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Brief Theory of Coherent Processor

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BRIEF THEORY OF COHERENT PROCESSOR

INTRODUCTION

THE PARTICULAR COHERENT DATA PROCESSOR WHICH IS TO BE DISCUSSED RELIES HEAVILY UPON SEVERAL BASIC CONCEPTS. THESE CONCEPTS REPRESENT A DEPARTURE FROM CONVENTIONAL PRACTICE AND HENCE THE THEORETICAL DESCRIPTION MUST AWAIT THEIR PRESENTATION IN ORDER TO GAIN SOME CONTINUITY. WE WILL ACCORDINGLY PRESENT THE UNDERLYING ASSUMPTIONS AND SIGNAL PHYSICS PRIOR TO DESCRIPTION OF THE PROCESSOR ITSELF.

PHYSICAL CONCEPTS

1. SIGNAL ENERGY DENSITY

NO MATTER WHAT MEANS IS USED, ANY INTERCEPTION OF A SIGNAL WHICH IS ITSELF CAPABLE OF DOING WORK, IMPLIES AN ENERGY DENSITY ASSOCIATED WITH THAT SIGNAL. IF THE INTERCEPT IS THAT OF A DISTANT ACOUSTIC EVENT, THEN THE STATE OF THE MEDIUM AT THE POINT OF PICKUP IS SUCH AS TO HAVE A TOTAL SIGNAL ENERGY DENSITY ATTRIBUTABLE TO THAT DISTANT EVENT. THE TOTAL ENERGY DENSITY, $E(s)$, IS EXPRESSED IN JOULES PER UNIT OF STATE VARIABLE s AND REPRESENTS THE AMOUNT OF WORK WHICH COULD BE PERFORMED AT ANY MOMENT IF THE AGENT OF PERCEPTION WERE SUFFICIENTLY CLEVER. FOR A NON-TURBULENT (VECTOR ONLY) MEDIUM IT CAN BE STATED THAT THE TOTAL ENERGY DENSITY, $E(s)$, IS PARTITIONED INTO A KINETIC ENERGY DENSITY, $T(s)$, AND A POTENTIAL ENERGY DENSITY, $V(s)$. FOR AN ACOUSTIC MEDIUM OF ENERGY PROPAGATION THE KINETIC ENERGY DENSITY IS PROPORTIONAL TO THE SQUARE OF PARTICLE VELOCITY WHILE THE POTENTIAL ENERGY DENSITY IS PROPORTIONAL TO THE SQUARE OF INSTANTANEOUS PRESSURE DEVIATION FROM EQUILIBRIUM.

The ocean does work on a hydrophone in the process of interaction of the water with the physical boundaries of the hydrophone. Part of this transfer of energy is manifest as an electrical energy density available to do work on the proper electrical termination. By intent a hydrophone is designed so as to maximize to some measure the transfer of local acoustic energy to electrical energy. Both a desired hydrophone signal due to a distant acoustic event of interest and presumed incoherent acoustic disturbances are made available as electrical signals. The maximum information concerning the desired acoustic event is obtained when all available energy density due to that event is processed. At the point where the electrical signal is made available the energy density representing the instantaneous capability to do work is $E(t)$ and has the dimensions of Joules per second. If this energy is converted to work the total work performed in a time T is ([1] page 265)([2] page 212)([3] page 197),

$$E = \int_T dE = \int_T E(t)\,dt \qquad \text{, Joules} \qquad (1)$$

It has been shown ([4] page 902) that when all components are considered the following relationship exists for energy density partitioning for quantities of class L^2, ([5] page 125, page 10) ([6] page 18)

$$\sqrt{E(s)} = \sqrt{T(s)} + i\sqrt{V(s)} \qquad (2)$$

where the kinetic and potential energy density terms are Hilbert Transforms of each other and s is the state variable under analysis. The important consequence of this is that if one attempts a catagorization of a signal based solely on one energy density component he has not utilized all the information necessary to specify that signal's total energy density without further mathematical manipulation. A further consequence of this relation is that by utilization of Hilbert Transformation it is possible

TO OBTAIN TOTAL ENERGY DENSITY FROM A MEASUREMENT MADE OF ONE COMPONENT. FOR THE SIMPLEST CLASS OF SIGNAL NORMALLY CONSIDERED WHERE ONE COMPONENT IS A POTENTIAL ENERGY VOLTAGE WITH SINE TIME DEPENDENCE THE HILBERT TRANSFORM IS A COSINE SIGNAL AND THE RESULTING OPTIMIZED ANALYSIS PROCEEDS AS A CONVENTIONAL FOURIER EXPANSION.

2. CORRELATION AS A VECTOR OPERATION

THE TERM CORRELATION, AS UTILIZED IN CONVENTIONAL SIGNAL ANALYSIS, RELATES TO THAT SCALAR PROPERTY OBTAINED AS THE FOLLOWING INTEGRAL OF TWO SCALAR SIGNALS, ([7] EQN 19-78, P 334)([8] EQN 134, P 36)

$$R_{fg}(\tau) = \int_T f(x)\, g(x+\tau)\, dx \tag{3}$$

BY THIS CONVENTION A MAXIMUM VALUE OF CORRELATION WILL OCCUR WHEN AN IDENTITY EXISTS BETWEEN THE TWO SIGNAL FUNCTIONS. WHEN THE FUNCTIONS TO BE CORRELATED ARE COMPLEX AND MAY EXIST FOR INDEFINITELY LARGE SEQUENCES OF COORDINATE IT IS CONVENTIONAL TO DEFINE THE AUTOCORRELATION FUNCTION AS THE NORMALIZED LIMITING INTEGRAL, ([9] P 5)

$$\phi(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau)\, f^*(t)\, dt \tag{3.1}$$

WHERE THE STAR INDICATES COMPLEX CONJUGATION. IT IS THE CORRELATION OF A SIGNAL WITH A DELAYED VERSION OF ITSELF AS A FUNCTION OF THE DELAY. BECAUSE THE INTENT OF AUTOCORRELATION IS THE GENERATION OF A SCALAR VALUE AT IDENTITY THE FUNCTION $\phi(\tau)$ HAS PRESERVED SOME INFORMATION WHILE DESTROYING OTHERS. THE INFORMATION PRESERVED IS THE AMPLITUDE AND FREQUENCY OF COMPONENTS, WHILE THE INFORMATION DESTROYED IS THE PHASE OF THE SIGNAL COMPONENTS ([9] P 6). THE TAKING OF AN AUTOCORRELATION FUNCTION IS THUS NOT A REVERSIBLE PROCESS AND HENCE IS NOT SIGNAL UNIQUE ([10] P 45).

SIMILARLY THE CORRELATION BETWEEN TWO FUNCTIONS, CALLED CROSS-CORRELATION IS DEFINED AS ([9] P 5),

$$\phi_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t+\tau)\, g^{*}(t)\, dt \qquad (3.2)$$

THIS IS DONE BY ANALOGY WITH (3.1) IN SUCH A MANNER AS TO BE CONSISTENT WITH AUTO CORRELATION SHOULD THE TWO FUNCTIONS WITHIN THE INTEGRAL COINCIDE. ACCORDING TO THIS DEFINITION THE CORRELATION BETWEEN A SINE WAVE AND A COSINE WAVE IS ZERO WHEN NO DELAY EXISTS TETWEEN THEM. THAT IS $\phi(\tau) = \frac{1}{2}\sin\omega\tau$ AND IS ZERO WHEN $\tau = 0$. ([11] P 254)

THERE IS ABSOLUTELY NOTHING WRONG WITH RELATIONS 3.1 AND 3.2 AS USED IN CONTEMPORARY ANALYSIS SO LONG AS ONE RECOGNIZES THE CONDITIONS UNDER WHICH THEY ARE TO BE EMPLOYED. WHEN ONE BECOMES CONCERNED WITH TIME AS A POSSIBLE SIGNAL PARAMETER HE MUST NOW RECOGNIZE THAT A TIME RETARDATION OR ADVANCEMENT OF A QUARTER PERIOD OF A SINE WAVE IS ANALYTICALLY IDENTICAL TO THE CREATION OF A COSINE WAVE ON THE ORIGINAL TIME BASE. THUS IN THIS CASE THE CONCEPTUAL CORRELATION IS THAT THE SIGNAL HAS CHANGED PHASE, WHICH IS QUITE DISTINCT FROM A NULL CORRELATION FOR THAT VALUE OF THE PARAMETER τ.

FOR THE PURPOSE OF OUR ANALYSIS AN ASSUMPTION IS MADE, BASED ON EQUATION (2), THAT A GENERAL SIGNAL MUST BE CONSIDERED A VECTOR AND NOT JUST A SCALAR. IT CAN BE SHOWN THAT A TIME DOMAIN REPRESENTATION OF A SIGNAL UNDER THESE GROUNDRULES MAY BE GIVEN AS THE VECTOR, $h(t)$, WHERE,

$$h(t) = f(t) + i\,g(t) \qquad (4)$$

THE SCALAR COMPONENTS $f(t)$ AND $g(t)$ ARE HILBERT TRANSFORMS OF EACH OTHER AND RELATE TO ENERGY DENSITY IN THE MANNER SHOWN IN EQUATION (2) ([12] P 739). THE VECTOR $h(t)$ HAS ALSO RECEIVED RECOGNITION IN NARROW BAND COMMUNICATION PROCESSES WHERE IT IS KNOWN AS THE ANALYTIC SIGNAL ([3] EQN 3-11, P 83)([13] EQN 7.6 - 10, P 407)([14] EQN 4.27, P 75)

([15] eqn 1-6-2, p 30)([16] eqn 2.40, p 18)([17] p 94). For this analysis the relation (4) is considered true for all processes. The result of this is that a signal must always be treated as a vector for complete characterization. The term vector, rather than phasor, is utilized to denote the concept of signal space ([14] p 20)([18] footnote page 253).

When one applies this concept it must follow that correlation must be considered to be a vector, not just a scalar. For the purpose of this analysis the process of equation (3.2), utilizing vector signal components, will be called correlation. Correlation will then refer to that complex measure of sameness between two complex signals. It is also of some merit to consider correlation between two signals as representing that operation which must be performed on one signal in order to produce an identity with another signal when evaluated by the integral process of (3.2). Thus the correlation between a sine wave and a cosine wave signal obtained from a like energy related process, such as pressure response, is the vector $i = \sqrt{-1}$.

3. Signal

For the purpose of discussion it will be assumed that the simplest signal of interest is a single frequency sinusoid of voltage time dependence $V(t)$. In terms of the energy which is available to do work when this voltage is impressed across a unit resistor, ([19] eqn 3a, p 244)

$$V(t) = \sqrt{2E(t)}\,\sin\omega t \quad , \text{ joule}^{\frac{1}{2}} - \text{sec}^{-\frac{1}{2}} \quad (5)$$

Note that we are allowing for signal energy fluctuations by this form.

A general signal of unit voltage dependence $S(t)$ may be expressed as, ([19] eqn 4, p 247)

$$V(t) = \sqrt{E(t)} \cdot S(t) \qquad , \text{JOULE}^{\frac{1}{2}} - \text{SEC}^{-\frac{1}{2}} \quad (6)$$

THE DEFINITION OF UNIT VOLTAGE DEPENDENCE IS SUCH THAT, FROM (1), THE AVERAGE ENERGY IN A ONE SECOND TIME IS

$$E = \int_0^1 E(t)\, s(t)\, s^*(t)\, dt \quad (6.1)$$

HENCE IF E(T) IS ESSENTIALLY CONSTANT AT VALUE E

$$\int_0^1 s(t)\, s^*(t)\, dt = 1 \quad (6.2)$$

THE POWER, OR RATE OF CHANGE OF ENERGY, MADE AVAILABLE WHEN THE VOLTAGE IS IMPRESSED ACROSS A UNIT RESISTOR IS,

$$P_s = V^2(t) \qquad , \text{JOULE} - \text{SEC}^{-1} = \text{WATT} \quad (7)$$

4. NOISE

FOR CONVENIENCE IT WILL BE ASSUMED THAT THE NOISE IS A ZERO MEAN GAUSSIAN WHITE RANDOM VARIABLE OF VARIANCE σ^2 AND WITH A DOUBLE-SIDED ENERGY SPECTRAL DENSITY OF $N_o/2$ JOULES PER HERTZ. THIS MEANS THAT A PERFECT FILTER WHICH PASSED ONLY FREQUENCIES FROM f TO $(f + 1)$ HERTZ WHEN MEASURED BY CONVENTIONAL OSCILLATORS AND VOLTMETERS WOULD HAVE AN OUTPUT ENERGY OF N_o JOULES. A TRUE RMS METER WOULD, IN THIS INSTANCE, MEASURE $\sqrt{N_o}$ VOLTS ACROSS A UNIT RESISTOR UNDER CONDITIONS OF MAXIMUM ENERGY TRANSFER.

5. SIGNAL TO NOISE

THE SIGNAL TO NOISE RATIO, α, WILL BE DEFINED AS THE AVERAGE SIGNAL POWER TO AVERAGE NOISE POWER IN A ONE HERTZ BANDWIDTH CENTERED ON THE FREQUENCY OF THE SIGNAL WHEN BOTH SIGNAL AND NOISE IS IMPRESSED ON THE SAME IMPEDANCE. THE ASSUMED DEFINITION OF SIGNAL AND NOISE ALLOWS THIS TO BE EXPRESSED ANALYTICALLY AS,

$$\alpha = \frac{E}{N_o} \qquad , \text{WATT/WATT} \quad (8)$$

Both signal and noise have double sided spectral dependence. The one Hertz bandwidth referred to is that which would pass only frequencies from f Hertz to $(f + 1)$ Hertz when measured by conventional oscillators and voltmeters.

6. MATCHED FILTER

If a signal be such that it may be expressed as $x(t)$ then a filter through which $x(t)$ is to be passed will be defined as matched to $x(t)$ for a time T if the impulse response of the filter is the complex conjugate of the time reversed signal ([20] p 62),

$$h(t) = x^*(T-t) \tag{9}$$

The output power available to a unit load from a filter matched to a signal of energy E Joules is, at the end of T seconds, ([21] p 163)

$$P_S = E^2 \quad \text{, Watt} \tag{10}$$

The output power available to a unit load from a filter matched to a signal of energy E but fed from a noise of double sided energy density $N_0/2$ Joules per Hertz is, ([21] p 161)

$$P_N = \frac{N_0 \cdot E}{T} \quad \text{, Watt} \tag{11}$$

The output signal to noise ratio available from a filter matched to a signal of energy E for a time T but fed both the above signal and noise is,

$$\alpha_0 = \frac{P_S}{P_N} = \frac{E \cdot T}{N_0} = \alpha T \quad \text{, Watt/Watt} \tag{12}$$

ENERGY PLANE PORTRAYAL

The information of interest is the total signal energy density as a function of time and/or frequency. There is a vector relationship between that signal component representing total energy density and the signal components representing kinetic and potential energy densities. Consequently a very useful portrayal of a signal involves a two dimensional complex plane with one axis corresponding to kinetic energy related values and the quadrature axis corresponding to potential energy related values. In this section we shall discuss signal trajectories on this complex energy plane when processed through a matched filter. The statistical decision concerning signal characterization will be made within the reference of this energy plane.

1. SIGNAL ONLY TRAJECTORY

Assume that the output of a matched filter is plotted as a function of time for a signal which starts at time zero. This output is that of the complex correlation integral of equation (~~3~~ 3.2). The signal trajectory is as shown in figure 1, curve 1.

Because we presume to know the complex energy partitioning, the trajectory commences at time zero from the origin and uniformly travels through points 2 and 3 and at the end of T seconds resides at point 4. If we did not know the precise energy partitioning the trajectory may be that of curve 2. In the case of curve 1 the correlation coefficient will be unity at an angle zero degrees. In the case of curve 2, the correlation coefficient will be unity at an angle θ degrees.

If the filter is matched to a signal slightly different in frequency

FROM THE ACTUAL SIGNAL, THE TRAJECTORY WILL BE AS SHOWN IN FIGURE 2.

IN THIS CASE THE TRAJECTORY WILL BE A SEGMENT OF A CIRCLE COMMENCING AT THE ORIGIN EMERGING WITH SOME ANGLE θ_1, REPRESENTING THE DIFFERENCE BETWEEN ASSUMED AND ACTUAL KINETIC ENERGY, AND DESCRIBE A CIRCULAR ARC TERMINATING AT AN ANGULAR SLOPE θ_2. THE CORRELATION COEFFICIENT IS THE VECTOR $\rho e^{i\varphi}$. THE FREQUENCY OFFSET IN HERTZ BETWEEN THE ACTUAL AND ASSUMED SIGNAL IS,

$$\Delta f = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{2\pi T} \tag{13}$$

2. NOISE ONLY TRAJECTORY

FOR NOISE ONLY WE HAVE A SIGNAL $n(t)$ OF ZERO MEAN GAUSSIAN CHARACTERISTIC. THIS NOISE, JUST AS ANY OTHER SIGNAL, HAS A KINETIC AND POTENTIAL ENERGY RELATED COMPONENT. THE FILTER ASSUMED FOR THIS PROCESS IS MATCHED TO THE SIGNAL,

$$s(t)\cdot\sqrt{E(t)} \tag{14}$$

FOR A TIME T. THE OUTPUT $y(t)$, FOR A SIGNAL $x(t)$ FED TO A MATCHED FILTER WITH A NORMALIZED GAIN OF 1/T IS, FROM (9),

$$\begin{aligned} y(t) &= \frac{1}{T}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \frac{1}{T}\int_{-\infty}^{\infty} x(\tau)s^*(T-t+\tau)\sqrt{E(T-t+\tau)}\,d\tau \end{aligned} \tag{15}$$

IN THE CASE OF THE SIGNAL (14) ONLY, THE OUTPUT AT THE END OF T SECONDS, CHOSEN SHORT ENOUGH THAT E(T) IS ESSENTIALLY CONSTANT AT VALUE E, IS,

$$\begin{aligned} y_s(t) &= \frac{1}{T}\int_0^T s(\tau)\sqrt{E}\,s^*(T-t+\tau)\sqrt{E}\,d\tau \\ &= E\cdot\frac{1}{T}\int_0^T s(\tau)s^*(T-t+\tau)\,d\tau \end{aligned} \tag{16}$$

WHICH FROM (3.1) BECOMES,

$$y_s(t) = E \cdot \varphi(T-t) \tag{16.1}$$

AT THE TIME t = T THE MAGNITUDE OF THE FILTER OUTPUT IS THUS FROM (6.2)

$$Y_s(t) = E \cdot \varphi(0) = E \tag{17}$$

THIS IS THE LENGTH OF THE VECTOR IN FIGURE 1.

FOR NOISE ONLY, $X(t)$ IS A ZERO MEAN GAUSSIAN AND SO,

$$\overline{y_N(t)} = 0 \tag{18}$$

BY DEFINITION.

THE SEPARATE ENERGY COMPONENTS BECOME,

$$\overline{a^2} = \overline{b^2} = \overline{y_N^2(T)}$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \overline{n(\lambda)n^*(r)}\, s(\lambda)\sqrt{E(\lambda)}\, s^*(r)\sqrt{E(r)}\, dr\, d\lambda \tag{19}$$

BECAUSE OF THE ASSUMED DEFINITION FOR THE NOISE,

$$\overline{n(\lambda)n^*(r)} = \frac{N_0}{2}\delta(\lambda - r) \tag{20}$$

IT THEN FOLLOWS THAT THE MEAN SQUARE LENGTH OF ENERGY COORDINATE REACHED AT THE END OF T SECONDS, SHOWN IN FIGURE 3, IS,

$$\overline{a^2} = \overline{b^2} = \overline{y_N^2(T)}$$

$$= \frac{1}{T^2}\int_0^T \frac{N_0}{2} s(\lambda)s^*(\lambda)\, E(\lambda)\, d\lambda = \frac{N_0 \cdot E}{2T} \tag{21}$$

THE ENERGY PLANE TRAJECTORY IS A TWO DIMENSIONAL RANDOM WALK FROM THE ORIGIN TO THE TERMINAL POINT OF EQUATION (21). THE TOTAL MEAN SQUARE DISTANCE FROM THE ORIGIN, WHICH RELATES TO TOTAL ENERGY DENSITY, IS,

$$\sigma^2 = c^2 = a^2 + b^2 = \frac{2N_0E}{2T} = \frac{N_0E}{T} \tag{22}$$

THE PARAMETER c^2 IS THE VARIANCE OF THE DISTANCE FROM THE ORIGIN FOR SAMPLES STARTING FROM THE ORIGIN AND TERMINATING T SECONDS AFTER INITIATION.

3. SIGNAL PLUS NOISE

The noise, by definition, has a zero mean. The statistical probability that a noise only vector will be of length r from the origin at the end of T seconds is the well known ([15] eqn 1-4-18, p 23),

$$p(r)\,dr = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\,dr \qquad (23)$$

It is apparent that the effect of adding a signal will be that of creating a mean to the distribution since the energy plane relates to matched filter output. This is sketched in Figure 4.

In terms of the definition previously assumed, the probability density function (PDF) for the distance from the origin for signal plus noise is ([15] eqn 1-4-26, p 26),

$$p(r) = \frac{r}{\sigma^2} e^{-\left(\frac{r^2+2\sigma^2\alpha}{2\sigma^2}\right)} I_0\left(\frac{r}{\sigma}\sqrt{2\alpha}\right) \qquad (24)$$

The relation of equation (24) is shown plotted in Figure 5 for noise only and several values of signal to noise ratio. The assumed time of integration is one second, which under the relation of equation (12) makes input and output signal to noise ratios equal for the energy plane. For a T second matched filter the numerical value of the signal to noise ratio in equation (24) and those following will be increased T times in accordance with equation (12). Thus, if the incoming signal has a measured α, in accordance with equation (8), of 0 dB then the value to use for a matched filter of 2 seconds length will be +3 dB. For a filter of 4 seconds length the α will be +6 dB, and so on.

The curves of Figure 5 are the probability density function (PDF) for the value of the vector length from the origin of the energy plane at the end of T seconds of filter processing. These relate to the wagering

ODDS ONE COULD USE TO MAKE AN ESTIMATE OF THE PROBABLE VECTOR LENGTH FOR A GIVEN SAMPLE. THE INTEGRAL OF THIS FUNCTION TO A POINT r, THE CUMULATIVE DISTRIBUTION, YIELDS THE PROBABILITY THAT ANY VECTOR LENGTH IS LESS THAN r.

WE OF COURSE ARE INTERESTED IN A VECTOR, NOT JUST A SCALAR LENGTH, SO THAT IT IS OF IMPORTANCE TO KNOW THE PDF FOR ANGLE. REFERRING TO FIGURE 4, ASSUME THAT THE MEAN OF THE DISTRIBUTION IS AT THE ANGLE θ. THE PDF FOR ANGULAR DISTRIBUTION AROUND THE DEFINED VALUE $\theta = 0$ IS, ([22] EQN 7-152)

$$p(\theta) = \frac{e^{-\alpha}}{2\pi} + \frac{1}{2}\sqrt{\frac{\alpha}{\pi}} \cos\theta \cdot e^{-\alpha \sin^2\theta}\left[1 + \mathrm{erf}\left(\sqrt{\alpha}\cos\theta\right)\right] \qquad (25)$$

AS CAN BE SEEN, THIS EQUATION IS MUCH MORE COMPLEX THAN THE SIMPLE DISTRIBUTION OF (24), AND CONSEQUENTLY IS DELETED FROM MOST ELEMENTARY TEXTS ON THE SUBJECT [23] [24]. UNFORTUNATELY, IT IS ALSO DELETED AS AN EQUATION FROM MANY MORE ADVANCED TEXTS [14] [15] [19] WITH THE CONSEQUENCE THAT MANY PEOPLE ARE UNAWARE OF THE GREATER SENSITIVITY OF THE ANGULAR PDF TO LOW VALUES OF α THAN THE VECTOR LENGTH PDF. WHEN ONE TALKS OF THE TOTAL POWER, SUCH AS IN (10), HE THROWS AWAY THIS ANGULAR PDF IN FAVOR OF (24). EQUATION (25) IS PLOTTED IN FIGURE 6.

IN FIGURE 7 ONE CAN SEE PICTORIALLY THE INTERPLAY BETWEEN THE ANGULAR AND VECTOR LENGTH PDF. THIS PICTORIAL DISPLAY SHOWS THE "ICE CREAM CONE" LOCUS OF THOSE VECTORS WITH A GIVEN CUMULATIVE DISTRIBUTION. FOR EXAMPLE WITH A GIVEN α IT MAY BE STATED THAT X PERCENT OF ALL VECTORS MUST LIE WITHIN THE CIRCULAR PORTION OF THE ICE CREAM CONE.

DESCRIPTION OF COHERENT PROCESSOR

THE PREVIOUS VERY CURSORY PRESENTATION OF THE PHYSICAL CONCEPTS ALLOW FOR A DISCUSSION AT THIS POINT OF THE INSTRUMENTATION THAT IS THE SUBJECT OF THIS REPORT. THIS DESCRIPTION WILL GO ONLY TO THE DETAIL OF A BLOCK DIAGRAM AT THIS POINT. TWO MAJOR MODES WILL BE DESCRIBED. THESE RELATE TO SIGNAL DETECTION AND SIGNAL TRACKING.

SIGNAL DETECTION

FROM THE FOREGOING DISCUSSION OF PHYSICAL CONCEPTS IT IS OBVIOUS THAT THE PROCESS OF DETECTION OF A COHERENT SIGNAL IS REDUCIBLE TO THAT OF STATISTICAL DETERMINATION OF A MEAN TO THE DISTRIBUTION OF VECTORS ON THE APPROPRIATE ENERGY PLANE. THREE BASIC SUBSETS OF THIS DETECTION PROBLEM ARISE WHEN CONSIDERING THAT CLASS OF SIGNAL ASSUMED. WE MAY KNOW (A) BOTH FREQUENCY AND PHASE EXACTLY, (B) ONLY FREQUENCY EXACTLY, OR (C) FREQUENCY ONLY APPROXIMATELY. THROUGHOUT IT WILL BE ASSUMED THAT NO KNOWLEDGE EXISTS OF THE EXACT SIGNAL TO NOISE RATIO OR WHETHER THERE IS A SIGNAL AT ALL.

A) KNOWN FREQUENCY AND PHASE

ASSUME THAT KNOWING THE FREQUENCY AND PHASE, THE ENERGY PLANE DECISION FOR DETERMINATION OF A MEAN IS MADE BY EVALUATION OF SUCCESSIVE VECTORS AND DETERMINING THE NUMBER OF VECTORS IN THE HALF PLANE CONTAINING THE MEAN. THIS IS SHOWN IN FIGURE 8 WITH THE DECISION CRITERION THE NUMBER OF HITS WITHIN THE ANDLE $|\phi|$.

FIGURE 9 SHOWS THE CUMULATIVE DISTRIBUTION FOR ANGLES FROM THE MEAN AS A FUNCTION OF SEVERAL SIGNAL TO NOISE RATIOS. AS AN EXAMPLE OF THE USE OF THIS CURVE ASSUME AN α OF -6 dB. FIFTY NINE PERCENT OF THE vectors will lie within sixty degrees of the known mean angle, and seventy five percent will lie within ninety degrees. It is a remarkable feature of equation (25) that the cumulative distribution for angles within ninety degrees simplifies to the expression ([7] eqn 18-27),

$$\int_{-\pi/2}^{\pi/2} p(\theta)\,d\theta = \frac{1}{2}\,\mathrm{cerf}\,\sqrt{\alpha} \qquad (26)$$

This is plotted in figure 10 as a function of α.

Figure 11 shows the equipment capable of presenting an oscilloscope display of the energy plane of figure 8. The incoming signal is split into two identical channels. In one channel it is multiplied by its complex conjugate (assumed to be a sine wave) and in the other channel by the Hilbert Transform of its complex conjugate (assumed to be a cosine wave). The output of each multipler feeds its own integrator and in turn is sent to an oscilloscope terminal. Prior to signal evaluation S_X and S_Y are closed initializing the integrators at zero and centering the oscilloscope plot. At time zero the switches S_X and S_Y are opened allowing integration to proceed and the CRT spot will commence its two dimensional random walk plus drift from the center of the screen. At the end of T seconds the spot will be at the position of the signal vector upon which the analysis will be based. Assume now that the circuitry of figure 12 is connected to that of figure 11.

The angular dependence, $\sin\theta$ and $\cos\theta$, is stripped out as shown along with the vector length ρ. A simple coordinate rotation may be made from the assumed θ coordinate system to a ϕ coordinate system defined such that $\phi = 0$ lies along the known signal-only trajectory in the energy plane. A simple determination of half plane statistics

MAY THEN BE MADE BY COUNTING A "HIT" AS A VECTOR WITH POSITIVE COSINE ϕ, AND AVERAGING HITS TO TOTAL POSSIBLE COUNTS FOR A RUNNING SAMPLE WHERE S_X AND S_Y ARE SET TO ZERO AFTER EACH TALLY THEN OPENED FOR THE NEXT SAMPLE. AN EXAMPLE OF CIRCUITRY CAPABLE OF DOING THIS IS SHOWN IN FIGURE 13.

SYMBOLICALLY THE VECTOR ROTATION IS MADE BY TWO MULTIPLIERS AND A SUMMER. THE DECISION TO CLOCK A UNIT STATE INTO A SHIFT REGISTER IS MADE ON THE BASIS OF THE ALGEBRAIC SIGN OF THIS ROTATED VECTOR. IF, FOR EXAMPLE, THE SHIFT REGISTER IS 100 STAGES IN LENGTH THEN A 100 PERCENT "HIT" WILL FILL ALL STAGES WITH ONES. A 75 PERCENT "HIT" RATIO WILL HAVE THE SAME PERCENTAGE OF ONES. BY SIMPLY SUMMING A UNIT CURRENT FROM EACH STAGE WHEN IT CONTAINS A ONE AND NO CURRENT WITH A ZERO, ONE NEED ONLY AVERAGE THE 100 STAGES IN ORDER TO OBTAIN AN ANALOG INDICATION OF "HITS" TO "MISSES", OR IF NEEDED A DIGITAL UPDATE CAN BE MAINTAINED BY CONVENTIONAL MEANS.

B) KNOWN FREQUENCY ONLY

IN THE CASE WHERE PHASE IS UNKNOWN, ONE HAS TWO ALTERNATIVES: HE MAY ELECT TO BASE A STATISTICAL SIGNAL ESTIMATE ON THE VECTOR LENGTH, OR HE MAY ACCUMULATE ENOUGH SAMPLES TO ESTIMATE PROBABLE PHASE AND ROTATE HIS PLANE OF REFERENCE IN ORDER TO USE THE STATISTICS OF KNOWN FREQUENCY AND PHASE. IN EITHER CASE SOME MEANS MUST BE FOUND TO ELIMINATE THE REQUIREMENT FOR INTIMATE KNOWLEDGE OF THE NOISE POWER. THIS MAY READILY BE DONE BY UTILIZING A VECTOR DECISION BASED ON UNIT AMPLITUDE VECTORS OF THE TYPE FED INTO FIGURE 13. THIS IS BECAUSE THE ANGLE STATISTICS WILL BE GAUSSIAN IF THE NOISE STATISTICS ARE GAUSSIAN, AND WILL HAVE A MEAN INDICATIVE OF THE EXISTENCE OF A SIGNAL.

IN THIS CASE THE PDF IS STILL THE MOST POWERFUL INDICATOR WITH A DECISION MATRIX MADE MORE DIFFICULT SINCE THE EXACT PHASE IS NOT KNOWN. THREE COURSES OF ACTION ARE AVAILABLE.

1) SINCE THE PURPOSE OF ANALYSIS IS DETECTION OF A FIXED, KNOWN FREQUENCY SIGNAL THIS IS EQUIVALENT TO DETERMINING THE EXISTENCE OF A MEAN TO THE ANGLE PDF. THIS MAY BE DONE BY EVALUATING A SLIDING ACCUMULATION OF 180° SEGMENTS OF THE PDF. THIS IS SHOWN IN FIGURE 14 AS AN EVALUATION OF THE AREA UNDER SEGMENT 1, THEN UNDER SEGMENT 2 AND SO ON, CONSTANTLY EVALUATING THE RATIO OF THE RESPECTIVE AREAS TO UNITY. THIS IS BECAUSE A FULL 360° SEGMENT MUST BE CONSIDERED TO HAVE A UNIT PROBABILITY OF A "HIT." A PLOT OF THE 180° AREAS VERSUS THE CENTRAL ANGLE OF EACH AREA SHOULD YIELD A PLOT SUCH AS SHOWN IN FIGURE 15. A SIMPLE DETERMINATION MAY THEN BE MADE OF THE PROBABILITY BY NOTING THE PEAK ON FIGURE 15 AND COMPARING WITH FIGURE 10.

2) A COMPLEX HARMONIC ANALYSIS OF THE ANGLE PDF SHOULD DISCLOSE THE EXISTENCE OF A FUNDAMENTAL AND HIGHER HARMONIC TERMS.

3) THE PDF ANGLES MAY HAVE A MEAN DISTRIBUTION AT ANY ANGLE RELATIVE TO LOCAL REFERENCE WHEN A SIGNAL IS PRESENT. A PDF OF DIFFERENCE ANGLES BETWEEN SUCCESSIVE SAMPLES, ON THE OTHER HAND, WILL ALWAYS HAVE ITS MEAN AT ZERO DEGREES. THIS DIFFERENCE ANGLE IS THAT OF $\Delta\theta$ IN FIGURE 16 WHERE A FIRST AND SECOND SAMPLE HAD ϕ_1 AND ϕ_2 RESPECTIVELY.

THE HALF PLANE STATISTICS OF $\Delta\theta$ IS GIVEN BY THE EXPRESSION,

$$\int_{-\pi/2}^{\pi/2} p(\Delta\theta)\, d(\Delta\theta) = \frac{1}{2} e^{-\alpha} \tag{27}$$

THIS IS SHOWN IN FIGURE 10.

C) FREQUENCY ONLY APPROXIMATELY

If one does not know the exact frequency, the angle pdf will deteriorate very rapidly. The only recourse for extended analysis with a fixed frequency local reference utilizing a single channel matched filter will be difference statistics. One must then choose the matched period of each sample, T, such that the probable signal trajectory does not accumulate more than perhaps a 90° shift in this period. Figure 17 shows the on-frequency pdf of difference angle $\Delta\theta$. If the offset between incoming and reference frequency is such as to accumulate not more than 45° in the period T, then to first order the statistics of Figure 17 will hold and one may assume the pdf of $\Delta\theta$ is shifted by just this offset angle. To give some comparison assume that the offset between incoming signal and reference is 0.1 Hz and one second samples are accumulated. Within ten samples the angle statistics would have traversed 360° and be useless. Each difference angle vector will on the average have a shift of 36° allowing an indefinite accumulation of one second samples for developing an estimator of the pdf and a determination of this mean.

There is another means available for signal determination if each vector calculation is maintained in memory and time-tagged. If the difference statistic, which is a running calculation, shows the probability of a signal slightly off frequency then one may bring the more powerful angle statistics to bear by drawing from memory each vector and rotating by the proper angle to bring each into the same phase alignment for the existing epoch of time. As in our example if a sufficient suspicion were to exist that a signal was present and had a probable frequency

OFFSET OF 0.1 Hz, ONE WOULD DRAW THE LATEST VECTOR FROM MEMORY AND ROTATE 36°, THE EARLIER VECTOR BY 72°, THE NEXT EARLIER VECTOR BY 108°, ETC. EACH PROPERLY ROTATED VECTOR WOULD THEN BE USED TO EVALUATE AN ANGLE PDF. IT IS OBVIOUS BY THIS MEANS THAT NOT ONLY CAN ONE RETAIN THE SIGNAL IMMEDIATE HISTORY SO AS TO CORRECT FOR OFFSET FREQUENCY, BUT BY USING A DICTIONARY OF FREQUENCY MANEUVER CORRECTIONS DETERMINE SIGNAL FREQUENCY BASSISTICS A POSTERIORI.

SIGNAL TRACKING

THE PROCESS OF TRACKING IN THIS REPORT MEANS THE ADAPTIVE CORRECTION OF A LOCAL SIGNAL ESTIMATOR SO AS TO MINIMIZE TO THE SMALLEST STATISTICAL UNCERTAINTY THE DIFFERENCE BETWEEN A PREDETERMINED SIGNAL ATTRIBUTE AND THE SAME ATTRIBUTE OF THE LOCAL SIGNAL ESTIMATOR. BECAUSE OF THE STATISTICAL UNCERTAINTY OF SIGNAL AMPLITUDE OR PHASE IN A REAL WORLD SIGNAL MEDIUM SUFFERING FROM MULTIPATH TRANSMISSIONS, THE SIGNAL ATTRIBUTE CHOSEN FOR TRACKING IS FREQUENCY. FREQUENCY IS DEFINED AS A TIME RATE OF CHANGE OF PHASE. IF IN A PERIOD OF T SECONDS A SIGNAL UNDERGOES A PHASE SHIFT OF $\Delta\theta$ RELATIVE TO OUR LOCAL REFERENCE, THEN THE FREQUENCY OFFSET IS,

$$\Delta f = \frac{\Delta\theta}{T} \tag{28}$$

IT IS OBVIOUS THAT THE SIMPLEST TRACKING STRATEGY IS TO OBTAIN THE DIFFERENCE PHASE FROM THE TWO MOST RECENT MATCHED FILTER CALCULATIONS THEN INCREMENT THE LOCAL REFERENCE FREQUENCY IN ACCORDANCE WITH RELATION (28) IMMEDIATELY PRIOR TO THE NEXT MATCHED FILTER INTEGRATION. ONE SIMPLE MEANS OF ACCOMPLISHING THIS IS SHOWN IN FIGURE 18.

THE VOLTAGE CONTROLLED OSCILLATOR (VCO) WHICH GENERATES THE LOCAL ESTIMATOR IS INCREMENTED BY THE SIGNAL IN THE ZERO ORDER HOLD (ZOH) CIRCUIT WHENEVER A CALCULATION HAS JUST TERMINATED. THE SINE AND COSINE OF THE MOST RECENT ANGLE CALCULATION IS CROSS-MULTIPLIED WITH THE STORED VALUES OF THE PREVIOUS ANGLES UNDER THE RELATION,

$$\begin{aligned} \sin\Delta\theta &= \sin(\theta_N - \theta_{N-1}) \\ &= \sin\theta_N \cdot \cos\theta_{N-1} - \cos\theta_N \cdot \sin\theta_{N-1} \end{aligned} \tag{29}$$

RATHER THAN MAKE A FULL CORRECTION OF THE MAGNITUDE OF EQUATION (28)

IT IS WISE TO REMEMBER THAT THE SIGNAL, IF PRESENT, MAY BE HIGHLY CORRUPTED WITH NOISE. THEREFORE IT IS ADVISABLE TO PROVIDE A VCO UPDATE WHICH IS A FRACTION OF THAT OF EQUATION (28). IT HAS BEEN FOUND EXPERIMENTALLY THAT A GOOD VCO UPDATE VOLTAGE PROFILE IS AS SHOWN IN FIGURE 19. A LINEAR RELATIONSHIP BETWEEN Δf UPDATE VOLTAGE AND $\Delta\theta$ IN ACCORDANCE WITH EQUATION (28) IS MAINTAINED FOR A RANGE ABOUT ZERO. BEYOND THAT THE Δf UPDATE IS MAINTAINED CONSTANT AT A VALUE,

$$\Delta f = \left(\frac{\Delta\Theta_M}{T}\right)\left(\mathrm{SGN}\,\Delta\theta\right) \tag{30}$$

WHERE SGN(X) IS THE SIGNUM FUNCTION.

FORTUNATELY SUCH A RELATION AS (30) MAY BE READILY GENERATED BY PASSING THE SIN$\Delta\theta$ VOLTAGE THROUGH A LINEAR CIRCUIT WHICH LIMITS IN THE MANNER SHOWN BY THE DASHED LINES IN FIGURE 19.

MORE ELABORATE TRACKING STRATEGIES MAY BE UTILIZED SUCH AS MAINTAINING A RUNNING RECORD OF SUCCESSIVE $\Delta\theta$'S AND UPDATING THE VCO EACH TIME BY A WEIGHTED SUM,

$$\Delta f_{UPDATE} = \sum_i m_i \Delta\theta_i \tag{31}$$

OR, AS CITED UNDER DETECTION, A FREQUENCY TRAJECTORY MAY BE INFERRED FROM A RUNNING MEMORY OF SUCCESSIVE VECTORS AND AN UPDATE STRATEGY CHOSEN SO AS TO BRING THE OBSERVED FREQUENCY TRAJECTORY INTO ALIGNMENT WITH A PARTICULAR TRAJECTORY FROM A GIVEN DICTIONARY. AS AN EXAMPLE OF THIS, FIGURE 20 IS THE FREQUENCY-TIME TRAJECTORY OF A FIXED FREQUENCY SOURCE MOUNTED ON A VEHICLE WITH UNIFORM VELOCITY IN A "FLYBY" MODE. THE FREQUENCY OF THE SOURCE IS f_0. AT LARGE NEGATIVE TIME THE RECEIVER PERCEIVES A FREQUENCY $f_0+\delta f$ DUE TO APPROACHING DOPPLER WITH CONVERSE RELATION AS THE VEHICLE RECEDES. IN THE EVENT THAT AN INTERCEPT

DISCLOSED A SET OF FREQUENCIES AT THE TIMES t_1, t_2, t_3, t_4 WHICH SHOWED A HIGH LIKELIHOOD OF BELONGING TO THIS TRAJECTORY, THEN THE BEST STRATEGY MIGHT BE TO WEIGHT THE OFFSET TO THE LOCAL FREQUENCY ESTIMATOR SO AS TO BE OF VALUE Δf AS SHOWN IN FIGURE 20. THIS PROCESS WOULD THEN CONTINUE SO LONG AS THE CONDITIONS OF TRACKING THIS TRAJECTORY WERE MET.

MANY VARIATIONS ON THESE METHODS ARE POSSIBLE. HOWEVER, THE STATISTICAL LIKELIHOOD OF A SIGNAL PHASE FLIP DUE TO THE PROPAGATION MEDIUM MUST BE ASSESSED FOR ANY LONG STATISTICAL SAMPLES.

ONE POWERFUL ADVANTAGE TO THE USE OF THE SIMPLE $\Delta\theta$ VALUES FOR FREQUENCY UPDATING IS THAT NO PENALTY IS ACCRUED SHOULD A SIGNAL DISAPPEAR. UNLIKE PHASE OR AMPLITUDE TRACKING MEANS, THERE IS NO TENDENCY TO INCUR UNCONTROLLED "OUT OF LOCK" BEHAVIOR WITH NOISE ONLY. THIS IS BECAUSE THE TRACKING METHOD IS SUCH AS TO MINIMIZE THE MEAN OF THE DIFFERENCE ANGLE STATISTICS. IF ONE IS JUDICIOUS IN HIS CHOICE OF VCO UPDATE, A NOISE ONLY SITUATION WITH ITS ZERO MEAN IS IDENTICAL IN EFFECT TO ON-FREQUENCY TRACKING. IF ONE HAD BEEN TRACKING A SIGNAL WHICH DISAPPEARED THE TREQUENCY TRACKER WILL "MARK TIME" AS IT WERE WITH A RANDOM FLUCTUATION ABOUT THE PREVIOUS SIGNAL FREQUENCY. WHEN THE SIGNAL AGAIN REAPPEARS THE MEAN OF THE DISTRIBUTION WILL INDICATE THE OFF-FREQUENCY CONDITION AND THE FREQUENCY TRACKER WILL BEGIN ITS STATISTICAL DRIFT TOWARD A CORRECTION. ALL OF THIS IS INDEPENDENT OF THE ABSOLUTE PHASE OF THE INCOMING SIGNAL.

ONE POINT MUST BE MADE CONCERNING THE SPECTRAL PURITY OF THE RECEIVED SIGNAL. THE METHOD DESCRIBED FOR FREQUENCY TRACKING MAKES USE OF THE EXISTENCE OF A STATISTICAL MEAN FOR INDICATION OF OFF-FREQUENCY CONDITIONS. IF THE SIGNAL IS BADLY SMEARED IN FREQUENCY IN A NOISE-LIKE

SENSE THEN THE FREQUENCY TRACKER WILL STILL FUNCTION, ALTHOUGH AT REDUCED EFFICIENCY, IF A LEGITIMATE COHERENT MEAN EXISTS. IF, ON THE OTHER HAND, AN OTHERWISE SHARP SPECTRAL LINE SPLITS INTO MULTIPLE LINES, THE TRACKER WILL FOLLOW THE MEAN FREQUENCY OF THESE LINES.

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