

Riccati differential equations with elliptic coefficients II

| | |
|-----|---|
| 著者 | Ishizaki Katsuya, Laine Ilpo, Shimomura Shun, Tohge Kazuya |
| 雑誌名 | 東北数学雑誌. Second series |
| 巻 | 55 |
| 号 | 1 |
| ページ | 99-108 |
| 発行年 | 2003-03 |
| URL | http://id.nii.ac.jp/1146/00008071/ |

RICCATI DIFFERENTIAL EQUATIONS WITH ELLIPTIC COEFFICIENTS II

KATSUYA ISHIZAKI¹⁾, ILPO LAINE, SHUN SHIMOMURA²⁾ AND KAZUYA TOHGE¹⁾

Abstract. We study a Riccati differential equation whose coefficient is expressible in terms of a special Weierstrass \wp -function. We show that all the solutions are meromorphic, and examine the periodicity of them.

1. Introduction. In our preceding paper [4], we studied the Riccati differential equation

$$(1.1) \quad w' + w^2 + \frac{1}{4}(1 - m^2)\wp(0, g_3; z) = 0,$$

where

- (1) m is a natural number such that $m \geq 2$, $m \notin 6\mathbb{N} = \{6n \mid n \in \mathbb{N}\}$;
- (2) $\wp(0, g_3; z)$ is the Weierstrass \wp -function satisfying

$$(v')^2 = 4v^3 - g_3, \quad g_3 \neq 0.$$

Let $\wp(z)$ be an arbitrary \wp -function satisfying $(v')^2 = 4v^3 - g_2v - g_3$, $g_2^3 - 27g_3^2 \neq 0$. As was explained in [4, Section 1], under a certain condition, if, for various values of a , an equation of the form $w' + w^2 + a\wp(z) = 0$ admits a plenty of meromorphic solutions, then it is either (1.1) or

$$(1.2) \quad w' + w^2 + \frac{1}{4}(1 - m^2)\wp_0(z) = 0,$$

where

- (1) m is a natural number such that $m \geq 2$, $m \notin 4\mathbb{N} = \{4n \mid n \in \mathbb{N}\}$;
- (2) $\wp_0(z) = \wp(g_2, 0; z)$ is the Weierstrass \wp -function satisfying

$$(1.3) \quad (v')^2 = 4v^3 - g_2v, \quad g_2 \neq 0.$$

Let ω_1^0, ω_2^0 be primitive periods of $\wp_0(z)$ satisfying $\text{Im}(\omega_2^0/\omega_1^0) > 0$ (cf. (2.3)).

The main results of this paper are stated as follows.

2000 *Mathematics Subject Classification.* Primary 34M05; Secondary 34A34.

Key words and phrases. Meromorphic function, Riccati equation, Weierstrass \wp function

¹⁾ Partly supported by the Grant-in-Aid for General Scientific Research, the Ministry of Education, Science, Sports and Culture 11640164 (Toda).

²⁾ Partly supported by the Grant-in-Aid for General Scientific Research, the Ministry of Education, Science, Sports and Culture 11640212.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - TEX

THEOREM 1.1. *All the solutions of (1.2) are meromorphic in the whole complex plane \mathbb{C} .*

THEOREM 1.2. *Suppose that m is even. Then,*

- (i) *every solution of (1.2) is a doubly periodic function with periods $(2\omega_1^0, 2\omega_2^0)$;*
- (ii) *there exist exactly two distinct solutions with periods $(\omega_1^0, 2\omega_2^0)$ (or with periods $(2\omega_1^0, \omega_2^0)$);*
- (iii) *there exists no solution with periods (ω_1^0, ω_2^0) .*

THEOREM 1.3. *For every odd integer m satisfying $m \geq 3$, the equation (1.2) admits no periodic solution except a doubly periodic one, which is expressible in the form:*

$$\begin{aligned} \psi_m(z) &= \frac{\wp_0'(z)}{2\wp_0(z)} + \sum_{h=1}^k \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, & \text{if } m = 8k + 3, \quad k = 0, 1, 2, \dots, \\ \psi_m(z) &= \frac{\wp_0''(z)}{\wp_0'(z)} - \frac{\wp_0'(z)}{2\wp_0(z)} + \sum_{h=1}^k \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, & \text{if } m = 8k + 5, \quad k = 0, 1, 2, \dots, \\ \psi_m(z) &= \frac{\wp_0''(z)}{\wp_0'(z)} + \sum_{h=1}^k \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, & \text{if } m = 8k + 7, \quad k = 0, 1, 2, \dots, \\ \psi_m(z) &= \sum_{h=1}^{k+1} \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, & \text{if } m = 8k + 9, \quad k = 0, 1, 2, \dots, \end{aligned}$$

where, for each (m, h) , $\theta_{m,h}$ is some complex constant.

Using the properties of $\wp_0(z)$ explained in Section 2, we prove these results in Sections 3 and 4. For a related result concerning linear systems with doubly periodic coefficients, see [1].

2. Properties of the elliptic function $\wp_0(z)$. We review basic facts concerning elliptic functions (see [6], [7]). The elliptic function $\wp_0(z) = \wp(g_2, 0; z)$ satisfies (1.3), which is written in the form

$$(2.1) \quad \begin{aligned} (v')^2 &= 4v(v - e_1)(v - e_2), \\ e_1 &= g_2^{1/2}/2, \quad e_2 = -g_2^{1/2}/2, \quad e_3 = 0, \quad g_2 \neq 0. \end{aligned}$$

Consider the expression of $\wp_0(z)$:

$$(2.2) \quad \wp_0(z) = \frac{1}{z^2} + \sum_{(p,q) \in \mathbb{Z}_*^2} \left(\frac{1}{(z - \Omega_{p,q})^2} - \frac{1}{\Omega_{p,q}^2} \right), \quad \mathbb{Z}_*^2 = \mathbb{Z}^2 - \{(0, 0)\},$$

where $\Omega_{p,q} = p\omega_1^0 + q\omega_2^0$, $(p, q) \in \mathbb{Z}_*^2$ constitute the lattice of poles. By (2.1) the periods ω_1^0, ω_2^0 of $\wp_0(z)$ may be given by

$$(2.3) \quad \omega_1^0 = \sqrt{2}g_2^{-1/4}\varepsilon_0, \quad \omega_2^0 = i\omega_1^0, \quad \varepsilon_0 = \int_{-1}^0 \frac{dt}{\sqrt{t^3 - t}}.$$

Then we have

PROPOSITION 2.1. $\wp_0(\omega_j^0/2) = e_j$ ($j = 1, 2$), $\wp_0(\omega_3^0/2) = 0$, where $\omega_3^0 = \omega_1^0 + \omega_2^0$.

Furthermore the Weierstrass ζ -function

$$\zeta_0(z) = \frac{1}{z} + \sum_{(p,q) \in \mathbb{Z}_*^2} \left(\frac{1}{z - \Omega_{p,q}} + \frac{1}{\Omega_{p,q}} + \frac{z}{\Omega_{p,q}^2} \right), \quad \zeta_0'(z) = -\wp_0(z)$$

has the properties:

$$(2.4) \quad \zeta_0(z + \omega_j^0) = \zeta_0(z) + 2\eta_j^0, \quad j = 1, 2, 3,$$

$$(2.5) \quad \eta_j^0 = \zeta_0(\omega_j^0/2) = \frac{1}{\omega_j^0/2} + \sum_{(p,q) \in \mathbb{Z}_*^2} \left(\frac{1}{\omega_j^0/2 - \Omega_{p,q}} + \frac{1}{\Omega_{p,q}} + \frac{\omega_j^0/2}{\Omega_{p,q}^2} \right),$$

$$(2.6) \quad \eta_1^0 \omega_2^0 - \eta_2^0 \omega_1^0 = \pi i.$$

Relation (2.6) implies $(\eta_1^0, \eta_2^0) \neq (0, 0)$. Observing that $-i\Omega_{p,q} = \Omega_{q,-p}$, from (2.3) and (2.5), we obtain

PROPOSITION 2.2. $\eta_1^0/\eta_2^0 = \zeta_0(\omega_1^0/2)/\zeta_0(\omega_2^0/2) = i$.

Around each lattice pole $z = \sigma_L = \Omega_{p(L),q(L)}$, the Laurent series expansion of $\wp_0(z)$ is given by the following

PROPOSITION 2.3. *For an arbitrary pole $z = \sigma_L$ of $\wp_0(z)$,*

$$\wp_0(z) = \sum_{n=0}^{\infty} b_{4n} (z - \sigma_L)^{4n-2}, \quad b_0 = 1,$$

around $z = \sigma_L$.

PROOF. It suffices to consider the case where $\sigma_L = 0$. We put $\wp_0(z) = \sum_{k=0}^{\infty} b_k z^{k-2}$, $b_0 = 1$, near $z = 0$. Then $-\wp_0(iz) = \sum_{k=0}^{\infty} i^k b_k z^{k-2}$. Since $-i\Omega_{p,q} = \Omega_{q,-p}$, we have $\wp_0(z) = -\wp_0(iz)$, which implies $b_k = 0$ for $k \notin 4\mathbb{N}$. \square

Let $\varpi_0(z) = \wp_0(z)^{1/2}$ be a branch such that $\lim_{z \rightarrow 0} z\varpi_0(z) = 1$. Then $\varpi_0(z)$ is a doubly periodic function with the periods $(2\omega_1^0, \omega_3^0)$, which has two simple poles with residues 1 and -1 in its period parallelogram. A simple computation leads us to the following.

PROPOSITION 2.4. *The functions $\varpi_0(z)$ and $W_0(z) = 2\varpi_0'(z) = \wp_0'(z)\wp_0(z)^{-1/2}$ satisfy*

$$(2.7) \quad \varpi_0'(z)^2 = \varpi_0(z)^4 - g_2/4,$$

and

$$(2.8) \quad W_0''(z) = 6\wp_0(z)W_0(z),$$

respectively.

3. Proofs of Theorems 1.1 and 1.2. Consider the linear differential equation

$$(3.1) \quad u'' + \frac{1-m^2}{4} \wp_0(z)u = 0,$$

which is associated with (1.2).

LEMMA 3.1. *Let $z = \sigma_L$ be an arbitrary lattice pole of $\wp_0(z)$. Then (3.1) admits linearly independent solutions expressed in the form*

$$U_1(z) = (z - \sigma_L)^{(1-m)/2} \sum_{j=0}^{\infty} \beta_j^{(1)} (z - \sigma_L)^{4j}, \quad \beta_0^{(1)} = 1,$$

$$U_2(z) = (z - \sigma_L)^{(1+m)/2} \sum_{j=0}^{\infty} \beta_j^{(2)} (z - \sigma_L)^{4j}, \quad \beta_0^{(2)} = 1,$$

around $z = \sigma_L$.

PROOF. Around $z = \sigma_L$, we have

$$\wp_0(z) = (z - \sigma_L)^{-2} P_0((z - \sigma_L)^4)$$

with

$$P_0(t) = \sum_{n=0}^{\infty} b_{4n} t^n, \quad b_0 = 1$$

(cf. Proposition 2.3). Consider the equation

$$(3.2) \quad t^2 \frac{d^2 u}{dt^2} + \frac{3}{4} t \frac{du}{dt} + \frac{1-m^2}{64} P_0(t)u = 0$$

around the regular singular point $t = 0$. The roots $\rho_1 = (1-m)/8$ and $\rho_2 = (1+m)/8$ of the indicial equation

$$\rho(\rho - 1) + \frac{3}{4}\rho + \frac{1-m^2}{64} = 0$$

satisfy $\rho_2 - \rho_1 = m/4 \notin \mathbb{Z}$. Hence, (3.2) admits local solutions of the form

$$\varphi_1(t) = t^{(1-m)/8} \sum_{j=0}^{\infty} \beta_j^{(1)} t^j, \quad \varphi_2(t) = t^{(1+m)/8} \sum_{j=0}^{\infty} \beta_j^{(2)} t^j, \quad \beta_0^{(1)} = \beta_0^{(2)} = 1,$$

around $t = 0$ (see [2], [3]). By the transformation $t = (z - \sigma_L)^4$, (3.2) becomes (3.1) admitting the solutions $U_1(z) = \varphi_1((z - \sigma_L)^4)$, $U_2(z) = \varphi_2((z - \sigma_L)^4)$. This completes the proof. \square

An arbitrary solution $w(z)$ of (1.2) is written in the form $w(z) = U'(z)/U(z)$, where $U(z)$ is a solution of (3.1). By Lemma 3.1, $w(z)$ is meromorphic in the whole complex plane \mathbb{C} , which completes the proof of Theorem 1.1.

Theorem 1.2 is proved by the same argument as that of the proof of [4, Theorem 3.1].

4. Proof of Theorem 1.3.

4.1. Case $m = 8k + 3$. When $m = 8k + 3$, $k = 0, 1, 2, \dots$, we write (3.1) in the form

$$(4.1) \quad L^*(u) = 0, \quad L^* = (d/dz)^2 - (4k + 1)(4k + 2)\wp_0(z).$$

In what follows, $\wp_0(z)^{1/2}$ denotes the branch given in Section 2. Then we have

PROPOSITION 4.1. *For every $k \in \mathbb{N} \cup \{0\}$, (4.1) admits a doubly periodic solution of the form*

$$X_m(z) = \wp_0(z)^{1/2} \prod_{h=1}^k (\wp_0(z)^2 - \theta_{m,h})$$

($m = 8k + 3$) with periods $(2\omega_1^0, \omega_3^0)$.

PROOF. Let $\Delta_0^{1/2}$ be the period parallelogram of $\wp_0(z)^{1/2}$ with vertices $(-\omega_1^0 - \omega_3^0)/2$, $(3\omega_1^0 - \omega_3^0)/2$, $(-\omega_1^0 + \omega_3^0)/2$ and $(3\omega_1^0 + \omega_3^0)/2$. The poles of $\wp_0(z)^{1/2}$ in $\Delta_0^{1/2}$ are $z = 0$ and $z = \omega_1^0$, whose residues are 1 and -1 , respectively. By Proposition 2.3, for every $q \in \mathbb{N} \cup \{0\}$, we have

$$(4.2.1) \quad \wp_0(z)^{1/2} \wp_0(z)^q = z^{-2q-1} \sum_{n=0}^{\infty} b_{4n}^{(q)} z^{4n}, \quad b_0^{(q)} = 1$$

around $z = 0$, and

$$(4.2.2) \quad \wp_0(z)^{1/2} \wp_0(z)^q = -(z - \omega_1^0)^{-2q-1} \sum_{n=0}^{\infty} b_{4n}^{(q)} (z - \omega_1^0)^{4n}$$

around $z = \omega_1^0$. Then, for $\nu = 0, 1, \dots, k-1, k$,

$$\begin{aligned} L^*(\wp_0(z)^{1/2} \wp_0(z)^{2\nu}) &= z^{-4\nu-3} \sum_{n=0}^{\infty} B_{4n}^{\nu,k} z^{4n} \\ &= B_0^{\nu,k} z^{-4\nu-3} + B_4^{\nu,k} z^{-4\nu+1} + \dots + B_{4\nu}^{\nu,k} z^{-3} + O(z), \end{aligned}$$

where $B_0^{\nu,k} = (4\nu + 1)(4\nu + 2) - (4k + 1)(4k + 2)$. Observing that $B_0^{k,k} = 0$, $B_0^{\nu,k} \neq 0$ ($\nu \neq k$), we can choose $C_{k,n} \in \mathbb{C}$, $n = 0, 1, \dots, k$, satisfying $C_{k,k} = 1$ in such a way that

$$L^* \left(\wp_0(z)^{1/2} \sum_{n=0}^k C_{k,n} \wp_0(z)^{2n} \right) = O(z)$$

near $z = 0$. Then, by (4.2.1) and (4.2.2),

$$L^* \left(\wp_0(z)^{1/2} \sum_{n=0}^k C_{k,n} \wp_0(z)^{2n} \right) = O(z - \omega_1^0)$$

also holds near $z = \omega_1^0$. By this fact and the Liouville theorem, we conclude that

$$X_m(z) = \wp_0(z)^{1/2} \sum_{n=0}^k C_{k,n} \wp_0(z)^{2n} = \wp_0(z)^{1/2} \prod_{h=1}^k (\wp_0(z)^2 - \theta_{m,h})$$

satisfies

$$L^*(X_m(z)) \equiv 0,$$

which implies the proposition. \square

It is easy to see that

$$(4.3) \quad \psi_m(z) = \frac{X'_m(z)}{X_m(z)} = \frac{\wp'_0(z)}{2\wp_0(z)} + \sum_{h=1}^k \frac{2\wp_0(z)\wp'_0(z)}{\wp_0(z)^2 - \theta_{m,h}}$$

is a solution of (1.2).

In order to verify that there exists no periodic solution of (1.2) other than $\psi_m(z)$, we examine another solution of (4.1). By the uniqueness of the solution of an initial value problem associated with (4.1), every zero of $X_m(z)$ is simple. Hence each constant $\theta_{m,h}$ satisfies $\theta_{m,h} \neq 0$, $\theta_{m,h} \neq \theta_{m,i}$ for $i \neq h$. It is easy to see that all zeros are located symmetrically with respect to $z = 0$. Furthermore, $X_m(z)^2$ is a doubly periodic function with periods (ω_1^0, ω_2^0) . It follows from these facts and the Liouville theorem, that

$$\frac{1}{X_m(z)^2} = \frac{1}{2} \sum_{\tau \in Z} \frac{1}{X'_m(\tau)^2} (\wp_0(z - \tau) + \wp_0(z + \tau) - 2\wp_0(\tau)),$$

where Z denotes the set of all zeros of $X_m(z)$ in

$$(4.4) \quad \Delta_0 = \{s_1\omega_1^0 + s_2\omega_2^0 \mid -1/2 < s_1 \leq 1/2, -1/2 < s_2 \leq 1/2\}.$$

Then we have another solution of (4.1) written in the form

$$Y_m(z) = X_m(z) \int_{z_0}^z \frac{dt}{X_m(t)^2} = -\frac{X_m(z)}{2} \sum_{\tau \in Z} \frac{1}{X'_m(\tau)^2} (\zeta_0(z - \tau) + \zeta_0(z + \tau) + 2\wp_0(\tau)z)$$

(see Section 2). For the linearly independent solutions $X_m(z)$ and $Y_m(z)$, we have the Floquet matrices

$$M_j = \begin{pmatrix} 1 & \delta_j \\ 0 & 1 \end{pmatrix}, \quad \delta_j = -\omega_j^0 \sum_{\tau \in Z} \frac{\wp_0(\tau)}{X'_m(\tau)^2} - 2\eta_j^0 \sum_{\tau \in Z} \frac{1}{X'_m(\tau)^2} \quad (j = 1, 2),$$

satisfying $[\omega_j^0](X_m(z), Y_m(z)) = (X_m(z), Y_m(z))M_j$, where $[\omega_j^0]$ denotes the analytic continuation along the segment $[z, z + \omega_j^0]$ (cf. Sections 2 and [4, Section 3]). Note that Z is written in the form

$$Z = Z_0 \cup \left(\bigcup_{h=1}^k Z_h \right)$$

with

$$Z_0 = \{\tau \in Z \mid \wp_0(\tau) = 0\},$$

$$Z_h = \{\pm\tau_{h,-}, \pm\tau_{h,+} \in Z \mid \wp_0(\pm\tau_{h,-}) = -\theta_{m,h}^{1/2}, \wp_0(\pm\tau_{h,+}) = \theta_{m,h}^{1/2}\}.$$

LEMMA 4.2. *We have*

$$\sum_{\tau \in Z} \frac{\wp_0(\tau)}{X'_m(\tau)^2} = 0.$$

PROOF. Since every zero of $X_m(z)$ is simple,

$$(4.5) \quad \sum_{\tau \in Z_0} \frac{\wp_0(\tau)}{X'_m(\tau)^2} = 0.$$

Observe that

$$\begin{aligned} X'_m(\tau_{h,\pm})^2 &= \wp_0(\tau_{h,\pm}) \cdot 4\wp_0'(\tau_{h,\pm})^2 \wp_0(\tau_{h,\pm})^2 \prod_{q \neq h} (\wp_0(\tau_{h,\pm})^2 - \theta_{m,q})^2 \\ &= 4\wp_0(\tau_{h,\pm})^4 (4\wp_0(\tau_{h,\pm})^2 - g_2) \prod_{q \neq h} (\wp_0(\tau_{h,\pm})^2 - \theta_{m,q})^2 \\ &= 4\theta_{m,h}^2 (4\theta_{m,h} - g_2) \prod_{q \neq h} (\theta_{m,h} - \theta_{m,q})^2 = \Gamma_h \neq 0, \end{aligned}$$

and that

$$X'_m(-\tau_{h,\pm})^2 = \Gamma_h \neq 0.$$

Hence we have

$$(4.6) \quad \sum_{\tau \in Z_h} \frac{\wp_0(\tau)}{X'_m(\tau)^2} = \Gamma_h^{-1} \left((\wp_0(\tau_{h,-}) + \wp_0(\tau_{h,+})) + (\wp_0(-\tau_{h,-}) + \wp_0(-\tau_{h,+})) \right) = 0.$$

From (4.5) and (4.6), the lemma immediately follows. \square

By Lemma 4.2, we have $\delta_j = -2\eta_j^0 \sum_{\tau \in Z} X'_m(\tau)^{-2}$ ($j = 1, 2$), which satisfy $(\delta_1, \delta_2) \neq (0, 0)$. Indeed, if $\delta_1 = \delta_2 = 0$, then (4.1) admits a nontrivial doubly periodic solution which vanishes at each pole of $\wp_0(z)$, contradicting the Liouville theorem. Let δ be the ratio

$$\delta = \begin{cases} \delta_1/\delta_2, & \text{if } \delta_2 \neq 0, \\ 0, & \text{if } \delta_2 = 0. \end{cases}$$

Now, we note the following criteria, which is proved by the same way as in the proof of [4, Proposition 4.5].

LEMMA 4.3. *If $\delta \notin \mathbb{Q}$, then there exists no periodic solution of (1.2) other than (4.3). If $\delta \in \mathbb{Q}$, then every solution of (1.2) is purely simply periodic.*

Since $\sum_{\tau \in Z} X'_m(\tau)^{-2} \neq 0$, using Proposition 2.2, we have

$$(4.7) \quad \delta = \eta_1^0/\eta_2^0 = i.$$

Hence, by Lemma 4.3, there exists no periodic solution other than (4.3).

4.2. Case $m = 8k + 7$. When $m = 8k + 7$, $k = 0, 1, 2, \dots$, we can construct a solution of (3.1) expressible in the form

$$\tilde{X}_m(z) = \wp'_0(z) \left(\wp_0(z)^{2k} + \sum_{n=0}^{k-1} \tilde{C}_n \wp_0(z)^{2n} \right) = \wp'_0(z) \prod_{h=1}^k (\wp_0(z)^2 - \theta_{m,h}),$$

by an argument analogous to that for the case $m = 8k + 3$ (see also [4, Section 4]). Then

$$(4.8) \quad \psi_m(z) = \frac{\tilde{X}'_m(z)}{\tilde{X}_m(z)} = \frac{\wp''_0(z)}{\wp'_0(z)} + \sum_{h=1}^k \frac{2\wp_0(z)\wp'_0(z)}{\wp_0(z)^2 - \theta_{m,h}}$$

is a periodic solution of (1.2). By the same argument as in Section 4.1, we obtain the Floquet matrices

$$\tilde{M}_j = \begin{pmatrix} 1 & \tilde{\delta}_j \\ 0 & 1 \end{pmatrix}, \quad \tilde{\delta}_j = -\omega_j^0 \sum_{\tau \in \tilde{Z}} \frac{\wp_0(\tau)}{\tilde{X}'_m(\tau)^2} - 2\eta_j^0 \sum_{\tau \in \tilde{Z}} \frac{1}{\tilde{X}'_m(\tau)^2}, \quad (j = 1, 2),$$

where \tilde{Z} denotes the set of all zeros of $\tilde{X}_m(z)$ in Δ_0 (cf. (4.4)). Decompose the set \tilde{Z} into

$$(4.9) \quad \begin{aligned} \tilde{Z} &= \tilde{Z}' \cup \left(\bigcup_{h=1}^k \tilde{Z}_h \right), \\ \tilde{Z}' &= \{ \tau \mid \wp'_0(\tau) = 0 \} = \{ \omega_1^0/2, \omega_2^0/2, \omega_3^0/2 \}, \\ \tilde{Z}_h &= \{ \tau \mid \wp_0(\tau)^2 = \theta_{m,h} \}. \end{aligned}$$

Using the formulas

$$\begin{aligned} \wp_0(\omega_1^0/2) &= g_2^{1/2}/2, & \wp_0(\omega_2^0/2) &= -g_2^{1/2}/2, & \wp_0(\omega_3^0/2) &= 0, \\ \tilde{X}'_m(\omega_j^0/2)^2 &= \wp''_0(\omega_j^0/2)^2 \prod_{h=1}^k \left(\wp_0(\omega_j^0/2)^2 - \theta_{m,h} \right)^2 \\ &= g_2^2 \prod_{h=1}^k (g_2/4 - \theta_{m,h})^2 \quad (j = 1, 2), \end{aligned}$$

we have

$$(4.10) \quad \sum_{\tau \in \tilde{Z}'} \frac{\wp_0(\tau)}{\tilde{X}'_m(\tau)^2} = 0.$$

Furthermore, by the same argument as in the proof of Lemma 4.2, we have

$$(4.11) \quad \sum_{\tau \in \tilde{Z}_h} \frac{\wp_0(\tau)}{\tilde{X}'_m(\tau)^2} = 0 \quad (h = 1, \dots, k).$$

From (4.9), (4.10), (4.11) and Proposition 2.2, it follows that $\tilde{\delta} = \tilde{\delta}_1/\tilde{\delta}_2 = i$. Hence, applying Lemma 4.3, we conclude that there exists no periodic solution of (1.2) other than (4.8).

4.3. Cases $m = 8k + 5$ and $m = 8k + 9$. When $m = 8k + 5$, (3.1) is written in the form

$$(4.12) \quad L_k(u) = 0, \quad L_k = (d/dz)^2 - (4k + 2)(4k + 3)\wp_0(z).$$

Then we have

PROPOSITION 4.4. *For every $k = 0, 1, 2, \dots$, (4.12) admits a solution expressed as*

$$(4.13) \quad W_k(z) = \wp_0'(z)\wp_0(z)^{-1/2} \sum_{n=0}^k \tilde{C}_{k,n} \wp_0(z)^{2n}$$

with $\tilde{C}_{k,k} = 1$.

Proof. We show the conclusion by induction on k . By (2.8) the function $W_0(z) = \wp_0'(z)\wp_0(z)^{-1/2}$ satisfies (4.12) with $k = 0$. Suppose that, for $k = 0, 1, \dots, \kappa - 1$, (4.12) admits a solution expressed as (4.13). By Proposition 4.1, for suitably chosen constants C_n , $n = 0, 1, \dots, \kappa$, the function

$$X(z) = \wp_0(z)^{1/2} \sum_{n=0}^{\kappa} C_n \wp_0(z)^{2n}, \quad C_{\kappa} = 1$$

satisfies

$$(4.14) \quad X''(z) = (4\kappa + 1)(4\kappa + 2)\wp_0(z)X(z).$$

Differentiate (4.14) and put $w_{\kappa}(z) = X'(z)$. Observing that

$$\frac{\wp_0'(z)}{\wp_0(z)}X(z) - \frac{w_{\kappa}(z)}{2\kappa + 1/2} = \wp_0'(z)\wp_0(z)^{-1/2} \sum_{n=0}^{\kappa-1} C'_n \wp_0(z)^{2n}, \quad C'_n \in \mathbb{C},$$

we have

$$L_{\kappa}(w_{\kappa}(z)) = (4\kappa + 2)(4\kappa + 3)\wp_0(z) \left(\wp_0'(z)\wp_0(z)^{-1/2} \sum_{n=0}^{\kappa-1} C''_n \wp_0(z)^{2n} \right), \quad C''_n \in \mathbb{C}.$$

By supposition,

$$\begin{aligned} L_{\kappa}(w_{\kappa}(z) + \gamma_{\kappa-1}W_{\kappa-1}(z)) &= L_{\kappa}(w_{\kappa}(z)) + \gamma_{\kappa-1}\rho_{\kappa,\kappa-1}\wp_0(z)W_{\kappa-1}(z), \\ \rho_{\kappa,\kappa-1} &= (4\kappa + 2)(4\kappa + 3) - (4\kappa - 2)(4\kappa - 1) \neq 0. \end{aligned}$$

Hence, if $\gamma_{\kappa-1} = -(4\kappa + 2)(4\kappa + 3)C''_{\kappa-1}/\rho_{\kappa,\kappa-1}$, then

$$L_{\kappa}(w_{\kappa}(z) + \gamma_{\kappa-1}W_{\kappa-1}(z)) = (4\kappa + 2)(4\kappa + 3)\wp_0(z) \left(\wp_0'(z)\wp_0(z)^{-1/2} \sum_{n=0}^{\kappa-2} C_n^{(3)} \wp_0(z)^{2n} \right).$$

Repeating this procedure, we may choose γ_j ($j = 0, \dots, \kappa - 1$) in such a way that $W_\kappa(z) = w_\kappa(z) + \sum_{j=0}^{\kappa-1} \gamma_j W_j(z)$ satisfies (4.12) with $k = \kappa$. Thus the proposition is verified. \square

We can write (4.13) in the form

$$W_k(z) = \wp_0'(z) \wp_0(z)^{-1/2} \prod_{h=1}^k (\wp_0(z)^2 - \theta_{m,h}),$$

which yields the solution $\psi_m(z) = W_k'(z)/W_k(z)$ of (1.2) with $m = 8k + 5$.

Next consider the case where $m = 8k + 9$, $k = 0, 1, 2, \dots$. It is easy to see that $V_0(z) = 6\wp_0(z)^2 - 9g_2/10$ satisfies

$$V_0''(z) = 20\wp_0(z)V_0(z),$$

which means that $V_0(z)$ is a solution of (3.1) with $m = 9$. Using this fact, from the solution of (3.1) with $m = 8k + 7$ given in Section 4.2, we can derive a solution of (3.1) with $m = 8k + 9$ written in the form

$$V_k(z) = \prod_{h=1}^{k+1} (\wp_0(z)^2 - \theta_{m,h}),$$

by the same argument as in the proof of Proposition 4.4. Then, $\psi_m(z) = V_k'(z)/V_k(z)$ is a doubly periodic solution of (1.2) with $m = 8k + 9$. Furthermore, in both cases $m = 8k + 5$ and $m = 8k + 9$, we can also verify the non-existence of periodic solutions of (1.2) other than $\psi_m(z)$ by the same way as in Section 4.1. This completes the proof.

REFERENCES

1. F. GESZTESY AND W. STICKA, On a theorem of Picard, Proc. Amer. Math. Soc. **126** (1998), 1089–1099.
2. H. HEROLD, Differentialgleichungen im Komplexen, Vanderhoeck & Ruprecht, Göttingen, 1975.
3. E. HILLE, Ordinary Differential Equations in the Complex Domain, Wiley-Interscience, New York, 1976.
4. K. ISHIZAKI, I. LAINE, S. SHIMOMURA AND K. TOHGE, Riccati differential equations with elliptic coefficients, Result. Math. **38** (2000), 58–71.
5. I. LAINE, Nevanlinna Theory and Complex Differential Equations, de Gruyter Stud. Math. 15, Walter de Gruyter, Berlin–New York, 1993.
6. J. NOGUCHI, Introduction to Complex Analysis, American Mathematical Society, Providence, RI, 1998.
7. E. T. WHITTAKER AND G. N. WATSON, Modern Analysis, Cambridge Univ. Press, 1927.

Department of Mathematics
Nippon Institute of Technology
4-1 Gakuendai Miyashiro
Minamisaitama Saitama 345-8501
Japan
E-mail address: ishi@nit.ac.jp

Department of Mathematics
Joensuu University
P.O.Box 111
FIN-80101 Joensuu
Finland
E-mail address: laine@ns.joensuu.fi

Department of Mathematics
Keio University
3-14-1 Hiyoshi Kohoku-ku
Yokohama 223-8522
JAPAN
E-mail address: shimomur@math.keio.ac.jp

Faculty of Technology
Kanazawa University
2-40-20 Kodatsuno
Kanazawa 920-8667
JAPAN
E-mail address: tohge@t.kanazawa-u.ac.jp