Paper

Geometrical representation of a monochromatic electromagnetic wave using the tangential vector approach

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Abstract — The aim of this work is to develop a coherent polarimetric model and to find a geometrical description of a monochromatic wave. The spinor form of the electrical field, its links to the coherency matrix and the Poincare' sphere are introduced with the aim to obtain a geometrical representation of the spinor. It consists, from the "polarization point of view", on the polarization vector and a tangential plane to the Poincare' sphere where it is possible to visualize the zero phase.

Keywords* — *polarimetric, coherent model, Poincare' sphere.

1. Introduction

Pulse radar has a very narrow band, so, to describe the state of the signal, it is possible to consider one single pulse like a monochromatic electromagnetic wave, which is completely polarized [1, 2]. A very useful representation of the electrical field is its spinor form which contains the complete information even the zero phase[1]. The aim of this work is to develop a coherent polarimetric description which has a geometrical representation.

2. Spinors and quadrivectors – the coherency matrix

The two-component complex field of the Jones representation may be treated as a spinor η^A:

$$\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} = \begin{pmatrix}E_x\\ E_y\end{pmatrix} = \begin{pmatrix}a_x e^{i\delta_x}\\ a_y e^{i\delta_y}\end{pmatrix}, \tag{1}$$

where a_x, a_y are the amplitudes and δ_x, δ_y are the phases of the phasor representation of a RF signal.
A quadrivector $x^\mu = (x^0, x^1, x^2, x^3)$ may be regarded as a Hermitian second-rank spinor. The spin matrix X [3]:

$$X = x^0 + (\vec{x}\cdot\vec{\sigma}) = \left\| \begin{matrix} x^0+x^4 & x^1-ix^2\\ x^1+ix^2 & x^0-x^4\end{matrix}\right\| = \\ = \left\| \begin{matrix} X^{1\dot{1}} & X^{1\dot{2}}\\ X^{2\dot{1}} & X^{2\dot{2}}\end{matrix}\right\| \tag{2}$$

is transformed like a second rank spinor namely the coefficients in the law for the transformation of the components of the spin matrix $X^{A\dot{V}}$ are identical with the coefficient in the law for the transformation of the second rank spinor $\chi^A\xi^{\dot{V}}$ (the dots are used for the conjugate complex, not transpose). In more compact form:

$$X^{A\dot{V}} = \left[x^0 + (\vec{x}\cdot\vec{\sigma})\right]^{A\dot{V}} = x^\mu \sigma_\mu^{A\dot{V}} \quad (\mu = 0, 1, 2, 3), \tag{3}$$

where σ_0 is the unit matrix and σ_1, σ_2, σ_3 are the Pauli matrices:

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3 \tag{4}$$

and cyclic permutations. In this way a geometric representation of the spinor η^A which the spinor form of the Jones vector, is possible. Then, if $X^{A\dot{V}}$ is calculated as

$$X^{A\dot{V}} = \eta^A(\overline{\eta})^{\dot{V}} \tag{5}$$

it results: $X^{1\dot{1}} = E_xE_x^*$, $X^{1\dot{2}} = E_xE_y^*$, $X^{2\dot{1}} = E_yE_x^*$, $X^{2\dot{2}} = E_yE_y^*$, which are the components of the coherency matrix J [4] (where E_i^* is the conjugate complex of the complex number E_i).
The correspondent 4-vector x^μ is obtained from the Eq. (3) and from Eq. (5):

$$\left\| \begin{matrix} x^0+x^1 & x^2-ix^3\\ x^2+ix^3 & x^0-x^1\end{matrix}\right\| = \left\| \begin{matrix} \eta^1\overline{\eta}^{\dot{1}} & \eta^1\overline{\eta}^{\dot{2}}\\ \eta^2\overline{\eta}^{\dot{1}} & \eta^2\overline{\eta}^{\dot{2}}\end{matrix}\right\|, \tag{6}$$

where the cyclic permutation: $\sigma_1 \to \sigma_2$, $\sigma_2 \to \sigma_3$, $\sigma_3 \to \sigma_1$ is considered. Substituting the components of the Jones vector, the components of the Stokes vector are found:

$$x^0 = \frac{1}{2}g^0, \quad x^1 = \frac{1}{2}g^1, \quad x^2 = \frac{1}{2}g^2, \quad x^3 = \frac{1}{2}g^3. \tag{7}$$

For a monochromatic wave, (g^0, g^1, g^2, g^3) is a real null 4-vector

$$(g^0)^2-(g^1)^2-(g^2)^2-(g^3)^2 = 0 \Rightarrow (x^0)^2-(x^1)^2+ \\ -(x^2)^2-(x^3)^2 = 0. \tag{8}$$

All the directions of the 4-vectors x^μ in the Minkowski space-time for which the components satisfy (8) are null directions and they build the null cone [5]. The space of the null directions can be represented in the Euclidean space by the intersections of the null cone with the hyper planes $x^0 = const$ and so $g^0 = const$ (with the same intensity of the electrical field, because $g^0 = I$). If the $const = \pm 1$, the intersection is a sphere which can be regarded as a Riemann sphere of an Argand plane, which is the Poincare'

[1] D. H. O. Bebbington, "Analytical foundations of polarimetry: I" – to be published.

sphere. But in general for any value of the constant, unless $g^0 = 0$, we get from the relation (8):

$$\left(\frac{g^1}{g^0}\right)^2 + \left(\frac{g^2}{g^0}\right)^2 + \left(\frac{g^3}{g^0}\right)^2 = 1 \qquad (9)$$

and we can define

$$p^1 = \frac{g^1}{g^0}, \quad p^2 = \frac{g^2}{g^0}, \quad p^3 = \frac{g^3}{g^0} \qquad (10)$$

which are the components of the polarization vector. The equation of the Poincare' sphere is in general:

$$(p^1)^2 + (p^2)^2 + (p^3)^2 = 1 . \qquad (11)$$

The exterior of the sphere represents space-like directions namely unpolarized or partially polarized light.

Multiplying the spinor η^A by a complex number $p = \lambda e^{i\theta}$ (λ and θ real) the 4-vector x^μ is stretched of λ^2 but is unchanged in direction (cfr. (5)), namely it is independent from the choice of the angle θ. The 4-vector is uniquely defined by the spinor but to a 4-vector correspond a lot of spinors, which differ by the multiplicative factor $e^{i\theta}$.

On the other side we want find a coherent description of a monochromatic wave, which contains the so-called "zero phase" $\alpha = \delta_x$ $(0 < \alpha < 2\pi)$. In order to do this, we look at the spinor in its polarization vector form [6]:

$$\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} = \sqrt{\frac{I}{2}}\, e^{i\alpha} \begin{pmatrix} (1+p^1)^{1/2} \\ (1+p^1)^{-1/2}(p^2+ip^3)\end{pmatrix} . \qquad (12)$$

This form of the spinor contains explicitily the zero phase and, as we have stated below, the corresponding 4-vector (g^0, g^1, g^2, g^3), is unaffected by the choice of the angle α.

3. The tangential plane and the angle α

Let us consider the spinor mate [3] ξ^B of η^A:

$$\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} = \sqrt{\frac{I}{2}}\, e^{-i\alpha} \begin{pmatrix} -(1+p^1)^{-1/2}(p^2-ip^3) \\ (1+p^1)^{1/2}\end{pmatrix} . \qquad (13)$$

The spinor and the spinor mate so defined satisfy the condition:

$$\eta_A \xi^A = I . \qquad (14)$$

They build a basis normalized to I and if we consider $I = 1$ the two spinor build a basis normalized to 1. The spinor and the spinor mate are linked by the equations:

$$\eta^A \xi^B - \xi^A \eta^B = \varepsilon^{AB} \quad (A, B = 1,\, 2) , \qquad (15)$$

where ε^{AB} is an antisymmetric symbol such that: $\varepsilon^{12} = \varepsilon_{12} = 1$, $\varepsilon^{AB} = -\varepsilon^{BA}$. The spinor and the spinor mate constitute a spinor basis.

As we have stated below that $X^{A\dot{V}}$ is transformed like a second rank spinor $\chi^A \overline{\xi}^{\dot{V}}$, we can calculate the component of $Q^{A\dot{V}}$ using the spinor mate:

$$Q^{A\dot{V}} = \eta^A \left(\overline{\xi}\right)^{\dot{V}} . \qquad (16)$$

Now multiplying the spinor η^A by a complex number $\rho = \lambda e^{i\theta}$, the vector is stretched but also it depends on the choice of the angle θ and in particular it depends on 2θ. The calculation of $Q^{A\dot{V}}$ gives:

$$Q^{1\dot{1}} = -\tfrac{I}{2} e^{2i\alpha}(p^2 + ip^3), \qquad Q^{1\dot{2}} = \tfrac{I}{2} e^{2i\alpha}(1+p^1),$$

$$Q^{2\dot{1}} = -\tfrac{I}{2} e^{2i\alpha}(1+p^1)^{-1}(p^2+ip^3)^2, \; Q^{2\dot{2}} = \tfrac{I}{2} e^{2i\alpha}(p^2+ip^3) \qquad (17)$$

which corresponds to a complex point. Infact, by the Eq. (3) the components of the corresponding 4-vector q^μ are:

$$\begin{aligned} q^0 &= 0, \\ q^1 &= -\frac{I}{2} e^{2i\alpha}(p^2+ip^3), \\ q^2 &= \frac{I}{4} e^{2i\alpha}\, \frac{(1+p^1)^2 - (p^2+ip^3)^2}{1+p^1}, \\ q^3 &= -\frac{I}{4i}\, \frac{(p^2+ip^3) - (1+p^1)^2}{1+p^1} . \end{aligned} \qquad (18)$$

If the real and imaginary parts are separated, the two real 4-vectors have components $q_R^\mu = (0, \vec{q}_R)$ and $q_I^\mu = (0, \vec{q}_I)$ which are:

$$\begin{aligned} q_R^0 &= 0, \\ q_R^1 &= I(-p^2\cos 2\alpha + p^3 \sin 2\alpha), \\ q_R^2 &= I\left(\frac{p^1(1+p^1)+(p^3)^2}{1+p^1}\cos 2\alpha + \frac{p^2p^3}{1+p^1}\sin 2\alpha\right), \\ q_R^3 &= I\left(-\frac{p^2p^3}{1+p^1}\cos 2\alpha - \frac{p^1(1+p^1)+(p^2)^2}{1+p^1}\sin 2\alpha\right). \end{aligned} \qquad (19)$$

$$\begin{aligned} q_I^0 &= 0, \\ q_I^1 &= I(-p^2\sin 2\alpha - p^3 \cos 2\alpha), \\ q_I^2 &= I\left(\frac{p^1(1+p^1)+(p^3)^2}{1+p^1}\sin 2\alpha - \frac{p^2p^3}{1+p^1}\cos 2\alpha\right), \\ q_I^3 &= I\left(-\frac{p^2p^3}{1+p^1}\sin 2\alpha + \frac{p^1(1+p^1)+(p^2)^2}{1+p^1}\cos 2\alpha\right). \end{aligned} \qquad (20)$$

The 4-vector q^μ is space-like and in particular of magnitude equal to I. The 4-vector $p^\mu(1, p^1, p^2, p^3)$, $q_R^\mu(0, \vec{q}_R)$, $q_1^\mu(0, \vec{q}_I)$ are orthogonal in the sense:

$$p^\mu (q_R)_\mu = 0, \quad p^\mu (q_I)_\mu = 0, \quad (q_I)^\mu (q_R)_\mu = 0 . \qquad (21)$$

And it is easy to see that even $\vec{g} = (g^1, g^2, g^3)$, $\vec{q}_R = (q_R^1, q_R^2, q_R^3)$ and $\vec{q}_I = (q_I^1, q_I^2, q_I^3)$ are orthogonal and of modul equal to I in the Euclidean space. So the vectors $\vec{q}_R$ and $\vec{q}_I$ provide basis vectors ($I = 1$) in the two-dimensional space which is the tangential plane at the point $\vec{p}$ on the Poincare' sphere. When the angle α varies, the vectors $\vec{q}_R$ and $\vec{q}_I$ rotate in the tangential plane.

The aim is now to visualize the angle α and to find a reference for $\alpha = 0$. For the horizontal polarization $\vec{p}_H = (1, 0, 0)$ and for $\alpha = 0$, $\vec{q}_R$ is the tangential vector to the equatorial great circle. If α increases, $\vec{q}_R$ rotates in the tangential plane clockwise through an angle of 2α. Keeping $\alpha = 0$, the fact that the point $\vec{p}_H$ moves into the point $\vec{p}$ corresponds to a rotation applied to the spinor η^A. This means a change of the basis, which means different $\vec{q}_{R(\alpha=0)}$ and $\vec{q}_{I(\alpha=0)}$. The rotation matrix, which preserves the angle α and which moves the point $\vec{p}_H$ to the point $\vec{p}$ is:

$$R = \left(\frac{1}{1+|\rho|^2}\right)^{-1/2} \begin{pmatrix} 1 & -|\rho|e^{-i\delta} \\ |\rho|e^{i\delta} & 1 \end{pmatrix}, \qquad (22)$$

where $\rho = \frac{E_y}{E_x} = |\rho|e^{i\delta}$ ($\delta = \delta_y - \delta_x$, cfr. (1)) is the polarization ratio. This is a rotation around the axis $\vec{n}(0, -\sin\delta, \cos\delta)$ through an angle such that $\cos\theta = \frac{1-|\rho|^2}{1+|\rho|^2} = p^1$. The rotation (22) preserves the angle between the directions but not the direction, so the vector $\vec{q}_{R(\alpha=0)}$ changes its direction. The direction r (cfr. Fig. 1), obtained by the intersection of the great circle through $\vec{p}_H$ and $\vec{p}$, forms with the vector $\vec{q}_{R(\alpha=0)}$ an angle δ and with the vector $\vec{q}_R$ the angle $2\alpha + \delta$. It is very important to find a reference for $\alpha = 0$ because $\vec{q}_R$ forms an angle δ with the direction r but δ is different for every point on the sphere. To solve this problem, let us consider $\vec{p}$ and $\vec{q}_R$ and $\vec{q}_I$ for any α, consider the correspondent spinor, apply the rotation which preserves the angle α and move the vector $\vec{p}$ in the point $\vec{p}_H$ to obtain the vector $\vec{q}_{RH}$ and the angle 2α is the angle between $\vec{q}_{RH}$ and $\vec{q}_{RH(\alpha=0)}$.

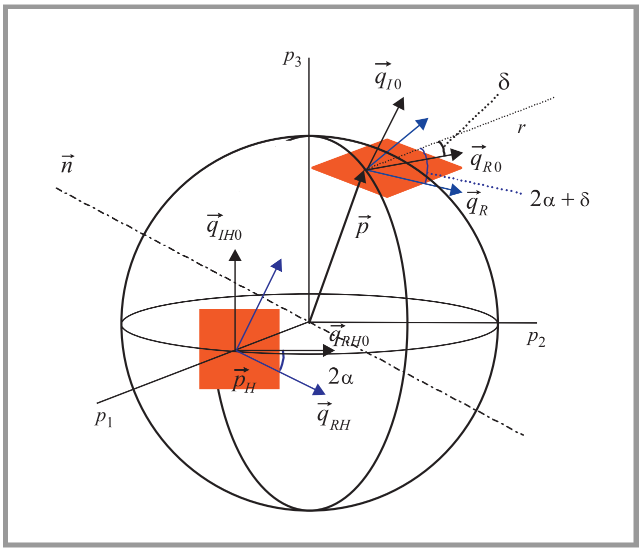

Fig. 1. The Poincare' sphere and the tangential planes in the point $\vec{p}_H$ and in the point $\vec{p}$.

The spinor and the spinor mate constitute a spinor basis. It is easy to see that the correspondent 4-vectors (cfr. Eq. (6)) fix on the Poincare' sphere two antipodal points $((p^1, P^2, p^3)$ and $(-p^1, -p^2, -p^3))$ which are the basis states of polarization [7]. If the corresponding $\vec{q}_R$ and $\vec{q}_I$ vectors are calculated, the result is:

$$\vec{p} \to \vec{q}_R, \vec{q}_I \qquad -\vec{p} \to -\vec{q}_R, \vec{q}_I\,. \qquad (23)$$

With the help of the spinor, the change of basis is easy because it corresponds to a unitary transformation of the spinor which corresponds to a rotation in the three dimensional space. Infact the group of two-dimensional special unitary transformations (with unit determinants), which preserve the invariants, are homomorphic to the three-dimensional rotation group [5]. The general form of the spin rotation matrix is:

$$R = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3)\,, \qquad (24)$$

where θ is the angle of rotation, (n_1, n_2, n_3) are the components of the axis $\vec{n}$ of rotation in the Euclidean space and σ_i are the Pauli matrices. The transformation law of a spinor is:

$$\eta \to \eta' = R\eta\,. \qquad (25)$$

It is possible to show that the rotation spin matrix is a unitary matrix and its determinant is necessarily unity.

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