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STACKELBERG SOLUTION OF FIRST-ORDER MEAN FIELD GAME WITH A MAJOR PLAYER

The paper is concerned with the study of the large system of identical players interacting with the environment. We model the environment as a major (exogenous) player. The main assumption of our model is that the minor players influence on each other and on the major (exogenous) player only via certain averaging characteristics. Such models are called mean field games with a major player. It is assumed that the game is considered in the continuous time and the dynamics of major and minor players is given by ordinary differential equations. We study the Stackelberg solution with the major player playing as a leader, i.e., it is assumed that the major player announces his/her control. The main result of the paper is the existence of the Stackelberg solution in the mean field game with the major player in the class of relaxed open-loop strategies.

Keywords: mean field game, Stackelberg solution, game with infinitely many players.

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Introduction

The mean field game theory studies the behavior of the large number of small players by examination the limit case when the number of players tends to infinity. The main assumptions of the mean field game approach are that: (*i*) the players are identical; (*ii*) the interaction between the players is performed by certain averaging characteristics. The study of mean field game theory starts with seminal papers by Lasry and Lions [23, 24] and Huang, Caines and Malhamé [17, 18].

There are several approaches in the mean field game theory. First one reduces the original game with infinitely many players to the initially-boundary value problem for the coupled system of PDEs. Within the framework of this approach the existence of solution is proved [23–25, 27]. Moreover, based on this approach one can construct an approximate equilibrium in the game with the finite number of players [27].

The second approach is called probabilistic [10–12, 20]. It regards a solution of mean field game as a Nash equilibrium in a infinite player game. This approach is fruitful in the analysis of the limit of open-loop Nash equilibria in finite player games [15, 21]. Furthermore, it was used to study the deterministic limit of stochastic mean field games [3].

The third approach is concerned with so called master equation which is a partial differential equation in the space of probabilities [6, 9, 27]. This approach was used to study the link between feedback Nash equilibria in the finite player game and mean field game [8, 19, 22]. Notice that the existence results for the master equation are obtained for the non-degenerate stochastic mean field games [8]. Additionally, the short time existence was delivered [11].

The mean field game methodology can be used to analyze the interaction of the large group of identical small players with the exogenous player. This problem is called mean field games with a major player [13, 14, 16, 26, 28, 29]. In this case we study the system of infinitely many identical minor players and one major players. It is assumed that the minor players can influence on each other and on the major player only via certain mean-filed characteristics.

There are several game designs suitable for the mean field game with a major player. First one assumes that the major and minor players choose their strategies simultaneously. It leads to the Nash solution for the mean field game with a major player. Another game design appears when we assume that the major player chooses his/her strategy first and announce it. This solution concept refers to the Stackelberg game with the minor players as followers and the major player as leader. Previously, the Stackelberg solution of the mean field game with a major player was studied only for the linear-quadratic games (see [4, 5, 30]).

In the paper we study the Stackelberg solution of the first order mean field game with a major player, i.e., we assume that the dynamics of the minor and the major players are given by ODEs.

The main result of the paper is the existence of the Stackelberg solution. The paper is organized as follows. First, we introduce the general notations. In Section 2 we define the mean field game with a major player and formulate the main result. In the last section we examine the properties of the motions and prove the main result.

§ 1. Preliminaries

Given a Polish space (X,ρ_X), denote by $\mathcal{P}(X)$ the set of Borel probabilities on X. We endow $\mathcal{P}(X)$ with the narrow convergence. Further, let $\mathcal{P}^1(X)$ be the set of probabilities $m\in\mathcal{P}(X)$ satisfying, for some (and, thus, any) $x_0\in X$,

$$\varsigma(m)\triangleq\int_X\rho_X(x,x_0)m(dx)<\infty.$$

We introduce on $\mathcal{P}^1(X)$ the 1-Wasserstein metric by the rule [2]: if $m_1,m_2\in\mathcal{P}^1(X)$, then

$$\begin{aligned}W_1(m_1,m_2)&=\inf_{\pi\in\Pi(m_1,m_2)}\int_{X\times X}\rho(x_1,x_2)\pi(d(x_1,x_2))\\&=\sup_{\phi\in\mathrm{Lip}_1(X)}\left[\int_X\phi(x)m_1(dx)-\int_X\phi(x)m_2(dx)\right].\end{aligned}$$

Here $\Pi(m_1,m_2)$ is the set of plans between m_1 and m_2 i.e.

$$\begin{aligned}\Pi(m_1,m_2)\triangleq\{\pi\in\mathcal{P}^1(X\times X):\\ \pi(\Gamma\times X)=m_1(X),\pi(X\times\Gamma)=m_2(X)\text{ for any measurable }\Gamma\subset X\};\end{aligned}$$

$\mathrm{Lip}_1(X)$ stands for the set of 1-Lipschitz continuous functions from X to $\mathbb{R}$. Note that the convergence in W_1 implies the narrow convergence [2].

If (Ω_1,Σ_1), (Ω_2,Σ_2) are measurable space, m is the probability on Σ_1, $h:\Omega_1\to\Omega_2$ is measurable, then denote by $h_\#m$ the probability on Σ_2 defined by the rule: for $\Gamma\in\Sigma_2$,

$$(h_\#m)(\Gamma)\triangleq m(h^{-1}(\Gamma)).$$

If (X,ρ_X), (Y,ρ_Y) are Polish spaces, η is a finite Borel measure on X, then denote by $\Lambda(X,\eta,Y)$ the set of measures on $X\times Y$ with the marginal on X equal to η. As above, we endow $\Lambda(X,\eta,Y)$ with the narrow convergence. Notice that $\Lambda(X,\eta,Y)$ is itself a Polish space. If $\mu\in\Lambda(X,\eta,Y)$, then $\mu(\cdot|x)$ stands for the disintegration of μ along η, i.e., for each x, $\mu(\cdot|x)$ is a probability on Y, for any $\phi\in C_b(X\times Y)$, the function

$$x\mapsto\int_Y\phi(x,y)\mu(dy|x)$$

is measurable and

$$\int_{X\times X}\phi(x,y)\mu(d(x,y))=\int_X\int_Y\phi(x,y)\mu(dy|x)\eta(dx).$$

The existence of the disintegration is proved in [7, Theorems 7.10.6].

Further, if (Z,ρ_Z) is a Polish space, $\mu\in\Lambda(X,\eta,Y)$, $\nu\in\Lambda(X,\eta,Z)$, then denote by $\mu*\nu$ the measure on $X\times Y\times Z$ defined by the rule: for any $\phi\in C_b(X\times Y\times Z)$,

$$\int_{X\times Y\times Z}\phi(x,y,z)(\mu*\nu)(d(x,y,z))=\int_X\int_X\int_Y\phi(x,y,z)\mu(dy|x)\nu(dz|x)\eta(dx).$$

The existence of $\mu\star\nu$ follows from [7, Theorem 10.6.6.]. Notice that $\mu*\nu\in\Lambda(X,\eta,Y\times Z)$.

Below we consider $\mathbb{R}^d$ as a phase space for the control problem of each player. Further, we regard $\mathbb{R}^{d+1}$ to be an extended phase space. We often denote the elements of $\mathbb{R}^{d+1}$ as pairs $w=(x,z)$, where $x\in\mathbb{R}^d$, $z\in\mathbb{R}$. If $w=(x,z)$, then set $\mathrm{p}(w)\triangleq x$, $\mathrm{q}(w)\triangleq z$. For a fixed $T>0$, let $\mathcal{C}$ stand for $C([0,T],\mathbb{R}^{d+1})$. If $t\in[0,T]$, then denote by e_t, $\hat{e}_t$ the operators from $\mathcal{C}$ onto $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$ respectively defined by the rules: for $w(\cdot)=(x(\cdot),z(\cdot))$,

$$e_t(w(\cdot))\triangleq x(t),\quad\hat{e}_t(w(\cdot))\triangleq w(t).$$

§ 2. Mean field game with a major player

We assume that the system consists of one major player and infinitely many minor player. Let the dynamics of the major player be given by

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u(t)), \tag{2.1}$$

whereas the dynamics of each minor player be given by

$$\frac{d}{dt}y(t) = g(t, x(t), y(t), m(t), u(t), v(t)). \tag{2.2}$$

Here $t \in [0, T]$; $x(t) \in \mathbb{R}^d$, $y(t) \in \mathbb{R}^d$ are states of major and minor players respectively; $u(t) \in U$, $v(t) \in V$ stand for the controls of major and minor players, $m(t) \in \mathcal{P}^1(\mathbb{R}^d)$ is a distribution of minors player at a time t. The initial distribution of the minor players m_0 and the initial state of the major player x_0 are assumed to be given.

It is assumed that the objective function of the major player is

$$I(x(\cdot), m(\cdot), u(\cdot)) \triangleq \xi(x(T), m(T)) + \int_0^T f^0(t, x(t), m(t), u(t))dt;$$

whereas the representative minor player wishes to maximize the outcome equal to

$$J(x(\cdot), y(\cdot), m(\cdot), u(t), v(t)) \triangleq \zeta(x(T), y(T), m(T)) + \int_0^T g^0(t, x(t), y(t), m(t), u(t), v(t))dt.$$

As it was mentioned above, we are seeking for the Stackelberg solution with major player as the leader. This means that the minor players solve their mean field game for the given control of the major player, whereas the major player maximizes his/her outcome subject to the solution of the mean field game of the minor players.

We impose the following conditions.

(C1) The sets U and V are metric compacts.

(C2) The functions f, f^0, g, g^0, ξ and ζ are continuous.

(C3) The functions f, f^0 are Lipschitz continuous w.r.t. x and m.

(C4) The functions g, g^0 are Lipschitz continuous w.r.t. x, y and m.

(C5) The functions f, f^0, g, g^0 are bounded.

(C6) The initial distribution of minor players m_0 lies in $\mathcal{P}^1(\mathbb{R}^d)$.

Now, let us introduce the extended dynamics. For $t \in [0, T]$, $\hat{x}, \hat{y} \in \mathbb{R}^{d+1}$, $m \in \mathcal{P}^1(\mathbb{R}^d)$, $u \in U$, $v \in V$, we put

$$\hat{f}(t, \hat{x}, m, u) \triangleq (f(t, \mathrm{p}(\hat{x}), m, u), f^0(t, \mathrm{p}(\hat{x}), m, u)),$$

$$\hat{g}(t, \hat{x}, \hat{y}, m, u, v) \triangleq (g(t, \mathrm{p}(\hat{x}), \mathrm{p}(\hat{y}), m, u, v), g^0(t, \mathrm{p}(\hat{x}), \mathrm{p}(\hat{y}), m, u, v)).$$

The extended dynamics for the major player is

$$\frac{d}{dt}\hat{x}(t) = \hat{f}(t, \hat{x}(t), m(t), u(t)), \quad \hat{x}(0) = (x_0, 0);$$

whereas the dynamics of the representative minor player is

$$\frac{d}{dt}\hat{y}(t) = \hat{g}(t, \hat{x}(t), \hat{y}(t), m(t), u(t), v(t)).$$

Here, $m(t)$ is a distribution of the minor players in the original phase space at the time t. Note that in this case the major player wishes to maximize the outcome

$$\hat{\xi}(\hat{x}(T), m(T)) \triangleq \xi(\mathrm{p}(\hat{x}(T)), m(T)) + \mathrm{q}(\hat{x}(T)).$$

Whereas the representative minor player wishes to maximize

$$\hat{\zeta}(\hat{x}(T), \hat{y}(T), m(T)) \triangleq \zeta(\mathrm{p}(\hat{x}(T)), \mathrm{p}(\hat{y}(T)), m(T)) + \mathrm{q}(\hat{y}(T)).$$

Obviously, the function $\hat{f}$ is Lipschitz continuous w.r.t. $\hat{x}$ and m, whereas $\hat{g}$ is Lipschitz continuous w.r.t. $\hat{x}$, $\hat{y}$ and m. Moreover, these functions are bounded. This means that, for some non-negative K and C_0,

$$\|\hat{f}(t, \hat{x}, m, u) - \hat{f}(t, \hat{x}', m', u)\| \leq K\|x - x'\| + KW_1(m, m'),$$

$$\|\hat{g}(t, \hat{x}, \hat{y}, m, u, v) - \hat{g}(t, \hat{x}', \hat{y}', m', u, v)\| \leq K\|\hat{x} - \hat{x}'\| + K\|\hat{y} - \hat{y}'\| + KW_1(m, m'),$$

$$\|\hat{f}(t, \hat{x}, m, u)\|, \|\hat{g}(t, \hat{x}, \hat{y}, m, u, v)\| \leq C_0. \tag{2.3}$$

As it was mentioned in Introduction, we consider the Stackelberg solution of the mean field game in the class of relaxed open-loop strategies. Put $\mathcal{U} \triangleq \Lambda([0, T], \lambda, U)$, $\mathcal{V} \triangleq \Lambda([0, T], \lambda, V)$. Notice that $\mathcal{U}$ and $\mathcal{V}$ are the sets of relaxed controls of the major and minor players. Moreover, it is convenient to consider the distributions of motions instead of the flow of probabilities on the phase space. Below let $\chi \in \mathcal{P}(\mathcal{C})$, $\mu \in \mathcal{U}$. Denote by $\hat{x}(\cdot, \chi, \mu)$ the solution of the equation

$$\hat{x}(t) = (x_0, 0) + \int_{[0,t]\times U} \hat{f}(\tau, \hat{x}(\tau), e_{\tau\#}\chi, u)\mu(d(\tau, u)).$$

If, additionally, $y_0 \in \mathbb{R}^d$, $\tilde{x} \in \mathcal{C}$, $\nu \in \mathcal{V}$, then denote by $\hat{y}(\cdot, y_0, \tilde{x}, \chi, \mu, \nu)$ the solution of the equation

$$\hat{y}(t) = (y_0, 0) + \int_{[0,t]\times U\times V} \hat{g}(\tau, \tilde{x}(\tau), \hat{y}(\tau), e_{\tau\#}\chi, u, v)(\mu * \nu)(d(\tau, u, v)).$$

Denote the mapping that assigns to $y_0, \tilde{x}, \chi, \mu$ and ν the motion $\hat{y}(\cdot)$ by Φ.

Further, set $\mathcal{A} \triangleq \Lambda(\mathbb{R}^d, m_0, \mathcal{U})$. Elements of $\mathcal{A}$ are distributions of the minor players' controls. A distribution of controls generates the distribution on $\mathcal{C}$ by the following rule: if $\chi \in \mathcal{P}^1(\mathcal{C})$, $\mu \in \mathcal{U}$, $\gamma \in \mathcal{A}$, $\tilde{x}$ is equal to $\hat{x}(\cdot, \chi, \mu)$, then $\Xi(\chi, \mu, \gamma) \triangleq \Phi(\cdot, \tilde{x}, \chi, \mu, \cdot)_{\#}\gamma$.

Further, we say that χ is a distribution of minor players generated by the distribution of the minor players' controls γ and the major player's control μ if

$$\Xi(\chi, \mu, \gamma) = \chi.$$

Below we denote by $\Psi(\mu, \gamma)$ the distribution of the minor players generated by μ and γ.

P r o p o s i t i o n 2.1. *Given a relaxed control of the major player, and a distribution of the controls of minor players γ, there exists a unique distribution of the minor players $\Psi(\mu, \gamma)$.*

This proposition is proved in Section 3.

D e f i n i t i o n 2.1. We say that γ is an equilibrium distribution of the minor players' controls corresponding to μ if, for $\chi^* = \Psi(\mu, \gamma)$, $\tilde{x} = x(\cdot, \chi^*, \mu)$, γ_*-a.e. (y_0, ν) and any $\nu' \in \mathcal{V}$,

$$\begin{aligned}\hat{\zeta}(\hat{x}(T, \chi^*, \mu), &\hat{y}(T, y_0, \tilde{x}, \chi^*, \mu, \nu), e_{T\#}\chi) \\ &\geq \hat{\zeta}(\hat{x}(T, \chi^*, \mu), \hat{y}(T, y_0, \tilde{x}, \chi^*, \mu, \nu'), e_{T\#}\chi).\end{aligned}$$

Denote by $\mathcal{E}(\mu)$ the set of equilibrium distributions of the minor players' controls corresponding to μ.

D e f i n i t i o n 2.2. We say that $\mu^* \in \mathcal{U}$ and $\gamma^* \in \mathcal{A}$ provide a Stackelberg solution of the mean field game with a major player if

(*a*) $\gamma^* \in \mathcal{E}(\mu^*)$;

(*b*) $\hat{\xi}(\hat{x}(T,\Psi(\mu^*,\gamma^*),\mu^*), e_{T\#}\Psi(\mu^*,\gamma^*)) \geq \sup\{\hat{\xi}(\hat{x}(T,\Psi(\mu,\gamma'),\mu), e_{T\#}\Psi(\mu^*,\gamma)) : \mu \in \mathcal{V}, \gamma \in \mathcal{E}(\mu)\}$.

The main result of the paper is the following.

T h e o r e m 2.1. *There exists at least one Stackelberg solution in the mean filed game with major player.*

This statement is proved in Section 3.

§ 3. Properties of the motions

This section is concerned with the continuous dependence of the motions introduced in the previous section on its parameters.

P r o p o s i t i o n 3.1. *The dependence* $(\chi,\mu) \mapsto \hat{x}(\cdot,\chi,\mu)$ *is continuous.*

P r o o f. Assume that $\mu_n \to \mu$ and $W_1(\chi_n,\chi) \to 0$ as $n \to \infty$. We shall prove that $\|\hat{x}(\cdot,\chi_n,\mu_n) - \hat{x}(\cdot,\chi,\mu)\| \to 0$.

Denote $\tilde{x}_n \triangleq \hat{x}(\cdot,\chi_n,\mu_n)$, $\tilde{x} \triangleq \hat{x}(\cdot,\chi,\mu)$. Recall that K is a Lipschitz constant for $\hat{f}$ and $\hat{g}$. For $t \in [0,T]$, we have that

$$
\begin{aligned}
&\|\tilde{x}_n(t) - \tilde{x}(t)\| \\
&\quad = \Bigg\| \int_{[0,t]\times U} \hat{f}(t,\tilde{x}_n(\tau), e_{\tau\#}\chi_n, u)\mu_n(d(\tau,u)) \\
&\qquad\qquad - \int_{[0,t]\times U} \hat{f}(t,\tilde{x}(\tau), e_{\tau\#}\chi, u)\mu(d(\tau,u)) \Bigg\| \\
&\quad \leq \int_{[0,t]\times U} \|\hat{f}(t,\tilde{x}_n(\tau), e_{\tau\#}\chi_n, u) - \hat{f}(t,\tilde{x}(\tau), e_{\tau\#}\chi, u)\|\mu_n(d(\tau,u)) \\
&\qquad\qquad + \left\| \int_{[0,t]\times U} \hat{f}(\tau,\tilde{x}(\tau), e_{\tau\#}\chi, u)(\mu_n(d(\tau,u) - \mu(d(\tau,u))) \right\| \\
&\quad \leq K\int_0^t \|\tilde{x}_n(\tau) - \tilde{x}(\tau)\|d\tau + KtW_1(\chi_n,\chi) \\
&\qquad\qquad + \left\| \int_{[0,t]\times U} \hat{f}(\tau,\tilde{x}(\tau), e_{\tau\#}\chi, u)(\mu_n(d(\tau,u) - \mu(d(\tau,u))) \right\|.
\end{aligned}
\tag{3.1}
$$

Since $\mu_n \to \mu$, we have that, for any $\theta \in [0,T]$,

$$
\left\| \int_{[0,\theta]\times U} \hat{f}(\tau,\tilde{x}(\tau), e_{\tau\#}\chi, u)(\mu_n(d(\tau,u) - \mu(d(\tau,u))) \right\|
$$

converges to 0 as $n \to \infty$. Let L be a natural number, $l \in \overline{1,L}$, $t_L^l = lT/L$. We have that

$$
r_L^n \triangleq \sup_{l\in\overline{1,L}} \left\| \int_{[0,t_L^l]\times U} \hat{f}(\tau,\tilde{x}(\tau), e_{\tau\#}\chi, u)(\mu_n(d(\tau,u) - \mu(d(\tau,u))) \right\|
$$

converges to 0 as $n \to \infty$ for any natural L. Further, we have that

$$
\left\| \int_{[0,t]\times U} \hat{f}(\tau,\tilde{x}(\tau), e_{\tau\#}\chi, u)(\mu_n(d(\tau,u) - \mu(d(\tau,u))) \right\| \leq r_L^n + 2C_0T/L.
$$

Here C_0 satisfies (2.3). This and (3.1) imply

$$\|\tilde{x}_n(t) - \tilde{x}(t)\| \leq K \int_0^t \|\tilde{x}_n(\tau) - \tilde{x}(\tau)\| d\tau + KtW_1(\chi_n, \chi) + r_L^n + 2C_0T/L.$$

Using the Gronwall's inequality, we get

$$\|\tilde{x}_n(t) - \tilde{x}(t)\| \leq e^{KT}(KTW_1(\chi_n, \chi) + r_L^n + 2C_0T/L).$$

Since $W_1(\chi_n, \chi) \to 0$ and $r_L^n \to 0$ as $n \to$ for any natural L, we conclude that

$$\lim_{n\to\infty} \|\tilde{x}_n - \tilde{x}\| \leq 2e^{KT}C_0T/L.$$

Further, letting $L \to \infty$, we get that

$$\lim_{n\to\infty} \|\tilde{x}_n - \tilde{x}\| = 0.$$

□

P r o p o s i t i o n 3.2. *The dependence* $(y_0, \tilde{x}, \chi, \mu, \nu) \mapsto \hat{y}(\cdot, y_0, \tilde{x}, \chi, \mu, \nu)$ *is continuous.*

This proposition is proved in the same way as the previous one.

Let $\mathcal{Y}$ be the set of probabilities χ on $\mathcal{C}$ satisfying the following properties:

1. $\hat{e}_{0\#}\chi$ is concentrated on $\mathbb{R}^d \times \{0\}$;
2. $e_{0\#}\chi = m_0$;
3. χ is concentrated on the set of C_0-Lipschitz continuous functions from $[0, T]$ to $\mathbb{R}^{d+1}$.

Since $m_0 \in \mathcal{P}^1(\mathcal{C})$, we have that $\mathcal{Y}$ is a compact set in $\mathcal{P}^1(\mathcal{C})$ [2]. Additionally, notice that, for any $\chi \in \mathcal{P}^1(\mathcal{C})$, $\mu \in \mathcal{U}$, $\gamma \in \mathcal{A}$,

$$\Xi(\chi, \mu, \gamma) \in \mathcal{Y}. \tag{3.2}$$

P r o p o s i t i o n 3.3. *The dependence* $(\mathcal{Y}, \mathcal{U}, \mathcal{A}) \ni (\chi, \mu, \gamma) \mapsto \Xi(\chi, \mu, \gamma) \in \mathcal{Y}$ *is continuous.*

P r o o f. Let $W_1(\chi_n, \chi_*) \to 0$, $\mu_n \to \mu_*$, $\gamma_n \to \gamma_*$ as $n \to \infty$. Set $\tilde{x}_n \triangleq \hat{x}(\cdot, \chi_n, \mu_n)$, $\tilde{x}_* \triangleq \hat{x}(\cdot, \chi_*, \mu_*)$. By Proposition 3.1,

$$\|\tilde{x}_n - \tilde{x}_*\| \to 0 \text{ as } n \to \infty. \tag{3.3}$$

Further, let $\varrho_n \in \mathcal{P}^1(\mathbb{R}^d \times \mathcal{U} \times \mathcal{V})$ be such that, for any $\phi \in C_b(\mathbb{R}^d \times \mathcal{U} \times \mathcal{V})$,

$$\int_{\mathbb{R}^d\times\mathcal{U}\times\mathcal{V}} \phi(x, \mu, \nu)\varrho_n(d(x, \mu, \nu)) = \int_{\mathbb{R}^d\times\mathcal{V}} \phi(x, \mu_n, \nu)\gamma_n(d(x, \nu)).$$

Analogously, define $\varrho_* \in \mathcal{P}^1(\mathbb{R}^d \times \mathcal{U} \times \mathcal{V})$ by the rule:

$$\int_{\mathbb{R}^d\times\mathcal{U}\times\mathcal{V}} \phi(x, \mu, \nu)\varrho_*(d(x, \mu, \nu)) = \int_{\mathbb{R}^d\times\mathcal{V}} \phi(x, \mu_*, \nu)\gamma_*(d(x, \nu)).$$

The existence of the probabilities ϱ_n and ϱ_* follows from [7, Theorem 7.10.6]. Notice that

$$\varrho_n \to \varrho_* \text{ as } n \to \infty. \tag{3.4}$$

We have that $\Xi(\chi_n, \mu_n, \gamma_n) = \Phi(\cdot, \tilde{x}_n, \chi_n, \cdot, \cdot)_{\#}\varrho_n$, $\Xi(\chi_*, \mu_*, \gamma_*) = \Phi(\cdot, \tilde{x}_*, \chi_*, \cdot, \cdot)_{\#}\varrho_*$. Thus,

$$\begin{aligned} W_1(\Xi(\chi_n, \mu_n, \gamma_n), \Xi(\chi_*, \mu_*, \gamma_*)) \leq{} & W_1(\Phi(\cdot, \tilde{x}_n, \chi_n, \cdot, \cdot)_{\#}\varrho_n, \Phi(\cdot, \tilde{x}_*, \chi_*, \cdot, \cdot)_{\#}\varrho_n) \\ & + W_1(\Phi(\cdot, \tilde{x}_*, \chi_*, \cdot, \cdot)_{\#}\varrho_n, \Phi(\cdot, \tilde{x}_*, \chi_*, \cdot, \cdot)_{\#}\varrho_*). \end{aligned} \tag{3.5}$$

Choosing the plan between $\Phi(\cdot,\tilde{x}_n,\chi_n,\cdot,\cdot)_{\#}\varrho_n$ and $\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n$ equal to $(\Phi(\cdot,\tilde{x}_n,\chi_n,\cdot,\cdot),\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot))_{\#}\varrho_n$, we get

$$\begin{aligned} W_1(\Phi(\cdot,\tilde{x}_n,&\chi_n,\cdot,\cdot)_{\#}\varrho_n,\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n) \\ &\leq \int_{\mathbb{R}^{d+1}\times\mathcal{U}\times\mathcal{V}} \|\Phi(y_0,\tilde{x}_n,\chi_n,\mu,\nu)-\Phi(y_0,\tilde{x}_*,\chi_*,\mu,\nu)\|\varrho_n(d(y_0,\mu,\nu)). \end{aligned} \tag{3.6}$$

To estimate $\|\Phi(y_0,\tilde{x}_n,\chi_n,\mu,\nu) - \Phi(y_0,\tilde{x}_*,\chi_*,\mu,\nu)\|$ recall that $\|\Phi(y_0,\tilde{x}_n,\chi_n,\mu,\nu) - \Phi(y_0,\tilde{x}_*,\chi_*,\mu,\nu)\|$ is equal to $\sup_{t\in[0,T]}\|\hat{y}_n(t,y_0,\tilde{x}_n,\chi_n,\mu,\nu)-\hat{y}(t,y_0,\tilde{x}_*,\chi_*,\mu,\nu)\|$. Further,

$$\begin{aligned} \|\hat{y}_n(t,y_0,\tilde{x}_n,\chi_n,\mu,\nu)-\hat{y}(t,y_0,\tilde{x}_*,\chi_*,\mu,\nu)\| \\ \leq K\int_0^t \|\hat{y}_n(\tau,y_0,\tilde{x}_n,\chi_n,\mu,\nu)-\hat{y}(\tau,y_0,\tilde{x}_*,\chi_*,\mu,\nu)\|d\tau + Kt\|\tilde{x}_n-\tilde{x}_*\| + KtW_1(\chi_n,\chi_*). \end{aligned}$$

The Gronwall's inequality yields that $\sup_{t\in[0,T]}\|\hat{y}_n(t,y_0,\tilde{x}_n,\chi_n,\mu,\nu)-\hat{y}(t,y_0,\tilde{x}_*,\chi_*,\mu,\nu)\|$ converges to zero uniformly w.r.t. y_0, μ, and ν. Thus, by (3.6) we have that

$$W_1(\Phi(\cdot,\tilde{x}_n,\chi_n,\cdot,\cdot)_{\#}\varrho_n,\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n) \to 0 \text{ as } n\to\infty. \tag{3.7}$$

To prove that $W_1(\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n,\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_*)\to 0$ as $n\to\infty$, recall that ϱ_n converges to ϱ_* narrowly. Consider the compact set $\mathcal{C}_R \triangleq \{\hat{y}\in\mathcal{C} : \|\hat{y}(0)\|\leq R\}$. We have that the restriction of $\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n$ on $\mathcal{C}_R$ converges narrowly to the restriction of $\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_*$ on $\mathcal{C}_R$. Further, for any $\phi\in C_b(\mathcal{C})$,

$$\left|\int_{\mathcal{C}\setminus\mathcal{C}_R}\phi(\hat{y})\chi_n(d\hat{y})\right|,\ \left|\int_{\mathcal{C}\setminus\mathcal{C}_R}\phi(\hat{y})\chi(d\hat{y})\right| \leq \|\phi\|m_0(\mathbb{R}^d\setminus B_R),$$

where B_R stands for the ball of radius R centered at the origin. Therefore, we conclude that $\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n$ converges narrowly to $\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_*$. Since $\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n,\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_*$ are concentrated on $\mathcal{Y}$ and $\mathcal{Y}$ is compact, we conclude that $W_1(\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_n,\Phi(\cdot,\tilde{x}_*,\chi_*,\cdot,\cdot)_{\#}\varrho_*)\to 0$ as $n\to\infty$.

Hence, taking into account (3.5) and (3.7), we get the consequence of the proposition. □

Proof of Proposition 2.1. The existence of the $\Psi(\mu,\gamma)$ follows from the compactness of $\mathcal{Y}$, (3.2), the continuity of the mapping $\mathcal{P}(\mathcal{C})\ni\chi\mapsto\Xi(\chi,\mu,\gamma)\in\mathcal{Y}$ for all μ and γ and the famous Schauder fixed-point theorem [1, Corollary 17.56].

To prove the uniqueness, let us consider, for $\theta\in[0,T]$, the space $\mathcal{C}^\theta \triangleq C([0,\theta],\mathbb{R}^{d+1})$. Notice that $\mathcal{C}=\mathcal{C}^T$. Further, denote by r^θ the natural projection from $\mathcal{C}$ on $\mathcal{C}^\theta$. Now assume that there exist two probabilities χ_1 and χ_2 such that $\chi_i=\Xi(\chi_i,\mu,\gamma)$. Let $\tilde{x}_i=\hat{x}(\cdot,\mu,\chi_i)$. Notice that there exists a constant c_1 such that

$$\|\tilde{x}_1(t)-\tilde{x}_2(t)\|\leq c_1 W_1(\mathrm{r}^t{}_{\#}\chi_1,\mathrm{r}^t{}_{\#}\chi_2). \tag{3.8}$$

Using the Lipschitz continuity of the function g and (3.8), we have that

$$\begin{aligned} \|\hat{y}(t,y_0,\tilde{x}_1,\chi_1,\mu,\nu)&-\hat{y}(t,y_0,\tilde{x}_2,\chi_2,\mu,\nu)\| \\ &\leq K\int_0^t\|\hat{y}(\tau,y_0,\tilde{x}_1,\chi_1,\mu,\nu)-\hat{y}(\tau,y_0,\tilde{x}_2,\chi_2,\mu,\nu)\|d\tau \\ &\qquad + K\int_0^t\|\tilde{x}_1(\tau)-\tilde{x}_2(\tau)\|d\tau + K\int_0^t W_1(\mathrm{r}^\tau{}_{\#}\chi_1,\mathrm{r}^\tau{}_{\#}\chi_2) \\ &\leq K\int_0^t\|\hat{y}(\tau,y_0,\tilde{x}_1,\chi_1,\mu,\nu)-\hat{y}(\tau,y_0,\tilde{x}_2,\chi_2,\mu,\nu)\|d\tau \\ &\qquad + K(c_1+1)\int_0^t W_1(\mathrm{r}^\tau{}_{\#}\chi_1,\mathrm{r}^\tau{}_{\#}\chi_2)d\tau. \end{aligned} \tag{3.9}$$

The Gronwall's inequality gives the following estimate, for some constant c_2:

$$\|\hat{y}(t, y_0, \tilde{x}_1, \chi_1, \mu, \nu) - \hat{y}(t, y_0, \tilde{x}_2, \chi_2, \mu, \nu)\| \leq c_2 \int_0^t W_1(\mathrm{r}^\tau{}_\#\chi_1, \mathrm{r}^\tau{}_\#\chi_2) d\tau.$$

Therefore, we have that, for any $\theta \in [0, T]$,

$$\sup_{t\in[0,\theta]} \|\hat{y}(t, y_0, \tilde{x}_1, \chi_1, \mu, \nu) - \hat{y}(t, y_0, \tilde{x}_2, \chi_2, \mu, \nu)\| \leq c_2 \int_0^\theta W_1(\mathrm{r}^\tau{}_\#\chi_1, \mathrm{r}^\tau{}_\#\chi_2) d\tau.$$

Integrating this inequality w.r.t. the probability γ, we get the estimate

$$W_1(\mathrm{r}^\theta{}_\#\chi_1, \mathrm{r}^\theta{}_\#\chi_2) \leq c_2 \int_0^\theta W_1(\mathrm{r}^\tau{}_\#\chi_1, \mathrm{r}^\tau{}_\#\chi_2) d\tau.$$

Using the Gronwall's inequality once more time, we conclude that, for any $\theta \in [0, T]$,

$$W_1(\mathrm{r}^\theta{}_\#\chi_1, \mathrm{r}^\theta{}_\#\chi_2) \leq 0.$$

Since $\mathrm{r}^T = \mathrm{id}$, we obtain the uniqueness. □

Proposition 3.4. *For any $\mu \in \mathcal{U}$, the set $\mathcal{E}(\mu)$ is nonempty. Moreover, the mapping $\mu \mapsto \mathcal{E}(\mu)$ is upper semicontinuous.*

Proof. Given $\gamma \in \mathcal{A}$, let us construct the multifunction $\Gamma[\mu, \gamma]$ taking values in $\mathcal{A}$ by the following rule. First, set $\chi \triangleq \Psi(\mu, \gamma)$. Further, let $\mathcal{B}[\mu, \gamma]$ be the set of pairs $(w, \nu_*) \in \mathbb{R}^d \times \mathcal{V}$ such that, for any $\nu \in \mathcal{V}$,

$$\hat{\zeta}(\hat{x}(T, \chi, \mu), \hat{y}(T, w, \chi, \mu, \nu_*), e_{T\#}\chi) \geq \hat{\zeta}(\hat{x}(T, \chi, \mu), \hat{y}(T, w, \chi, \mu, \nu), e_{T\#}\chi). \quad (3.10)$$

Put $\Gamma[\mu, \gamma]$ be equal to the set of all distributions $\gamma' \in \mathcal{A}$ such that $\mathrm{supp}(\gamma') \subset \mathcal{B}[\mu, \gamma]$.

Notice that $\gamma \in \mathcal{E}(\mu)$ iff

$$\gamma \in \Gamma[\mu, \gamma].$$

Let us show that Γ is upper semicontinuous. Indeed, assume that $\mu_n \to \mu_*$, $\gamma_n \to \gamma_*$, $\gamma'_n \in \Gamma[\mu_n, \gamma_n]$, $\gamma'_n \to \gamma'_*$. We shall prove that $\mathrm{supp}(\gamma'_*) \subset \mathcal{B}[\mu_*, \gamma_*]$. Pick $(w_*, \nu_*) \in \mathrm{supp}(\gamma'_*)$. By [2, Proposition 5.1.8] we have that there exists $(w_n, \nu_n) \in \mathrm{supp}(\gamma'_n)$ such that $w_n \to w_*$, $\nu_n \to \nu_*$. Since $\mathrm{supp}(\gamma'_n) \subset \mathcal{B}[\mu_n, \gamma_n]$, we have that, for $\chi_n \triangleq \Psi(\mu_n, \gamma_n)$, and any $\nu \in \mathcal{V}$,

$$\begin{aligned}\hat{\zeta}(\hat{x}(T, \chi_n, \mu_n), &\hat{y}(T, w_n, \chi_n, \mu_n, \nu_n), e_{T\#}\chi_n) \\ &\geq \hat{\zeta}(\hat{x}(T, \chi_n, \mu_n), \hat{y}(T, w_n, \chi_n, \mu_n, \nu), e_{T\#}\chi_n).\end{aligned}$$

Using the continuity of all functions and passing to the limit, we get that, for any $\nu \in \mathcal{V}$,

$$\begin{aligned}\hat{\zeta}(\hat{x}(T, \chi_*, \mu_*), &\hat{y}(T, w_*, \chi_*, \mu_*, \nu_*), e_{T\#}\chi_*) \\ &\geq \hat{\zeta}(\hat{x}(T, \chi_*, \mu_n), \hat{y}(T, w_n, \chi_*, \mu_*, \nu), e_{T\#}\chi_*).\end{aligned}$$

This means that $\gamma'_* \in \Gamma[\mu_*, \gamma_*]$.

Further, let us show that $\Gamma[\mu, \gamma]$ is nonempty. To this end, given $w \in \mathbb{R}^d$, pick

$$\bar{\nu}(w) \in \mathrm{Argmax}\{\hat{\zeta}(\hat{x}(T, \chi, \mu), \hat{y}(T, w, \chi, \mu, \nu), e_{T\#}\chi) : \nu \in \mathcal{V}\}. \quad (3.11)$$

Here we denote $\chi \triangleq \Psi(\mu, \gamma)$. Without loss of generality one can choose the function $\bar{\nu}$ to be measurable. Define γ' by the rule: for any $\phi \in C_b(\mathbb{R}^d \times \mathcal{V})$,

$$\int_{\mathbb{R}^{d+1}\times\mathcal{V}} \phi(w, \nu)\gamma'(d(w, \nu)) \triangleq \int_{\mathbb{R}^d} \phi(w, \bar{\nu}(w)) m_0(dw).$$

Obviously, $\mathrm{supp}(\gamma') = \{(w, \bar{\nu}(w)) : w \in \mathrm{supp}(m_0)\}$. Therefore, $\gamma' \in \mathcal{A}$. Furthermore, (3.11) implies that inequality (3.10) is fulfilled for every $(w, \nu_*) \in \mathrm{supp}(\gamma')$ and any ν. Hence, $\gamma' \in \Gamma[\mu, \gamma]$. Additionally, the set $\Gamma[\mu, \gamma]$ is convex by the construction. Thus, by the Fan–Gliksberg theorem [1, Corollary 17.55], for each $\mu \in \mathcal{U}$, there exists a fixed point of the multifunction $\gamma \rightrightarrows \Gamma[\mu, \gamma]$. Since $\mathcal{E}(\mu) = \{\gamma \in \mathcal{A} : \gamma \in \Gamma[\mu, \gamma]\}$ and the multifunction Γ is upper semicontinuous, we have that the $\mathcal{E}$ is also upper semicontinuous. □

P r o o f o f T h e o r e m 2.1. By Definition 2.2, we have that (μ^*, γ^*) is the Stackelberg solution iff

$$(\mu^*, \gamma^*) \in \mathrm{Argmax}\{\hat{\hat{\xi}}(\hat{x}(T, \Psi(\mu, \gamma), \mu), e_{T\#}\Psi(\mu, \gamma)) : \mu \in \mathcal{U}, \gamma \in \mathcal{E}(\mu)\}.$$

Notice that, by Proposition 3.4, the set

$$\mathrm{gr}(\mathcal{E}) \triangleq \{(\mu, \gamma) : \mu \in \mathcal{U}, \gamma \in \mathcal{E}\}$$

is compact. Additionally, by construction, we have the set $\mathcal{U}$ is also compact. Thus, using the continuity of the functions Ψ, $\hat{x}$ and $\hat{\hat{\xi}}$, we get the existence result for the Stackelberg solution in the first-order mean filed game with a major player. □

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Решение по Штакельбергу игры среднего поля с ведущим игроком

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Статья посвящена исследованию системы большого числа однотипных игроков, взаимодействующих с внешним окружением. Мы моделируем это окружение как ведущего (внешнего) игрока. Основное предположение нашей модели состоит в том, что малые игроки могут влиять друг на друга и на ведущего игрока лишь через те или иные усредненные характеристики. Подобные модели носят название игр среднего поля с ведущим игроком. Мы предполагаем, что время непрерывно и динамика ведущего и малых игроков описывается обыкновенными дифференциальными уравнениями. Мы рассматриваем решение по Штакельбергу с ведущим игроком–лидером, то есть предполагается, что ведущий игрок объявляет заранее свое управление малым игрокам. Основной результат статьи состоит в доказательстве существования решения по Штакельбергу игры среднего поля с ведущим игроком в классе обобщенных программных управлений.

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