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A New Approach of Bernoulli Sub-ODE Method to Solve Nonlinear PDEs

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Abstract

In this paper, a new approach of the Bernoulli Sub-ODE method is proposed and this method is applied to solve the modified Liouville equation and the regularized long wave equation. As a result some new traveling wave solutions for them are successfully established. When the parameters are taken as special values, the solitary wave solutions are originated from these traveling wave solutions. Further, graphical representation of some solutions are given to visualize the dynamics of the equation. The results reveal that this method may be useful for solving higher order nonlinear partial differential equations.

Keywords: Modified Liouville equation; regularized long wave equation; traveling wave solutions.

1. Introduction

The investigation of traveling wave solutions(exact solutions) of nonlinear partial differential equations(PDEs) plays an important role not only in theoretic research but also in the applications. They describe different types of physical systems, ranging from gravitation to fluid dynamics. The interest of finding travelling wave solutions of nonlinear PDEs is increasing day by day and has now become a hot topic to researchers. In recent years, many researchers who are interested in the nonlinear physical phenomena have investigated exact solutions of nonlinear PDEs.

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With the development of soliton theory and the application of computer symbolic system such as Maple and Mathematica, many powerful methods for obtaining exact solutions of nonlinear evolution equations are presented, such as the tanh-method [1-3], the extended tanh method [4-5], the Jacobi elliptic function expansion [6-8], the Bucland transformation[9-12], the homogeneous balance method[13], the inverse scattering method [14], the variational iteration method [15], the exp-function method [16], (G'/G)-expansion method [17], modified simple equation method [18], F-expansion method[19-20] and so on. In 2011, Ben Jing proposed Bernoulli Sub-ODE method for finding exact solutions of nonlinear PDEs. After reducing the nonliner PDEs $P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots\dots) = 0$ to nonlinear ODEs $P(u, -cu', u', c^2u'', -cu'', u'', \ldots) = 0$ by choosing $u(x,t) = u(\xi)$, where $\xi = x - ct$, he assumed the solution of nonlinear ODEs in the form $u = \sum_{i=0}^{m} a_i H^i$, where $H(\xi)$ can be determined from the first order ODE $H' = H^2 - H$ [21-22]. But in our proposed method, we consider a set of first order ODEs in the form: $H'(\xi) \pm (H(\xi))^n = 0$ (where $n \geq 2$) instead of $H' = (H^2 - H)$ for getting a set of exact solutions and then a set of solitary wave solutions in a sequential manner.

Here $H'(\xi) \pm (H(\xi))^n = 0$ give the following

$$H(\xi) = \begin{cases} \dfrac{1}{[c_1 \pm (n-1)\xi]^{1/(n-1)}}, & n \text{ is even} \\ \dfrac{1}{[c_1 \pm (n-1)\xi]^{1/(n-1)}}, \dfrac{-1}{[c_1 \pm (n-1)\xi]^{1/(n-1)}}, & n \text{ is odd} \end{cases} \quad (1.1)$$

In this method, we will get the solutions in terms of $\dfrac{1}{[c_1 \pm (n-1)\xi]}, \xi = x - ct$, where the singularities occur in the case of $c_1 \pm (n-1)(x - ct) = 0.$

2. Methodology

Suppose that a nonlinear partial differential equation in two independent variables *x* and *t*, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots\dots\dots) = 0 \quad (2.1)$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, the highest order derivatives and nonlinear terms are involved. The outline of the method is given below:

Step-1: Combine the independent variables *x* and *t* into one variable ξ, by choosing

$$u(x,t) = u(\xi), \text{ where } \xi = x - ct \quad (2.2)$$

The traveling wave transformation (2.2) permits us to transform equation (2. 1) to the following ODE:

$$P(u,-cu',u',c^2u'',-cu'',u'',\cdots)=0, \tag{2.3}$$

where the prime denotes the differential with respect to ξ.

Step-2: We suppose that equation (2. 3) has the solution of the form in the finite series:

$$u(\xi)=\sum_{i=0}^{m}a_iH^i,\quad H=H(\xi) \tag{2.4}$$

where a_i, $a_m\neq 0$ are constants to be determined, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (2.3), and $H=H(\xi)$ satisfies the equation :

$$H'(\xi)\pm (H(\xi))^n=0,\ \text{where } n\geq 2 \tag{2.5}$$

Step-3: We substitute equation (2.4) into equation (2.3) and use equation (2.5) and then we account the function $H(\xi)$. As a result of this substitution, we get a polynomial of $H(\xi)$. We equate all the coefficients of same power of $H(\xi)$ to zero. This procedure yields a system of algebraic equations whichever can be solved to find a_i.

Step-4: Substituting the values a_i into equation (2.4) along with general solutions of equation (2.5) complete the determination of the solution of equation (2.1).Finally particular choice of unknown parameters in exact solutions gives the desired solitary wave solutions.

3. Applications of the method

To illustrate the idea of the proposed method, we have selected two nonlinear PDEs, such as the modified Liouville equation and the regularized long wave equation which arise in mathematical physics.

Example-1: Solution of modified Liouville equation

$$w_{tt}=a^2w_{xx}+be^{\beta w} \tag{3.1}$$

that arises in hydrodynamics, where $w(x,t)$ is the stream function and a, b, β are nonzero constants[23-24].

We first use the Painleve transformation $u(x,t)=e^{\beta w}$, so that

$$w=\frac{1}{\beta}lnu. \tag{3.2}$$

Now the wave transformation equations $u(x,t)=u(\xi),\xi=x-ct$ and equation (3.2) reduces equation (3.1) into the following ODE:

$$uu'' - u'^2 + ku^3 = 0 \quad \text{where} \quad k = \frac{b\beta}{a^2 - c^2} \quad \text{and} \quad c \neq \pm a \tag{3.3}$$

Let (3.3) has the solution of the form: $u(\xi) = \sum_{i=0}^{m} a_i H^i, \; H = H(\xi)$ (3.4)

Hence for different for values of n in (2.5), the corresponding exact solutions of (3.1) are obtained below:

(a) $n = 2$ **:** By considering the homogeneous balance between u''' and uu'appearing in eq.(3.3), we get $m = 2$. As a result, (3.4) takes the form:

$$u(\xi) = a_0 + a_1 H + a_2 H^2 \tag{3.5}$$

where a_0, a_1, a_2 are unknown constants to be determined and $H(\xi)$ satisfies the eq.(2.5) and this function is determined from eq. (1.1) by setting $n = 2$. Substituting (3.5) in the reduced ODE (3.3) and collecting the coefficients of various power of $H(\xi)$ yields the following system of algebraic equations.

H^6 : $\quad ka_2{}^3 + 2a_2{}^2 = 0$

H^5 : $\quad 4a_1a_2 + 3ka_1a_2{}^2 = 0$

H^4 : $\quad a_1{}^2 + 3ka_1{}^2a_2 + 6a_0a_2 + 3ka_0a_2{}^2 = 0$

H^3 : $\quad 2a_0a_1 + 6ka_0a_1a_2 + ka_1{}^3 = 0$

H^2 : $\quad 3ka_0a_1{}^2 + 3ka_0{}^2a_2 = 0$

H^1 : $\quad 3ka_0{}^2a_1 = 0$

H^0 : $\quad ka_0{}^3 = 0$

Solving the above equations, we get $a_0 = 0, a_1 = 0, a_2 = \frac{-2}{k}$. (3.6)

Hence the solution of (3.4) takes the form

$$u(\xi) = \frac{-2}{k(c_1 \pm \xi)^2} \tag{3.7}$$

Finally putting $\xi = x - ct,$ and using equation (1.1), we get the following desired exact solution of (3.1)

$$w_1(x,t) = \frac{1}{\beta}\ln\left\{\frac{-2(a^2-c^2)}{b\beta[c_1 \pm (x-ct)]^2}\right\} \tag{3.8}$$

Similarly, for $n = 3,4,5,6,7,....$ the corresponding exact solutions of eq. (3.1) are

(b) $n=3$, $w_2(x,t) = \frac{1}{\beta}\ln\left\{\frac{-8(a^2-c^2)}{b\beta[c_1 \pm 2(x-ct)]^2}\right\}$

(c) $n=4$ $w_3(x,t) = \frac{1}{\beta}\ln\left\{\frac{-18(a^2-c^2)}{b\beta[c_1 \pm 3(x-ct)]^2}\right\}$

(d) $n=5$ $w_4(x,t) = \frac{1}{\beta}\ln\left\{\frac{-32(a^2-c^2)}{b\beta[c_1 \pm 4(x-ct)]^2}\right\}$

(e) $n=6$ $w_5(x,t) = \frac{1}{\beta}\ln\left\{\frac{-50(a^2-c^2)}{b\beta[c_1 \pm 5(x-ct)]^2}\right\}$

(f) $n=7$: $w_6(x,t) = \frac{1}{\beta}\ln\left\{\frac{-72(a^2-c^2)}{b\beta[c_1 \pm 6(x-ct)]^2}\right\}$

In general, the solution of (3.1) is

$$w_{n-1}(\xi) = \frac{1}{\beta}\ln\left\{\frac{-2(n-1)^2(a^2-c^2)}{b\beta[c_1 \pm (n-1)(x-ct)]^2}\right\}, \text{ where } \frac{-2(n-1)^2(a^2-c^2)}{b\beta[c_1 \pm (n-1)(x-ct)]^2} > 0 \tag{3.9}$$

Singularity: From the above obtained solutions we observe that the solutions have singularities for the values x and t satisfying $c_1 \pm (n-1)(x-ct) = 0$, $n \geq 2$ or $\frac{2(n-1)^2(a^2-c^2)}{b\beta[c_1 \pm (n-1)(x-ct)]^2} > 0.$

Justification: Here, $w_i(x,t), i = 1,2,3,4,5,6$ obtained in different cases are exact solutions of the modified Liouville equation because they fully satisfy the equation (3.1). This justification is checked by MAPLE-13 and the corresponding MAPLE code is given below:

*** Justification of the solutions of the modified Liouville equation***

> $restart;$

> $PDE := a^2 \left(\frac{\partial^2}{\partial x^2} w(x,t) \right) - \left(\frac{\partial^2}{\partial t^2} w(x,t) \right) + b\, e^{\beta\, w(x,t)} = 0$

> $solution := \; w_{n-1}(\xi) = \frac{1}{\beta} \ln\left\{ \frac{-2(n-1)^2(a^2-c^2)}{b\beta[c_1 \pm (n-1)(x-ct)]^2} \right\}$

> $pdetest(solution, PDE)$

0

Figure 3

Velocity profile of $w_1(x,t)$ with wave speed, c=2

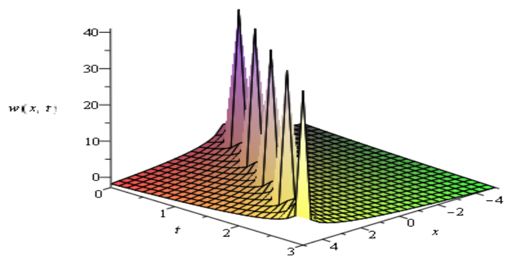

Figure 3.1: (3D Plot): Profile of (3.9), when $n = 2, a = 1, c_1 = 1, b = 1, \beta = 1$

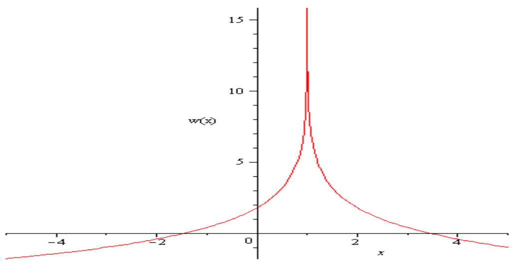

Figure 3.2: (2D Plot): Profile of (3.9), when $n = 2, a = 1, c_1 = 1, b = 1, \beta = 1$ and time t=1.

Velocity profile of $w_1(x,t)$ with wave speed, c=3

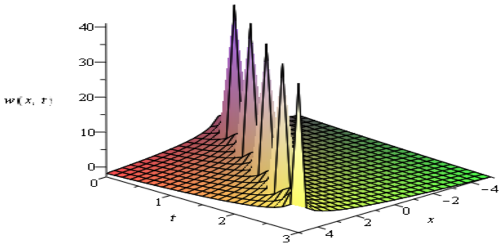

Figure 3.3: (3D Plot): Profile of (3.9), when $n = 2, a = 1, c_1 = 1, b = 1, \beta = 1$

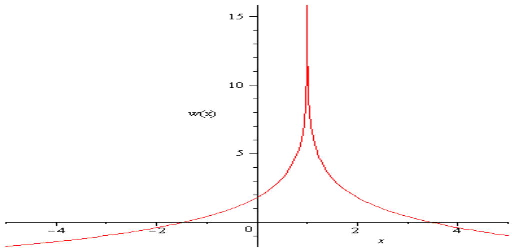

Figure 3.4: (2D Plot): Profile of (3.9), when $n = 2, a = 1, c_1 = 1, b = 1, \beta = 1$ and time t=1.

Example-2: Solution of regularized long-wave equation

The regularized long-wave equation is

$$u_t + au_x - 6uu_x - bu_{xxt} = 0, \quad a, b > 0 \tag{3.10}$$

where a,b are real constants[25]. In the above procedure, we can also find the solutions of this equation and the general form of the solution of (3.10) is:

$$u_{n-1}(x,t) = \frac{a-c}{6} + \frac{2(n-1)^2 bc}{[c_1 \pm (n-1)(x-ct)]^2}, \; c_1 \pm (n-1)(x-ct) \neq 0 \tag{3.11}$$

<u>Velocity profile of $u_1(x,t)$ with wave speed, *c*=2</u>

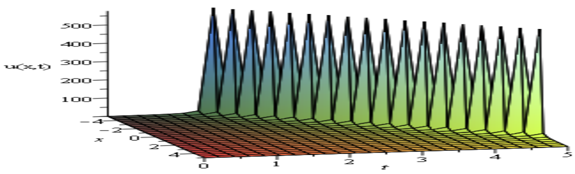

Figure 3.5: (3D Plot): Soliton profile of (3.11), when $n = 2, a = 1, b = 1, c_1 = 1$

<u>Velocity profile of $u_1(x,t)$ with wave speed, *c*=3</u>

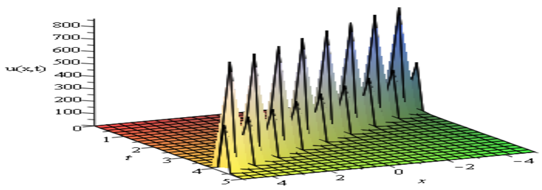

Figure 3.6: (3D Plot): Soliton profile of (3.11), when $n = 2, a = 1, b = 1, c_1 = 1$

Singularity: From the above obtained solutions we observe that the solutions (3.11) have singularities for the values x and t satisfying $c_1 \pm (n-1)(x-ct) = 0$, $n \geq 2$. For example, the solution (3.11) has a singularity at $x = -6$, when $c_1 = 8, n = 2, c = 2, t = 1$.

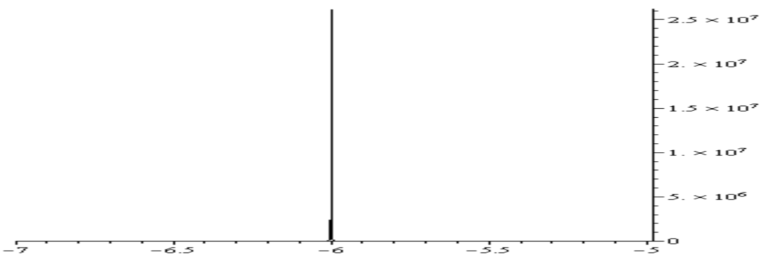

Figure 3.7: Singularity of $u_1(x,t)$ at $x = -6$

4. Results and Discussions

In case of our proposed method, we have got many exact solutions for differetnt values of n. So we will discuss some solutions only.

Equation (3.8) represents the soliton type solutions (shown in Figure 3.1(3D plot) and Figure 3.4(3D plot)) of the modified Liouville equation. The corresponding two-dimensional plot are given in Figure 3.2(2D plot) and Figure 3.4(2D plot) respectively.

And finally Figure 3.5(3D plot) and Figure 3.6(3D plot) give also the soliton profile for the solutions of the equation regularized long wave equation where the waves move with 2 and 3 respectively. A singularity has been shown in Figure 3.7.

5. Conclusion

The main achievement of this work is to explore a new approach of the Bernoulli Sub-ODE method, which is capable of producing more fruitful and new solitary wave solutions of several nonlinear evolution equations. Using this method, we have successfully obtained exact solutions and then solitary wave solutions in terms of fractional functions. Besides, there are several avenues for further investigation:

- Seeking exact solutions to NEEs having (2+1) or (3+1) dimension through the proposed method is another possible study.
- Applications of the obtained solutions can be discussed.

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