" of sets of weights resulting from subsequent reductions. In general, reference point methods do not incur a similar natural convergence measure.

Only by the simple modification proposed above the variant of reference point methods acquires this property. Indeed, any two outcomes from two subsequent iterations give rise to a cut (6) or (7) and in consequence to a reduction of the set of weights. Though the DM would have to answer questions "which of two outcomes do you prefer?" those are kind of technical questions of no influence on the course of the decision process, which relies in, we recall, selecting a reservation point with the aspiration point fixed at y^*.

7. Concluding remarks

The fact that weight methods and the variant of reference point methods can be realized in the same technical framework has an important practical consequence. Namely, as shown in companion papers of the author [5, 6], with weight methods it is possible to calculate bounds, lower and upper, on values of criteria (outcome components) prior to explicitly decision determining. This possibility is of utmost practical importance for applications of MCDM methods because using bounds instead of exact values one can avoid determining decisions explicitly and hence solving optimization problems. Since, as shown above, weight and the variant of reference point methods can be reduced to (and implemented in) the same technical framework realizing a "fan" type decisional process, reference point methods also enjoy this property. This aspect will be a topic for further research.

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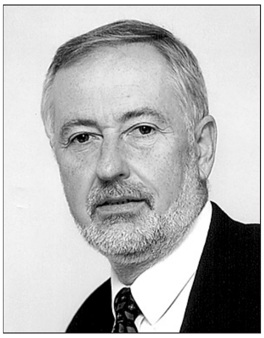

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