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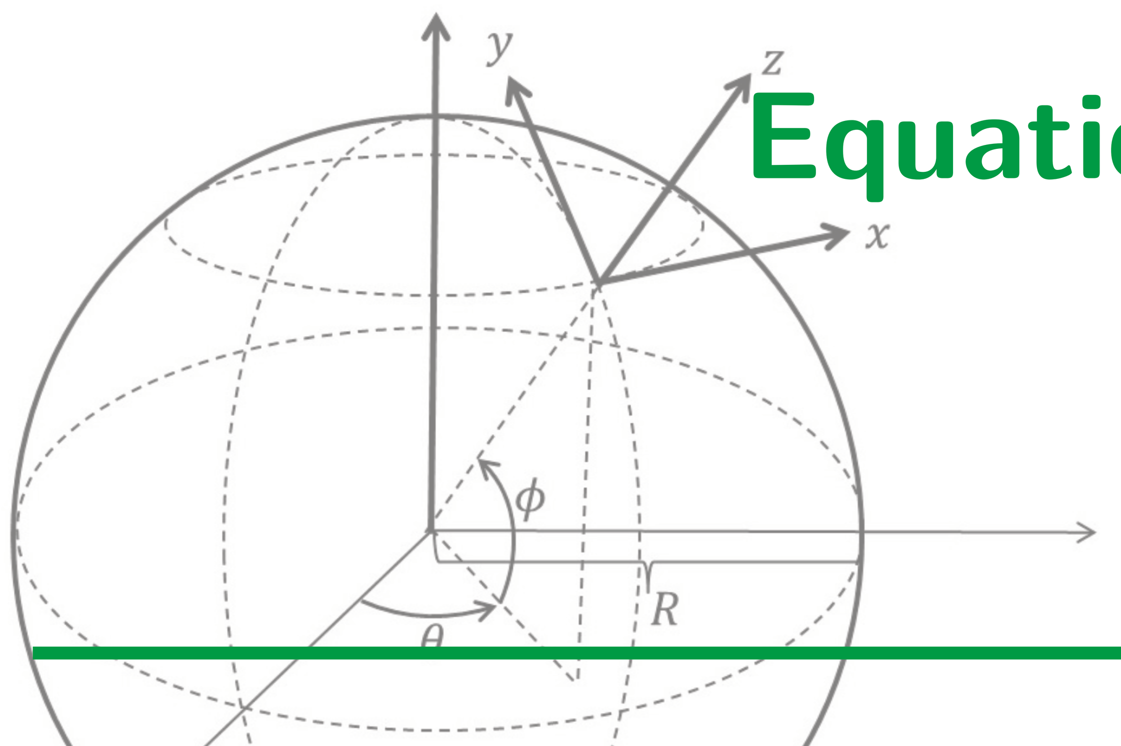

Equations of Motion in a Rotating Noninertial Reference Frame

The Coriolis Force

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Objectives

To demonstrate how "fictitious" forces arise from a frame of reference that isn't in a state of inertia, a familiar model is constructed in the form of a rotating planet.

- Describe the motion of a sphere rotating about a stationary axis
- Determine the equations of motion of an object moving in the frame of the planet's surface
- Test the solutions with expectations under different parameters

Introduction

Newton's first law of mechanics states that *a body remains at rest or in uniform motion unless acted upon by a force.* Though not explicitly stated, this law defines an inertial reference frame. If a reference frame is subject to acceleration intrinsic to its motion, like the surface of a rotating sphere, it is a *noninertial* frame of reference. Seemingly measurable forces that manifest from this frame are termed *fictitious* forces and are artificial corrections required due to attempts to extend Newton's equations to a noninertial system [1].

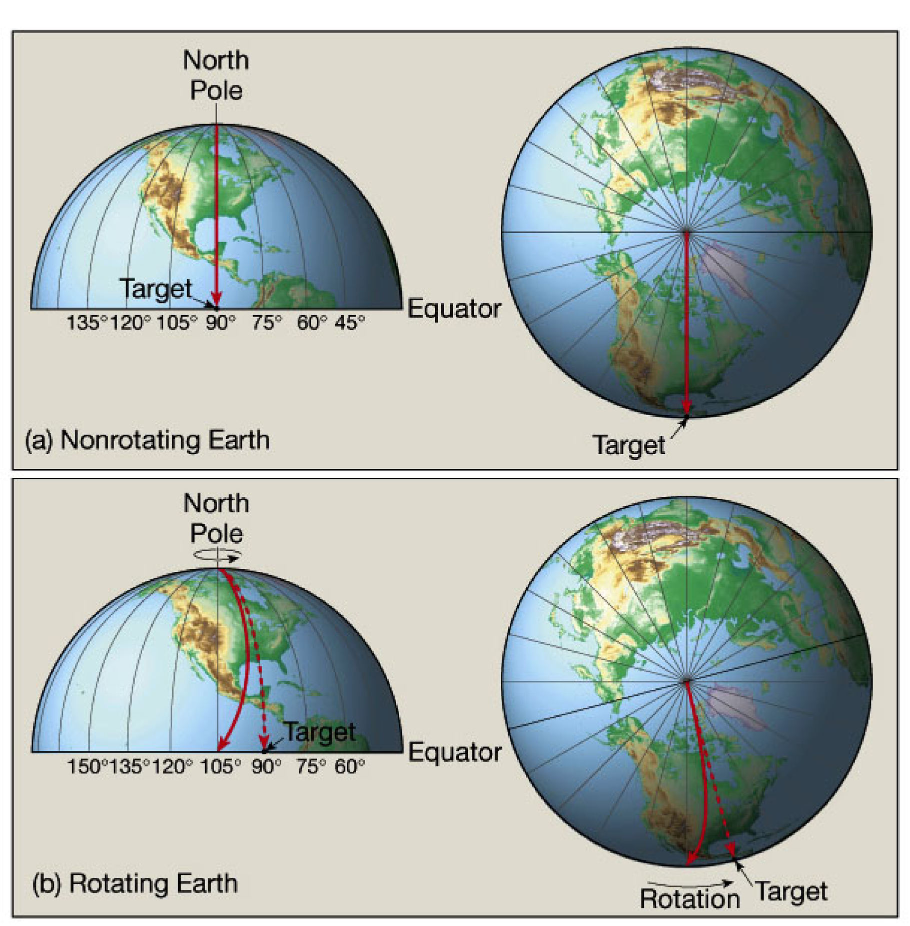

Figure 1: The path an object traces changes when experiencing "fictitious" forces induced by a noninertial frame of reference. A target due south on a globe (a) is deflected from a straight path (b) by the rotational motion of the globe. © Encyclopædia Britannica

Homogeneous Solution

When solving differential equations, a general solution for the equation $\frac{d\vec{v}}{dt} = \mathbf{A}\vec{v}$ is,

$$\vec{v}_h = e^{\mathbf{A}t}\vec{v}_o\,. \tag{1}$$

As a differential vector equation, $\mathbf{A}$ must be a matrix. Interpreting the equation as written can be done by diagonalization and power series expansion such that,

$$e^{t\mathbf{A}} = \mathbf{V}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t}\end{bmatrix}\mathbf{V}^{-1}\,, \tag{2}$$

where matrix $\mathbf{V}$ is a composite of the eigenvectors of $\mathbf{A}$, $\mathbf{V}^{-1}$ is its inverse, and λ_n are the eigenvalues of $\mathbf{A}$.

Particular Solution

The particular solution can be represented with a linear operator such that $L_{op}\vec{v}_p = \vec{g}(t)$, and solved using a Green's function.

$$L_{op}G(t,t_o) = \delta(t-t_o) \tag{3}$$

A function can be rewritten with a Dirac delta so that,

$$\begin{aligned} L_{op}\vec{v}_p &= \int_{-\infty}^{+\infty} \delta(t-t_o)\mathbf{I}\,\vec{g}(t_o)\,dt_o \\ \vec{v}_p &= \int_{-\infty}^{+\infty} \mathbf{G}(t,t_o)\,\vec{g}(t_o)\,dt_o \\ &= \int_0^t e^{\mathbf{A}(t-t_o)}\vec{g}(t_o)\,dt_o \end{aligned}$$

Complete Solution

$$\begin{aligned} v_x &= g\frac{s_\alpha \sin^2\omega t}{\omega} + v_{xo}\cos 2\omega t + v_{yo}c_\alpha \sin 2\omega t - v_{zo}s_\alpha \sin 2\omega t & \alpha &= 90^\circ - \phi \\ v_y &= g\frac{s_{2\alpha}(\sin 2\omega t - 2\omega t)}{4\omega} - v_{xo}c_\alpha \sin 2\omega t + v_{yo}\left(c_\alpha^2\cos 2\omega t + s_\alpha^2\right) + v_{zo}s_{2\alpha}\sin^2\omega t & c_\alpha &= \cos\alpha \\ v_z &= -gt\left(c_\alpha^2 + s_\alpha^2 \operatorname{sinc} 2\omega t\right) + v_{xo}s_\alpha \sin 2\omega t + v_{yo}s_{2\alpha}\sin^2\omega t + v_{zo}\left(c_\alpha^2 + s_\alpha^2\cos 2\omega t\right) & s_\alpha &= \sin\alpha \end{aligned}$$

Change of Frame Transformation

If the rotating sphere is embedded in a "fixed" frame the equation that relates measurements from an observer rotating on the surface to that of a celestial, "fixed", observer is as follows:

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{r}}{dt}\right)_{\text{rotating}} + \vec{\omega}\times\vec{r} \tag{4}$$

The velocity, $\frac{d\vec{r}}{dt}$, as measured by the fixed observer is dependant on the angular velocity of the sphere. To derive the *fictitious* forces, the same process can be carried out to determine acceleration, $\frac{d\vec{v}}{dt}$, corrections between the frames.

$$\vec{F}_{\text{effective}} = \vec{F} - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_r \tag{5}$$

Each term in equation 5 can be interpreted physically as,

$\vec{F}$: sum of the forces acting on the object as measured in the fixed system

$-m\dot{\vec{\omega}}\times\vec{r}$: result of rotational acceleration

$-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$: centrifugal force

$-2m\vec{\omega}\times\vec{v}_r$: Coriolis force .

Results

Explicit values for the aforementioned equations are as follows:

$$\vec{\omega} = \begin{pmatrix} 0 \\ \omega\sin\alpha \\ \omega\cos\alpha \end{pmatrix} \qquad \vec{\omega}\times\vec{v}_r = \omega\begin{bmatrix} 0 & -c_\alpha & s_\alpha \\ c_\alpha & 0 & 0 \\ -s_\alpha & 0 & 0 \end{bmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$e^{t\mathbf{A}} = \begin{bmatrix} \cos 2\omega t & c_\alpha \sin 2\omega t & -s_\alpha \sin 2\omega t \\ -c_\alpha \sin 2\omega t & c_\alpha^2\cos 2\omega t + s_\alpha^2 & s_{2\alpha}\sin^2\omega t \\ s_\alpha \sin 2\omega t & s_{2\alpha}\sin^2\omega t & c_\alpha^2 + s_\alpha^2\cos 2\omega t \end{bmatrix} \quad L_{op} = \left(\frac{d}{dt} + 2\vec{\omega}\times\vec{v}_r\right)$$

To test whether these solutions agree with expectations, α and ω can be altered. If an object is dropped over a pole ($\alpha = 0$) it should only be effected by gravity. If dropped at the equator ($\alpha = \frac{\pi}{2}$) an additional easterly velocity should occur. If there is no rotation, only the gravity term should survive. Additionally, *in the Northern Hemisphere a particle projected in a horizontal plane will be directed towards the right of the particle's motion* [1]. All deflections in the Southern Hemisphere are opposite to the Northern. For the velocity vector function, $\vec{v}(\vec{v_o},\alpha,\omega)$, these constraints result in:

$$\vec{v}(0,0,\omega) = \begin{pmatrix} 0 \\ 0 \\ -gt \end{pmatrix} \qquad \vec{v}\left(\frac{\pi}{2},0,\omega\right) = \begin{pmatrix} g\omega^{-1}\sin^2\omega t \\ 0 \\ -gt\operatorname{sinc} 2\omega t \end{pmatrix}$$

$$\vec{v}(\vec{v_o},\alpha,0) = \begin{pmatrix} v_{xo} \\ v_{yo} \\ v_{zo} - gt \end{pmatrix}$$

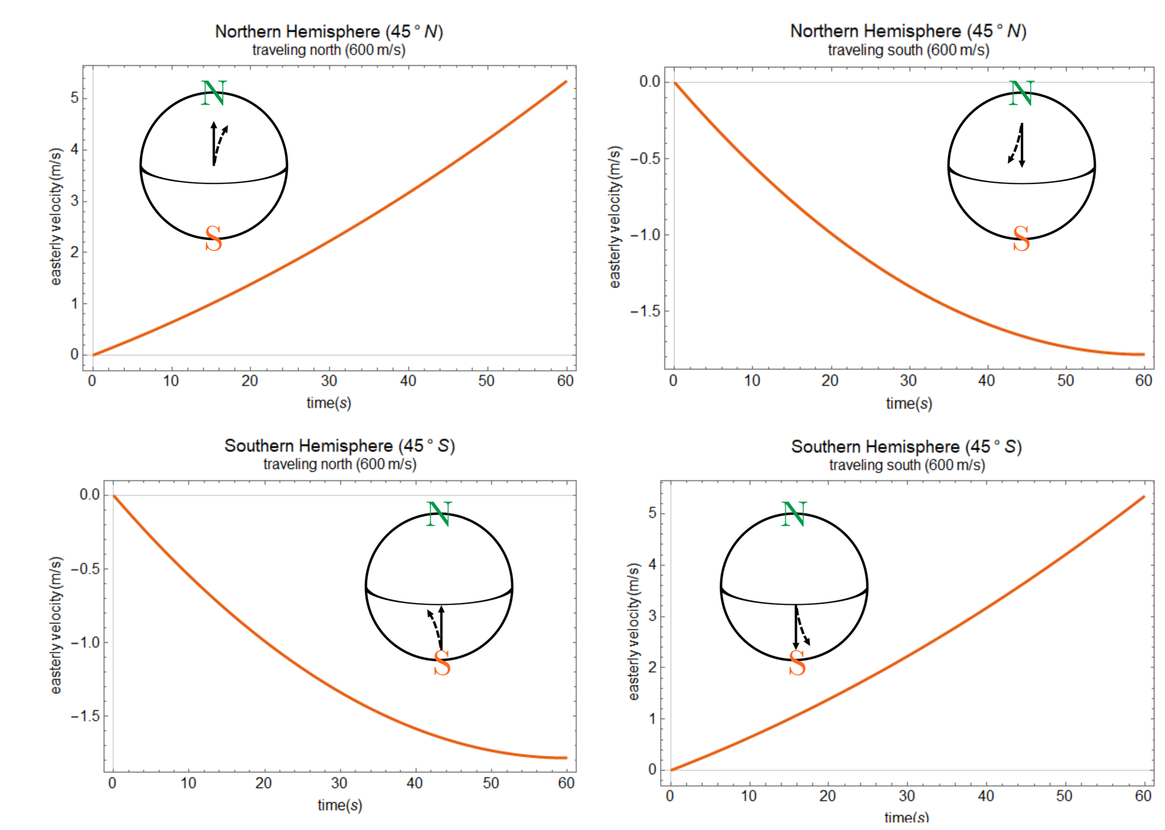

Figure 2: Easterly deflection demonstrated by the change in velocity, v_x, over one minute. Motion in the Northern Hemisphere deflects to the right, while motion in the southern hemisphere deflects left.

Additional Information

The southerly deflection is on the order of a million times smaller than the easterly deflection. Despite many attempts, no credible evidence that the southerly deflection has been detected has been correctly measured [2].

References

[1] S. T. Thornton and J. B Marion. *Classical Dynamics of Particles and Systems.* Brooks/Cole, Cengage Learning, 5th edition, 2008.

[2] M. S. Tiersten and H. Soodak. Dropped objects and other motions relative to the noninertial earth. *American Journal of Physics*, 68(2):129–142, February 2000.

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