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Properties of the odd Generalized Exponential-Exponential Distribution

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Abstract

The main aim of this paper is to construct and study a new continuous distribution, called the odd generalized exponential-exponential distribution. Some of the mathematical properties and graphical description of the new distribution are obtained and discussed. In addition, the density functions of the smallest and largest order statistics of the odd generalized exponential-exponential distribution are obtained.

Keywords: Generalized exponential distribution; Odd generalized distribution.

1. Introduction

It is well known that the generalization of a probability distribution makes it richer and more flexible for modeling data. In addition, It is, also, well known that the exponential distribution can have only constant hazard rate function whereas and generalized exponential distributions can have monotone failure rate functions. Several developments according to the generalization of the exponential distribution are pioneered by [1, 2, 3, 4, 5, 6] among others. Reference [7] proposed a new family of continuous distributions referred to as the odd generalized exponential family, whose hazard rate function might be increasing, decreasing. It includes as a special case the widely known exponentiated-Weibull distribution and mentioned three special models within the family density function is expressed as a combination of exponentiated densities supported constant baseline distribution. They derived specific expressions for the ordinary and incomplete moments, quantile and generating functions, Bonferroni and Lorenz curves, Shannon and Rényi entropies and order statistics for the primary time obtained the generating function of the Fréchet distribution and projected characterizations of the family.

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Moreover, they developed the odd generalized exponential family (OGE); this is done by considering the cumulative distribution function (CDF) of the GE model, given by:

$$F_{GE}(x; \alpha, \lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha}; \qquad (1)$$

Where α and λ are positive parameters; [7]replaced x by $W(x; \eta)/\,\overline{W}(x; \eta)$ wherever $W(x; \eta)$ and $\overline{W}(x; \eta)$ are the CDF and the reliability function of a parent distribution with a parameter vector η. That is, the CDF of the OGE family of distributions ca be written as:

$$F_{OGE}(x; \alpha, \lambda, \eta) = \left(1 - e^{-\lambda \frac{W(x;\eta)}{\overline{W}(x;\eta)}}\right)^{\alpha}. \qquad (2)$$

Hence, the probability density function (PDF) of the OGE family of distributions is then given by:

$$f_{OGE}(x; \alpha, \lambda, \eta) = \frac{\alpha\lambda r(x;\eta)}{\overline{W}(x;\eta)} e^{-\lambda \frac{W(x;\eta)}{\overline{W}(x;\eta)}} \left(1 - e^{-\lambda \frac{W(x;\eta)}{\overline{W}(x;\eta)}}\right)^{\alpha-1}. \qquad (3)$$

Wherever $r(x; \eta)$ is the baseline PDF.

The rest of the paper is organized as follows: In Section 2, the cumulative distribution function, density, reliability and hazard rate functions of the odd generalized exponential-exponential (OGE-E(Θ)) where Θ is a vector of parameters distribution are defined. Moreover, graphical description of the PDF, CDF and hazard rate funciton of the OGE-E(Θ) disrtibution are provied. In Section 3, some of the statistical properties including the quantile and mode function, skewness, kurtosis, moments and moment generating function are studied. In Section 4, the distribution of the order statistics for the OGE-E(Θ) distribution is developped.Finally, the conclusions are discussed in Section 5.

2. The OGE-E(Θ) distribution

In this section, the construction of the new three parameters distribution called Odd Generalized Exponential-Exponential OGE-E(Θ) where the vector Θ is defined in the form Θ = (γ, λ, α) is considered; where the PDF, CDF, reliability and hazard rate functions are presented. Moreover, the graphical of the OGE-E(Θ) distribution for different values of the parameters is studied.

2.1. Specifications

A random variable X is said to have the OGE-E(Θ) distribution if its cumulative distribution function CDF given as**:**

$$F_{OGE-E}(x; \theta) = \left(1 - e^{-\lambda(e^{\gamma x}-1)}\right)^{\alpha}, x > 0 \qquad (4)$$

Which is obtained from the CDF of the OGE family of distributions, given by equation (2), after putting

$W(x;\eta) = 1 - e^{-\gamma x}$, $\overline{W}(x;\eta) = 1 - W(x;\eta) = e^{-\gamma x}$, which are the CDF and the survival function of the exponential distribution, where $\eta = \gamma$.

The corresponding PDF of the OGE-E(Θ) distribution is then given by

$$f_{OGE-E}(x;\Theta) = \frac{\alpha\lambda\gamma}{e^{-\gamma x}} e^{-\lambda(e^{\gamma x}-1)}\left(1 - e^{-\lambda(e^{\gamma x}-1)}\right)^{\alpha-1}, x > 0 \quad (5)$$

Where the parameters Θ = (γ, λ, α) are positive.

Hence, the survival function is given by

$$S(x) = 1 - \left(1 - e^{-\lambda(e^{\gamma x}-1)}\right)^{\alpha}. \quad (6)$$

The hazard rate function of the OGE-E(Θ) distribution is

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha\lambda\gamma e^{-\lambda(e^{\gamma x}-1)+\gamma x}\left(1 - e^{-\lambda(e^{\gamma x}-1)}\right)^{\alpha-1}}{1 - (1 - e^{-\lambda(e^{\gamma x}-1)})^{\alpha}}. \quad (7)$$

Also, the reversed hazard function of the OGE-E(Θ) distribution is

$$r(x) = \frac{f(x)}{F(x)} = \frac{\alpha\lambda\gamma e^{-\lambda(e^{\gamma x}-1)+\gamma x}}{1 - e^{-\lambda(e^{\gamma x}-1)}} \quad (8)$$

a. Graphical description

This subsection provides several plots of the PDF, CDF and the hazard rate functions of the OGE-E(Θ) distribution for different values of their parameters, in order to show the flexibility of the new OGE-E(Θ) distribution.

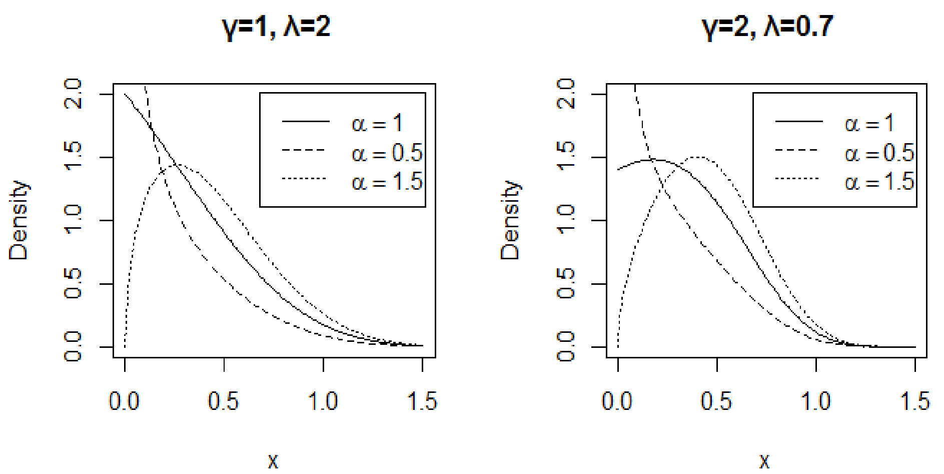

Figure 1: The PDF of various OGE-E(Θ) distribution for different values of parameters.

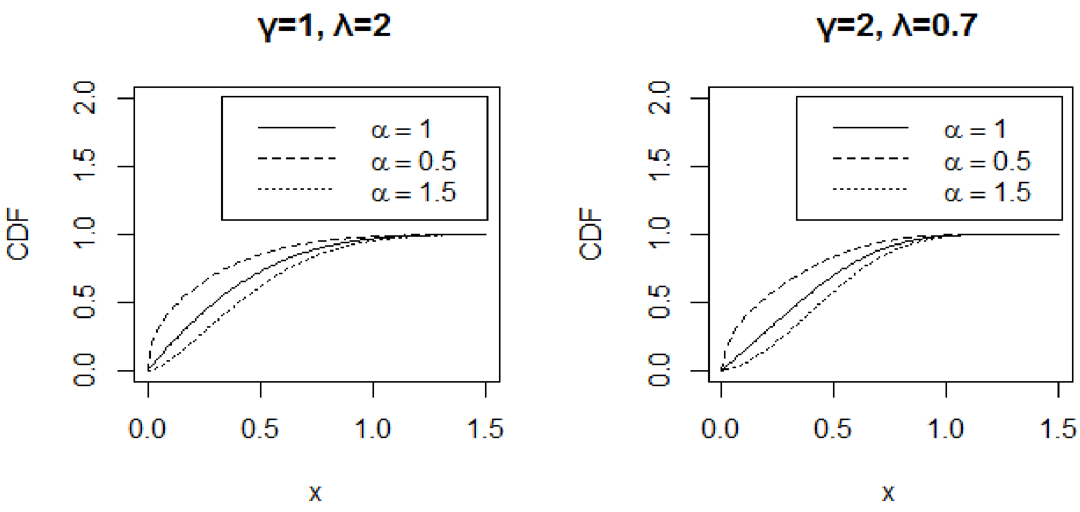

Figure 2: The CDF of various OGE-E(Θ) distribution for different values of parameters.

One can see from Figure 1, that when the parameters (γ, λ) are fixed to the values (1, 2) respectively, the PDF has the exponential curve for α = 0.5, 1. However, when α = 1.5, the PDF curve is right skewed. On the other hand, when the parameters (γ, λ) are fixed to the values (2, 0.7), the PDF curve is exponential when α=0.5. Although, the PDF curve is right skewed when α=1; and it is almost symmetric when α =1.5.

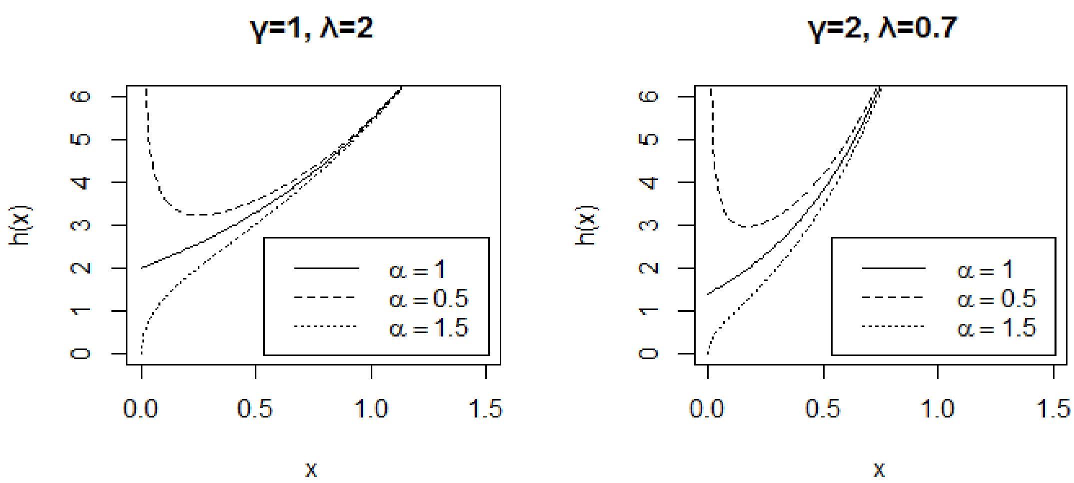

Figure 3: The Hazard function of various OGE-E(Θ) distribution for different values of parameters.

Figure 3 shows the different monotonic shapes of the hazard rate function of the OGE-E(Θ) distribution. That is, the hazard rate function is monotonic increasing when (γ, λ) are fixed to (1, 2) and (2, 0.7) respectively, and α = 0.5, 1. However, the hazard rate function take the bath tub shape when α = 0.5. (i. e. α < 1).

3. Some Statistical Properties

This section is devoted for studying some statistical properties of the OGE-E(Θ) distribution, specifically quantiles, mode, moments, skewness and kurtosis.

a. Quantiles and mode

Theorem 1.

The quantile x_q of the OGE-E(Θ), where Θ= $(\alpha; \lambda; \gamma)$ random variable X is given by

$$(x_q)_{OGE-E(\Theta)} = \frac{1}{\gamma}\ln\left[1 + \ln\left(\frac{1}{(1-q^{\frac{1}{\alpha}})^{\frac{1}{\lambda}}}\right)\right], \quad 0 < q < 1 \qquad (9)$$

Proof.

Starting with the well known definition of the 100 qth quantile, which is simply the solution of the following equation, with respect to $x_q, 0 < q < 1$

$$q = P(X \leq x_q) = F(x_q), \quad x_q > 0$$

Using the distribution function of (4) of OGE-E(Θ) distribution we have

$$q = F(x_q) = (1 - e^{-\lambda(e^{\gamma x_q}-1)})^{\alpha}$$

That is

$$q^{\frac{1}{\alpha}} = 1 - e^{-\lambda(e^{\gamma x_q}-1)}, \Rightarrow$$

$$1 - q^{\frac{1}{\alpha}} = e^{-\lambda(e^{\gamma x_q}-1)}$$

Therefore, by taking the log of both sides of the above equation, we have

$$\ln(1 - q^{\frac{1}{\alpha}}) = -\lambda(e^{\gamma x_q} - 1), \Rightarrow$$

$$\ln\frac{1}{(1-q^{\frac{1}{\alpha}})^{\frac{1}{\lambda}}} = e^{\gamma x_q} - 1, \Rightarrow$$

$$e^{\gamma x_q} = 1 + \ln\frac{1}{(1-q^{\frac{1}{\alpha}})^{\frac{1}{\lambda}}}$$

Hence, again taking the log of both sides of the above equation, one gets

$$\gamma x_q = \ln\left[1 + \ln\frac{1}{(1 - q^{\frac{1}{\alpha}})^{\frac{1}{\lambda}}}\right], \Rightarrow$$

$$x_q = \frac{1}{\gamma}\ln\left[1 + \ln\frac{1}{(1 - q^{\frac{1}{\alpha}})^{\frac{1}{\lambda}}}\right] \qquad (10)$$

The Median of the distribution is obtained by using q=0.5 in (10)

$$\text{Median} = x_{0.5} = \frac{1}{\gamma}\ln\left[1 + \ln\frac{1}{(1 - ½^{\frac{1}{\alpha}})^{\frac{1}{\lambda}}}\right] \qquad (11)$$

The mode of the OGE-E(Θ) distribution is obtained by differentiating the PDF, given by (5),

$$f'(x; \alpha; \lambda; \gamma) = \alpha\lambda\gamma\, e^{\gamma x - \lambda e^{\gamma x} + \lambda}(1 - e^{-\lambda(e^{\gamma x} - 1)})^{\alpha - 1}\left[\gamma - \lambda\gamma e^{\gamma x} + \frac{\lambda\gamma(\alpha - 1)e^{\gamma x - \lambda e^{\gamma x} + \lambda}}{1 - e^{-\lambda(e^{\gamma x} - 1)}}\right], \qquad (12)$$

Equate (12) to zero, then

$$\alpha\lambda\gamma\, e^{\gamma x - \lambda e^{\gamma x} + \lambda}\left(1 - e^{-\lambda(e^{\gamma x} - 1)}\right)^{\alpha - 1}\left[\gamma - \lambda\gamma e^{\gamma x} + \frac{\lambda\gamma(\alpha - 1)e^{\gamma x - \lambda e^{\gamma x} + \lambda}}{1 - e^{-\lambda(e^{\gamma x} - 1)}}\right] = 0 \qquad (13)$$

Hence, either

$$e^{\gamma x - \lambda e^{\gamma x} + \lambda} = 0 \text{ or } (1 - e^{-\lambda(e^{\gamma x} - 1)})^{\alpha - 1} = 0 \text{ or } \gamma - \lambda\gamma e^{\gamma x} + \frac{\lambda\gamma(\alpha - 1)e^{\gamma x - \lambda e^{\gamma x} + \lambda}}{1 - e^{-\lambda(e^{\gamma x} - 1)}} = 0.$$

However, it is not possible to get an explicit solution of (13) in the general case. So that numerical methods can be used. Here, a program code is constructed in order to solve this problem.

Table 1: provides the computed mode values for OGE-E(Θ) distribution with different parameters values.

Θ=(α, λ, γ)	Θ= (5,2,1)	Θ= (2,2,1)	Θ=(1.5,2,1)	Θ= (1,2,1)	Θ= (1,1,1)
mode	0.6565784	0.3798513	0.2604954	0	0

b. The Moments

Moments are necessary and important in any statistical ananlysis, especially in applications. It can be used to study the most important features and characteristics of a distribution.

If X ~ OGE-E(Θ), then the rt non-central moment of X is given by

$$\mu_r' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{\lambda^{j+1}\alpha(i+1)^j\, r!}{j!\ \gamma^r(j-k+1)^{r+1}}.$$

Hence when r=1, the mean of the OGE-E(Θ) distribution is

$$E(X) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{\lambda^{j+1}\alpha(i+1)^j}{j!\ \gamma(j-k+1)^2}.$$

and where putting r=2, then the second moment is

$$E(X^2) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{2\lambda^{j+1}\alpha(i+1)^j}{j!\ \gamma^2(j-k+1)^3}.$$

Therefore, the variance of the OGE-E(Θ), can be easily shown to be

$$\begin{aligned}\mathrm{var}(X) &= E(X^2) - [E(X)]^2 \\ &= \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{2\lambda^{j+1}\alpha(i+1)^j}{j!\ \gamma^2(j-k+1)^3} \\ &\quad -\left(\sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{\lambda^{j+1}\alpha(i+1)^j}{j!\ \gamma(j-k+1)^2}\right)^2\end{aligned}$$

The third and fourth non-central moments are, respectively, given by

$$\mu_3' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{\lambda^{j+1}\alpha(i+1)^j\, 3!}{j!\ \gamma^3(j-k+1)^4}.$$

and

$$\mu_4' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{\lambda^{j+1}\alpha(i+1)^j\, 4!}{j!\ \gamma^4(j-k+1)^5}.$$

Based on the first four non-central moments of the OGE-E(Θ) distribution, the classical measures of the coefficient of skewness and kurtosis can be obtained as:

The skewness (S1) of the OGE-E(Θ) is given by:

$$S1 = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{\sigma^2}$$

The kurtosis (K1) of the OGE-E(Θ) is given by:

$$k1 = \frac{\mu_4' - \mu_3'\mu + 6\mu_2' - 3\mu^4}{\sigma^4}$$

$$\text{Where } \sigma^2 = var(x).\mu = \mu_1'$$

Moment generating function of a random variable X having an OGE-E(Θ), distribution was derived. If X has OGE-E(Θ), distribution, then the moment generating function (MGF) $M_X(t)$ has the form

$$M_X(t) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{\lambda^{j+1}\alpha(i+1)^j t^r}{j!\,\gamma^r(j-k+1)^{r+1}}.$$

The Cumulant generating function (CGF) is:

$$K_x(t) = \ln_e(M_X(t))$$

$$K_x(t) = \ln_e\left[\sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{i+j+k}\binom{\alpha-1}{i}\binom{j}{k}\frac{\lambda^{j+1}\alpha(i+1)^j t^r}{j!\,\gamma^r(j-k+1)^{r+1}}\right] \qquad (14)$$

4. Order Statistics

Suppose $X_1, X_2, X_3. \dots\dots\dots, X_n$ is a random sample of size n from the OGE-E(Θ) distribution with CDF F(x) and PDF $f(x)$ given by (4) and (5) respectively. Let $X_{1:n}, X_{2:n}. \dots\dots., X_{n:n}$ denote the order statistics obtained from a random sample $X_1, X_2, X_3, \dots\dots\dots, X_n$ then the probability density function of Xr:n is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \qquad (15)$$

For r=1,2,3,….,n

The PDF of the rth order statistic of the OGE-E(Θ) distribution is:

$$f_{r:n}(x) = \frac{\alpha\,\lambda\,\gamma\,n!}{e^{-\gamma x}(r-1)!\,(n-r)!}\left[1-e^{-\lambda(e^{\gamma x}-1)}\right]^{\alpha r-1}\left[1-\left(1-e^{-\lambda(e^{\gamma x}-1)}\right)^{\alpha}\right]^{n-r} e^{-\lambda(e^{\gamma x}-1)} \qquad (16)$$

The PDF of the smallest order statistic $X_{(1)}$ is:

$$f_{1:n}(x) = \frac{n\alpha\,\lambda\,\gamma}{e^{-\gamma x}}\left[1-e^{-\lambda(e^{\gamma x}-1)}\right]^{\alpha-1}\left[1-\left(1-e^{-\lambda(e^{\gamma x}-1)}\right)^{\alpha}\right]^{n-1} e^{-\lambda(e^{\gamma x}-1)} \qquad (17)$$

The PDF of the largest order statistic $X_{(n)}$ is:

$$f_{r:n}(x) = \frac{\alpha\,\lambda\,\gamma\,n}{e^{-\gamma x}}\left[1 - e^{-\lambda(e^{\gamma x}-1)}\right]^{\alpha N-1} e^{-\lambda(e^{\gamma x}-1)} \quad (18)$$

5. Conclusions

In this article, proposed a new model OGE-E(ϴ) distribution and developed its various properties of the new distribution such as moments, moment generating function, quantile function, skewness and kurtosis have been derived. However, the graphical description of this new distribution, OGE-E(ϴ), show the flexibility of this model.

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