Paper

Hybrid Models for the OWA Optimization

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Abstract—When dealing with multicriteria problems, the aggregation of multiple outcomes plays an essential role in finding a solution, as it reflects the decision-maker's preference relation. The Ordered Weighted Averaging (OWA) operator provides a flexible preference model that generalizes many objective functions. It also ensures the impartiality and allow to obtain equitable solutions, which is vital when the criteria represent evaluations of independent individuals. These features make the OWA operator very useful in many fields, one of which is location analysis. However, in general the OWA aggregation makes the problem nonlinear and hinder its computational complexity. Therefore, problems with the OWA operator need to be devised in an efficient way. The paper introduces new general formulations for OWA optimization and proposes for them some simple valid inequalities to improve efficiency. A hybrid structure of proposed models makes the number of binary variables problem type dependent and may reduce it significantly. Computational results show that for certain problem types, some of which are very useful in practical applications, the hybrid models perform much better than previous general models from literature.

Keywords—location problem, mixed integer (linear) programming, multiple criteria, ordered weighted averaging.

1. Introduction

In many practical problems we have to deal with multiple conflicting criteria. Typically, there does not exist unique optimal solution for such problems and we need to use decision-maker's preference to solve them. We have to be able to compare different alternatives and decide, which one is better from a decision-maker point of view. A common approach is to aggregate all original criteria by some scalarizing function into one overall objective function. In this solution concept an aggregation is crucial, as it provides the preference model, and thus determines preference relation between alternatives. The so-called Ordered Weighted Averaging (OWA) operator provides a parameterized aggregation function that generalized many scalarizing functions, including the most popular the average and the maximum (minimum) along with many other cases. The OWA operator, introduced by Yager [1], is a special weighted average, where weights are assigned to the ordered values of outcomes (i.e. to the largest value, the second largest and so on) rather than to the specific outcomes.

The OWA operator not only generalizes various objective functions, but also ensures impartial and in some circumstances equitable solutions. It plays an essential role when the distribution of outcomes is more important than values' assignment to specific outcomes. It is the case when we deal with outcomes that express, for example, the evaluation of multiple independent users or scenarios. Thus, the OWA aggregation has been widely applied in different domains [2]–[4]. However, when we aggregate the variable criteria by the OWA operator, we get the nonlinear problem, even if the original problem has a linear formulation. Yager [5] showed that this type of nonlinearity can be transformed into a Mixed Integer Linear Programming (MILP) problem. Furthermore, in [6] it was proofed that the OWA optimization with appropriate monotonic weights can be expressed as linear programming (LP) problem of higher dimension, allowing to improve solution techniques for many related problems [7].

The outcomes distribution is important in location analysis when we deal with independent clients [8]. Within this field the so-called Ordered Median (OM) function was developed and analyzed [9], which in fact corresponds to the OWA operator. Thus, several models and some dedicated solution methods for the OWA optimization were developed within the location analysis, including branch and bound [10] or branch and cut [11], [12] approaches. A significant improvement in computational efficiency has been made. However, the solution times are still not satisfactory. Besides, some of these formulations take advantage of specific assumptions such as free self-service.

In this paper a new general MILP model for the OWA optimization is introduced, which is the extension of the LP formulation [6] and can be applied to any non-negative preference weights **w**. Some similar concept of LP formulation extension has been recently applied for the weighted OWA aggregation [13]. Due to hybrid structure with the linear and the mixed integer linear parts, the number of binary variables in our new formulations depend on problem type and can be substantially reduced for some of them. We evaluate new models for the discrete location problems, but we do not exploit any specific structure and assume only the non-negativity of the outcomes for some results. We also propose some simple valid inequalities to improve the computational performance of new formulations, which we set together with one of the most efficient general model for OWA optimization (see comparison in [14]).

The paper is organized as follows. In Sections 2 and 3 the problem is formally defined and the hybrid models for the OWA optimization are developed. In Section 4 the experimental procedure is described and results are presented. Section 5 concludes and proposes some further research directions.

2. Problem Formulations

We consider uncapacitated discrete location problem [15], which can also be defined as network location problem, where facilities are allowed to be placed only on vertices (or subset of vertices) of the underlying network. Given a set of m clients and a set of potential facility locations, which without loss of generality can be assumed to be identical sets, we have to place n facilities ($n \leq m$) and assign them to clients to meet the demand. We aim at optimizing a given objective function, which is usually based on abstract distances (e.g. geographic distances, service costs, service times) between the clients and the facilities. We assume no capacity limit of facilities, so each client is assigned the closest facility. The model can be formally expressed in the following form:

$$\min \quad (y_1, y_2, \ldots, y_m), \tag{1a}$$

$$\text{s.t.} \quad y_i = \sum_{j=1}^{m} c_{ij} x'_{ij} \qquad \forall i, \tag{1b}$$

$$\sum_{j=1}^{m} x_j = n, \tag{1c}$$

$$\sum_{j=1}^{m} x'_{ij} = 1 \qquad \forall i, \tag{1d}$$

$$x'_{ij} \leq x_j \qquad \forall i,j, \tag{1e}$$

$$x_j \in \{0,1\} \qquad \forall i,j, \tag{1f}$$

$$x'_{ij} \geq 0 \qquad \forall i,j, \tag{1g}$$

where c_{ij} denotes the cost of satisfying the total demand of client i from facility j. The main decisions are described by binary variables x_j ($j = 1,2,\ldots,m$) equal 1 if a facility is placed at site j and equal 0 otherwise. Additional binary variables represent allocation decisions: x'_{ij} ($i,j = 1,2,\ldots,m$) is equal to 1 if the demand of client i is satisfied by facility j and 0 otherwise. Due to lack of capacity restriction, each client will be assigned to the closest facility, and therefore variables x'_{ij} can be relaxed to continuous variables. The auxiliary variable y_i (1b) expresses the cost of satisfying the demand of client i. Constraint (1c) enforces that exactly n facilities are placed. Constraint (1d) limits each client to be assigned only one facility and constraint (1e) ensures that the assignment is done to the existing facilities. Thus, constraints (1c)–(1g) define the set of feasible solutions Q, which is mapped into the set of attainable outcome (cost) vectors $\mathbf{y}$ by constraint (1b).

We want to obtain efficient solutions of problem (1) in the sense of outcomes $y_i = f_i(\mathbf{x})$ for $i = 1,2,\ldots,m$ using the OWA operator. To define the OWA aggregation of a vector $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ more formally, let us introduce the ordering map $\Theta\colon R^m \to R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$ satisfies $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \ldots \geq \theta_m(\mathbf{y})$ and there exist a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1,2,\ldots,m$. Then for a given preference weight vector $\mathbf{w} = (w_1, \ldots, w_m)$ with $w_i \geq 0$ for all i, the OWA operator takes the form

$$A_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}). \tag{2}$$

Finally, we apply formula (2) to problem (1) and receive the following optimization problem

$$\min\{A_{\mathbf{w}}(\mathbf{y}) \colon \ \mathbf{y} = \mathbf{f}(\mathbf{x}), \ \mathbf{x} \in Q\}.$$

3. Optimization Models

At first we recall the LP formulation for the OWA optimization [6] that can be used with appropriate monotonic weights (non-increasing in case of minimization). Then we extend it by mixed integer linear part, and thus making it valid to any non-negative preference weights.

3.1. LP Model for OWA

The ordering operator Θ in the OWA aggregation is nonlinear and, in general, it leads to complex optimization models. However, in special case with non-increasing weights the OWA aggregation is piecewise linear convex function and can be minimized using the linear programming form [6]. This so-called deviational model exploits the linear programming representation of the cumulated ordered outcomes $\bar{\Theta}(\mathbf{y}) = (\bar{\theta}_1(\mathbf{y}), \bar{\theta}_2(\mathbf{y}), \ldots, \bar{\theta}_m(\mathbf{y}))$, where

$$\bar{\theta}_k(\mathbf{y}) = \sum_{i=1}^{k} \theta_i(\mathbf{y}) \quad \forall k$$

expresses the total of the k largest outcomes. These quantities can be determined as

$$\bar{\theta}_k(\mathbf{y}) = k\theta_k(\mathbf{y}) + \sum_{i=1}^{k-1} (\theta_i(\mathbf{y}) - \theta_k(\mathbf{y})) \quad \forall k,$$

where the k-th largest outcome $\theta_k(\mathbf{y})$ is treated as a reference value to which its deviations from greater outcomes are added. Provided we introduce the explicit variables d_{ik} for deviations, each $\bar{\theta}_k(\mathbf{y})$ for any given $\mathbf{y} \in R^m$ and $k = 1,\ldots,m$ can be found by solving the following LP problem:

$$\bar{\theta}_k(\mathbf{y}) = \min_{t_k, d_{ik}} \quad (kt_k + \sum_{i=1}^{m} d_{ik}), \tag{3a}$$

$$\text{s.t.} \quad d_{ik} \geq y_i - t_k \qquad \forall i, \tag{3b}$$

$$d_{ik} \geq 0 \qquad \forall i. \tag{3c}$$

Variable t_k corresponds to k-th largest outcome (strictly speaking in optimal solution $\theta_{k+1}(\mathbf{y}) \leq t_k^* \leq \theta_k(\mathbf{y})$ for $k = 1,\ldots,m-1$ and $t_m^* \leq \theta_m(\mathbf{y})$; and $t_k^* = \theta_k(\mathbf{y})$ provided that at most $k-1$ variables $d_{ik} > 0$).

The ordered outcomes can be determined as differences $\theta_k(\mathbf{y}) = \bar{\theta}_k(\mathbf{y}) - \bar{\theta}_{k-1}(\mathbf{y})$ for $k = 2,\dots,m$ and $\theta_1(\mathbf{y}) = \bar{\theta}_1(\mathbf{y})$. Hence the OWA aggregation $\sum_{k=1}^{m} w_k \theta_k(\mathbf{y})$ with weights w_k can be replaced by $\sum_{k=1}^{m} w'_k \bar{\theta}_k(\mathbf{y})$, where $w'_m = w_m$ and $w'_k = w_k - w_{k+1}$ for $k = 1,2,\dots,m-1$. Therefore, as shown in [6], in case of non-increasing and non-negative original weights ($w_1 \geq w_2 \geq \ldots \geq w_m \geq 0$), the OWA optimization problem can be formulated as follows:

$$\min_{t_k, d_{ik}, y_i} \quad \sum_{k=1}^{m} w'_k (k t_k + \sum_{i=1}^{m} d_{ik}), \tag{4a}$$

$$\text{s.t.} \quad d_{ik} \geq y_i - t_k \qquad \forall i,k, \tag{4b}$$

$$d_{ik} \geq 0 \qquad \forall i,k, \tag{4c}$$

$$\mathbf{y} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in Q. \tag{4d}$$

3.2. Hybrid Model for OWA

In LP formulation (4) we minimize the upper bound of function $k t_k + \sum_{i=1}^{m} d_{ik}$ for each $k = 1,2,\dots,m$. We then multiply these bounds by modified weights w'_k in the objective function (4a). If original weights w_k do not satisfy monotonicity condition (non-increasing), some weights w'_k are negative and then the problem (4) is unbounded. To make it valid for general case, we need to apply a lower bound for function $k t_k + \sum_{i=1}^{m} d_{ik}$.
The cumulative sum of the k-th largest outcomes $\bar{\theta}_k(\mathbf{y})$ can be determined in a similar way like in (3) by using a lower bound of function $k t_k + \sum_{i=1}^{m} d_{ik}$. However, it requires binary variables. For any $\mathbf{y} \in R^m$ and $k = 1,\dots,m$ the problem is as follows:

$$\bar{\theta}_k(\mathbf{y}) = \max_{\rho_k, t'_k, d'_{ik}, z_{ik}} \quad \rho_k, \tag{5a}$$

$$\text{s.t.} \quad \rho_k \leq k t'_k + \sum_{i=1}^{m} d'_{ik}, \tag{5b}$$

$$t'_k + d'_{ik} \leq y_i + M(1 - z_{ik}) \qquad \forall i, \tag{5c}$$

$$d'_{ik} \leq M z_{ik} \qquad \forall i, \tag{5d}$$

$$\sum_{i=1}^{m} z_{ik} = k, \tag{5e}$$

$$z_{ik} \in \{0,1\} \qquad \forall i. \tag{5f}$$

For any $k = 1,\dots,m$ there are m binary variables z_{ik}. They determine which constraints (5c) are relaxed by adding large constant M and which variables d'_{ik} are non-zero according to (5d). If $z_{ik} = 1$, then for respective i constraint (5c) becomes active and d'_{ik} may take positive values. Thus, according to maximization, $t'_k + d'_{ik}$ becomes equal to y_i. Solving this problem amounts to selecting k variables z_{ik} ($i = 1,\dots,m$), which take value 1 in order to make the respective sums $t'_k + d'_{ik}$ as large as possible. Therefore, in optimal solution the value 1 is taken by k variables z_{ik} that correspond to the k largest outcomes y_i. Variable t'_k is not greater than k-th largest outcome due to k active constraints (5c); and k respective variables d'_{ik} complete t'_k to the k largest outcomes.

Formulation (5) can be simplified by removing the binary component from formula (5c). The character of t'_k changes a little bit, but the optimal value still equals the sum of the k largest outcomes.

Proposition 1: For any given vector $\mathbf{y} \in R^m$, the sum of its k largest components $\bar{\theta}_k(\mathbf{y})$ can be found as the optimal value of the following MILP problem:

$$\bar{\theta}_k(\mathbf{y}) = \max_{\rho_k, t'_k, d'_{ik}, z_{ik}} \quad \rho_k, \tag{6a}$$

$$\text{s.t.} \quad \rho_k \leq k t'_k + \sum_{i=1}^{m} d'_{ik}, \tag{6b}$$

$$t'_k + d'_{ik} \leq y_i \qquad \forall i, \tag{6c}$$

$$d'_{ik} \leq M z_{ik} \qquad \forall i, \tag{6d}$$

$$\sum_{i=1}^{m} z_{ik} = k, \tag{6e}$$

$$z_{ik} \in \{0,1\} \qquad \forall i. \tag{6f}$$

Proof: In order to proof the proposition, we will show that the optimal value of problem (6) is the same as that of problem (5). First of all, we may notice that problem (6) is a restriction of problem (5). Every feasible solution of (6) is also a feasible solution of (5). Consider any feasible solution of (5). Let $I_k^1 = \{i \colon z_{ik} = 1\}$ be the subset of $i \in I$ for which $z_{ik} = 1$ and respectively $I_k^0 = I \setminus I_k^1$. For each $i \in I_k^1$, constraint (5c) simplifies to (6c), and thus any feasible solution of (5) satisfies (6c) for $i \in I_k^1$. If feasible solution of (5) additionally satisfies (6c) for $i \in I_k^0$, then it is also a feasible solution of (6). Otherwise, there is some number s of $i \in I_k^0$ for which $t'_k + d'_{ik} > y_i$. According to (5d), as $d'_{ik} \leq 0$ for $i \in I_k^0$ then $t'_k > y_i$ for some s of $i \in I_k^0$. However, each such solution can be replaced by equally good or better alternative, which violates at most $s - 1$ constraints (6c) for $i \in I_k^0$. We can determine

$$\delta = \min_{\substack{i \in I_k^0, \\ y_i < t'_k}} (t'_k - y_i), \tag{7}$$

replace t'_k by $\breve{t}'_k = t'_k - \delta$ and d'_{ik} by $\breve{d}'_{ik} = d'_{ik} + \delta$ for these i's for which $\breve{d}'_{ik} \leq M z_{ik}$ is satisfied. It holds at least for all $i \in I_k^1$ as follows

$$\begin{aligned}
\breve{d}'_{ik} = d'_{ik} + \delta &\leq y_i - t'_k + \delta && \text{by (5c)}, \\
&\leq y_i - \theta_m(\mathbf{y}) && \text{by (7)}, \\
&\leq \theta_1(\mathbf{y}) - \theta_m(\mathbf{y}) \leq M.
\end{aligned}$$

So $\breve{d}'_{ik} \leq M z_{ik}$ is satisfied at least for $i \in I_k^1$, and thus at least $r \geq k$ variables d'_{ik} can be replaced by $\breve{d}'_{ik}$, obtaining

$$k \breve{t}'_k + \sum_{i=1}^{m} \breve{d}'_{ik} = k t'_k + \sum_{i=1}^{m} d'_{ik} + (r - k)\delta \geq k t'_k + \sum_{i=1}^{m} d'_{ik}.$$

It means that for any feasible solution of (5), reducing the number of violated constraints (6c) for $i \in I_k^0$, we can obtain not worse corresponding feasible solution of (6). In conclusion, the optimal value of problem (6), similarly like the optimal value of problem (5), determines the sum of the k largest components of any given vector $\mathbf{y} \in R^m$. ■

From proposition (1), we can also conclude that for any $k = 1, \ldots, m$, one of the optimal solution of problem (6) is one with $t_k'^* = \theta_m(\mathbf{y})$ and $d_{ik}'^* = y_i - \theta_m(\mathbf{y})$ for these i's that correspond to the k largest outcomes y_i, and $d_{ik}'^* = 0$ for the rest of i's. In general, the following holds:

Lemma 2: For any given vector $\mathbf{y} \in R^m$ and any given value $\zeta \leq \theta_m(\mathbf{y})$, there exists the optimal solution of problem (6) with $t_k'^* = \zeta$.

Proof: As stated above, problem (6) has the optimal solution with $t_k'^* = \theta_m(\mathbf{y})$. We will show that the value of the objective function is constant for $t_k' \leq \theta_m(\mathbf{y})$ for any $k = 1, \ldots, m$. The objective function of problem (6) can be expressed as

$$g(t_k') = kt_k' + \sum_{i \in I_k^1} \min(y_i - t_k', M) + \sum_{i \in I_k^0} \min(y_i - t_k', 0), \quad (8)$$

where $I_k^1 = \{i \colon z_{ik} = 1\}$ with $|I_k^1| = k$ and $I_k^0 = I \setminus I_k^1$ with $|I_k^0| = m - k$. For any given $t_k' \leq \theta_m(\mathbf{y})$, provided that constant M is large enough, function (8) can be simplified to

$$g(t_k') = kt_k' + \sum_{i \in I_k^1} (y_i - t_k') = \sum_{i \in I_k^1} y_i, \quad (9)$$

and thus the value of the $g(t_k')$ is constant in the considered interval. ■

Using problem (6), we can now propose the optimization model of OWA for any weights $w_k \geq 0$, which is based on function $kt_k + \sum_{i=1}^m d_{ik}$.

$$\min_{\rho_k, t_k, d_{ik}, t_k', d_{ik}', z_{ik}, y_i} \quad \sum_{k=1}^m w_k' \rho_k, \quad (10a)$$

$$\text{p.o.} \quad kt_k + \sum_{i=1}^m d_{ik} \leq \rho_k \quad \forall k, \quad (10b)$$

$$t_k + d_{ik} \geq y_i, \; d_{ik} \geq 0 \quad \forall i, k, \quad (10c)$$

$$\rho_k \leq kt_k' + \sum_{i=1}^m d_{ik}' \quad \forall k, \quad (10d)$$

$$t_k' + d_{ik}' \leq y_i \quad \forall i, k, \quad (10e)$$

$$d_{ik}' \leq Mz_{ik} \quad \forall i, k, \quad (10f)$$

$$\sum_{i=1}^m z_{ik} = k \quad \forall k, \quad (10g)$$

$$z_{ik} \in \{0, 1\} \quad \forall i, k, \quad (10h)$$

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \; \mathbf{x} \in Q. \quad (10i)$$

Formulation (10) is a valid optimization model for OWA. However, we do not need to use both lower and upper bound for each $k = 1, \ldots, m$. We need constraints (10b) and (10c) only for k for which $w_k' \geq 0$. What is more important, we need constraints (10d)–(10h) only for k for which $w_k' < 0$. It significantly reduces the number of variables too. Variables t_k, d_{ik} are defined only for $w_k' \geq 0$, whereas t_k', d_{ik}' and z_{ik} only for $w_k' < 0$. Taking advantage of above observations, the OWA optimization problem takes the following form:

$$\min_{\substack{\rho_k, t_k, d_{ik}, \\ t_k', d_{ik}', z_{ik}, y_i}} \quad \sum_{k=1}^m w_k' \rho_k, \quad (11a)$$

$$\text{s.t.} \quad kt_k + \sum_{i=1}^m d_{ik} \leq \rho_k \quad \forall k; w_k' \geq 0, \quad (11b)$$

$$t_k + d_{ik} \geq y_i, \; d_{ik} \geq 0 \quad \forall i, k; w_k' \geq 0, \quad (11c)$$

$$\rho_k \leq kt_k' + \sum_{i=1}^m d_{ik}' \quad \forall k; w_k' < 0, \quad (11d)$$

$$t_k' + d_{ik}' \leq y_i \quad \forall i, k; w_k' < 0, \quad (11e)$$

$$d_{ik}' \leq Mz_{ik} \quad \forall i, k; w_k' < 0, \quad (11f)$$

$$\sum_{i=1}^m z_{ik} = k \quad \forall k; w_k' < 0, \quad (11g)$$

$$z_{ik} \in \{0, 1\} \quad \forall i, k; w_k' < 0, \quad (11h)$$

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \; \mathbf{x} \in Q. \quad (11i)$$

It is clear now that problem (11) has hybrid structure and consists of linear part with constraints (11b)–(11c), and of mixed integer linear part with constraints (11d)–(11h). The linear part, for $w_k' \geq 0$, is in fact the LP problem (4). The mixed integer linear part is much more computationally expensive. However, it is worth to notice that the number of binary variables is proportional to the number of negative weights w_k'. If we define this set as $K^- = \{k \colon w_k' < 0, k = 1, \ldots, m\}$, the total number of binary variables equals $|K^-|m$. Taking into account that preference weights $w_k \geq 0$ and $w_m' = w_m$, $w_k' = w_k - w_{k+1}$, the cardinality of set K^- may be at most $m - 1$. Therefore, it seems that hybrid model can be an interesting alternative for other general MILP formulations for OWA optimization, where the number of binary variables is of order m^2 independently of problem type (for instance, models M1 and M2, compared in [14]). On the other hand, here we have $O(m^2)$ additional continuous variables, whereas other MILP formulations usually introduce $O(m)$ continuous variables. Moreover, the formulation has more constraints – in both cases the number of constraints is of order m^2, but in our formulation the proportional factor is greater. Due to these observations, formulation (11) seems interesting for problems with small number of negative weights w_k'. It is especially true for trimmed mean problems, where only one weight w_k' is negative. Trimmed mean problems are used when we want to discard some of the largest and smallest outcomes, and are ones of the most useful among problems with non-monotonic weights.

To improve the computational efficiency of formulation (11), we consider some simple valid inequalities for the mixed integer linear part. From now on, we also assume that the outcome vector is non-negative, i.e. $\mathbf{y} \geq 0$. This is very general assumption that holds not only in location problems but also in many others.

Proposition 3: There exists an optimal solution of (11) that satisfies the following constraints:

(i) non-negativity of d'_{ik}

$$d'_{ik} \geq 0 \qquad \forall i,k; w'_k < 0, \tag{12}$$

(ii) non-negativity of t'_k

$$t'_k \geq 0 \qquad \forall k; w'_k < 0, \tag{13}$$

(iii) non-decreasing order of binary variables z_{ik} for each i

$$z_{ik} \leq z_{ik'} \qquad \forall i,k; k \in \{K^- \setminus \max\{K^-\}\}; k' = suc(k), \tag{14}$$

where $suc(k) = \min\{k' \colon k' \in K^- \wedge k' > k\}$ is the successor function within set K^-.

Proof: There exists an optimal solution of (11) that satisfies constraints (12)–(13) if for any $k = 1,\dots,m$ for which $w'_k < 0$ there exists an optimal solution of (6) that satisfies constraints (12)–(13). We will consider a specific optimal solution of (6) for any $k = 1,\dots,m$. To show constraint (14) holds, we will consider relation between optimal solutions of (6) for different $k,k' = 1,\dots,m$ ($k' > k$).
Due to Lemma 2 for any $\mathbf{y} \in R^m$ and any value $\zeta \leq \theta_m(\mathbf{y})$, there exists the optimal solution of (6) with $t'^*_k = \zeta$. Since we have assumed $\mathbf{y} \geq 0$, our consideration can be limited to $\zeta \in [0, \theta_m(\mathbf{y})]$. Lets consider the optimal solution with $t'^*_k = \theta_m(\mathbf{y})$. It follows directly that the optimal solution satisfies (13). Then, for any $k = 1,\dots,m$

$$d'^*_{ik} = \begin{cases} \min(y_i - \theta_m(\mathbf{y}), 0) = 0 & \text{for } \; i \in I^0_k, \\ \min(y_i - \theta_m(\mathbf{y}), M) = y_i - \theta_m(\mathbf{y}) \geq 0 & \text{for } \; i \in I^1_k, \end{cases}$$

and the optimal solution satisfies (12).
Inequality (14) is defined only if $|K^-| \geq 2$. Due to formula (9) of the objective function, we know that for any k, indices $i \in I^1_k$ will correspond to k largest components of outcome vector $\mathbf{y}$. So, if y_i is one of the k largest outcomes, then $z^*_{ik} = 1$. If we consider formulation (6) for the same $\mathbf{y}$ and any $k' > k$, it follows that y_i is also one of the k' largest outcomes, and thus $z^*_{ik'} = 1$. The reverse implication is analogous. If y_i is not one of the k' largest outcomes, then it is not one of the $k < k'$ largest outcomes, and thus when $z^*_{ik'} = 0$, it follows that $z^*_{ik} = 0$. So, for the same $\mathbf{y}$ and any $k,k' = 1,\dots,m$ ($k' > k$), the optimal solutions of (6) satisfies inequality $z^*_{ik} \leq z^*_{ik'}$ for any $i = 1,\dots,m$. Thus, the optimal solution of (11) with $t'^*_k = \theta_m(\mathbf{y}^*)$ satisfies (14) for all $k \in K^-$.
We can conclude that there exists the optimal solution of (11) that satisfies constraints (i)–(iii). ■

Formulation (6) can be further modified by reducing the number of variables and simplifying some constraints.

Proposition 4: For any given vector $\mathbf{y} \geq 0$, the sum of its k largest components $\bar{\theta}_k(\mathbf{y})$ can be found as the optimal value of the following MILP problem:

$$\bar{\theta}_k(\mathbf{y}) = \max_{\rho_k, y'_{ik}, z_{ik}} \quad \rho_k, \tag{15a}$$

$$\text{s.t.} \quad \rho_k \leq \sum_{i=1}^{m} y'_{ik}, \tag{15b}$$

$$y'_{ik} \leq y_i \qquad \forall i, \tag{15c}$$

$$y'_{ik} \leq M z_{ik} \qquad \forall i, \tag{15d}$$

$$\sum_{i=1}^{m} z_{ik} = k, \tag{15e}$$

$$z_{ik} \in \{0,1\} \qquad \forall i. \tag{15f}$$

Proof: We will show that the optimal value of problem (15) is the same as that of problem (6).
By Lemma 2, we know that for any value $\zeta \leq \theta_m(\mathbf{y})$ there exists the optimal solution of problem (6) with $t'^*_k = \zeta$. As $\mathbf{y} \geq 0$, we may set t'_k equal to any value from interval $[0, \theta_m(\mathbf{y})]$, and the optimal value of problem (6) still equals the sum of the k largest components of outcome vector $\mathbf{y}$. Let $t'_k = 0$ in problem (6), then we get problem (15), where variables d'_{ik} are replaced by y'_{ik}. This change in notation follows the change in variables interpretation. Variables d'_{ik} stand for deviations of k largest outcomes from reference value. When we set the reference value to 0, these variables represent in fact the k largest outcomes.
In conclusion, the optimal value of problem (15), similarly like that of problem (6), is equal to the sum of the k largest components of outcome vector $\mathbf{y}$. ■

Analyzing problem (15), we can see now that consraints (15c) and (15d) form a linearization of formula

$$y'_{ik} \leq y_i z_{ik} \qquad \forall i.$$

We can use problem (15), similarly like problem (6), to determine the sum of the k largest components of outcome vector $\mathbf{y}$ for k for which $w'_k < 0$. The general model for the OWA optimization is as follows:

$$\min_{\substack{\rho_k, t_k, d_{ik}, \\ y'_{ik}, z_{ik}, y_i}} \quad \sum_{k=1}^{m} w'_k \rho_k, \tag{16a}$$

$$\text{p.o.} \quad k t_k + \sum_{i=1}^{m} d_{ik} \leq \rho_k \qquad \forall k; w'_k \geq 0, \tag{16b}$$

$$t_k + d_{ik} \geq y_i, \; d_{ik} \geq 0 \qquad \forall i,k; w'_k \geq 0, \tag{16c}$$

$$\rho_k \leq \sum_{i=1}^{m} y'_{ik} \qquad \forall k; w'_k < 0, \tag{16d}$$

$$y'_{ik} \leq y_i \qquad \forall i,k; w'_k < 0, \tag{16e}$$

$$y'_{ik} \leq M z_{ik} \qquad \forall i,k; w'_k < 0, \tag{16f}$$

$$\sum_{i=1}^{m} z_{ik} = k \qquad \forall k; w'_k < 0, \tag{16g}$$

$$z_{ik} \in \{0,1\} \qquad \forall i,k; w'_k < 0, \tag{16h}$$

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \; \mathbf{x} \in Q. \tag{16i}$$

Problem (16) is another version of hybrid model and in comparison to (11) has slightly fewer continuous variables (by the number of negative weights w'_k). The structure of

Table 1
Problem types defined by the vector of preference weights **w** with respect to the number of clients m and the number of facilities n ($\lceil a \rceil$, $\lfloor a \rfloor$ denote the ceil and floor of a, respectively)

Type	Name/description	Weighting vector **w**
T1	$k_1 + k_2$-trimmed mean	$(\underbrace{0,\ldots,0}_{k_1},1,\ldots,1,\underbrace{0,\ldots,0}_{k_2})$ $k_1 = \lceil \frac{m}{10} \rceil$, $k_2 = \lceil n + \frac{m}{10} \rceil$
T2	Alternating 0's and 1's, beginning with 1	$(1,0,1,0,1,0,\ldots)$
T3	Alternating 0's and 1's, beginning with 0	$(0,1,0,1,0,1,\ldots)$
T4	Repeating the sequence $(1,1,0)$	$(1,1,0,1,1,0,\ldots)$
T5	Repeating the sequence $(1,0,0)$	$(1,0,0,1,0,0,\ldots)$
T6	From 1 increasing by 1	$(1,2,\ldots,m-1,m)$
T7	Ending with $3m$ and decreasing towards beginning in a piecewise linear manner, k weights by 3, next k weights by 2 and rest by 1	$(\ldots,3m-5k-2,3m-5k-1,$ $\underbrace{3(m-k)-2k,\ldots,3(m-k)-2}_{k},$ $\underbrace{3(m-k),\ldots,3(m-1)}_{k},3m)$ $k = \lfloor \frac{m}{3} \rfloor$

constraints (16d) and (16e) is simpler than that of (11d) and (11e), respectively.
For problem (16), we also consider an impact of valid inequality (14). The proof that it is a valid inequality for (16) is analogous to that in Proposition 3.

4. Computational Tests

To investigate the computational performance of the proposed formulations, we have applied them to various location problems and compared their results. We have used CPLEX solver to solve problems. The experimental scheme has been analogous to that presented in [16]. We have considered some parameters of location problems and have defined the set of their possible values. Then we have generated various testing instances as the combinations of parameters' values. We have taken into account the following parameters: the number of sites (clients), the number of facilities to be placed, and the type of problem defined by the vector of preference weights.
The number of sites (clients) m determines the size of the problem. Six different values are considered $m \in \{8,10,12,15,20,25\}$. The second parameter, the number of facilities n, is defined as proportional to the problem size, and the following cases are examined: $\lceil \frac{m}{4} \rceil$, $\lceil \frac{m}{3} \rceil$, $\lceil \frac{m}{2} \rceil$ and $\lceil \frac{m}{2} + 1 \rceil$, where $\lceil a \rceil$ is the smallest integer value not smaller than a. The last parameter is the vector of preference weights **w**, which defines the problem type (the objective function) and determines the structure, and thus the complexity of the problem. We consider seven problem types with non-monotonic or increasing weights, which are defined in Table 1. In case of non-increasing weights, our hybrid models simplify to LP formulation (4), which performs much better than MILP models and was studied for location problems in [14]. The trimmed mean problem (T1) discard some of the largest and smallest outcomes and is considered as a robust objective. Problem types T2–T5 are artificial and are mainly used to check the computational efficiency for very irregular preference weights. The last two problem types, T6 and T7, represent increasing weights and can be treated as extended version of difficult min-min problems.
We have generated 15 cost matrices, for each size case, with zero on the main diagonal (Free Self-Service, FSS) and remaining entries randomly generated from a discrete uniform distribution on the interval $[1,100]$. These matrices have been combined with each combination of parameters with the corresponding problem size. Thus, we have received 15 problem instances differing in cost matrices for each combination of the number of sites, the number of facilities and the problem type. To solve them, we have applied the CPLEX 12.4 from IBM ILOG CPLEX Optimization Studio [17]. Computation have been carried out on a machine with the Intel Core2 Duo 2.53 GHz (mobile) and 4 GB of RAM. A time limit of 600 s has been imposed on maximum solution time for a single problem instance.
We have investigated the following formulations:

- $MH1_1$ – basic problem (11),
- $MH1_2$ – problem (11) with inequality (12),
- $MH1_3$ – problem (11) with inequality (13),
- $MH1_4$ – problem (11) with inequality (14),
- $MH1_5$ – problem (11) with inequalities (13), (14),
- $MH2_1$ – basic problem (16),
- $MH2_2$ – problem (16) with inequality (14).

Table 2
Average solution times of hybrid models and one of the most efficient general previous MILP model M1
(upper index depicts the number of instances out of 15 that reached the limit of 600 s;
"–" means that all 15 instances reached the time limit)

Problem			CPU[s]							
Type	m	n	M1	MH1$_1$	MH1$_2$	MH1$_3$	MH1$_4$	MH1$_5$	MH2$_1$	MH2$_2$
T1	8	2	0.35	0.05	0.05	0.06	0.05	0.06	0.08	0.08
		3	0.29	0.05	0.05	0.07	0.05	0.07	0.07	0.06
		4	0.15	0.05	0.05	0.06	0.05	0.06	0.06	0.06
		5	0.06	0.05	0.04	0.05	0.05	0.05	0.05	0.05
	10	3	1.85	0.08	0.09	0.15	0.09	0.15	0.15	0.15
		4	1.33	0.09	0.09	0.14	0.09	0.14	0.13	0.13
		5	0.67	0.11	0.11	0.11	0.11	0.12	0.11	0.11
		6	0.28	0.10	0.10	0.10	0.10	0.10	0.09	0.10
T2	8	2	0.09	52.48	51.39	10.22	3.97	2.69	10.98	2.62
		3	0.07	53.16	54.28	7.02	3.63	2.39	7.58	2.10
		4	0.04	50.14	53.76	5.06	3.61	1.48	5.74	1.38
		5	0.04	52.74	52.91	4.68	4.26	1.23	4.91	1.20
	10	3	0.32	–	–	–	167.56	103.15	–	108.89
		4	0.15	–	–	–	160.29	83.34	–	96.73
		5	0.12	–	–	–	157.81	72.70	–	82.83
		6	0.06	–	–	–	152.06	56.88	–	58.92
T3	8	2	0.28	148.20	151.27	18.73	6.36	2.53	19.88	2.73
		3	0.20	177.86	189.57	14.01	6.42	2.02	14.05	1.88
		4	0.14	170.15	199.93	11.29	6.65	2.25	11.41	2.17
		5	0.06	163.80	159.00	8.66	6.44	1.32	7.13	1.30
	10	3	0.94	–	–	–	294.08	91.91	–	96.32
		4	0.84	–	–	–	321.08	84.96	–	90.87
		5	0.51	–	–	–	313.26	72.61	–	74.96
T4	8	6	0.39	–	–	–	297.26	68.36	–	70.64
		2	0.10	1.41	1.72	0.21	0.68	0.19	0.19	0.18
		3	0.09	2.06	2.29	0.19	0.63	0.19	0.17	0.17
		4	0.05	1.71	1.81	0.13	0.53	0.13	0.13	0.12
		5	0.03	1.50	1.49	0.11	0.53	0.11	0.11	0.11
	10	3	0.24	^{1}346.19	^{2}393.42	12.78	22.12	6.25	10.21	6.11
		4	0.18	^{1}327.38	^{1}364.44	6.66	20.24	4.35	6.34	4.61
		5	0.13	310.17	260.00	5.54	23.00	3.08	4.32	3.49
		6	0.07	207.92	226.05	2.03	19.51	1.53	1.93	1.63
T5	8	2	0.10	2.57	2.85	0.83	0.80	0.65	0.76	0.62
		3	0.08	2.33	2.30	0.93	0.69	0.58	0.89	0.59
		4	0.06	2.17	2.40	0.64	0.63	0.49	0.62	0.45
		5	0.03	1.90	1.99	0.33	0.54	0.33	0.38	0.32
	10	3	0.28	^{3}416.97	^{2}416.28	45.31	25.34	11.81	43.93	12.12
		4	0.15	^{1}287.87	311.06	36.27	24.31	9.77	31.67	9.37
		5	0.12	291.73	289.73	34.71	25.30	9.44	30.62	9.10
		6	0.09	293.54	310.41	26.53	24.76	8.06	27.82	7.68
T6	8	2	0.30	–	–	0.28	8.88	0.22	0.26	0.21
		3	0.22	–	–	0.18	14.44	0.15	0.16	0.14
		4	0.11	–	–	0.11	19.69	0.09	0.09	0.09
		5	0.04	–	–	0.06	24.00	0.05	0.05	0.04
	10	3	1.81	–	–	2.68	^{4}490.57	0.68	3.36	0.66
		4	0.79	–	–	0.65	^{14}593.68	0.40	0.54	0.35
		5	0.46	–	–	0.30	–	0.21	0.26	0.19
		6	0.16	–	–	0.15	–	0.13	0.13	0.11
T7	8	2	0.12	^{12}536.72	^{11}528.55	0.13	3.47	0.11	0.12	0.10
		3	0.08	–	–	0.09	7.50	0.07	0.06	0.08
		4	0.04	–	–	0.05	11.19	0.05	0.05	0.05
		5	0.03	–	–	0.03	15.50	0.03	0.03	0.03
	10	3	0.42	–	–	0.38	195.78	0.32	0.52	0.27
		4	0.19	–	–	0.18	^{1}361.19	0.17	0.16	0.14
		5	0.10	–	–	0.09	^{7}518.86	0.09	0.09	0.09
		6	0.05	–	–	0.07	^{14}589.66	0.06	0.06	0.06

Table 3
Average solution times of the most efficient hybrid and previous MILP models for selected problem types (upper index depicts the number of instances out of 15 that reached the limit of 600 s; "–" when all 15 instances reached the time limit)

Problem			CPU[s]	
Type	m	n	M1	$MH2_2$
T1	12	3	11.80	0.64
		4	7.86	0.39
		6	2.11	0.37
		7	1.33	0.33
	15	4	162.75	0.90
		5	94.90	0.81
		8	11.70	0.77
		9	5.56	0.73
	20	5	–	2.47
		7	–	2.31
		10	^{12}551.73	3.03
		11	^{4}305.99	2.93
	25	7	–	24.37
		9	–	32.14
		13	–	43.77
		14	–	44.54
T6	12	3	11.79	3.64
		4	10.95	1.61
		6	1.77	0.46
		7	0.54	0.20
	15	4	156.83	37.00
		5	101.02	7.54
		8	11.19	0.91
		9	3.05	0.54
	20	5	–	^{13}574.93
		7	–	^{5}297.77
		10	–	10.45
		11	^{7}476.59	5.67
	25	7	–	^{14}595.39
		9	–	^{12}532.15
		13	–	64.75
		14	–	37.10
T7	12	3	1.47	1.29
		4	0.71	0.46
		6	0.20	0.16
		7	0.10	0.12
	15	4	5.59	6.18
		5	2.61	1.85
		8	0.27	0.53
		9	0.15	0.35
	20	5	100.52	^{8}356.07
		7	29.00	^{1}73.42
		10	4.97	4.08
		11	1.71	3.12
	25	7	^{8}480.75	^{11}480.85
		9	^{3}252.69	^{4}277.26
		13	24.44	14.56
		14	12.89	8.78

4.1. Results

Table 2 presents solution times for instances with 8 and 10 locations. Solution times for model MH1 vary widely between different types of problem and inclusion of some valid inequalities. In general, valid inequalities allow to improve the performance and reduce the solution time. An exception is inequality (12), which hardly influence the computational performance. Inequalities (13) and (14) in most cases shorten the solution time of one or two orders of magnitude. The best results are obtained for trimmed mean problems (T1), and it is consistent with our expectation. In this case, valid inequalities do not influence the solution time significantly. In fact, inequality (14) is not defined for T1 problems as there is only one negative weight w'_k. The valid inequalities significantly improve results for problems T2–T5. However, the solution times for these problems are much longer than for other types. Interesting situation is for problems T6 and T7, which in theory are the most difficult as there are the maximum number of negative weights w'_k. Basic model $MH1_1$, indeed, performs very poorly. However, formulations with valid inequality (14) and especially with (13) achieve much shorter solution times, even of three orders of magnitude.

Analyzing model MH2, we see that its basic formulation $MH2_1$ performs much better than basic formulation $MH1_1$ of model MH1. In fact, basic formulation $MH2_1$ performs similar to formulation $MH1_3$, and formulation $MH2_2$ achieves similar solution times to formulation $MH1_5$. Generally, formulations $MH1_5$ and $MH2_2$ seem to be the best ones from examined hybrid models.

We have also compared the hybrid models with one of the most efficient MILP model for the OWA optimization from literature (see model $M1_3$ from [14]). Table 2 shows clearly that the hybrid models perform much worse for problem types T2–T5. On the other hand, for problem types T6 and T7 they obtain similar or shorter solution times. For trimmed mean problem (T1) the results show the advantage of hybrid models even more (formulation $MH2_2$). To investigate it a little more, Table 3 presents the results for problems T1, T6 and T7 with 12–25 locations. It reveals that hybrid model has much better performance for trimmed mean problems, presents the substantial advantage for T6 problems and has similar efficiency for T7 problems. So the hybrid models are specially useful for trimmed mean problems, which seems to be one of the most important for practical application from all non-monotonic problem types, to which linear model can not be applied.

5. Conclusions

The paper analyzes the OWA optimization models for discrete location problems. The OWA operator provides a parametrized preference model that generalizes many objective functions and allows to obtain impartial solution, what is important when we consider independent clients. Unfortunately, the ordering operator hinders the problem increasing its computational complexity. Therefore, the efficient formulations for OWA optimization are sought. We introduce general MILP models that can be applied for any non-negative preference weights. It extends the LP formulation, adding the mixed integer part. We also propose some simple valid inequalities to improve the computational performance. The results show the advantage of proposed new hybrid formulations over other general MILP models from literature for some specific problem types. The greatest improvement is obtained for trimmed mean problems. This is particularly important as trimmed mean problems seems to be one of the most useful in practical applications from all other problem types with non-monotonic preference weights, which can not be solved by LP model. On the other hand, hybrid models perform very poorly for problem types T2–T5. However, these types represent rather artificial objective functions (preferences) with little practical value. The proposed models perform surprisingly well for problems with increasing weights, which require the largest number of binary variables. For example, considering problems T6, the hybrid models obtain much shorter solution times than previous general formulations.

The presented new models shorten solution times for some specific problems, but there is still room for improvement. As mentioned before, presented formulations are general and can be applied to various multicriteria problems (for MH2 and inequality (13) we only require the non-negativity of the outcomes). Some modifications and valid inequalities that exploit specific problem structure (such as free self-service assumption) may increase the computational efficiency.

As the location problem with OWA objective is an *NP*-hard problem, heuristic methods seems a reasonable approach. Despite this fact, the literature on approximation algorithms for these problems is rather limited. Thus, it is also the area where we are currently carrying out some research.

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