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LQR and H_2 Controllers Design Using State Derivative Feedback for Multivariable Systems

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Abstract

This paper presents the design of LQR (linear quadratic regular) and H_2 controller using state derivative feedback. This design is solvable for all controllable systems. The state derivative feedback is used instead of state feedback in many mechanical systems because the main sensors of vibration are accelerometers. A multivariable active suspension system is used in this paper to show the effectiveness of the proposed controllers. The obtained results are compared to the same approaches when a state feedback is used. It is shown that the design using state derivative feedback can achieve a better performance.

***Keywords*:** LQR control; H_2 control; state derivative feedback; multivariable systems; active suspension.

1. Introduction

The state derivative feedback is very useful and essential for achieving a desired specification for some control problems. The motivation of using state derivative feedback comes from controlled vibration suspension of mechanical systems where the accelerometers represent the main sensors of vibration [1]. Different approaches that are based on state feedback have been extended to be designed using state derivative feedback. Linear quadratic regulator (LQR) is considered one of the well-known approaches that provide practical feedback gains. This method has adopted either feedback or derivative feedback controller and it provides a perfect stabilization for an active suspension system [2]. The LQR approach can achieve an acceptable performance of the system by minimizing the performance index [3]. Based on LQR, some of new control algorithms have been derived such as in [1,4].

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The H_2 is used to find the optimal gain matrices the achieve the desired performance. The H_2 optimal control is used in the design of state feedback control by minimizing a quadratic performance index of the system and attenuating the effect of disturbances. Reference [5] have used the state derivative feedback for direct algorithm for the pole placement for multi input linear system. Reference [6] used state derivative feedback for Pole-placement for single input single output systems. Cardim and his colleagues [7] used state derivative feedback for linear control systems. Kataria and his colleagues [8] used state derivative feedback for Pole-placement problem. Wang and his colleagues [9] have used the state feedback H_2 control with regional pole assignment. Reference [10] presented a technique based on state derivative for robust vibration control of dynamical systems.

In this paper, the design of LQR and H_2 controllers are presented using state derivative feedback. The proposed controllers are applied to a multivariable active suspension.

2. Controllers Design

In this section, the solutions of LQR optimal control and H_2 robust control using state derivative feedback are presented.

2.1. LQR State Derivative Feedback Problem Formulation

Consider a continuous, time-invariant, linear system:

$$\dot{x}(t)= Ax(t) + Bu(t) \qquad (1)$$

The objective is to stabilize the system by means of a linear state derivative feedback expressed by:

$$u(t) = -K\dot{x}(t) \qquad (2)$$

The control law in equation (2) is to stabilize the system with a desired performance. The closed-loop system dynamics is:

$$\dot{x}(t)=A_c\, x(t) \qquad (3)$$

where

$$A_c=(I + BK)^{-1}A \qquad (4)$$

The stabilizing control with good dynamic behavior is achieved by minimizing a quadratic cost or performance index of the type [1]:

$$J(\dot{x}(t), u(t)=\int_0^{\infty}(\dot{x}(t)Qx(t) + u^T(t)Ru(t))dt \qquad (5)$$

Substituting equation (2) into J, the performance index is:

$$J = \int_0^\infty (\dot{x}Qx + (K\dot{x})^T R(K\dot{x}))dt = \int_0^\infty \dot{x}^T (Q + K^T RK)\dot{x}dt \qquad (6)$$

Suppose that a constant positive semidefinite symmetric matrix P that satisfy equation (6) can be obtained, thus

$$\dot{x}^T(Q + K^T RK)\dot{x} = -\frac{d}{dt}(x^T Px) = -\dot{x}^T Px - x^T P\dot{x} \qquad (7)$$

then, equation (7) can be rewritten as:

$$\dot{x}^T(Q + K^T RK)\dot{x} = -\dot{x}^T(PA_c^{-1} + A_c^{-T}P)\dot{x} \qquad (8)$$

where

$$A_c^{-1} = A^{-1}(I + BK) = A^{-1} + A^{-1}BK \qquad (9)$$

Comparing both sides of equation (8),

$$PA_c^{-1} + A_c^{-T}P + K^T RK + Q = 0 \qquad (10)$$

where

$$A_c^{-T} = K^T B^T A^{-T} + A^{-T} \qquad (11)$$

Substituting equation (9) and (11) in equation (10),

$$P(A^{-1} + A^{-1}BK) + (K^T B^T A^{-T} + A^{-T})P + K^T RK + Q = 0 \qquad (12)$$

then, equation (12) can be rewritten as:

$$PA^{-1} + PA^{-1}BK + K^T B^T A^{-T}P + A^{-T}P + K^T RK + Q = 0 \qquad (13)$$

Since R is positive-definite symmetric matrix, then

$$R = T^T T \qquad (14)$$

where T is a nonsingular matrix. Substituting equation (14) in equation (13), yields:

$$PA^{-1} + A^{-T}P + (TK + T^{-T}B^T A^{-T}P)^T(TK + T^{-T}B^T A^{-T}P) - PA^{-1}BT^{-1}T^{-T}B^T A^{-T}P + Q = 0 \qquad (15)$$

The minimization of J requires the minimization of the following:

$$\dot{x}^T(TK + T^{-T}B^T A^{-T}P)^T(TK + T^{-T}B^T A^{-T}P)\dot{x} \qquad (16)$$

Since the last expression is nonnegative, the minimum occurs when it is zero, then

$$TK = -T^{-T}B^TA^{-T}P \tag{17}$$

The optimal gain matrix K is:

$$K = -R^{-1}B^TA^{-T}P \tag{18}$$

Finally, the optimal stabilizing control law is given by:

$$u(t) = R^{-1}B^TA^{-T}P\dot{x}(t) \tag{19}$$

The matrix P in equation (19) must satisfy equation (13) or the following algebraic Riccati equation (ARE):

$$PA^{-1} + A^{-T}P - PA^{-1}BR^{-1}B^TA^{-T}P + Q = 0 \tag{20}$$

2.2. H_2 State Derivative Feedback Problem Formulation

Consider a linear time invariant system expressed by:

$$\dot{x}(t) = Ax(t) + B_1d(t) + B_2u(t) \tag{21}$$

$$e(t) = C_1\dot{x}(t) + D_{12}u(t) \tag{22}$$

$$Z(t) = \dot{x}(t) \tag{23}$$

The following assumptions are made:

1. The system matrix A is of full rank.
2. (A, B_1) and (A, B_2) are stabilizable.
3. (C_1, A) is detectable.
4. All state derivative measurements are possible.

The objective of this work is to obtain a scalar state derivative feedback control law described by:

$$u(t) = -K\dot{x}(t) \tag{24}$$

Assuming that $d(t)$ is the white noise vector with unity intensity, then [11]:

$$\|T_{ed}\|_{H_2}^2 = E(e^T(t)e(t)) \tag{25}$$

where T_{ed} represents the overall transfer function $d(t)$ to $e(t)$, then

$$e^T e = \dot{x}^T {C_1}^T C_1 \dot{x} + 2\dot{x}^T {C_1}^T D_{12} u + u^T {D_{12}}^T D_{12} u \tag{26}$$

The minimization of $\|T_{ed}\|_{H_2}^2$ is equivalent to the solution of the stochastic regulator problem by setting:

$Q = {C_1}^T C_1$, $N = {C_1}^T D_{12}$, $R = {D_{12}}^T D_{12}$

then

$$E\big(e^T(t)e(t)\big) = J\big(\dot{x}(t), u(t)\big) = \int_0^\infty (\dot{x}^T(t) Q x(\mathrm{t}) + 2\dot{x}^T(\mathrm{t}) N u(\mathrm{t}) + u^T(\mathrm{t}) R u(\mathrm{t}))dt \tag{27}$$

and

$$J\big(\dot{x}(t), v(t)\big) = \int_0^\infty (\dot{x}^T(t) Q_m \dot{x}(\mathrm{t}) + v^T(\mathrm{t}) R v(\mathrm{t}))dt \tag{28}$$

where

$$Q_m = Q - NR^{-1}N^T \tag{29}$$

$$v(\mathrm{t}) = u(\mathrm{t}) + R^{-1}N^T \dot{x}(t) \tag{30}$$

Consequently, the system in equation (21) will be rewritten as:

$$\dot{x}(t) = A_m x(t) + B_1 d(t) + B_2 v(t) \tag{31}$$

where

$$A_m = A - B_2 R^{-1} N^T \tag{32}$$

In term of $v(t)$ and from equation (30), the optimal state derivative feedback is:

$$v(t) = -K_m \dot{x}(t) \tag{33}$$

where

$$K_m = K - R^{-1}N^T \tag{34}$$

Substitute equation (33) in equation (31), the system equation will be:

$$\dot{x}(t) = A_m x(t) + B_1 d(t) - B_2 \mathrm{K}_m \dot{x}(t) = A_n x(t) + B_1 d(t) \tag{35}$$

where

$$A_n = (I + B_2 \mathrm{K}_m)^{-1} A_m \tag{36}$$

Substitute equation (33) in equation (28), the objective function will be:

$$J\big(\mathrm{x}(t), v(t)\big) = \int_0^\infty (\dot{x}^T(t)(Q_m + K_m^T R K_m)x(\mathrm{t}))dt \qquad (37)$$

Suppose that, it can be found a constant positive simidefinite symmetric P that satisfy equation (37),

$$\dot{x}^T(t)(Q_m + K_m^T R \mathrm{K}_m)\dot{x}(\mathrm{t}) = -\frac{d}{dt}\big(x^T(t)Px(t)\big) = -\dot{x}^T(t)Px(t) - x^T(t)P\dot{x}(t) \qquad (38)$$

Therefore, the performance index can be obtained as:

$$J\big(\dot{x}(t), v(t)\big) = \int_0^\infty (\dot{x}^T(t)Q_m\dot{x}(\mathrm{t}) + (K_m\dot{x}(\mathrm{t}))^T R(K_m\dot{x}(\mathrm{t})))dt = -x^T(t)Px(t)|_0^\infty = -x^T(\infty)PX(\infty) + x^T(0)Px(0) \qquad (39)$$

Assume that the closed loop system is asymptotically stable, then $x(\infty) \rightarrow 0$. Therefore the performance index can be obtained in terms of initial conditions and matrix P as:

$$J = x^T(0)Px(0) \qquad (40)$$

From equation (35), the following relationship can be obtained:

$$x(t) = A_n^{-1}\dot{x}(t) + B_1 d(t) \qquad (41)$$

where

$$A_n^{-1} = A_m^{-1}(I + B_2 K_m) \qquad (42)$$

Then equation (38) can be rewritten as:

$$\dot{x}^T(t)(Q_m + K_m^T R K_m)\dot{x}(\mathrm{t}) = -\dot{x}^T(t)(PA_n^{-1} + A_n^{-T}P)\dot{x}(\mathrm{t}) \qquad (43)$$

By comparing the two sides of equation (43), we obtain:

$$PA_n^{-1} + A_n^{-1}P + K_m^T R K_m + Q_m = 0 \qquad (44)$$

Substituting equation (42) in equation (44), one can obtain:

$$P(A_m^{-1}(I + B_2 K_m)) + (A_m^{-1}(I + B_2 K_m))^T P + K_m^T R K_m + Q_m = 0 \qquad (45)$$

then, equation (45) can be rewritten as:

$$PA_m^{-1} + A_m^{-T}P + PA_m^{-T}B_2 K_m + K_m^T B_2^T A_m^{-T}P + K_m^T R K_m + Q_m = 0 \qquad (46)$$

Since R is positive definite symmetric matrix, then $R = T^T T$, where T is nonsingular matrix. Equation (46) can be rewritten as:

$$PA_m^{-1} + A_m^{-T}P + PA_m^{-1}B_2K_m + K_m^TB_2^TA_m^{-T}P + K_m^TT^TTK_m + Q_m = 0 \quad (47)$$

By reformulating equation (47), the following equation can be obtained:

$$PA_m^{-1} + A_m^{-T}P\ (TK_m + T^{-T}B_2^TA_m^{-T}P)^T(TK_m + T^{-T}B_2^TA_m^{-T}P) - PA_m^{-1}B_2R^{-1}B_2^TA_m^{-T}P + Q_m = 0 \quad (48)$$

The minimization of J requires the minimization of

$$\dot{x}^T(TK + T^{-T}B^TA^{-T}P)^T(TK + T^{-T}B^TA^{-T}P)\dot{x} \quad (49)$$

Since the last expression is nonnegative, the minimum occurs when it is zero

$$TK_m + T^{-T}B_2^TA_m^{-T}P = 0 \quad (50)$$

The optimal gain matrix K_m is:

$$K_m = -T^{-1}T^{-T}B_2^TA_m^{-T}P = -R^{-1}B_2^TA_m^{-T}P \quad (51)$$

Finally, the optimal stabilizing control is:

$$v(t) = -K_m\dot{x}(t) = R^{-1}B_2^TA_m^{-T}P\dot{x}(t) \quad (52)$$

Substituting equation (52) in equation (30) yields:

$$u(t) = R^{-1}B_2^TA_m^{-T}P\dot{x}(t) - R^{-1}N^T\dot{x}(t) \quad (53)$$

then

$$K = -R^{-1}[B_2^T(A - B_2R^{-1}N^T)^{-T}P - N^T] \quad (54)$$

The equations of the closed loop system using state derivative feedback H_2 control are:

$$\dot{x}(t) = A_cx(t) + B_1d(t) + B_cr(t) \quad (55)$$

$$y(t) = Cx(t) \quad (56)$$

where

$$A_c = (I + B_2K)^{-1}(A - B_2C) \quad (57)$$

$$B_c = (I + B_2K)^{-1}B_2 \quad (58)$$

3. Illustrative Example

A multivariable active suspension system, shown in Figure 1 is used to show the effectiveness of the proposed controllers. The system dynamics can be represented by a state space model as [7]:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{M_c} & \frac{k_2}{M_c} & \frac{-b_1 - b_2}{M_c} & \frac{b_2}{M_c} \\ \frac{k_2}{m_s} & \frac{-k_2}{m_s} & \frac{b_2}{m_s} & \frac{-b_2}{m_s} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{M_c} & \frac{-1}{M_c} \\ 0 & \frac{1}{m_s} \end{bmatrix} u(t) \qquad (59)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \qquad (60)$$

where M_c represents a car mass, m_s represents the driver plus seat mass. The stiffness k_1 and the damping b_1 represent the shock absorbers by which the vertical vibration caused by a street may be partially attenuated. The stiffness k_2 and the damping b2 represent the car seat suspension elements by which the undesirable vibrations subjected to the driver can be reduced. The control inputs $u_1(t)$ and $u_2(t)$.can be changed to increase the damping of vibration of the masses M_c and m_s.

The accelerations signals $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$ are only available for feedback because they are measured by accelerometers sensors. Depending on their measured time derivatives, the velocities $\dot{x}_1(t)$ and $\dot{x}_2(t)$ can be estimated. Now, the accelerations and velocities signals are available and the proposed method can be used to solve the problem.

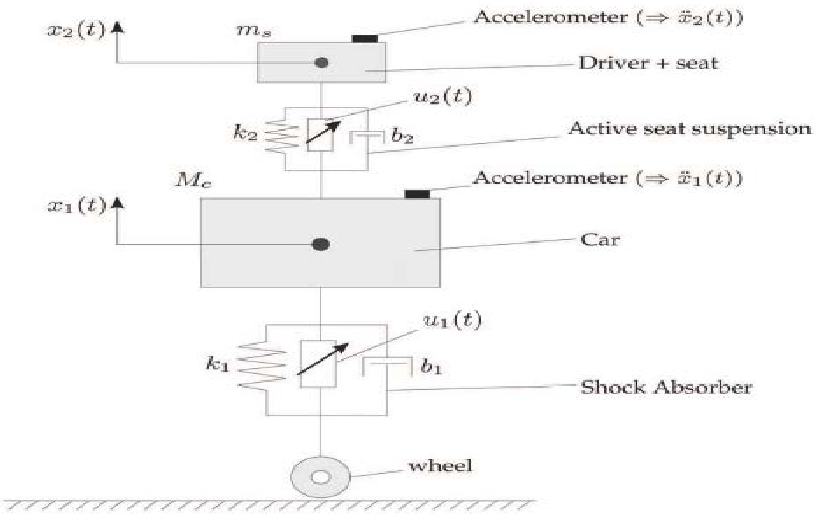

Figure 1: Active suspension of a car seat [7].

The nominal system parameters are taken as follows [7]: b_1(damping) $= 4 \times 10^3$ *Ns/m*, b_2(damper of the seat suspension) $= 5 \times 10^2$ *Ns/m*, k_1 (stiffness)$= 4 \times 10^4$ *N/m*, k_2 (stiffness)$= 5 \times 10^3$ *N/m*, M_C (mass of the car)$=$ 1500 kg, m_s(mass of the driver) $=$ 70 kg.

3.1. LQR Controller Results

Figure 2 shows the system states trajectories when state feedback LQR control and state derivative feedback LQR control are applied. It shows that the response obtained using state derivative feedback LQR control is fast with small oscillation amplitudes in comparison to that obtained using state feedback LQR control. The performance index weighting matrices Q and R for state feedback LQR control and state derivative feedback LQR control are chosen as $\boldsymbol{Q}$ = diag$\{8 \times 10^7$, 0.568475001, 0.2×10^{-8}, 0.8521111$\}$ and $\boldsymbol{R}$ = diag$\{1, 1\}$. The resulting feedback gain matrices in cases, state feedback LQR control and state derivative feedback LQR control respectively are:

$$K = \begin{bmatrix} 935.1726 & 50.8625 & 343.2930 & 18.9456 \\ -417.4379 & -0.2854 & 62.8865 & 6.4995 \end{bmatrix}$$

$$K = \begin{bmatrix} 4.6024\times10^3 & -0.2590\times10^3 & -1.4695\times10^3 & -0.0582\times10^3 \\ 1.1000\times10^3 & 0.0629\times10^3 & 0.1273\times10^3 & -0.0072\times10^3 \end{bmatrix}$$

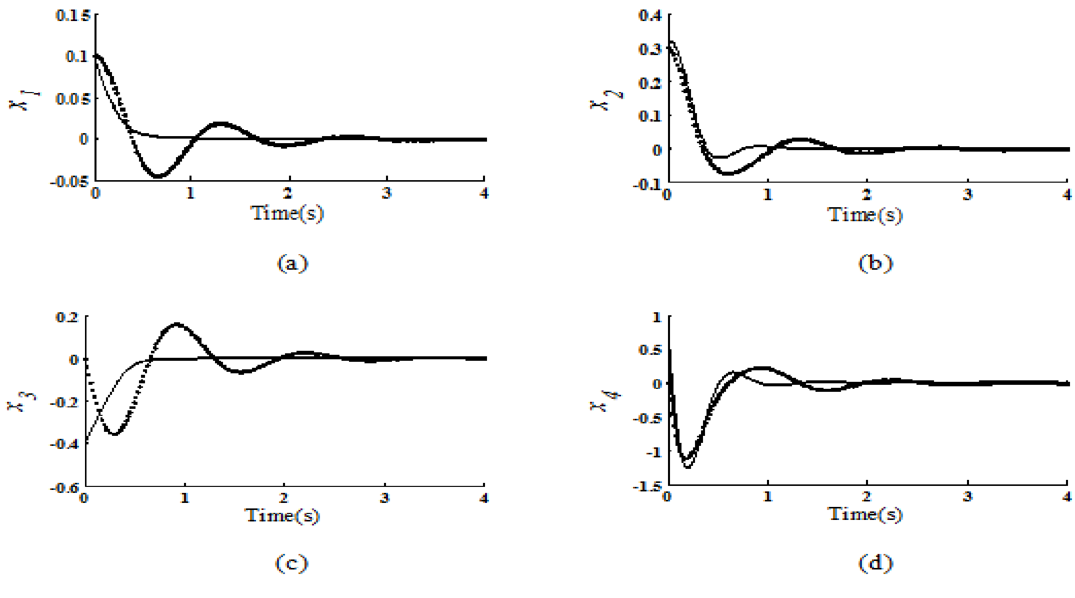

Figure 2: System trajectories using state feedback LQR control (dotted line) and state derivative feedback LQR control (solid line).

3.2. H_2 Controller Results

Figure 4 shows the system states trajectories when state feedback H_2 control and state derivative feedback H_2 control are applied. It shows that the response obtained using state derivative feedback H_2 control is fast with small oscillation amplitudes in comparison to that obtained using state feedback H_2 control. The performance index weighting matrices Q and R for state feedback H_2 control and state derivative feedback H_2 control are chosen as $\boldsymbol{Q}$ = diag{100,100, 1, 1} and $\boldsymbol{R}$ = diag{0.01, 100}. The resulting feedback gain matrices in cases, state feedback H_2 control and state derivative feedback H_2 control respectively are:

$$K = \begin{bmatrix} 6.4641\times10^{3} & 9.3209\times10^{3} & 0.9903\times10^{3} & 0.1164\times10^{3} \\ 0.0021\times10^{3} & 0.0011\times10^{3} & 0.0012\times10^{3} & 0.0001\times10^{3} \end{bmatrix}$$

$$K = \begin{bmatrix} 2.9414\times10^{3} & 9.6049\times10^{3} & -1.1078\times10^{3} & -0.2162\times10^{3} \\ -0.0101\times10^{3} & -0.0119\times10^{3} & 0.0008\times10^{3} & -0.0005\times10^{3} \end{bmatrix}$$

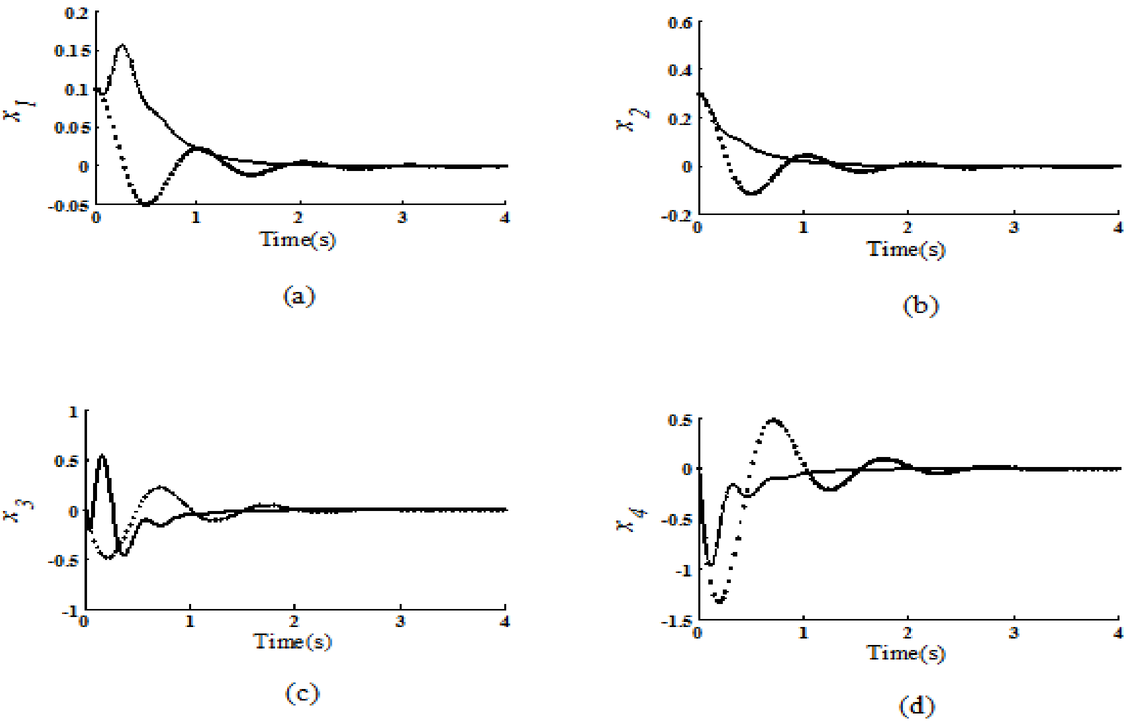

Figure 4: System trajectories using state feedback H_2 control (dotted line) and state derivative feedback H_2 control (solid line).

4. Conclusion

In this paper the LQR and H_2 controllers have been designed using state derivative feedback. The H_2 optimal control has been derived using state derivative feedback similar to LQR to find the optimal gain matrices that achieve the desired performance. The two designed approaches were applied to a multivariable active suspension system. It was found that the designed LQR and H_2 controllers using state derivative feedback can given a better performance in comparison to the same approaches using state feedback.

References

[1]. T. H. S. Abdelaziz and M. Valask, “State Derivative Feedback By LQR For Linear Time-Invariant System”, Proceedings of the 16^{th} IFAC World Congress, Czech Republic, 2005, pp. 933-938.

[2]. M. Pourebrahim and A. S. Ghafari, “Designing a LQR Controller for an Electro-Hydraulic-Actuated-Clutch Model”, International conference on Control Science and Systems Engineering, 2016.

[3]. H. I. Ali, “Mixed LQR/H-Infinity Controller Design For Uncertain Multivariable Systems”, Emirates Journal for Engineering Research, 2015, Vol. 20, No. 1, PP. 79-85.

[4]. Rodrigues C. R. and Kuiava R. and Ramos R.A., “Design of a linear quadratic regulator for nonlinear systems modeled via norm bounded linear differential inclusions”, Proceedings of the 18th World Congress, Milano(Italy), 2011.

[5]. T. H. S. Abdelaziz and M.Valask, “Direct Algorithm For Pole Placement By State-Derivative Feedback For Multi-Input Linear Systems - Nonsingular Case”, Kybernetika, 2005, Vol. 41, No. 5, pp. 637-660.

[6]. T. H. S. Abdelaziz and M.Valask, “Pole-placement for SISO linear systems by state-derivative feedback”, Acta Polytechnica, 2003, Vol. 43, No. 6, pp. 52-60.

[7]. R. Cardim, M. C. M. Teixeira, E. Assuncao and F. A. Faria, “Control Designs for Linear Systems Using State-Derivative Feedback”, Systems Structure and Control, Pert Husek, 2008.

[8]. J. Kataria, M. K. Madhav and A. Kumar, “State Derivative Feedback Control Application for Pole Placement Problem”, International Journal of Emerging Technology, 2014, Vol. 4, No. 4, pp. 79-85.

[9]. G. S. Wang, B. Liang and G. R. Duan, “H2-Optimal Control with Regional Pole Assignment via State Feedback, International Journal of Control”, Automation, and Systems, 2006, Vol. 4, No. 5, pp. 653-659.

[10]. E. Reithmeier and G. Leitmann, “Robust Vibration Control of Dynamical Systems based on Derivative of the State”, Archive Appl. Mechanics,, 2003, Vol. 72, PP. 856-864.

[11]. A. Sinha, “Linear Systems Optimal and Robust Control”, Taylor and Francis Group, LLC, 2007.