Paper

Estimation of Network Disordering Effects by In-depth Analysis of the Resequencing Buffer Contents in Steady-state

Alexander Pechinkin and Rostislav Razumchik

Institute of Informatics Problems, Federal Research Center "Computer Science and Control", Russian Academy of Sciences, Moscow, Russia

Abstract—The paper is devoted to the analytic analysis of resequencing issue, which is common in packet networks, using queueing-theoretic approach. The authors propose the mathematical model, which describes the simplest setting of packet resequencing, but which allows one to make the first step in the in-depth-analysis of the queues dynamics in the resequencing buffer. Specifically consideration is given to N-server queueing system ($N > 3$) with single infinite capacity buffer and resequencing, which may serve as a model of packet reordering in packet networks. Customers arrive at the system according to Poisson flow, occupy one place in the buffer and receive service from one of the servers, which is exponentially distributed with the same parameter. The order of customers upon arrival has to be preserved upon departure. Customers, which violated the order are kept in resequencing buffer which also has infinite capacity. It is shown that the resequencing buffer can be considered as consisting of n, $1 \leq n \leq N-1$, interconnected queues, depending on the number of busy servers, with i-th queue containing customers, which have to wait for i service completions before they can leave the system. Recursive algorithm for computation of the joint stationary distribution of the number of customers in the buffer and servers, and each queue in resequencing buffer are being obtained. Numerical examples, which show the dynamics of the characteristics of the queues in resequencing buffer are given.

Keywords—infinite capacity, joint distribution, queueing system, resequencing.

1. Introduction

It is well-known that performance of multi-node simultaneous processing systems can suffer from the resequencing issue, i.e. when the order of arriving customers (packets, jobs, items, etc.) is violated due to disordering, which may be introduced by service process or other external/internal factors. As a consequence of disordering, some customers have to wait for other customers before they are allowed to leave the system. So far various analytical methods and models have been proposed to study the impacts of resequencing. Survey on the resequencing problem that covers the early period up to 1997 and review of queueing theoretic methods and early models for the modeling and analysis of parallel and distributed systems, including network systems, with resequencing can be found in [1] and [2]. Queueing-theoretic approach to the resequencing problem implies that the system under consideration is represented as interconnected queueing systems/networks, where the disordering of customers takes place. The system is followed with resequencing buffer, where the order of customers is recovered. When the system under consideration is the packet network, then the disordering may take place in the core network and the resequencing buffer is, for example, the de-jitter buffer in the end node. In [3] there was proposed to group existing papers on resequencing into two categories: papers that characterize the disordering process using single queueing system with several servers sharing a single queue (see e.g. [4]) and papers where disordering is modeled by a queueing system with several parallel servers and queues, and each server has its own dedicated queue (see e.g. [5]). Paper [3] contains the survey of papers belonging to these two categories.

In this paper, authors consider the system belonging to the first one. Up to now various problems setting have been considered and solved including calculation of the distribution of number of packets in resequencing buffer and in system under different assumptions about arrival and service process, calculation of the distribution of the resequencing delay, and optimal allocation of customers (see e.g. [1], [2], [5]–[15]). The resequencing effects can be estimated by calculation one or several parameters of the resequencing buffer (say, mean buffer size). Clearly the less mean buffer size is observed, the less packet resequencing is required in the system.

Here authors propose to dig deeper in the resequencing issue by giving a more thorough analysis of the resequencing buffer. It is probably the simplest problem setting but it gives a general view of the approach and method of the analysis. It is important to notice that the proposed method heavily relies on the fact that the servers are homogeneous and its extension to the heterogeneous case is a question of further research.

Specifically the network is modeled, where disordering takes place, as a $M|M|N|\infty$ queue ($N > 3$). Here each server may represent the link (or group of links) in the network. Transmission times (service times) are exponentially distributed with the same parameter. The elimination of the disordering effect (i.e. recovery of the packets' sequence) takes place in the resequencing buffer. The sketch of the system can be seen in Fig. 1. Packets arrive according to Poisson flow and are stored in the infinite capacity buffer before entering the network, where from they are chosen for transmission according to First Come First Served (FCFS) or Last Come First Served (LCFS) or Random discipline. Customers, which violated the arrival order are kept in the resequencing buffer (RB) of infinite capacity before each of them can leave the system. As it was noticed in [16], in such $M|M|N|\infty$ resequencing queue with $N > 2$ servers, the resequencing buffer can be thought of either as a single queue, where all customers which violated arrival order reside together (Fig. 1a) or as a collection of several separate interconnected queues (Fig. 1b). In the latter case i-th queue contains those customers, which have to wait for i service completions before they can leave the system. Notice that the number of service completions needed by a customer in the RB to leave the system cannot be greater than $N-1$.

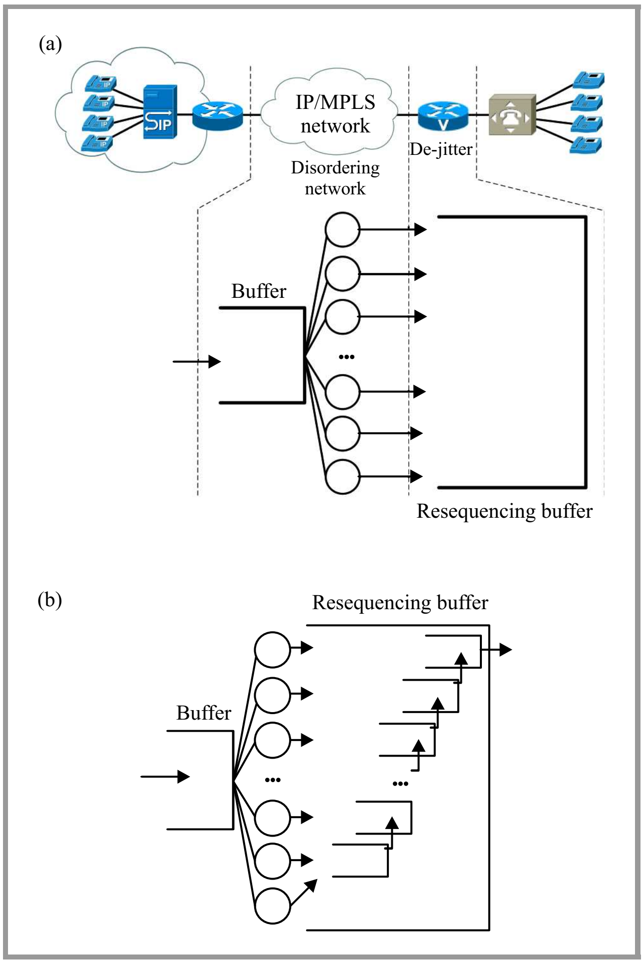

Fig. 1. (a) example of the resequencing issue in the VoIP scenario, (b) sketch of the multiserver resequencing queue with separate interconnected queues in the resequencing buffer.

The proposed point of view of the in-depth-dynamics of the RB can be probably best described by an example. Consider the network modeled by $M|M|4|\infty$ queueing system (where disordering takes place) and a resequencing queue at the exit from the network (see Fig. 1a). Without loss of generality authors suppose that packets (customers) upon entering the network (system) obtain a sequential number. The sequence starts from 1 and coincides with the row of natural numbers. Let us assume that at some time instant network occupancy is as depicted in Fig. 2a. Each square represents one packet and number in the square is its sequential number.

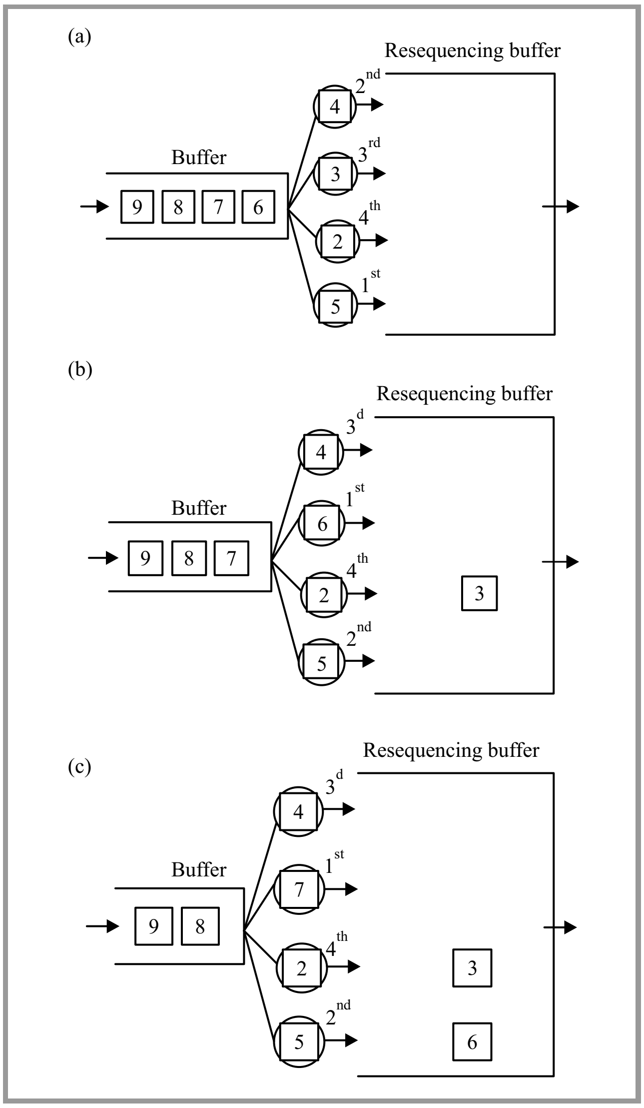

Fig. 2. Example of how resequencing system's content may evolve in one step.

After each service completion let one label customers in servers according to the order in which they occupied servers. Let us refer to the customer, which was the last to enter server as the 1st level customer. Customer which entered server just before the 1st level customer is referred to as the 2nd level level customer. The 3rd level customer is the one which entered server just before the 2nd level customer. Finally the 4th level customer was the first (among other three in service) to enter server. In Fig. 2a one can see the corresponding labeling.

Assume the next service to happen is the completion of service of the 3rd level customer. It will not leave the system but occupy one place in the resequencing buffer, and customer from the buffer with the sequential number 6 will occupy free server (Fig. 2b). At this time instant authors have to re-label customers in servers because the order in which they occupied servers had changed. Now customer with the sequential number 6 becomes the 1st level customer. New labeling can be seen in Fig. 2b. If the next service completion is the service completion of the 1st level customer, then it joins the resequencing buffer and customer from the buffer with sequential number 7 occupies free server. From Fig. 2c it can be seen, that though two customers reside together in the resequencing buffer and can constitute a single queue, time until each of them leaves the system is different. Indeed customer with the sequential number 3 has to wait only for one customer (one service completion) before it can leave the system and customer with the sequential number 6 has to wait for three customers before it may depart from the system. By this attribute – number of service completions, which customer residing in resequencing buffer has to wait for before it can leave the system – by which the single queue in resequencing buffer can be partitioned into several separate interconnected queues (see Fig. 1b).

One may continue the example further and arrive, for example, to the network occupancy as depicted in Fig. 3. In figure one can see how packets in RB are distributed among different queues. Partitioning of the RB into several queues gives a more detailed view of its dynamics and leads to number of interesting questions:

- what is the joint stationary distribution of all queues in the system?
- are there any dependencies between queues' sizes?
- what happens with queues in the RB if N grows without bound?
- what influence does service rate (distribution) has on queues' sizes in the RB, etc?

In this paper the authors focus on the first two questions.

In system with $N \geq 2$ servers, if all of them are busy, then resequencing buffer can be partitioned into $N-1$ queues (see Fig. 3 as example for $N = 4$). If the number of busy servers is less than N, then the number of queues in the resequencing buffer is equal to the number of busy servers. The analysis of the joint stationary distribution of number of customers even in simple cases with Poisson flow and homogeneous exponential servers turns out to be a challenging task. In [16] for $M/M/3/\infty$ queue followed with infinite resequencing buffer one obtains expressions for joint stationary distribution of number of customers in buffer and servers, and number of customers in each of two queues in resequencing buffer both in explicit form and in terms

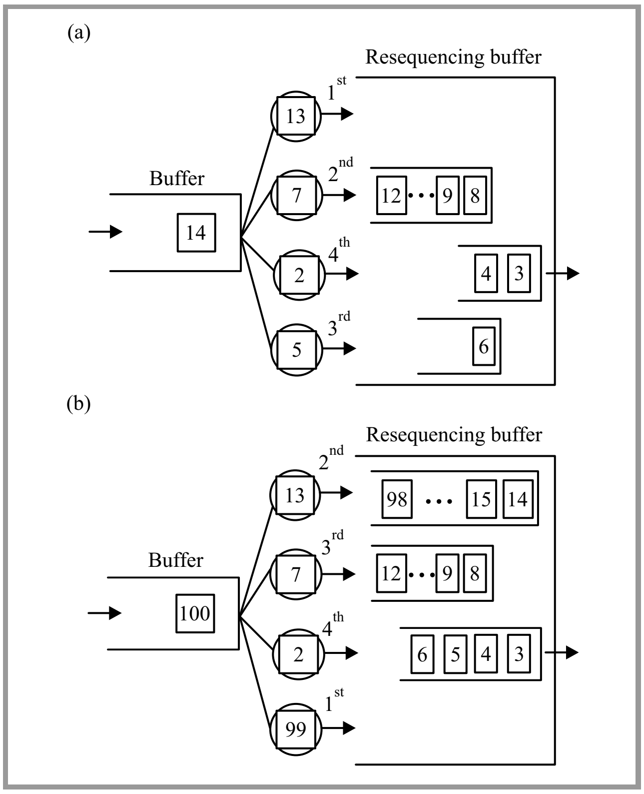

Fig. 3. Examples of resequencing system's contents at two different time instants.

of generating functions. In [17] for $M/M/N/\infty$ queue followed with infinite resequencing buffer there was obtained algorithm for recursive computation joint stationary distribution of number of customers in buffer and servers, and sum of number of customers in two, three, ..., and $N-1$ queues in resequencing buffer.

In this paper by modeling the disordering of packets by $M/M/N/\infty$ queue followed with the RB of infinite capacity we propose the methodology for computation of joint stationary distribution of number of customers in buffer and servers, and number of customers in each queue in the RB. Here it is shown that in the general case $N > 3$ the joint stationary distribution can be computed recursively. The special case of this methodology has already been used in [16]. The authors note that the joint distribution for the general case can be also obtained algorithmically in terms of the generating functions (as it is shown in [18]), but that results are, as usual, hardly applicable for the computation of the joint distribution itself.

The next Section 2 is devoted to the description of the system and the necessary notation. In Section 3 it is shown how one can obtain the system of equilibrium equations for joint stationary distribution of number of customers in buffer and servers, and number of customers in each queue in resequencing buffer. The description of the solution algorithm comes after. Several numerical examples are given in Section 4. In the conclusion, one provides a short discussion of obtained results and outlines possible directions of further research.

2. System Description and Notation

Consider a queueing system with $3 < N < \infty$ servers, infinite capacity buffer, incoming Poisson flow of customers of intensity λ, exponential service time distribution in each server with parameter μ and RB of infinite capacity. Customer upon entering the system obtains a sequential number and joins the buffer. Without the loss of generality authors suppose that the sequence starts from 1 and coincides with the row of natural numbers, i.e. customer upon entering the empty system receives number 1, the next one – number 2 and so on and so forth. Customers leave the system strictly in the order of their arrival. Thus, after customer's arrival it enters server (if there are any idle) or remains in the buffer for some time and then receives service from one of the servers. If at the moment of its service completion there are no customers in the system or all other customers present at that moment in the buffer and in all other servers have greater sequential numbers it leaves the system. Otherwise, it occupies a place in the RB. Each customer from the RB leaves it if and only if its sequential number is less than sequential numbers of all other customers present in the system. It may be noticed that the customers may leave the RB in groups. For example, in Fig. 3a if customer with sequential number 2 is the next to finish service then it leaves the system at one together with customer number 3 and 4.

In order to correctly define the partitioning of the RB into several queues the following approach is used. Assume there are n, $n=\overline{1,N}$, busy servers in the system. Each time any server becomes free or busy the customers in servers are labeled according to the order in which they occupied servers. Let us refer to the customer which was the last to enter server as the 1st level customer. Customer, which entered server right before the 1st level customer, is referred to as the 2nd level level customer. The 3rd level customer is the one which entered server before the 2nd level customer. Proceeding in similar manner customer, which was the first (among n) to enter server, is referred to as the n^{th} level the customer. Customers which reside in the RB form $(n-1)$ separate queues in the following way. Customers which entered the RB between the 1st level and the 2nd level customer form queue #1. Customers which entered the RB between the 2nd level and the 3rd level customer form queue #2 and so on. Customers which entered the RB between the $(n-1)$ level and the n^{th} level customer form queue #$(n-1)$. Example of such partitioning of the RB into separate queues in case when $N=4$ is given in Fig. 3.

Let us denote by $\xi(t)$ – the number of customers in buffer and servers at instant t, and by $\eta_i(t)$ — the number of customers in i-th queue in resequencing buffer at instant t. Then the Markov process $\zeta(t)$, describing the stochastic behavior of the system, is

$$\zeta(t) = \{(\xi(t), \eta_1(t), \eta_2(t), \ldots, \eta_{N-1}(t)), \quad t \geq 0\}.$$

In case $\xi(t)=0$, all components of the process $\zeta(t)$ except for the first one are omitted; in case $\xi(t)=n$, $n=\overline{1,N-2}$, last $N-1-n$ components are omitted. The state space of the process $\zeta(t)$ has the form

$$\mathscr{X} = \{0\} \cup \{(1,i_1),\ i_1 \geq 0\} \cup \{(2,i_1,i_2),\ i_1,i_2 \geq 0\} \cup \ldots \cup \{(n,i_1,i_2,\ldots,i_{N-1}),\ n \geq N-1, i_1,i_2,\ldots,i_{N-1} \geq 0\}.$$

Let us denote by p_n, $n \geq 0$, the stationary probabilities of the fact, that there are n customer in buffer and servers (customers in the RB are not taken into account), i.e.

$$p_n = \lim_{t\to\infty} \mathbf{P}\{\xi(t)=n\}.$$

One can notice that p_n, $n \geq 0$, are determined by the same equations as in the simple $M/M/N/\infty$ queue (see e.g. [19]):

$$p_0 = \left(\sum_{i=0}^{N-1} \frac{\rho^i}{i!} + \frac{\rho^N}{(N-1)!\,(N-\rho)}\right)^{-1}, \ \rho = \lambda/\mu, \quad (1)$$

$$p_i = \frac{\rho^i}{i!} p_0, \quad i=\overline{1,N}, \quad (2)$$

$$p_i = \frac{\rho^i}{N!\,N^{i-N}} p_0 = \tilde{\rho}^{i-N} p_N,\ \tilde{\rho} = \rho/N, \quad i \geq N+1. \quad (3)$$

It can be observed that for the stationary probabilities of the considered system with resequencing to exist it is necessary and sufficient that the condition (necessary and sufficient) for the existence of probabilities p_n is fulfilled, i.e. $\rho/N < 1$ must hold.

Let us denote by $p_{n;i_1,\ldots,i_m}$, $m=\overline{1,N-1}$, $i_1,\ldots,i_m \geq 0$, the stationary probability of the fact that there are $n \geq N$ customers in buffer and servers, and in the RB there are i_1 customers in queue #1, i_2 customers in queue #2, …, i_m customers in queue #m, that is

$$p_{n;i_1,\ldots,i_m} = \lim_{t\to\infty} \mathbf{P}\{\xi(t)=n, \eta_1(t)=i_1,\ldots,\eta_m(t)=i_m\},$$
$$m=\overline{1,N-1},\ n \geq N,\ i_1,\ldots,i_m \geq 0.$$

If the number of busy servers is $n<N$, then we denote by $p_{n;i_1,\ldots,i_m}$, $m=\overline{1,n}$, $i_1,\ldots,i_m \geq 0$, the stationary probability of same fact, that is

$$p_{n;i_1,\ldots,i_m} = \lim_{t\to\infty} \mathbf{P}\{\xi(t)=n, \eta_1(t)=i_1,\ldots,\eta_m(t)=i_m\},$$
$$n=\overline{1,N-1},\ m=\overline{1,n},\ i_1,\ldots,i_m \geq 0.$$

The only difference between cases $n \geq N$ and $n<N$ is that in the former case number of queues in RB may vary from 1 to $N-1$ and in the latter case it may vary only from 1 to n. From the definition of the joint probabilities it follows that the stationary distribution p_n, $n \geq 1$, can be calculated from $p_{n;i_1,\ldots,i_m}$ by summation

$$p_n = \mathbf{P}\Big\{\zeta(t) \in \bigcup_{i_1,\ldots,i_n \geq 0}^{\infty} (n,i_1,i_2,\ldots,i_n)\Big\} = \sum_{i_1,\ldots,i_n=0}^{\infty} p_{n;i_1,\ldots,i_n},\ n=\overline{1,N-2},$$

$$p_n = \mathbf{P}\Big\{\zeta(t) \in \bigcup_{i_1,\ldots,i_{N-1} \geq 0}^{\infty} (n,i_1,i_2,\ldots,i_{N-1})\Big\} = \sum_{i_1,\ldots,i_{N-1}=0}^{\infty} p_{n;i_1,\ldots,i_{N-1}},\ n \geq N-1.$$

3. System of Equilibrium Equations

In order to obtain the balance equations let us consider step-by-step different partitions of the state space and use rate-in-rate-out principle (local balance). Notice that if one sums up, say the probability $p_{N;i_1,\ldots,i_{N-1}}$, over all possible values of $i_2,\ldots,i_{N-1}$, then one obtains probability of the state set

$$\bigcup_{i_2,\ldots,i_{N-1}\geq 0}^{\infty} (N,i_1,i_2,\ldots,i_{N-1}) \,,$$

i.e. probability of the fact that there are N customers in buffer and servers, and queue #1 contains i_1 customers (irrespectively of the number of customer in the queues #2, #3 … #$(N-1)$ in the RB). For the probabilities of such state sets it is possible to analyse one-step transitions and write out the balance equations, that eventually lead to the determination of the whole joint distribution.

Denote by $p_{n;i_1,\ldots,i_m}$, $n \geq 2$, $m = \overline{1,\min(n-1,N-2)}$, $i_1,\ldots,i_m \geq 0$, the probability of the fact that there are n customers in the queue and servers, and in the RB there are i_1 customers in queue #1, i_2 customers in queue #2, …, i_m customers in queue #m, that is

$$p_{n;i_1,\ldots,i_m} = \sum_{i_{m+1},\ldots,i_n=0}^{\infty} p_{n;i_1,\ldots,i_m,i_{m+1},\ldots,i_n} \,, \quad n=\overline{2,N-2},\ m=\overline{1,n-1},\ i_1,\ldots,i_m \geq 0, \tag{4}$$

$$p_{n;i_1,\ldots,i_m} = \sum_{i_{m+1},\ldots,i_{N-1}=0}^{\infty} p_{n;i_1,\ldots,i_m,i_{m+1},\ldots,i_{N-1}} \,, \quad n \geq N-1,\ m=\overline{1,N-2},\ i_1,\ldots,i_m \geq 0 \,. \tag{5}$$

Notice that Eqs. (4) and (5) define the probabilities not of a single state of the system but of the set of states. For example, probability $p_{N-2;i_1}$ defined by (4) is the probability of the fact that there are $N-2$ busy servers, the buffer is empty, and there are $i_1 \geq 0$ customers in queue #1 in RB.

Balance equations for $p_{n;i_1,\ldots,i_m}$ will be written out in the following way. Firstly, one establishes equations for $p_{n;i_1}$, $n \geq N$, $i_1 \geq 0$ and then for $p_{n;i_1}$, $n = \overline{N-1,1}$, $i_1 \geq 0$. Secondly, one finds equations for $p_{n;i_1,i_2}$, $n \geq N$, $i_1,i_2 \geq 0$ and then for $p_{n;i_1,i_2}$, $n = \overline{N-1,2}$, $i_1,i_2 \geq 0$. After that one proceeds to $p_{n;i_1,i_2,i_3}$, $n \geq N$, $i_1,i_2,i_3 \geq 0$ and $p_{n;i_1,i_2,i_3}$, $n = \overline{N-1,3}$, $i_1,i_2,i_3 \geq 0$. This procedure continues until one arrives to $p_{n;i_1,\ldots,i_m}$, $n \geq N$, $i_1,\ldots,i_m \geq 0$ and $p_{N-1;i_1,\ldots,i_m}$, $i_1,\ldots,i_m \geq 0$.

For probabilities $p_{n;i_1}$, $n \geq N$, $i_1 \geq 0$, the following equations hold

$$p_{n;0}(\lambda+N\mu) = p_{n-1;0}\lambda+p_{n+1}(N-1)\mu, \quad n \geq N, \tag{6}$$

$$p_{n;i_1}(\lambda+N\mu) = p_{n-1;i_1}\lambda+p_{n+1;i_1-1}\mu, \quad n \geq N,\ i_1 \geq 1 \,. \tag{7}$$

Equation (6) is derived as follows. Assume that the system is in one of the states when there are $n \geq N$ customers in the buffer and servers and queue #1 in the RB is empty. The considered state set is $\bigcup_{i_2,\ldots,i_{N-1}\geq 0}^{\infty} (n,0,i_2,\ldots,i_{N-1})$ and the probability of this state set is $p_{n;0}$ according to Eq. (5). The system can leave this state set if the service completion or arrival occurs, i.e. the rate-out flow is $p_{n;i_1}(\lambda+N\mu)$. The system can enter this state set if:

- there were $n+1$ customers in the buffer and servers (which happens with probability p_n) and service completion of any of the N customers except for the 1^{st} *level* customer occurred, which happens with rate $(N\mu)\frac{(N-1)}{N} = (N-1)\mu$;
- there were $n-1$ customers in the buffer and servers and queue #1 in the RB was empty, which happens with the probability $p_{n+1;0}$ according to Eq. (5), and an arrival occurred.

Thus the rate-in flow is $p_{n-1;0}\lambda + p_{n+1}(N-1)\mu$. By equating rate-out and rate-in flows one obtains Eq. (6).

In order to explain Eq. (7) assume that the system is in one of the states when there are $n \geq N$ customers in the buffer and servers and there are $i_1 \geq 1$ customers in queue #1 in the RB. The considered state set is $\bigcup_{i_2,\ldots,i_{N-1}\geq 0}^{\infty} (n,i_1,i_2,\ldots,i_{N-1})$ and the probability of this state set is $p_{n;i_1}$ according to Eq. (5). The rate-out flow from this state set equals $p_{n;i_1}(\lambda+N\mu)$. The system can enter this state set with an arrival if there were $n-1$ customers in the buffer and servers and i_1 customers in queue #1 in the RB, which happens with the probability $p_{n-1;i_1}$ according to Eq. (5). The system can also enter this state set with a service completion from state set when there were $n+1$ customers in the queue and servers, and queue #1 in the RB contained i_1-1 customers, which happens with the probability $p_{n+1;i_1-1}$ according to Eq. (5) and service completion of the 1st level customer occurred (which happens with rate $(N\mu)\frac{1}{N} = \mu$). By equating rate-out and rate-in flows one obtains Eq. (7).

Probabilities $p_{N-1;i_1}$, $i_1 \geq 0$, are governed by the following equations

$$p_{N-1;0}\big[\lambda+(N-1)\mu\big] = p_{N-2}\lambda+p_N(N-1)\mu \,, \tag{8}$$

$$p_{N-1;i_1}\big[\lambda+(N-1)\mu\big] = p_{N;i_1-1}\mu, \quad i_1 \geq 1 \,. \tag{9}$$

Probabilities $p_{n;i_1}$, $n=\overline{1,N-2}$, $i_1 \geq 0$, are given by

$$p_{n;0}(\lambda+n\mu) = p_{n-1}\lambda+p_{n+1;0}n\mu, \quad n=\overline{1,N-2}, \tag{10}$$

$$p_{n;i_1}(\lambda+n\mu) = p_{n+1;i_1}n\mu + \sum_{j=0}^{i_1-1} p_{n+1;i_1-j-1,j}\mu \,, \quad n=\overline{1,N-2},\ i_1 \geq 1 \,. \tag{11}$$

For probabilities $p_{n;i_1,\ldots,i_m}$, $m = \overline{2,N-1}$, $n \geq m$, $i_1,\ldots,i_{N-1} \geq 0$, one can write out the system of balance equations in the general form. It holds

$$p_{n;0,i_2,\ldots,i_m}(\lambda+N\mu) = p_{n-1;0,i_2,\ldots,i_m}\lambda + p_{n+1;i_2,\ldots,i_m}(N-m)\mu + \sum_{j=0}^{i_2-1} p_{n+1;j,i_2-j-1,i_3,\ldots,i_m}\mu + \ldots + \sum_{j=0}^{i_m-1} p_{n+1;i_2,\ldots,i_{m-1},j,i_m-j-1}\mu, \quad n \geq N,\ i_2,\ldots,i_m \geq 0\,, \tag{12}$$

$$p_{n;i_1,\ldots,i_m}(\lambda+N\mu)=p_{n-1;i_1,\ldots,i_m}\lambda+p_{n+1;i_1-1,i_2,\ldots,i_m}\mu,$$
$$n\geq N,\quad i_1\geq 1,\quad i_2,\ldots,i_m\geq 0\,, \qquad (13)$$

$$p_{N-1;0,i_2,\ldots,i_m}\left[\lambda+(N-1)\mu\right]=p_{N-2;i_2,\ldots,i_m}\lambda+$$
$$+p_{N;i_2,\ldots,i_m}(N-m)\mu+\sum_{j=0}^{i_2-1}p_{N;j,i_2-j-1,i_3,\ldots,i_m}\mu+\ldots$$
$$+\sum_{j=0}^{i_m-1}p_{N;i_2,\ldots,i_{m-1},j,i_m-j-1}\mu,\quad i_2,\ldots,i_m\geq 0\,, \qquad (14)$$

$$p_{N-1;i_1,\ldots,i_m}\left[\lambda+(N-1)\mu\right]=p_{N;i_1-1,i_2,\ldots,i_m}\mu\,,$$
$$i_1\geq 1,\quad i_2,\ldots,i_m\geq 0\,, \qquad (15)$$

$$p_{n;0,i_2,\ldots,i_m}(\lambda+n\mu)=p_{n+1;0,i_2,\ldots,i_m}(n-m+1)\mu+$$
$$+p_{n-1;i_2,\ldots,i_m}\lambda+\sum_{j=0}^{i_2-1}p_{n+1;0,j,i_2-j-1,i_3,\ldots,i_m}\mu+\ldots+$$
$$+\sum_{j=0}^{i_m-1}p_{n+1;0,i_2,\ldots,i_{m-1},j,i_m-j-1}\mu\,,$$
$$m\neq N-1,\quad n=\overline{m,N-2},\quad i_2,\ldots,i_m\geq 0\,, \qquad (16)$$

$$p_{n;i_1,\ldots,i_m}(\lambda+n\mu)=p_{n+1;i_1,\ldots,i_m}(n-m+1)\mu+$$
$$+\sum_{j=0}^{i_1-1}p_{n+1;j,i_1-j-1,i_2,\ldots,i_m}\mu+\ldots$$
$$+\sum_{j=0}^{i_m-1}p_{n+1;i_1,\ldots,i_{m-1},j,i_m-j-1,}\mu\,,$$
$$m\neq N-1, n=\overline{m,N-2},\quad i_1\geq 1,\quad i_2,\ldots,i_m\geq 0\,. \qquad (17)$$

In Eqs. (12)–(17) for the sake of brevity agreement is used that $\sum_{i=0}^{-1}a_i=0$. The system of Eqs. (12)–(17) is derived using the same argumentation, which is used above for Eqs. (6)–(7).

For the fixed value of N system, Eqs. (6)–(17) can be solved recursively. Computation of $p_{n;i_1,\ldots,i_m}$ consists of $N-1$ steps. The first step consists of the following sequential computations. Firstly one computes probabilities p_n, $n\geq 0$ using Eqs. (1)–(3). Then one finds probability $p_{N-1;0}$ from Eq. (8), probabilities $p_{n;0}$, $n=\overline{N-2,1}$, from Eq. (10) and then probabilities $p_{n;0}$, $n\geq N$, from Eq. (6). Secondly one computes probability $p_{N-1;0,0}$ from Eq. (14), probabilities $p_{n;0,0}$, $n\geq N$, from Eq. (12), and probabilities $p_{n;0,0}$, $n=\overline{N-2,2}$ from Eq. (16). Thirdly for each $i\geq 1$ using Eqs. (9) and (7) one finds probabilities $p_{n;i}$, $n\geq N-1$.

The second step starts with computation of probability $p_{N-2-k;1-k}$, $k=\overline{0,\min(0,N-1)}$ from Eq. (11). Then starting from $i_1=0$ one computes probabilities $p_{N-1;i_1,i_2}$, $i_1+i_2=1$, from Eqs. (14) and (15). Finally, starting from $i_1=1$, one finds probabilities $p_{n;i_1,i_2}$, $n\geq N$, $i_1+i_2=1$, from Eq. (13).

The third step starts with computation of probabilities $p_{N-2-k;2-k}$, $k=\overline{0,\min(1,N-2)}$, from Eq. (11). Then starting from $i_1=0$, using Eqs. (14) and (15) one finds probabilities $p_{N-1;i_1,i_2}$, $i_1+i_2=2$. After that starting from $i_1=2$, one computes probabilities $p_{n;i_1,i_2}$, $n\geq N$, $i_1+i_2=2$, from Eq. (13). Finally using Eqs. (16) and (17) one obtains probabilities $p_{N-2;i_1,i_2}$, $i_1+i_2=1$, starting from $i_1=0$ and then from Eqs. (12) and (13), firstly, one computes probabilities $p_{N-1;i_1,i_2,i_3}$, $n\geq N-1$, $i_1+i_2+i_3=1$, starting from $i_3=1$ and, secondly, one computes probabilities $p_{n;i_1,i_2,i_3}$, $n\geq N$, $i_1+i_2+i_3=1$, starting from $i_3=1$.

The fourth step starts with computation of probabilities $p_{N-2-k;3-k}$, $k=\overline{0,\min(2,N-3)}$, from Eq. (11), which is followed by computation of probabilities $p_{N-1;i_1,i_2}$, $i_1+i_2=3$, starting from $i_1=0$, etc.

The algorithm for the computation of the whole joint stationary distribution, wherefrom the general pattern can be seen, is given below in pseudo code.

Algorithm 1: Computation of the joint stationary distribution

for $c\geq 0$ **do**
- Compute $p_{N-2-k;c+1-k}$, $k=\overline{0,\min(c,N-c-1)}$ using Eq. (11).
- Compute $p_{N-1;i_1,i_2}$, $i_1+i_2=c+1$, starting from $i_2=c+1$ using Eqs. (14) and (15).
- Compute $p_{n;i_1,i_2}$, $n\geq N$, $i_1+i_2=c+1$, from Eq. (13).
- **if** $c=1$ **then**
  - Compute $p_{N-2;i_1,i_2}$, $i_1+i_2=c$, starting from $i_2=c$ using Eqs. (16) and (17).
  - Compute $p_{N-1;i_1,i_2,i_3}$, $i_1+i_2+i_3=c$, starting from $i_3=c$ using Eqs. (12) and (13).
  - Compute $p_{n;i_1,i_2,i_3}$, $n\geq N$, $i_1+i_2+i_3=c$, starting from $i_3=c$ using Eqs. (12) and (13).
- **end if**
- **if** $c=2$ **then**
  - Compute $p_{N-3;i_1,i_2}$, $i_1+i_2=c-1$, using Eq. (16) and (17).
  - Compute $p_{N-2;i_1,i_2,i_3}$, $i_1+i_2+i_3=c-1$, starting from $i_3=c-1$ using Eq. (16) and (17).
  - Compute $p_{N-1;i_1,i_2,i_3,i_4}$, $i_1+i_2+i_3+i_4=c-1$, starting from $i_4=c-1$ using Eq. (12) and (13).
  - Compute $p_{n;i_1,i_2,i_3,i_4}$, $n\geq N$, $i_1+i_2+i_3+i_4=c-1$, starting from $i_4=c-1$ using Eqs. (12) and 13).
- **end if**
- **if** $c=3$ **then**
  - $\ldots$
- **end if**
- $\ldots$

end for

4. Numerical Examples

Extensive numerical experiments were carried out with recursive algorithm described in the previous section, which involved computation of the joint stationary distribution of number of customers in buffer and servers, and number of customers in queues in the RB, as well as several important performance characteristics. The complexity of the algorithm grows very fast as number of servers increases the computation of the whole joint stationary distribution becomes very slow.

Below several numerical results are given, which show different aspects of the in-depth-behavior of the queues in the RB.

It is assumed that number of servers is $N = 4$ and the service rate is $\mu = 1$. The mean and variance of the number of customers in the RB and correlation coefficient of the number of customers in the buffer and each queue in the RB, as functions of the system's load ρ/N, are depicted in Figs. 4 and 5.

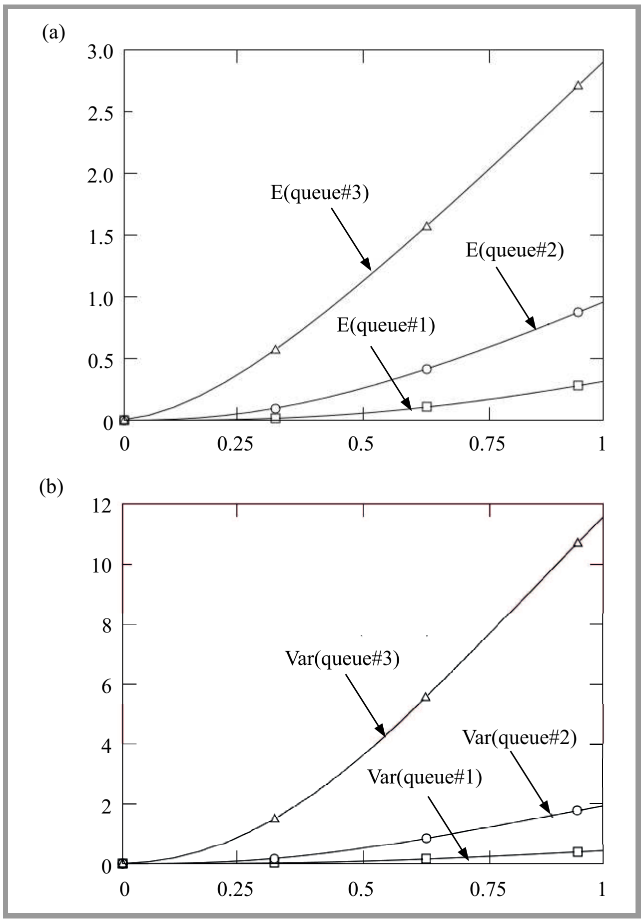

Fig. 4. Dependency of: (a) mean number of customers in each queue in the RB, (b) variance of the number of customers in each queue in the RB, on the system's load ρ/N.

From Fig. 5 it follows that number of customers in queues are weakly correlated and become uncorrelated as the value of load approaches critical value of 1. Conducted experiments show that the same result holds when one considers more general model with MAP arrivals and PH service times (for $N = 2$). From Fig. 4 it can be also observed that the mean lengths of queues in the RB are finite which follows from Little's law. In fact all the moments of the lengths of the queues in the RB are finite. The mean queue sizes in the RB are related to each other by inequalities E(queue #3) > E(queue #2) > E(queue #1). The same holds for the variances and, in general, for any $N \geq 3$ such inequalities hold. Intuitively this can be explained by the fact that queue #1 exists in the RB only when the number of busy servers is at least $N-1$, whereas queue #$(N-1)$ already appears when two servers become busy. The mean queue size in the RB if one sees it as a single queue is the sum of mean queue sizes of queue #1, queue #2, … and queue #$(N-1)$. This suggests that the moments of queue #$(N-1)$ size, say mean, may serve as another performance characteristic of the system with resequencing because eventually its dynamics shows how much disordering is incurred by the network.

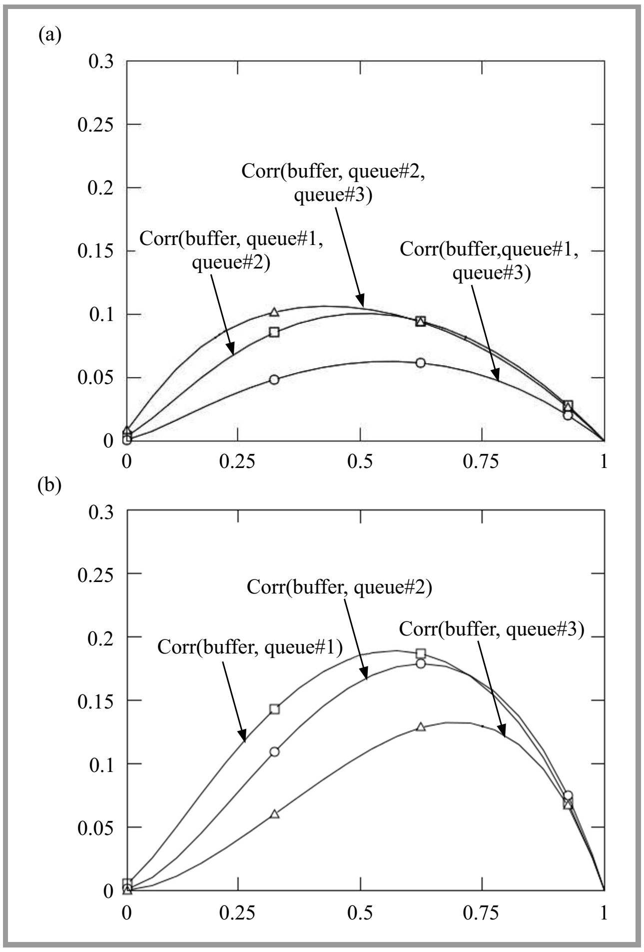

Fig. 5. Dependency of correlation coefficient: (a) number of customers in queues in the RB (pairwise), (b) number of customers in the queue and *each* queue in the RB (pairwise), on the system's load ρ/N.

5. Conclusion

In this paper the authors have considered probably the simplest model for the resequencing issue using queueing-theoretic approach, which allowed one to look "deeper" into the dynamics of the RB. It turns out that the joint stationary distribution of all queues can be computed recursively and, as expected, queues in the RB are not equivalent, although surprisingly weakly correlated. The mechanism according to which the queues in the RB are built allows one to use such characteristic as queue-size moments of the queue $\#(N-1)$ in the RB as another performance indicator of the whole system with resequencing. There are many possible ramifications of the system, which may make it more suitable for practical needs. Probably the Poisson arrival (and exponential service) assumption should not be the first ones to be relaxed, because, for example, in $MAP|PH|2|\infty$ queue followed with resequencing buffer joint stationary distribution can be also found in recursive way and the weak correlation of queue-sizes is preserved. The introduction of heterogeneity and rule for choosing idle servers (say, i-th server with probability p_i, or i-th server with probability $p_{i,j}$ if j servers are busy) is the more promising direction of research.

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Alexander Pechinkin (1946–2014) held the Ph.D. of Sciences in Physics and Mathematics and has principal scientist at the Institute of Informatics Problems of the Russian Academy of Sciences. He held a Professor position at the Peoples' Friendship University of Russia. He was the author of more than 200 papers in the field of theoretical and applied probability theory.

Rostislav Razumchik received his Ph.D. in Physics and Mathematics in 2011. At present he is a senior scientist at Institute of Informatics Problems of FRC CSC RAS and also holds associate professor position at the Peoples' Friendship University of Russia. His current research activities focus on queueing theory and scheduling.

E-mail: rrazumchik@ipiran.ru
Institute of Informatics Problems
Federal Research Center "Computer Science and Control"
of the Russian Academy of Sciences,
Vavilova, 44-2
119333 Moscow, Russia