

Even Degree Deficient Spline Interpolation

V. N. Jha^{a*}, K. S. Nisar^b, Arun Kumar^c

^{a,b}Department of Mathematics, College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Aldwasir, Kingdom of Saudi Arabia

^cDepartment of Mathematics & Computer Science, RD University, Jabalpur, India

Abstract

In this present paper we study the general even degree spline $S \in C^m[0,1]$, i.e. the spline of degree $2m$, where m is the positive integer, matches derivatives upto the order of m at the knots of uniform partition. Reference[6], have been constructed an interpolating quartic spline with matching first and second derivative of a given function at the knots. A similar study was made by [5], for even degree splines upto degree 10. Further, it was conjectured by [5], that higher degree splines can be obtained in a similar way. They also raised a question for getting a proof for general degree splines. We provide a proof for general degree spline of degree $2m$. Explicit formula for these splines are obtained. Error estimation to these splines in terms of Chebyshev norm is also represented by using the result due to [2]. On combining the result of this paper and the result obtained by [4] with some modification we get deficient spline of general degree for approximation. The deficient splines are found useful because of the fact that, in this case we require less continuity requirements (see De Boor [3], P. 125). The restrictions of smoothness are compensated by considering additional interpolatory conditions.

Keywords: Even degree; Deficient; Spline interpolation.

1. Introduction

Let $\Delta: 0 = x_0 < x_1 < \dots < x_n = 1$, with $x_i - x_{i-1} = h, i = 1, 2, \dots, n$. If π_m denotes the polynomials of degree not greater than m . We define the class of all $2m$ degree deficient splines over Δ with deficiency of order r

$$S(2m, \Delta, r) = \{S : S(x) \in C^{2m-r}[0,1] \cap \pi_{2m}, x \in [x_i, x_{i+1}], i = 1, 2, \dots, n-1\}.$$

Here we consider deficient splines with deficiency m . We claim the following:

* Corresponding author.

2. Existence and Uniqueness

We write $(x - x_i) / h = q$, for $i \in \{0, 1, \dots, n\}$, $0 \leq q \leq 1$; the value of $s(x)$ as s_i at x_i and $s^{(j)}(x_i) = f_i^{(j)}$.

Theorem 2.1 There exists a function of unique splines $s(x) \in C^m[0, 1]$, of degree $2m$, with matching conditions,

$$s_i^{(r)} = f_i^{(r)}, r = 1, 2, \dots, m, \tag{1}$$

For $i = 0, 1, \dots, n$ and the boundary condition $s_0 = f_0$. It has the following representation in the sub interval $[x_i, x_{i+1}]$, $i = 0, 1, \dots, n - 1$,

$$s(x) = s_i T_{m,1}(q) + s_{i+1} T_{m,2}(q) + \sum_{j=1}^{m-1} h^j \{ f_i^{(j)} T_{m,2j+1}(q) + f_{i+1}^{(j)} T_{m,2j+2}(q) \} + h^m f_i^{(m)} T_{m,2m+1}(q), \tag{2}$$

where $T_{m,2j+1}(q) = \frac{1}{j!} \sum_{k=0}^{m-j} \binom{m-1+k}{k} \left[\sum_{i=0}^m (-1)^i \binom{m}{i} q^{j+k+i} \right]$, $j = 0, 1, \dots, m$,

$$T_{m,2j+2}(q) = \frac{1}{j!} \sum_{k=0}^{m-1-j} (-1)^k \binom{m+k}{k} \binom{2m-j}{m-1-j-k} \left[\sum_{i=0}^j (-1)^i \binom{j}{i} q^{m+1+k+i} \right], j = 0, 1, \dots, m - 1.$$

We observe that the functions $T_{m,1}, T_{m,2}, \dots, T_{m,2m+2}$, are linearly independent polynomials of π_{2m} .

We need the following lemma for proof of the theorem.

Lemma. We have

- (i) $\sum_{k=0}^{r-j} (-1)^k \binom{m-1+k}{k} \binom{m}{r-j-k} = 0,$
- (ii) $\sum_{k=0}^{m-1-j} (-1)^k \binom{m+k}{k} \binom{2m-r}{m-1-j-k} = 1, \text{ for } j = r$
 $= 0, \text{ for } j < r.$
- (iii) $\sum_{k=0}^{m-j} \binom{m-1+k}{k} = \binom{2m-j}{m-j},$
- (iv) $(-1)^{m-j-i} \sum_{k=0}^{m-i-j} (-1)^k \binom{m}{m-i-j-k} \binom{m+k}{k} = 1, \text{ for } m \geq i + j.$

Proof of Lemma. We use the notation that $EB(1-x)^q$ is the binomial expansion.

For (i) we consider $EB(1-x)^{-m} EB(1-x)^m$ and collecting the coefficients for x^{r-j} , we get the result.

We get part (ii) if we compare the coefficients of x^{m-i-j} in the expansion $EB(1-x)^{-m-1} EB(1-x)^{2m-r}$.

For (iii) we compare the coefficients of x^{m-j} in the identity,

$$(1-x^{m+1})EB(1-x)^{-m-1} = (1+x+x^2+\dots+x^m)EB(1-x)^{-m}.$$

The last result follows when we compare the coefficients of x^{m-1-j} in the expansion of

$$EB(1-x)^{-m-1} = (1-x)^m = EB(1-x)^{-1}.$$

Proof of Theorem 2.1 Let D denote the differential operator with respect to x , $D^r = D(D^{r-1}), r < 0, (D^0 = 1)$. We observe that $D(q) = 1/h$. We have for $r = 1, 2, \dots, m-1$,

$$D^r T_{m,2j+1}(q) = \frac{1}{j!} \sum_{k=0}^{m-j} \binom{m-1+k}{k} \sum_{i=0}^m (-1)^i \binom{m}{i} D^r q^{j+k+i}.$$

In case $r < j$, at $q = 0$,

$$D^r T_{m,2j+1}(q) = 0. \tag{3}$$

For the case $r = j$, the term $D^r q^{j+k+1}$, at $q = 0$ is not zero if and only if $k = i = 0$, we get

$$D^r T_{m,2j+1}(q) = h^{-r}, \text{ at } q = 0. \tag{4}$$

For the case $r > j$, and $i = r - j - k$. But $i \geq 0$, i.e. $k \leq r - j$ we obtain

$$D^r T_{m,2j+1}(q) = (-1)^{r-j} \frac{r!}{j! h^r} \sum_{k=0}^{r-j} (-1)^k \binom{m-1+k}{k} \binom{m}{r-j-k} = 0 \tag{5}$$

by the result (i) of the lemma.

It is direct to see that

$$D^r T_{m,2j+2}(q) = 0, \text{ at } q = 0. \tag{6}$$

From the foregoing observations (2), yields

$$s^{(r)}(x_i) = f^{(r)}(x_i), \quad r = 1, 2, \dots, m-1, \quad i = 0, 1, \dots, n,$$

from the definition of $s(x)$ in $[x_i, x_{i+1}]$. Now we claim

$$D^r T_{m,2j+1}(q) = 0, \quad \text{at } q = 1. \tag{7}$$

We can write

$$D^r T_{m,2j+1}(q) = \frac{1}{j!} \sum_{k=0}^{m-j} \binom{m-1+k}{k} q^{j+k} (1-q)^m.$$

Using Leibnitz's formula for successive differentiation, we get

$$D^r T_{m,2j+1}(q) = \frac{1}{j!} \sum_{k=0}^{m-j} \binom{m-1+k}{k} \sum_{l=0}^r \binom{r}{l} D^{r-l} (1-q)^m D^l q^{j+k},$$

which establishes the claim. Writing

$$\sum_{i=0}^j (-1)^i \binom{j}{i} q^i = (1-q)^j,$$

we get

$$D^r T_{m,2j+2}(q) = \frac{1}{j!} \sum_{k=0}^{m-1-j} (-1)^k \binom{m+k}{k} \binom{2m-j}{m-1-j-k} \sum_{l=0}^r \binom{r}{l} D^{r-l} (q-1)^j D^l q^{m+1+k}.$$

If $j > r$, the above expression is zero at $q = 1$. Now let $j \leq r$ the term $D^{r-l} (q-1)^j$ at $q = 1$ is not zero if $j = r - l$ i.e. $l = r - j$. We have

$$\begin{aligned} D^r T_{m,2j+2}(q) &= \binom{r}{r-j} \frac{(2m-j)!}{(2m-r)! h^r} \sum_{k=0}^{m-1-j} (-1)^k \binom{m+k}{k} \binom{2m-r}{m-1-j-k}, \\ &= \frac{1}{h^r}, \quad \text{for } j = r, \quad \text{at } q = 1; \end{aligned} \tag{8}$$

and $D^r T_{m,2m+2}(q) = 0$, for $j < r$ by lemma (ii).

Whence $s^{(r)}(x_{i+1}) = f^{(r)}(x_{i+1})$, from the definition of $s(x)$ in $[x_i, x_{i+1}]$.

Hence we get $s(x) \in C^{m-1}[0,1]$.

Now we proceed to find the recurrence relation for explicit computation of s_i , by using the condition that $s(x) \in C^{m-1}[0,1]$. The m -th differential coefficient of $T_{m,2j+1}(q)$ and $T_{m,2j+2}(q)$ at $q = 0, j = 0, 1, \dots, m$, are given below:

$$D^r T_{m,2j+1}(q) = \frac{1}{j!} \sum_{k=0}^{m-j} \binom{m-1+k}{k} \sum_{i=0}^m (-1)^i \binom{m}{i} D^r q^{j+k+i},$$

$$= \frac{1}{h^m}, \text{ for } j = m. \tag{9}$$

and $T_{m,2j+2}(q) = 0$, at $q = 0$. (10)

The above differential coefficients at $q=1$, i.e.

$$D^r T_{m,2j+1}(q) = \frac{(-1)^m m!}{j! h^m} \sum_{k=0}^{m-j} \binom{m-1+k}{k}$$

$$= \frac{(-1)^m (2m-j)!}{j!(m-j)! h^m}, \text{ at } q = 1. \tag{11}$$

By lemma (iii), and at $q = 1$.

$$D^m T_{m,2j+2}(q) = \frac{1}{j!} \sum_{k=0}^{m-1-j} (-1)^k \binom{m+k}{k} \binom{2m-j}{m-1-j-k} \sum_{l=0}^m \binom{m}{l} D^{m-l} q^{m+1+k} D^l (q-1)^j.$$

The term $D^l (q-1)^j$ is not zero only if $l = j$; hence

$$D^m T_{m,2j+2}(q) = \frac{(2m-j)!}{j!(m-j)! h^m} \sum_{k=0}^{m-1-j} (-1)^k \binom{m+k}{k} \binom{m}{m-1-j-k},$$

$$= \frac{(-1)^{m-1-j} (2m-j)!}{j!(m-j)! h^m}, \tag{12}$$

by lemma (iv).

Thus the condition get $s(x) \in C^m[0,1]$, that is $s_i^{(m)}(x_i) = s_{i-1}^{(m)}(x_i), i = 1, 2, \dots, n$ gives

$$s_i - s_{i-1} = \frac{m!}{(2m)!} \sum_{j=1}^m (2m-j)! h^j \frac{\{f_{i-1}^{(j)} + (-1)^{j+1} f_i^{(j)}\}}{j!(m-j)!},$$

$$= F_{m,i} \text{ (say)} \tag{13}$$

Using the boundary condition $s_0 = f_0$, we obtain

$$s_i = \sum_{r=1}^i F_{m,r} + f_0, i = 1, 2, \dots, n.$$

3. Error Bounds

Now we shall obtain error bound for the above splines and its derivatives. For this we use the following result due to Cirlet, Schultz and Varga [2].

Let $f \in C^{2m}[0, h]$ be given. Let Q_{2m-1} be the unique Hermite interpolant polynomial of degree $2m-1$ that matches g and its first $m-1$ derivatives $g^{(r)}$ at 0 and h . Then the error $e^{(r)} = g^{(r)} - Q_{2m-1}^{(r)}$, for $0 \leq x \leq h$ ([5], P. 161)

$$|e^{(r)}(x)| = \frac{h^r [x(h-x)]^{m-r} F}{r!(2m-2r)!}, r = 0, 1, \dots, m, \tag{14}$$

where $F = \max_{0 \leq x \leq h} |g^{(2m)}(x)|, 0 \leq x \leq h$.

The bounds in (14) are best possible for $r = 0$ only. For some values of m ($m=2$ and $m = 3$) optimal error bounds on the derivatives $e^{(r)}(x)$ do exist due to [1], (see also Varma and Howell [7]). Let $s'(x)$ be Hermite interpolation polynomial of degree $2m-1$ matching $f^{(r)}, r = 1, 2, \dots, m$ at x_i and x_{i+1} , therefore for any $x \in [x_i, x_{i+1}]$, we have by (14), with $g = f', Q_{2m-1} = s'$,

$$|s^{(r+1)} - f^{(r+1)}| \leq \frac{h^r [(x-x_i)(x_{i+1}-x)]^{m-r}}{r!(m-2r)!} \|f^{(2m+1)}\|_{\infty}, r = 0, 1, \dots, m.$$

By the simple calculation we get

$$|s^{(r+1)} - f^{(r+1)}| \leq \frac{h^r [h^2 q(1-q)]^{m-r}}{r!(m-2r)!} \|f^{(2m+1)}\|_\infty, \quad r = 0, 1, \dots, m.$$

Where $x = x_i + qh$. Then for $0 \leq x \leq 1$, we have for $q=1/2$,

$$|s^{(r+1)} - f^{(r+1)}| \leq \frac{h^{2m-r}}{4^{m-r} r!(2m-r)!} \|f^{(2m+1)}\|_\infty, \quad r = 0, 1, \dots, m. \tag{15}$$

Provided $f \in C^{2m+1}[0,1]$, using $s_0 = f_0$, and integrating equation (15) over $[0, x]$ for $r = 0$, we obtain

$$|s(x) - f(x)| \leq \frac{h^{2m}}{4^m (2m)!} \|f^{(2m+1)}\|_\infty.$$

Thus we have proved the following:

Theorem 3.1 If $s \in C^m[0,1]$ is the Hermite interpolating polynomial of degree $2m-1$, matching the derivatives $f^{(r)}, r = 0, 1, \dots, m$ at the knots x_i and x_{i+1} . Then for $f \in C^{2m+1}[0,1]$, and $x \in [x_i, x_{i+1}]$,

$$|s^{(r+1)} - f^{(r+1)}| \leq K(r) h^{2m-r} \|f^{(2m+1)}\|_\infty,$$

Where $K(r) = [4^{m-r} r!(2m-r)!]^{-1}$, $r = 0, 1, \dots, m$ is constant depending on r .

4. Conclusion

In this paper we studied the existence and uniqueness of a class of general even degree deficient spline the matching derivatives at the knots to a given order. We have also obtained explicit formula with error estimation of approximation. In study of statistical distributions we are getting such type of integrals

$$f(x) = \int_a^x f'(t)dt, \text{ for } x \in [a, b]$$

which allow us to approximate and the above obtained spline establish a new class of numerical quadrature rules.

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