Variable Structure Controller for Surface Ship Steering

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Abstract— A Lyapunov approach to constructing switching surfaces for variable structure systems is investigated in this paper. The method guarantees sliding mode for any initial condition of the state vector and asymptotic stability is always achieved during sliding motion. An application for the design of a variable structure ship steering controller is carried out and simulation results are presented. The designed controller exhibits robustness as applied to a linear time-invariant ship model and a time varying non-linear ship model operating in an uncertain and time-varying environment.

Keywords—Lyapunov Function, Reaching Phase, Switching Function, Sliding Mode, Sliding Surface, Variable Structure Systems.

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1 INTRODUCTION

The theory of Variable Structure Systems (VSS) is based on the concept of switching the structure of the controller at a high speed. Such a controller exhibits the property of invariance to changes in plant parameters and external disturbances once the system is in the so called sliding mode (generally reached after a measurable time duration). The conditions under which these properties hold are investigated in (Drazenovic,1969), recalled in (Utkin, 1977) and (Zinober, 1981).

Conventional controllers for ship steering generally require a knowledge of plant parameters. In practice, ship models are subject to variations as time goes by, aside the fact that those methods cannot be efficiently implemented due to their inability to track changes. In contrast, this method, namely the VSC, assumes that only the ranges of the variation of the plant parameters are known. In this paper, proof is first given (Section 2) as in (Zinober, 1981) of the fact that parametric invariance and disturbance rejection conditions in sliding mode are always met for systems in phase canonic form. Section 3 is about the Lyapunov theory and the Lyapunov approach to switching surface design. Section 4 is about ship models and ship steering, where a ship proposed in (Mort & Linkens, 1981), is considered. The state vector of the ship is driven in sliding mode, onto the designed surface, and results are presented in Section 5. Finally, Section 6 is dedicated to discussion of the results.

2 PHASE CANONIC FORM
2.1 Systems in Phase Canonic

Such systems are described by a set of linear ordinary differential equations (ODEs):

$$\dot{x}_i = x_{i+1} \ldots \qquad i = 1,2\ldots n-1 \qquad (1)$$
$$\dot{x}_n = -a_1 x_1 - a_2 x_2 + \cdots + a_n x_n + b_n u(t) + f(t).$$

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Here, $u = u(x,t)$ is a scalar control; $f(t)$. is a disturbance term due to environmental changes ; a_i are constant or time-varying plant parameters and b_n is constant.

The function $f(t)$, and the plant parameters a_i may be unknown, but their ranges of variation are known. The control input $u = u(x,t)$ is a function of the state vector $x(t) \in \mathbb{R}^n$ and u undergoes discontinuities on the switching surface G.

$$G: \{x | Sx = 0\} \qquad (2)$$

Here, the switching surface matrix S is defined as in equation (3) below.

$$S = [S_1 \ S_2 \ \ldots \ S_{n-1} \ S_n] \qquad (3)$$

The so called switching function $g = g(x)$ is defined as in equation (4).

$$g = Sx \qquad (4)$$

The following inequalities in (5) are sufficient conditions for sliding mode to exist (Zinober, 1981).

$$\lim_{g \to -0} \dot{g} > 0 \quad \text{and} \quad \lim_{g \to +0} \dot{g} < 0 \qquad (5)$$

The velocity vector $\dot{x} = \dot{x}(t) = \frac{dx}{dt}$ undergoes discontinuities on the same plane (Drazenovic,1969), Zinober (1981), Zinober (1993) and (Utkin, Guldner & Shi ,1999).

2.2 The Invariance Proof

To prove the invariance of the sliding mode with respect to the plant parameters a_i and the disturbance $f(t)$, we solve equation (1) for the variable x_n and substitute into equation (2). Such a substitution of the value of x_n from the last but one of the set of equations (6) into the equation of the switching surface , is carried out as in (Zinober, 1981).

From equation (2) and equation (3), we have $S_1 x_1 + S_2 x_2 + \cdots + S_n x_n = 0$. Then, from equation (1), we have $\dot{x}_{n-1} = x_n$.

Direct substitution gives $S_1 x_1 + S_2 x_2 + \cdots + S_n \dot{x}_{n-1} = 0$ and finally we arrive at equation (6).

$$\dot{x}_{n-1} = -\frac{1}{S_n}(S_1 x_1 + S_2 x_2 + \cdots + S_{n-1} x_{n-1}) \tag{6}$$

This describes the dynamic system, independent of a_i. Therefore, the resulting equations (7) of the sliding mode are independent of the plant parameters, but depend only on the switching surface matrix. The state x_n is discarded as suggested in (Zinober, 1981) and (Utkin, Guldner & Shi ,1999). There is an order reduction of the system from n to $n-1$.

$$\dot{x}_i = x_{i+1} \qquad i = 1,2....n-1 \tag{7}$$
$$\dot{x}_{n-1} = -\frac{1}{S_n}(S_1 x_1 + S_2 x_2 + \cdots + S_{n-1} x_{n-1})$$

3 REVIEW OF LYAPUNOV THEORY
3.1 Definitions and Preliminaries
Let $x(t) \in \mathbb{R}^n$, $\dot{x} = \frac{dx}{dt} = f(x)$, $f(x_0) = 0$, $t_0 \in \mathbb{R}$.
and $\quad O_d(x_0) \equiv \{x: \|x - x_0\| < d\}$. $\quad x_0 = x(t_0)$, $\quad \|x\| = \sqrt{x^T x}$

Definition 1:
An equilibrium point x_0 is stable in the sense of Lyapunov if :
$\forall \varepsilon > 0, \exists \delta > 0 : x(t_0) \in O_\delta(x_0) \Rightarrow x(t) \in O_\varepsilon(x_0)$,
$\forall t > t_0$.
If x_0 is an equilibrium point , then $x(t) = x_0$ is a trajectory of the system.
Any trajectory starting close to the equilibrium point remains close to it.

Definition 2:
An equilibrium point $x_0 = x(t_0)$ is asymptotically stable in region D if :
$x(t_0) \in D \Rightarrow x(t) \to x_0$ as $t \to +\infty$.
Any trajectory starting sufficiently close to the equilibrium point, will eventually approach it.

Theorem 1. (Lyapunov).
Given a linear autonomous system of the form $\dot{x} = Ax$, the existence of a Lyapunov function $V(x) = x^T P x \geq 0$ where P is a symmetric positive definite matrix $(P > 0)$, guarantees stability.

Proof:
$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = (Ax)^T P x + x^T P A x = x^T(A^T P + PA x \leq 0$ for all $x \neq 0$.
So, $x^T(A^T P + PA)x = -x^T Q x$ for some positive definite matrix Q, provides necessary and sufficient conditions for stability. That is $A^T P + PA = -Q$ (Lyapunov equation).
Given a particular matrix $Q > 0$, the symmetric matrix $P > 0$, that solves the Lyapunov equation is unique.
If A is stable, there is an explicit formula for solution of Lyapunov equation:
$P = \int_0^\infty e^{A^T t} Q e^{At} dt$.

$$A^T P + PA = \int_0^\infty (A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A)dt =$$
$$\int_0^\infty \left(\frac{d}{dt} e^{A^T t} Q e^{At}\right) dt = \left. e^{A^T t} Q e^{At}\right|_0^\infty = -Q$$

3.2 Lyapunov's Second Method (Direct Method)
The Lyapunov's First Method developed the solution in a series which was then proved convergent within limits (Wikipédia, 2017) and a number of other websites. The Lyapunov's Second Method, which is now referred to as the Lyapunov Stability Criterion, makes use of a Lyapunov function to check the stability of an equilibrium point of a system.

To apply the Lyapunov's Second Method to asymptotically stabilize a dynamic system, the following assumptions are made:

(i) There exists a stabilizing feedback Kx so that $\dot{x} = (A - BK)x$ is stable;

(ii) A Lyapunov function $V = V(x,t)$ exists and satisfies $\dot{V}(x,t) = \frac{dV}{dt} = \nabla V.\dot{x} + \frac{\partial V}{\partial t} < 0$;

(iii) $V = V(x,t) \neq 0$ if $x \neq 0$ in the state trajectories, except at the origin ; $V = V(x,t) = 0 \Leftrightarrow x = 0$;

(iv) $V(x,t) \to \infty$ if $\|x\| \to \infty$; $\|x\| = \sqrt{x^T x}$.

The proposed approach is to synthesize the Lyapunov function into the design of the switching surface to achieve the same performance as if stabilizing feedback was employed. In other words, asymptotic stability is realized in sliding mode (Wu et al., 1996).

3.3 The Lyapunov Approach to VSC Design
Considering the linear plant given by equation (8)

$$\dot{x} = Ax + Bu \tag{8}$$
Assume that (A, B) is controllable, that is ,
$\det([B \ AB \ A^2 B \ldots A^{n-2}B \ A^{n-1}B]) \neq 0$. There exists a stabilizing feedback gain K, as in equation (9),

$$Kx = [k_1 \, k_2 \, \ldots \, k_n][x_1 \, x_2 \, \ldots \, x_n]^T \tag{9}$$
such that the autonomous system given in equation (10) is asymptotically stable.

$$\dot{x} = [A - BK]x \tag{10}$$
The eigenvalues of $A_s = A - BK$ can be arbitrary assigned, as shown in (Nagrath & Gopal, 1982) . There exists a unique symmetric matrix P , $(P > 0)$ that solves the Lyapunov equation (11).

$$PA_s + A_s^T P = -Q \tag{11}$$
Here, $Q > 0$, is a given matrix.
It is suggested that the switching surface matrix S be chosen as in equation (12).

$$S = WB^T P \tag{12}$$
The sliding surface is G such that
$$G : \{ x \mid Sx = 0\} \tag{13}$$

Here W is an $(m \times n)$ nonsingular matrix (it has full rank). There are two fundamental results: lemma1 and lemma 2.

Lemma 1 :

The system $\dot{x} = Ax + Bu_s$ *where* $u_s = u - Kx$ *is a fictitious feedback to the initial system in (8), when undergoing sliding mode on G, is asymptotically stable.*
Proof:
Choosing the following Lyapunov function $V(x)$,
$V(x) = x^T P x \geq 0$;   and
$\dot{x} = Ax + Bu_s = (A - BK)x + Bu = A_s x + Bu$
where A_s , the system matrix of the stabilized ship is defined as in equation (14).

$$A_s = A - BK \tag{14}$$

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$; $\dot{x}^T = x^T A_s^T + u^T B^T$ that is,

$\dot{V}(x) = \frac{dV}{dt}$ can be written as

$\dot{V} = x^T A_s^T P x + u^T B^T P x + x^T P A_s x + x^T P B u$.
On the sliding surface the equalities $x^T P B u = 0$
    and
$u^T B^T P x = 0$ hold (since W is of full rank); and
$\dot{V}(x) = x^T (A^T P + PA)x = -x^T Q x$. Therefore $\dot{V}(x) < 0$, since $Q > 0$.

The feedback control law u_s has no effect on the sliding surface G, therefore, it does not affect the sliding motion.

Lemma 2 :

The equality $x^T(A^T P + PA)x = -x^T Q x$ *holds on the sliding surface* G.
Proof:
Substituting A_s from equation (14) yields:

$-x^T Q x = -x^T (PA_s + A_s^T P) x$
$\qquad\qquad = -x^T(PA - PBK + A^T P - K^T B^T P)x.$
On the sliding surface where $x^T PB = 0$ and $B^T Px = 0$ (since W is of full rank), the equality is satisfied. As pointed out in (Wu et al. ,1996) , the last result reveals that the stabilizing linear feedback $-Kx$ is realized on the switching surface instead of being applied explicitly in the control. The following Theorem 2, also in (Wu et al. ,1996), results from the above two lemmas.

Theorem 2 (Wu et al. ,1996).
The system $\dot{x} = Ax + Bu$ *undergoing mode on the surface* $G : \{ x \mid Sx = 0 \}$ *is asymptotically stable.*

3.4 Algorithm for Switching Surface and Controller Design
i) Given a Controllable Linear Plant
$$\dot{x} = Ax + Bu + f \tag{15}$$

$$f = [0 \quad 0 \quad f(t)]^T \quad \text{(disturbance vector)} \tag{16}$$

ii) Stabilization by State Feedback
$$A_s = A - BK \tag{17}$$

iii) Solving the Lyapunov Equation
$$PA_s + A_s^T P = -Q \tag{18}$$

iv) The Switching Surface Matrix
$$S = B^T P \tag{19}$$
Having set $W = I$ (identity matrix).

v) The Reaching Phase Control Function
$$u(x,t) = -\alpha_x(|x_1| + |x_2| + |x_3| + \alpha_f)sign(g) \tag{20}$$

It is generally sufficient, in the expression of the control function in equation (20), to consider only $|x_1|$, when dealing with linear systems, Drazenovic (1969). There is a minimum value α_{min} of α_x such that no sliding mode on the prescribed surface occurs if $\alpha_x < \alpha_{min}$. On the other hand, α_f must satisfy $\alpha_f > \max (\|f\|)$. More on this discussion is found in (Rimbe, 2005).

The *sign* function expression in the control function is defined as in equation (21).

$$sign\,(g) = \begin{cases} 1 & if \quad g > 0 \\ 0 & if \quad g = 0 \\ -1 & if \quad g < 0 \end{cases} \tag{21}$$

4 SHIP STEERING
A ship considered as a single-input, single-output system (SISO) is represented by the schematic in Fig. 1.
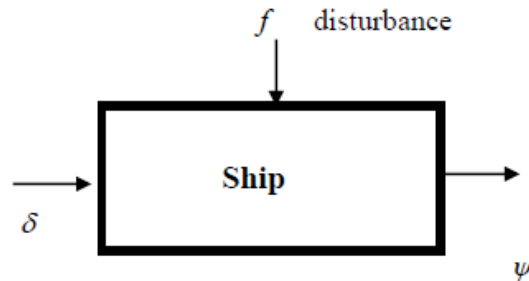δ = rudder angle (Input) ; ψ = heading angle (Output).

Fig. 1. The Ship : Input and Output

The following equation suggested in (Amerogen & Udink, 1975) and in (Mort & Linkens, 1981), is considered :

$$\dddot{\psi} + \left(\frac{1}{T_1} + \frac{1}{T_2}\right)\ddot{\psi} + \left(\frac{K_s}{T_1 T_2}\right)H(\dot{\psi}) = \frac{K_s}{T_2 T_1}\left(\delta + T_3\,\dot{\delta}\right) \tag{22}$$

4.1 Linear Model
A linear state model in phase canonic form is easily derived by setting the so called spiral function $H(\dot{\psi})$ equal to $\dot{\psi}$ and neglecting the rate of change $\dot{\delta}$ of the rudder angle.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\frac{1}{T_1 T_2}x_2 - \left(\frac{1}{T_1} + \frac{1}{T_2}\right)x_3 + \delta + f(t) \end{aligned} \tag{23}$$

- $x = [x_1 \quad x_2 \quad x_3]^T$ is the state vector of the ship's dynamics ; $x_1 = \text{Output} = \psi$ is the heading angle (output) ;
- $\delta = \text{Input}$, the input (a notation from literature) is the rudder angle ;
- f is disturbance vector;
- $f(t)$ represents the disturbance function (scalar);
- T_1, T_2, K_s are parameters of the ship .

4.2 Non-Linear Model

We consider the type of nonlinearity suggested in (Mort & Linkens, 1981) and in (Utkin et al., 1999) , characterized by the spiral curve H in equation (25) below.

$$H(x) = ax_2^3 + bx_2 \qquad (25)$$

Here, a is a varying, but always positive parameter, whereas b is also varying but can take on to both negative and positive values. In this simulation, parameter a is chosen to be constant positive, say $a = 1$, and $b(t) = 5\sin 3\pi t$.

$S_1 x_1 + S_2 x_2 + \cdots + S_3 x_3 = 0$ yields $\dot{x}_2 = x_3 = -\frac{S_1}{S_3}x_1 - \frac{S_2}{S_3}x_2$

The reduced order ship dynamics in sliding mode are given by the following equations :

$$\dot{x} = \quad A_{sl}x \quad \Leftrightarrow \quad \begin{array}{l} \dot{x}_1 = \quad x_2 \\ \dot{x}_2 = \quad -c_1 x_1 - c_2 x_2 \end{array} \qquad (26)$$

$$A_{sl} = \begin{bmatrix} 0 & 1 \\ -c_1 & -c_1 \end{bmatrix} \qquad (27)$$

$$c_1 = \frac{S_1}{S_3}; \qquad c_1 = \frac{S_2}{S_3} \qquad (28)$$

$$S = [S_1 \quad S_2 \quad S_3] \qquad (29)$$

The matrix A_{sl} is the system matrix of the closed-loop motion in sliding mode. Equation (29) is the constant switching surface matrix.

5 RESULTS

Two types of results are considered ; namely the computed switching surface matrix S and the simulations of the ship models that are steered by the controller.

5.1 Computing the Switching Surface Matrix

The values of the ship parameters for various water depth are from a table provided in (Mort & Linkens, 1981). The so called depth to draft factor (H/T) characterizes the changes in the environment that affect the ship parameters. The deeper the water, the greater that factor is. We use the first column corresponding to infinity. The table in appendix provides only data for two values of that factor. More data is found in (Mort &

Linkens, 1981) and a table is reproduced in (Rimbe, 2005).

$T_1 = 102.8$; $T_2 = 8.92$; $T_3 = 19.51$; $K_s = -0.102$.

The system in equation (23) is not stable and will be stabilized by state feedback as shown in Section 3. The following notations are used in the results.

A	is the unstabilized system matrix
B	is the system input matrix
N_U	is the vector of poles of the unstabilized ship
N_S	is the vector of poles of the stabilized ship
A_S	is the stabilized system matrix
P	is the solution to the Lyapunov equation
N_P	is the vector of the poles in sliding mode
K	is the stabilizing gain feedback

$$A = \begin{bmatrix} 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \\ 0 & -0.0011 & -0.1218 \end{bmatrix};$$

$B = [0\ 0\ 1]^T$; $Q = I$;

$N_U\ = [0\ \ -0.0097\ \ -0.1121\ ];$

$N_S\ = [-0.3000\ \ -0.6000\ \ -1.8000\ ];$

$K = [0.3240\ \ 1.7989\ \ 2.5782\ ];$

$$A_s = \begin{bmatrix} 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \\ -0.3240 & -1.8000 & -2.7000 \end{bmatrix};$$

$$P = \begin{bmatrix} 3.6893 & 4.5643 & 1.5432 \\ 4.5643 & 8.2622 & 2.8135 \\ 1.5432 & 2.8135 & 1.2272 \end{bmatrix};$$

$S = [1.5432\ \ 2.8135\ \ 1.2272]$;

$$A_{sl} = \begin{bmatrix} 0 & 1.0000 \\ -1.2575 & -2.2926 \end{bmatrix};$$

$N_p = [-0.9086\ \ -1.3840].$

5.2 Simulations

In the following results, the states $x_1 = \psi$ (heading angle or output) and x_3 , are plotted alongside the switching function $g = g(x)$. The time t_s when hitting occurs is also computed. The reaching control function $u = u(t, x)$ on a separate graph.

5.2.1 Disturbance Function

The disturbance function is plotted first, as it is the same for all situations. The disturbance function is $f(t) = 10\sin\beta t$. Such a function is chosen because it distorts very distinctly the response of the system (ship), so that its effect can be easily observed.
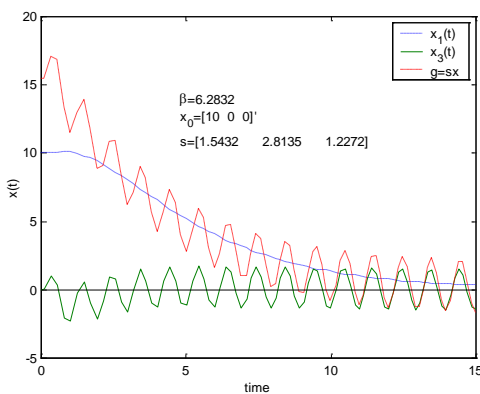
Fig. 7. Controlled Nonlinear Ship
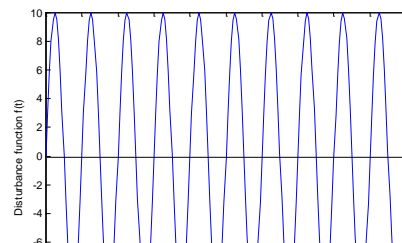
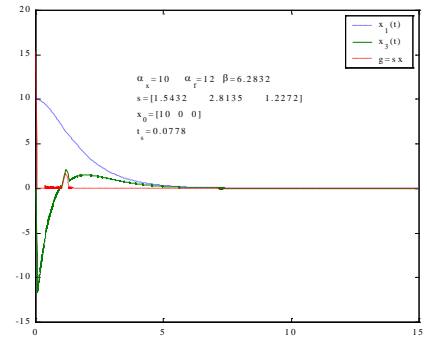## 5.2.2 The Linear Time-Invariant Ship Model
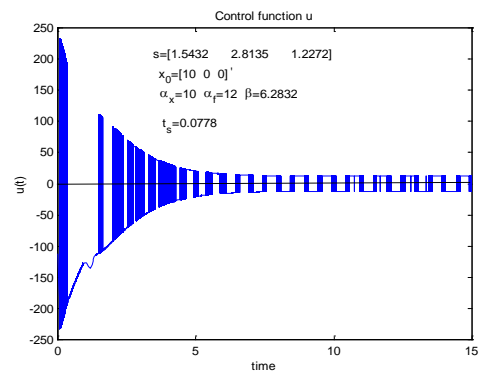

Fig. 3. Uncontrolled Time Invariant Ship

Fig. 8. The Control Function (Nonlinear Ship)

6 DISCUSSION

A practical advantage of this method of constructing switching surface is that the coefficients of the matrix S which are not freely chosen, automatically satisfy the conditions stated in (Utkin, 1977).

6.1 Limitations of the Controller

The simulation results show the efficiency of the variable structure algorithm even when both system and environment are time varying. As soon as the state reaches the sliding surface G (after the hitting time t_s), the effect of the disturbances and parameter variations are either stopped immediately or soon after. When α_f is small, the controller may not yield the expected property. Oscillations set in and carry on, soon after hitting occurs. However, that constraint is weakened by the choice of the controller defined by equation (20).

Fig.5. The Control Function (Time Invariant Ship)

Fig.4. Controlled Linear Time Invariant Ship

5.2.3 The non-Linear Time-Varying Ship

Although this observation has not been plotted, when α_f and α_x are close to their critical values respectively, hitting occurs more than once before sliding regime set in. Finally, when the two coefficients are simultaneously below their critical values, the controller fails completely. A main shortcoming of VSC is the chattering
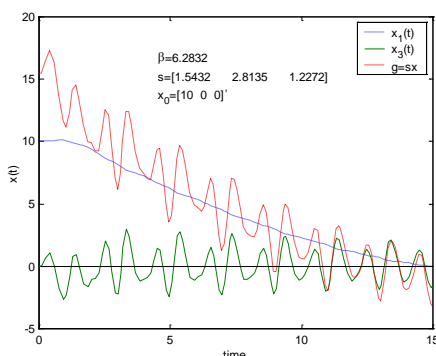
Fig. 6. Uncontrolled Nonlinear Ship

phenomenon due to the high frequency switching of the controller.

6.2 Stability
During sliding mode, stability is always achieved since all states converge to zero. Eigen values of the reduced order system matrix A_{sl}, have negative real parts.

6.3 Robustness
The concept of robustness of the controller refers to its ability to cope with changes in system parameters (uncertainties), system dynamics (nonlinearities) and in the environment (disturbances). Robustness is intrinsic to VSC. This is seen in figures Fig. 4. and Fig. 7., once sliding mode sets in.

6.4 Appendix
Table 1. Mariner Ship Parameters (Mort & Linkens,981)

Water Depth $(H/T) \rightarrow$	∞	2.50
T_1 (s)	102.8	83.45
T_2 (s)	8.92	9.57
T_3 (s)	19.51	17.71
$T = T_1 + T_2 - T_3$	92.22	75.32
K_s (s)	-0.102	-0.095
$-\dfrac{1}{T_2 T_1}$	-0.001091	-0.001252
$-\dfrac{1}{T_1} - \dfrac{1}{T_2}$	-0.121835	-0.116476
$\lambda_{o1} = -\dfrac{1}{T_1}$	-0.01	-0.012
$\lambda_{o2} = -\dfrac{1}{T_2}$	-0.112	-0.105
λ_{03}	0	0

7 CONCLUSION

In this paper, a procedure for deriving a variable structure controller that relies on the Lyapunov direct method of constructing switching surfaces has been presented. Provided a system has a stabilizing feedback with a known Lyapunov function, a sliding surface can be obtained directly. The method has been used to simulate the steering of a nonlinear ship and to show features of variable structure control. Disturbance rejection and invariance to parameter variations have also been exhibited.

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