American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS)
ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

http://asrjetsjournal.org/

HPM Approximations for Trajectories: From a Golf Ball Path to Mercury's Orbit

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Abstract

In this work, we propose the approximated analytical solutions for two highly nonlinear problems using the homotopy perturbation method (HPM). We obtained approximations for a golf ball trajectory model and a Mercury orbit's model. In addition, to enlarge the domain of convergence of the first case study, we apply the Laplace-Padé resummation method to the HPM series solution. For both case studies, we were able to obtain approximations in good agreement with numerical methods, depicting the basic nature of the trajectories of the phenomena.

Keywords: Nonlinear differential equations; Homotopy; perturbation method; Resummation method.

1. Introduction

Nonlinear ordinary differential equations (ODEs) models a wide variety of physical phenomena. This type of models can be solved using standard numerical methods. However, it is known that these algorithms can give some problems, such as numerical instabilities, oscillations, among others.

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This means that the numerical solution may not correspond to the real solution of the original ODEs [16]. Therefore, analytical approximations can be an interesting alternative to the pure numerical solutions. Among the different methods that have been developed to obtain approximations for the solution of ODEs, the homotopy perturbation method is widely used, by its simplicity and accuracy of approximations.

In this work, we propose HPM approximations for a golf ball trajectory model and a Mercury orbit's model. Furthermore, to enlarge the domain of convergence of the first case study, we apply the Laplace-Padé after-treatment to the HPM series solution. For the second case study, we propose an approximation in good agreement with numerical results.

This paper is organized as follows. In Section 2, we provide a brief review of HPM method. Section 3 presents the basic concept of Laplace-Padé resummation method. In section 4, we introduce the mathematical models of both cases study and the HPM approximations. Next, Section 5 shows results and discusses our findings. Finally, a concluding remark is given in Section 6.

2. Basic idea of HPM method

In the HPM method [29, 30, 10, 9, 8, 31, 14, 12, 32, 5, 26, 27, 24, 1, 15, 25, 33, 23, 7] is considered that a non-linear differential equation can be expressed as

$$A(u) - f(r) = 0, \qquad \text{where} \qquad r \in \Omega, \tag{1}$$

with the boundary condition

$$B(u, \frac{\partial u}{\partial \eta}) = 0, \qquad \text{where} \qquad r \in \Gamma, \tag{2}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary operator, and Γ is the boundary of the domain Ω. The A operator, generally, can be divided into two operators, L and N, where L is the linear operator and N is the nonlinear operator. Hence, (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Now, the homotopy function is

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p(L(v) + N(v) - f(r)) = 0, \quad p \in [0,1], \tag{4}$$

where u_0 is the initial approximation of (3) which satisfies the boundary conditions and p is known as the perturbation homotopy parameter. Analysing (4) can be concluded that

$$H(v,0) = L(v) - L(u_0), \qquad (5)$$

$$H(v,1) = L(v) + N(v) - f(r). \qquad (6)$$

For $p \to 0$, the homotopy map (4) is reduced to the linear problem (5) that possesses a trivial solution u_0. Moreover, for $p \to 1$, the homotopy map (4) is transformed into the original nonlinear problem (6) that possesses the sought solution. Furthermore, we assume that the solution of (4) can be written as a power series of p

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots. \qquad (7)$$

Adjusting $p = 1$ results that the approximate solution for (1) is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \qquad (8)$$

The series (8) is convergent on most cases, nevertheless, the convergence depends of the nonlinear operator N [29, 30, 10, 9, 8, 27].

3. Laplace-Padé Resummation Method

Several approximate methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lack of large domains of convergence. Therefore, Laplace-Padé [11, 21, 19, 13, 2, 18, 17, 22, 3, 28] method is used in literature to enlarge the domain of convergence of solutions. The procedure can be described as follows:

1. First, Laplace transformation is applied to power series (8).
2. Next, s is substituted by $1/t$ in the resulting equation.
3. After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller value than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.
4. Then, t is substituted by $1/s$.
5. Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution.

This process is known as the Laplace-Padé homotopy perturbation method (LPHPM).

4. Cases Study

In this Section, we will describe and solve the governing equations for the trajectories of a golf ball and Mercury.

4.1 Golf ball trajectory

The dynamics for a golf ball trajectory (see Figure 1) can be modelled by the following system of ODEs [20]

$$\begin{aligned} &x'' + Kv(C_D x' + C_L y') = 0, \\ &y'' + Kv(C_D y' - C_L x') + g = 0, \\ &v = \sqrt{x'^2 + y'^2}, \\ &K = \rho A / (2m), \end{aligned} \tag{9}$$

where the prime denotes derivative with respect to t, v is the speed of the ball, x and y are the horizontal and vertical displacements, respectively; C_D and C_L are the drag and lift coefficients, g is the gravitational acceleration, ρ the air density, $A = \pi(d/2)^2$, d and m are the cross-sectional area, diameter and mass of the ball, respectively.

Now, the initial conditions of the problem are

$$\begin{aligned} &x(0) = X_0, \\ &y(0) = Y_0, \\ &x'(0) = v(0)\cos(\theta(0)), \\ &y'(0) = v(0)\sin(\theta(0)). \end{aligned} \tag{10}$$

where $[X_0, Y_0]$, $\theta(0)$, and $v(0)$ are the initial: position, angle, and speed of the ball, respectively.

According to the HPM (relation (4)), we can construct the homotopy map as follows

$$\begin{aligned} &(1-p)(v_1'' - x_0'') + p\left(v_{1''} + K\sqrt{v_{1'}^2 + v_{2'}^2}\,(C_D v_{1'} + C_L v_{2'})\right) = 0, \\ &(1-p)(v_2'' - y_0'') + p\left(v_{2''} + K\sqrt{v_{1'}^2 + v_{2'}^2}\,(C_D v_{2'} - C_L v_{1'}) + g\right) = 0 \end{aligned} \tag{11}$$

where the dots denote differentiation with respect to t, and the initial approximation is

$$v_{1,0}(t) = x_0(t) = x(0) + x'(0)t,$$
$$v_{2,0}(t) = y_0(t) = y(0) + y'(0)t, \tag{12}$$

which satisfy the initial conditions of the problem.

From (7), we assume that the solution of (11) can be written as a power series of p as follows

$$v_i = \sum_{j=0}^{26}(v_{i,j}p^i), \qquad i = 1,2 \tag{13}$$

where $v_{i,j}$ $(i, j = 1,2,3,\ldots)$ are functions yet to be determined.

Following the HPM protocol, we substitute (13) into (11), nonetheless, we observe that the square root term avoids the rearranging of the coefficients of p powers as required by HPM. Then, we expand such term by Taylor with respect to p, resulting

$$\sqrt{v_{1'}^2 + v_{2'}^2} \approx \sqrt{v_{1,0}'^2 + v_{2,0}'^2} + \frac{v_{1,0}v_{1,1} + v_{2,0}v_{2,1}}{\sqrt{v_{1,0}'^2 + v_{2,0}'^2}}p + \cdots + (\cdots)p^{15}, \tag{14}$$

Now, it is possible to rearrange of the coefficients of p powers, resulting

$$\begin{aligned}
&v_{1,1}'' + KC_L v_0 v_{2,0}' \\
&\quad + KC_D v_0 v_{1,0}' = 0, \qquad v_{1,1}(0) = 0, v_{1,1}'(0) = 0, \\
\\
&v_{1,2}'' + KC_L v_0 v_{2,1}' \\
&\quad + KC_L (v_{2,0}')^2 v_{2,1}' / v_0 + KC_L v_{1,0}' v_{1,1}' v_{2,0}' / v_0 + KC_D (v_{1,0}')^2 v_{1,1}' / v_0 \\
&+ KC_D v_0 v_{1,1}' + KC_D v_{2,0}' v_{2,1}' v_{1,0}' / v_0 = 0, \qquad v_{1,2}(0) = 0, v_{1,2}'(0) = 0, \\
&\quad \vdots \\
&v_{2,1}'' + g + KC_D v_0 v_{2,0}' - KC_L v_0 v_{1,0}' = 0, \qquad v_{2,1}(0) = 0, v_{2,1}'(0) = 0, \\
&v_{2,2}'' - KC_L v_0 v_{1,1}' - KC_L v_{2,0}' v_{2,1}' v_{1,0}' / v_0 + KC_D (v_{2,0}')^2 v_{2,1}' / v_0 \\
&+ KC_D v_{1,0}' v_{1,1}' v_{2,0}' / v_0 + KC_D v_0 v_{2,1}' \\
&\quad - KC_L (v_{1,0}')^2 v_{1,1}' / v_0 = 0, \qquad v_{2,2}(0) = 0, v_{2,2}'(0) = 0, \\
&\quad \vdots
\end{aligned} \tag{15}$$

where $v_0 = \sqrt{(v_{1,0}')^2 + (v_{2,0}')^2}$.

Therefore,

$$
\begin{aligned}
v_{1,1}(t) &= -(1/2)lt^2K(C_D x'(0) + C_L y'(0)),\\
v_{1,2}(t) &= ((1/3)(l(-(1/2)C_L^2 + C_D^2)(x'(0))^3\\
&\quad + (3/2)C_D ly'(0)C_L(x'(0))^2 + l(-(1/2)C_L^2 + C_D^2)(y'(0))^2 x'(0)\\
&\quad + (3/2)C_D l(y'(0))^3 C_L)K^2/l + (1/3)((1/2)C_L(x'(0))^2 g\\
&\quad + (1/2)C_D x'(0)y'(0)g + C_L(y'(0))^2 g)K/l)t^3,\\
&\vdots\\
v_{2,1}(t) &= (-(1/2)K(C_D y'(0) - C_L x'(0))l - (1/2)g)t^2,\\
v_{2,2}(t) &= (1/3)(lK(-(1/2)C_L^2 + C_D^2)(y'(0))^3 - (3/2)C_D(KC_L lx'(0)\\
&\quad - (2/3)g)(y'(0))^2 + x'(0)(lK(-(1/2)C_L^2 + C_D^2)x'(0)\\
&\quad - (1/2)C_L g)y'(0) - (3/2)(x'(0))^2 C_D(KC_L lx'(0)\\
&\quad - (1/3)g))t^3K/l,\\
&\vdots
\end{aligned}
\tag{16}
$$

where $l = \sqrt{(x'(0))^2 + (y'(0))^2}$.

We obtained $v_{1,3}, v_{2,3}$, and the succeeding terms using Maple software, nevertheless, because they were too cumbersome, we skip them and use them only in the final results. Now, from (7), we obtain a 26-th order approximation, then considering $p \to 1$ yields the approximate solution of (9) as

$$
\begin{aligned}
x(t) &= \lim_{p\to 1} v_1(t) = \sum_{j=0}^{26} v_{1,j}(t),\\
y(t) &= \lim_{p\to 1} v_2(t) = \sum_{j=0}^{26} v_{2,j}(t).
\end{aligned}
\tag{17}
$$

In order to perform the Laplace-Padé after-treatment, we set the values of the parameters as in [20]: $C_D = 0.28$, $C_L = 0.28$, $g = 9.81$ m/s ^{2}, $\rho = 1.21$ kg/m ^{3}, $d = 4.267$ m, $m = 4.593$ kg, $v(0) = 70$ m/s, and $\theta(0) = 16^\circ$.

First, Laplace transformation is applied to (17) and then $1/t$ is written in place of s in the equation.

Afterwards, Padé approximant ($[5/6]$ for $x(t)$ and $[8/8]$ for $y(t)$) is applied and $1/s$ is written in place of t. Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution

$$\begin{aligned} x(t) = {} & 0.2029509828\times10^{-3}\exp(-4.999529628t) \\ & +0.01810030329\exp(-3.058610449t) \\ & -5.165390824\exp(-1.171860096t) \\ & -85.79724064\exp(-0.5491271940t) \\ & +90.94432822\exp(0.06739850672t)\cos(0.08603945021t) \\ & +93.54394846\exp(0.06739850672t)\sin(0.08603945021t), \end{aligned}$$

$$\begin{aligned} y(t) = {} & 0.1934806557\times10^{-5}\exp(-6.379116675t) \\ & +0.1169527020\times10^{-2}\exp(-4.250858281t) \\ & +0.1269162478\exp(-2.681336898t) \\ & +4.992816897\exp(-1.470140005t) \\ & -14.64408668\exp(-0.3883152650t)\cos(0.4888655127t) \\ & +14.37291850\exp(-0.3883152650t)\sin(0.4888655127t) \\ & +9.553182056\exp(0.05600508947t)\cos(0.3571684824t) \\ & +38.44703880\exp(0.05600508947t)\sin(0.3571684824t), \end{aligned} \qquad (18)$$

4.2 Mercury's Orbit

The dynamics of Mercury's orbit can be described by the following equation

$$u'' + u - \delta u^2 - \frac{1}{\alpha} = 0, \qquad u(0) = 0, u'(0) = 0, \qquad (19)$$

where the prime denotes derivative with respect to the angular displacement θ and

$$\begin{aligned} \delta &= \frac{3GM}{c^2}, \\ \alpha &= \frac{\ell}{Gm^2M}, \\ \ell &= \sqrt{\mu GmM\alpha(1-\varepsilon^2)}, \\ \mu &= \frac{mM}{m+M}. \end{aligned} \qquad (20)$$

The parameters of the model are: the gravitational constant $G = 6.6726\times10^{-11}$, the mass of the sun $M = 1.99\times10^{30}$, the speed of light $c = 3\times10^{8}$, $\alpha = 5.787145\times10^{10}$, the eccentricity $\varepsilon = 0.2056$ and the mas $m = 3.285\times10^{23}$ of the planet.

According to the HPM (relation (4)), we can construct the homotopy map as follows

$$(1-p)(v''+v-\frac{1}{\alpha})+p\left(v''+v-\delta v^{2}-\frac{1}{\alpha}\right)=0, \tag{21}$$

From (7), we assume that the solution of (21) can be written as a power series of p as follows

$$v=\sum_{i=0}^{3}(v_i p^i), \tag{22}$$

Next, we substitute (22) into (21), rearrange the coefficients of p powers, and equating them to zero, resulting

$$\begin{aligned}
&v_{0''}+v_0-\frac{1}{\alpha}=0, && v_0(0)=0, v_{0'}(0)=0,\\
&v_{1''}-\delta v_0^2+v_1=0, && v_1(0)=0, v_{1'}(0)=0,\\
&v_{2''}+v_2-2\delta v_0 v_1=0, && v_2(0)=0, v_{2'}(0)=0,\\
&v_{3''}-\delta v_1^2-2\delta v_0 v_2+v_3=0, && v_3(0)=0, v_{3'}(0)=0,
\end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned}
v_0 &= -\frac{1}{\alpha}(\cos(\theta)-1),\\
v_1 &= -\frac{1}{6\alpha^2}\delta(-9+\cos(2\theta)+8\cos(\theta)+6\sin(\theta)\theta),\\
v_2 &= -\frac{1}{144\alpha^3}\delta^2(509\cos(\theta)-624+112\cos(2\theta)+468\sin(\theta)\theta+3\cos(3\theta)\\
&\quad +48\theta\sin(2\theta)-72\cos(\theta)\theta^2),\\
v_3 &= \frac{1}{432\alpha^4}\delta^3(-5048\cos(\theta)-27\theta\sin(3\theta)-5163\sin(\theta)\theta-948\theta\sin(2\theta)\\
&\quad +72\sin(\theta)\theta^3-\cos(4\theta)-1512\cos(2\theta)+6633-72\cos(3\theta)\\
&\quad +144\theta^2\cos(2\theta)+1116\cos(\theta)\theta^2).
\end{aligned} \tag{24}$$

Now, from (7), we obtain a third order approximation, then considering $p \to 1$ yields the approximate solution of (9) as

$$u(t) = \lim_{p \to 1} v(t) = \sum_{i=0}^{3} v_i, \qquad (25)$$

5. Numerical Simulation and Discussion

Figures 1-2 show a comparison between the HPM approximations (18) and (25) of (9) and (19), resulting a good agreement with numerical results. The numerical algorithm used is Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant (RKF45) [4, 6] solution (built-in function of Maple software. Furthermore, in order to obtain a good numerical reference the accuracy of RKF45 was set to an absolute error of 10^{-12} and relative error of 10^{-12}.

For the first case study, the HPM solution (17) was easily obtained by a straightforward procedure. However, the resulting power series solution diverge for large periods of time. Therefore, in order to enlarge the convergence, the Laplace-Padé resummation method was successfully applied to (17) to obtain (18), resulting a good agreement with RKF45 results for the complete trajectory of the golf ball. Using the approximations, we obtain a predicted impact point of 223.2487023 m with a relative error of 5×10^{-3}. This result exhibited the high accuracy of the proposed solution.

For the second case study, we obtained the HPM approximation (25) of (19), resulting a good agreement ($\theta < 10$) with numerical results as depicted in figure 2 and figure 3. The accuracy of the proposed approximation decrease as θ increases due to the secular terms of the approximation.

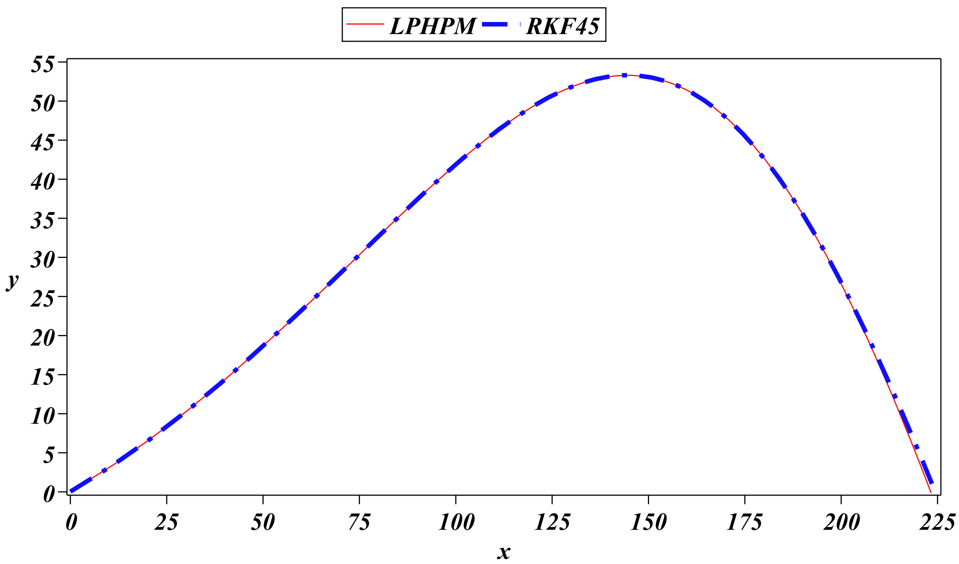

Figure 1: LPHPM solution (18) (solid-line) for (9), and its RKF45 solution (dash-dot)

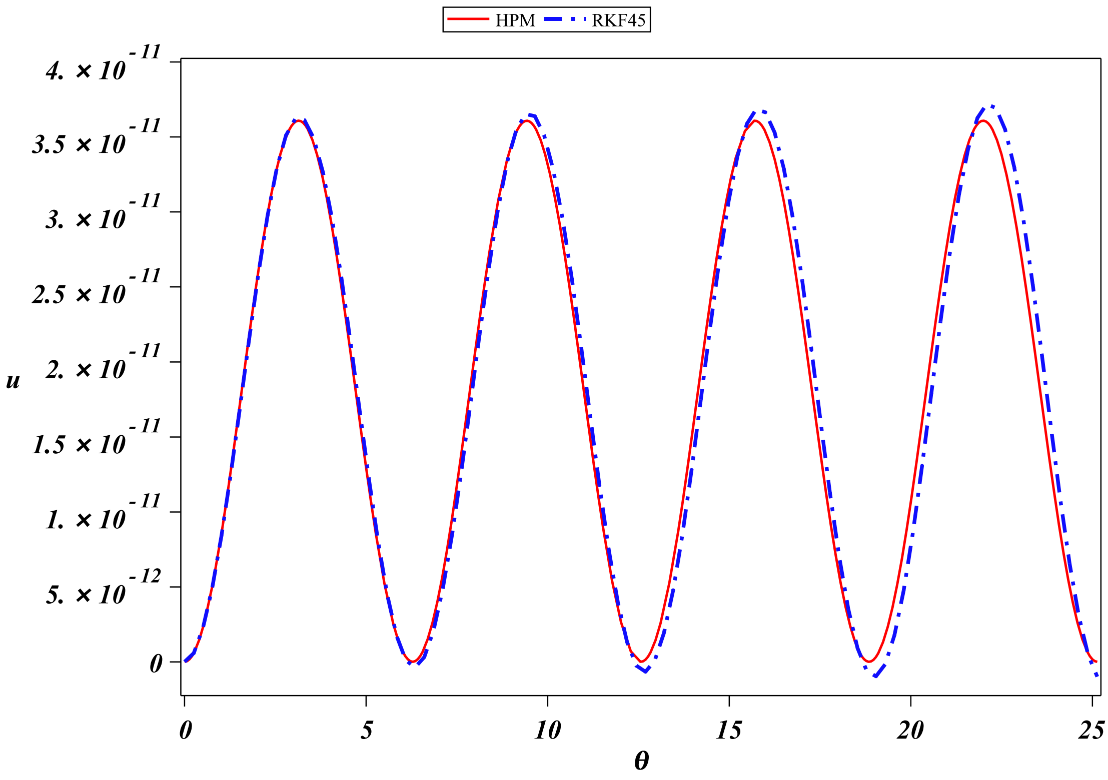

Figure 2: HPM third order solution, (solid-line); RKF45 solution, (dash-dot).

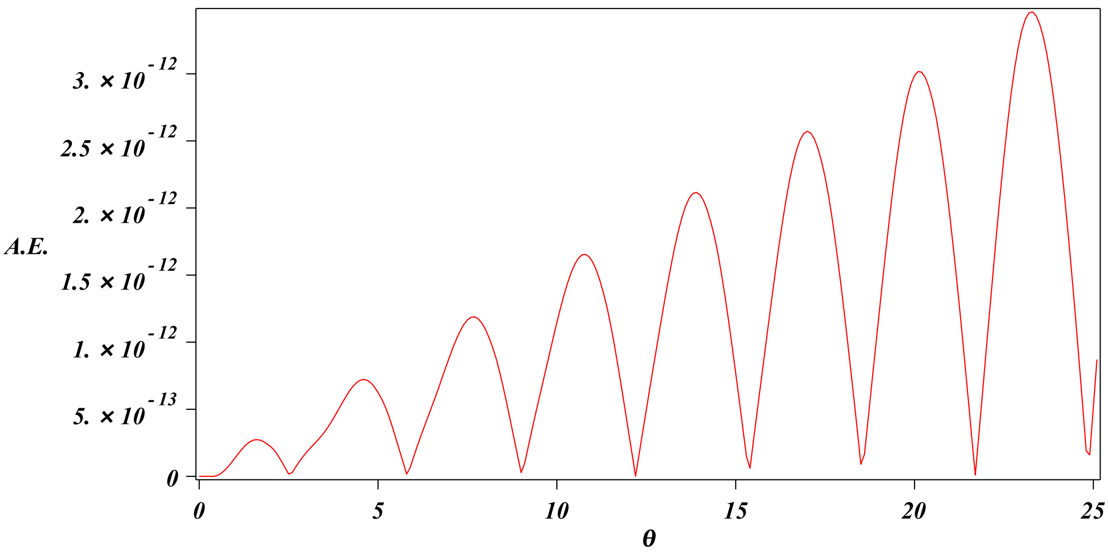

Figure 3: Absolute error for HPM with respect to RKF45 numerical solution

6. Concluding remarks

In this paper, HPM method is applied to construct approximated analytical solutions for the model for a golf ball trajectory and Mercury's orbit. The numerical experiments and error analysis are presented to support the theoretical results. Our solutions agree well with the pure numerical solutions.

Acknowledgements

All the authors express our appreciation to the late Professor Jose Antonio Agustin Perez-Sesma whose contribution to this work was of great significance. The authors would like to express their gratitude to Roberto Ruiz-Gomez for his contribution to this paper.

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