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Improved Rotated Finite Difference Method for Solving Fractional Elliptic Partial Differential Equations

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Abstract

Real life problems with fractional partial differential equations (FPDE's) are of great importance, since fractional differential equations accumulate the whole information of the function in a weighted form. This has many applications in physics, chemistry, engineering, etc. For that reason, we need a method for solving such equations, effectively, easy use and applied for different problems. The objective of this paper is to solve fractional elliptic partial differential equations, by using new accelerated version of rotated five point's approximation method. Experiment results of the test problem are given in order to confirm the superiority of our proposed method.

Keywords: Rotated Finite Difference Approximation Method; Fractional Elliptic Partial Differential Equations.

1. Introduction

Fast computational methods for solving partial differential equations using finite difference schemes derived from skewed (rotated) difference operators have been extensively investigated over the years. These Iterative methods based on the rotated finite difference approximations have been shown to be much faster than the methods based on the standard five-point formula in solving the partial differential equations which is due to the formers' overall lower computational complexities ([1,2,3,4,5]). Fractional Partial Differential Equations (FPDE's) can be seen as a generalization of the classical partial differential equations (PDE's) in the sense that it takes into account the memory and hereditary properties of the physical phenomena ([6,7]). As it was in the classical PDE's there is no general method that can be used in solving FPDE's. Numerical solution of FPDE's has received great progress in the recent years ([8,9]).

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The time and space-fractional partial differential equation describe transport dynamics in complex systems governed by anomalous dispersion and non-exponential relaxation [10].

Because of complexity in the theoretic analysis of numerical approximation of fractional systems, the common approach is to apply the finite difference method to discretize fractional derivative operators, and then obtain the numerical solutions of the fractional partial differential equations. Furthermore, Goloviznin and his colleagues [11] developed a numerical method for solving some 1-D equations with fractional derivatives.

The paper is organized in five sections: Section 2 describes the formulation of the Rotated Point Iterative Method for solving the fractional Poisson's equation. In Section 3, the proposed accelerated version of rotated five point's approximation method will be given. In Section 4, the numerical results are presented in order to show the efficiency of the new proposed method. Finally, the conclusion is given in Section 5.

2. Formulation of the Rotated Point Iterative Method for solving the Fractional Poisson's equation

Consider the Poisson's equation in the form:

$$-(\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\frac{\partial^{\alpha}}{\partial y^{\alpha}})u(x,y)=F(x,y),\qquad (x,y)\in D \tag{2.1}$$

Where: $D=\{(x,y):(x,y)\in[0,L]\times[0,L]\}$.

Subject to the Dirichlet-boundary conditions:

$$u(0,y)=u(x,0)=u(L,y)=0, \text{ and } u(x,L)=g(x)$$

Beibalaev and his colleagues [8] considered the fractional Poisson's equation in the form:

$$-(\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\frac{\partial^{\alpha}}{\partial y^{\alpha}})u(x,y)=F(x,y), \text{ where } 1\le\alpha\le 2, \tag{2.2}$$

subject to the same Dirichlet-boundary conditions of equation (2.1). Now, we consider the corresponding fractional order Elliptic by the form:

$$-(\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\frac{\partial^{\beta}}{\partial y^{\beta}})u(x,y)=F(x,y), \text{ where } (1\le\alpha,\beta\le 2) \tag{2.3}$$

It can be seen that equation (2.3) is a generalization to equations (2.1) and (2.2).

The simplest standard five-point finite difference approximation of the Laplacian is

$$\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2}+\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{h^2}-\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)-O(h^4)=f_{ij} \tag{2.4}$$

Here, $u_{ij}=u(x_i,y_j)$. Another approximation to equation (2.1) can be derived from the rotated five-point finite difference approximation to give [1]

$$\begin{aligned}&\frac{u_{i+1,j+1}+u_{i-1,j-1}+u_{i+1,j-1}+u_{i-1,j+1}-4u_{ij}}{2h^2}\\&-h^2\left[\frac{1}{2}\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{1}{12}\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)\right]-O(h^4)=f_{ij}.\end{aligned} \tag{2.5}$$

In order to obtain the finite difference approximation of the fractional order equation (2.2), we use the treatment introduced in [8] for approximate Caputo's fractional derivative of order $\alpha,\ (1\le\alpha\le 2)$ in the form:

$${}^{c}D^{\alpha}_{x_i^+}u(x_i,y_j)\approx\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{(3-\alpha)h^{\alpha}}, \tag{2.6a}$$

$${}^{c}D^{\alpha}_{y_j^+}u(x_i,y_j)\approx\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{(3-\alpha)h^{\alpha}} \tag{2.6b}$$

We can observe that for equation (2.1), the corresponding finite difference approximation of the Caputo's fractional order derivative of order α is:

$$\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{(3-\alpha)h^{\alpha}}\approx{}^{c}D^{\alpha}_{x_i^+}u(x_i,y_j)+\frac{h^{4-\alpha}\max\left|u^{(4)}(x_i)\right|}{12(3-\alpha)}$$

(2.7a)

and

$$\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{(3-\alpha)h^{\alpha}}\approx{}^{c}D^{\alpha}_{y_j^+}u(x_i,y_j)+\frac{h^{4-\alpha}\max\left|u^{(4)}(x_i)\right|}{12(3-\alpha)} \tag{2.7b}$$

Therefore, the standard five-point finite difference approximation of equation (2.2) can be written as:

$$\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{(3-\alpha)h^{\alpha}}+\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{(3-\alpha)h^{\alpha}}=-f_{ij}$$

which can be rearranged as in the form

$$4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = (3-\alpha)h^{\alpha} f_{ij} \tag{2.8}$$

By the same manner, the rotated five-point finite difference approximation can be written as:

$$4u_{ij} - u_{i+1,j+1} - u_{i-1,j-1} - u_{i-1,j+1} - u_{i+1,j-1} = (3-\alpha)h^{\alpha} f_{ij} \tag{2.9}$$

There are two ways to approximating equation (2.3), the first one by using Caputo's formula which is replaced by a finite sum of integrals at the discretization points, and approximate the second order derivative by using the standard five-point finite difference formula (2.8) or rotated five-point finite difference formula (2.9). If the standard five-point finite difference is used, then the finite difference formula of Caputo's fractional derivative will take the form:

$${}_0^c D_x^{\alpha} u(x,y) \cong \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{i-1} \int_{x_k}^{x_{k+1}} \frac{\partial^2 u(s,y)}{\partial s^2} (x-s)^{2-\alpha-1} ds$$

$${}_0^c D_x^{\alpha} u_{i,j} \cong \frac{1}{h^{\alpha}\Gamma(3-\alpha)} \sum_{k=0}^{i-1} b_k^{\alpha} (u_{i-k+1,j} - 2u_{i-k,j} + u_{i-k-1,j}) \tag{2.10}$$

where $b_k^{\alpha} = [(k+1)^{2-\alpha} - k^{2-\alpha}]$, let $b_s^{*} = \dfrac{-b_s^{\beta}}{h^{\beta}\Gamma(3-\beta)}$, $b_k = \dfrac{-b_k^{\alpha}}{h^{\alpha}\Gamma(3-\alpha)}$

Then, the finite difference scheme for equation (2.3) will be given in the form:

$$\sum_{k=0}^{i-1} b_k (u_{i-k+1,j} - 2u_{i-k,j} + u_{i-k-1,j}) + \sum_{s=0}^{j-1} b_s^{*} (u_{i,j-s+1} - 2u_{i,j-s} + u_{i,j-s-1}) = f_{ij}$$

where $f_{ij} = f(x_i, y_j)$. If $\alpha = \beta$, we can see that: $b_s^{*} = b_k = \dfrac{-1}{h^{\alpha}\Gamma(3-\alpha)}\ b^{\alpha}$.

The second way to approximating equation (2.3) by using Grunwald-Letnikov (G-L) approximation [12] as the following:

$${}_0^{R-L} D_x^{\alpha} u(x_i, y_j) = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)} \sum_{k=0}^{N} w_k^{\alpha} u_{N-k,j}, \qquad w_k^{\alpha} = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}$$

$${}_0^c D_x^{\alpha} u(x,y) = {}_0^{R-L} D_x^{\alpha} u(x,y) - \sum_{k=0}^{N} \frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)} \frac{\partial^k u(x,y)\big|_{x=0}}{\partial x^k},$$

$$ {}_{0}^{G-L}D_{x}^{\alpha}u(x_i,y_j)=\frac{1}{h^{\alpha}}\sum_{k=0}^{i+1}g_k^{\alpha}u_{i-k+1,j}+o(h). $$

By using the standard five-point finite difference approximation (2.8), equation (2.3) can be written in the form:

$$ \sum_{k=0}^{i+1}g_k^{\alpha}u_{i-k+1,j}+\sum_{s=0}^{j+1}g_s^{\beta}u_{j-s+1,i}=f_{ij} \qquad (2.11) $$

where $g_k^{\alpha}=-r_1z_k^{\alpha};\ g_s^{\beta}=-r_2z_k^{\beta};\ r_1=\frac{1}{h^{\alpha}};\ r_2=\frac{1}{h^{\beta}}z_k^{\beta};\ z_0^{\gamma}=1;\ z_1^{\gamma}=-\gamma;$

$$ z_k^{\gamma}=\frac{\gamma(\gamma-1)...(\gamma-k+1)}{k!};\quad k>1 $$

Furthermore, by using the rotated five-point finite difference approximation (2.9), equation (2.3) can be written in the form:

$$ \sum_{k=0}^{i+1}g_k^{\alpha}u_{i-k+1,j+1}+\sum_{s=0}^{j+1}g_s^{\beta}u_{j-s+1,i+1}=f_{ij} \qquad (2.12) $$

where $g_k^{\alpha}=-r_1z_k^{\alpha};\ g_s^{\beta}=-r_2z_k^{\beta};\ r_1=\frac{1}{h^{\alpha}};\ r_2=\frac{1}{h^{\beta}}z_k^{\beta};\ z_0^{\gamma}=1;\ z_1^{\gamma}=-\gamma;$

$$ z_k^{\gamma}=\frac{\gamma(\gamma-1)...(\gamma-k+1)}{k!};\quad k>1 $$

3. The proposed accelerated version of rotated five point's approximation method

It's well known that in the finite difference treatment the PDE's or the FPDE's are replaced by an algebraic system of equations which can be written as the form

$$ A\bar{u}=\bar{f}, \qquad (3.1) $$

where, A is $(\mathrm{N}-1)^2\times(\mathrm{N}-1)^2$ coefficients matrix, $\bar{u}$ and $\bar{f}$ are two $(\mathrm{N}-1)^2\times 1$ matrices, where $\bar{u}=[u_{j,1},u_{j,2},...,u_{j,N-1}]^T$ and $\bar{f}=[f_{j,1},f_{j,2},...,f_{j,N-1}]^T,\ \ j=1,2,...,N-1.$

It is well known that the computational molecule of standard finite difference approximation for the classical (integer) case of PDE's can be represented as in figure 1 whereas the computational molecule of standard finite difference approximation for FPDE's can be represented as in figure 2.

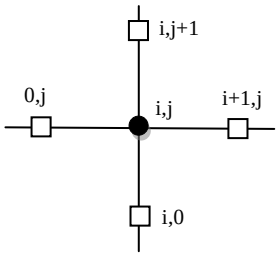

Figure 1: Computational molecule of Eq. (2.4)

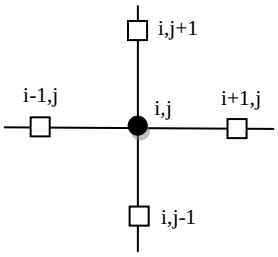

Figure 2: Computational molecule of Eq. (2.8)

Also, we can observe that for the rotated five-point finite difference approximation the following transformations take place

$$i\,,\, j \pm 1 \rightarrow i \pm 1,\, j \pm 1$$

$$i \pm 1,\, j \rightarrow i \pm 1,\, j \mp 1$$

$$h \rightarrow \sqrt{2}h.$$

Therefore, the computational molecule of the rotated five-point finite difference approximation for PDE and FPDE can be shown in figure 3 and figure 4 respectively.

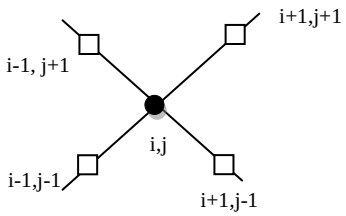

Figure 3: Computational molecule of Eq. (2.5)

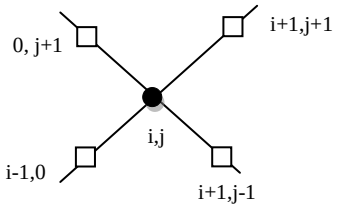

Figure 4: Computational molecule of Eq. (2.9)

It is clear that the coefficients matrix for the fractional order case 0f equations (2.8) and (2.9) have the same structure as in equations (2.4) and (2.5) except the free column $\overline{f}$ in the right side of system (3.1). Theoretically, it can be seen that the coefficients matrices resulting from systems (2.4) and (2.5) are non-singulars, so the system (3.1) has unique solution ([5]). In addition to that the coefficients matrix A is strictly diagonal dominant, then A is non-singular and systems (2.8) and (2.9) have a unique solution for $(1 < \alpha, \beta < 2)$.

Since it is well known that preconditioners play a vital role in accelerating the convergence rates of iterative methods, several preconditioned strategies have been used for improving the convergence rate of the iterative methods derived from the standard and skewed (rotated) finite difference operators ([4], [5]). A well-designed

preconditioning of the PDE and FPDE problems reduces the number of iterations to reach convergence. Dramatic improvements are possible, but the difficulty is to construct the suitable preconditioner. In general, a good preconditioner should satisfy the following prosperities: the first one is that, the preconditioned system should be easy to solve and the second one is that the preconditioner should be cheap to construct and apply. Usually the system (3.1) is large and the matrix A is sparse. Furthermore, matrix A can be write as

$$A = D - L - U \tag{3.2}$$

where *D* is diagonal matrix *A*, $-L$ is strictly lower triangular parts of *A* and $-U$ is

strictly upper triangular parts of *A*. A preconditioner $(I + ML)$ where $0 \le M < 2$ is used to modify the original system (3.1) to the following system:

$$(I + ML)A\bar{u} = (I + ML)\overline{f} \tag{3.3}$$

A preconditioner $P = (I + ML)$ is a matrix that transforms the original system (3.1) into new system (3.3) that is equivalent in the sense that it has the same solution, but that has more favourable spectral properties.

4. Numerical Results and Discussion

In the first part of this section, we have compared between the spectral radiuses of the iteration matrix corresponding to the resulting system of rotated five-point finite difference (original system) and the preconditioned system for different values of α, β such that $(1 \le \alpha, \beta \le 2)$. Table 1 shows the comparison of the spectral radius between the original and the preconditioned systems. Clearly it can be seen that the spectral radius of the preconditioned system is smaller compared to the original system, thus justifying the superiority of the preconditioned system against the original system which is quite agreeing with the results obtained in the previous work ([3,4,8]).

Table 1: Comparison of the spectral radiuses of the iteration matrix corresponding to the original and the preconditioned systems

N	order		Original system	Preconditioned system
	α	β	$\rho(A)$	$\rho(PA)$
42	1.2	1.4	0.053	0.041
74	1.4	1.4	0.091	0.083
114	1.6	1.4	0.141	0.112
186	1.8	1.4	0.184	0.171

The second part of this section has discussed the numerical solution of the following modal problem using the proposed iterative method:

$${}^{c}D^{\beta}_{x^{+}}u(x,y)+{}^{c}D^{\beta}_{y^{+}}u(x,y)=-10[4+\pi^{2}(x-2x^{2})]\sin\pi y$$

defined in $D=\{0<x<0.5,\ 0<y<1\}$ with the boundary condition $u_{\Gamma}=0$ such that: Γ is the boundary of the area D, and the fractional order is $1<\beta\leq 2$.

Table 2: Solution of the modal problem for $\beta=1.5$ by the original and the preconditioned systems

	Original system						**Preconditioned system**					
y \ x	0	0.1	0.2	0.3	0.4	0.5	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	1.846	2.831	3.046	1.984	0	0	1.561	2.503	2.883	1.552	0
0.2	0	2.692	5.411	5.662	3.784	0	0	2.341	4.854	5.404	3.334	0
0.3	0	3.147	7.604	7.934	4.527	0	0	2.733	6.931	7.663	4.416	0
0.4	0	4.533	7.599	7.962	5.681	0	0	3.883	7.201	7.533	5.334	0
0.5	0	5.703	8.104	8.544	6.571	0	0	4.914	7.995	8.212	6.204	0
0.6	0	6.342	8.212	8.704	7.364	0	0	5.683	8.113	8.673	6.677	0
0.7	0	5.425	6.425	6.225	6.225	0	0	4.842	5.524	5.791	5.875	0
0.8	0	3.843	4.511	4.341	4.542	0	0	2.661	3.641	3.973	4.341	0
0.9	0	1.846	2.831	3.046	1.984	0	0	1.543	2.492	2.785	1.473	0
1	0	0	0	0	0	0	0	0	0	0	0	0

Numerical data of the original and the preconditioned systems are summarized in tables (2-3) for two fractional orders for $\beta=1.5$ and $\beta=2$. It can be observed that in all cases illustrated the increase in the fractional order $\beta=2$ reduces the magnitudes of the peaks.

Throughout the two sections of our experiments, the results reveal that the proposed preconditioned is superior to the original system in solving fractional elliptic partial differential equations.

Table (3): Solution of the modal problem for $\beta = 2$ by the original and the preconditioned systems

	Original system						**Preconditioned system**					
y \ x	0	0.1	0.2	0.3	0.4	0.5	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	0.246	0.387	0.365	0.234	0	0	0.203	0.324	0.324	0.179	0
0.2	0	0.383	0.522	0.534	0.461	0	0	0.334	0.462	0.475	0.403	0
0.3	0	0.527	0.924	0.964	0.625	0	0	0.503	0.856	0.922	0.556	0
0.4	0	0.533	1.024	1.013	0.691	0	0	0.481	0.987	0.989	0.603	0
0.5	0	0.564	1.171	1.142	0.717	0	0	0.524	1.043	1.031	0.601	0
0.6	0	0.531	1.141	1.122	0.677	0	0	0.499	1.044	0.987	0.662	0
0.7	0	0.424	0.993	0.976	0.504	0	0	0.375	0.985	0.921	0.478	0
0.8	0	0.363	0.651	0.691	0.463	0	0	0.322	0.622	0.584	0.357	0
0.9	0	0.231	0.373	0.362	0.234	0	0	0.203	0.323	0.304	0.178	0
1	0	0	0	0	0	0	0	0	0	0	0	0

5. Conclusion and Future Works

In this paper, we have formulated new preconditioned iterative method based on rotated finite difference method for solving fractional elliptic partial differential equations. From observation of all experimental results, it can be concluded that the proposed scheme may be a good alternative to solve fractional elliptic partial differential equation and many other numerical problems. The idea of this proposed method can be extended to group iterative solver which will be reported separately in the future.

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References

[1] A. R. Abdullah, “The Four Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver” International Journal of Comp. Math. vol. 38, pp. 61-70, 1991.

[2] A. Gunawardena, S. Jain, L. Snyder, ”Modified iterative methods for consistent linear systems”. Linear Algebra Appl.vol. 154/156, pp 123–143, 1991.

[3] A. Ibrahim and A. R. Abdullah, "Solving the Two Dimensional Diffusion Equation by the Four Point Explicit Decoupled Group (EDG) Iterative Method". ,International Journal of Comp. Math. vol. 58, pp 253-263, 1994.

[4] A. M. Saeed and N. H. M. Ali, "Preconditioned Modified Explicit Decoupled Group Method In The Solution Of Elliptic PDEs". Applied Mathematical Sciences.vol. 4 (24), pp. 1165-1181, 2010.

[5] A. M. Saeed, N. H. M. Ali, "On the Convergence of the Preconditioned Group Rotated Iterative Methods In The Solution of Elliptic PDEs", Applied Mathematics &Information Sciences, vol. 5(1), pp. 65-73, 2011.

[6] S. I. Muslih, O. P. Agrawal, "Riesz Fractional Derivatives and Dimensional Space", Int J Theor Phys, vol. 49(2), pp. 270-275, 2010.

[7] J. Hristov, "Approximate Solutions to Fractional Subdiffusion Equations", European Physical Journal, vol. 193(1) pp. 229-243, 2011.

[8] V. D. Beibalaev, P. Ruslan, Meilanov, "The Dirihlet Problem for The Fractional Poisson's Equation With Caputo Derivatives: Afinite Difference Approximation and A Numerical Solution", Thermal Science, vol. 16(2), pp.385-394, 2012.

[9] Z. B. Li, J. H. He, "Fractional Complex Transform for Fractional Differential Equations", Mathematical and Computational Applications, vol. 15(5), pp. 970-973, 2010.

[10] R. Metzler, J. Klafter, "The Random Walk's Guide to Anomalous Diffusion: A Fractional Dynamics Approach". Phys. Rep., vol.339(1) , pp.1-77, 2000.

[11] V. M. Goloviznin, I. A. Korotkin, "Methods of the Numerical Solutions Some One-Dimensional Equations with Fractional Derivatives", Differential Equations, vol.42(7), pp. 21-130, 2006.

[12] K. B. oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, USA, 1974.